Characterizations of rings and modules by means of lattices.

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CHARACTERIZATIONS OF RINGS AND MODULES BY MEANS OF LATTICES

A thesis for the Ph.D. degree of the University of London

by

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ABSTRACT

In this thesis we study the relationship between the lattice of submodules and the algebraic structure of a module. The key remark in our study will be the fact that the homomorphisms between two independent submodules of a module can be 'represented' by elements of its lattice of submodules. Exploiting this fact we show that the endomorphism ring of a module which is the direct sum of more than three isomorphic submodules is determined up to isomorphism by its lattice of submodules.

Lattice isomorphisms arise naturally in two ways, viz., through category equivalences and semi-linear isomorphisms. Any lattice isomorphism between a free module of infinite rank and a module containing at least one free submodule is shown to be induced by a category equivalence. This result is used to give new characterizations of Morita equivalence.

If certain mild conditions are satisfied a lattice isomorphism between a free module of rank ≥ 3 and a faithful module is shown to give rise to a semi-linear isomorphism between the modules. If both modules are free of rank $n \geq 3$ then the question of whether there is a semi-linear isomorphism between them is equivalent to asking when an isomorphism of matrix rings $R_n \cong S_n$ implies a ring isomorphism $R \cong S$.

We study rings R with this property for any n and any ring S. The following are shown to be of this type (1) commutative rings (2) p-trivial rings (3) matrix rings over strongly regular rings (4) left self-injective rings.

Applying these results we give new **examples** of regular rings which uniquely co-ordinatize a complemented modular lattice of order ≥ 3. In particular we show such a co-ordinatization is always unique to within injective hull.

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CONTENTS

INTRODUCTION

Associated with any module there are a number of important algebraic structures and it is of independent mathematical interest to study the interrelationship between these structures and the part they play in determining the structure of the module itself. In this thesis we attempt to carry out this programme for one of the most important structures of a module namely its lattice of submodules.

Our method of attack will be to study the consequences of assuming that two modules have isomorphic lattices of submodules. It will turn out (see chap.2) that in very general situations this will imply that their endomorphism rings are isomorphic. Our main interest however will be in the relationship between the module structures of the modules.

We consider two main ways in which lattice isomorphisms arise, viz., category equivalences and semi-linear isomorphisms. These will be our canonical lattice isomorphisms - so to speak our yard-stick. Our aim will be to find conditions on our modules to ensure that any lattice isomorphism between them is canonical. However before going into further details we will need some notation and definitions.

Notation

Unless otherwise stated all rings will be assumed to contain an identity element $1 \neq 0$ and all modules will be assumed to be unital.

We shall use the notation ${}_RM$ for modules where the suffix R indicates that R is a ring and its position on the left of M indicates that M is to be considered as a left R-module. Likewise we shall denote a right R-module by M_R. For a ring R we denote the category of all left R-modules by ${}_R\mathcal{M}$.

We adopt the convention of letting the endomorphisms of a module act on the opposite side to the ring of operators. Hence if ${}_RM$ is a module with $S = \mathrm{End}_R(M)$ then S acts on the right of M and we (as we often shall) can consider M as a right S-module.

If ${}_RM$ is a module we shall denote its lattice of submodules by $L({}_RM)$ and by the notation $\Sigma : L({}_RM) \cong L({}_SN)$ we shall understand that Σ is a lattice isomorphism between the lattices of submodules of the modules ${}_RM$ and ${}_SN$. The lattice isomorphism $L({}_RM) \cong L({}_RM)$ defined by $P \longrightarrow P$ is called the identity lattice isomorphism and will be denoted by $1_{L({}_RM)}$.

If R is a ring and I a non-empty set then R^I will denote the direct product of I copies of R and ${}^IR$ the direct sum, i.e., R^I is the set of all maps $I \longrightarrow R$ and ${}^IR$ is the subset consisting of all maps which are zero on all but a finite number of elements of I. If I is finite with n elements then R^I and ${}^IR$ coincide and will be denoted by R^n. We will write R_n for the matrix ring of rank n over R.

Throughout we use the convention that integral domains need not necessarily be commutative. References will be listed in numerical order under each author.

Category equivalences

Let R and S be rings and ${}_R\mu$, ${}_S\mu$ be the categories of all left R- and S-modules respectively. The categories ${}_R\mu$ and ${}_S\mu$ are said to be equivalent if there are functors $F: {}_R\mu \longrightarrow {}_S\mu$, $G: {}_S\mu \longrightarrow {}_R\mu$ and natural equivalences $FG \simeq 1$, $GF \simeq 1$. The rings R and S are said to be Morita equivalent, $R \underset{M}{\sim} S$, if the categories ${}_R\mu$ and ${}_S\mu$ are equivalent.

Now suppose that R and S are rings such that $R \underset{M}{\sim} S$ where $F: {}_R\mu \longrightarrow {}_S\mu$ is the corresponding category equivalence. Then if ${}_RM$ is any R-module and ${}_SN = M^F$ it is clear that F induces a lattice isomorphism $L({}_RM) \cong L({}_SN)$.

In chap.3 the converse problem is considered, i.e., given modules ${}_RM$ and ${}_SN$ such that $\Sigma: L({}_RM) \cong L({}_SN)$, we study the circumstances under which it is possible to deduce that $R \underset{M}{\sim} S$ and that the corresponding category equivalence induces Σ. Our investigation of this problem will lead to new characterizations of Morita equivalence and, as far as the author is aware, this is the first time such an investigation has been made.

Semi-linear isomorphisms

Let ${}_RM$ and ${}_SN$ be modules. Suppose $\ell : R \cong S$ is a ring isomorphism and $s : (M, +) \cong (N, +)$ is an abelian group isomorphism then (ℓ, s) is called a semi-linear isomorphism if for any $r \in R$ and any $m \in M$ $(rm)^s = r^\ell m^s$. We will write $(\ell, s) : (R, M) \cong (S, N)$.

It is clear that any semi-linear isomorphism $(\ell, s) : (R, M) \cong (S, N)$ induces a lattice isomorphism $L({}_RM) \cong L({}_SN)$. In chapters 4 - 7 the converse problem is considered, i.e., given modules ${}_RM$ and ${}_SN$ such that $\Sigma : L({}_RM) \cong L({}_SN)$, we study the circumstances under which we can deduce that there is a semi-linear isomorphism $(\ell, s) : (R, M) \cong (S, N)$ and that (ℓ, s) induces Σ.

This aspect of our problem has received rather more attention, particularly in the case when ${}_RM$ and ${}_SN$ are the free modules ${}_RR^n$ and ${}_SS^n$ for some integer n. The question in this case turns out to be equivalent to asking when the co-ordinatization of a geometry (or equivalently a lattice) is unique. For example if R and S are division rings and $\Sigma : L({}_RR^n) \cong L({}_SS^n)$ for some integer $n \geq 3$ then the first fundamental theorem of projective geometry states that there is a semi-linear isomorphism $(\ell, s) : (R, R^n) \cong (S, S^n)$ which induces Σ (see p.44 of Baer (1)).

Von Neumann considered regular rings as a generalization of division rings and showed that the same theorem holds for a particular type of regular ring, the so-called continuous regular rings (see von Neumann (1)). The question as to whether this theorem holds when R and S are any regular rings is unsolved. In chap.6 we consider this problem and extend von Neumann's result and similar results in chap.7 of Skornyakov (1).

In von Neumann (1) there are a number of interesting results which are not explicitly stated (see chap.2 of this thesis for proofs and generalizations of these). For example let R and S be rings and n an integer ≥ 3 then the following results hold.

(1) If $\Sigma : L({}_{R_n}R_n) \cong L({}_S S)$ then $R_n \cong S$.

(2) If $\Sigma : L({}_R R^n) \cong L({}_S S^n)$ then $R_n \cong S_n$.

Result (2) shows that the 'uniqueness of co-ordinatization' problem can be reduced to one of considering isomorphisms of matrix rings, i.e., when does $R_n \cong S_n$ imply $R \cong S$. Several results are known for this problem.

(1) The uniqueness part of the Artin-Wedderburn theorem (or the first fundamental theorem of projective geometry) gives the result if R is a division ring and hence for semi-simple Artinian rings. These results can be generalized to rings which are division rings and Artinian rings modulo their Jacobson radical, i.e., local and semi-primary rings (see e.g. pp.56-59 of Jacobson (1)).

(2) The results in Baer (1) have been generalized in Wolfson (1) to give the result if R and S are principal left ideal domains. We extend results (1) and (2) in chap.5 and chap.6.

In the general case when ${}_RM$ is not free the problem is much harder. There are results in Baer (2) and Baer (3) on abelian groups and vector spaces. The results in Skornyakov (2) generalize those in Baer (2) and the first fundamental theorem of projective geometry. In chap.4 we generalize Skornyakov's results.

CHAPTER I

FINITELY GENERATED MODULES AND INFINITE MATRIX RINGS

In this chapter we consider two separate topics. In section 1 we show that the lattice of submodules of a module is determined by the partially ordered set of its finitely generated submodules. In section 2 we give generalizations of a theorem of von Neumann. In particular we show that there are lattice isomorphisms between the lattices of submodules of direct products and direct sums of copies of a ring R and the lattices of left and right ideals of the various possible infinite matrix rings definable over R.

1. Finitely generated modules

The results in this section are part of the folklore of universal algebra. Lemma 1.1 is given as exercise 7 on p.85 of Cohn (1). Lemma 1.3 does not appear explicitly in the literature, as far as I know, but the construction used in it appears in Birkhoff and Frink (1).

Definition. Let ${}_RM$ be a module. A subset $D \subset L({}_RM)$ is called additively closed if (1) $D \neq \phi$ (2) if $P, Q \in D$ then $P + Q \in D$. D is called an ideal if it satisfies further (3) if $P \in D$ and $Q \subset P$, $Q \in L({}_RM)$, then $Q \in D$.

Lemma 1.1. Let ${}_RM$ be a module and P a submodule. Then P is finitely generated if and only if P is not the sum of elements of an additively closed set D of $L({}_RM)$ which does not contain P.

Proof

Let P be a finitely generated submodule of M generated by the finite set of elements $(a_i)_1^n \in M$. Suppose further $P = \sum_{Q\in D} Q$ where D is an additively closed set of $L({}_RM)$ with $P \notin D$. Then $(a_i)_1^n$ are contained in a finite sum of elements of D and hence in an element Q' of D. Thus $P = \sum_1^n Ra_i \subset Q' \in D$. But $\sum_{Q\in D} Q = P$ and so $Q' \subset P$. Therefore $P = Q' \in D$ - a contradiction.

Suppose conversely that P cannot be written in the form $\sum_{Q\in D} Q$ for any additively closed set D with $P \notin D$. Let D = the additively closed set of finitely generated submodules of P. Then $P = \sum_{Q\in D} Q$ and so by hypothesis $P \in D$, i.e., P is finitely generated.

Definition. Let ${}_RM$ be a module. Then we denote the partially ordered set of finitely generated submodules of M by $F({}_RM)$. By the notation $\Sigma : F({}_RM) \cong F({}_SN)$ we shall understand that Σ is an order preserving set isomorphism between the partially ordered sets of finitely generated submodules of the modules ${}_RM$ and ${}_SN$.

Cor.1. Let ${}_RM$ and ${}_SN$ be modules with $\Sigma : L({}_RM) \cong L({}_SN)$. Then Σ induces $F({}_RM) \cong F({}_SN)$.

Proof

The characterization of finitely generated submodules given in lemma 1.1 is preserved under lattice isomorphism.

It is worth remarking that cyclic modules are not necessarily preserved under lattice isomorphism as the following example shows.

Example 1.2. There exist a non-cyclic module ${}_RM$ and a cyclic module ${}_SN$ such that $L({}_RM) \cong L({}_SN)$.

Proof

Let R be a division ring and n an integer. Then anticipating section 2 we know that $L({}_RR^n) \cong L({}_{R_n}R_n)$. But R_n is a cyclic R_n-module and if $n > 1$ R^n is non-cyclic.

Lemma 1.3. Let ${}_RM$ and ${}_SN$ be modules. Then $L({}_RM) \cong L({}_SN)$ if and only if $F({}_RM) \cong F({}_SN)$.

Proof

(1) If $L({}_RM) \cong L({}_SN)$ then by cor.1 of lemma 1.1 $F({}_RM) \cong F({}_SN)$.

(2) Suppose on the other hand that $F({}_RM) \cong F({}_SN)$. Let $D({}_RM)$ be the set of all ideals of $F({}_RM)$. If S is any subset $\subset D({}_RM)$ then $\bigcap_{s \in S} s$ is an ideal $\in D({}_RM)$ and this is clearly the greatest lower bound of S with respect to the order relation of set inclusion. Hence by a well-known result (see e.g. prop.4.1, chap.1 of Cohn (1)) $D({}_RM)$ is a complete lattice.

Clearly $D({}_RM)$ is completely determined by $F({}_RM)$ and the isomorphism $F({}_RM) \cong F({}_SN)$ can be extended to a lattice isomorphism $D({}_RM) \cong D({}_SN)$. To complete the lemma we need only show that $L({}_RM) \cong D({}_RM)$.

Let $F: L({}_RM) \longrightarrow D({}_RM)$ be defined by $P^F = F({}_RP)$ and $G: D({}_RM) \longrightarrow L({}_RM)$ by $D^G = \sum_{Q \in D} Q$. Then $P^{FG} = \sum_{Q \in F({}_RP)} Q$ = sum of all finitely generated submodules of P = P and D^{GF} = set of all finitely generated submodules of $\sum_{Q \in D} Q$. If Q' is a finitely generated submodule of $\sum_{Q \in D} Q$ then $Q' \subset$ some $Q \in D$. Since D is an ideal this means that $Q' \in D$ and so $D^{GF} = D$.

As F, G are order preserving and FG = 1, GF = 1 they are inverse lattice isomorphisms. Hence $L({}_RM) \cong D({}_RM)$ and $F({}_RM) \cong F({}_SN)$ can be extended to $L({}_RM) \cong L({}_SN)$.

<u>Cor.1.</u> Let ${}_RM$ and ${}_SN$ be modules and $K({}_RM)$ and $K({}_SN)$ be subsets of $L({}_RM)$ and $L({}_SN)$ respectively. Suppose that $F({}_RM) \subset K({}_RM)$ and $F({}_SN) \subset K({}_SN)$. If $\Sigma: K({}_RM) \cong K({}_SN)$ as partially ordered sets then Σ can be extended to an isomorphism $L({}_RM) \cong L({}_SN)$.

<u>Proof</u>

By lemma 1.1 it is clear that $\Sigma: K({}_RM) \cong K({}_SN)$ induces $\Sigma: F({}_RM) \cong F({}_SN)$. By lemma 1.3 we can extend $\Sigma: F({}_RM) \cong F({}_SN)$ to a lattice isomorphism $:L({}_RM) \cong L({}_SN)$. It is not difficult to see that this induces $\Sigma: K({}_RM) \cong K({}_SN)$.

2. Infinite matrix rings

Let R be a ring and n an integer. Then von Neumann showed (chap.1, part 2 of von Neumann (1)) that there is a lattice isomorphism $L({}_RR^n) \cong L({}_{R_n}R_n)$ (actually von Neumann's proof is stated for division rings but goes through without change for rings with a 1 e.g. see theorem 2 of Skornyakov (1)).

An easy way to see this theorem is to note that for any integer n $R \underset{M}{\sim} R_n$ under the category equivalences $\mathrm{Hom}_R(R^n, -)$: ${}_R\mu \longrightarrow {}_{R_n}\mu$ and $R^n \underset{P_n}{\otimes} - : {}_{R_n}\mu \longrightarrow {}_R\mu$. Now R^n and R_n correspond under these equivalences and so they induce $L({}_RR^n) \cong L({}_{R_n}R_n)$. Though at first sight the lattice isomorphism constructed by von Neumann seems to depend on the basis chosen for R^n, this is not so. In fact his lattice isomorphism is exactly the same as the one given above.

Using the idea of our proof we can now generalize von Neumann's theorem to a certain class of finitely generated projective modules.

Definition. A module ${}_RP$ is called a generator if every left R-module is a homomorphic image of a direct sum of copies of P. We call P a self-generator if every submodule of P is a homomorphic image of a direct sum of copies of P. If P is both a generator and a finitely generated projective module then we call P a progenerator.

We note the following characterization of generators. Namely, ${}_RP$ is a generator if and only if some finite direct sum of copies of P contains R as a direct summand. This follows since if P is a generator then R is a homomorphic image of a direct sum of copies of P and so is a direct summand of this direct sum, which may be taken as finite as R is finitely generated. Conversely it is easily seen that P is a generator if a direct sum of copies of P contain R as a direct summand.

<u>Theorem 1.4</u>. Let ${}_RP$ be a module and $S = End_R(P)$. Define the maps $F: L({}_RP) \longrightarrow L({}_SS)$ and $G: L({}_SS) \longrightarrow L({}_RP)$ by $Q^F = Hom_R(P, Q)$ and $A^G = PA$ for $Q \in L({}_RP)$ and $A \in L({}_SS)$. Then

(1) if P is a self-generator then $FG = 1$

(2) if P is a finitely generated projective then $GF = 1$

(3) if P is a finitely generated projective self-generator then F and G are inverse lattice isomorphisms giving $L({}_RP) \cong L({}_SS)$.

<u>Proof</u>

(1) If Q is a submodule of P then $Q^{FG} = P\ Hom_R(P, Q)$. Suppose P is a self-generator then there is an epimorphism from a direct sum of copies of P to Q. This is equivalent to saying that $Q = \Sigma Pf$ where f runs over $Hom_R(P, Q)$ i.e. $Q = P\ Hom_R(P, Q) = Q^{FG}$. Thus $FG = 1$.

(2) Let $P^* = Hom_R(P, R)$. If $x \in P$ and $u \in P^*$ then define

(a) (x, u) as the element $\in R$ obtained by applying u to x

(b) $[u, x]$ as the element of $S = \mathrm{End}_R(P)$ defined by $p[u, x] = (p, u)x$ for $p \in P$. It is easily verified that $[u, x]$ does belong to S and that for $x, y \in P$ and $a \in S$ satisfies $[u, x + y] = [u, x] + [u, y]$ and $[u, x]a = [u, xa]$.

Suppose now that P is a finitely generated projective module. Then by the dual basis lemma (see e.g. prop.3.1 of chap.7 of Cartan and Eilenberg (1)) there are $u_i \in P^*$, $x_i \in P$, where i runs over some finite set, such that $\Sigma[u_i, x_i] = 1$.

Let A be a left ideal of S. Then $A^{GF} = \mathrm{Hom}_R(P, PA)$ and clearly $A \subset A^{GF}$. If $a \in A^{GF}$ then $Pa \subset PA$ and for each i there are $p_k^i \in P$, $a_k^i \in A$ with $x_i a = \sum_k p_k^i a_k^i$. Hence

$$a = 1.a = \sum_i [u_i, x_i]a = \sum_i [u_i, x_i a]$$

$$= \sum_i [u_i, \sum_k p_k^i a_k^i]$$

$$= \sum_i \sum_k [u_i, p_k^i a_k^i]$$

$$= \sum_i \sum_k [u_i, p_k^i] a_k^i \quad .$$

But $[u_i, p_k^i]a_k^i \in A$ as A is a left ideal. Hence a is the sum of elements of A and so $a \in A$. Thus $A^{GF} \subset A$ and $A = A^{GF}$.

(3) Combining (1) and (2) and noting that F and G are order preserving it follows that F and G are inverse lattice isomorphisms: $L({}_RP) \cong L({}_SS)$ if P is a finitely generated projective self-generator.

We note that von Neumann's theorem follows by putting $P = R^n$ and then $R_n \cong End_R(P)$.

Cor.1. If ${}_RP$ and ${}_SQ$ are finitely generated projective self generators with $U = End_R(P)$ and $V = End_S(Q)$ then $L({}_RP) \cong L({}_SQ)$ if and only if $L({}_UU) \cong L({}_VV)$.

Proof

By theorem 1.4 $L({}_RP) \cong L({}_UU)$ and $L({}_SQ) \cong L({}_VV)$.

Cor.2. If R and S are rings and n, m integers then $L({}_RR^n) \cong L({}_SS^m)$ if and only if $L({}_{R_n}R_n) \cong L({}_{S_m}S_m)$.

Proof

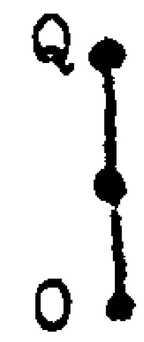
Put $P = R^n$ and $Q = S^m$ in cor.1 of theorem 1.4.

Whether the conditions in theorem 1.4 can be weakened in some way is an open question. However we give examples to show that the theorem does not hold if one or other of the conditions in (1) or (2) are dropped.

Example 1.5. There exists a ring R and a finitely generated generator ${}_RP$ such that $L({}_RP) \not\cong L({}_SS)$, where $S = End_R(P)$.

Proof

Let $R = \mathbb{Z}$ and $P = \mathbb{Z} \oplus \mathbb{Z}/p$, where p is a prime. Then ${}_RP$ is a finitely generated generator (but is not projective). The order of an element of P is either infinite or p. Thus P cannot have a submodule Q with submodule lattice of the form



For Q would have to be cyclic and so isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p$, which is impossible.

Suppose S is any ring and e an idempotent ε S. Then consider matrices of the form $\begin{bmatrix} e\,s\,e & e\,t(1-e) \\ (1-e)\,u\,e & (1-e)v(1-e) \end{bmatrix}$ where s, t, u, v ε S.

These form a ring with respect to the usual matrix addition and multiplication and this ring is isomorphic to S by the map

$$s \longrightarrow \begin{bmatrix} e\,s\,e & e\,s(1-e) \\ (1-e)s\,e & (1-e)s(1-e) \end{bmatrix} .$$

Now consider the case when $S = \mathrm{End}_R(P)$ and e is the projection $P \longrightarrow \mathbb{Z}/p$. Clearly e is an idempotent and

$$e\,S\,e \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p \text{ (as rings)}$$

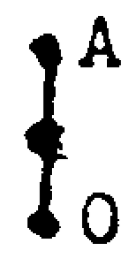
$$e\,S(1-e) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) = 0$$

$$(1-e)S(1-e) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \text{ (as rings)}$$

$$(1-e)S\,e \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/p) \cong \mathbb{Z}/p \text{ (as } \mathbb{Z}\text{-modules)} .$$

Hence we can consider S to be matrices of the form

$\begin{bmatrix}\mathbb{Z}/p & 0\\ \mathbb{Z}/p & \mathbb{Z}\end{bmatrix}$. Let $A = S\begin{bmatrix}1 & o\\ o & o\end{bmatrix}$ then A contains only one non-trivial left ideal namely $S\begin{bmatrix}o & o\\ 1 & o\end{bmatrix}$. Hence A has a submodule lattice of the form



. Therefore $L({}_RP) \not\cong L({}_SS)$.

<u>Example 1.6.</u> There exist finitely generated projective modules ${}_RA$, ${}_SB$, ${}_TC$ such that if $U = \mathrm{End}_R(A)$, $V = \mathrm{End}_S(B)$, $W = \mathrm{End}_T(C)$ then

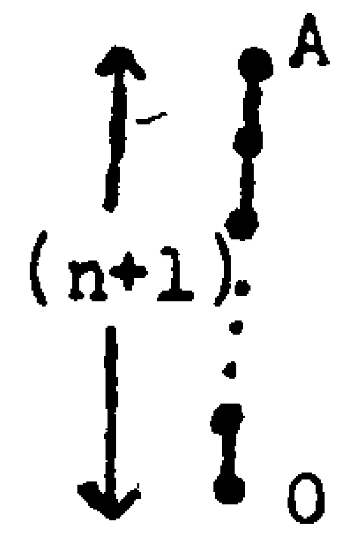
(1) $L({}_RA) \not\cong L({}_UU)$

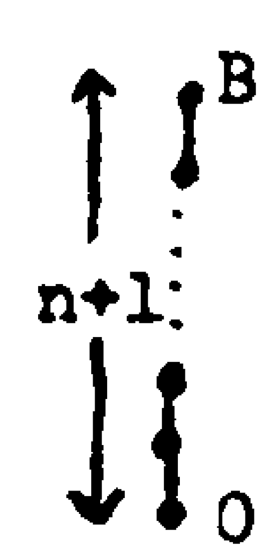
(2) $L({}_RA) \cong L({}_SB)$ but $L({}_UU) \not\cong L({}_VV)$

(3) $U \cong W$ but $L({}_RA) \not\cong L({}_TC)$.

<u>Proof</u>

(1) Let D be a division ring and n an integer > 1. Define $T_n(D)$ to be the $n \times n$ triangular matrices (with zeros above the main diagonal) over D. Let $R = T_n(D)$ and let $A = Re$ where e is the element of R with 1 in the $(1, 1)^{th}$ place and zeros elsewhere. Then e is an idempotent and A is a direct summand of R and so A is a finitely generated projective module. An easy calculation then shows that $L({}_RA)$ is of the

form  . On the other hand $U = End_R(A) \cong eRe \cong D$. So as $n > 1$ it is clear that $L(_RA) \ncong L(_UU)$.

(2) Let $S = B = \mathbb{Z}/p^n$ for some prime p. Then $_SB$ is certainly a finitely generated projective module and $L(_SB)$ is of the form  .
Hence $L(_RA) \cong L(_SB)$. But
$V = End_S(B) \cong \mathbb{Z}/p^n$ and $L(_UU) \ncong L(_VV)$.

(3) Let $T = C = D$. Then $_TC$ is a finitely generated projective module and $W = End_T(C) \cong D \cong U$. But $L(_TC) \ncong L(_RA)$.

We now consider a generalization of von Neumann's theorem in another direction. Firstly we need to define the various possible infinite matrix rings over a ring R.

<u>Definition.</u> Let R be a ring and I a non-empty set. Denote by R_I the set of maps from $I \times I$ to R. If $f, g \in R_I$ then we can define an addition and multiplication on R_I by

$(f+g)(i, j) = f(i, j) + g(i, j)$ and $(f g)(i, j) = \sum_{k \in I} f(i, k)g(k, j)$ for $(i, j) \in I \times I$. The multiplication of course is only well defined if $f(i, k)g(k, j) = o$ for almost all $k \in I$ i.e. for all but a finite number of $k \in I$.

The various subsets of R_I we now define are easily seen to be rings with respect to this multiplication and addition (not strictly rings as some of them do not contain a 1). However with this abuse of language we call them the matrix rings of rank I over R.

Define:

(1) $R_{fI} = \{f \in R_I: f(i, j) = o \text{ for almost all } (i, j)\}$. This is the ring of I × I matrices over R with only a finite number of non-zero entries - the finite matrices of rank I over R.

(2) $R_{rbI} = \{f \in R_I:$ there is a finite subset $D(f) \subset I$ with $f(i, j) = o$ if $j \notin D\}$. This is the ring of I × I matrices whose columns are ~~zero almost everywhere~~ almost all zero - the row bounded matrices of rank I over R.

(3) $R_{cbI} = \{f \in R_I:$ there is a finite subset $D(f) \subset I$ with $f(i, j) = o$ if $i \notin D\}$. This is the ring of I × I matrices whose rows are ~~zero almost everywhere~~ almost all zero - the column bounded matrices of rank I over R.

(4) $R_{rfI} = \{f \in R_I:$ for each $i \in I$ $f(i, j) = o$ for almost all $j\}$. This is the ring of I × I matrices each of whose rows has only a finite number of non-zero entries - the row finite matrices of rank I over R.

(5) $R_{cfI} = \{f \in R_I:$ for each $j \in I$ $f(i, j) = o$ for almost all $i\}$. This is the ring of I × I matrices each of whose columns has only a finite number of non-zero entries - the column finite matrices of rank I over R.

We note that if I is finite with n elements then the rings defined in (1) to (5) all coincide and we get the usual matrix ring R_n.

Lemma 1.7. Let R be a ring and I a set. If ${}_RM(M_R)$ is a free left (right) module of rank I then

(1) $R_{rfI} \cong \mathrm{End}({}_RM)$ and $R_{cfI} \cong \mathrm{End}(M_R)$

(2) $R_{rbI} \cong \{a \in \mathrm{End}({}_RM) : Ma \subset$ finitely generated submodule of ${}_RM\}$

$R_{cbI} \cong \{a \in \mathrm{End}(M_R) : aM \subset$ finitely generated submodule of $M_R\}$

(3) if $(e_i)_{i \in I}$ is a basis for ${}_RM$ then $R_{fI} \cong \{a \in \mathrm{End}({}_RM) : e_i a = o$ for almost all $i \in I\}$.

Proof

(1) Let $(e_i)_{i \in I}$ be a basis for ${}_RM$. If $a \in \mathrm{End}({}_RM)$ then $e_i a = \sum_j a(i, j)e_j$ for some $a(i, j) \in R$ and where $a(i, j) = o$ for almost all $j \in I$. The matrix $[a(i, j)]$ whose $(i, j)^{th}$ entry is $a(i, j)$ is $\in R_{rfI}$ and it is easily verified that the map $a \longrightarrow [a(i, j)]$ gives the required isomorphism. Exactly similarly we can prove $R_{cfI} \cong \mathrm{End}(M_R)$.

(2) Under the isomorphism $R_{rfI} \cong \mathrm{End}({}_RM)$ an element $[a(i, j)] \in R_{rbI}$ is mapped to the endomorphism $a: e_i \longrightarrow \sum_j a(i, j)e_j$ of ${}_RM$. But $a(i, j) = o$ for all j outside some finite set $D \subset I$. Hence $Ma \subset \sum_{j \in D} Re_j$ which is a finitely generated submodule of M.

