The SQ universality of some small cancellation groups

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THE SQ-UNIVERSALITY

OF SOME

SMALL CANCELLATION GROUPS

BY

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ABSTRACT

A group G is a small cancellation group if, roughly, it has a presentation

$$G = \langle A;R \rangle$$

with the property that for any pair r,s of elemets of R either $r \equiv s^{-1}$ or there is very little free cancellation in forming the product rs. The classical example of such a group is the fundamental group of a closed orientable 2-manifold of genus k which has a presentation

$$G = \langle a_1, b_1, \ldots, a_k, b_k;\ \prod_{i=1}^{k} [a_i, b_i] \rangle .$$

A countable group G is SQ-universal if every countable group can be embedded in some quotient of G. The obvious example of SQ-universal group is the free group of rank $\aleph_0$.

This work is a study of the SQ-universality of some small cancellation groups. A theory of diagrams is investigated in some detail to be used as a tool in this study. The main achievement in this work is the following two results:

(1) With few exceptions a small cancellation group contains non-abelian free subgroups. (The emphasis here is on the nature of the free generators.)

(2) A characterization of the SQ-universality of some small cancellation groups.

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M.A.R. AL-JANABI

CONTENTS

CHAPTER I

DEFINITIONS AND PRELIMINARY RESULTS

The reader is presumed to have some background in group theory, general topology and graph theory to the extent, at least, of feeling at home with the basic concepts. Since the work of Lyndon (10) and Weinbaum (25) the study of small cancellation groups has been based to a considerable extent on geometric techniques, specifically the study of planar maps; in section 1 we set forth some conventions in notation and terminology, and record some preliminary results on planar maps. In section 2 we give the basic definitions and results concerning R-diagrams and C(6)-groups. We conclude the chapter in section 3 with some general and historical remarks.

In what follows we rely, mainly, on the following papers. Lyndon (10), Miller and Schupp (12), Schupp (17,18,19,20). We are glad to acknowledge our debt to these works.

Section 1 - Maps

Let Π denote the Euclidean plane. If $S \subseteq \Pi$, then $S^{\bullet}$ will denote the boundary of S, the topological closure of S will be denoted by $\bar{S}$, and $-S$ will denote $\Pi - S$. A vertex is a point of Π. An edge is a bounded subset of Π homomorphic to the open unit interval. A region is a bounded set homeomorphic to the open unit disc. A map $\mathcal{M}$ is a finite collection of vertices, edges, and regions which are pairwise disjoint and satisfy:

(i) If E is an edge of $\mathcal{M}$, there are vertices v_1 and v_2 (not necessarily distinct) in $\mathcal{M}$ such that $\bar{E} = E \cup \{v_1\} \cup \{v_2\}$.

(ii) The boundary $D^{\bullet}$ of each region D of $\mathcal{M}$ is connected and there is a set of edges $E_1, \ldots, E_n$ in $\mathcal{M}$ such that $D^{\bullet} = \bar{E}_1 \cup \ldots \cup \bar{E}_n$.

If E is an edge with $\bar{E} = E \cup \{v_1\} \cup \{v_2\}$, the vertices v_1 and v_2 are called the end points of E. A closed edge is an edge E together with its end points. Since we usually want to consider closed edges, the word "edge" will, ambiguously, often mean a closed edge.

Let $\mathcal{M}$ be a map. We shall use $\mathcal{V}_{\mathcal{M}}$, $\mathcal{E}_{\mathcal{M}}$, $\mathcal{F}_{\mathcal{M}}$, to denote the set of vertices, edges and regions in $\mathcal{M}$ respectively. When no confusion seems possible we may write $\mathcal{V}$, $\mathcal{E}$, $\mathcal{F}$. The support of $\mathcal{M}$ is the set theoretic union of $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$. Since $\mathcal{M}$ is finite, its support is bounded. We shall use $\mathcal{M}$ to denote the support of $\mathcal{M}$. $\mathcal{M}$ is called connected, simply connected if its support is. A bounded component of $-\mathcal{M}$ is called a hole, while the unbounded component of $-\mathcal{M}$ constitutes the exterior of $\mathcal{M}$. We write $\mathcal{M}^{\bullet}$ for the boundary of the support of $\mathcal{M}$. A vertex (or edge) is a boundary vertex (or edge) if it lies in $\mathcal{M}^{\bullet}$. A region D is a boundary region if $D^{\bullet} \cap \mathcal{M}^{\bullet} \neq \emptyset$, where $\emptyset$ is the empty set. A vertex, edge or region of $\mathcal{M}$ which is not a boundary vertex, edge or region is called interior. The 1-skeleton $\mathcal{M}'$ of $\mathcal{M}$ (the edges and vertices) does not determine $\mathcal{M}$, since it does not determine which components of $-\mathcal{M}'$ are regions of $\mathcal{M}$.

Since $\mathcal{M}$ is planar, it is possible to orient the regions of $\mathcal{M}$ and the components of $-\mathcal{M}$ so that in traversing the boundaries of regions of $\mathcal{M}$ and the components of $-\mathcal{M}$, each edge of $\mathcal{M}$ is traversed twice, once in each of its possible orientations. We shall consider $\mathcal{M}$ as being oriented anti-clockwise. With the orientation in mind, it is often convenient to consider each edge of $\mathcal{M}$ as consisting of an unordered pair of oriented edges. To do this formally one replaces $\mathcal{E}$ by the disjoint union $\vec{\mathcal{E}} \cup \overleftarrow{\mathcal{E}}$ where $\overleftarrow{\mathcal{E}}, \vec{\mathcal{E}}$ are both in (1-1) correspondence with $\mathcal{E}$. We write $\mathcal{M} = \mathcal{V} \cup \vec{\mathcal{E}} \cup \overleftarrow{\mathcal{E}} \cup \mathcal{F}$ if we wish to emphasise the orientation. If an edge E lies in $\overleftarrow{\mathcal{E}}$, running from end point v_1 to end point v_2, then v_1 is the initial vertex of E, denoted by $\lambda(E)$, and v_2 is the terminal vertex of E, denoted by $\mu(E)$. If v is a vertex in $\mathcal{M}$, $d(v)$ will denote the number of oriented edges having v as an initial vertex, and will be called the degree of v. If E lies in $\overleftarrow{\mathcal{E}}$ (or $\vec{\mathcal{E}}$), then we shall denote the corresponding edge in $\vec{\mathcal{E}}$ (or $\overleftarrow{\mathcal{E}}$) by E^{-1}. Hence for each E in $\mathcal{M}$, $\lambda(E) = \mu(E^{-1})$ and $\mu(E) = \lambda(E^{-1})$. A path is a sequence of oriented (closed) edges $E_1, \ldots, E_n$ such that $\mu(E_i) = \lambda(E_{i+1})$. The end points are $\lambda(E_1)$ and $\mu(E_n)$. A path whose end points coincide is called a closed path or cycle. A path is reduced if it does not contain a successive pair of edges of the form E, E^{-1}. A path $E_1, \ldots, E_n$ is simple if for every pair i,j of distinct indices, $\lambda(E_j) \neq \lambda(E_i^{-1})$. If D is a region of $\mathcal{M}$ with a given orientation, any cycle $E_1, \ldots, E_n$ of minimal length which includes all edges of $D^{\cdot}$, and in which the edges are oriented in accordance with the orientation of D, is called a boundary cycle of D, and denoted by $\delta(D)$. Note that distinct boundary cycles are just cyclic permutations of one another and that we use $\delta(D)$ for any suitable boundary cycle of D. We write $\delta(D) = (E_1, \ldots, E_n)$, and call $n = d(D)$ the degree of D. If $\mathcal{M}$ is a connected simply connected map, a boundary cycle $\delta(\mathcal{M})$ of $\mathcal{M}$ is a cycle of minimal length which contains all the edges in $\mathcal{M}^{\cdot}$; and note that $\delta(\mathcal{M})$ stands for any suitable boundary cycle of $\mathcal{M}$.

The symbol $i(D)$ will denote the number of interior edges in $\delta(D)$. (When necessary, a subscript naming the map will be added.) We define

$$\beta(D) = \{E, E^{-1}, \lambda(E) \mid E \text{ occurs in } \delta(D)\},$$

and

$$\beta(\mathcal{M}) = \{E, E^{-1}, \lambda(E) \mid E \text{ occurs in } \delta(\mathcal{M})\}.$$

If E is an edge in $\mathcal{M}$ and D is a region in $\mathcal{M}$ with E (or E^{-1}) in $\delta(D)$, we write $D = \rho(E)$ (or $\sigma(E)$). As a consequence of this notation, if E is an interior edge, then $\rho(E)$ and $\sigma(E)$ are defined and $\rho(E) = \sigma(E^{-1})$, $\sigma(E) = \rho(E^{-1})$. (Possibly $\rho(E) = \rho(E^{-1})$.)

A submap $\mathcal{N}$ of $\mathcal{M}$ is called an <u>extremal disc</u> if $\mathcal{N}$ is bounded by a simple closed path $E_1, \ldots, E_k$ in $\beta(\mathcal{M})$, where edges $E_1, \ldots, E_k$ occur consecutively on some boundary cycle of $\mathcal{M}$ and no proper subpath of $E_1, \ldots, E_k$ is closed. Since a bounded closed set with connected boundary is connected, any extremal disc is connected.

(1.1) <u>Lemma</u> (12) Let $\mathcal{M}$ be a connected, simply connected map without vertices of degree 1. If $\delta(\mathcal{M})$ is not simple closed, then $\mathcal{M}$ has at least two extremal discs.

Let D be a boundary region of $\mathcal{M}$. We say that D is a <u>boundary connected</u> region (a bc-region) if either

(a) $\beta(D) \cap \beta(\mathcal{M})$ is a single vertex, or

(b) $\delta(D) = (E_1, \ldots, E_k, E_{k+1}, \ldots, E_n)$ where $E_i \in \beta(\mathcal{M})$, $1 \le i \le k$ and $E_j \notin \beta(\mathcal{M})$, $k+1 \le j \le n$.

If (a) occurs we say that D is a <u>weakly boundary connected</u> region (a wbc-region) and if (b) occurs, we say D is a strongly boundary connected region (a sbc-region). We say that the boundary region D is boundary disconnected (a bdc-region) if it is not boundary connected. We say that D is a semi-interior region if $\beta(D) \cap \beta(\mathcal{M}) \cap \mathcal{E}_{\mathcal{M}} = \emptyset$. Thus wbc-regions and interior regions are semi-interior and sbc-regions are not semi-interior.

(1.2) Lemma (12) (Consecutive Boundary Lemma)

Let $\mathcal{M}$ be a connected, simply connected map with more than one region. Assume that if D is any region of $\mathcal{M}$, then $\delta(D)$ is a simple closed path. Further, assume that $\mathcal{M}$ has no vertices of degree 1. Then $\mathcal{M}$ has at least two distinct sbc-regions.

Let $\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ be a map ($\mathcal{E}$ consisting of unoriented edges). Let $\mathcal{V}^{\#} = \text{card}\,\mathcal{V}$, $\mathcal{E}^{\#} = \text{card}\,\mathcal{E}$ and $\mathcal{F}^{\#} = \text{card}\,\mathcal{F}$. Plain summation signs Σ will refer to sums indexed by all vertices or all regions of $\mathcal{M}$. Thus $\Sigma\, d(v)$ is the sum of the degrees of all vertices of $\mathcal{M}$ and $\Sigma\, d(D)$ is the sum of the degrees of all regions of $\mathcal{M}$. The notation $\Sigma^{\cdot}$ denotes summation restricted to boundary vertices or regions and Σ^{o} summation over interior vertices or regions.

Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$; the only positive integral solution are (3,6), (4,4), and (6,3). If $\mathcal{M}$ is a non-empty map such that each interior vertex in $\mathcal{M}$ has degree at least p and each interior region in $\mathcal{M}$ has degree at least q, $\mathcal{M}$ will be called a (p,q) map.

Let Q be the number of components of $\mathcal{M}$ and h the number of holes of $\mathcal{M}$.

(1.3) Lemma (10) Let $\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ be an arbitrary map, and let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then

$$(1)\quad p(Q-h) = \sum{}^{\bullet} [p-d(v)] + \sum{}^{o} [p-d(v)] + \frac{p}{q} \sum [q - d(D)] - \frac{p}{q} (\mathcal{E}^{\#})^{\bullet} ,$$

where $(\mathcal{E}^{\#})^{\bullet} = \text{card } \{E \mid E$ is an unoriented boundary edge in $\mathcal{M}\}$.

$$(2)\quad p(Q-h) = \sum{}^{\bullet} \left[\frac{p}{q} + 2 - d(v)\right] + \sum{}^{o} [p - d(v)] + \frac{p}{q} \sum [q - d(D)] + \frac{p}{q} [(\mathcal{V}^{\#})^{\bullet} - (\mathcal{E}^{\#})^{\bullet}] ,$$

where $(\mathcal{V}^{\#})^{\bullet} = \text{card } \{v \mid v$ is a boundary vertex in $\mathcal{M}\}$.

(1.4) Theorem (10) Let $\mathcal{M}$ be a connected, simply connected (q,p) map with (q,p) one of the pairs (3,6), (4, 4), (6,3). Assume that $\mathcal{M}$ has more than one region. Then

$$\sum{}^{\bullet} \left[\left(\frac{p}{q} + 2\right) - i(D)\right] \geq p .$$

(1.5) Lemma (18) Let $\mathcal{M}$ be a connected, simply connected (q,p) map with (q,p) one of (3,6), (4,4) or (6,3). Assume that if D is a semi-interior region then $d(D) \geq p$.

Then $\delta(D)$ is a simple closed path.

Schupp in (18) derived an inequality as in Theorem (1.4), where the summation is restricted to the sbc-regions as follows.

(1.6) Theorem Let $\mathcal{M}$ be a connected, simply connected (q,p) map. Suppose that $\mathcal{M}$ has more than one region, and no vertices of degree one. Suppose further that if D is a region of $\mathcal{M}$, then $d(D) \geq p$, for D semi-interior. Then

$$\sum \left[\frac{p}{q} + 2 - i(D)\right] \geq p ,$$

where the summation runs through all sbc-regions of $\mathcal{M}$.

Section 2 - R-diagrams and C(6) groups

Let F be a free group on a set A of generators. A letter is an element of the set X consisting of generators and inverses of generators. A word w is finite string of letters, $w = x_1 \ldots x_m$. We shall not always distinguish between w and the element of F it represents. We denote the identity of F by 1. Each element of F other than the identity has a unique representation as a reduced word $w = x_1 \ldots x_n$ in which no successive letters $x_i\ x_{i+1}$ form an inverse pair $a_j\ a_j^{-1}$ or $a_j^{-1}\ a_j$. The integer n is the length of w, which we denote by $|w|$. A reduced word w is called cyclically reduced if x_n is not the inverse of x_1. If there is no cancellation in forming the product $z = y_1 \ldots y_n$, we write $z \equiv y_1 \ldots y_n$.

A diagram over F is an oriented map $\mathcal{M}$ and a function ϕ assigning to each (oriented) edge E of $\mathcal{M}$, an element $\phi(E)$ of F, called the label on E, such that for any (oriented) edge, $\phi(E^{-1}) = \phi(E)^{-1}$. If $E_1, \ldots, E_k$ is a path in $\mathcal{M}$, we define $\phi(E_1, \ldots, E_k) = \phi(E_1) \ldots \phi(E_k)$.

(2.1) Lyndon-Van Kampen Theorem (10)

Let F be a free group and $u_1 r_1 u_1^{-1}, \ldots, u_n r_n u_n^{-1}$ a sequence of words with each r_i cyclically reduced. Then there exists a connected, simply connected diagram $\mathcal{M}$ over F such that

(i) for each edge E of $\mathcal{M}$, $\phi(E) \not\equiv 1$,

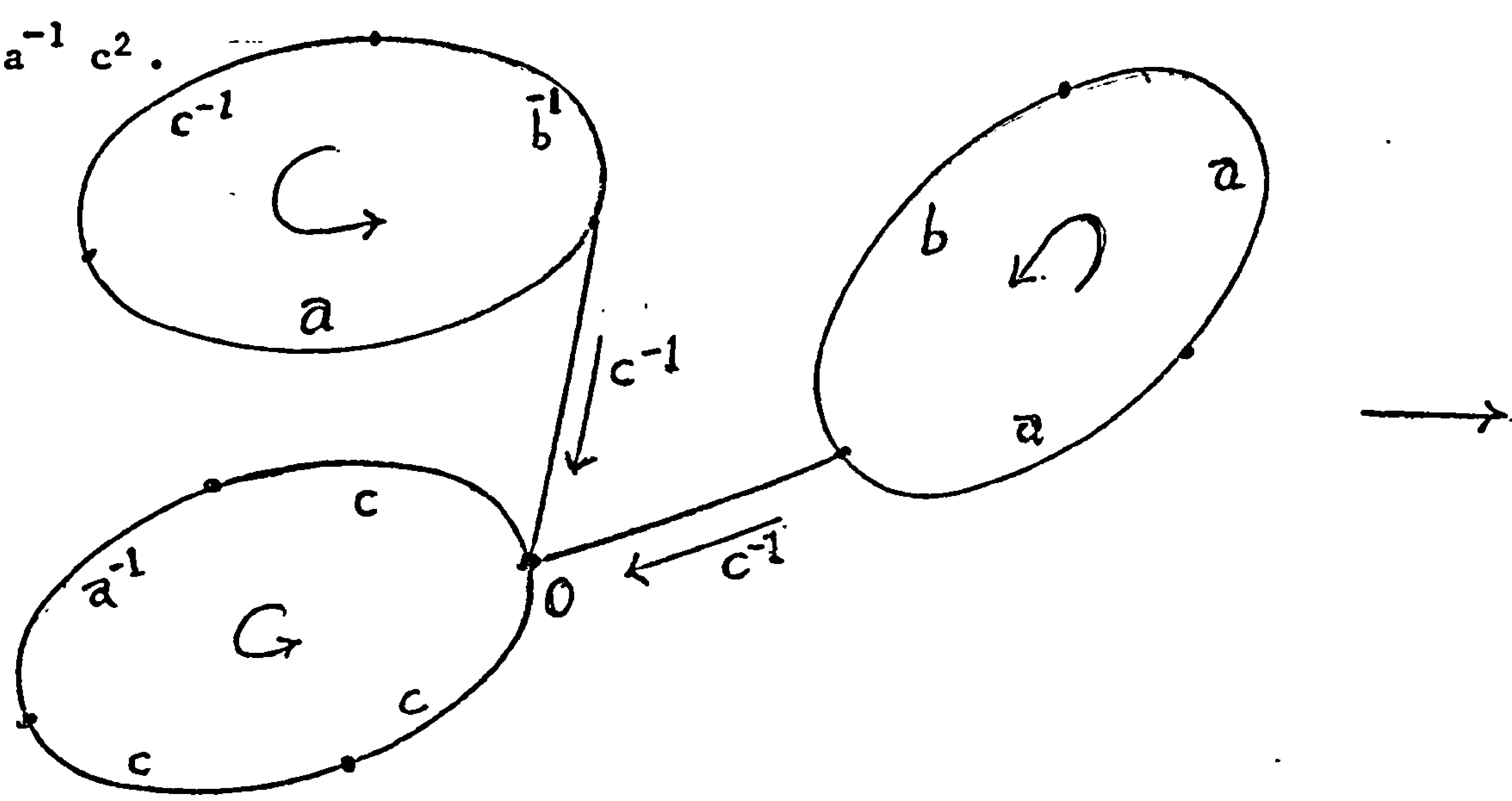
(ii) for each interior vertex v in $\mathcal{M}$, $d(v) \geq 3$,

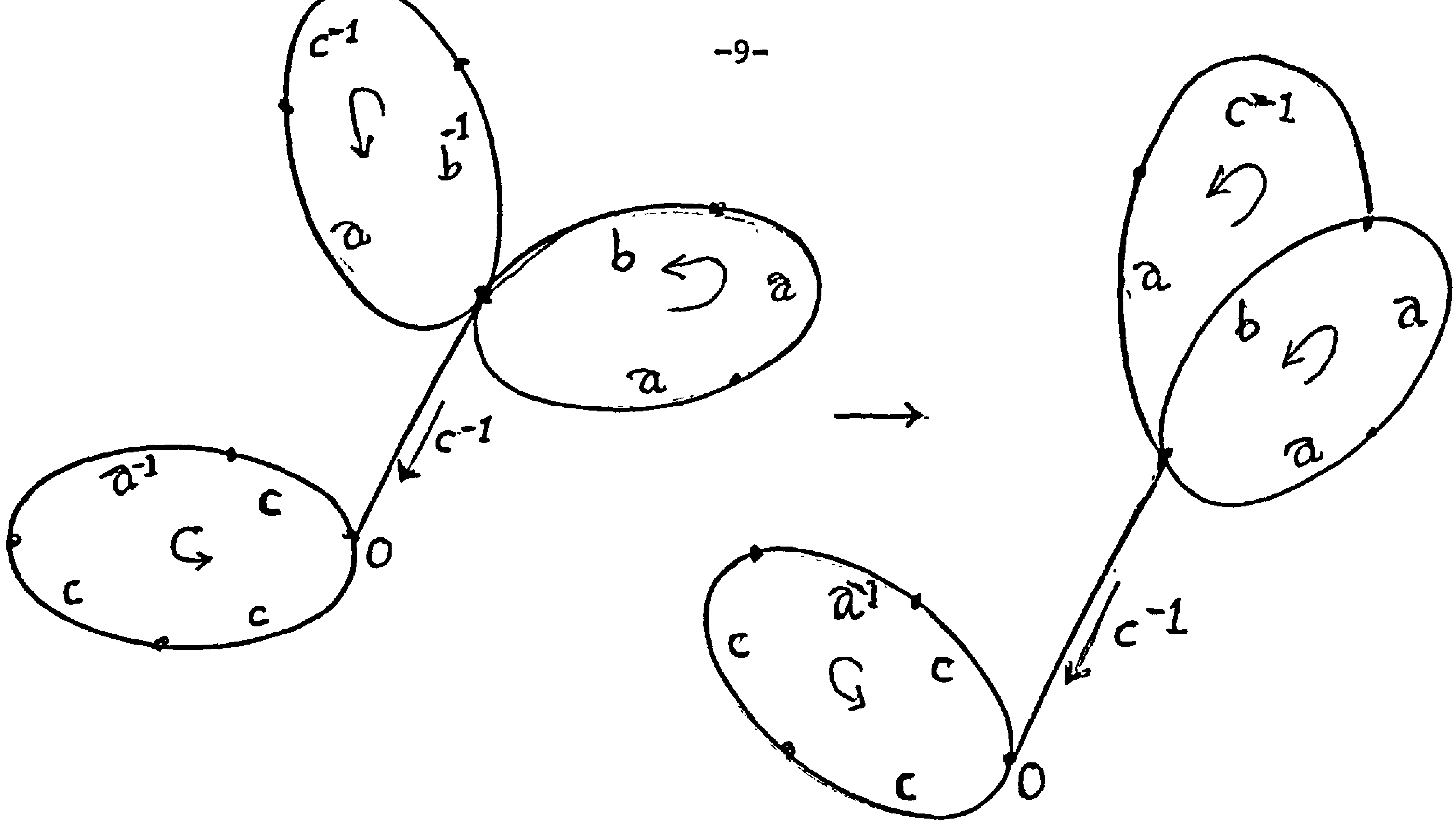
(iii) there is a boundary vertex 0 in $\mathcal{M}$ such that if $\delta(\mathcal{M})$ starts at 0 then $\phi(\delta(\mathcal{M}))$ is a freely reduced word and equal in F to $\left[\prod_{i=1}^{n} u_i r_i u_i^{-1}\right]^{-1}$.