Conversely if $a \in \mathrm{End}({}_RM)$ and $Ma \subset$ some finitely generated submodule of M then $Ma \subset \sum_{j \in D} Re_j$ for some finite set $D \subset I$. Hence the matrix representation of a is $\in R_{rbI}$. Thus the isomorphism $R_{rfI} \cong \mathrm{End}({}_RM)$ induces the required isomorphism. Similarly we get the result for R_{cbI}.

(3) Under the isomorphism $R_{rfI} \cong \mathrm{End}({}_R M)$ an element $[a(i, j)] \in R_{fI}$ is mapped to the endomorphism $a: e_i \longrightarrow \sum_j a(i, j) e_j$ of ${}_R M$. As $a(i, j) = o$ for almost all (i, j) we have that $e_i a = o$ for almost all i.

Conversely if $a \in \mathrm{End}({}_R M)$ and $e_i a = o$ for almost all i then the matrix representation of $a \in R_{fI}$. Thus the isomorphism $R_{rfI} \cong \mathrm{End}({}_R M)$ induces the required isomorphism.

Lemma 1.8. Let R be a ring and I an infinite set. If J is a set with $|J| \leq |I|$ then $R_{xI} \cong (R_{xI})_{xJ} \cong (R_{xJ})_{xI}$ ~~where x stands for one of f, rf, cf, rb, cb respectively~~.

Proof

Since $|J| \leq |I|$ and I is infinite we have $|J||I| = |I| = |I||J|$. Hence I may be divided into (1) $|I|$ parts of $|J|$ elements or (2) $|J|$ parts of $|I|$ elements. To each of these partitions of I there corresponds a 'block' dissection of any matrix $\in R_{xI}$. Omitting the details it is easy to see that these lead to

(1) $R_{xI} \cong (R_{xJ})_{xI}$ (2) $R_{xI} \cong (R_{xI})_{xJ}$.

Cor.1. If n is an integer and R, I are as in lemma 1.8 then $R_{xI} \cong (R_{xI})_n \cong (R_n)_{xI}$ where $x = f, rf, cf, rb, cb$.

Proof

In lemma 1.8 take J to be a finite set with n elements.

Using von Neumann's method of 'vector set representation' in chap.1 of part 2 of von Neumann (1) we prove the following theorem.

Theorem 1.9. Let R be a ring and I a set. Then we have lattice isomorphisms

(1) $L({}^{I}_{R}R) \cong L({}_{R_{fI}}R_{fI})$

(2) $L({}^{I}R_{R}) \cong L(R_{fI}\ {}_{R_{fI}})$

(3) $L({}_{R}R^{I}) \cong L({}_{R_{cbI}}R_{cbI})$

(4) $L(R^{I}_{R}) \cong L(R_{rbI}\ {}_{R_{rbI}})$

Proof

Let Q be a submodule of ${}^{I}_{R}R$. Define $F: L({}^{I}_{R}R) \longrightarrow L({}_{R_{fI}}R_{fI})$ by $Q^F = (x \in R_{fI}$: the rows of $x \in Q)$. It is easily verified that Q^F is a left ideal of R_{fI}.

Let A be a left ideal of R_{fI}. Define $G: L({}_{R_{fI}}R_{fI}) \longrightarrow L({}^{I}_{R}R)$ by $A^G = (y \in {}^{I}_{R}R$: y is the row of some $x \in A)$. Given $i, j \in I$ define $e(i, j) \in R_{fI}$ to be the matrix with 1 in the $(i, j)^{th}$ place and zeros elsewhere. If $y \in A^G$ is the j^{th} row of an element $x \in A$ then $e(i, j)x \in A$ and has y for its i^{th} row and zero rows otherwise. Using this fact it is easily shown that A^G is a submodule of ${}^{I}_{R}R$.

For any left ideal A of R_{fI} we have $A \subset A^{GF}$. Suppose $x \in A^{GF}$ then x has only a finite number of non-zero rows $(x_i)_{i \in D}$ and each

$x_i \in A^G$, i.e., x_i is the j^{th} row of a matrix $b_j \in A$. Consider $e(i, j)b_j$. This has i^{th} row x_i and zero rows elsewhere and so $x = \sum_{i \in D} e(i, j)b_j$. But $b_j \in$ left ideal A and so $x \in A$ and $A = A^{GF}$.

It is worth noting that this proof would break down if we did not know that x had only a finite number of non-zero rows (otherwise we could not form the sum $\sum_{i \in D} e(i, j)b_j$). F and G could equally well be defined for $L({}^I_R R)$ and $L({}_{R_{rfI}} R_{rfI})$ but we would not get $A^{GF} = A$ in this case. However as can easily be seen in both cases we do get $Q^{FG} = Q$ for $Q \in L({}^I_R R)$.

Thus F and G are order preserving maps such that FG = 1, GF = 1 and so are inverse lattice isomorphisms giving $L({}^I_R R) \cong L({}_{R_{fI}} R_{fI})$.

(2), (3) and (4) are proved in an analogous manner. For the right R-modules ${}^I R_R$ and R^I_R we have to use a column representation and so get right ideals instead of left ideals. When we have the direct product R^I we have to allow vectors to be infinite and so the appropriate rings in these cases are R_{cbI} and R_{rbI} .

Finally we note that if I is finite we get von Neumann's theorem.

CHAPTER 2

ENDOMORPHISM RINGS

In this chapter we prove our basic results. For a given module we study the relationship between its lattice of submodules and its endomorphism ring. In particular we show that if a module is a direct sum of more than three isomorphic submodules then its endomorphism ring is determined up to isomorphism by its lattice of submodules.

The methods used in this chapter are generalizations of those used in chap.4 of part 2 of von Neumann (1) and are closely related to the results in chap.3 of Baer (1) and chap.7 of Skornyakov (1).

Definition. Let ${}_RM$ be a module and A, B, C, submodules. We say A is perspective to B with axis C, $A \underset{C}{\sim} B$, if $A \cap C = B \cap C = 0$ and $A \oplus C = B \oplus C$.

Lemma 2.1. Let ${}_RM$ be a module and $(M_i)_{i \in I}$ an independent set of submodules of M. Suppose i, j are distinct elements of I then define $\overline{M}_{i,j}$ = (all submodules $P \subset M$: $P \underset{M_j}{\sim} M_i$). For any $a \in Hom_R(M_i, M_j)$ define $(a) M_{i,j} = (m - m^a : m \in M_i)$ then

(1) the map $a \longrightarrow (a)M_{i,j}$ is a set isomorphism : $Hom_R(M_i, M_j) \cong \overline{M}_{i,j}$.

(2) if $a \in Hom_R(M_i, M_j)$ then $ker(a) = M_i \cap (a) M_{i,j}$ and a is a monomorphism if and only if $M_i \cap (a) M_{i,j} = 0$.

(3) if $a \in Hom_R(M_i, M_j)$ then image $(a) = [M_i + (a) M_{i,j}] \cap M_j$ and a is an epimorphism if and only if $[M_i + (a) M_{i,j}] \cap M_j = M_j$.

(4) if $a \in \mathrm{Hom}_R(M_i, M_j)$ then a is an isomorphism if and only if $(a)\,M_{i,j} \in \overline{M}_{j,i}$ and in this case $(a)\,M_{i,j} = (a^{-1})M_{j,i}$.

(5) if $a, b \in \mathrm{Hom}_R(M_i, M_j)$ and $M_i^a \cap M_i^b = 0$ then $(a + b)M_{i,j} = [(a)\,M_{i,j} + M_i^b] \cap [(b)M_{i,j} + M_i^a]$.

(6) if i, j, k are distinct elements of I and if $a \in \mathrm{Hom}_R(M_i, M_j)$ and $b \in \mathrm{Hom}_R(M_j, M_k)$ then $(ab)\,M_{i,k} = [M_i + M_k] \cap [(a)M_{i,j} + (b)M_{j,k}]$.

Proof

(1) For any $a \in \mathrm{Hom}_R(M_i, M_j)$ it is clear that $(a)M_{i,j}$ is a submodule of M. Suppose $m \in M_i$ then $m = (m - m^a) + m^a \in (a)M_{i,j} + M_j$. Hence $M_i \subset (a)M_{i,j} + M_j$ and so $M_i \oplus M_j = (a)\,M_{i,j} + M_j$.

Now suppose $z \in (a)\,M_{i,j} \cap M_j$ then $z = m - m^a = n$ for some $m \in M_i$ and $n \in M_j$. Since $M_i \cap M_j = 0$ we have $m = o$ and so $m^a = o$ and $z = o$. Therefore $(a)M_{i,j} \cap M_j = 0$ and $(a)\,M_{i,j} \in \overline{M}_{i,j}$. Hence the map $M_{i,j}$ sends $\mathrm{Hom}_R(M_i, M_j) \longrightarrow \overline{M}_{i,j}$.

Suppose $a, b \in \mathrm{Hom}_R(M_i, M_j)$ and $(a)\,M_{i,j} = (b)\,M_{i,j}$. Let $m \in M_i$; then there is a $n \in M_i$ with $m - m^a = n - n^b$. As $M_i \cap M_j = 0$ we have $m = n$ and $m^a = n^b$. Hence for any $m \in M_i$ $m^a = n^b = m^b$ and so $a = b$ and $M_{i,j}$ is a set monomorphism.

Suppose $P \in \overline{M}_{i,j}$ then $P \oplus M_j = M_i \oplus M_j = H$ say. Consider the natural homomorphism $e : H \longrightarrow H/P$. Clearly $H/P \cong M_j$ and $M_i \subset H$. So, via the isomorphism $H/P \cong M_j$, e induces a homomorphism

$a: M_i \longrightarrow M_j$. If $m \in M_i$ then there is a unique $p \in P$ and $n \in M_j$ such that $m = n + p$. From the definition of a we have $n = m^a$ and so $(m - m^a) = p \in P$. Thus $(a)M_{i,j} \subset P$. But $P = P \cap H = P \cap [(a)M_{i,j} + M_j]$ = (applying modular law) $(a)M_{i,j} + P \cap M_j = (a)M_{i,j}$ since $P \cap M_j = 0$. Hence $M_{i,j}$ is a set epimorphism and thus a set isomorphism.

(2) Suppose $a \in \mathrm{Hom}_R(M_i, M_j)$ and that $z \in M_i \cap (a)M_{i,j}$. Then $z = m = n - n^a$ for some $m, n \in M_i$. As $M_i \cap M_j = 0$ we have $n^a = o$ and $m = n$ and so $z \in \ker(a)$. On the other hand if $m \in \ker(a)$ then $m^a = o$ and so $m = (m - m^a) \in M_i \cap (a)M_{i,j}$.

Now a is a monomorphism if and only if $\ker(a) = 0$ i.e. if and only if $M_i \cap (a)M_{i,j} = 0$.

(3) Suppose $a \in \mathrm{Hom}_R(M_i, M_j)$ then $M_i + (a)M_{i,j} = M_i + M_i^a$. For if $z \in M_i + M_i^a$ then $z = m + n^a$ for some $m, n \in M_i$ and so $z = (m + n) - (n - n^a) \in M_i + (a)M_{i,j}$. On the other hand if $z \in M_i + (a)M_{i,j}$ then for some $m, n \in M_i$ $z = m + (n - n^a) = (m + n) - n^a \in M_i + M_i^a$.

Hence we have
$$[M_i + (a)M_{i,j}] \cap M_j = (M_i + M_i^a) \cap M_j$$
$$= M_i \cap M_j + M_i^a$$
$$= M_i^a = \text{image } (a) \quad .$$

Now a is an epimorphism if and only if $M_i^a = M_j$ i.e. if and only if $[M_i + (a)M_{i,j}] \cap M_j = M_j$.

(4) Combining (2) and (3) we see that $a \in Hom_R(M_i, M_j)$ is an isomorphism if and only if $M_i \cap (a) M_{i,j} = 0$ and $[M_i + (a)M_{i,j}] \cap M_j = M_j$. But the last condition holds if and only if $M_j \subset M_i + (a)M_{i,j}$ i.e. if and only if $M_i + M_j = M_i + (a)M_{i,j}$. Hence a is an isomorphism if and only if $M_j \underset{M_i}{\sim} (a) M_{i,j}$ i.e. if and only if $(a)M_{i,j} \in \overline{M}_{j,i}$.

If a is an isomorphism then $(a)M_{i,j} = [m - m^a : m \in M_i] = [(-m^a) - (-m^a)^{a^{-1}} : m \in M_i]$. As m runs through M_i, $(-m)^a$ runs through M_j. Hence $(a)M_{i,j} = [n - n^{a^{-1}} : n \in M_j] = (a^{-1})M_{j,i}$.

(5) Suppose $a, b \in Hom_R(M_i, M_j)$. If $z \in (a)M_{i,j} + M_i{}^b$ then for some $m, n \in M_i$ $z = (m - m^a) + n^b = (m - m^a - m^b) + (n^b + m^b)$

$$= (m - m^{(a+b)}) + (m + n)^b \in (a+b)M_{i,j} + M_i{}^b.$$

On the other hand if $z \in (a + b)M_{i,j} + M_i{}^b$ then for some $m, n \in M_i$ $z = m - m^{(a+b)} + n^b = (m - m^a) + (n - m)^b \in (a)M_{i,j} + M_i{}^b$. Hence we have $(a)M_{i,j} + M_i{}^b = (a + b)M_{i,j} + M_i{}^b$ and similarly $(b)M_{i,j} + M_i{}^a = (a + b)M_{i,j} + M_i{}^a$. Therefore

$$[(a)M_{i,j}+M_i^b] \cap [(b)M_{i,j}+M_i^a] = [(a+b)M_{i,j}+M_i^b] \cap [(a+b)M_{i,j}+M_i^a]$$

$$= [(a+b)M_{i,j}]+M_i^a \cap [(a+b)M_{i,j}+M_i^b] \quad .$$

Now if $M_i^a \cap M_i^b = 0$ then, since $(a + b)M_{i,j} \cap M_j = 0$, we have that $[M_i^a, M_i^b, (a + b)M_{i,j}]$ is an independent set of submodules of M. Hence $M_i^a \cap [(a + b)M_{i,j} + M_i^b] = 0$ and so $(a + b)M_{i,j} =$

$[(a)M_{i,j} + M_i^b] \cap [(b)M_{i,j} + M_i^a]$.

(6) Suppose i, j, k are distinct elements of I and $a \in Hom_R(M_i, M_j)$ and $b \in Hom_R(M_j, M_k)$. If $z \in [M_i + M_k] \cap [(a)M_{i,j} + (b)M_{j,k}]$ then for some m_i, $n_i \in M_i$, $n_j \in M_j$ and $m_k \in M_k$ we have $z = m_i + o + m_k =$ $n_i + (-n_i^a + n_j) + (-n_j^b)$. As (M_i, M_j, M_k) are independent we have $m_i = n_i$, $n_j = n_i^a$, $m_k = -n_j^b$ and so $m_k = -n_j^b = -n_i^{ab}$. Therefore $z = n_i - n_i^{ab} \in (ab)M_{i,k}$.

Conversely suppose $z \in (ab)M_{i,k}$; then for some $m \in M_i$ $z = m - m^{ab} = (m - m^a) + [m^a - (m^a)^b] \in [M_i + M_k] \cap [(a)M_{i,j} + (b)M_{j,k}]$.

(1) of lemma 2.1 is a key remark. It shows that if $M_i \cap M_j = 0$ then $Hom_R(M_i, M_j)$ can be represented by elements of $L(_RM)$. In the next lemma we show how, using (5) and (6) of lemma 2.1 we can get at the multiplicative and additive structure of $End_R(M)$.

<u>Lemma 2.2</u>. Let $_RM$ be a module and $(M_i)_{i \in I}$ an independent set of submodules of M. Suppose that $_SN$ is a module with $\Sigma : L(_RM) \cong L(_SN)$.

If $N_i = M_i^{\Sigma}$ and i, j, k are distinct elements of I then

(1) there is a set isomorphism $\ell_{i,j}$: $\mathrm{Hom}_R(M_i, M_j) \cong \mathrm{Hom}_S(N_i, N_j)$

(2) if $a \in \mathrm{Hom}_R(M_i, M_j)$ then $[\ker(a)]^{\Sigma} = \ker(a\ell_{i,j})$ and $[\mathrm{image}\ (a)]^{\Sigma} = \mathrm{image}\ (a\ell_{i,j})$; if a is an isomorphism so is $(a)\ell_{i,j}$ and in this case $[(a)\ell_{i,j}]^{-1} = (a^{-1})\ell_{j,i}$.

(3) if $a \in \mathrm{Hom}_R(M_i, M_j)$ and $b \in \mathrm{Hom}_R(M_j, M_k)$ then $(ab)\ell_{i,k} = (a)\ell_{i,j}\ (b)\ell_{j,k}$

(4) if $a, b \in \mathrm{Hom}_R(M_i, M_j)$ and $M_i^a \cap M_i^b = 0$ then $(a+b)\ell_{i,j} = (a)\ell_{i,j} + (b)\ell_{i,j}$

(5) if there is a monomorphism $s: M_i \longrightarrow M_k$ then $\ell_{i,j}$ is a homomorphism.

<u>Proof</u>

(1) Define $\overline{M}_{i,j}$ and $\overline{N}_{i,j}$ as in lemma 2.1. Then Σ induces a set isomorphism $\overline{M}_{i,j} \cong \overline{N}_{i,j}$. But by (1) of lemma 2.1 $M_{i,j}: \mathrm{Hom}_R(M_i, M_j) \cong \overline{M}_{i,j}$ and $N_{i,j}: \mathrm{Hom}_S(N_i, N_j) \cong \overline{N}_{i,j}$ are set isomorphisms. Hence the map $\ell_{i,j} = M_{i,j}\ \Sigma\ N_{i,j}^{-1}$ is a set isomorphism: $\mathrm{Hom}_R(M_i, M_j) \cong \mathrm{Hom}_S(N_i, N_j)$.

(2) Suppose $a \in \mathrm{Hom}_R(M_i, M_j)$; then $[(a)\ M_{i,j}]\Sigma = (a)\ell_{i,j}\ N_{i,j}$. By (2) and (3) of lemma 2.1 $\ker(a) = M_i \cap (a)M_{i,j}$ and image $(a) = [M_i + (a)M_{i,j}] \cap M_j$. Applying Σ we get $[\ker(a)]^{\Sigma} = N_i \cap (a)\ell_{i,j}\ N_{i,j}$

and $[\text{image }(a)]^{\Sigma} = [N_i + (a)\ell_{i,j}\, N_{i,j}] \cap N_j$. By (2) and (3) of lemma 2.1 applied to $(N_i)_{i\varepsilon I}$ we have $[\ker(a)]^{\Sigma} = \ker(a\ell_{i,j})$ and $[\text{image }(a)]^{\Sigma} = \text{image}(a\ell_{i,j})$.

If a is an isomorphism then by (4) of lemma 2.1 $(a)M_{i,j} \varepsilon \overline{M}_{j,i}$. Therefore $(a)M_{i,j}\Sigma \varepsilon \overline{N}_{j,i}$. Hence $(a\ell_{i,j})N_{i,j} \varepsilon \overline{N}_{j,i}$ and so applying (4) of lemma 2.1 again we have that $a\ell_{i,j}$ is an isomorphism.

In this case by (4) of lemma 2.1 $(a)M_{i,j} = (a^{-1})M_{j,i}$. Applying Σ this gives $(a)\ell_{i,j}\, N_{i,j} = (a)M_{i,j}\Sigma = (a^{-1})M_{j,i}\Sigma =$ $(a^{-1})\ell_{j,i}\, N_{j,i}$. Applying (4) of lemma 2.1 once again we have $(a)\ell_{i,j}\, N_{i,j} = (a^{-1})\ell_{j,i}\, N_{j,i} = [(a^{-1})\ell_{j,i}]^{-1}\, N_{i,j}$. Cancelling $N_{i,j}$ we get $(a^{-1})\ell_{j,i} = [(a)\ell_{i,j}]^{-1}$.

(3) Suppose $a \varepsilon \text{Hom}_R(M_i, M_j)$ and $b \varepsilon \text{Hom}_R(M_j, M_k)$. By (6) of lemma 2.1 $(ab)M_{i,k} = [M_i + M_k] \cap [(a)M_{i,j} + (b)M_{j,k}]$. Applying Σ we get $(ab)\ell_{i,k}\, N_{i,k} = [N_i + N_k] \cap [(a)\ell_{i,j}\, N_{i,j} + (b)\ell_{j,k}\, N_{j,k}] =$ $[(a)\ell_{i,j}\,(b)\ell_{j,k}]N_{i,k}$ applying (6) of lemma 2.1 again. Hence cancelling $N_{i,k}$ $(ab)\ell_{i,k} = (a)\ell_{i,j}\,(b)\ell_{j,k}$.

(4) Suppose $a, b \varepsilon \text{Hom}_R(M_i, M_j)$ and $M_i{}^a \cap M_i{}^b = 0$; then by (5) of lemma 2.1 $(a + b)M_{i,j} = [(a)M_{i,j} + M_i{}^b] \cap [(b)M_{i,j} + M_i{}^a]$. Applying Σ

we get $(a + b)\ell_{i,j}\, N_{i,j} = (a + b)M_{i,j}\Sigma$

$$= [(a)M_{i,j}\Sigma + (M_i^b)\Sigma] \cap [(b)M_{i,j}\Sigma + (M_i^a)\Sigma]$$

$$= [(a)\ell_{i,j}N_{i,j} + N_i^{(b)\ell_{i,j}}] \quad [(b)\ell_{i,j}N_{i,j} + N_i^{(a)\ell_{i,j}}] .$$

But, since $M_i^a \cap M_i^b = 0$, $N_i^{(a)\ell_{i,j}} \cap N_i^{(b)\ell_{i,j}} = 0$ and so by (5) of lemma 2.1 $(a + b)\ell_{i,j}\, N_{i,j} = [(a)\ell_{i,j}\, N_{i,j} + N_i^{(b)\ell_{i,j}}]$

$$\cap [(b)\ell_{i,j}\, N_{i,j} + N_i^{(a)\ell_{i,j}}]$$

$$= [(a)\ell_{i,j} + (b)\ell_{i,j}]N_{i,j} .$$

Cancelling $N_{i,j}$ we get $(a + b)\ell_{i,j} = (a)\ell_{i,j} + (b)\ell_{i,j}$.

(5) Consider the submodules M_i and $(M_j + M_k)$ then just as in (1) there is a map $M(i;\, j,\, k): \mathrm{Hom}_R(M_i,\, M_j + M_k) \longrightarrow \overline{M}(i;\, j,\, k)$ where $\overline{M}(i;\, j,\, k) =$ (all submodules P: $P \underset{M_j + M_k}{\sim} M_i$). Defining $N(i;\, j,\, k)$ and $\overline{N}(i;\, j,\, k)$ in exactly the same way gives a set isomorphism $l(i;\, j,\, k)$: $\mathrm{Hom}_R(M_i,\, M_j + M_k) \cong \mathrm{Hom}_S(N_i,\, N_j + N_k)$. $\mathrm{Hom}_R(M_i,\, M_j)$ and $\mathrm{Hom}_S(N_i,\, N_j)$ are naturally embedded in $\mathrm{Hom}_R(M_i,\, M_j + M_k)$ and $\mathrm{Hom}_S(N_i,\, N_j + N_k)$ respectively. If $a \in \mathrm{Hom}_R(M_i,\, M_j)$ then by (2) of lemma 2.2 $(a)l(i;\, j,\, k) \in \mathrm{Hom}_S(N_i,\, N_j)$ and recalling the definitions of $M(i;\, j,\, k)$ and $N(i;\, j,\, k$ we see that $l(i;\, j,\, k)$ induces $l_{i,j}$.

Suppose that a, b ε $\mathrm{Hom}_R(M_i, M_j)$ and that there is a monomorphism s ε $\mathrm{Hom}_R(M_i, M_k)$. If z ε $M_i^{s+a} \cap M_i^{b}$ then for some m, n ε M_i $z = m^s + m^a = n^b$. Hence $m^s = n^b - m^a = o$ $(M_j \cap M_k = 0)$ and as s is a monomorphism m = o. Thus $M_i^{s+a} \cap M_i^{b} = 0$. But $M_i^{s} \cap M_i^{a+b} = 0$, $M_i^{s} \cap M_i^{a} = 0$ as well and so repeatedly applying (4) we get, writing ℓ for $\ell(i; j, k)$,

$$\begin{aligned}(s)\ell + (a+b)\ell &= (s+a+b)\ell \\ &= (s+a)\ell + (b)\ell \\ &= (s)\ell + (a)\ell + (b)\ell .\end{aligned}$$

Hence $(a+b)\ell = (a)\ell + (b)\ell$ and so ℓ is a homomorphism for elements of $\mathrm{Hom}_R(M_i, M_j)$. But ℓ induces $\ell_{i,j}$ and so $\ell_{i,j}$ is a homomorphism.

Another way of proving (5) (based on the methods of von Neumann (see equation 17 on p.111 of von Neumann (1)) is to use the fact that if a, b and s are defined as in (5) then

$$(a+b)M_{i,j} = \{[(aM_{i,j}+sM_{i,k})\cap(M_j+M_k)]+[(bM_{i,j}+M_k)\cap(M_j+sM_{i,k})]\}\cap[M_i+M_j] .$$

Our method based on that of Baer (see p.47 of Baer (1)) brings out clearly the partial additivity of $\ell_{i,j}$ for maps whose images have zero intersection and shows how the existence of the monomorphism s is sufficient to ensure full additivity.

<u>Theorem 2.3</u>. Let ${}_RM$ be a module which is the direct sum of an independent set of submodules $(M_i)_{i\varepsilon I}$ where I is an index set containing at least <u>three</u> elements. Suppose that ${}_RP$ is a module and that for each

$i \in I$ there is a submodule $P_i \subset M_i$ with $P \cong P_i$ as R-modules. If ${}_SN$ is a module such that $\Sigma : L({}_RM) \cong L({}_SN)$ and if $Q_i = P_i^{\Sigma}$ then

(1) there is an S-module ${}_SQ$ such that for each $i \in I$ $Q \cong Q_i$ as S-modules

(2) there is a ring isomorphism $\ell : \mathrm{End}_R(P) \cong \mathrm{End}_S(Q)$

(3) there is an abelian group isomorphism $s : \mathrm{Hom}_R(P, M) \cong \mathrm{Hom}_S(Q, N)$

(4) considering $\mathrm{Hom}_R(P, M)$ as a left $\mathrm{End}_R(P)$-module and $\mathrm{Hom}_S(Q, N)$ as a left $\mathrm{End}_S(Q)$-module (ℓ, s) is a semi-linear isomorphism : $(\mathrm{End}_R(P), \mathrm{Hom}_R(P, M)) \cong (\mathrm{End}_S(Q), \mathrm{Hom}_S(Q, N))$.

Proof

(1) Let $a_i : P \cong P_i$ for $i \in I$. Define $a_{i,j} = a_i^{-1} a_j : P_i \cong P_j$. The $a_{i,j}$'s satisfy $a_{i,j}\, a_{j,k} = a_{i,k}$; $a_{i,i} = 1_{P_i}$; $a_{i,j}\, a_{j,i} = 1_{P_i}$ for any $i, j, k \in I$.

If $i \neq j$ then by (1) of lemma 2.2 applied to $(P_i)_{i \in I}$ there are set isomorphisms $\ell_{i,j} : \mathrm{Hom}_R(P_i, P_j) \cong \mathrm{Hom}_S(Q_i, Q_j)$. Define $b_{i,j} = a_{i,j}\ell_{i,j}$ if $i \neq j$ and $b_{i,j} = 1_{Q_i}$ if $i = j$. Then by (2) of lemma 2.2 $b_{i,j}$ is an isomorphism for any $i, j \in I$. Let ${}_SQ$ be any module isomorphic to Q_t for some fixed $t \in I$ i.e. let $b_t : Q \cong Q_t$. Hence for any $i \in I$ $b_i = b_t b_{t,i}$ is an isomorphism: $Q \cong Q_i$.

(2) Suppose that i, j, k are distinct elements of I then by (3) of lemma 2.2 $(a_{i,k})\ell_{i,k} = (a_{i,j}\, a_{j,k})\ell_{i,k}$

$$= (a_{i,j})\ell_{i,j}(a_{j,k})\ell_{j,k}.$$

Hence for distinct $i, j, k \varepsilon I$ $b_{i,k} = b_{i,j}\, b_{j,k}$. Now if i, j are distinct elements εI $a_{j,i} = (a_{i,j})^{-1}$ and by (2) of lemma 2.2 $(a_{j,i})\ell_{j,i} = ((a_{i,j})\ell_{i,j})^{-1}$ and so $b_{i,j}\, b_{j,i} = b_{i,i}$ for any $i \neq j \varepsilon I$. But as we defined $b_{i,i} = 1_{Q_i}$ we have that for any $i, j, k \varepsilon I$ $b_{i,k} = b_{i,j}\, b_{j,k}$.

Let $f \varepsilon \mathrm{End}_R(P)$ then define $\ell:\mathrm{End}_R(P) \longrightarrow \mathrm{End}_S(Q)$ by $(f)^{\ell} = b_i(a_i^{-1} f a_j)\ell_{i,j} b_j^{-1} = f(i, j)$ say, for any $i \neq j \varepsilon I$. Firstly we must show that the definition of ℓ is independent of the choice of i and j.

Suppose that $i \neq k \varepsilon I$. If $k = j$ then $f(i, j) = f(i, k)$. If $k \neq j$ then $f(i, k) = b_i(a_i^{-1} f a_k)\ell_{i,k}\, b_k^{-1}$

$$= b_i[(a_i^{-1} f a_j)(a_j^{-1} a_k)]\ell_{i,k}\, b_k^{-1}$$

$$= b_i[(a_i^{-1} f a_j)\ell_{i,j}(a_j^{-1} a_k)\ell_{j,k}]b_k^{-1} \text{ by (3) of lemma 2.2}$$

$$= [b_i(a_i^{-1} f a_j)\ell_{i,j}\, b_j^{-1}][b_j(a_{j,k})\ell_{j,k}\, b_k^{-1}]$$

$$= f(i, j) b_j b_{j,k} b_k^{-1}$$

$$= f(i, j) \text{ since } b_j b_{j,k} b_k^{-1} = b_t b_{t,j} b_{j,k} b_{k,t} b_t^{-1} = 1 .$$

Thus if $i, j, k \in I$ and $i \neq j$ and $i \neq k$ then $f(i, j) = f(i, k)$ .... (A) and similarly with the same conditions on i, j, k we can show that

$$f(j, i) = f(k, i) \quad .... \text{ (B)} .$$

Suppose that $i \neq j \in I$. As I has more than three elements there is an element $k \in I$ distinct from i and j. Then

$$\begin{aligned} f(i, j) &= f(i, k) \quad \text{by (A)} \\ &= f(j, k) \quad \text{by (B)} \\ &= f(j, i) \quad \text{by (A)} . \end{aligned}$$

Hence if $i \neq j \in I$ then $f(i, j) = f(j, i)$ .... (C) .

Now suppose $i, j, i', j' \in I$ and $i \neq j$, $i' \neq j'$. Then

(a) if $i' = i$ $f(i, j) = f(i', j)$

$$= f(i', j') \text{ by (A)}$$

(b) if $i' \neq i$ $f(i, j) = f(i, i')$ by (A)

$$= f(i', i) \text{ by (C)}$$

$$= f(i', j') \text{ by (A)} .$$

Hence in every case $f(i, j) = f(i', j')$ and the definition of ℓ is independent of i and j.

It is clear that ℓ is a set isomorphism: $\text{End}_R(P) \cong \text{End}_S(Q)$ so we have only to show that ℓ preserves addition and multiplication.

Let $f_1, f_2 \in \mathrm{End}_R(P)$ and let i, j, k be distinct elements of I. Then $(f_1 + f_2)^{\ell} = b_i(a_i^{-1}(f_1 + f_2)a_j)\ell_{i,j}\ b_j^{-1}$

$$= b_i(a_i^{-1}f_1a_j + a_i^{-1}f_2a_j)\ell_{i,j}\ b_j^{-1} \quad .$$

But $a_{i,k}$ is a monomorphism: $P_i \longrightarrow P_k$ and so by (5) of lemma 2.2 $\ell_{i,j}$ is a homomorphism. Therefore

$$(f_1+f_2)^{\ell} = b_i(a_i^{-1}f_1a_j)\ell_{i,j}\ b_j^{-1} + b_i(a_i^{-1}f_2a_j)\ell_{i,j}\ b_j^{-1}$$

$$= f_1^{\ell} + f_2^{\ell} \text{ .}$$ Hence ℓ preserves addition.

Now $(f_1f_2)^{\ell} = b_i(a_i^{-1}f_1f_2a_k)\ell_{i,k}\ b_k^{-1}$

$$= b_i[(a_i^{-1}f_1a_j)(a_j^{-1}f_2a_k)]\ell_{i,k}\ b_k^{-1}$$

$$= [b_i(a_i^{-1}f_1a_j)\ell_{i,j}\ b_j^{-1}][b_j(a_j^{-1}f_2a_k)\ell_{j,k}\ b_k^{-1}]$$ by (3) of lemma 2.2

$$= f_1^{\ell}f_2^{\ell} \text{ .}$$

Hence ℓ preserves multiplication and addition and so is a ring isomorphism.

(3) Firstly we assume that I is finite with $n \geq 3$ elements. Fix $j \in I$ and apply lemma 2.2 to the modules $P_1, \ldots, P_{j-1}, M_j, P_{j+1}, \ldots, P_n$.

This gives maps $h_{s,t}$ for $s \neq t \in I$ where

(1) $h_{s,j}: \mathrm{Hom}_R(P_s, M_j) \cong \mathrm{Hom}_S(Q_s, N_j)$ if $t = j$

(2) $h_{s,t} = \ell_{s,t}$ if $t \neq j$, $s \neq j$.

For $d \in \mathrm{Hom}_R(P, M_j)$ define s_j by $ds_j = b_i(a_i^{-1}d)h_{i,j}$ where $i \neq j$. We show firstly that the definition of s_j is independent of the choice of i. Let $i' \in I$ and $i' \neq j$, $i' \neq i$. Then

$$b_i(a_i^{-1}d)h_{i,j} = b_i[(a_i^{-1}a_{i'})(a_{i'}^{-1}\ d)]\ h_{i,j}$$

$$= b_i(a_i^{-1}a_{i'})h_{i,i'}(a_{i'}^{-1}d)h_{i',j} \quad \text{by (3) of lemma 2.2}$$

$$= b_i b_{i,i'}(a_{i'}^{-1}d)h_{i',j} \quad \text{as } \left.\begin{matrix} i' \neq j \\ i \neq j \end{matrix}\right) \text{ implies } h_{i,i'} = \ell_{i,i'}$$

$$= b_{i'}(a_{i'}^{-1}d)h_{i',j} \ .$$

Hence s_j is independent of the choice of i. Clearly s_j is a set isomorphism: $\mathrm{Hom}_R(P, M_j) \cong \mathrm{Hom}_S(Q, N_j)$.

Now $a_{i,k}: P_i \longrightarrow P_k$ is a monomorphism and so by (5) of lemma 2.2 $h_{i,j}$ is a homomorphism. This implies that for

$$d_1, d_2 \in \mathrm{Hom}_R(P, M_j) \quad (d_1+d_2)s_j = b_i(a_i^{-1}(d_1+d_2))h_{i,j}$$

$$= b_i[(a_i^{-1}d_1) + (a_i^{-1}d_2)]h_{i,j}$$

$$= b_i(a_i^{-1}d_1)h_{i,j} + b_i(a_i^{-1}d_2)h_{i,j}$$

$$= d_1 s_j + d_2 s_j \ .$$

Therefore s_j is also a homomorphism and thus an isomorphism.

As I is finite $\mathrm{Hom}_R(P, M) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_R(P, M_i)$

$\mathrm{Hom}_S(Q, N) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_S(Q, N_i)$ as abelian groups.

If $d \in \mathrm{Hom}_R(P, M)$ then $d = \sum_1^n d_i$ for unique $d_i \in \mathrm{Hom}_R(P, M_i)$. Define the map $s: \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}_S(Q, N)$ by $ds = \sum_1^n d_i s_i$. This gives the required isomorphism.

The case when I is infinite can be reduced to the finite case. For let $F = (1, \ldots, n)$ be any finite subset of I with $n \geq 3$. Let $I' = (1, \ldots, n, n+1)$ and define $M_{i'} = M_i$ for $1 \leq i' \leq n$ and $M_{i'} = \bigoplus_{j \notin F} M_j$ for $i' = n+1$. Then $\bigoplus_{i' \in I'} M_{i'} = M$ and so we can apply the previous arguments to $(M_{i'})_{i' \in I'}$.

We note that the map $s(F)$ obtained from taking the finite set F is in fact independent of F. It is sufficient to show that if F, G are finite subsets of I with $|F| \geq 3$, $|G| \geq 3$ and $F \subset G$ then $s(F) = s(G)$. The general case will then follow since $F \subset F \cup G$ and $G \subset F \cup G$ imply $s(F) = s(F \cup G) = s(G)$. By induction we can assume F has n elements and G has n+1. We get sets $(M_1, \ldots, M_n, \bigoplus_{j>n} M_j)$ and $(M_1, \ldots, M_{n+1}, \bigoplus_{j>n+1} M_j)$ giving rise to **maps** $(s_1, \ldots, s_n, s_{n+1})$ and $(s_1', \ldots, s_{n+1}', s_{n+2}')$ where $\sum_1^{n+1} s_i = s(F)$ and $\sum_1^{n+2} s_i' = s(G)$.

But it is clear that $s_i = s_i'$ for $1 \leq i \leq n$. On the other hand by arguments similar to those used in (5) of lemma 2.2 we have that s_{n+1} induces s'_{n+1} and s'_{n+2} and so $s_{n+1} = s'_{n+1} + s'_{n+2}$. Hence $s(F) = \sum_1^{n+1} s_i = \sum_1^{n+2} s_i' = s(G)$. Thus the definition of s is independent of the choice of F.

(4) Let $f \in \mathrm{End}_R(P)$ and $d \in \mathrm{Hom}_R(P, M)$ where $d = \sum_1^n d_i$ and $d_i \in \mathrm{Hom}_R(P, M_i)$ and where again we reduce the infinite case to the finite case as in (3).

Then $(fd)s = (\sum_1^n fd_i)s$

$$= \sum_1^n (fd_i)s$$

$$= \sum_1^n (fd_i)s_i \quad .$$

Fix $i \in I$ and choose (as we may) distinct $j, k \in I$ such that $i \neq j$ and $i \neq k$. Then

$$f^{\ell}(d_i)s_i = [b_j(a_j^{-1}fa_k)\ell_{j,k}\, b_k^{-1}][b_k(a_k^{-1}d_i)h_{k,i}]$$

$$= b_j(a_j^{-1}fa_k)h_{j,k}(a_k^{-1}d_i)h_{k,i}$$

$$= b_j(a_j^{-1}fa_ka_k^{-1}d_i)h_{j,i} \quad \text{by (3) of lemma 2.2}$$

$$= b_j(a_j^{-1}fd_i)h_{j,i} = (fd_i)s_i \quad .$$

Therefore $(fd)s = \sum_{1}^{n} (fd_i)s_i$

$$= \sum_{1}^{n} f^{\ell}(d_i)s_i$$

$$= f^{\ell} \sum_{1}^{n} (d_i)s_i$$

$= f^{\ell}(d)s.$ Hence (ℓ, s) is a semi-linear isomorphism: $(End_R(P), Hom_R(P, M)) \cong (End_S(Q), Hom_S(Q, N))$.

Remark 1. A look at the proofs of lemmas 2.1 and 2.2 and theorem 2.3 will show that the results proved so far in this chapter remain true if the conditions, that all modules are unital and all rings have a 1, are dropped. The reason why these conditions are not necessary is basically because the elements of a ring R which act trivially on a module $_RM$ do not affect the endomorphism ring $End_R(M)$. If we are hoping for stronger results say involving semi-linear isomorphisms we shall see in chap. 4 that it is not possible to drop these conditions.

Remark 2. Suppose A is a submodule of M and for some $i \in I$ $A \subset \bigoplus_{j \neq i} M_j$ then $[Hom_R(P, A)]\, s = Hom_S(Q, A^{\Sigma})$.

Proof

If I is infinite we can, without affecting s, choose as in the last part of (3) the finite subset $F \subset I$ so that it contains i. Suppose $a \in Hom_R(P, A)$ then $as_j = b_i(a_i^{-1}a)h_{i,j}$ hence

$$Qas_j = Qb_i(a_i^{-1}a)h_{i,j} = Q_i(a_i^{-1}a)h_{i,j}$$

$$= [P_i(a_i^{-1}a)]^{\Sigma} \text{ by (2) of lemma 2.2}$$

$$= (Pa)^{\Sigma} \subset A^{\Sigma} .$$

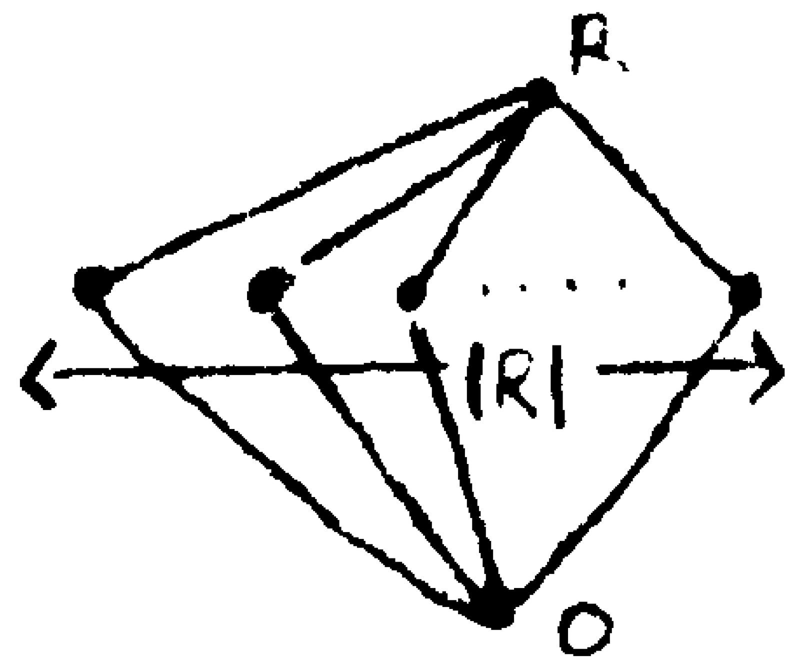
Therefore $as_j \in \mathrm{Hom}_S(Q, A^{\Sigma})$. But $as = \Sigma as_j$ and so $Qas = Q \, \Sigma \, as_j \subset A^{\Sigma}$. Hence $[\mathrm{Hom}_R(P,A)]s \subset \mathrm{Hom}_S(Q, A^{\Sigma})$. A symmetrical argument gives $[\mathrm{Hom}_S(Q, A^{\Sigma})]s^{-1} \subset \mathrm{Hom}_R(P, A)$. Thus $\mathrm{Hom}_S(Q, A^{\Sigma}) \subset [\mathrm{Hom}_R(P, A)]^s$ and so we get $\mathrm{Hom}_S(Q, A^{\Sigma}) = [\mathrm{Hom}_R(P, A)]s$.

The condition that I has at least three elements in theorem 2.3 is necessary as the following example shows.

<u>Example 2.4.</u> Let R and S be non-isomorphic division rings such that $|R| = |S|$ then (1) $L(_RR) \cong L(_SS)$ (2) $L(_RR^2) \cong L(_SS^2)$ but $\mathrm{End}_R(R) \not\cong \mathrm{End}_S(S)$ and $\mathrm{End}_R(R^2) \not\cong \mathrm{End}_S(S^2)$.

<u>Proof</u>

For any division ring R $L(_RR)$ is of the form  and $L(_RR^2)$ is of the form



Hence we have $L(_RR) \cong L(_SS)$ and $L(_RR^2) \cong L(_SS^2)$. But $R \not\cong S$ so $\mathrm{End}_R(R) \not\cong \mathrm{End}_S(S)$ and $\mathrm{End}_R(R^2) \not\cong \mathrm{End}_S(S^2)$. For if R and S are division rings by the Artin-Wedderburn theorem (see e.g. isomorphism theorem of chap. 3.5 of Jacobson (1)) $R_2 \cong S_2$ implies $R \cong S$.

Cor.1. Let I be a set with at least three elements and ${}_RP$, ${}_SN$ be modules. If ${}_RM = {}^I_RP$ and $\Sigma : L({}_RM) \cong L({}_SN)$ then $End_R(M) \cong End_S(N)$.

Proof

Let $Q_i = (P_i)^\Sigma$ for $i \in I$ and where $M = \bigoplus_{i \in I} P_i$. Putting $M_i = P_i$ $N_i = Q_i$ in theorem 2.3 we see that there is a module ${}_SQ$ with $Q \cong Q_i$ for all $i \in I$ and $End_R(P) \cong End_S(Q)$. By lemma 1.7 $End_R(M) \cong (End_R(P))_{rfI}$ and $End_S(N) \cong (End_S(Q))_{rfI}$. Hence the isomorphsim $End_R(P) \cong End_S(Q)$ induces $End_R(M) \cong End_S(N)$.

The converse is not true. For in example 1.6 we constructed modules ${}_RA$, ${}_TC$ such that $End_R(A) \cong End_T(C)$ but $L({}_RA) \not\cong L({}_TC)$. Hence if ${}_RM = {}^I_RA$ and ${}_TN = {}^I_TC$ then $End_R(M) \cong End_T(N)$. It is easy to see that in this case $L({}_RM) \not\cong L({}_TN)$.

Cor.2. Let I be a set with at least three elements and ${}_RM$, ${}_SN$ be modules. If ${}_RM$ is a free module of rank I and $\Sigma : L({}_RM) \cong L({}_SN)$ then $End_R(M) \cong End_S(N)$.

Proof

Put $P = R$ in cor.1.

Cor.3. Let R and S be rings and n an integer ≥ 3. If $\Sigma : L({}_{R_n}R_n) \cong L({}_SS)$ then $R_n \cong S$.

Proof

Let $e_{i,i}$ be the matrix of R_n with 1 in the $(i, i)^{th}$ place and zeros elsewhere. Then $R_n = \bigoplus_1^n R_n e_{ii}$ and $R_n e_{ii} \cong R_n e_{jj}$ for $1 \leq i,j \leq n$.

Putting $P_i = R_n e_{ii}$ in cor.1 we get that $End_{R_n}(P_n) \cong End_S(S)$. But R_n and S are rings with a 1 and so are isomorphic to their own endomorphism rings. Hence $R_n \cong S$.

Example 2.4 shows that the condition $n \geq 3$ is necessary. For there we saw that there are rings with $L(_R R) \cong L(_S S)$ and $L(_R R^2) \cong L(_S S^2)$ but $R \ncong S$ and $R_2 \ncong S_2$. But by von Neumann's theorem $L(_{R_2} R_2) \cong L(_R R^2)$ $L(_S S^2) \cong L(_{S_2} S_2)$.

The result in cor.3 is due to von Neumann (see theorem 4.2 of chap.4 of part 2 of von Neumann (1)). Although his theorem is stated for regular rings it goes through unchanged for rings with a 1. Von Neumann however proves more. He shows that the isomorphism $s: R_n \cong S$ induces the lattice isomorphism Σ. The proof of this depends heavily on the fact that R is regular. Using remark 2 to theorem 2.3 we can show that if A is a left ideal of R_n such that for some $1 \leq i \leq n$ $A \subset \bigoplus_{j \neq i} R_n e_{j,j}$ then $A^s = A^\Sigma$. We have been unable to show that this is true for any left ideal A. The difficulty is to know how to deal with proper principal left ideals which have non-zero intersection with every other left ideal i.e. large principal left ideals in the terminology of chap.6. In the case when R is regular this case is excluded.

Cor. 4. Let R and S be rings and I a set containing at least three elements. If

(1) $\Sigma: L({}_{R_{rfI}}R_{rfI}) \cong L({}_{S}S)$ then $R_{rfI} \cong S$

(2) $\Sigma: L({}_{R_{cfI}}R_{cfI}) \cong L({}_{S}S)$ then $R_{cfI} \cong S$.

Proof

If I is finite then the result follows from cor.3. If I is infinite then by cor.1 to lemma 1.8 for any integer n $R_{rfI} \cong (R_{rfI})_n$ and $R_{cfI} \cong (R_{cfI})_n$. Take $n \geq 3$ and, noting R_{rfI} and R_{cfI} both have identity element, apply cor.3. This gives $R_{rfI} \cong (R_{rfI})_n \cong S$ and $R_{cfI} \cong (R_{cfI})_n \cong S$.

Stated in another way we can say that the endomorphism ring of a free left (right) module of rank ≥ 3 is determined up to isomorphism by its lattice of left ideals.

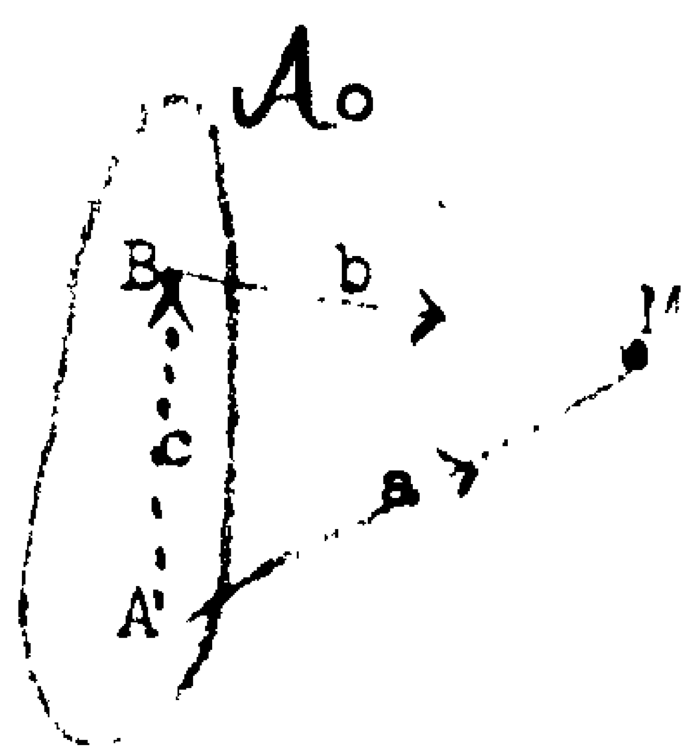
CHAPTER 3

CATEGORY EQUIVALENCES

In this chapter we consider lattice isomorphisms which are induced by category equivalences. In particular we show that any lattice isomorphism between the lattices of submodules of a free module of infinite rank and a module containing at least one free element (see definition preceding lemma 3.3) is of this type. Using this result we give new conditions for rings R and S to be Morita equivalent in terms of infinite matrices over R and S and in terms of the lattices of submodules of direct sums and direct products of copies of R and S.

Firstly we recall some basic facts about categories. Let $\mathcal{A}$ be a category. Consider the equivalence classes of objects of $\mathcal{A}$ under the equivalence relation of isomorphism. Let $\mathcal{A}_o$ be a set of representatives of these classes plus all the morphisms between them. Then $\mathcal{A}_o$ is called a <u>skeleton</u> for $\mathcal{A}$ and is a full subcategory which is equivalent to $\mathcal{A}$. It is easily seen that any two skeletons of $\mathcal{A}$ are isomorphic and that any isomorphism between two such skeletons can be extended to a category auto-equivalence of $\mathcal{A}$. More generally suppose that $\mathcal{A}$ and $\mathcal{B}$ are two categories and $\mathcal{A}_o$ and $\mathcal{B}_o$ are skeletons for $\mathcal{A}$ and $\mathcal{B}$ respectively. Then it is easy to show that any isomorphism of $\mathcal{A}_o$ and $\mathcal{B}_o$ can be extended to an equivalence of $\mathcal{A}$ and $\mathcal{B}$. Conversely any equivalence between $\mathcal{A}$ and $\mathcal{B}$ induces an isomorphism of $\mathcal{A}_o$ and $\mathcal{B}_o$.

Suppose that R is a ring and $\mathcal{A}_o$ is a skeleton for ${}_R\mu$ the category of all left R-modules. Let ${}_RM$ be a module and consider the monomorphisms from objects in $\mathcal{A}_o$ to M. We can pre-order these by defining $a \leq b$ if there is a map c such that $a = cb$ or diagrammatically



Consider the equivalence relation $a \sim b$ if $a \leq b$ and $b \leq a$, i.e., $a = cb$ where c is an isomorphism. Then the equivalence classes form an ordered set, which is a lattice, lattice isomorphic to $L({}_RM)$ by the map $a \longrightarrow$ image (a).

Suppose that S is another ring and $F: {}_R\mu \longrightarrow {}_S\mu$ is a category equivalence. Clearly F maps $\mathcal{A}_o$ to a skeleton of ${}_S\mu$ and F is an order preserving set isomorphism between the monomorphisms from $\mathcal{A}_o$ to M and the monomorphisms from $\mathcal{A}_o^F$ to M^F. This gives us a lattice isomorphism $L({}_RM) \cong L({}_S(M)^F)$ and we say that F induces this lattice isomorphism. We also note that this lattice isomorphism is independent of the choice of skeleton $\mathcal{A}_o$.

We now collect together as a theorem a number of Morita's results on category equivalences. These are all in Morita (1) in one form or another. We present them in the form given in Bass (1).

<u>Theorem 3.1</u>

(A) Let R and S be rings and suppose $F: {}_R\mu \longrightarrow {}_S\mu$ and $G: {}_S\mu \longrightarrow {}_R\mu$ are inverse category equivalences. Then

(1) $(R)^F$ is a S-R bimodule ${}_SQ_R$ and $(S)^G$ is a R-S bimodule ${}_RP_S$

(2) P and Q are progenerators both as left and right modules

(3) $R \cong \text{End}_S(Q)$ and $S \cong \text{End}_R(P)$

(4) $P \otimes_S Q \cong R$ and $Q \otimes_R \quad \cong S$

(5) $F \simeq Q \otimes_R - \simeq \text{Hom}_R(P,-)$ and $G \simeq P \otimes_S - \simeq \text{Hom}_S(Q,-)$.

(B) If ${}_SQ$ is a progenerator and $R \cong \text{End}_S(Q)$ and ${}_RP_S = \text{Hom}_S(Q,S)$ then $\text{Hom}_R(P,-) \simeq Q \otimes_R - : {}_R\mu \longrightarrow {}_S\mu$ and $\text{Hom}_S(Q,-) \simeq P \otimes_S - : {}_S\mu \longrightarrow {}_R\mu$ are inverse category equivalences.

(C) $R \underset{M}{\sim} S$ if and only if there is a progenerator ${}_SQ$ such that $R \cong \text{End}_S(Q)$.

<u>Lemma 3.2.</u> Suppose ${}_RM$ and ${}_SN$ are modules and $(\ell, s):(R, M) \cong (S, N)$ is a semi-linear isomorphism. Let $\Sigma: L({}_RM) \cong L({}_SN)$ be the lattice isomorphism induced by (ℓ, s). Then there exists a category equivalence $F: {}_R\mu \longrightarrow {}_S\mu$ such that $M^F = N$ and F induces Σ.

<u>Proof</u>

Firstly suppose that ${}_RM$ and ${}_RN$ are modules such that $s: {}_RM \cong {}_RN$ is a linear isomorphism. Let Σ be the lattice isomorphism: $L({}_RM) \cong L({}_RN)$ induced by s. Consider two skeletons for ${}_R\mu$ which are the same except that in one we choose ${}_RM$ as representative and in the other ${}_RN$. The isomorphism $s:{}_RM \cong {}_RN$ induces an isomorphism between these two skeletons which can be extended to a category equivalence $F:{}_R\mu \longrightarrow {}_R\mu$. Clearly F induces Σ.

More generally suppose ${}_RM$ and ${}_SN$ are modules and (ℓ, s): $(R, M) \cong (S, N)$ is a semi-linear isomorphism. Now $\ell:R \cong S$ induces a category isomorphism $F_1: {}_R\mu \longrightarrow {}_S\mu$ by $({}_RP)^{F_1} = {}_SP$ where P is made into an S-module by $tp = t^{\ell^{-1}}p$ for $t \in S$ and $p \in P$. Now $s:(M)^{F_1} \longrightarrow N$ is an S-isomorphism. For if $m \in M$ and $t \in S$ then $(tm)^s = (t^{\ell^{-1}}m)s = t^{\ell^{-1}\ell}m^s = tm^s$. By the first part we can find a category equivalence $F_2:{}_S\mu \longrightarrow {}_S\mu$ which induces the lattice isomorphism:$L({}_S(M)^{F_1}) \cong L({}_SN)$ induced by s. Hence $F = F_1F_2$ is a category equivalence:${}_R\mu \longrightarrow {}_S\mu$ which induces the lattice isomorphism:$L({}_RM) \cong L({}_SN)$ induced by (ℓ, s).

This lemma shows that any lattice isomorphism induced by a linear or semi-linear isomorphism can be induced by a category equivalence.

<u>Definition.</u> Let ${}_RM$ be a module and A a subset of M. Then the <u>left annihilator of A</u>, $\ell_R(A)$, is defined to be $(r \in R:rA = o)$. Similarly if M is a right R-module we define the <u>right annihilator of A</u>, $r_R(A)$. When the ring R is obvious from the context we will omit the suffix R. We call M <u>faithful</u> if $\ell(M) = 0$ and an element $m \in M$ is called <u>free</u> if $\ell(m) = 0$.

<u>Lemma 3.3.</u> Let R and S be rings and F_1, F_2 category equivalences: ${}_R\mu \longrightarrow {}_S\mu$. Suppose ${}_RM$ is a module such that $M^{F_1} = M^{F_2} = {}_SN$ then we have

(1) if $T:F_1 \simeq F_2$ is a natural equivalence and $T(M):N \longrightarrow N$ induces $1_{L({}_SN)}$ then F_1 and F_2 both induce the same lattice isomorphism: $L({}_RM) \cong L({}_SN)$.

(2) if M has at least one free element and F_1 and F_2 both induce the same lattice isomorphism: $L({}_RM) \cong L({}_SN)$ then there is a natural equivalence $T:F_1 \simeq F_2$ such that $T(M)$ induces $1_{L({}_SN)}$.

<u>Proof</u>

(1) Let $\mathcal{A}_o$ be a skeleton for ${}_R\mu$ and f be a monomorphism: $A \longrightarrow M$ for some $A \in \mathcal{A}_o$. Then we have a commutative diagram

$$\begin{array}{ccc} A^{F_2} & \xrightarrow{(f)^{F_2}} & M^{F_2} = N \\ \uparrow T(A) & & \uparrow T(M) \\ A^{F_1} & \xrightarrow{(f)^{F_1}} & M^{F_1} = N \end{array}$$

and so $A^{F_1}T(A)(f)^{F_2} = A^{F_1}(f)^{F_1}T(M)$. Now $M^{F_2} = M^{F_1} = N$ and $T(M)$ induces the identity lattice isomorphism on $L({}_SN)$. Hence as $T(A)$ is an isomorphism and so certainly an epimorphism we get $(A)^{F_2}(f)^{F_2} = (A)^{F_1}(f)^{F_1}$ for any $A \in \mathcal{A}_o$ and monomorphism $f:A \longrightarrow M$. Therefore F_1 and F_2 induce the same lattice isomorphism: $L({}_RM) \cong L({}_SN)$.