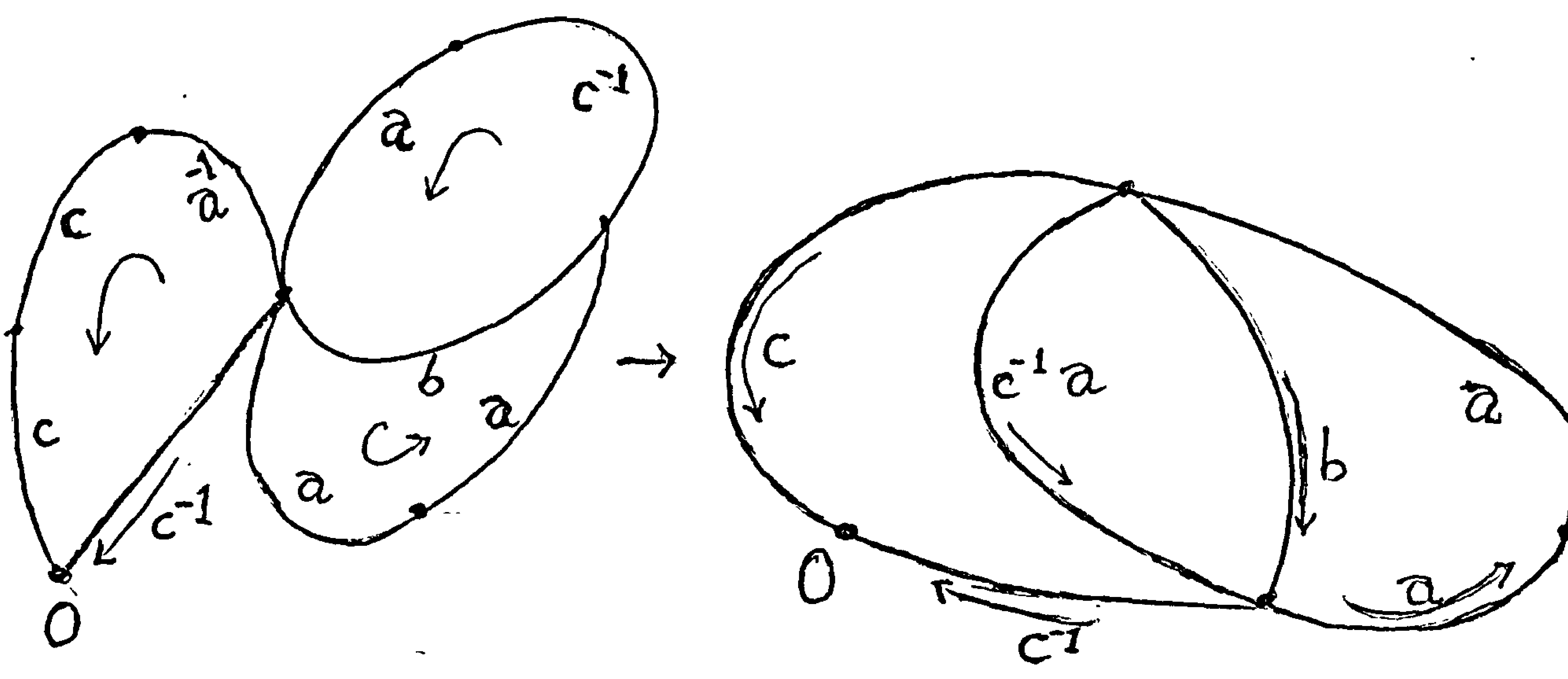
(iv) For each region D of $\mathcal{M}$, $\phi(\delta(D))$ is a cyclic permutation of some r_i.

(v) If D and D' are any two distinct regions in $\mathcal{M}$, with $\phi(\delta(D)) \equiv r_i$ and $\phi(\delta(D')) \equiv r_j$ then $i \neq j$.

Note the variation from Lyndon-Schupp which is due to the fact that their orientation of $\mathcal{M}$ is according to the orientation of regions rather than $\delta(\mathcal{M})$. We illustrate the construction of Theorem (2.1) for the sequence $c a^2 b c^{-1}$, $c b^{-1} c^{-1} a c^{-1}$, $c a^{-1} c^2$.







Note that the region corresponding to $(b^{-1}\ c^{-1}\ a)$ has no edges on the boundary of the final diagram.

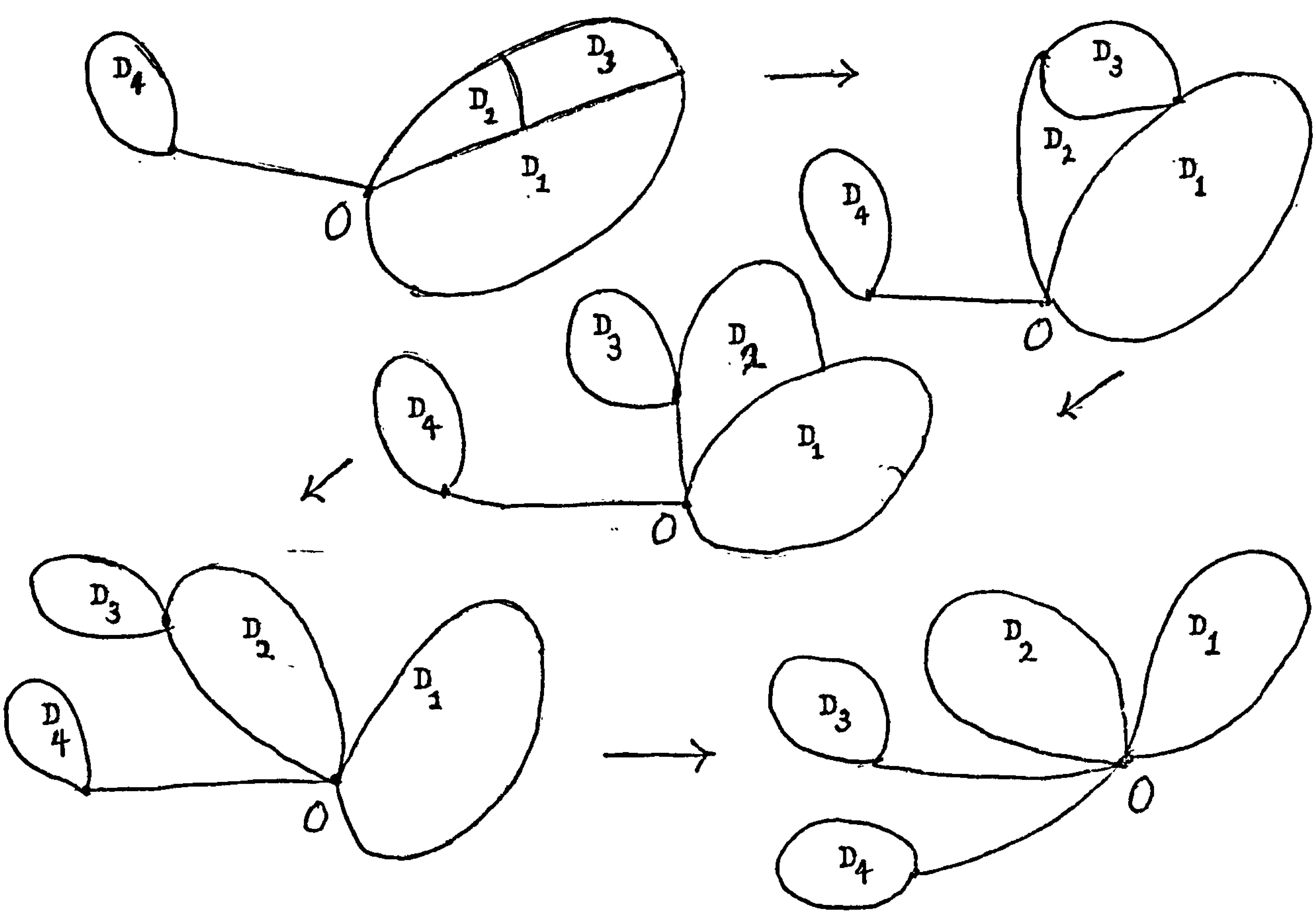
Let $K_{\mathfrak{M}} = \sum_{j=1}^{\ell} |\phi(E_j)|$, where the sum is over all the oriented edges of $\mathfrak{M}$.

(2.2) Lemma (17) (Normal Subgroup Lemma)

Let $\mathcal{M}$ be a connected, simply connected diagram over F with regions $D_1, \ldots, D_m$. Let $\delta(\mathcal{M})$ be a boundary cycle of $\mathcal{M}$ beginning at a vertex $v_o \in \beta(\mathcal{M})$, and let $w \equiv (\phi(\delta(\mathcal{M})))^{-1}$. Then there exist elements $u_1, \ldots, u_m$ of F such that
$w = u_1 r_1 u_1^{-1} \ldots u_m r_m u_m^{-1}$, where $r_j = \phi(\delta(D_j))$, and $|u_j| \le m K_{\mathcal{M}}$, $j = 1, \ldots, m$.

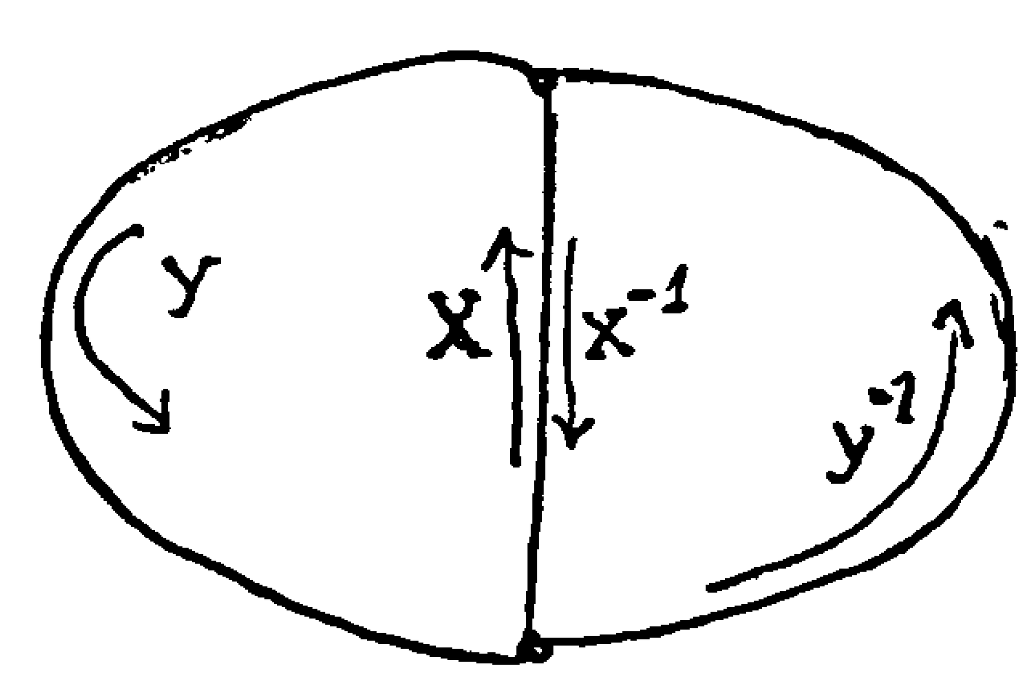
Note that $\phi(\delta(D_j))$ need not be reduced.

Hint for the proof: If $\mathcal{M}$ has no region then $\mathcal{M}$ is a tree and there is nothing to prove. In general "unstitch" $\mathcal{M}$. For example:



A subset R of F is called symmetrized if all elements of R are cyclically reduced and, for each r in R, all cyclically reduced conjugates of both r and r^{-1} also belong to R. An R-diagram is a diagram $\mathcal{M}$ over F such that if D is a region in

$\mathcal{M}$ then $\phi(\delta(D)) \in R$. As a consequence of Theorem (2.1) and Lemma (2.2), we have the following fact. Let N be the normal closure of R in F. For any element w in F, $w \in N$ if and only if there is a connected, simply connected R-diagram $\mathcal{M}$ such that the label on the boundary of $\mathcal{M}$ is w. Connected simply connected diagrams are therefore an adequate tool for studying membership in normal subgroups. Now, if w is in N, then w can be written as a product $w = c_1 \ldots c_n$, for some $n \geq 0$, of conjugates c_i of elements $r^{\pm 1}$ for r in R. A sequence $c_1, \ldots, c_n$ of conjugates of elements of R will be called a minimal R-sequence if the product $w = c_1 \ldots c_n$ cannot be written as a product of fewer than n conjugates of elements of R. Let $\mathcal{M}$ be an arbitrary diagram over F. $\mathcal{M}$ is reduced unless there exist D, D', regions in $\mathcal{M}$ such that $\delta(D) = (E, E_1, \ldots, E_n)$, $\delta(D') = (E'_1, \ldots, E'_m, E^{-1})$ and $\phi(E_1, \ldots, E_n)^{-1} \equiv \phi(E'_1, \ldots, E'_m)$.