(2) Let m be a free element $\in M$. Then there is a monomorphism $f: R \longrightarrow M$ defined by $1 \longrightarrow m$. As F_1 and F_2 induce the same lattice isomorphism: $L({}_RM) \cong L({}_SN)$ we have $(R)^{F_1}(f)^{F_1} = (R)^{F_2}(f)^{F_2}$. Now f is a monomorphism and hence so are $(f)^{F_1}$ and $(f)^{F_2}$. Thus we get an isomorphism $(R)^{F_1} \cong (R)^{F_2}$. Let $Q_i = (R)^{F_i}$ for $i = 1, 2$. Then by (A) of theorem 3.1 ${}_SQ_i$ is a progenerator and there is a natural equivalence $T_i: F_i \simeq Q_i \otimes_R -$. But as ${}_SQ_1 \cong {}_SQ_2$ there is a natural equivalence $U: Q_1 \otimes_R - \simeq Q_2 \otimes_R -$. Hence $T = T_1UT_2^{-1}: F_1 \simeq F_2$ is a natural equivalence.

Let Q be a submodule of ${}_SN$ and $\mathcal{A}_o$ a skeleton for ${}_R\mathcal{M}$. Hence for some $A \in \mathcal{A}_o$ and monomorphism $f: A \longrightarrow M$ we get $Q = (A)^{F_1}(f)^{F_1} = (A)^{F_2}(f)^{F_2}$. Now we have the commutative diagram

$$\begin{array}{ccc} A^{F_2} & \xrightarrow{(f)^{F_2}} & M^{F_2} = N \\ \uparrow{\scriptstyle T(A)} & & \uparrow{\scriptstyle T(M)} \\ A^{F_1} & \xrightarrow{(f)^{F_1}} & M^{F_1} = N \end{array}$$

and so as in the first part of the lemma we get $(A)^{F_1}(f)^{F_1}T(M) = (A)^{F_2}(f)^{F_2}$ i.e. $Q = QT(M)$ for any submodule Q of N. Therefore $T(M)$ induces $1_{L({}_SN)}$.

Theorem 3.4. Let ${}_RM$ be a free module of infinite rank and ${}_SN$ a module with $\Sigma:L({}_RM) \cong L({}_SN)$. Then, if N contains at least one free element, Σ is induced by a category equivalence $F:{}_R\mu \longrightarrow {}_S\mu$. Furthermore a category equivalence $G:{}_R\mu \longrightarrow {}_S\mu$ induces Σ if and only if there is a natural equivalence $T:F \simeq G$, such that $T(M)$ induces $1_{L({}_SN)}$.

Proof

Let $(e_i)_{i \in I}$ be a basis for ${}_RM$ and let $Q_i = (Re_i)^{\Sigma}$. By theorem 2.3 there is an S-module ${}_SQ$ such that $Q \cong Q_i$ for all $i \in I$ and a semi-linear isomorphism $(\ell, s):[End_R(R), Hom_R(R, M)] \cong [End_S(Q), Hom_S(Q, N)]$ i.e. $(\ell, s):[R, M] \cong [End_S(Q), Hom_S(Q, N)]$.

Let $y \in N$ be free. Now $N = \bigoplus_{i \in I} Q_i$ and so there is a finite subset $F \subset I$ with n elements, say, such that $y \in \bigoplus_{i \in F} Q_i$. Hence for every finite subset $G \subset I$ with n elements $\bigoplus_{i \in G} Q_i$ has a free element y_G.

As Re_i is finitely generated, by cor.1 to lemma 1.1 so is Q_i, say by m elements. Let H be any finite subset of I containing mn elements such that $i \notin H$. Then $H = \bigcup_1^m G_j$ of disjoint subsets G_j of H where each G_j contains n elements. We have $Q_i \cap \bigoplus_{k \in H} Q_k = 0$ and $\bigoplus_{k \in G_j} Q_k$ contains a free element y_{G_j}. Hence there is a free submodule U of rank m, namely $\bigoplus_1^m Sy_{G_j}$, such that $U \cap Q_i = 0$. As Q_i is generated by m elements there is an epimorphism $f:U \longrightarrow Q_i$. By (2) of lemma 2.2 there is an epimorphism $g:U^{\Sigma^{-1}} \longrightarrow Q_i^{\Sigma^{-1}} = Re_i$. But Re_i

is free and so projective. Hence $Re_1 \cong$ a direct summand of $U^{\Sigma^{-1}}$. Hence by (2) of lemma 2.2 we get $Q_1 \cong$ direct summand of U. But U is free and hence Q_1, and therefore Q, is projective.

Let $P = (Sy)^{\Sigma^{-1}}$; then P is finitely generated. Exactly as before there is a finite subset $D \subset I$ such that if $V = \bigoplus_{k \in D} Re_k$ then $V \cap P = 0$ and there is an epimorphism $g: V \longrightarrow P$. By (2) of lemma 2.2 there is an epimorphism $f: V^{\Sigma} \longrightarrow Sy$. As before this means that S is isomorphic to a direct summand of $V^{\Sigma} = \bigoplus_{k \in D} Q_k$. Thus a direct sum of copies of Q contains S as a direct summand and so Q is a generator. Therefore ${}_SQ$ is a progenerator. But $R \cong End_S(Q)$ and so by (C) of theorem 3.1 $R \underset{M}{\sim} S$.

Now the functor $Hom_S(Q, -): {}_S\mu \longrightarrow {}_R\mu$ is a category equivalence by (B) of theorem 3.1, considering Q as a right R-module via the isomorphism $\ell: R \cong End_S(Q)$. Now saying that (ℓ, s) is a semi-linear isomorphism: $[R, M] \cong [End_S(Q), Hom_S(Q, N)]$ is equivalent to saying that $s: {}_RM \cong {}_RHom_S(Q, N)$ is a R-isomorphism. Let $G_1 = Hom_S(Q, -)$ and let $G_2: {}_R\mu \longrightarrow {}_R\mu$ be a category equivalence inducing the lattice isomorphism induced by s^{-1}. Let $G = G_1G_2$ then G is a category equivalence: ${}_S\mu \longrightarrow {}_R\mu$.

If A is a finitely generated submodule of M then for some finite subset $E \subset I$ we have $A \subset \bigoplus_{i \in E} Re_i$. Hence by remark 2 to theorem 2.3 $Hom_S(Q, A^{\Sigma})s^{-1} = A$ i.e. $A\Sigma G = A$. Let F_1 be an equivalence: ${}_R\mu \longrightarrow \mu$

such that there is a natural equivalence $T: GF_1 \simeq 1$. Let F_2 be a category equivalence: ${}_S\mu \longrightarrow {}_S\mu$ inducing the same lattice isomorphism: $L({}_SN) \cong L({}_SN)$ as the isomorphism $T(M): {}_SN \cong {}_SN$. Define $F = F_1F_2$. Then F is a category equivalence: ${}_R\mu \longrightarrow {}_S\mu$ such that the equivalence GF induces $1_{L({}_SN)}$. Since $A\Sigma\, G = A$ for all finitely generated submodules A of M we have that $A\Sigma = AF$ for all finitely generated submodules A of M. Thus by lemma 1.3 F induces the lattice isomorphism Σ.

The last part of the theorem follows from lemma 3.3. Later (cor.1 lemma 4.1) we will show that the isomorphism $T(M)$ inducing the identity lattice isomorphism on $L({}_SN)$ must in fact be left multiplication by some unit contained in the centre of S.

It is not clear whether the condition that N has at least one free element can be weakened. If however ${}_SN$ is not faithful the theorem need not hold as the following example shows.

<u>Example 3.5</u>. There exist a free module ${}_RM$ of infinite rank, a non-faithful module ${}_SN$ and a lattice isomorphism $\Sigma: L({}_RM) \cong L({}_SN)$ where $R \not\approx_M S$.

<u>Proof</u>

Let R be a Noetherian ring and let S = the direct product of R with a non-Noetherian ring T. Then R is Noetherian while S is not and so $R \not\approx_M S$. Let ${}_RM$ be any free module of infinite rank. We can consider M as a S-module by letting the component T of S act trivially on M.

Denote this S-module by ${}_S N$. It is easily seen that ${}_S N$ is unital and that $L({}_R M) \cong L({}_S N)$ since every R-submodule of M is a S-submodule of M and conversely.

<u>Cor.1.</u> Suppose R and S are rings and I is an infinite set. Then

(a) $L({}^I_R R) \cong L({}^I_S S)$ implies $R \underset{M}{\sim} S$

(b) $L({}_R R^I) \cong L({}_S S^I)$ implies $R \underset{M}{\sim} S$.

<u>Proof</u>

(a) This follows immediately from theorem 3.4 as ${}^I S$ contains a free element.

(b) Let ${}_R M = {}_R R^I$ and ${}_S N = {}_S S^I$ and suppose $\Sigma : L({}_R M) \cong L({}_S N)$. We have $R^I \cong R^I \oplus R^I \oplus R^I \oplus R^I$. Hence there are submodules P_1 and P_2 of M such that $P_1 \cong P_2 \cong R^I \oplus R^I \cong R^I$ and submodules P_{11}, P_{12}, P_{21} and P_{22} such that $P_1 = P_{11} \oplus P_{12}$ and $P_2 = P_{21} \oplus P_{22}$ where $P_1 \cong P_{21} \cong P_{22} \cong R^I$ and $P_2 \cong P_{11} \cong P_{12} \cong R^I$. Let $Q_{ij} = P^\Sigma_{ij}$ and $Q_i = P^\Sigma_i$ where $1 \leq i, j \leq 2$. Then by (2) of lemma 2.2 $Q_{21} \cong Q_{22} \cong Q_1$, $Q_{11} \cong Q_{12} \cong Q_2$ and $Q_1 \cong Q_2$. Hence $Q_1 = Q_{11} \oplus Q_{12} \cong Q_2 \oplus Q_2 \cong Q_1 \oplus Q_2 = N$ and similarly $Q_2 \cong N$.

Now $P_1 \cong R^I$ so P_1 contains a free module of infinite rank. Let $(e_i)_{i \in I}$ be a basis for this free module. Then as $Q_1 = P^\Sigma_1$ we have $P^\Sigma_1 \cap Q_2 = 0$ and $Q_2 \cong S^I$ contains a free module of infinite rank. The arguments used in theorem 3.4 then show that $(Re_i)^\Sigma$ is a progenerator and $R \underset{M}{\sim} S$.

Definition. A ring R is called <u>subcommutative</u> if every left ideal of R is a two-sided ideal i.e. for any given elements a, x ϵ R there is an element y $\in$ R with ax = ya. The notation subcommutative has been used by Barbilian in another context.

Let R be a subcommutative ring and Rx a cyclic left R-module on generator x. Then $\ell(x)$ is a two sided ideal of R and so for any element r ϵ R $\ell(x)r \subset \ell(x)$. Hence $\ell(x)rx \subset \ell(x)x = 0$ i.e. $\ell(x) \subset \ell(Rx)$. But $\ell(Rx) \subset \ell(x)$ and so $\ell(x) = \ell(Rx)$.

Exploiting this fact we show in our next corollary that the condition that ${}_SN$ has a free element in theorem 3.4 can be dropped if S is subcommutative.

Cor.2. Let R be a ring and I an infinite set. Suppose that S is a subcommutative ring and ${}_SN$ is a faithful module with $\Sigma : L({}^I_RR) \cong L({}_SN)$. Then $R \underset{M}{\sim} S$.

Proof

Let $(e_i)_{i \epsilon I}$ be a basis for ${}^IR$. Let 1 be a fixed element of I. Then by theorem 2.3 there are isomorphisms $s_i : Q_1 \cong Q_i$ where $Q_i = (Re_i)^{\Sigma}$. Now Q_1 is finitely generated by n elements $x_1, \ldots, x_n$ say. As $Q_1 \cong Q_i$ $\ell(Q_1) = \ell(Q_i)$ and hence $\ell(Q_1) = \bigcap_{i \in I} \ell(Q_i) = \ell(N) = 0$, since ${}_SN$ is faithful. But since S is subcommutative we have that $\ell(x_i) = \ell(Sx_i)$ and so $\bigcap_1^n \ell(x_i) = \bigcap_1^n \ell(Sx_i) = \ell(Q_1) = 0$.

Now consider the element $y \in N$ where $y = \sum_1^n x_i s_i$ and $x_i s_i \in Q_i$. As $(Q_i)_{i \in I}$ is an independent set of submodules of N, we have $\ell(y) = \bigcap_1^n \ell(x_i s_i) = \bigcap_1^n \ell(x_i) = 0$ since the s_i's are all isomorphisms. Hence N has a free element and the result follows from theorem 3.4.

Cor.3. Let R, S be commutative rings and I an infinite set. Suppose ${}_S N$ is a faithful module with $\Sigma : L({}^I_R R) \cong L({}_S N)$. Then $R \cong S$.

Proof

S is obviously subcommutative. Hence by cor.2 $R \underset{M}{\sim} S$. But this implies (see e.g. (7) of Morita 1 of Bass (1)) that centre (R) $\cong$ centre (S) i.e. $R \cong S$.

Theorem 3.6.* Let R and S be rings and I an infinite set then the following are equivalent

(1) $R \underset{M}{\sim} S$

(2) $R_{fI} \cong S_{fI}$

(3) $R_{rbI} \cong S_{rbI}$

(4) $R_{cbI} \cong S_{cbI}$

(5) $L({}^I_R R) \cong L({}^I_S S)$

(6) $L({}^I R_R) \cong L({}^I S_S)$

(7) $L({}_R R^I) \cong L({}_S S^I)$

(8) $L(R^I_R) \cong L(S^I_S)$.

*Part of this theorem was communicated to the author as a conjecture due to Lawvere in the form $R \underset{M}{\sim} S$ if and only if "the infinite matrices over R and S are isomorphic".

Further if any one of the equivalent conditions (1) to (8) hold then $R_{rfI} \cong S_{rfI}$ and $R_{cfI} \cong S_{cfI}$. ~~The converse need not hold~~.

<u>Proof</u>

Let ${}_SQ$ be a progenerator. Then for some integers m and n there are modules ${}_SF$ and ${}_SG$ such that $Q^n = S \oplus F$ and $S^m = Q \oplus G$. Hence we get the following isomorphisms

$${}^IQ \cong {}^I(Q^n) \cong {}^IS \oplus {}^IF \cong {}^IS \oplus {}^IS \oplus {}^IF \cong {}^IS \oplus {}^I(Q^n) \cong {}^IS \oplus {}^IQ \cong {}^I(S^m) \oplus {}^IQ \cong$$

${}^IQ \oplus {}^IG \oplus {}^IQ \cong {}^IQ \oplus {}^IG \cong {}^I(S^m) \cong {}^IS$. Thus there is an isomorphism $s: {}^IQ \cong {}^IS$. Similarly noting that $(A \oplus B)^I \cong A^I \oplus B^I$ for S-modules A and B we can also prove $Q^I \cong S^I$.

Suppose that $R \underset{M}{\sim} S$ then there is a category equivalence $F: {}_R\mu \longrightarrow {}_S\mu$ and for some progenerator ${}_SQ$ we have $R^F = Q$. But F preserves direct sums and direct products and so $({}^IR)^F = {}^IQ \cong {}^IS$ and $(R^I)^F \cong Q^I \cong S^I$. Hence we get lattice isomorphisms $L({}_R^IR) \cong L({}_S^IS)$ and $L({}_RR^I) \cong L({}_SS^I)$ and so (1) implies (5) and (7). By symmetry we get (1) implies (6) and (8).

By (A)(3) of theorem 3.1 we have that $R \cong End_S(Q)$. Now $s: {}^IQ \cong {}^IS$ and so there is a ring isomorphism $\ell: End_S({}^IQ) \cong End_S({}^IS)$ defined by $f^{\ell} = s^{-1}fs$ for $f \in End_S({}^IQ)$. Hence by (1) of lemma 1.7 $(End_S(Q))_{rfI} \cong S_{rfI}$ i.e. $R_{rfI} \cong S_{rfI}$. By symmetry we also have $R_{cfI} \cong S_{cfI}$.

Now suppose $a \in End_S({}^IQ)$ and image (a) is contained in a finitely generated submodule. Then it is clear that image (a^ℓ) is also contained in a finitely generated submodule and conversely. By (2) of lemma 1.7 we see that ℓ induces $(End_S(Q))_{rbI} \cong S_{rbI}$ i.e. $R_{rbI} \cong S_{rbI}$. By symmetry we also have $R_{cbI} \cong S_{cbI}$.

Let ${}^IQ = \bigoplus_{i \in I} Q_i$ where $Q_i \cong Q$ and ${}^IS = \bigoplus_{i \in I} S_i$ and $S_i \cong S$. Let $a \in End_S({}^IQ)$ then a careful look at the constituent parts of the isomorphism $s : {}^IQ \cong {}^IS$ shows that if $Q_i a = o$ for almost all i then it follows that $S_i a = o$ for almost all i and conversely. Hence by (3) of lemma 1.7 ℓ induces $(End_S(Q))_{fI} \cong S_{fI}$ i.e. $R_{fI} \cong S_{fI}$.

Hence (1) implies (2) to (8) and $R_{cfI} \cong S_{cfI}$ and $R_{rfI} \cong S_{rfI}$. Since any ring isomorphism of two rings certainly induces a lattice isomorphism between their lattices of left (right) ideals we have by theorem 1.9 that

(2) implies (5) and (6)

(3) implies (8)

(4) implies (7)

and by cor.1 of theorem 3.4

(5) implies (1) and by symmetry so does (6)

(7) implies (1) and by symmetry so does (8) .

Hence the conditions (1) to (8) are equivalent. ~~We conclude the proof with the following example~~.

~~Example 3.7. There are rings R and S such that for any infinite set I $R_{rfI} \cong S_{rfI}$ but $R \not\approx_M S$.~~

Proof

~~Let R be a Noetherian ring and $S = R_{rfI}$ (note S has a 1). As R is Noetherian and S is not then $R \not\approx_M S$. But by lemma 1.8 $R_{rfI} \cong (R_{rfI})_{rfI} \cong S_{rfI}$.~~

Cor.1. Let R be a ring and I a set containing at least three elements. If $\Sigma : L({}_{R_{xI}} R_{xI}) \cong L({}_{S_{xI}} S_{xI})$ where x = f, rf, cf, rb, cb then $R_{xI} \cong S_{xI}$.

Proof

If I is finite this is cor.3 of theorem 2.3.

If I is infinite and x = rf or cf this is cor.4 of theorem 2.3.

If I is infinite and x = f, rb or cb then by theorem 1.9 it follows that

(1) $L({}_R^I R) \cong L({}_S^I S)$ if x = f

(2) $L({}_R R^I) \cong L({}_S S^I)$ if x = cb

(3) $L(R_R^I) \cong L(S_S^I)$ if x = rb .

In any of the cases (1) to (3) it follows from theorem 3.6 that $R_{xI} \cong S_{xI}$.

CHAPTER 4

SEMI-LINEAR ISOMORPHISMS

In this chapter we consider lattice isomorphisms which give rise to semi-linear isomorphisms. In the first part of the chapter we assume that our modules can be decomposed into a direct sum of more than 3 submodules each containing a free element. In the second part of the chapter we impose restrictions on our modules similar to (but more general than) those in Skornyakov (2). We also consider cyclic preserving lattice isomorphisms i.e. lattice isomorphisms under which the image and the inverse image of a cyclic module is again a cyclic module. In particular we show that if there is a cyclic preserving lattice isomorphism between the lattices of submodules of a free module of rank ≥ 3 over an inverse symmetric ring (a fairly mild ring condition) and a faithful module then there is a semi-linear isomorphism between them. A generalization along similar lines is given of a theorem of Skornyakov.

In remark 1 to theorem 2.3 we pointed out that a number of theorems in chapter 2 were true without imposing the restrictions that all rings have a 1 and all modules are unital. We now give some very general examples to show that some sort of restrictions are necessary to get theorems on semi-linear isomorphisms.

Suppose that S is a ring without a 1. We can adjoin a 1 by making the abelian group $S \oplus \mathbb{Z}$ into a ring with a 1 where we define multiplication by $(s, n)(t, m) = (st + nt + ms, nm)$ for any $n, m \in \mathbb{Z}$ and $s, t \in S$. Denote this ring by S^1. The map $s \longrightarrow (s, o)$ is a ring monomorphism:$S \longrightarrow S^1$ i.e. S is embedded in S^1 and S^1 has a 1 namely (o, 1).

Suppose ${}_SN$ is a module; then N can also be considered as a S^1-module by defining $(s, n)p = sp + np$ for $s \in S$, $n \in \mathbb{Z}$ and $p \in N$. Clearly every left S^1-submodule of N is also a left S-submodule and conversely. Hence $L({}_{S^1}N) = L({}_SN)$.

Suppose that ${}_SN$ is a module which is not necessarily faithful. Now $\ell(N)$ is a two-sided ideal of S and so $S/\ell(N)$ is a ring. Denote it by $\bar{S}$. There is a natural ring epimorphism $p:S \longrightarrow \bar{S}$ and we can consider N as a left $\bar{S}$-module by defining for $t \in \bar{S}$ and $x \in N$ $tx = sx$ where $s^p = t$. It is easily shown that this definition does not depend on the choice of s and gives us a well defined $\bar{S}$-module, which we note is faithful. Clearly every S-submodule of N is also a $\bar{S}$-submodule and conversely. Hence $L({}_SN) = L({}_{\bar{S}}N)$.

Let ${}_RM$ be a module and suppose that we want to prove a theorem of the form: if ${}_SN$ is any module with $\Sigma:L({}_RM) \cong L({}_SN)$ then there is a semi-linear isomorphism:$(R, M) \cong (S, N)$. If S does <u>not</u> have a 1 we know $L({}_S N) \cong L({}_{S^1} N)$ and so there are semi-linear isomorphisms $(R, M) \cong (S, N)$ and $(R, M) \cong (S^1, N)$. Hence there is a semi-linear isomorphism: $(S, N) \cong (S^1, N)$. This is impossible as S^1 has a 1 while S does not. Similarly if ${}_S N$ is not faithful then we know that $L({}_S N) \cong L({}_{\bar{S}} N)$ and so there is a semi-linear isomorphism: $(S, N) \cong (\bar{S}, N)$. But this is impossible as ${}_S N$ is not faithful while ${}_{\bar{S}} N$ is. These examples show that to get theorems about semi-linear isomorphisms we must assume that all rings have a 1 and that all modules are faithful.

Definition. Let ${}_R M$ be a module and c a unit ε R. Let $\ell : R \cong R$ be the ring isomorphism defined by $r \longrightarrow crc^{-1}$ and $s : (M, +) \cong (M, +)$ be the abelian group isomorphism defined by $m \longrightarrow cm$. It is easily seen that $(\ell, s) : (R, M) \cong (R, M)$ is a semi-linear isomorphism. We say that (ℓ, s) is the unit semi-linear isomorphism defined by c. Any unit semi-linear isomorphism induces the lattice isomorphism $1_{L({}_R M)}$ and our next lemma shows that the converse is also true for free modules of rank ≥ 2. The proof follows that of prop.3 of chapter 3.1 of Baer (1).

Lemma 4.1. Let ${}_R M$ be a free module of rank ≥ 2. A semi-linear isomorphism $(\ell, s) : (R, M) \cong (R, M)$ induces the lattice isomorphism $1_{L({}_R M)}$ if and only if (ℓ, s) is a unit semi-linear isomorphism.

Proof

Let (ℓ, s) be a unit semi-linear isomorphism: $(R, M) \cong (R, M)$ given by some unit c ε R. If m ε M then $(Rm)^s = R^\ell m^s = (cRc^{-1})cm = cRm = Rm$. Hence if P is a submodule of M then $P = \sum_{p \varepsilon P} Rp$ and so

$P^s = \sum_{p \varepsilon P} (Rp)^s = \sum_{p \varepsilon P} Rp = P$. Thus (ℓ, s) induces $1_{L({}_RM)}$.

Suppose conversely that (ℓ, s) induces $1_{L({}_RM)}$. Let x be a free element ε M. Then $(Rx)^s = (Rx)1_{L({}_RM)} = Rx$ and so $Rx^s = Rx$. Hence there are a, b ε R with $x = bx^s$ and $x^s = ax$. Since x is free so is x^s and we get $ab = ba = 1$. Hence for any free element x ε M there is a unit $f(x)$ ε R such that $x^s = f(x)x$.

Let $(e_i)_{i \varepsilon I}$ be a basis for M. Consider a fixed basis element e_1 and any other distinct basis element e_i. Then $e_1{}^s = f(e_1)e_1$ and $e_i{}^s = f(e_i)e_i$. Now $(e_1 + e_i)$ is a free element and so $f(e_1)e_1 + f(e_i)e_i = e_1{}^s + e_i{}^s = (e_1 + e_i)^s = f(e_1 + e_i)(e_1 + e_i) = f(e_1 + e_i)e_1 + f(e_1 + e_i)e_i$. Thus $f(e_1) = f(e_1 + e_i) = f(e_i)$. Hence for any i ε I $f(e_1) = f(e_i) = c$ where c is a unit. Let $1 \neq i \varepsilon I$ and r ε R then $e_1 + re_i$ is free. Thus $ce_1 + (re_i)^s = f(e_1)e_1 + (re_i)^s = e_1{}^s + (re_i)^s = (e_1 + re_i)^s = f(e_1 + re_i)(e_1 + re_i) = f(e_1 + re_i)e_1 + f(e_1 + re_i)re_i$. Hence $c = f(e_1 + re_i)$ and so $(re_i)^s = f(e_1 + re_i)re_i = cre_i$. Similarly since rank of $M \geq 2$ we can show that $(re_1)^s = cre_1$. Now for any element $x = \Sigma r_i e_i$ ε M we have $x^s = (\Sigma r_i e_i)^s = \Sigma (r_i e_i)^s = \Sigma c r_i e_i = c \Sigma r_i e_i = cx$. Hence s is left multiplication by the unit c. Suppose r ε R then $r^\ell e_1{}^s = (re_1)^s = cre_1 = crc^{-1}ce_1 = crc^{-1}e_1{}^s$. But $e_1{}^s$ is free and so $r^\ell = crc^{-1}$ and (ℓ, s) is the unit semi-linear isomorphism defined by c.

<u>Cor.1.</u> If ${}_RM$ is a free module of rank ≥ 2 then a linear isomorphism $s: {}_RM \cong {}_RM$ induces $1_{L({}_RM)}$ if and only if s is left multiplication by a unit $c \in$ centre of R.

<u>Proof</u>

Left multiplication by a unit $c \in$ centre of R is clearly a linear isomorphism inducing $1_{L({}_RM)}$. If $s: {}_RM \cong {}_RM$ is a linear isomorphism inducing $1_{L({}_RM)}$ then (1, s) is a semi-linear isomorphism inducing $1_{L({}_RM)}$, where 1 is the identity ring isomorphism $r \longrightarrow r$. Hence by lemma 4.1 (1, s) is a unit semi-linear isomorphism for some unit $c \in R$ and so for any $r \in R$ $r = r^1 = crc^{-1}$ i.e. $c \in$ centre of R. Thus s is left multiplication by a unit $c \in$ centre of R. This proves the remark made at the end of theorem 3.4.

It is easy to see that the condition rank $M \geq 2$ cannot be weakened. Let D be any division ring with a ring automorphism $s: D \cong D$ which is not inner (e.g. the complex numbers where s is conjugation). Consider D as a left D-module then $(s, s):(D, D) \cong (D, D)$ is a semi-linear isomorphism inducing $1_{L({}_DD)}$. As s is not an inner automorphism (s, s) is not a unit semi-linear isomorphism.

<u>Theorem 4.2.</u> Let ${}_RM$ be a module which is the direct sum of an independent set of submodules $(P_i)_{i \in I}$, where I is an index set containing at least three elements and where for each $i \in I$ there is a free element $e_i \in P_i$. Suppose ${}_SN$ is a module with $\Sigma: L({}_RM) \cong L({}_SN)$ and such that for some $i \in I$ and free element $f_i \in N$ $(Re_i)^{\Sigma} = Sf_i$. Then

(1) there is a semi-linear isomorphism $(\ell, s):(R, M) \cong (S, N)$

(2) defining $P_i^* = \sum_{j \neq i} P_j$ and $Q_i^* = P_i^{*\Sigma}$ then (ℓ, s) induces $\Sigma: L(_R P_i^*) \cong L(_S Q_i^*)$

(3) (ℓ, s) is unique to within unit semi-linear isomorphism with respect to property (2)

(4) if I is infinite then (ℓ, s) induces $\Sigma: L(_R M) \cong L(_S N)$.

Proof

(1) Putting $P = R$, $Q = S$ in (4) of theorem 2.3 we get a semi-linear isomorphism $(\ell, s):[End_R(R), Hom_R(R, M)] \cong [End_S(S), Hom_S(S, N)]$ i.e. $(\ell, s):(R, M) \cong (S, N)$.

(2) By remark 2 to theorem 2.3 (ℓ, s) induces $\Sigma: L(_R P_i^*) \cong L(_S Q_i^*)$.

(3) Suppose (ℓ', s') is another lattice isomorphism inducing $\Sigma: L(_R P_i^*) \cong L(_S Q_i^*)$ then $(\ell, s)(\ell', s')^{-1}$ induces $1_{L(_R P_i^*)}$. Since $_R P_i^*$ has a free module of rank ≥ 2 we have by lemma 4.1 that (ℓ, s) and (ℓ', s') differ by a unit semi-linear isomorphism.

(4) If I is infinite then any finitely generated submodule A of M is contained in P_i^* for some $i \in I$. Thus we see that (ℓ, s) induces $\Sigma: F(_R M) \cong F(_S N)$. Hence by lemma 1.3 (ℓ, s) induces $\Sigma: L(_R M) \cong L(_S N)$.

Cor.1. Let $_R M$ be a free module of rank ≥ 3 on free generators $(e_i)_{i \in I}$. Suppose $_S N$ is a module with $\Sigma: L(_R M) \cong L(_S N)$ where, for some $i \in I$ and free element $f_i \in N$, $(Re_i)^\Sigma = Sf_i$. Then there is a semi-linear isomorphism $(\ell, s):(R, M) \cong (S, N)$ which for any $j \in I$ induces $\Sigma: L(_R P_j^*) \cong L(_S Q_j^*)$ where $P_j^* = \sum_{i \neq j} Re_i$ and $Q_j^* = P_j^{*\Sigma}$.

Proof

This follows immediately from theorem 4.2 putting $P_i = Re_i$.

We have already noted in example 2.4 that this corollary fails if the rank of $M < 3$. Our next theorem shows that even if the rank of $M = 2$ the rings R and S are closely related.

Theorem 4.3. Let ${}_RM$ be a free module of rank 2 on free generators e_1 and e_2. Suppose ${}_SN$ is a module with $\Sigma : L({}_RM) \cong L({}_SN)$ such that, for some free element $f_1 \in N$, $(Re_1)^\Sigma = Sf_1$. Then

(1) there is a set isomorphism $t : R \longrightarrow S$ with $1^t = 1$

(2) if $a \in R$ then $(Ra)^t = Sa^t$ and $(\ell_R(a))^t = \ell_S(a^t)$

(3) t induces the lattice isomorphism $L({}_RR) \cong L({}_SS)$ induced by $\Sigma_i : L({}_RRe_i) \cong L({}_SSf_i)$ for $i = 1, 2$.

(4) if $U(R)$, $U(S)$ are the groups of units of R and S respectively then $U(R)^t = U(S)$

(5) if $a, b \in R$ and $Ra \cap Rb = 0$ then $(a + b)^t = a^t + b^t$.

Proof

These results are basically translations of the results of lemma 2.2 to our particular case.

By (1) of lemma 2.2 there is a set isomorphism $\ell_{1,2}$: $Hom_R(Re_1, Re_2) \longrightarrow Hom_S(Sf_1, (Re_2)^\Sigma)$. Now there is an isomorphism $Re_1 \cong Re_2$ defined by $e_1 \longrightarrow e_2$ and so by (2) of lemma 2.2 there is an isomorphism $Rf_1 \cong (Re_2)^\Sigma$ defined by $f_1 \longrightarrow f_2$ for some $f_2 \in N$ with $\ell(f_2) = \ell(f_1) = 0$.

There is a set isomorphism $x: R \longrightarrow \mathrm{Hom}_R(Re_1, Re_2)$ defined by $r \longrightarrow (e_1 \longrightarrow re_2)$. Similarly there is a set isomorphism $y: S \longrightarrow \mathrm{Hom}_S(Sf_1, Sf_2)$. Thus we have a set isomorphism $t = x\ell_{1,2}y^{-1}: R \longrightarrow S$ with $1^t = 1$.

Since we have that f_1 and f_2 are free we get lattice isomorphisms $\Sigma_i : L(_RR) \cong L(_SS)$ induced by $\Sigma_i : L(_RRe_i) \cong L(_SSf_i)$ for $i = 1, 2$. If A is a left ideal of R then $Ae_1 + A(e_1 + e_2) = Ae_1 + Ae_2 = Ae_2 + A(e_1 + e_2)$. Applying Σ we get $A^{\Sigma_1}f_1 + B(f_1 + f_2) = A^{\Sigma_1}f_1 + A^{\Sigma_2}f_2 = A^{\Sigma_2}f_2 + B(f_1 + f_2)$ where B is a left ideal of S and $A(e_1 + e_2)^{\Sigma} = B(f_1 + f_2)$. If $t \in A^{\Sigma_1}$ then $tf_1 = sf_2 + b(f_1 + f_2)$ where $b \in B$. Thus $t = b$ and $A^{\Sigma_1} \subset B$. Similarly $A^{\Sigma_2} \subset B$. If on the other hand $b \in B$ then $b(f_1 + f_2) = s_1f_1 + s_2f_2$ where $s_i \in A^{\Sigma_i}$ for $i = 1, 2$. Hence $b = s_1 = s_2 \in A^{\Sigma_1} \cap A^{\Sigma_2}$ and so $B \subset A^{\Sigma_1} \cap A^{\Sigma_2}$. Therefore $A^{\Sigma_1} = B = A^{\Sigma_2}$. Thus $\Sigma_1 = \Sigma_2 = \Delta$ say.

For any $a \in R$ there is a $f \in \mathrm{Hom}_R(Re_1, Re_2)$ defined by $e_1 \longrightarrow a e_2$. By (2) of lemma 2.2 we have (A) image $(f)^{\Sigma}$ = image $(f\ell_{1,2})$ (B) $\ker(f)^{\Sigma} = \ker(f\ell_{1,2})$.

From (A) we deduce that $(Rae_2)^{\Sigma} = Sa^te_2$ i.e. $(Ra)^{\Delta} = Sa^t$. If $b \in Ra$ and $g \in \mathrm{Hom}_R(Re_1, Re_2)$ is defined by $e_1 \longrightarrow be_2$ then image$(g) \subset$ image (f). Hence $Sb^t \subset Sa^t$ for any $b \in Ra$ and so $(Ra)^t \subset Sa^t$. Symmetrically $(Sa^t)^{t^{-1}} \subset Ra$ and thus $Sa^t \subset (Ra)^t$ and $(Ra)^t = Sa^t = (Ra)^{\Delta}$. If A is a left ideal of R then clearly $A^t \subset A^{\Delta}$ and by symmetry $(A^{\Delta})^{t^{-1}} \subset (A^{\Delta})^{\Delta^{-1}} = A$. Hence $A^{\Delta} \subset A^t$ and $A^t = A^{\Delta}$. Thus t induces Δ and $(Ra)^t = Sa^t$.

From (B) we deduce that $(\ell_R(a)e_1)^\Sigma = \ell_S(a^t)f_1$ i.e. $\ell_R(a)^\Delta = \ell_S(a^t)$. Therefore $\ell_R(a)^t = \ell_R(a)^\Delta = \ell_S(a^t)$. Now $U(R) = (a \in R: \ell_R(a) = 0$ and $Ra = R)$. Hence $U(R)^t = U(S)$. If $a, b \in R$ and $Ra \cap Rb = 0$ then it is clear from (4) of lemma 2.2 that $(a + b)^t = a^t + b^t$.

We now show how we can drop the assumption that f_i is free in theorem 4.2 by imposing suitable restrictions on R, S and Σ.

<u>Lemma 4.4.</u> Let $_RM$ be a module and X, Y submodules such that $X \cap Y = 0$. Suppose $_SN$ is a module where $\Sigma: L(_RM) \cong L(_SN)$ and for some $x', y' \in N$ and $x \in X, y \in Y$ $X^\Sigma = Sx'$, $Y^\Sigma = Sy'$ and $R(x + y) = (S(x' + y'))^{\Sigma^{-1}}$. Then $X = Rx$ and $Y = Ry$.

<u>Proof</u>

We have $Sx' + Sy' = S(x' + y') + Sx' = S(x' + y') + Sy'$. Applying Σ^{-1} we get $X + Y = R(x + y) + X = R(x + y) + Y$. Hence

$$\begin{aligned} X + Y &= R(x + y) + Y \\ &= R(x + y) + Ry + Y \\ &= Rx + Ry + Y \\ &= Rx + Y . \end{aligned}$$

Intersecting X with both sides we get $X = Rx$. Similarly $Y = Ry$. This lemma is prop.9.1 of Baer (2).

<u>Definition.</u> Let R be a ring such that any elements $x, y \in R$ satisfying $xy = 1$ also satisfy $yx = 1$. Then R is called an <u>inverse symmetric ring</u>.

A ring R is not inverse symmetric if and only if it contains a copy of itself as a proper direct summand. In fact it is not difficult to see that, for any module ${}_RM$, $\mathrm{End}_R(M)$ is inverse symmetric if and only if M does not contain a copy of itself as a proper direct summand.

If R is not inverse symmetric then R contains an infinite direct sum of isomorphic left ideals generated by idempotents (see Jacobson (2)). Hence any sort of minimum or maximum condition on principal left (right) ideals or on left (right) annihilator ideals is sufficient to ensure a ring is inverse symmetric. Other obvious examples of such rings are commutative rings and integral domains. We shall see in chapter 6 that this condition also arises naturally in the study of regular rings.

Lemma 4.5. Let ${}_RM$ be a module and $x, y \in M$ with $Rx \cap Ry = 0$. Suppose ${}_SN$ is a module with $\Sigma : L({}_RM) \cong L({}_SN)$ where for some $x', y' \in N$ $(Rx)^{\Sigma} = Sx'$ and $(Ry)^{\Sigma} = Sy'$. Then if

either (1) $(S(x' + y'))^{\Sigma^{-1}}$ is cyclic, $\ell(x) = 0$ and R is inverse symmetric

or (2) $\ell(x) \subset \ell(y)$ and S is subcommutative

then $\ell(x') \subset \ell(y')$.

Proof

(1) Suppose $(S(x' + y'))^{\Sigma^{-1}}$ is cyclic and $= R(x_1 + y_1)$ for some $x_1 \in Rx$ and $y_1 \in Ry$. By lemma 4.4 $Rx = Rx_1$ and $Ry = Ry_1$ and so there are $a, b \in R$ with $x_1 = ax$ and $x = bx_1$. Hence $x = bax$ and if $\ell(x) = 0$ we get $ba = 1$. If R is inverse symmetric then $ab = 1$ and so a is a unit and $x_1 = ax$ is a free element.

Since x_1 is free there is a homomorphism $f: Rx_1 \longrightarrow Ry_1$ defined by $x_1 \longrightarrow -y_1$. We can "represent" f as in lemma 2.1 by $(m - m^f : m \in Rx_1) = R(x_1 + y_1)$. By (1) of lemma 2.2 there is a homomorphism $g: (Rx_1)^\Sigma \longrightarrow (Ry_1)^\Sigma$ which is "represented" by $(R(x_1 + y_1))^\Sigma = S(x' + y')$ i.e. $x'^g = -y'$. Hence $\ell(x') \subset \ell(-y') = \ell(y')$.

(2) Suppose that $\ell(x) \subset \ell(y)$. Then there is an epimorphism $f: Rx \longrightarrow Ry$ defined by $x \longrightarrow y$. Hence by (2) of lemma 2.2 there is an epimorphism $g: Sx' \longrightarrow Sy'$ defined by $x' \longrightarrow ty'$ for some $t \in S$ such that $Sty' = Sy'$. We have then that $\ell(x') \subset \ell(ty')$. If S is subcommutative then $\ell(y') = \ell(Sy') = \ell(ty')$. Hence $\ell(x') \subset \ell(y')$.

<u>Definition.</u> Let ${}_RM$ and ${}_SN$ be modules such that $\Sigma: L({}_RM) \cong L({}_SN)$. Define the following conditions on Σ.

(C_1) For any $x \in M$ there is a $y \in N$ with $(Rx)^\Sigma = Sy$.

(C_2) For any $y \in N$ there is a $x \in M$ with $(Sy)^{\Sigma^{-1}} = Rx$.

If Σ satisfies C_1 and C_2 we call Σ <u>a cyclic preserving lattice isomorphism</u> and we write $L({}_RM) \overset{c}{\cong} L({}_SN)$.

<u>Theorem 4.6.</u> Let ${}_RM$ be a module which is the direct sum of an independent set of submodules $(P_i)_{i \in I}$, where I is an index set containing at least three elements and where for each $i \in I$ there is a free element $e_i \in P_i$. Suppose that ${}_SN$ is a faithful module with $\Sigma: L({}_RM) \overset{c}{\cong} L({}_SN)$. If

either (1) R is inverse symmetric

or (2) S is subcommutative

then there is a semi-linear isomorphism: $(R, M) \cong (S, N)$.

Proof

Let e_1 be a fixed element of $(e_i)_{i \in I}$. As Σ is cyclic preserving $(Re_1)^\Sigma = Sf_1$ for some $f_1 \in N$. By theorem 4.2 we need only show that f_1 is a free element.

By theorem 2.3 $Sf_1 \cong (Re_i)^\Sigma$ for each $i \in I$ and so $(Re_i)^\Sigma = Sf_i$ for some $f_i \in N$ with $\ell(f_i) = \ell(f_1)$. Let $Q_i = P_i^\Sigma$, $P_i^* = \sum_{j \neq i} P_j$ and $Q_i^* = P_i^{*\Sigma}$. If $q \in Q_i^*$ then since Σ is cyclic preserving there is a $p \in P_i^*$ with $(Rp)^\Sigma = Sq$.

Now $Re_i \cap P_i^* = 0$, $(Re_i)^\Sigma = Sf_i$, $(Rp)^\Sigma = Sq$ and $0 = \ell(e_i) \subset \ell(p)$. Furthermore $(S(f_i + q))^{\Sigma^{-1}}$ is cyclic since Σ is cyclic preserving. Hence if either (1) or (2) hold it follows from lemma 4.5 that $\ell(f_i) \subset \ell(q)$. But q was any element $\in Q_i^*$ and so $\ell(f_1) = \ell(f_i) \subset \ell(Q_i^*)$. Therefore $\ell(f_1) \subset \bigcap_{i \in I} \ell(Q_i^*) = \bigcap_{i \in I} \ell(Q_i) = \ell(N) = 0$ as ${}_SN$ is faithful. Thus f_1 is free and the result follows.

Cor.1. Let ${}_RM$ be a free module of rank ≥ 3 and ${}_SN$ a faithful module such that $\Sigma : L({}_RM) \overset{c}{\cong} L({}_SN)$. If R is inverse symmetric then there is a semi-linear isomorphism: $(R, M) \cong (S, N)$.

Proof

Put $P_i = Re_i$ in theorem 4.6 where $(e_i)_{i \in I}$ are a basis for ${}_RM$.

Cor.2. Let ${}_RM$ be a free module of rank ≥ 3 and ${}_SN$ a faithful module with $\Sigma : L({}_RM) \cong L({}_SN)$. If Σ satisfies condition C_1 and S is subcommutative then there is a semi-linear isomorphism: $(R, M) \cong (S, N)$.

Proof

Let $(e_i)_{i \in I}$ be a basis for M and $(Re_i)^{\Sigma} = Sf_i$ for elements $(f_i)_{i \in I}$ of N with $\ell(f_i) = \ell(f_j)$ for all $i, j \in I$. By theorem 4.6 we need only show that $\ell(f_i) \subset \ell(q)$ for any $q \in \sum_{j \neq i} Sf_j$. Suppose $q = \sum_{j \neq i} s_j f_j$ for some $(s_j)_{j \in I} \in S$. Then $\ell(q) = \bigcap_{j \neq i} \ell(s_j f_j)$. But $\ell(s_j f_j) = [t \in S : ts_j \in \ell(f_j)] = [t \in S : ts_j \in \ell(f_i)] = \ell(s_j f_i) \supset \ell(f_i)$ since S is subcommutative. Hence $\ell(f_i) \subset \bigcap_{j \neq i} \ell(s_j f_j) = \ell(q)$ and the result follows.

The following example shows that the conditions on Σ in cor.1 and cor.2 cannot be dropped.

Example 4.7. There exist free modules ${}_RM$, ${}_SN$ of rank ≥ 3 such that $\Sigma : L({}_RM) \cong L({}_SN)$ where (1) Σ satisfies C_2 (2) R is inverse symmetric (3) S is commutative (4) $R \not\cong S$.

Proof

Let S by a commutative field and n an integer ≥ 3. Then since $(S_3)_n \cong (S)_{3n}$ we have by von Neumann's theorem a lattice isomorphism $\Sigma : L({}_{S_3}S_3{}^n) \cong L({}_SS^{3n})$. If we put $R = S_3$ then $R \not\cong S$ and is inverse symmetric (it is Noetherian for example) and ${}_RM = {}_{S_3}S_3{}^n$ is a free module of rank ≥ 3. S is commutative and ${}_SN = {}_SS^{3n}$ is a free module of rank ≥ 3.

But any cyclic module over a field S must be $\cong S$ and so is simple. Hence any lattice isomorphic image of a cyclic module over S is again simple and so cyclic. Hence Σ satisfies C_2 (but not C_1!).

The lattice isomorphisms $\Sigma:L(_RM) \cong L(_SN)$ and $\Sigma^{-1}:L(_SN) \cong L(_RM)$ show that cor.1 need not hold if Σ does not satisfy one of the conditions C_1 or C_2. The lattice isomorphism $\Sigma:L(_RM) \cong L(_SN)$ shows that cor.2 need not hold if Σ does not satisfy C_1, even if it satisfies C_2.

So far we have assumed a rather explicit form for our modules viz. that they can be split into a direct sum of more than three submodules. We now impose rather different restrictions allowing us to study modules which are not necessarily of this type. The conditions we consider are slightly weaker versions of the following conditions, which appear in Skornyakov (2).

Definition. A module $_RM$ is called admissible if the following properties hold.

(M_1) For any $x, y, z \in M$ there is a free element w with $Rw \cap (Rx+Ry+Rz) = 0$

(M_2) If $t \in M$ and x, y, u are free elements $\in M$ such that $Ru \cap Rt \neq 0$ and $Rx \cap Ry \neq 0$ then there is a free element $w \in M$ with $Rw \cap Rx = Rw \cap Ry = Rw \cap Rt = Rw \cap Ru = 0$.

Definition. Let $_RM$ and $_SN$ be modules and $K(_RM)$ and $K(_SN)$ sublattices (with respect to the operations $+$ and $\cap$) of $L(_RM)$ and $L(_SN)$ respectively.

Suppose further $F({}_RM) \subset K({}_RM)$ and $F({}_SN) \subset K({}_SN)$. A lattice isomorphism $\Sigma : K({}_RM) \cong K({}_SN)$ is called a projective mapping if

(1) Σ is cyclic preserving

(2) there are free elements $u \in M$, $u' \in N$ with $(Ru)^{\Sigma} = Su'$.

Theorem (Skornyakov). Let R be an inverse symmetric ring and ${}_RM$ an admissible module. Suppose ${}_SN$ is a module and $\Sigma : K({}_RM) \cong K({}_SN)$ is a projective mapping. Then there is a semi-linear isomorphism: $(R, M) \cong (S, N)$ inducing Σ.

Remarks

(1) Cor.1 to lemma 1.3 shows that no generality is gained by assuming $K({}_RM) \cong K({}_SN)$ rather than $L({}_RM) \cong L({}_SN)$.

(2) From condition M_1 for admissibility there is a free element w_1 such that $Ro \cap Rw_1 = 0$, a free element w_2 such that $Rw_1 \cap Rw_2 = 0$, a free element w_3 such that $(Rw_1 \oplus Rw_2) \cap Rw_3 = 0$ and a free element w_4 such that $(Rw_1 \oplus Rw_2 \oplus Rw_3) \cap Rw_4 = 0$. Hence condition M_1 implies that M has a free submodule of rank 4. So far we have seen that the existence of a free module of rank 3 is usually all that is needed to get theorems on semi-linear isomorphisms. For example a vector space of dimension 3 does not satisfy M_1 and so is not admissible. Skornyakov's theorem thus fails to generalize the first fundamental theorem of projective geometry (see chap.3.1 of Baer (1)) for dimension 3.

In our theorem we replace conditions M_1, M_2 by weaker conditions S_1, S_2. We will show in chap.7 that any left module of rank ≥ 3 over a left Ore domain satisfies S_1 and S_2. Hence our theorem gives a true generalization of the first fundamental theorem of projective geometry.

(3) We show (c.f. theorem 4.6) that if R is inverse symmetric, Σ cyclic preserving and $_SN$ faithful then Σ is a projective mapping. Alternatively if we only assume that there are free elements $u \in M$, $u' \in N$ with $(Ru)^\Sigma = Su'$ i.e. drop the conditions that R is inverse symmetric and Σ is cyclic preserving then the conclusions of Skornyakov's theorem still hold (c.f. theorem 4.2).

(4) Finally we note that in this case the lattice isomorphism Σ <u>is</u> induced by the semi-linear isomorphism.

<u>Definition.</u> Let $_RM$ be a module then we define the following conditions on M.

(S_1) For any x, y, z $\in$ M with $Rx \cap Ry = 0$ there is a free element w such that $(Rx + Ry) \cap Rw = (Ry + Rz) \cap Rw = (Rz + Rx) \cap Rw = 0$.

(S_2) If $t \in M$ and u, x, y are free elements $\in M$ with $(Ru + Rt) \cap Rx = (Ru + Rt) \cap Ry = 0$ and $Rx \cap Ry \neq 0$, $Ru \cap Rt \neq 0$ then there is a free element $w \in M$ such that $Ru \cap Rw = Rt \cap Rw = Rx \cap Rw = Ry \cap Rw = 0$.

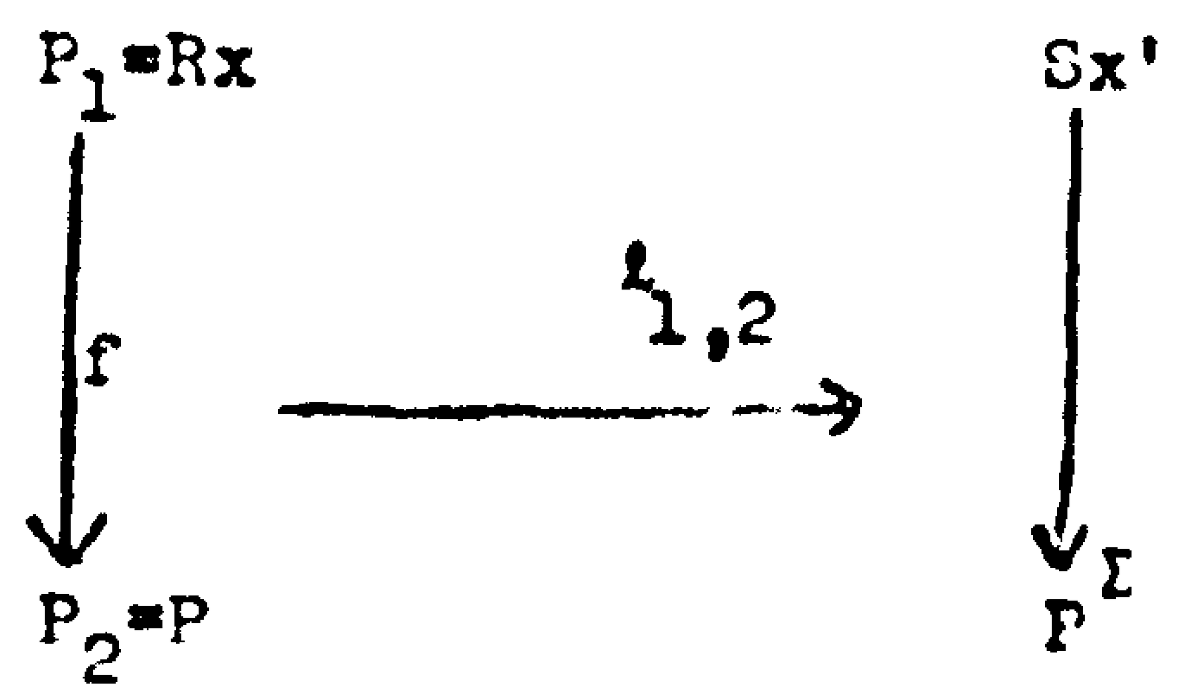
We note that condition M_i implies S_i for i = 1, 2.

<u>Theorem 4.8.</u> Let $_RM$ be a module satisfying conditions S_1 and S_2 and $_SN$ a module with $\Sigma : L(_RM) \cong L(_SN)$. If there are free elements $u \in M$, $u' \in N$ such that $(Ru)^\Sigma = Su'$ then there is a semi-linear isomorphism: $(R, M) \cong (S, N)$ inducing Σ.

Proof.

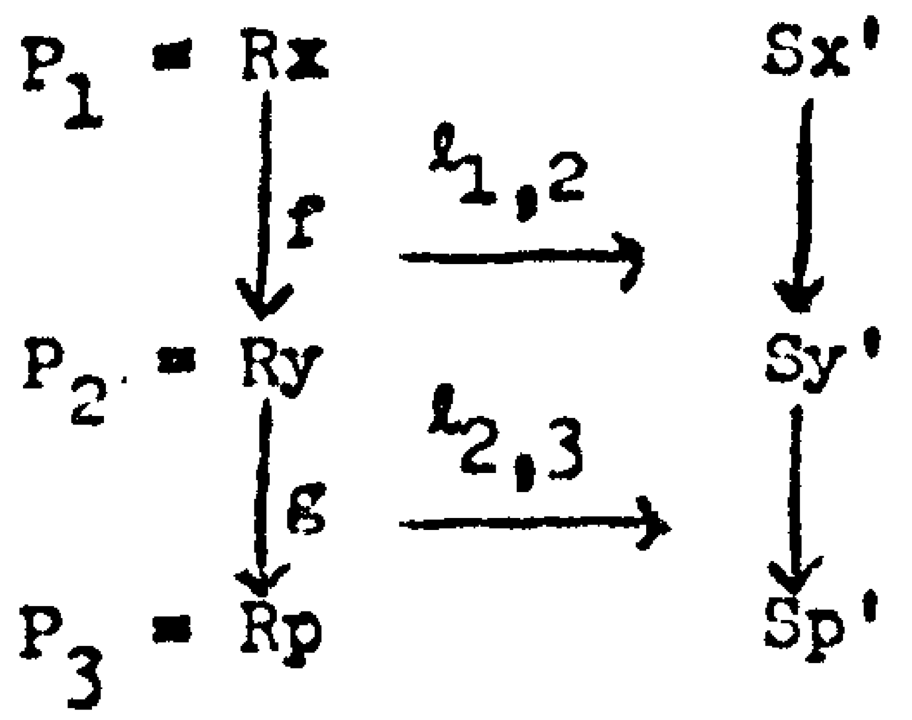
We follow Skornyakov's proof making the necessary modifications.

(1) Suppose that x is a free element with $(Rx)^{\Sigma} = Sx'$ for some $x' \in N$ and that P is a submodule with $Rx \cap P = 0$. Putting $P_1 = Rx$ $P_2 = P$ we have by (1) of lemma 2.2 a set isomorphism $\ell_{1,2}: \mathrm{Hom}_R(Rx, P) \cong \mathrm{Hom}_S(Sx', P^{\Sigma})$. As x is free this gives rise to a map $h(x, x'): P \longrightarrow P^{\Sigma}$ defined as follows. If $p \in P$ and $f \in \mathrm{Hom}_R(Rx, P)$ is defined by $x \longrightarrow p$ then we define $p\, h(x, x')$ to be $x' f \ell_{1,2} = p'$ say. By (2) of lemma 2.2 we note that $(Rp)^{\Sigma} = \mathrm{image}(f)^{\Sigma} = \mathrm{image}(f\ell_{1,2}) = Sp'$.



If y is a free element $\in P$ then the map $f: Rx \longrightarrow Ry$ defined by $x \longrightarrow y$ is an isomorphism. Hence by (2) of lemma 2.2 $(f)\ell_{1,2}$ is an isomorphism. If $y' = yh(x, x')$ then we have $\ell(x') = \ell(y')$ and since by (2) of lemma 2.2 $(f\ell_{1,2})^{-1} = (f^{-1})\ell_{2,1}$ we get $xh(y, y') = x'$.

Suppose $p \in P$ and $Ry \cap Rp = 0$. Then we have two maps $h(x, x')$ and $h(y, y')$ mapping $Rp \longrightarrow (Rp)^{\Sigma}$. Put $P_1 = Rx$, $P_2 = Ry$, $P_3 = Rp$ in lemma 2.2. Let $f: Rx \longrightarrow Ry$ and $g: Ry \longrightarrow Rp$ be defined by $x \longrightarrow y$ and $y \longrightarrow p$ respectively.



By (3) of lemma 2.2 $(fg)\ell_{1,3} = (f)\ell_{1,2}$ $(g)\ell_{2,3}$. Now $xfg = p$ and so $x'(fg)\ell_{1,3} = ph(x, x')$. By definition of y', $x'(f)\ell_{1,2} = y'$ and so $x'(f)\ell_{1,2}\,(g)\ell_{2,3} = y'(g)\ell_{2,3} = p\,h(y, y')$. Hence $ph(x,x') = ph(y, y')$.

(2) Let $a, b, c \in M$ and $a' \in N$. Suppose a, b are free and that $(Ra)^{\Sigma} = Sa'$ and $Ra \cap Rb = Rb \cap Rc = Rc \cap Ra = 0$. If $b' = bh(a, a')$ then $ch(a, a') = ch(b, b')$.

<u>Proof</u>

By S_1 there is a free element $d \in M$ such that

(A) $(Ra \oplus Rb) \cap Rd = 0$

(B) $(Rb \oplus Rc) \cap Rd = 0$

(C) $(Rc \oplus Ra) \cap Rd = 0$.

Let $d' = d\,h(a, a')$. Applying (1) we get from

(A) that $b' = bh(a, a') = bh(d, d')$

(B) that $ch(d, d') = ch(b, b')$

(C) that $ch(d, d') = ch(a, a')$.