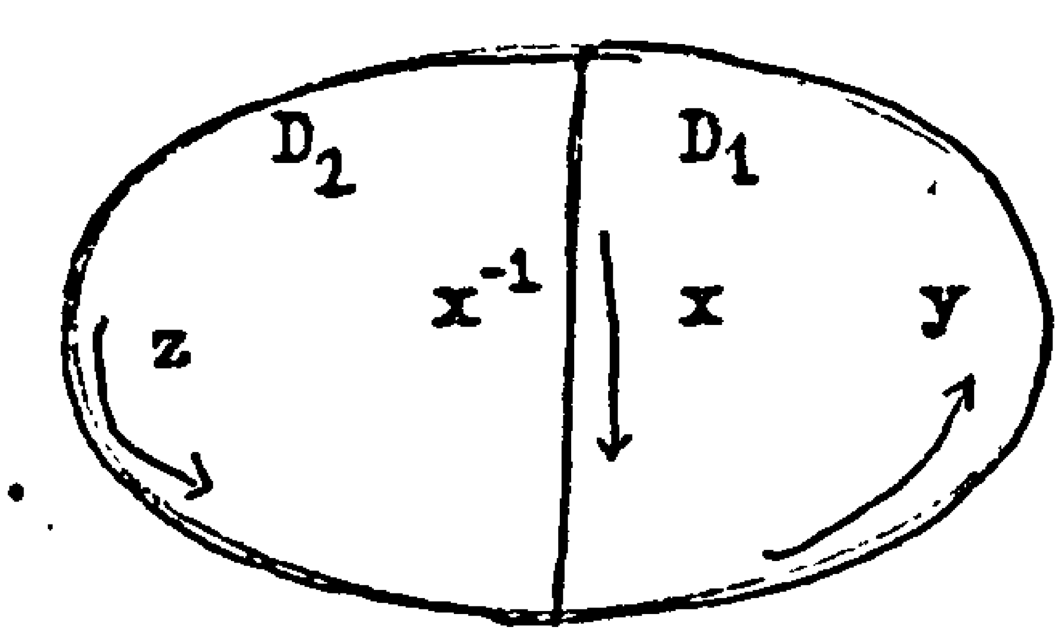
As a consequence of Normal Subgroup Lemma we have the following:

(2.3) Lemma (10,17) If $\mathcal{M}$ is a diagram of a minimal R-sequence, then $\mathcal{M}$ is reduced.

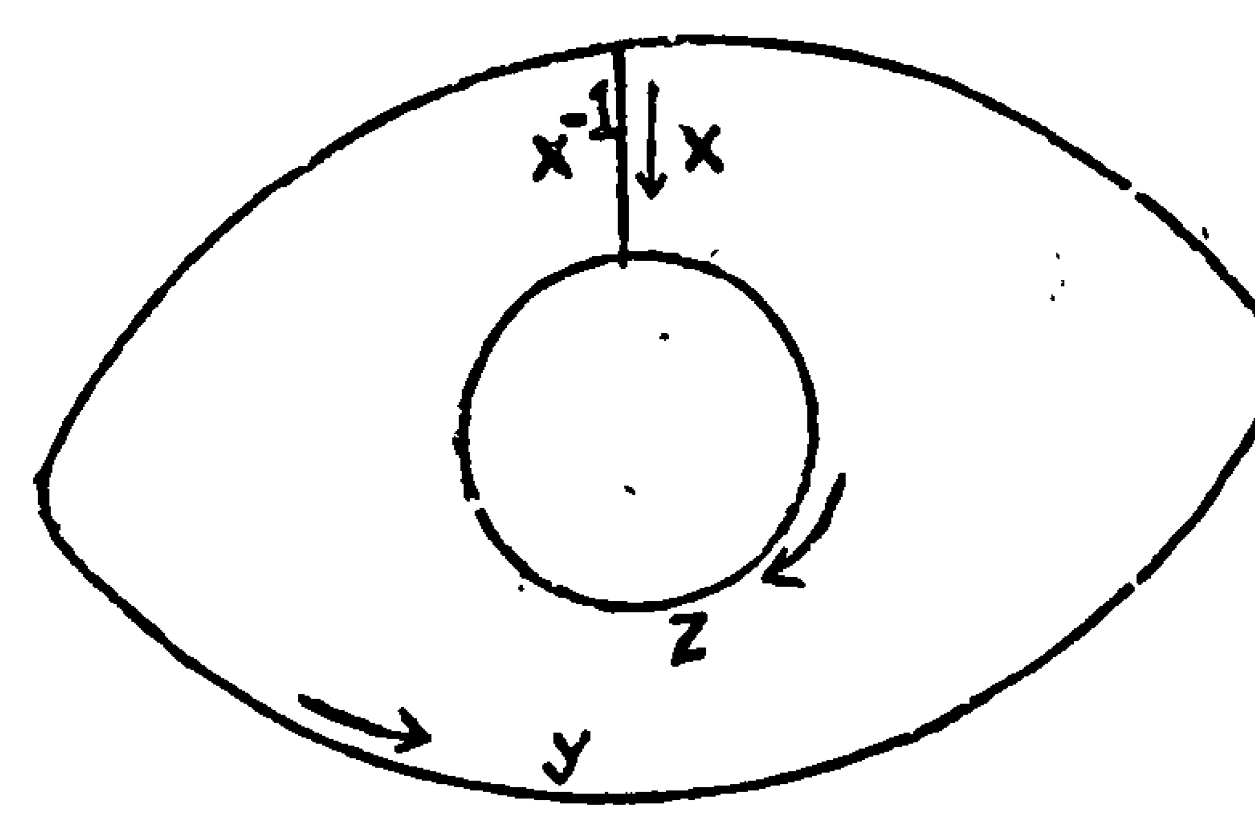
Suppose that r_1 and r_2 are distinct elements of R with $r_1 \equiv xz_1$ and $r_2 \equiv xz_2$. Then x is called a piece relative to R. Since x is cancelled in the product $r_1^{-1}r_2$, and R is a symmetrized, a piece is simply a subword of an element of R which can be cancelled by multiplication of two non-inverse elements of R. Note that any subword of a piece relative to R is a piece relative to R; in particular, any generator that occurs in a piece relative to R is a piece relative to R.

(2.4) Lemma (10) Let $\mathfrak{M}$ be a reduced R-diagram. If E is an interior edge of $\mathfrak{M}$, then $\phi(E)$ is a piece relative to R.

Proof: Let D_1 and D_2 be any two regions in $\mathfrak{M}$ such that $\beta(D_1) \cap \beta(D_2)$ has an edge.



Suppose first that $D_1 \neq D_2$. Let $\phi(\delta(D_1)) \equiv xy$ and $\phi(\delta(D_2)) \equiv zx^{-1}$. Since $\mathfrak{M}$ is reduced, $yz \not\equiv 1$; and so x is a piece.



Now suppose $D_1 = D_2 = D$.
$\phi(\delta\ (D)) \equiv xzx^{-1}y \equiv r$.
Let $r' \equiv xz^{-1}x^{-1}y^{-1}$. Since $z^{-1}x^{-1}y^{-1} \not\equiv zx^{-1}y$, x is a piece.

Let $\mathfrak{M}$ be a reduced R-diagram. Then the condition that $d(D) \geq p$, for any interior region D will follow from the following condition on R.

Condition C(p) No element of R is a product of fewer than p pieces relative to R.

Note that an element of R need not be a product of pieces relative to R.

Let v be an interior vertex such that $d(v) = q-1$, $q \geq 4$. Let $E_1, \ldots, E_{q-1}$ be the edges in $\mathcal{M}$ in anticlockwise order such that $\lambda(E_j) = v$, $1 \leq j \leq q-1$; and let $D_j = \rho(E_j)$. Then $\phi(\delta(D_j)) \equiv r_j \in R$. Assume that there is a cancellation in each of the products $r_1 r_2, \ldots, r_{q-2} r_{q-1}, r_{q-1} r_1$. To exclude such vertices we have the following condition on R.

Condition T(q) Let $3 \leq \ell \leq q-1$. Suppose $r_1, \ldots, r_\ell$ are elements of R with no successive elements $r_i r_{i+1}$ an inverse pair. Then at least one of the pairs $r_1 r_2, \ldots, r_{\ell-1} r_\ell, r_\ell r_1$ is reduced without cancellation.

(2.5) Lemma (10) Let R be a symmetrized set of elements of a free group F, and let $\mathcal{M}$ be a reduced R-diagram.

(1) If R satisfies $T(q)$, then for each interior vertex v in $\mathcal{M}$, $d(v) \geq q$.

(2) If R satisfies $C(p)$, then for each semi-interior region D of $\mathcal{M}$, $d(D) \geq p$.

A symmetrized set R which satisfies the hypotheses $C(p)$ and $T(q)$ for (q,p) one of $(3,6)$, $(4,4)$, or $(3,6)$ is called a small cancellation set (a (q,p)-set).

The factor group of F by the normal closure of such a set is called a small cancellation group (a (q,p)-group). Since condition $T(3)$ is vacuous, a $(3,6)$ group is actually a group that satisfies $C(6)$, and hence we call such a group a $C(6)$-group.

(2.6) Lyndon's Theorem (10) Let F be a free group, R a symmetrized subset of F, N the normal closure of R in F, and w a non-trivial freely reduced word in N. If R is a C(6)-set, then w contains some r in R with three pieces relative to R missing, i.e. w and r have reduced forms $w \equiv bac$ and $r \equiv ax_1x_2x_3$ where x_1, x_2 and x_3 are pieces relative to R.

Section 3 - General and Historical Comments

A condition closely related to C(p) is the condition $C'(\zeta)$, where ζ is a real positive number.

Condition $C'(\zeta)$ If an element r of R has reduced form $r \equiv xz$, where x is a piece relative to R, then $|x| < \zeta|r|$.

Note that $C'(\zeta)$ implies C(p) for $\zeta \le \frac{1}{p-1}$. The classical example is the fundamental group of closed, orientable 2-manifold of genus g which has a presentation

$G = \langle a_1, b_1, \ldots, a_g, b_g ; \prod_{i=1}^{g} [a_i, b_i] \rangle$. Let R be the set of all cyclic permutation of r and r^{-1}, where r is $\prod_{i=1}^{g} [a_i, b_i]$. Clearly, pieces relative to R are single letters and R satisfies $C'(1/4g-1)$ and $C(4g)$.

The cancellation conditions can be extended naturally to the case where F is a free product or a free product with amalgamation by using the appropriate normal forms and associated length functions. The definitions are then essentially the same as for F a free group.

The following Lemma due to Lipschutz (9):

(3.1) Lemma Let F be a free group on a set A. Let R be a symmetrized subset of F. Let $G = F/\langle F \langle R\rangle$, where $\langle F \langle R\rangle$ is the normal closure of R in F. Let $r \in R$. If $r \equiv x^{\ell}z$ and $z \not\equiv xz'$, where $\ell \geq 2$, $x,z,z' \in F$, then either $x^{\ell-1}$ (and hence x) is a piece relative to R or x and z are powers (in F) of a common subword.

A countable group G is called SQ-universal if every countable group can be embedded in a quotient group of G. The obvious example of an SQ-universal group is the free group of rank χ_0. The property of being SQ-universal may, in a very rough sense, be considered as an indication of "largeness" or "freeness".

(3.2) Theorem (19) If $G = H * K$ is any non-trivial free product except $C_2 * C_2$, where C_2 is the cyclic group of order two, then G is SQ-universal.

P. Neumann(13) has shown that a subgroup H of a group G, with $[G : H] < \infty$, is SQ-universal if and only if G is SQ-universal; and also he has proved that a finitely generated Fuchsian group which does not have an abelian subgroup of finite index is SQ-universal.

Let U be a subgroup of the group K. Let $\{x_1,x_2\}$ be a pair of distinct elements of K, neither of which is in U. We say that $\{x_1,x_2\}$ is a blocking pair for U in K if the following condition is satisfied:

(1) If $u \in U$, $u \neq 1$, then $x_i^{\varepsilon}\, u\, x_j^{\eta} \notin U$, $1 \leq i, j \leq 2$,

$$\varepsilon = \pm 1,\ \eta = \pm 1.$$

Note that (1) implies that $x_i^{\varepsilon}\, x_j^{\eta} \notin U$, $1 \leq i, j \leq 2$, $\varepsilon = \pm 1$,

$\eta = \pm 1$, unless $x_i^{\varepsilon} x_j^{\eta} = 1$.

(3.3) Theorem (19) Let $G = \langle K * L ; U = V \rangle$ be a free product amalgamating the proper subgroups U and V of K and L. If there is a blocking pair for U in K, then G is SQ-universal.

In (8), G. Higman used the group

$$H = \langle a,b,c,d ; \quad a^b = a^2 , \quad b^c = b^2 , \quad c^d = c^2 , \quad d^a = d^2 \rangle$$

to establish the existence of finitely generated infinite simple groups. He showed that H is infinite but has no non-trivial finite quotients.

(3.4) Corollary (20) The Higman group H is SQ-universal.

(3.5) Theorem (15) If $G = \langle A ; r \rangle$ is a group with a presentation having one defining relator and A has at least three generators, then G is SQ-universal.

For completeness, we include a brief historical sketch of small cancellation theory. What follows in this section is due to Schupp (19).

In 1911, M. Dehn (3) posed the word and conjugacy problems for groups in general and provided algorithms which solves these problems for the fundamental groups of closed orientable two-dimensional manifolds. A crucial feature of these groups is that (with trivial exceptions) they are defined by a single relator r with the property that if s is any cyclic conjugate of r or r^{-1} , $s \neq r^{-1}$, there is very little cancellation in forming the product rs . Dehn's algorithms have been extended to large classes of groups possessing presentations in which the defining relations have a similar "small cancellation" property. At first, investigations were concerned with the solution of the word problem for groups G presented as "small cancellation" quotient of a free group F. The theory was subsequently extended to the case where F is a free product of a free product with amalgamation. Moreover, strong results were obtained about algebraic properties; for example, one can classify torsion elements and commuting elements in "small cancellation" quotients.

Dehn's methods (3,4) were geometric. He used the fact that with the fundamental group G of an orientable closed 2-manifold there is an associated regular tessellation of the hyperbolic plane which is composed of transforms of a fundamental region for G. Using the hyperbolic metric, Dehn inferred that a non-trivial word w equal to 1 in G contained more than half of an element of R. Reidemeister (14) pointed out that Dehn's conclusion followed from the combinatorial properties of the tessellation, without metric considerations.

In 1949, V. A. Tartakovskii (21,22,23) initiated the algebraic study of small cancellation theory. Tartakovskii solved the word problem for finitely presented quotients of free products of cyclic groups by symmetrized R satisfying C(7). J. Britton (1), in 1957, independently investigated quotient groups of arbitrary free products by R satisfying C'(1/6). The triangle condition, Condition T(4), was introduced in 1956 by Schiek (16), who solved the word problem for R satisfying C'(1/4) and T(4). Greenlinger (5,6), in 1960, solved the conjugacy problem for C'(1/8), gave a new proof of the solvability of the word problem for C'(1/6), and obtained several other important results. Greenlinger (7) subsequently also investigated the C'(1/4) and T(4) hypothesis.

Very few group presentations have Cayley diagrams which are embeddable in the plane. However, it turns out that for any group G = <A; R>, if w is in N, (the normal closure of R in the free group F on A), there exists a finite planar diagram $\mathfrak{M}$, each edge of which is labelled by an element of F, and such that each region (face) D of $\mathfrak{M}$ has a label on its boundary an element of R, while the label on the boundary of the entire diagram is the reduced word w.

The existence of such a diagram $\mathfrak{M}$ was observed by Van Kampen (24) in 1933. Van Kampen's paper seems to have been totally ignored until Weinbaum (25), 1966, used the ideas to prove some of the results of Greendlinger. The above ideas were rediscovered independently by R. C. Lyndon (10) in his 1966 paper "On Dehn's Algorithm", which provided a unification, simplification, and generalization of many previous results.

Lyndon observed that the condition C(p) asserts that every interior region of the diagram $\mathfrak{M}$ borders on at least p other regions. Condition T(q) expresses the dual condition that each interior vertex of $\mathfrak{M}$ (excluding vertices of degree two) has at least q incident edges. Lyndon solved the word problem for finite R satisfying one of the hypotheses C(p) and T(q) where (q,p) is one of the pairs (3,6), (4,4), or (6,3). (Condition T(3) is vacuous.) These hypotheses correspond naturally to the three regular tessellations of Euclidean plane. For example, the hypotheses C(4) and T(4) correspond to the regular tessellation of the plane by squares. In the regular tessellation, all vertices and all regions have degree four. In the diagrams considered under the hypotheses

C(4) and T(4), all interior regions and interior vertices have degree greater than or equal to four.

CHAPTER II

FREE SUBGROUPS OF C(6)-GROUPS

D. J. Collins (2) has shown that, with a few exceptions, a small cancellation group whose presentation satisfies conditions C(4) and T(4) contains a free subgroup of rank two.

In this chapter we prove the analogous result for the class of finitely related C(6)-groups and determine free generators for the free subgroups.

We shall usually work with a finitely related C(6)-group $G = \langle A ; R \rangle$ where R is a symmetrized subset of the free group on A.

For each $r \in R$, the relator cycle of r is the set of all permutations of r and r^{-1}. Since R will usually be fixed in advance, we often refer simply to a piece rather than a piece relative to R. We say the generator $a \in A$ occurs in R if a occurs in some relator of R.

In section 1, we shall consider the situation in which there is a generator $a \in A$ that is not a piece. In section 2, we develop some further results on (3,6)-maps which we shall employ in section 3, when we turn to the case that all generators in A are pieces. In section 3, we shall see that for $w = 1$ in G, an "associated R-diagram for w" possesses a certain "nice" configuration, which we call a T-path. In section 4, we shall use this configuration to consider the situation in which every generator in A is a piece. Our method of proof will be by contradiction.

Section 1

Let $G = \langle A ; R \rangle$. In this section we consider the situation in which there is a generator $a \in A$ which is not a piece. This situation is in many ways untypical and we choose to dispose of it first.

Lyndon's Theorem (I.2.6) immediately gives rise to the following:

(1.1) Lemma If $w = 1$ in G, then there exists $r \in R$ such that $r \equiv yz$, where y is a subword of w and z is a product of at most three pieces (possibly $z \equiv 1$). Moreover, if y is a product of pieces, then it cannot be written as a product of less than three pieces.

At the outset, we note some $C(6)$-groups that do not contain a free subgroup of rank two. These are:

(1) $G_1 = \langle a ; \phi \rangle$,

(2) $G_2 = \langle a ; a^m, a^{-m} \rangle$, $m \neq 0$,

(3) $G_3 = \langle a,b ; a^2, a^{-2}, b^2, b^{-2} \rangle$.

In each case, the $C(6)$ condition is satisfied because no generator is a piece and hence no relator can be a product of pieces. We shall prove that these are essentially the only $C(6)$-groups that do not contain a free subgroup of rank two.

(1.2) Lemma Let $G = \langle A ; R \rangle$ and let a be a generator which is not a piece. Then either a does not occur in any relator in R or there is a unique relator cycle, in which a appears, which is defined by a relator $(az)^m$, $m \geq 1$ with z a-free.

Proof: If $a^m \in R$, there is nothing to prove. Now, assume that no relator in R has the form $a^{\pm m}$. If a^n, $n \geq 2$, is a subword of some relator in R, then by Lemma (I.3.1), a is a piece. This is a contradiction. Since a is not a piece, no relator cycle r in R is defined by $az_1a^{-1}z_2$.

If a is not a piece and occurs in the relator cycle defined by az where z is a-free, we may remove the generator a and the relator cycle of az, preserving both the abstract group defined, and the $C(6)$ property by this process. Henceforth, we assume that if a is not a piece then either a does not occur in any relator in R or else a occurs only in $(az)^m$, $m \geq 2$, with z a-free, (possibly $z \equiv 1$).

(1.3) Proposition Let $G = \langle A ; R \rangle$, with card $A \geq 2$, be a $C(6)$-group. Let a be a generator which does not occur in R and $b \neq a$ be another generator.

(i) If b does not occur in R, then a and b are free generators.

(ii) If b occurs in R, then a and bab^{-1} are free generators.

Proof: (i) We may either note that the free group on a and b is clearly a homomorphic image of G or appeal to Lemma (1.1).

(ii) Let w be freely reduced as a word in a and bab^{-1}. Cancelling some occurrences of b gives rise to a freely reduced word w^* in a and b. If $w = 1$ in G, then there exists r in R such that $r \equiv yz$, where y is a subword of w^* and z is a product of at most three pieces, by Lemma (1.1). Clearly, $y \neq a^{\pm m}$, for any m. Hence y must be either b or b^{-1}. By Lemma (1.1), b cannot be a piece. Hence there is a relator cycle $(bz')^m$, $m \geq 2$, where z' is b-free. Therefore either b or b^{-1} appears in z; it follows that b is a piece which is absurd.

(1.4) Proposition Let $G = \langle A ; R \rangle$ with Card $A \geq 2$, and let a be a generator that occurs in R but is not a piece. Let $b \neq a$ be another generator. Suppose that $G \neq \langle a,b ;\ a^2, a^{-2}, b^2, b^{-2} \rangle$.

(i) If $a^2 \in R$ and Card $A = 2$, then $aba^{-1}b$ and $b^{-1}abab^2$ are free generators.

(ii) If $a^2 \in R$ and A has three distinct members $a,b,c,$ then $abac$ and $baca^{-1}$ are free generators.

(iii) If $a^m \in R$ with $m \geq 3$, then $bab^{-1}a$ and $a^{-1}baba^2$ are free generators.

(iv) If no relator in R is of the form $a^{\pm m}$, and b is not a piece, then $(aba^{-1}b)^2 a$ and $b^{-1}(a^{-1}bab)^2 a^{-1}b$ are free generators.

(v) If no relator in R is of the form $a^{\pm m}$ and b is a piece, then $(aba^{-1}b)^2 a$ and $b(a^{-1}bab)^2 a^{-1}b^q$ are free generators, where q is the largest integer such that b^q is a subword of some relator r in R, $r \neq b^{\pm n}$.

Proof: (i) Suppose that some relation in $aba^{-1}b$ and $b^{-1}abab^2$ is valid in G. When cancelled to reduced form as a word in a and b this relation gives a cyclically reduced word w with the property that it contains $a^{\pm 1}$ and $b^{\pm 2}$ but no higher powers. We have $a^2 \in R$ and a is not a piece. So a appears in no other relator cycle. Thus every other element of R is of the form b^m, $m \neq 0$. Since C(6) condition is satisfied there is at most one such relator cycle $b^{\pm m}$, $m \geq 3$, and b is not a piece, i.e. $G = \langle a,b \,;\, a^2, a^{-2}, b^m, b^{-m} \rangle$. Either the normal form for free products or Lemma (1.1) gives a contradiction.

(ii) Let w be freely reduced as a word in $aba^{-1}c$ and $b^{-1}aca^{\pm 1}$. Then w can be reduced to a freely reduced word w* in a,b and c. If w = 1 in G then w* = 1 in G and hence there exists r in R such that $r \equiv yz$, where y is a subword of w* and z is a product of at most three pieces, by Lemma (1.1). By our choice of generators, it follows that $y \equiv a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, (cb)^{\pm 1}$,

Since $a^2 \in R$ and a is not a piece, $y \not\equiv a^{\pm 1}$. If $y \equiv b^{\pm 1}$, then by Lemma (1.1), b cannot be a piece. Hence there exists a relator cycle $(b^{\varepsilon}z')^m$, where $m \geq 2$, z' is b -free, and $\varepsilon = \pm 1$. It follows that b is a piece, contradiction. Similarly, $y \not\equiv c^{\pm 1}$. If $y \equiv (cb)^{\pm 1}$, then again b and c are pieces; and this is a contradiction since $(cb)^{\pm 1}$ is then a product of two pieces and y is a product of at least three pieces. Hence $y \not\equiv (cb)^{\pm 1}$. Therefore $w \neq 1$ in G.

(iii) The argument is similar to that for (i) and (ii). Here the possibilities for y are $b^{\pm 1}$, $a^{\pm 1}$ and $a^{\pm 2}$.

(iv) Let w be freely reduced as a word in $(aba^{-1}b)^2 a$ and $b^{-1}(a^{-1}bab)^2 a^{-1}b$. Then w can be reduced to a freely reduced word w* in a and b. If w = 1 in G (and hence w* = 1 in G), then there exists r in R such that $r \equiv yz$, where y is a subword of w* and z is a product of at most three pieces. Here the possibilities for y are $a^{\pm 1}$ and $b^{\pm 1}$. Since a and b are not pieces, $y \not\equiv a^{\pm 1}$ and $y \not\equiv b^{\pm 1}$. Therefore $w \neq 1$ in G.