Hence $ch(a, a') = ch(b, b')$.

From now on we assume that $u \in M$, $u' \in N$ are free elements with $(Ru)^{\Sigma} = Su'$.

(3) Suppose $x, y \in M$ are free elements and $(Ru + Rt) \cap Rx = (Ru + Rt) \cap Ry = 0$ for some $t \in M$. If $x' = xh(u, u')$ and $y' = yh(u, u')$ then $th(x, x') = th(y, y')$.

<u>Proof</u>

(a) <u>Suppose $Rx \cap Ry = 0$</u>. Putting $a = u$ $a' = u'$ $b = x$ $c = y$ in (2) we get $y' = yh(u, u') = yh(x, x')$.

Putting $a = x$, $a' = x'$, $b = y$, $c = t$ in (2) we get $th(x, x') = th(y, y')$.

(b) Suppose $Ru \cap Rt = 0$. Putting $a = u$, $a' = u'$, $b = x$, $c = t$ in (2) we get $th(u, u') = th(x, x')$. Similarly $th(u, u') = th(y, y')$ and so $th(x, x') = th(y, y')$.

(c) Suppose $Rx \cap Ry \neq 0$, $Ru \cap Rt \neq 0$. Since $(Ru + Rt) \cap Rx = (Ru + Rt) \cap Ry = 0$ we have by S_2 (precisely where this condition is needed) a free element $w \in M$ such that $Ru \cap Rw = Rt \cap Rw = Rx \cap Rw = Ry \cap Rw = 0$. Let $w' = wh(u, u')$.

Putting $a = u$, $a' = u'$, $b = w$, $c = x$ in (2) we get $xh(u, u') = xh(w, w') = x'$.

Putting $a = w$, $a' = w'$, $b = x$, $c = t$ in (2) we get $th(w, w') = th(x, x')$. Similarly $th(w, w') = th(y, y')$ and so $th(x, x') = th(y, y')$.

(4) We now define a map $s: M \longrightarrow N$ as follows. Let $t \in M$. By S_1 there is a free element $x \in M$ with $(Ru + Rt) \cap Rx = 0$. Let $x' = xh(u, u')$ and define $t^s = th(x, x')$. If y is another free element with $(Ru + Rt) \cap Ry = 0$ then by (3) $th(y, y') = th(x, x')$ and so s is well defined.

We note that if $Ru \cap Rw = 0$ for some free element $w \in N$ then $u^s = uh(w, w')$ where $w' = wh(u, u')$. Hence by (1) $u^s = u'$. Suppose $t \in M$ and $Ru \cap Rt = 0$. If w is a free element with $(Ru \oplus Rt) \cap Rw = 0$ then by (1) we get $th(u, u') = th(w, w') = t^s$.

We note that if $t \in M$ and w is a free element with $(Ru + Rt) \cap Rw = 0$ then $t^s = th(w, w')$ and so by (1) $St^s = (Rt)^\Sigma$. Hence Σ satisfies C_1.

(5) We now show s is a homomorphism. Suppose $t \in M$ and w is a free element $\in M$ with $Ru \cap Rw = Rt \cap Rw = 0$. Then there is a free element $w_1 \in M$ with $(Ru + Rt) \cap Rw_1 = (Ru \oplus Rw) \cap Rw_1 = (Rt \oplus Rw) \cap Rw_1 = 0$. Suppose $w' = w^s = wh(u, u')$ and $w_1' = w_1^s = w_1h(u, u')$. Then by (1) $w_1' = w_1h(u, u') = w_1h(w, w')$ and $th(w, w') = th(w_1, w'_1) = t^s$. Thus we have shown that if $Ru \cap Rw = Rt \cap Rw = 0$ then $t^s = th(w, w^s)$.

Now suppose $a, b \in M$ and $Ra \cap Rb = 0$. By S_1 there is a free element w such that $(Ra \oplus Rb) \cap Rw = (Ru + Ra) \cap Rw = (Ru + Rb) \cap Rw = 0$. Thus $(a + b)^s = (a + b)h(w, w^s)$, $a^s = ah(w, w^s)$ and $b^s = bh(w, w^s)$.

Put $P_1 = Rw$, $P_2 = Ra \oplus Rb$ in (4) of lemma 2.2. Let $\ell_a : Rw \longrightarrow Ra$ and $\ell_b : Rw \longrightarrow Rb$ be defined by $w \longrightarrow a$ and $w \longrightarrow b$ respectively. Then $(\ell_a + \ell_b)\ell_{1,2} = (\ell_a)\ell_{1,2} + (\ell_b)\ell_{1,2}$. Hence $(a + b)^s = (a + b)h(w, w^s) = w^s[(\ell_a + \ell_b)\ell_{1,2}] = w^s[(\ell_a)\ell_{1,2} + (\ell_b)\ell_{1,2}] = a\, h(w, w^s) + b\, h(w, w^s) = a^s + b^s$.

Now suppose $Ra \cap Rb \neq 0$. Then by S_1 there is a free element $w \in M$ with $(Ra + Rb) \cap Rw = 0$. We note then that $R(a + b) \cap Rw = Ra \cap R(b + w) = Rb \cap Rw = 0$. Hence $(a + b)^s + w^s = (a + b + w)^s = a^s + (b + w)^s = a^s + b^s + w^s$. Therefore $(a + b)^s = a^s + b^s$ and s is a homomorphism.

(6) We now show that s is an isomorphism: $(M, +) \cong (N, +)$. So far we have not used the fact that u' is free. This fact will be essential in proving that s is an epimorphism.

If $t^s = o$ for some $t \in M$ then $(Rt)^\Sigma = St^s = 0$ and so $t = o$. Hence S is a monomorphism.

Let $q \in N$ then by cor.1 of lemma 1.1 $(Sq)^{\Sigma^{-1}}$ is finitely generated $= Rp_1 + \dots + Rp_n$, say. Then $Sq = \sum_1^n (Rp_i)^{\Sigma} = \sum_1^n Sp_i^s$ and so $q = \sum_1^n t_i p_i^s$ for some $(t_i)_1^n \in S$.

By S_1 there is a free element $w_i \in M$ with $(Rp_i + Ru) \cap Rw_i = 0$. Now by (1) $\ell(w_i^s) = \ell(u^s) = \ell(u') = 0$ as u' is free. Hence $Sw_i^s \cap Sp_i^s = 0$ and w_i^s is free. Put $P_1 = Rw_i$ and $P_2 = Rp_i$. Then by (1) of lemma 2.2 there is a map $f: Rw_i \longrightarrow Rp_i$ such that $f\iota_{1,2} = g$ where g is defined by $w_i^s \longrightarrow t_i p_i^s$. If $p_i' = w_i f$ then $p_i' h(w_i, w_i^s) = t_i p_i^s$, i.e., $(p_i')^s = t_i p_i^s$. Hence $q = \sum_1^n t_i p_i^s = (\sum_1^n p_i')^s$ as s is a homomorphism. s is thus an epimorphism and so an isomorphism.

(7) Let $r \in R$ then $(Rru)^{\Sigma} = S(ru)^s \subset Su'$. Hence $(ru)^s = tu'$ for some unique $t = r^{\ell} \in S$. Then as in Skornyakov's proof or as in (14) on p.49 of Baer (1) it follows that ℓ is in fact a ring isomorphism: $R \cong S$ and that (ℓ, s) is a semi-linear isomorphism: $(R, M) \cong (S, N)$.

If P is a submodule of M then P^s is a submodule of N. Hence $P^s = (\sum_{p \in P} Rp)^s = \sum_{p \in P} Sp^s = \sum_{p \in P} (Rp)^{\Sigma} = (\sum_{p \in P} Rp)^{\Sigma} = P$. Hence (ℓ, s) induces Σ. We note that condition S_1 ensures that for any $p \in M$ there is a free element $w \in M$ with $Rp \cap Rw = 0$, i.e., cyclic modules are not large in M.

Since M contains a free module of rank ≥ 2 we have by lemma 4.1 that (ℓ, s) is unique up to unit semi-linear isomorphism. The ambiguity can be thought of as arising because of the ambiguity in the choice of u' which is only determined to within unit by the equation $(Ru)^{\Sigma} = Su'$.

We remarked in the proof that the fact that u' was free was used only once in the proof. The theorem however is not true in general if we do not make this hypothesis as the lattice isomorphism $\Sigma^{-1}: L({}_SN)$ $L({}_RM)$ of example 4.7 shows.

Theorem 4.9. Let ${}_RM$ be a module satisfying conditions S_1 and S_2 and ${}_SN$ a faithful module with $\Sigma: L({}_RM) \overset{c}{\cong} L({}_SN)$. If either (1) R is inverse symmetric or (2) S is subcommutative then there is a semi-linear isomorphism: $(R, M) \cong (S, N)$ inducing Σ.

Proof

Let u be a free element ε M and u' ε N be such that $(Ru)^{\Sigma} = Su'$. Suppose q ε N and p ε M with $Rp = (Sq)^{\Sigma^{-1}}$. By S_1 there is a free element w ε M such that $(Ru + Rp) \cap Rw = 0$ and where $Sw' = (Rw)^{\Sigma}$ for some w' ε N.

$$\text{Now } 0 = \ell(w) \subset \ell(p) \quad \text{and} \quad Rw \cap Rp = 0 \quad \text{and}$$

$$0 = \ell(w) = \ell(u) \quad \text{and} \quad Rw \cap Ru = 0 .$$

By lemma 4.5 $\ell(w') \subset \ell(q)$ and $\ell(w') = \ell(u')$. Therefore $\ell(u') \subset \ell(q)$. But q was any element ε N and so $\ell(u') \subset \ell(N) = 0$ as N is faithful. Thus u' is free and the result follows from theorem 4.8.

CHAPTER 5

UNIQUE CO-ORDINATIZATION RINGS

In this chapter we consider lattice isomorphisms between the lattices of submodules of free modules of the same rank. This is shown to be equivalent (if rank $n \geq 3$) to the problem of considering when a ring isomorphism $R_n \cong S_n$ implies a ring isomorphism $R \cong S$. A ring R with this property for all rings S and integers n is called a unique co-ordinatization ring. We study these and associated rings giving their elementary properties as well as a number of examples.

Theorem 5.1. Let R and S be rings and n an integer ≥ 3. Then $L({}_R R^n) \cong L({}_S S^n)$ if and only if $R_n \cong S_n$.

Proof

By cor.1 of theorem 2.3 $L({}_R R^n) \cong L({}_S S^n)$ implies that $\mathrm{End}_R(R^n) \cong \mathrm{End}_S(S^n)$ i.e. $R_n \cong S_n$.

By cor.2 to theorem 1.4 if $R_n \cong S_n$ then $L({}_R R^n) \cong L({}_S S^n)$.

Definition. A ring R is called a unique co-ordinatization ring (u.c. ring) if for any ring S and integer n $R_n \cong S_n$ always implies $R \cong S$.

In analogy with the co-ordinatization theorems for projective geometry and more generally for complemented modular lattices we say that a lattice L is co-ordinatized by a ring R if $L \cong L({}_R R^n)$ for some integer n. Theorem 5.1 shows that for fixed $n \geq 3$ any co-ordinatization by a u.c. ring is unique up to ring isomorphism - hence the terminology.

Let ${}_RM$ and ${}_SN$ be free modules of rank n with bases $(e_i)_1^n$ and $(f_i)_1^n$ respectively. Suppose $\ell : R \cong S$ is a ring isomorphism. Then $s : \sum_1^n r_i e_i \longrightarrow \sum_1^n r_i^{\ell} f_i$ is an abelian group isomorphism: $(M, +) \cong (N, +)$ and $(\ell, s) : [R, M] \cong [S, N]$ is a semi-linear isomorphism. This gives us the following corollary.

__Cor.1.__ Let R be a u.c. ring, S a ring and n an integer ≥ 3. If $L({}_RR^n) \cong L({}_SS^n)$ then there is a semi-linear isomorphism: $(R, R^n) \cong (S, S^n)$.

__Proof__

By theorem 5.1 $R_n \cong S_n$ and as R is a u.c. ring this implies that $R \cong S$. The result then follows from the remarks above.

We now note the following criterion for a ring to be a direct product of rings.

A ring R is the direct product of a set of rings $(R_i)_{i \in I}$ if and only if there is a set of central idempotents $(e_i)_{i \in I}$ of R such that

(1) $R_i \cong e_i R e_i$ for all $i \in I$

(2) if $r \in R$ and $re_i = o$ for all $i \in I$ then $r = o$

(3) given a set of elements $(a_i)_{i \in I}$ of R then there is an element $a \in R$ with $ae_i = a_i e_i$ for all $i \in I$. We will write $R \cong \prod_{i \in I} R_i$.

__Lemma 5.2.__ Let R be a ring and n an integer. Then $R_n \cong \prod_{i \in I} T_i$ for some ~~set~~ family of rings $(T_i)_{i \in I}$ if and only if $R \cong \prod_{i \in I} R_i$ for ~~some set~~ some family of rings $(R_i)_{i \in I}$ with $T_i \cong (R_i)_n$.

Proof

If $R_n \cong \prod_{i \varepsilon I} T_i$ then there are central idempotents $(f_i)_{i \varepsilon I} \varepsilon R_n$ with $T_i \cong f_i R_n f_i$. For $e \varepsilon R$ write diag(e) for the matrix $\begin{bmatrix} e_e & & 0 \\ & \ddots & \\ 0 & & e \end{bmatrix}$.

The central idempotents of R_n are then precisely the elements diag(e) where e is a central idempotent ε R.

Let f_i = diag(e_i) where $e_i \varepsilon R$ is a central idempotent and define $R_i = e_i R e_i$. It is easily seen that since $(f_i)_{i \varepsilon I}$ satisfy conditions (2) and (3) for direct products so do $(e_i)_{i \varepsilon I}$ and so $R \cong \prod_{i \varepsilon I} R_i$. But $(R_i)_n = (e_i R e_i)_n = f_i R_n f_i \cong T_i$.

Conversely suppose $R \cong \prod_{i \varepsilon I} R_i$ and $(e_i)_{i \varepsilon I}$ are the associated central idempotents. Defining f_i = diag(e_i) it is easily seen that $R_n \cong \prod_{i \varepsilon I} f_i R_n f_i = \prod_{i \varepsilon I} (e_i R e_i)_n \cong \prod_{i \varepsilon I} (R_i)_n$.

Lemma 5.3. The class of all u.c. rings is closed under direct products.

Proof

Let $(R_i)_{i \varepsilon I}$ be a ~~set~~ family of u.c. rings and suppose $R \cong \prod_{i \varepsilon I} R_i$. Suppose further that for some ring S and integer n $R_n \cong S_n$. By lemma 5.2 $S_n \cong \prod_{i \varepsilon I} (R_i)_n$ and so $S \cong \prod_{i \varepsilon I} S_i$ where $(S_i)_n \cong (R_i)_n$. But R_i is a u.c. ring and so $R_i \cong S_i$. Hence $R \cong \prod_{i \varepsilon I} R_i \cong \prod_{i \varepsilon I} S_i \cong S$ and R is a u.c. ring.

Definition. A ring R is called a strong unique co-ordinatization ring (s.u.c. ring) if for any R-module $_R P$ and integer n $_R P^n \cong {}_R R^n$ always implies $_R P \cong {}_R R$.

We will show later that any s.u.c. ring is a u.c. ring.

A useful way of describing s.u.c.rings is in terms of the semi-group of the isomorphism types of finitely generated projective modules.

For a ring R the set $\mathfrak{J}_R$ of isomorphism types $\langle P \rangle$ of finitely generated projective modules ${}_R P$ is an additive semigroup under the operation $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$. We call an element a of an additive semi-group <u>torsion free</u> if for any integer n and element b na = nb implies a = b. In this terminology we see that R is an s.u.c. ring if and only if $\langle R \rangle$ is a torsion free element in $\mathfrak{J}_R$.

<u>Lemma 5.4</u>

The class of s.u.c. rings is closed under direct products.

<u>Proof</u>

Suppose $(R_i)_{i \in I}$ is a ~~set~~ family of s.u.c. rings and $R \cong \prod_{i \in I} R_i$ where $(e_i)_{i \in I}$ are the associated central idempotents. Suppose ${}_R P$ is a module and n an integer with $R^n \cong P^n$.

For each $i \in I$ $e_i R^n \cong e_i P^n$ as R-modules and we have therefore that $(e_i R e_i)^n \cong (e_i P)^n$ as $e_i R e_i$-modules. Since $e_i R e_i \cong R_i$ and R_i is an s.u.c. ring we have that $e_i R e_i \cong e_i P$ as $e_i R e_i$-modules and hence as R-modules.

Since $P \subset R^n$ every element of P can be written as a vector $(r_1, \dots, r_n)$. The map $p \longrightarrow (e_i p)_{i \in I}$ of $P \longrightarrow \prod_{i \in I} e_i P$ is therefore an isomorphism $P \cong \prod_{i \in I} e_i P$. Therefore $P \cong \prod_{i \in I} e_i P \cong \prod_{i \in I} e_i R e_i \cong R$ as R-modules. Hence R is an s.u.c. ring.

Lemma 5.5. Let R be a ring then the following are equivalent

(1) R is an s.u.c. ring

(2) For any integer n and left ideals A, B of R_n $A^n = B^n = R_n$ implies $A \cong B$.

Proof

For any integer n there are inverse category equivalences $F: {}_R\mu \longrightarrow {}_{R_n}\mu$ and $G: {}_{R_n}\mu \longrightarrow {}_R\mu$ where $(R^n)^F = R_n$ and $(R_n)^G = R^n$.

Suppose R satisfies condition (2) and $P^n \cong R^n$ for some module ${}_RP$. Applying F we get $(P^F)^n \cong (R^F)^n = R_n$. Hence by condition (2) $P^F \cong R^F$. Applying the inverse equivalence G we get $P \cong R$. Hence R is an s.u.c. ring.

Suppose R is an s.u.c. ring and $A^n = B^n = R_n$ for left ideals A, B of R_n. Applying G we get $(A^G)^n = (B^G)^n = R^n$. As R is an s.u.c. ring we get $A^G \cong R \cong B^G$. Applying F we have then $A \cong B$ and so R satisfies condition (2).

Cor.1. Any s.u.c. ring is a u.c. ring.

Proof

Let R be an s.u.c. ring and suppose n is an integer and S a ring with $\ell: R_n \cong S_n$. Let $e_{i,i}$ and $f_{i,i}$ be the matrices of R_n and S_n respectively with 1 in the $(i,i)^{th}$ place and zeros elsewhere. Then $R_n = \bigoplus_1^n R_n e_{i,i}$ and for any $1 \leq i, j \leq n$ $R_n e_{i,i} \cong R_n e_{j,j}$. Similarly $S_n = \bigoplus_1^n S_n f_{i,i}$ and $S_n f_{i,i} \cong S_n f_{j,j}$.

Applying ℓ we have $R_n = \bigoplus_1^n R_n e_{i,i} = \bigoplus_1^n R_n (f_{i,i})^{\ell}$. By condition (2) of lemma 5.5 $R_n e_{i,i} \cong R_n (f_{i,i})^{\ell}$. Taking endomorphism rings we have $R \cong e_{i,i} R_n e_{i,i} \cong f_{i,i}^{\ell} R_n f_{i,i}^{\ell} \cong (f_{i,i} S_n f_{i,i})^{\ell} \cong f_{i,i} S_n f_{i,i} \cong S$. Therefore R is a u.c. ring.

The converse to cor.1 does not hold i.e. there are u.c. rings which are not s.u.c. rings. This will be proved later - see cor.1 of theorem 5.9.

Lemma 5.6. Let R be a ring and J(R) its Jacobson radical. If R/J(R) is an s.u.c. ring then R is an s.u.c. ring.

Proof

We recall the following lemma which is prop.1 on p.53 of Jacobson (1). If e_1, e_2 are non-zero idempotents and $\bar{e}_1$, $\bar{e}_2$ are their images under the natural ring homomorphism $R \longrightarrow R/J(R) = \bar{R}$ then $Re_1 \cong Re_2$ as left ideals if and only if $\bar{R}\,\bar{e}_1 \cong \bar{R}\,\bar{e}_2$ are isomorphic as left ideals.

Suppose $\bar{R}$ is an s.u.c. ring and that for some integer n and independent sets of isomorphic left ideals $(A_i)_1^n$, $(B_i)_1^n$ of R_n we have $\bigoplus_1^n A_i = \bigoplus_1^n B_i = R_n$. As $(A_i)_1^n, (B_i)_1^n$ are direct summands of R_n they are generated by idempotents $(e_i)_1^n, (f_i)_1^n$ respectively. Furthermore since $(A_i)_1^n$ and $(B_i)_1^n$ are independent we can take the idempotents $(e_i)_1^n$ and $(f_i)_1^n$ to be orthogonal i.e. $e_i e_j = f_i f_j = 0$ if $i \neq j$.

Now consider the natural ring homomorphism $R_n \longrightarrow R_n/J(R_n) = \overline{R}_n$ where $J(R_n)$ is the Jacobson radical of R_n. Let $(\overline{A}_i)_1^n$, $(\overline{B}_i)_1^n$, $(\overline{e}_i)_1^n$, $(\overline{f}_i)_1^n$ be the images of $(A_i)_1^n$, $(B_i)_1^n$, $(e_i)_1^n$, $(f_i)_1^n$ respectively. Since $(\overline{e}_i)_1^n$ and $(\overline{f}_i)_1^n$ are orthogonal sets of idempotents and $\overline{R}_n\overline{e}_i = \overline{A}_i$ and $\overline{R}_n\overline{f}_i = \overline{B}_i$ we see that $(\overline{A}_i)_1^n$ and $(\overline{B}_i)_1^n$ are independent sets of left ideals of $\overline{R}_n$ and $\bigoplus_1^n \overline{A}_i = \bigoplus_1^n \overline{B}_i = \overline{R}_n$. Further by the lemma in Jacobson (1) they are sets of mutually isomorphic left ideals.

But $J(R_n) = [J(R)]_n$ and so $\overline{R}_n = R_n/J(R_n) = R_n/[J(R)]_n \cong [R/J(R)]_n = (\overline{R})_n$. By hypothesis $\overline{R}$ is an s.u.c. ring. Hence by lemma 5.5 $\overline{A}_1 \cong \overline{B}_1$. Therefore by the lemma in Jacobson (1) $A_1 \cong B_1$ and so R is an s.u.c. ring.

We now prove a result which subsumes the known results on s.u.c. rings.

<u>Definition.</u> A ring R is called <u>p-trivial</u> if there is a finitely generated projective module $_RP$ such that every other finitely generated projective R-module is of the form P^k for some unique integer k. Such rings have been studied in Cohn (2).

<u>Definition.</u> A ring R is said to have <u>invariant basis number</u> (I.B.N.) if any two bases for a free left R-module always have the same number of elements.

<u>Definition.</u> A ring R is called a (local) <u>P.F. ring</u> (projective-free) if every (finitely generated) projective left R-module is free.

Suppose R is p-trivial and ${}_RP$ is the associated finitely generated projective module. Then for some unique integer k $R = P^k$ and so $R \cong S_k$ where $S = \text{End}_R(P)$. But P is a progenerator and thus $R \underset{M}{\sim} S$. By the p-triviality of R it is clear that S is a local P.F. ring with I.B.N. It is now easy to see that p-trivial rings are precisely the class of all matrix rings over local P.F. rings with I.B.N.

We can give another characterization of p-trivial rings, viz., a ring R is p-trivial if and only if the additive semi-group $\mathcal{S}_R$ of isomorphism types of finitely generated projectives is isomorphic to the additive semigroup of non-negative integers. If R is p-trivial it is clear that every element of $\mathcal{S}_R$ is torsion free and so in particular $\langle R \rangle$ is torsion free. By the remarks following the definition of s.u.c. rings we have the following theorem.

<u>Theorem 5.7</u>. Every p-trivial ring is an s.u.c. ring.

<u>Definition</u>. A ring R is called semi-primary if $R/J(R)$ is an Artinian ring (see chap.3.9 of Jacobson (1)).

<u>Cor.1</u>. A semi-primary ring is an s.u.c. ring.

<u>Proof</u>

A division ring is certainly a local P.F. ring and so any simple Artinian ring is p-trivial and hence an s.u.c. ring. By lemma 5.4 a direct product of simple Artinian rings is an s.u.c. ring and so in particular a semi-simple Artinian ring is an s.u.c. ring.

But if R/J(R) is Artinian it is certainly also semi-simple and so by lemma 5.6 R is an s.u.c. ring. We have in fact proved more than we needed. We have shown that if R/J(R) is the direct product of simple Artinian rings then R is an s.u.c. ring.

The following rings were already well known to be s.u.c. rings. Our corollary includes all these as special cases.

(1) Division rings - the first fundamental theorem of projective geometry (see theorem 1 of chap.5.4 of Baer (1)).

(2) Semi-simple Artinian rings - the uniqueness part of the Artin-Wedderburn theorem (see e.g. theorem 2 of chap.3.4 and the isomorphism theorem of chap.3.5 of Jacobson (1)).

(3) Artinian rings - Krull-Schmidt theorem.

(4) Matrix rings over local rings (see e.g. theorem 3 of chap.3.10 of Jacobson (1)). This result also follows directly from the well-known result that a local ring is a P.F. ring with I.B.N. (see Kaplansky (1)).

Definition. A ring R is called a semi free ideal ring (semi-fir) if (1) R has I.B.N. (2) every finitely generated left ideal of R is free.

It can be shown that a semi-fir must in fact be an integral domain and that it is in fact a P.F. ring with I.B.N. and so p-trivial (see Cohn (2) and Cohn (3)).

Cor.2. A semi-fir is an s.u.c. ring.

Clearly a principal left ideal domain is a semi-fir. Wolfson has shown that if R and S are principal left ideal domains and $R_n \cong S_n$ for some integer n then $R \cong S$ (see Wolfson (1)). Our corollary includes this as a special case. The full generalization of Wolfson's results in the infinite case are given in chap.7.

We cannot drop the condition that a semi-fir has I.B.N. in cor.2 as the following example due to P.M.Cohn shows.

Example 5.8. There is an integral domain R all of whose left ideals are free and a ring S such that $R_3 \cong S_3$ but $R \not\cong S$.

Proof

Leavitt has considered integral domains which do not have I.B.N. In particular in Leavitt (1) an example of an integral domain R is given such that $R^2 \cong R^3$. Thus $R^3 \cong R^4 \cong R^5 \cong R^6$ and taking endomorphism rings we get $R_3 \cong R_6 \cong (R_2)_3$.

Put $S = R_2$. Then $R_3 \cong S_3$ but $R \not\cong S$ since R being an integral domain cannot have any proper direct summands. Furthermore theorem 3.1 of Skornyakov (3) shows that every left ideal of R is free. Our next theorem is an unpublished result due to Kaplansky.

Theorem 5.9 (Kaplansky). Every commutative ring is a u.c. ring.

Proof

Let R be commutative and suppose for some integer n and ring S that $R_n \cong S_n$.

For any integer t define $S_t(x_1, \ldots, x_t)$ to be the polynomial (in non-commuting indeterminates), $\Sigma(-1)^s x_{s_1} x_{s_2} \ldots x_{s_t}$ where s runs over all permutations of $(1, 2, \ldots, t)$ and $(-1)^s = +1$ or -1 according as s is an even or an odd permutation.

Now it is shown in Amitsur and Levitzki (1) that the elements of R_n satisfy the identity $S_{2n}(x_1, x_2, \ldots, x_{2n}) = o$ and hence so do the elements of S_n.

Let $e_{i,j}$ be the matrix of S_n with 1 in the $(i, j)^{th}$ place and zeros elsewhere. If $t \in S$ then by $te_{i,j}$ we mean the matrix with t in the $(i, j)^{th}$ place and zeros elsewhere.

Let $a, b \in S$ and consider the 2n elements $(ae_{1,1}, be_{1,1}, e_{1,2}$ $e_{2,2}, e_{2,3} \ldots\ldots e_{n-2,n-1}, e_{n-1,n-1}, e_{n-1,n}, e_{n,1})$. These satisfy $S_{2n}(x_1, x_2, \ldots, x_{2n}) = o$. Substituting we see that the only non-zero terms arise from cyclic permutations and interchanging $ae_{1,1}$ and $be_{1,1}$. Multiplying on the left and right by $e_{n,n}$ (and noting $e_{i,j} \cdot e_{\ell,m} = o$) if $j \neq \ell$ we get $(ab - ba)e_{n,n} = o$ i.e. $ab - ba = o$. Therefore S is commutative. But $R \cong$ centre $R \cong$ centre $R_n \cong$ centre $S_n \cong$ centre $S \cong S$. Hence R is a u.c. ring.

<u>Cor.1.</u> There are u.c. rings which are not s.u.c. rings.

<u>Proof</u>

Let R be a Dedekind domain and suppose I_1, I_2 are ideals of R. Then $I_1 \oplus I_2 \cong R \oplus I_1 I_2$ (see e.g. theorem 2(a) of Kaplansky (2)).

Suppose R has an ideal A which is not principal but such that A.A is principal. Then $A \oplus A \cong R \oplus A.A \cong R \oplus R$. But A is not principal and so $A \not\cong R$. Therefore R is not an s.u.c. ring but since it is commutative it is a u.c. ring.

An example of such a ring is $\mathbb{Z}[x]/(x^2 + 5) = R$ where $A = (2, 1+x)$. It is not difficult to show that A is not principal and that $A.A = R.2$. It can also be shown that R is the integral closure of $\mathbb{Z}$ in the quotient field K of R. But K is a finite algebraic extension of $\mathbb{Q}$ the quotient field of $\mathbb{Z}$. Since $\mathbb{Z}$ is a Dedekind domain we have by a well known theorem (see e.g. theorem 19 of chap.5 of Zariski and Samuel (1)) that R is a Dedekind domain.