(v) This again is similar. The possibilities for y are $a^{\pm 1}$ and $b^{\pm k}$, $1 \leq k \leq q+1$. The situation when $y \equiv b^{\pm k}$ is ruled out by using Lemma (I.3.1).

Section 2

Here we develop some further results on (3,6) maps which we shall employ in section 3, when we turn to the case that all generators in A are pieces.

Throughout this section, let $\mathcal{M}$ be a connected, simply connected map with the set $\mathcal{F}$ of regions, the set $\mathcal{E}$ of (unoriented) edges and the set $\mathcal{V}$ of vertices satisfying the following conditions:

(M1): For each $v \in \mathcal{V}$, $d(v) \geq 3$.

(M2): If $D \in \mathcal{F}$ is a semi-interior region, then $i(D) \geq 6$.

(M3): For each D in $\mathcal{M}$, $i(D) \geq 2$.

(M4): $\mathcal{F}^{\#} > 1$.

Then we have the following:

(2.1) Proposition (i) $\mathcal{M}$ is a (3,6)-map.

(ii) For each region D in $\mathcal{M}$, $\delta(D)$ is a simple closed path.

(iii) $\Sigma^{*} [4 - i(D)] \geq 6$, where D runs through the boundary regions in $\mathcal{M}$.

Proof: (i) This follows directly from conditions (M1) and (M2). (ii) and (iii) follow from Lemma (I.1.5) and Theorem (I.1.4) respectively.

We shall call an interior edge E, where either $\lambda(E)$ or $\mu(E)$ is a boundary vertex v_o and the other is interior, a bv-edge at v_o.

Let D_1 and D_2 be any two distinct regions in $\mathfrak{M}$ and suppose v is a boundary vertex such that $v \in \beta(D_1) \cap \beta(D_2)$. Then D_1 and D_2 are said to be adjacent if $\beta(D_1) \cap \beta(D_2)$ contains exactly one bv-edge at v. Let $D_1, \ldots, D_m$, $m \geq 2$, be a sequence of bc-regions in $\mathfrak{M}$. We say $D_1, D_2, \ldots, D_m$ is an adjacent sequence if for each $1 \leq j \leq m-1$, D_j and D_{j+1} are adjacent.

(2.2) Lemma There exists an adjacent sequence $D_1, \ldots, D_n$, $n > 1$, in $\mathfrak{M}$ such that $\sum_{j=1}^{n} [4 - i(D_j)] \geq 4$. Moreover, there exist r and t, where $0 \leq r < t \leq n$, such that $D_{r+1}, \ldots, D_t$ is an adjacent sequence with the following properties:

(i) $$\sum_{j=r+1}^{t} [4 - i(D_j)] \geq 4,$$

and

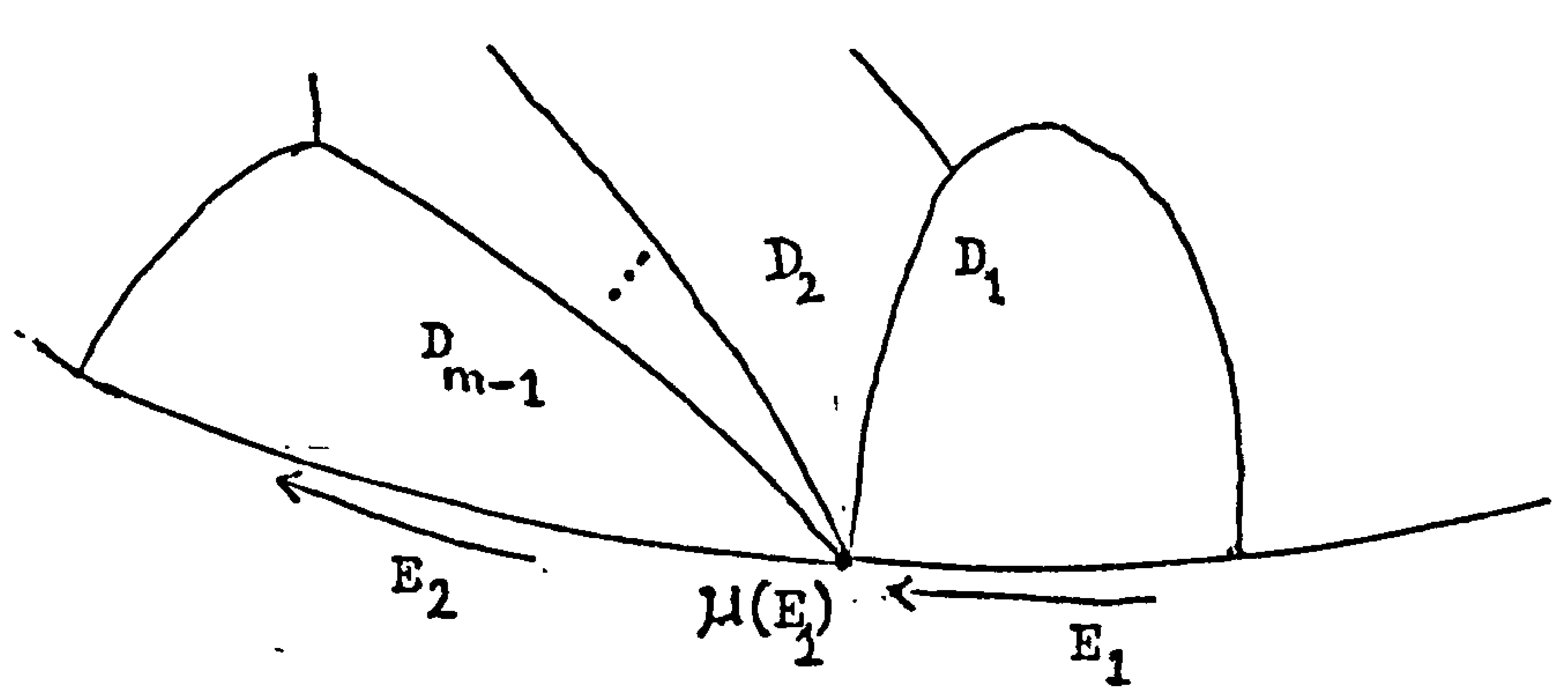
(ii) for each $r + 1 \leq k < t$,

$$0 < \sum_{j=r+1}^{k} [4 - i(D_j)] < 4.$$

Proof: For each region D in $\mathfrak{M}$, $\delta(D)$ is a simple path by Proposition (2.1). Then there exist at least two sbc-regions by Lemma (I.1.2). We assume first that $\delta(\mathfrak{M})$ is a simple closed path.

If every region in $\mathfrak{M}$ is a bc-region in $\mathfrak{M}$, then we have $\Sigma^{\bullet}$ $[4 - i(D)] \geq 6$, where D runs through the boundary regions in $\mathfrak{M}$ by Theorem (I.1.4); and hence there exists an adjacent sequence $D_1, \ldots, D_n$ with $\sum_{j=1}^{n} [4 - i(D_j)] \geq 6$, because the set of all boundary regions can then be formed into an adjacent sequence as follows:

Pick E_1 in $\delta(\mathfrak{M})$ and put $D_1 = \sigma(E_1)$. Now let $d(\mu(E_1)) = m$. Since every boundary region is a bc-region whose boundary cycle is simple closed, there exist $(m-1)$ boundary regions with $\mu(E_1)$ as a vertex and they can be formed into the adjacent sequence $D_1, D_2, \ldots, D_{m-1}$. See, for example:



The operation can be repeated in the successive edges $E_1, \ldots, E_{m-1}$ that occur in $\delta(\mathfrak{M})$. By condition (M3), $n > 1$.

Now, suppose that there exists a bdc-region D' in $\mathfrak{M}$.

choose two vertices v_1, $v_2 \in (D') \cap \beta(\mathcal{M})$ with the property that if $(E_\ell,\dots,E_k)$ is the subpath of $\delta(\mathcal{M})$ from v_1 to v_2, (i.e. $\lambda(E_\ell) = v_1$ and $\mu(E_k) = v_2$), then

(i) $E_\ell \notin (D')$, and

(ii) $\lambda(E_j) \notin \beta(D'), \ell + 1 \leq j \leq k$.

Now, consider the path $(E'_{\ell'},\dots,E'_{k'})$ in $\delta(D')$ from v_2 to v_1. Let $\gamma = (E_\ell,\dots,E_k,E'_{\ell'},\dots,E'_{k'})$. Then γ is a simple closed path. Let $\mathcal{C}$ be the submap of $\mathcal{M}$ consisting of γ all vertices, edges and regions interior to γ. We call $\mathcal{C}$ a " Corresponding component " for the bdc-region D'.

Among all bdc-regions in $\mathcal{M}$ there exists one D', say, such that one of the corresponding submap $\mathcal{C}$ has a minimal number of regions. Put $\mathcal{L} = \mathcal{C} \cup D'$. Then every boundary region D in $\mathcal{L}$ is a bc-region in $\mathcal{L}$, by the minimality. By Theorem (I.1.6), $\Sigma^\cdot\, [4 - i_{\mathcal{L}}(D)] \geq 6$, where D runs through the regions of $\mathcal{L}$. We claim that $i_{\mathcal{L}}(D') \geq 2$. By our construction of $\mathcal{L}$, $i_{\mathcal{L}}(D') \neq 0$. Suppose that $i_{\mathcal{L}}(D') = 1$. Let E be the unique interior edge in $\mathcal{L}$ such that E is in $\delta(D')$. Then $\lambda(E)$ and $\mu(E)$ are boundary vertices in $\mathcal{L}$. Since E is an interior edge and $\delta(D')$ is simple closed path, there exists a region D", say, in $\mathcal{L}$ such that E^{-1} is in $\delta(D'')$. We claim that D" is a bdc-region of $\mathcal{L}$. Suppose not; then there must exist a boundary edge E' of $\mathcal{L}$ such that $\delta(D'') = (E',E^{-1})$ and so $i_{\mathcal{L}}(D'') = 1$. But if D is a region of $\mathcal{L}$ with $D \neq D'$ then $i_{\mathcal{L}}(D) = i_{\mathcal{M}}(D) \geq 2$ which is contradictory. This gives $i_{\mathcal{L}}(D') \geq 2$. It follows that

$$\sum{}^\cdot\, [4 - i_{\mathcal{L}}(D)] = \sum{}^\cdot\, [4 - i_{\mathcal{M}}(D)] \geq 4\,,$$

where D runs through all boundary regions of $\mathcal{L}$ such that $D \neq D'$. Call the boundary regions D in $\mathcal{L}$, with $D \neq D'$, $D_1, \dots, D_n$.

Then we have an adjacent sequence $D_1, \ldots, D_n$ by the same method as we used when every boundary region in $\mathfrak{M}$ is a bc-region in $\mathfrak{M}$. So $\sum_{j=1}^{n} [4 - i(D_j)] \geq 4$; and (M3) ensures that $n > 1$.

If for each $1 \leq \ell \leq n$, $\sum_{j=1}^{\ell} [4 - i(D_j)] > 0$, let $r = 0$, and $t = s+1$, where s is the largest integer $1 \leq s < n$ such that $\sum_{j=1}^{s} [4 - i(D_j)] < 4$. Note that such s exists since $[4 - i(D_j)] \leq 2$, by condition (M3). Suppose that there exists $1 \leq \ell < n$ such that $\sum_{j=1}^{\ell} [4 - i(D_j)] \leq 0$. Let r be the largest integer s, $1 \leq s < n$ such that $\sum_{j=1}^{s} [4 - i(D_j)] \leq 0$. Then for each $r < k \leq n$, $\sum_{j=r+1}^{k} [4 - i(D_j)] > 0$. Let $t = s+1$, where s is the largest integer $r+1 \leq s < n$ such that $\sum_{j=r+1}^{s} [4 - i(D_j)] < 4$, (such s exists since $[4 - i(D_{r+1})] \leq 2)$. Note that $D_{r+1}, \ldots, D_t$ is still an adjacent sequence.

Now, if $\delta(\mathfrak{M})$ is not a simple closed path, then $\mathfrak{M}$ has at least two extremal discs and none of them can be a single region, by conditions (M1), (M3) and Lemma (I.1.1). Hence, we can apply the same argument as before to one of these discs and get the desired result.