CHAPTER 6

REGULAR RINGS

Von Neumann showed (theorem 14.1 of von Neumann (1)) that any complemented modular lattice L of order $n \geq 4$ is lattice isomorphic to $F({}_RR^n)$ for some regular ring R. Further implicit in his proof is the fact that if L is upper and lower continuous then the co-ordinatizing ring R is unique up to isomorphism (see e.g. chap.7 of Skornyakov (1)). In this chapter we consider the uniqueness of the co-ordinatizing ring R if the continuity conditions on L are weakened. In particular we show that any two co-ordinatizing rings for L have isomorphic injective hulls and so in some sense the co-ordinatization of L is unique up to 'quotient ring'. We also show that the following classes of regular rings are s.u.c. rings:

(1) direct products of matrix rings over strongly regular rings

(2) upper continuous regular rings. The co-ordinatization of L by such rings is therefore unique. An interesting corollary to (2) is that every left self-injective ring is an s.u.c. ring.

<u>Definition</u>. A ring R is called <u>regular</u> if for every $a \in R$ there is an element $x \in R$ such that $a = axa$.

We shall assume a number of well-known facts about regular rings. The proofs of these may be found in von Neumann (1) or Skornyakov (1).

Let R be a regular ring then the following results hold.

(A1) Every finitely generated left (right) ideal of R can be generated by an idempotent. Further $F({}_RR)$ and $F(R_R)$ are complemented modular lattices with respect to the usual operations + and $\cap$.

(A2) The maps $A \longrightarrow \ell(A)$, $B \longrightarrow r(B)$ are inverse lattice anti-isomorphisms $F(R_R) \longrightarrow F({}_RR)$ and $F({}_RR) \longrightarrow F(R_R)$ respectively. In particular if $a \in R$ then $\ell(a) \in F({}_RR)$ and $r(a) \in F(R_R)$. Let n be an integer.

(A3) A ring R is regular if and only if R_n is regular.

(A4) If R is a regular ring then $F({}_RR^n)$ is a complemented modular lattice and for any $x \in R^n$ $\ell(x) \in F({}_RR)$.

(A5) Any projective left R-module over a regular ring is a direct sum of elements of $F({}_RR)$ (see Kaplansky (1)).

<u>Definition.</u> A regular ring R is called (countably) <u>complete</u> if the lattice $F({}_RR)$ is (countably) complete, i.e., every (countable) subset $A \subset F({}_RR)$ has a least upper bound (l.u.b.(A)) and a greatest lower bound (g.l.b.(A)). As completeness is a self-dual concept for lattices, it is a left-right concept for regular rings.

<u>Definition.</u> A ring R is called a <u>Baer ring</u> if, for every subset $B \subset R$, $\ell(B)$ is a principal left ideal generated by an idempotent. This definition is left-right symmetric since $r(B) = r\ell r(B) = (1-e)R$ where e is an idempotent such that $\ell(r(B)) = Re$. These rings have been studied in Kaplansky (3).

<u>Lemma 6.1.</u> A regular ring R is complete if and only if it is a Baer ring. In this case if $(A_i)_{i\varepsilon I}$ is a subset $\subset F({}_RR)$ then g.l.b.$(A_i)_{i\varepsilon I}$ = $\bigcap_{i\varepsilon I} A_i$ and l.u.b.$(A_i)_{i\varepsilon I} = \ell r(\sum_{i\varepsilon I} A_i)$.

<u>Proof</u>

Suppose R is a complete regular ring and $(A_i)_{i\varepsilon I}$ is a subset $\subset$ $F({}_RR)$. If A = g.l.b.$(A_i)_{i\varepsilon I}$ then $A \subset \bigcap_{i\varepsilon I} A_i$. If however $x \varepsilon \bigcap_{i\varepsilon I} A_i$ then $Rx \varepsilon F({}_RR)$ and so $Rx \subset$ g.l.b.$(A_i)_{i\varepsilon I} = A$. Hence $\bigcap_{i\varepsilon I} A_i \subset A$ and so g.l.b.$(A_i)_{i\varepsilon I} = A = \bigcap_{i\varepsilon I} A_i$. Suppose B is a subset of R then $\ell(B) = \bigcap_{b\varepsilon B} \ell(b) \varepsilon F({}_RR)$ since $\ell(b) \varepsilon F({}_RR)$ for all $b \varepsilon B$. Thus R is a Baer ring.

Conversely let R be a regular Baer ring and $(A_i)_{i\varepsilon I} \subset F({}_RR)$. Then for some idempotent $e \varepsilon R$, $r(\sum_{i\varepsilon I} A_i) = (1-e)R$ and so $\sum_{i\varepsilon I} A_i \subset \ell r(\sum_{i\varepsilon I} A_i) = \ell((1-e)R) = Re \varepsilon F({}_RR)$. Hence $\ell r(\sum_{i\varepsilon I} A_i)$ is an upper bound for $(A_i)_{i\varepsilon I}$. Suppose Rf (f an idempotent) is another upper bound for $(A_i)_{i\varepsilon I}$. Then $A_i \subset Rf$ for all $i \varepsilon I$ and so $(1-f)R \subset r(\sum_{i\varepsilon I} A_i)$. Hence $\ell r(\sum_{i\varepsilon I} A_i) \subset Rf$ and therefore $\ell r(\sum_{i\varepsilon I} A_i) =$ l.u.b.$(A_i)_{i\varepsilon I}$. As is well-known the existence of l.u.b's in a lattice implies the lattice is complete. Thus R is complete and g.l.b.$(A_i)_{i\varepsilon I} = \bigcap_{i\varepsilon I} A_i$ and l.u.b.$(A_i)_{i\varepsilon I} = \ell r(\sum_{i\varepsilon I} A_i)$.

Let R be a regular ring.

(A6) The set of elements of $F({}_RR)$ which are two-sided ideals are precisely those generated by central idempotents. This subset of $F({}_RR)$

is denoted by $C({}_RR)$. $C({}_RR)$ is a complemented distributive lattice. The elements of $C({}_RR)$ can also be characterized as those elements of $F({}_RR)$ having unique complements. An element of $C({}_RR)$ is generated by a unique central idempotent.

(A7) Suppose R is a complete regular ring. Then $C({}_RR)$ is a complete lattice and g.l.b's and l.u.b's calculated in $C({}_RR)$ are the same as if they were calculated in $F({}_RR)$. If $A \in F({}_RR)$ then there is a least element $C(A)$ of $C({}_RR)$ containing A i.e. $C(A) = \bigcap(B \in C({}_RR) : B \supset A)$. $C(A)$ is called the <u>central envelope of A</u>.

<u>Lemma 6.2</u>. If R is a complete regular ring and $A \in F({}_RR)$ then $C(A) = r\ell(A)$.

<u>Proof</u>

As A is a left ideal $\ell(A)$ is a two-sided ideal. Since R is complete $\ell(A) \in F({}_RR)$ and hence $\in C({}_RR)$. Thus $r\ell(A) \in C({}_RR)$ and $A \subset r\ell(A)$. Therefore $C(A) \subset r\ell(A)$.

Suppose $C(A) = Re$ where e is a central idempotent. Then $(1 - e)A = 0$ and $R(1 - e) \subset \ell(A)$. Hence $r\ell(A) \subset r(R(1-e)) = Re = C(A)$. Therefore $C(A) = r\ell(A)$.

<u>Definition</u>. Let L be a lattice with least element 0. Suppose $a, b, c \in L$. Then we say that a is in <u>perspective</u> with b with axis c, $a \underset{c}{\sim} b$, if $a \wedge c = b \wedge c = 0$ and $a \vee c = b \vee c$ (c.f. chap.2).

Definition. Let L be a complete lattice and $(a_i)_{i\varepsilon I}$ a set of elements of L. If $J \subset I$ define $a_J = \bigvee_{i\varepsilon J} a_i$. $(a_i)_{i\varepsilon I}$ is called independent if $a_F \wedge a_G = 0$ for all finite subsets $F, G \subset I$ with $F \cap G = \phi$. $(a_i)_{i\varepsilon I}$ is called strongly independent if $\bigwedge_j a_{J_j} = 0$ for any subsets $J_j \subset I$ with $\bigcap J_j = \phi$.

Definition. Let L be a lattice with least element 0. An element ε L is called finite if it contains no infinite sequence of independent pairwise perspective elements. Otherwise it is called infinite.

Amemiya and Halperin have studied finiteness in complete complemented modular lattices. We collect together a number of their results which we shall need later on.

Lemma 6.3. Let L be a complete complemented modular lattice then

(1) if $(a_i)_1^\infty$ is an independent sequence of pairwise perspective elements of L then there is a strongly independent sequence $(b_i)_1^\infty$ of pairwise perspective elements of L such that $\bigvee_1^\infty b_i \leq \bigvee_1^\infty a_i$ and $a_1 = b_1$.

(2) if $(a_i)_{i\varepsilon I}$ and $(b_i)_{i\varepsilon I}$ are two strongly independent families of L such that $a_i \sim b_i$ for all $i \varepsilon I$ and $(\bigvee_{i\varepsilon I} a_i) \wedge (\bigvee_{i\varepsilon I} b_i) = 0$ then $\bigvee_{i\varepsilon I} a_i \sim \bigvee_{i\varepsilon I} b_i$.

(3) if $(a_i)_1^n$ is a finite set of elements of L such that for each i a_i is finite then $\bigvee_1^n a_i$ is finite.

<u>Proof</u>

(1) is [illegible]. 6.1 }
(2) is cor.1 of theorem 3.4 } of Amemiya and Halperin (1) .
(3) is cor.2 of theorem 6.3 }

<u>Definition</u>. A regular ring R is called <u>finite</u> if $F({}_R R)$ is finite (as a lattice) and <u>infinite</u> otherwise. R is called <u>properly infinite</u> if every non-zero element of $C({}_R R)$ (considered as an element of $F({}_R R)$) is infinite.

<u>Lemma 6.4</u>. A countably complete regular ring R is finite if and only if it is inverse symmetric.

<u>Proof</u>

As we remarked in chap.4 the results of Jacobson (2) show that if R is not inverse symmetric then there is an infinite direct sum of isomorphic left ideals generated by idempotents. But if $A, B \in F({}_R R)$ and $A \cap B = 0$ then by (4) of lemma 2.1 ${}_R A \cong {}_R B$ implies $A \sim B$. Hence if R is finite R must be inverse symmetric.

Conversely suppose (if possible) that a countably complete regular ring R is both inverse symmetric and infinite. By (1) of lemma 6.3 there is a strongly independent sequence $(A_i)_1^\infty$ of pairwise perspective elements of $F({}_R R)$. The sets $(A_{2i})_1^\infty$, $(A_{2i-1})_1^\infty$, $(A_{2i+1})_1^\infty$ are all strongly independent and $(\bigvee_1^\infty A_{2i}) \wedge (\bigvee_1^\infty A_{2i-1}) = (\bigvee_1^\infty A_{2i}) \wedge (\bigvee_1^\infty A_{2i+1}) = 0$. But $A_{2i} \sim A_{2i-1}$ and $A_{2i} \sim A_{2i+1}$ for $i = 1, 2, 3 \dots$.

Hence by (2) of lemma 6.3 we have $\overset{\infty}{\underset{1}{\bigvee}} A_{2i} \sim \overset{\infty}{\underset{1}{\bigvee}} A_{2i-1}$ and $\overset{\infty}{\underset{1}{\bigvee}} A_{2i} \sim \overset{\infty}{\underset{1}{\bigvee}} A_{2i+1}$. Thus if $A = \overset{\infty}{\underset{1}{\bigvee}} A_{2i-1}$ and $B = \overset{\infty}{\underset{1}{\bigvee}} A_{2i+1}$ then $A \cong B$ and $A \supsetneq B$. But $B \in F({}_R R)$ and is a direct summand of R and so of A. Hence we have $C, D \in F({}_R R)$ with $B \oplus C = A$ and $A \oplus D = R$. Therefore $R = B \oplus C \oplus D$ and, since $A \cong B$, $B \oplus D \cong R$ and $C \neq 0$. R then has a copy of itself as a proper direct summand and so cannot be inverse symmetric. This is a contradiction.

<u>Remark.</u> Kaplansky calls a Baer ring finite if it is inverse symmetric (see Kaplansky (3)). Our lemma shows that for complete regular rings the two definitions of finiteness coincide.

<u>Definition.</u> A regular ring is called <u>strongly regular</u> if every idempotent of R is central i.e. $F({}_R R) = C({}_R R)$. Using (A6) it is easy to see that a regular ring is strongly regular if and only if it is subcommutative.

<u>Examples</u>

(1) Direct products of division rings

(2) Boolean rings

<u>Theorem 6.5.</u> Let R and S be Boolean rings (not necessarily containing identity elements). If $\Sigma : L({}_R R) \cong L({}_S S)$ then there is a ring isomorphism $R \cong S$ inducing Σ.

<u>Proof</u>

Every finitely generated left ideal of a Boolean ring is generated by a unique element of the ring (even if the ring does not have a 1). Suppose that $a \in R$ then $(Ra)^{\Sigma}$ is a finitely generated left ideal of S and so $= Sa^t$ for some uniquely defined $a^t \in S$.

Further we have that $Ra \cap \ell(a) = 0$ and $Ra \oplus \ell(a) = R$ and $\ell(a)$ is the unique left ideal with this property. But applying Σ we get $(Ra)^{\Sigma} \cap \ell(a)^{\Sigma} = 0$ and $(Ra)^{\Sigma} \oplus \ell(a)^{\Sigma} = S$ and so as $(Ra)^{\Sigma} = Sa^t$ we have $\ell(a)^{\Sigma} = \ell(a^t)$.

Suppose $a, b \in R$.

(1) $Rab = Ra \cap Rb$ and so $(Rab)^{\Sigma} = (Ra)^{\Sigma} \cap (Rb)^{\Sigma} = Sa^t \cap Sb^t = Sa^t b^t$. Hence $(ab)^t = a^t b^t$.

(2) $R(a+b) = R(a-ab)+R(b-ab)$ (using the fact that $x+x=o$ for any $x \in R$)

$= [Ra \cap \ell(b)] + [Rb \cap \ell(a)]$. Therefore applying Σ we get

$$R(a+b)^{\Sigma} = [(Ra)^{\Sigma} \cap \ell(b)^{\Sigma}] + [(Rb)^{\Sigma} \cap \ell(a)^{\Sigma}]$$
$$= [Sa^t \cap \ell(b^t)] + [Sb^t \cap \ell(a^t)]$$
$$= S(a^t + b^t).$$

Hence $(a+b)^t = a^t + b^t$ and so t is a ring homomorphism.

If $a^t = o$ for some $a \in R$ then $(Ra)^{\Sigma} = Sa^t = 0$. Thus $Ra = 0$ and so $a = a^2 = o$.

If $c \in S$ then $(Sc)^{\Sigma^{-1}}$ is a finitely generated left ideal of R and so $= Ra$ for some $a \in R$. Therefore $Sc = (Ra)^{\Sigma} = Sa^t$ and so $c = a^t$. Thus t is a ring isomorphism. Since for any $a \in R$ $(Ra)^{\Sigma} = (Ra)^t$ t induces Σ.

This theorem is not true for strongly regular rings in general e.g. take R and S to be non isomorphic division rings. This is because we have no way of getting at the structures of the groups of units of R and S (the only possible unit for a Boolean ring is the identity element). The following remark shows that for arbitrary rings we can however still get at the structure of the central idempotents.

Remark.

If for any ring R (containing a 1 now) we denote the set of central idempotents by C(R) then C(R) is a Boolean ring with respect to ring multiplication and addition "$\oplus$" defined by $e \oplus f = e + f - ef - ef$ for $e, f \in C(R)$. A similar proof to that of theorem 6.5 shows that if $L({}_R R) \cong L({}_S S)$ for any rings R and S then there is a ring isomorphism: $C(R) \cong C(S)$.

Lemma 6.6. Let R be a strongly regular ring.

(1) If A and B are left ideals and ${}_R A \cong {}_R B$ then $A = B$

(2) For any integer n R_n is a finite regular ring.

Proof

(1) Let $s: {}_R A \cong {}_R B$ and let $b \in B$. Then for some $a \in A$ $b = a^s$ and $Ra = Re$, where e is a central idempotent $\in R$. Thus $Rb = Ra^s = (Ra)^s = (Re)^s = (Re.e)^s = Re.e^s = (Re^s)e \subset Re \subset A$. Hence $b \in A$ and so $B \subset A$. Similarly $A \subset B$ and thus $A = B$.

(2) In a strongly regular ring every left ideal is a two-sided ideal. Hence the maximal left ideals of R are exactly the maximal two-sided ideals of R. But the Jacobson radical of a regular ring is zero and

hence $\bigcap$ (all maximal two-sided ideals of R) = 0. Therefore $\bigcap$ (all maximal two-sided ideals of R_n) = 0.

Suppose if possible that R_n is infinite for some integer n. Then there is an infinite direct sum of isomorphic left ideals $(R_n e_i)_1^\infty$ generated by idempotents $(e_i)_1^\infty$. Since $e_1 \neq 0$ there is a maximal two-sided ideal of R_n not containing e_1. This must be of the form $(M)_n$ where M is a maximal two-sided ideal of R.

Now $R_n e_1 \cong R_n e_i$ for i = 1, 2, 3 ... and so there are elements $p_i \in R_n e_i$ $q_i \in R_n e_1$ with $e_1 = p_i q_i$ and $e_i = q_i p_i$. If $e_i \in M_n$ then $p_i \in M_n$ and so $e_1 \in M_n$ - a contradiction. Hence $e_i \notin M_n$.

As $(R_n e_i)_1^\infty$ form a direct sum we can choose without loss of generality the first (n + 1) e_i's to be orthogonal. Now consider the natural ring homomorphism $a: R_n \longrightarrow R_n/M_n \cong (R/M)_n$. Then as $e_i \notin M_n$ $e_i^a \neq 0$.

Now M is not only a maximal two-sided ideal but also a maximal left ideal. Hence R/M is a division ring and so $(R/M)_n$ does not contain any direct sum of non-zero left ideals with more than n members. But $(e_i^a)_1^{n+1}$ are (n+1) non-zero orthogonal idempotents of $(R/M)_n$ and so $[(R/M)_n e_i^a]_1^{n+1}$ is a direct sum of (n+1) non-zero left ideals - a contradiction. Hence R_n is finite.

As finiteness always implies inverse symmetry we have the following corollary.

Cor.1. If R is a strongly regular ring R_n is inverse symmetric for any integer n.

Cor.2. A strongly regular ring has I.B.N.

Proof

Suppose R is strongly regular and $R^m \cong R^n$ for integers m, n with $m \geq n$. Then if $m > n$ R^n has a copy of itself as a proper direct summand. Thus R_n cannot be inverse symmetric contradicting cor.1. Hence m = n and R has I.B.N.

Theorem 6.7. A direct product of matrix rings over strongly regular rings is an s.u.c. ring.

Proof

Let R be a strongly regular ring and $T = R_m$ for some integer m. Suppose ${}_TP$ is a module with $P^n \cong T^n$, where n is an integer. Now there is a category equivalence $F: {}_T\mu \longrightarrow {}_R\mu$ such that $T^F = R^m$. If ${}_RQ = P^F$ then $Q^n \cong R^{mn}$.

Q is a finitely generated projective and so by (A5) is a finite direct sum of cyclic submodules, say $Q = \bigoplus_{i=1}^{t} Rx_i$. Now R is strongly regular and so subcommutative. Therefore if $y_1 = \sum_1^t x_i$ then $\ell(y_1) = \bigcap_1^t \ell(x_i) = \bigcap_1^t \ell(Rx_i) = \ell(Q) = \ell(Q^n) = \ell(R^{mn}) = 0$. Thus $Q = Q_1 \oplus Ry_1$ where y_1 is free.

Therefore $Q_1{}^n \oplus R^n \cong R^{mn}$. Suppose $\ell(Q_1) \neq 0$ then there is a central idempotent $e \in R$ such that $eQ_1 = 0$ and so $(eRe)^n \cong (eRe)^{mn}$.

But eRe is strongly regular and so by cor.2 of lemma 6.6 it has I.B.N. Hence either $m = 1$ or $\ell(Q_1) = 0$. If $m \neq 1$ we can repeat the process until we get $Q = Q_m \oplus R^m$. Then $Q_m^n \oplus R^{mn} \cong R^{mn}$. But by cor.1 of lemma 6.6 R_{mn} is inverse symmetric and so R^{mn} does not contain a copy of itself as a proper direct summand. Hence $Q_m = 0$ and $Q = R^m$.

Applying the inverse category equivalence to F we get $P \cong T$. Therefore T is an s.u.c. ring and the theorem follows by lemma 5.4.

Definition. Let L be a complete complemented modular lattice. Then L is called upper continuous if for every directed set I and subset $(A_i)_{i \in I} \subset L$ such that $i_1 \leq i_2$ implies $A_{i_1} \subseteq A_{i_2}$ and for any $B \in L$ then $B \wedge \text{l.u.b.}(A_i)_{i \in I} = \text{l.u.b.}(B \wedge A_i)_{i \in I}$. L is called lower continuous if the dual condition holds. If L is both upper and lower continuous L is called continuous. We note that in an upper continuous lattice the notions of strong independence and independence coincide (see e.g. prop.75 of Skornyakov (1)).

Definition. A regular ring R is called upper continuous, lower continuous or continuous according as $F({}_R R)$ is upper continuous, lower continuous or continuous.

Upper continuous regular rings are closely related to left self-injective regular rings as the following results of Utumi show.

Lemma 6.8

(1) Any left self-injective regular ring is upper continuous

(2) Any upper continuous regular ring is the direct product of a strongly regular ring R_1 and a left self-injective regular ring R_2.

Proof

(1) See cor.1 of theorem of Utumi (1).

(2) See cor.1 of theorem 4 of Utumi (2).

Not every upper continuous regular ring is left self-injective. Utumi has remarked (p.604 of Utumi (2)) that the example given on p.526 of Kaplansky (4) is such a ring, viz., the ring of all sequences of complex numbers for which all but a finite number of entries are real.

Definition. Let ${}_R M$ be a module and P a submodule. If for every non-zero submodule $Q \subset M$ $P \cap Q \neq 0$ then P is called large in M and M is said to be an essential extension of P. We denote this by writing $P \subset' M$.

Definition. An element r of a ring R is called singular if $\ell(r) \subset' R$. The set of all singular elements of R form a two-sided ideal S(R) called the singular ideal of R. We quote a number of facts about such rings all of which may be found in Johnson (1) or Lambek (1).

If R is a ring with S(R) = 0 and A a left ideal of R then there is a unique maximal left ideal E(A) of R such that $A \subset' E(A)$. The operator E has the following properties

(1) $E(0) = 0$
(2) $E(E(A)) = E(A)$
(3) $E(A \cap B) = E(A) \cap E(B)$ where B is another left ideal.

A left ideal of R is called <u>closed</u> if $E(A) = A$. The set of all closed left ideals is denoted by $E({}_RR)$ and is an upper continuous complemented modular lattice with respect to the operations $A \wedge B = A \cap B$ and $A \vee B = E(A + B)$.

For any module ${}_RM$, as is well known, there is a unique minimal injective module ${}_RI(M)$ containing M. I(M) is called the <u>injective hull</u> of M. I(M) can also be characterized as the unique maximal essential extension of M so we always have $M \subset' I(M)$ (see Eckmann and Schopf (1) for details).

Suppose R is a ring with $S(R) = 0$ and that $Q = I(R)$, the injective hull of R. In this case Q can be given a ring structure (compatible with the structure of R) and Q is then a left self-injective regular ring. Further $E({}_RR) \cong F({}_QQ)$ (using (1) of lemma 6.8 this shows incidentally that $E({}_RR)$ is an upper continuous complemented modular lattice). Q can also be regarded as the maximal ring of quotients of R in the sense of Utumi (see e.g. Lambek (1)).

Examples of rings with zero singular ideal are

(1) simple rings

(2) integral domains

(3) regular rings (by (A2)).

<u>Lemma 6.9</u>. A regular ring R is upper continuous if and only if $F({}_RR) = E({}_RR)$. In this case for any left ideal A $E(A) = \ell r(A)$.

Proof

Noting that a direct summand of R is always closed the lemma follows immediately from theorem 2 of Utumi (3).

Cor.1. If R is an upper continuous regular ring and A, B are left ideals then $E(A+B) = E(A) + E(B)$.

Proof

By lemma 6.9 $E(A), E(B) \in F({}_RR)$ and so $E(A) + E(B) \in F({}_RR) = E({}_RR)$. Hence $E(A) + E(B)$ is a closed left ideal containing $A + B$ and so $E(A+B) \subset E(A) + E(B)$. But $A \subset A+B$ and $B \subset A+B$ and we have $E(A) \subset E(A+B)$ and $E(B) \subset E(A+B)$. Therefore $E(A+B) = E(A) + E(B)$.

Lemma 6.10. Suppose R is an upper continuous regular ring and A, B are left ideals of R with $f:A \cong B$. Then there is an isomorphism $g:E(A) \cong E(B)$ and if $A \cap B = 0$ then g can be chosen so that $g|A = f$.

Proof

By (2) of lemma 6.8 there are central idempotents e_1, e_2 such that $e_1 + e_2 = 1$ and $R_1 = e_1Re_1$ is a strongly regular ring and $R_2 = e_2Re_2$ is a left self-injective regular ring.

The isomorphism $f:A \cong B$ splits into two isomorphisms $f_1:Ae_1 \cong Be_1$ and $f_2:Ae_2 \cong Be_2$. By (1) of lemma 6.6 $Ae_1 = Be_1$ and so $E(Ae_1) = E(Be_1)$ and we can take g_1 as the identity map:$E(Ae_1) \cong E(Be_1)$.

Now R_2 is a left self-injective regular ring and so $E(Ae_2)$ and $E(Be_2)$ are the injective hulls of Ae_2 and Be_2 respectively. Hence the isomorphism $f_2:Ae_2 \cong Be_2$ can be extended to $g_2:E(Ae_2) \cong E(Be_2)$ i.e. $g_2|Ae_2 = f_2$.

By cor.1 of lemma 6.9 $E(A) = E(Ae_1) \oplus E(Ae_2)$ and $E(B) = E(Be_1) \oplus E(Be_2)$. Combining g_1 and g_2 we get $g: E(A) \cong E(B)$. If $A \cap B = 0$ then $Ae_1 = Be_1 \subset A \cap B = 0$. Hence $f = f_2$ and $g = g_2$ and so $g|A = f$.

Lemma 6.11. Let R be a complete regular ring and A, B ε $F({}_R R)$. Then $C(A) \cap C(B) = 0$ (see (A7)) if and only if there are no non-zero A_1, B_1 ε $F({}_R R)$ with $A_1 \subset A$ and $B_1 \subset B$ such that $A_1 \cong B_1$.

Proof

Let $C(A) = Re$ and $C(B) = Rf$ where e and f are central idempotents. Suppose A_1, B_1 ε $F({}_R R)$ with $A_1 \subset A$ and $B_1 \subset B$ and that $s: A_1 \cong B_1$. Then $A_1 = A_1 e$ and so $B_1 = A_1{}^s = (A_1 e)^s = (eA_1)^s = eA_1{}^s = A_1{}^s e \subset C(A)$. Hence $B_1 \subset C(A) \cap C(B) = 0$ and so $A_1 = B_1 = 0$.

Suppose there are no A_1, B_1 ε $F({}_R R)$ with $0 \neq A_1 \subset A$, $0 \neq B_1 \subset B$ and $A_1 \cong B_1$. Let a ε A and b ε B and let t be the R-homomorphism: $Ra \longrightarrow Rab$ defined by right multiplication by b. Now $\ker(t) = Ra \cap \ell(b)$ ε $F({}_R R)$ and so is a direct summand of Ra i.e. $\ker(t) \oplus C = A$ for some C ε $F({}_R R)$.

But $C \subset A$ and $C \cong Rab \subset B$ and so by hypothesis $ab = o$. Hence $AB = 0$ and so $A \subset \ell(B)$ giving by lemma 6.? $C(B) \subset r(A)$. Thus $A\,C(B) = 0$ and so $C(B)A = 0$ and $C(B) \subset \ell(A)$. We get therefore that $C(A) \subset \ell(C(B))$ and so $C(A)C(B) = 0$. Since C(A) and C(B) are generated by central idempotents we have then $C(A) \cap C(B) = 0$.

This is a ring version of a well known lattice result (see e.g. prop.66 of Skornyakov (1)).

Lemma 6.12. Let R be an upper continuous regular ring and A, B ε $F({}_R R)$. Then there are elements A_1, A_2, B_1, B_2 ε $F({}_R R)$ such that (1) $A = A_1 \oplus A_2$ (2) $B = B_1 \oplus B_2$ (3) $A_1 \cong B_1$ (4) $C(A_2) \cap C(B_2) = 0$.

Proof

Without loss of generality we can assume $A \cap B = 0$. By Zorn's lemma we can pick left ideals $A_1 \subseteq A$ $B_1 \subseteq B$ such that $f: A_1 \cong B_1$ and if there are left ideals A_2, B_2 with $A_1 \subset A_2 \subset A$, $B_1 \subset B_2 \subset B$ and $g: A_2 \cong B_2$ and $g|A_1 = f$ then $A_1 = A_2$ and $B_1 = B_2$.

By lemma 6.10 since $A \cap B = 0$ we have $A_1 = E(A_1)$ and $B_1 = E(B_1)$ i.e. A_1 and B_1 are closed left ideals. By lemma 6.9 we get A_1, B_1 ε $F({}_R R)$.