(2.3) <u>Corollary</u> Let $D_{r+1}, \ldots, D_t$ be the adjacent sequence in Lemma (2.2). Then

(i) D_{r+1} and D_t are sbc-regions such that $i(D_k) = 2$ or 3, where $k = r+1$ or t.

(ii) $\Sigma\,[4 - i(D_j)] \geq 4$, where D_j runs through the sbc-regions in the sequence $D_{r+1}, \ldots, D_t$.

Proof. (i) By our choice of r and t.

(ii) If D_j is a wbc-region, $r+1 < j < t$, then $[4 - i(D_j)] \leq -2$, by condition (M2). It follows that the only regions in the adjacent sequence $D_{r+1}, \ldots, D_t$ which contribute positively are the sbc-regions.

(2.4) Remark In the adjacent sequence $D_{r+1}, \ldots, D_t$ of Lemma (2.2), let D_ℓ, $r+1 < \ell < t$, be a wbc-region at v. Then there exist exactly $[d(v) - 3]$ weakly boundary connected regions at v, since $\delta(D)$ is simple closed for each region D in $\mathfrak{M}$.

Let $D_{r+1} = D_1^*$. By induction, for each $j \geq 2$, let $k(j)$ be the least integer, $k(j-1) < k(j) \leq t-r$, such that $D_{r+k(j)}$ is an sbc-region and put $D_j^* = D_{r+k(j)}$. Let L be the number of all sbc-regions D_j, $r+1 \leq j \leq t$. Then $L > 1$, by condition (M3), and $\sum_{j=1}^{L} [4 - i(D_j^*)] \geq \sum_{j=r+1}^{t} [4 - i(D_j)] \geq 4$. And as a consequence of Corollary (2.3), $i(D_k^*) = 2$ or 3, for $k = 1$ or L.

For each $1 \leq j \leq L$, let $\delta(D_j^*) = (X_{j,1}, X_{j,2}, \ldots, X_{j,n(j)})$. W.l.o.g. we assume that $X_{j,1}$ is the unique boundary edge and denote it by Y_j.

(2.5) Definition (1) Let $1 \leq j \leq L-1$. Let $m(j) = [d(\mu(Y_j) - 3]$. We call $[4 - i(\rho(Y_j)) - 2\,m(j)]$ the weight of Y_j and denote it by $h(Y_j)$. (Clearly $m_j = 0$ or 1.)

(2) Let $h(Y_L) = 4 - i(\rho(Y_L))$.

Now, Lemma (2.2) can be reformulated as follows:

(2.6) Lemma There exists a simple path $Y_1, \ldots, Y_L$, $L \geq 2$, in $\beta(\mathfrak{M})$ such that

(i) $\rho(Y_j)$ is a region in $\mathfrak{M}$,

(ii) $h(Y_1) = 1$ or 2,

(iii) $\sum_{j=1}^{L} h(Y_j) \geq 4$, and

(iv) for each $1 \leq k < L$, $0 < \sum_{j=1}^{k} h(Y_j) < 4$.

Proof: Let $D_1^*, \ldots, D_L^*$ be the sequence of sbc-regions in the adjacent sequence $D_{r+1}, \ldots, D_t$ and let Y_j be the unique boundary edge of D_j^*. Let $D_j^* = D_k$, $D_{k+1}, \ldots, D_{k+p} = D_{j+1}^*$ be the adjacent sequence of regions with common vertex $\mu(Y_j)$. Then

$h(Y_j) \geq \sum_{\ell=k}^{k+p-1} [4 - i(D_\ell)]$. By Definition (2.5),

$h(Y_L) = 4 - i(D_L^*)$.

The condition (M3) directly gives rise to the following proposition.

(2.7) Proposition For each $1 \leq j \leq L$, $h(Y_j) \leq 2$.

Section 3

Let $F(A)$ be the free group on a set A. Let R be a symmetrized subset of $F(A)$. Let $G = F(A)/<F(A) <R>$, where $<F(A) <R>$ is the normal closure of R in $F(A)$. Let w be a freely reduced word in $F(A)$ such that $w \not\equiv 1$. Assume that w is a product of pieces relative to R. We define $\theta_R(w)$ to be the least number of pieces $P_1, \ldots, P_{\theta_R(w)}$ relative to R such that $w \equiv P_1 \ldots P_{\theta_R(w)}$. Let $r \in R$. Assume that r is a product of pieces relative to R. Define $\theta_R^*(r) = \min\{\theta_R(r^*) \mid r^*$ is a cyclic permutation of r or $r^{-1}\}$.

Throughout this section we assume that R is finite and satisfies the condition $C(6)$. Further we assume that every generator in A is a piece relative to R. Since we only work with one symmetrized set at a time we shall omit the phrase "relative to R" and put $\theta_R(w) = \theta(w)$ and $\theta_R^*(r) = \theta^*(r)$.

The following Lemma is a consequence of the Lyndon-Van Kampen Theorem (I.2.1).

(3.1) <u>Lemma</u> Let w be a non-trivial freely reduced word in $F(A)$. If $w = 1$ in G, then there exists a connected, simply connected R-diagram $\mathfrak{M}$ over $F(A)$ such that

(i) for each edge E in $\mathfrak{M}$, $\phi(E) \not\equiv 1$,

(ii) $\phi(\delta(\mathfrak{M})) \equiv w^{-1}$, and

(iii) for each region D in $\mathfrak{M}$, $\phi(\delta(D)) \in R$.

We shall call an R-diagram which satisfies Lemma (3.1), an <u>associated R-diagram</u> for w.

(3.2) <u>Lemma</u> Let w be a non-trivial freely reduced word in $F(A)$. Suppose that $w = 1$ in G. Let $\mathfrak{M}$ be an associated R-diagram for w.

(i) If E is an interior edge then $\theta(\phi(E)) = 1$.

(ii) For each region D in $\mathfrak{M}$, $\theta^*(\phi(\delta(D))) \geq 6$.

Proof: (i) This follows from Lemma (I.2.4) and the definition of $\theta(w)$.

(ii) Since R satisfies $C(6)$ and every element in A is a piece, $\phi(\delta(D))$ is a product of pieces. Hence $\theta^*(\phi(\delta(D)) \geq 6$.

(3.3) Lemma Let w be a non-trivial cyclically reduced word in $F(A)$ such that $w = 1$ in G. Let $\mathfrak{M}$ be an associated R-diagram for w. Assume that for each boundary edge E in $\beta(\mathfrak{M})$, $\theta(\phi(E)) \leq 4$. Then $\mathfrak{M}$ satisfies conditions (M1) - (M4) of section 2.

Proof: Since w is a cyclically reduced word in $F(A)$, there is no vertex of degree 1 in $\mathfrak{M}$; and we may assume that $\mathfrak{M}$ has no vertices of degree 2 (if $d(v) = 2$ and $v = \mu(E) = \lambda(E')$ replace E, E' by a single edge E'' with $\phi(E'') \equiv \phi(E)\phi(E')$). $\mathfrak{M}$ under these procedures is a connected, simply connected R-diagram over $F(A)$ which satisfies Lemma (3.1). Hence condition (M1) is valid. Condition (M2) follows from Lemma (I.1.5); and conditions (M3) and (M4) follow from the assumption that for each boundary edge E, $\theta(\phi(E)) \leq 4$ and Lemma (3.2).

Now, from Lemmas (3.3) and (2.6), we have the following:

(3.4) Lemma Let w be a non-trivial cyclically reduced word in $F(A)$ such that $w = 1$ in G. Let $\mathfrak{M}$ be an associated R-diagram for w. Suppose that for each edge E in $\beta(\mathfrak{M})$, $\theta(\phi(E)) \leq 4$. Then there exists a simple path $Y_1, \ldots, Y_L$, $L \geq 2$ in $\beta(\mathfrak{M})$ labelled by a subword of w^{-1}, such that

(i) for each $1 \leq j \leq L$, $\rho(Y_j)$ is a region in $\mathfrak{M}$,

(ii) $h(Y_1) = 1$ or 2,

(iii) $\sum_{j=1}^{L} h(Y_j) \geq 4$, and

(iv) for each $1 \leq k < L$, $0 < \sum_{j=1}^{k} h(Y_j) < 4$.

We shall call a simple path, in a map $\mathfrak{M}$, satisfying (i) - (iv) of Lemma (3.4) a T-path.

We shall, in various cases, consider a C(6) group $G = \langle A ; R \rangle$ in which every generator is a piece. Depending on the case, we shall select a pair of words w_1, w_2 in the generators in A whose images in G are to be shown to be free generators. Our method of proof will be by contradiction. We shall suppose that some relation in w_1 and w_2 is valid in G. This relation can also be viewed as a relation in the generators of F(A) and thus gives rise to some cyclically reduced word w of F(A) with an associated R-diagram $\mathfrak{M}$. So we shall always be able to apply Lemma (3.4).

(3.5) Convention In the sequence of results which follow we shall assume that we have a fixed R-diagram $\mathfrak{M}$ with $\phi(\delta(\mathfrak{M}))$ cyclically reduced and a T-path $Y_1, \ldots, Y_L$ in $\mathfrak{M}$. We also use the following associated notation. For each $1 \leq j \leq L$,

(1) $D_j = \rho(Y_j)$.

(2) $\delta(D_j) = (X_{j,1}, \ldots, X_{j,n(j)})$, where $X_{j,1} = Y_j$. Note that $n(j) = d(D_j)$; and if $d(\mu(Y_j)) = 3$, then $X_{j,2} = X^{-1}_{j+1,n(j+1)}$.

(3) $\phi(Y_j) \equiv y_j$ and $\phi(X_{j,k}) \equiv x_{j,k}$, $2 \leq k \leq n(j)$. Hence $\theta(y_j) \leq 4$ and $\theta(x_{j,k}) = 1$.

(4) $r_j \equiv \phi(\delta(D_j)) \equiv y_j\, x_{j,2} \cdots x_{j,n(j)}$. If $d(\mu(Y_j)) = 3$, then we may write $r_j \equiv y_j\, x^{-1}_{j+1,n(j+1)}\, x_{j,3} \cdots x_{j,n(j)}$.

As a consequence of the above notation, Lemma (3.2) gives rise to the following Corollary:

(3.6) Corollary For each $1 \le j \le L$

(i) $\theta^*(r_j) \ge 6$

(ii) $\theta^*(r_j) \le \theta(y_j) + n(j) - 1$.

(3.7) Lemma For each $1 \le j \le L$. $h(Y_j) \le \theta(y_j) - 2$.

Proof: $\theta(y_j) + i(D_j) \ge 6$, since R satisfies condition $C(6)$. Then $h(Y_j) \le 4 - i(D_j) \le 4 - 6 + \theta(y_j)$.

(3.8) Lemma $\theta(y_1) = 3$ or 4 .

Proof: By Lemma (3.4), $h(Y_1) = 1$ or 2 ; and hence $d(\mu(Y_1)) = 3$. Now, if $\theta(y_1) \le 2$, then by Lemma (3.7), $h(Y_1) \le 0$, absurd.

Let $a \in A$. Recall that each element in A is a piece. Let p be the maximal integer such that a^p appears in some relator r in R , where r is not a proper power of a . If a^p is not a piece then there exists a unique r in R which involves a^p, and r has the form $(a^p u)^m$, $m \ge 1$, where u has no occurrences of $a^{\pm p}$.

Lipschutz's Lemma (I.2.7) gives rise to the following:

(3.9) Corollary $a^{\pm p}$ is not a piece if and only if

(i) $\theta(a^{\pm p}) = 2$, and

(ii) $\theta(a^{\pm\ell}) = 1$, $\ell < p$.

(3.10) <u>Lemma</u> Assume that $\theta(a^{\varepsilon p}) = 2$, $\varepsilon = \pm 1$. Let $y_j \equiv a^{\varepsilon\ell}$, $\ell < p$ and $j \geq 2$. Let $y_{j+1} \equiv a^{\varepsilon p}$. Suppose that $x_{j,n(j)}$ does not end in a^{ε}. Further assume that one of the following is true.