Let $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where A_2, B_2 ε $F({}_R R)$. Now there are no non-zero A_3, B_3 ε $F({}_R R)$ with $A_3 \subset A_2$ and $B_3 \subset B_2$ with $A_3 \cong B_3$. Otherwise we could extend $f: A_1 \cong B_1$. Hence by lemma 6.11 $C(A_2) \cap C(B_2) = 0$.

This is the ring form of a well known lattice result (see e.g. satz 1.1 of chap.4 of Maeda (1)).

Cor.1. Let R be an upper continuous regular ring and e, f idempotents $\in$ R. Then there are central idempotents h_1, h_2 with $h_1 + h_2 = 1$ and $Reh_1 \cong$ direct summand of Rfh_1 and $Rfh_2 \cong$ direct summand of Reh_2.

Proof

By lemma 6.12 there are idempotents e_1, e_2, f_1, f_2 ε R such that

(1) $Re = Re_1 \oplus Re_2$

(2) $Rf = Rf_1 \oplus Rf_2$

(3) $Re_1 \cong Rf_1$

(4) $C(Re_2) \cap C(Rf_2) = 0.$

Let $C(Rf_2) = Rh_1$ where h_1 is a central idempotent and let $h_2 = 1 - h_1$. Then $C(Re_2) \subset \ell(C(Rf_2)) = Rh_2$ and so $Re_2h_1 = 0$. Similarly $Rf_2h_2 = 0$.

Therefore $Reh_1 = Re_1h_1 \oplus Re_2h_1 = Re_1h_1 \cong Rf_1h_1$ which is a direct summand of Rfh_1. Similarly $Rfh_2 \cong$ direct summand of Reh_2.

This result is lemma 3.4 of Kaplansky (5).

<u>Theorem 6.13.</u> Let R be a finite upper continuous regular ring and $A, B \in F(_RR)$. If n is an integer and $A^n \cong B^n \cong R$ then $A \cong B$.

<u>Proof</u>

By lemma 6.12 there are $A_1, A_2, B_1, B_2 \in F(_RR)$ with (1) $A = A_1 \oplus A_2$ (2) $B = B_1 \oplus B_2$ (3) $A_1 \cong B_1$ (4) $C(A_2) \cap C(B_2) = 0$. Let $C(B_2) = Rh$ where h is a central idempotent. If $A^n \cong B^n \cong R$ then multiplying by h and noting $A_2h = 0$ and $B_2 = B_2h$ we get $(A_1h)^n \cong (B_1h)^n \oplus B_2^n \cong (hRh)$. But $A_1h \cong B_1h$ and (hRh) is a finite regular ring. But hRh cannot contain a copy of itself as a proper direct summand hence $B_2 = 0$. Similarly $A_2 = 0$ and we get $A = A_1 \cong B_1 = B$.

The corresponding theorem for complemented modular lattices is usually stated with the assumption of upper and lower continuity (see e.g. prop.81 of Skornyakov (1)). However, a careful look at the proof shows that the assumption of lower continuity can be replaced by the assumption of finiteness. Our theorem is the corresponding ring version of this modified theorem.

Lemma 6.14. A ring R is left self-injective if and only if for some integer n R_n is left self-injective.

Proof

A ring S is left self-injective if and only if there is an injective progenerator for ${}_S\mathcal{U}$. Hence any ring Morita equivalent to S is also left self-injective. In particular since for any integer n $R \underset{M}{\sim} R_n$ R is left self-injective if and only if R_n is left self-injective.

I am indebted to P.M.Cohn for this elegant proof which shortens earlier proofs in the literature (see e.g. theorem 8.3 of Utumi (4)).

Theorem 6.15. A finite left self-injective regular ring is an s.u.c. ring.

Proof

If R is a left self-injective regular ring then by (A3) and lemma 6.14 for any integer n R_n is also a left self-injective regular ring. Hence by (1) of lemma 6.8 R_n is upper continuous and in particular is complete.

Now $F({}_RR^n) \cong F({}_{R_n}R_n)$ is complete and by (3) of lemma 6.3 (since R is finite) we have $F({}_RR^n)$ is finite. Hence $F({}_{R_n}R_n)$ is finite and so R_n is an upper continuous finite regular ring. If A, B are left ideals with $A^n \cong B^n \cong R_n$ then by theorem 6.13 $A \cong B$ and R is an s.u.c. ring.

<u>Remark.</u> If R is an upper continuous regular ring then R_n need not be upper continuous. Indeed if R is an upper continuous regular ring which is not left self-injective then R_n cannot be complete if $n > 1$. For if it were then by theorem 4.3 of Amemiya and Halperin (1) R_n would be upper continuous. Then by corollary of theorem 8.3 of Utumi (4) R_n is left self-injective and so R is left self-injective, which is not so. Hence we cannot use the methods of theorem 6.15 to prove directly that a finite upper continuous regular ring is an s.u.c. ring. This however is true as the following corollary shows.

<u>Cor.1.</u> A finite upper continuous regular ring is an s.u.c. ring.

<u>Proof</u>

Let R be a finite upper continuous regular ring. Then by (2) of lemma 6.8 R is the direct product of a strongly regular ring R_1 and a left self-injective regular ring R_2.

By theorem 6.7 R_1 is an s.u.c. ring and by theorem 6.15 so is R_2. Hence by lemma 5.4 R is an s.u.c. ring.

<u>Theorem 6.16.</u> A properly infinite upper continuous regular ring R is an s.u.c. ring.

<u>Proof</u>

Since R is properly infinite there is an infinite independent set $(A_i)_{i\epsilon I}$ of ~~mutually~~ Pairwise isomorphic left ideals $\epsilon F(_R R)$. By Zorn's lemma we can take this to be maximal among such sets. Let $(e_i)_{i\epsilon I}$ and e be idempotents with $A_i = Re_i$ and Re = l.u.b.$(A_i)_{i\epsilon I}$ and let Re_1 be a fixed member of $(A_i)_{i\epsilon I}$.

By cor.1 of lemma 6.12 there is a central idempotent h with

(1) $Re_1(1-h) \cong$ direct summand of $Re(1-e)(1-h)$

(2) $R(1-e)h \cong$ direct summand of Re_1h.

Now $h \neq o$ for if $h = o$ then $Re_1 \cong$ direct summand of $R(1-e)$ and we could extend the set $(Re_i)_{i\epsilon I}$ of ~~mutually~~ Pairwise isomorphic independent elements of $F(_RR)$ thus contradicting the maximality of $(Re_i)_{i\epsilon I}$.

Since R is upper continuous

$$Reh = Rh \cap Re = Rh \cap \text{l.u.b.}(Re_i)_{i\epsilon I}$$

$$= \text{l.u.b.}(Rh \cap Re_i)_{i\epsilon I}$$

$$= \text{l.u.b.}(Re_ih)_{i\epsilon I} \quad .$$

Therefore $0 \neq Rh = Reh \oplus R(1-e)h$

$$= \text{l.u.b.}[(Re_ih)_{i\epsilon I}, R(1-e)h] \; .$$

Let $d_o = (1-e)h$ then $Rd_o \cong$ direct summand of Re_1h. Hence we can write for each $i \epsilon I$ $Re_ih = Rf_i \oplus Rd_i$ where $Rd_o \cong Rd_i$ and $(f_i)_{i\epsilon I}$ and $(d_i)_{i\epsilon I}$ are idempotents. Hence $Rh = \text{l.u.b.}(Rd_o \oplus Rf_1, Rd_1 \oplus Rf_2, \ldots)$. Putting $B_1 = Rd_o \oplus Rf_1$, $B_2 = Rd_1 \oplus Rf_2$, ... we get $Rh = \text{l.u.b.}(B_i)_{i\epsilon I}$ where $(B_i)_{i\epsilon I}$ is an infinite independent set of ~~mutually~~ Pairwise isomorphic elements of $F(_RR)$.

Since I is infinite it can be written as the disjoint union $\bigcup_{n=1}^{\infty} I_n$ of a countable number of sets $(I_n)_1^{\infty}$ each with the same cardinal as I. Define $B_{I_n} = \text{l.u.b.}(B_i)_{i \in I_n}$ then $Rh = \text{l.u.b.}(B_{I_n})_1^{\infty}$. But $\sum_{i \in I_n} B_i \cong \sum_{i \in I_1} B_i$ and so by lemma 6.10 $E(\sum_{i \in I_n} B_i) \cong E(\sum_{i \in I_1} B_i)$. Thus by lemma 6.1 and lemma 6.9 $B_{I_n} \cong B_{I_1}$. Hence we have that $(B_{I_n})_1^{\infty}$ is an independent countable set of ~~mutually~~ pairwise isomorphic elements of $F({}_RR)$.

Since R is properly infinite we may repeat the argument on R(1-h). By transfinite induction we get that R = l.u.b. of a countable independent set $(C_j)_{j \in J}$ of ~~mutually~~ pairwise isomorphic elements of $F({}_RR)$.

For any integer n J can be written as the disjoint union $\bigcup_{k=1}^{n} J_k$ of countable subsets J_k of J. Define $C_{J_k} = \text{l.u.b.}(C_j)_{j \in J_k}$. Now since J and J_k are both countable we have $\sum_{j \in J} C_j \cong \sum_{j \in J_k} C_j$ and so arguing as before $\text{l.u.b.}(C_j)_{j \in J} \cong \text{l.u.b.}(C_j)_{j \in J_k}$ i.e. $C_{J_k} \cong R$. But $R = \text{l.u.b.}(C_j)_{j \in J} = \text{l.u.b.}\ (C_{J_k})_1^n = \bigoplus_{k=1}^{n} C_{J_k} \cong R^n$.

Now suppose ${}_RP$ is a module with $P^n \cong R^n \cong R$. Let $S = \text{End}_R(P)$ and taking endomorphism rings we get $S_n \cong R$. Since R is upper continuous and regular then so is S_n. By (A3) S is regular and the lattice isomorphism $F({}_{S_n}S_n) \cong F({}_SS^n)$ shows that S is upper continuous.

Suppose f is a central idempotent of S then there is a central idempotent e of R such that $eRe \cong (fSf)_n$. As P is properly infinite eRe is infinite and hence so is $(fSf)_n$. But $F({}_{fSf}fSf^n)$ is a complete lattice and infinite. Hence by (3) of lemma 6.3 fSf must be infinite. Otherwise $F({}_{fSf}fSf^n)$ would be finite. Therefore S is properly infinite.

Now S is also upper continuous and so by the first part $S \cong S^n$. But ${}_RP$ is a progenerator for ${}_R\mu$ and so by (B) of theorem 3.1 there is a category equivalence $F: {}_S\mu \longrightarrow {}_R\mu$ such that $(S)^F = P$. But $S^n \cong S$ and so $(S^n)^F \cong S^F$ i.e. $P^n \cong P$. Therefore $P \cong P^n \cong R^n \cong R$ and so R is an s.u.c. ring.

The first part of this proof is closely modelled on lemmas 3.5 and 4.5 of Kaplansky (5).

<u>Remark.</u> In the proof we showed that for any integer ${}_RR \cong {}_RR^n$. In fact it can be shown that ${}_RR_R \cong {}_RR^I_R$ where I is a countable set.

<u>Theorem 6.17.</u> An upper continuous regular ring is an s.u.c. ring.

<u>Proof</u>

Any complete regular ring R is the direct product of a finite regular ring R_F and a properly infinite regular ring R_I (see e.g. prop.2 on p.9 of Kaplansky (3)).

If R is upper continuous then so are R_F and R_I. By cor.1 of theorem 6.15 and theorem 6.16 R_F and R_I are s.u.c. rings. Hence by lemma 5.4 R is an s.u.c. ring.

<u>Cor.1.</u> Any left self-injective ring R is an s.u.c. ring.

Proof

By theorem 4.3 of Utumi (4) if R is left self-injective then so is R/J(R). But lemma 8 of Utumi (1) shows that R/J(R) is regular and so R/J(R) is an s.u.c. ring. Therefore by lemma 5.6 R is an s.u.c. ring.

Utumi has made the following definitions (see Utumi (4)).

Definition. A ring R is called left continuous if

(1) for every left ideal A of R there is an idempotent $e \in R$ with $A \subset' Re$

(2) if B is a left ideal and f an idempotent $\in R$ and $B \cong Rf$ then B is generated by an idempotent.

Theorem 4.6 of Utumi (4) shows that if R is a left continuous ring then R/J(R) is an upper continuous regular ring and so R/J(R) is an s.u.c. ring. This gives us the following corollary.

Cor.2. A left continuous ring is an s.u.c. ring.

Theorem 6.18. Let R and S be rings with zero singular ideal and with injective hulls I(R) and I(S). If $R_n \cong S_n$ for some integer n then $I(R) \cong I(S)$.

Proof

Let F be the category equivalence: ${}_R\mu \longrightarrow {}_{R_n}\mu$ with $(R^n)^F = R_n$. Then $[I(R^n)]^F = I((R^n)^F) = I(R_n)$. Now $I(R^n) = I(R)^n$ and so taking endomorphism rings and noting $\mathrm{End}_R(I(R)) \cong I(R)$ as rings we have that $(I(R))_n \cong \mathrm{End}_{R_n}(I(R_n))$.

But $(I(R))_n$ is a regular ring and hence semi-simple (in the sense of Jacobson). So it follows (see e.g. sections 5 and 9 of Lambek (1)) that R_n has zero singular ideal. Hence $I(R_n)$ is a left self-injective regular ring and $I(R_n) \cong End_{R_n}(I(R_n)) \cong (I(R))_n$. Similarly $I(S_n) \cong (I(S))_n$.

If $R_n \cong S_n$ then $I(R_n) \cong I(S_n)$ and so $(I(R))_n \cong (I(S))_n$. But $I(R)$ and $I(S)$ are left self-injective regular rings and so by theorem 6.17 $I(R) \cong I(S)$.

Cor.1. Let R and S be regular rings such that $R_n \cong S_n$ for some integer n. Then $I(R) \cong I(S)$.

Proof

Both R and S have zero singular ideal and so the result follows from theorem 6.18.

This corollary shows that any co-ordinatization by a regular ring of a complemented modular lattice order ≥ 3 is unique up to injective hull or left quotient ring. The problem as to whether the co-ordinatization is in general unique seems difficult. Our results give some indication that it might be possible to prove this if the lattice is complete.

CHAPTER 7

INTEGRAL DOMAINS

We conclude by applying our results to the particular case of integral domains.

Definition. An integral domain is called a left Ore domain if for any non-zero elements a, b ϵ R $Ra \cap Rb \neq 0$. Otherwise R is called a non-Ore domain.

If R is a non-Ore domain then there are non-zero elements a, b ϵ R with $Ra \cap Rb = 0$. It is clear then that $[Rba^i]_0^\infty$ is a countably infinite independent set of principal left ideals of R. Thus R contains a free module of infinite rank.

As is well known a left Ore domain R can be embedded in a division ring Q such that every element of Q can be written in the form $a^{-1}b$ for some a, b ϵ R. Q is called the quotient ring of R. If $(a_i^{-1}b_i)_1^n$ are any finite set of elements of Q where $(a_i)_1^n$, $(b_i)_1^n$ ϵ R then they "can be put over a common denominator" i.e. there are elements $(c_i)_1^n$, a ϵ R such that $a_i^{-1}b_i = a^{-1}c_i$ for i = 1, ..., n.

Definition. Let R be a left Ore domain with quotient ring Q. Suppose $_RM$ is a module. Then the rank of M is defined to be the dimension of $Q \otimes_R M$ as a vector space over Q. The set $T(M) = \{m : \ell(m) \neq 0\}$ is a submodule of M called the torsion submodule. It can easily be shown that m ϵ T(M) if and only if $1 \otimes m = o$ in $Q \otimes_R M$.

Lemma 7.1. Let R be a left Ore domain and ${}_RM$ a module of rank ≥ 3. Then ${}_RM$ satisfies conditions S_1 and S_2 of theorem 4.8.

Proof

Let Q be the quotient ring of R. Any element $u \in Q \otimes_R M$ is of the form $\sum_{i=1}^{n} (r_i^{-1}t_i \otimes m_i)$ for some integer n and elements $r_i, t_i \in R$ and $m_i \in M$. Now for some $r, c_i \in R$ $r_i^{-1}t_i = r^{-1}c_i$ and so $u = \sum_{i=1}^{n} r^{-1}c_i \otimes m_i = r^{-1}(1 \otimes m)$ where $m = \sum_{i=1}^{n} c_i m_i$.

Suppose $x, y, z \in M$ and $N = Rx + Ry + Rz$. Then $Q \otimes_R N = Q(1 \otimes x) + Q(1 \otimes y) + Q(1 \otimes z)$. Now suppose either (1) one of $x, y, z \in T(M)$ or (2) x, y, z are free elements but (Rx, Ry, Rz) is not an independent set of submodules.

In case (1) one of $1 \otimes x$, $1 \otimes y$, $1 \otimes z$ is zero and so $\dim(Q \otimes N) \leq 2$.

In case (2) there are elements $a, b, c \in R$ two of which at least are non-zero such that $ax + by + cz = o$. Then if $a \neq o$ $(1 \otimes x) = a^{-1}(a \otimes x) = a^{-1}(1 \otimes ax)$

$$= a^{-1}[1 \otimes -(by + cz)] = -a^{-1}b(1 \otimes y) + -a^{-1}c(1 \otimes z).$$

Hence $Q(1 \otimes x) \subset Q(1 \otimes y) + Q(1 \otimes z)$ and so $\dim(Q \otimes N) \leq 2$.

But $\dim(Q \otimes M) \geq 3$ so in either case (1) or (2) there is a free element $w \in M$ with $Q(1 \otimes w) \cap [Q \otimes (Rx + Ry + Rz)] = 0$. Hence $Rw \cap [Rx + Ry + Rz] = 0$.

Suppose neither case (1) or (2) holds then x, y, z are free elements and (Rx, Ry, Rz) is an independent set of submodules. If $w = x + y + z$ then w is free and $Rw \cap (Rx + Ry) = Rw \cap (Ry + Rz) = Rw \cap (Rz + Rx) = 0$.

Hence M satisfies condition S_1. As in lemma 2 of Skornyakov (2) in this case S_2 is a consequence of S_1.

Theorem 7.2. Let R be a left Ore domain and $_RM$ a module of rank ≥ 3. If $_SN$ is a faithful module such that $\Sigma: L(_RM) \overset{c}{\cong} L(_SN)$ then there is a semi-linear isomorphism inducing Σ.

Proof

By lemma 7.1 M satisfies S_1 and S_2. Further R is an integral domain and so is inverse symmetric. The result then follows from theorem 4.9.

We note in this case the semi-linear isomorphism induces the lattice isomorphism. This is also true for our results on free modules if the rings considered are integral domains. First we need a lemma.

Lemma 7.3. Let $_RM$ be a module and a, b, c ε M such that $\ell(a) \subset \ell(b)$ and (Ra, Rb, Rc) are independent. Then $R(a + b + c) = [R(a + b) \oplus Rc] \cap [R(a + c) \oplus Rb]$.

Proof

$$R(a + b) + Rc = R(a + b + c) + Rc$$

$$R(a + c) + Rb = R(a + b + c) + Rb \quad .$$

Hence $[R(a+b) \oplus Rc] \cap [R(a+c) \oplus Rb] = [R(a+b+c)+Rc] \cap [R(a+b+c)+Rb]$

$= R(a+b+c) + [R(a+b+c)+Rc] \cap Rb$ (applying the modular law)

$= R(a+b+c) + [R(a+b) \oplus Rc] \cap Rb$

$= R(a+b+c) + R(a+b) \cap Rb$ (since (Ra, Rb, Rc) are independent)

$= R(a+b+c) + \ell(a)b$

$= R(a+b+c)$ (since $\ell(a) \subset \ell(b)$).

Theorem 7.4. Let R be an integral domain and $_RM$ a free module of rank ≥ 3. If $_SN$ is a faithful module with $\Sigma : L(_RM) \overset{c}{\cong} L(_SN)$ then there is a semi-linear isomorphism: $(R, M) \cong (S, N)$ inducing Σ.

Proof

Let $(e_i)_{i \in I}$ be a basis for M. Define $P_i^* = \sum_{j \neq i} Re_j$ and $Q_i^* = (P_i^*)^\Sigma$. By theorem 4.6 there is a semi-linear isomorphism (ℓ, s): $(R, M) \cong (S, N)$ which for each $i \in I$ induces $\Sigma : L(_RP_i^*) \cong L(_SQ_i^*)$.

Let $m = \sum_{i \in I} r_i e_i \in M$. If $r_i = o$ for some $i \in I$ then $m \in P_i^*$ and so $(Rm)^\Sigma = Sm^s$. Assume $r_i \neq o$ for any $i \in I$.

Since I has at least three elements we can write m in the form $x + y + z$ where $x = r_1e_1$ $y = r_2e_2$ and $z = \sum_{i \neq 1,2} r_i e_i$. But x is free and (Rx, Ry, Rz) are independent and so by lemma 7.3 we get $Rm = R(x + y + z) = [Ry \oplus R(x + z)] \cap [Rz \oplus R(x + y)]$. Now $Rx, R(x + z) \subset P_2^*$ and $Ry, R(y + z) \subset P_1^*$ and so $(Rx)^\Sigma = Sx^s$, $R(x + z)^\Sigma = S(x^s + z^s)$, $(Ry)^\Sigma = Sy^s$, $R(y + z)^\Sigma = S(y^s + z^s)$.

Therefore $(Rm)^{\Sigma} = [Sy^s \oplus S(x^s+z^s)] \cap [Sz^s \oplus S(x^s + y^s)]$

$= S(x^s + y^s + z^s)$ since $\ell(x^s) = 0$ and (Sx^s, Sy^s, Sz^s) are independent. Hence $(Rm)^{\Sigma} = Sm^s = (Rm)^s$ and it is now clear that (ℓ, s) induces Σ.

We now prove a generalization of a result in Wolfson (1). Wolfson shows that if R and S are both principal left ideal domains and I is a non-empty set then $R_{rfI} \cong S_{rfI}$ implies $R \cong S$.

Definition. A ring R is called indecomposable if R contains no idempotents other than o or 1 i.e. R has no proper left ideals as direct summands.

Theorem 7.5. Let R be an indecomposable ring and S a ring. Suppose I and J are non-empty sets and that $R_{rfI} \cong S_{rfJ}$.

If either (1) S is a local P.F.ring and I and J are finite

or (2) S is a P.F. ring

then $R \cong S$.

Proof

(1) Suppose I and J are finite with n and m elements respectively. We have then $R_n \cong S_m$ and without loss of generality can assume n, m are both ≥ 3. By von Neumann's theorem we have a lattice isomorphism $\Sigma : L({}_R R^n) \cong L({}_S S^m)$.

Let $(e_i)_1^n$ be a basis for R^n and $Q = (Re_1)^{\Sigma}$. Now Q is finitely generated projective. If S is a local P.F. ring then Q is free.

But Re_1 has no proper direct summands and hence neither does Q. Hence Q must be free on one generator i.e. $Q \cong S$. Therefore by cor.1 of theorem 4.2 $R \cong S$.

(2) Suppose $\ell : R_{rfI} \cong S_{rfJ}$. Let ${}_R M = {}_R^I R$ and ${}_S N = {}_S^J S$. By lemma 1.7 $R_{rfI} = End_R(M)$ and $S_{rfJ} = End_S(N)$. Let e be the idempotent of R_{rfI} with 1 in the $(1, 1)^{th}$ place and zeros elsewhere. If $f = e^{\ell}$ then $R \cong e\, R_{rfI}\, e \cong f\, S_{rfJ}\, f$. Since R is indecomposable $f\, S_{rfJ}\, f$ contains no idempotents other than o or 1. But $f\, S_{rfJ}\, f \cong End_S(Nf)$ where Nf is the image of f. Hence Nf has no proper direct summands.

Now $N = Nf \oplus N(1 - f)$ and so Nf is projective. If S is a P.F. ring then Nf is free. Therefore as Nf has no proper direct summands Nf is free on one generator and so $R \cong End_S(Nf) \cong S$.

<u>Cor.1</u>. Let R be an integral domain and S a ring. Suppose I and J are non-empty sets. If either (1) S is a local P.F. ring and I and J are finite or (2) S is a P.F. ring then $R_{rfI} \cong S_{rfJ}$ implies $R \cong S$.

<u>Proof</u>

Since R is an integral domain it is indecomposable. The result then follows from theorem 7.5.

As a principal left ideal domain is a P.F. ring cor.1 includes the results of Wolfson as a special case.

<u>Theorem 7.6</u>. Let R and S be non-Ore domains with $\Sigma : L({}_R R) \cong L({}_S S)$. Then

(1) if Σ satisfies C_1 or C_2 then $R \cong S$

(2) $R \underset{M}{\sim} S$

(3) if S is a local P.F. ring $R \cong S$

(4) $I(R) \cong I(S)$

Proof

Since R is a non-Ore domain it contains a free module of infinite rank. Let $(e_i)_1^\infty$ be a basis for it.

(1) Suppose Σ satisfies C_1. Then $(Re_i)^\Sigma = Sf_i$ for some $f_i \in S$.

Now Σ induces $L({}_R\bigoplus_{i=1}^{\infty} Re_i) \cong L({}_S \bigoplus_{i=1}^{\infty} Sf_i)$. As S is an integral domain f_i is a free element and so by cor.1 to theorem 4.2 $R \cong S$. By symmetry the result holds if Σ satisfies C_2.

(2) Let ${}_RM = \bigoplus_{i=1}^{\infty} Re_i$ and ${}_SN = M^\Sigma$. Then ${}_SN$ contains free elements and so $L({}_RM) \cong L({}_SN)$ implies by theorem 3.4 that $R \underset{M}{\sim} S$.

(3) From (2) $R \cong End_S(Q)$ where ${}_SQ$ is a progenerator. If S is a local P.F. ring then Q is free. Since R is indecomposable Q has no proper direct summands and so is free on one generator. Therefore $R \cong S$.

(4) R and S both have zero singular ideal and so I(R) and I(S) are left self-injective regular rings and $E({}_RR) \cong F({}_{I(R)}I(R))$, $E({}_SS) \cong F({}_{I(S)}I(S))$.

Now $A \in E({}_RR)$ if and only if for any left ideal B, $A \subset' B$ implies $A = B$. Hence it is clear that $\Sigma : L({}_RR) \cong L({}_SS)$ induces $E({}_RR) \cong E({}_SS)$ and so $F({}_{I(R)}I(R)) \cong F({}_{I(S)}I(S))$.

Suppose A is a left ideal of I(R). Then A is a R-submodule of ${}_R I(R)$ and so $A \cap R \neq 0$. Let $o \neq a \in A \cap R$ and suppose there is an element $q \in I(R)$ with $qa = o$. If $q \neq o$ then are non-zero elements $r, s \in R$ such that $rq = s$. But $sa = rqa = o$ and this is impossible since $s \neq o$ and $a \neq o$. Hence $q = o$ and $\ell_{I(R)}(a) = 0$.

We have shown that every left ideal A of I(R) contains a free element i.e. contains a copy of I(R) as a direct summand. Clearly I(R) is properly infinite and upper continuous so by theorem 6.16 $I(R) \cong I(R)_n$ for any integer n.

Take $n \geq 3$ then by cor.3 of theorem 2.3 and lemma 1.3 $I(R) \cong I(R)_n \cong I(S)$.

Remark 1. In (4) we could have taken S to be any ring with zero singular ideal.

Remark 2. The results of this theorem would seem to indicate that for any non-Ore domains R and S $\Sigma : L({}_R R) \cong L({}_S S)$ implies $R \cong S$. We have been unable to prove this.

Finally we note that the injective hull of a non-Ore domain has some rather remarkable properties.

Example 7.6. Let R be a non-Ore domain and Q its (left) injective hull. Then

(1) Q is left self-injective but not right self-injective

(2) Q is upper continuous but not lower continuous

(3) Q is simple

(4) for any countable set I $Q \cong Q^I$ as ~~left~~ right Q-modules

(5) for any integer n $Q \cong Q_n$

(6) every finitely generated left ideal of Q is free on one generator

(7) Q is a left and right P.F. ring.

Proof

(1) Q is left self injective and so upper continuous. If Q were also right self injective then Q would be lower continuous and hence continuous. But Q is infinite and as is well known a continuous regular ring is finite (see e.g. prop.80 of Skornyakov (1)). Hence Q cannot be right self injective.

(2) The arguments above show that Q cannot be lower continuous.

(3) Any non-zero left ideal A of Q contains an element a with $\ell_Q(a) = 0$ (see proof of (4) of theorem 7.5). But a Q = e Q for some idempotent $e \in Q$ and so $(1 - e)a = o$. Therefore e = 1 and aQ = Q. Thus AQ = Q for all non-zero left ideals A and so Q is simple.

(4) and (5) follow from theorem 6.16 since Q is properly infinite and upper continuous

(6) Let Q be any ring and ${}_QA$ and ${}_QB$ injective modules. Suppose there are Q-monomorphisms $A \longrightarrow B$ and $B \longrightarrow A$. Then Bumby has shown (see Bumby (1)) that $A \cong B$.

Now if $A \in F({}_Q Q)$ then A is a direct summand of Q and so is Q-injective. But A contains Q as a direct summand (see proof of (4) of theorem 7.5). Hence by Bumby's result $A \cong Q$ i.e. every finitely generated left ideal of Q is free on one generator.

(7) By a result of Bass (corollary of theorem 3 of Bass (2)) Q is a right P.F. ring. But by Kaplansky (1) every projective left Q-module over a regular ring Q is a direct sum of finitely generated left ideals of Q. Hence by (6) Q is also a left P.F. ring.

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