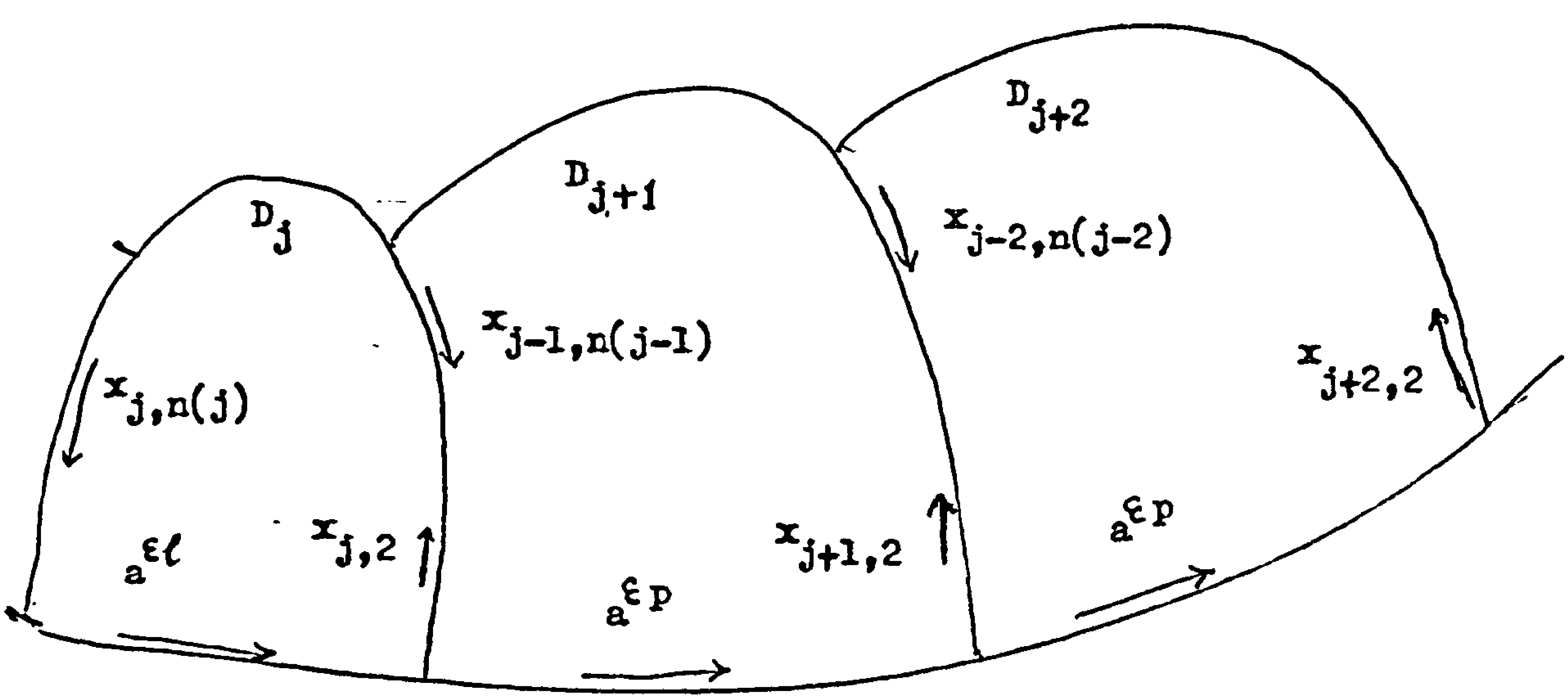
(1) $h(Y_{j+1}) = 0$, $y_{j+2} \equiv a^{\varepsilon p}$, and $d(\mu(Y_j)) = d(\mu(Y_{j+1})) = 3$.

(2) $y_s \equiv y_{s+1} \equiv a^{\varepsilon p}$, and $h(Y_s) = 0$, where either $j + 3 \leq s \leq L - 1$ or $2 \leq s \leq j - 2$.

(3) $y_s \equiv y_{s+1} \equiv a^{-\varepsilon p}$, $d(\mu(Y_s)) = 3$ and $h(Y_{s+1}) = 0$, where $j + 3 \leq s \leq L - 1$ or $2 \leq s \leq j - 3$.

Then $h(Y_j) \leq -2$.

<u>Proof:</u> Assume (1) is valid. By Corollary (3.9), $\theta(y_j) = 1$. Hence $h(Y_j) \leq -1$, by Lemma (3.7).



If $h(Y_j) = -1$, then $r_j \equiv a^{\varepsilon\ell} x_{j,2} x_{j,3} x_{j,4} x_{j,5} x_{j,6}$. Now $r_{j+1} \equiv a^{\varepsilon p} x_{j+1,2} x_{j+1,3} x_{j+1,4} x_{j,2}^{-1}$ since $h(Y_{j+1}) = 0$; and

$r_{j+2} \equiv a^{\varepsilon p}\, x_{j+2,2} \cdots x_{(j+2),n(j+2)-1}\, x_{j+1,2}^{-1}$. We shall compare r_{j+1} and r_{j+2}, noting that $a^{\varepsilon p}$ is not a piece. Either $|x_{j,2}| \geq |x_{j+1,2}|$ or $|x_{j+1,2}| \geq |x_{j,2}|$.

If $|x_{j,2}| \geq |x_{j+1,2}|$, then $r_j \equiv a^{\varepsilon \ell}\, x_{j+1,2}\, z\, x_{j,3}\, x_{j,4}\, x_{j,5}\, x_{j,6}$, where $x_{j,2} \equiv x_{j+1,2}\, z$. Since $x_{j,6}$ does not end in a^{ε}, comparing r_j and r_{j+1} shows that $a^{\varepsilon \ell}\, x_{j+1,2}$ must be a piece. Hence $\theta^*(r_{j+1}) \leq 5$, contradicting Corollary (3.6). Therefore $|x_{j+1,2}| > |x_{j,2}|$. Then $r_{j+1} \equiv a^{\varepsilon p}\, x_{j,2}\, z'\, x_{j+1,3}\, x_{j+1,4}\, x_{j,2}^{-1}$, where $x_{j,2}\, z' \equiv x_{j+1,2}$. Since $x_{j,6}$ does not end in a^{ε}, comparing r_j and r_{j+1} shows that $a^{\varepsilon \ell}\, x_{j,2}$ is a piece. Hence $\theta^*(r_j) \leq 5$, contradicting Corollary (3.6). Then $h(Y_j) \leq -2$.

Assume (2) is valid. By Corollary (3.9) and Lemma (3.7), $h(Y_j) \leq -1$. If $h(Y_j) = -1$, then $r_j \equiv a^{\varepsilon \ell}\, x_{j,2}\, x_{j,3}\, x_{j,4}\, x_{j,5}\, x_{j,6}$. Since $h(Y_s) = 0$, $r_s \equiv a^{\varepsilon p}\, x_{s,2}\, x_{s,3}\, x_{s,4}\, x_{s,5}$. Further, we can write

$r_{j+1} \equiv a^{\varepsilon p}\, x_{j+1,2} \cdots x_{j+1,n(j+1)-1}\, x_{j,2}^{-1}$ and

$r_{s+1} \equiv a^{\varepsilon p}\, x_{s+1,2} \cdots x_{s+1,n(s+1)-1}\, x_{s,2}^{-1}$. We shall compare r_{j+1} and r_{s+1}, noting that $a^{\varepsilon p}$ is not a piece. Now, either $|x_{j,2}| \geq |x_{s,2}|$ or $|x_{s,2}| \geq |x_{j,2}|$. If $|x_{j,2}| \geq |x_{s,2}|$, then $r_j \equiv a^{\varepsilon \ell}\, x_{s,2}\, z\, x_{j,3}\, x_{j,4}\, x_{j,5}\, x_{j,6}$, where $x_{j,2} \equiv x_{s,2}\, z$. Since $x_{j,6}$ does not end in a^{ε}, comparing r_j and r_s shows that $a^{\varepsilon \ell}\, x_{s,2}$ must be a piece. Hence $\theta^*(r_s) \leq 5$, contradicting Corollary (3.6). Therefore $|x_{s,2}| > |x_{j,2}|$. Then $r_s \equiv a^{\varepsilon p}\, x_{j,2}\, z'\, x_{s,3}\, x_{s,4}\, x_{s,5}^{-1}$, where $x_{j,2}\, z' \equiv x_{s,2}$. Since $x_{j,6}$ does not end in a^{ε}, comparing r_j and r_s shows that $a^{\varepsilon \ell}\, x_{j,2}$ is a piece. Hence $\theta^*(r_j) \leq 5$, contradicting Corollary

(3.6). Then $h(Y_j) \le -2$.

Assume (3) is valid. By a similar argument, $h(Y_j) \le -2$.

(3.11) <u>Lemma</u> Let $\theta(a^{\varepsilon p}) = 2$, $\varepsilon = \pm 1$. Let $y_j \equiv a^{\varepsilon \ell}$, $\ell < p$ and $j \ge 3$. Let $y_{j-1} \equiv a^{\varepsilon p}$. Assume that $x_{j,2}$ does not begin with a^{ε} . Further, assume that one of the following is true.

(1) $y_{j-2} \equiv a^{\varepsilon p}$, and $h(Y_{j-1}) = 0$ and $d(\mu(Y_{j-2})) = 3$.

(2) $y_s \equiv y_{s+1} \equiv a^{\varepsilon p}$, $d(\mu(Y_s)) = d(\mu(Y_{j-1})) = 3$ and $h(Y_{s+1}) = 0$, where $1 \le s \le j-3$ or $j+1 \le s \le L-1$.

(3) $y_s \equiv y_{s+1} \equiv a^{-\varepsilon p}$, $d(\mu(Y_{j-1})) = 3$ and $h(Y_s) = 0$, where $j+2 \le s \le L-1$ or $2 \le s \le j-4$.

Then $h(Y_j) \le -2$.

<u>Proof:</u> By an argument similar to that in Lemma (3,10).

(3.12) <u>Lemma</u> Let $\theta(a^{\varepsilon p}) = 2$. Let $d(\mu(Y_j)) = 3$ and $y_j \equiv u\, a^{\varepsilon \ell}$, $\ell < p$, $j \ge 1$. Let $y_{j+1} \equiv a^{\varepsilon p}$. Let $\theta(u) = m$, where $1 \le m \le 3$. Assume that u does not end in a^{ε} . Further assume that one of the following is true:

(1) $y_{j+2} \equiv a^{\varepsilon p}$ and $h(Y_{j+1}) = 0$.

(2) $y_s \equiv y_{s+1} \equiv a^{\varepsilon p}$, $d(\mu(Y_s)) = 3$ and $h(Y_s) = 0$, where $2 \le s \le j-2$ or $j+2 \le s \le L-1$.

(3) $y_s \equiv y_{s+1} \equiv a^{-\varepsilon p}$ and $h(Y_{s+1}) = 0$, where either $2 \le s \le j-3$ or $j+2 \le s \le L-1$.

Then $h(Y_j) \leq m-2$.

<u>Proof</u>: Assume (1) is valid. By Corollary (3.9), $\theta(a^{\varepsilon\ell}) = 1$. Hence $\theta(u\ a^{\varepsilon\ell}) \leq m+1$. By Lemma (3.7), $h(Y_j) \leq m-1$. Let $h(Y_j) = m-1$. We can write $r_j \equiv u\ a^{\varepsilon\ell}\ x_{j,2} \cdots x_{j,n(j)}$. We shall consider the case when $m = 3$, i.e. $h(Y_j) = 2$. So, we write $r_j \equiv u\ a^{\varepsilon\ell}\ x_{j,2}\ x_{j,3}$. Since $h(Y_{j+1}) = 0$ and $d(\mu(Y_j)) = 3$, $r_{j+1} \equiv a^{\varepsilon p}\ x_{j+1,2}\ x_{j+1,3}\ x_{j+1,4}\ x_{j,2}^{-1}$. We write $r_{j+2} \equiv a^{\varepsilon p}\ x_{j+2,2} \cdots x_{j+2,n(j+2)-1}\ x_{j+1,2}^{-1}$. We shall compare r_{j+1} and r_{j+2} , noting that $a^{\varepsilon p}$ is not a piece. Now either $|x_{j,2}| \geq |x_{j+1,2}|$ or $|x_{j+1,2}| \geq |x_{j,2}|$. If $|x_{j,2}| \geq |x_{j+1,2}|$, then $r_j \equiv u\ a^{\varepsilon\ell}\ x_{j+1,2}\ z\ x_{j,3}$ where $x_{j,2} \equiv x_{j+1,2}\ z$. Since u does not end in a^{ε} , comparing r_j and r_{j+1} shows that $a^{\varepsilon\ell}\ x_{j+1,2}$ must be a piece. Hence $\theta^*(r_{j+1}) \leq 5$, contradicting Corollary (3.6). Therefore $|x_{j+1,2}| > |x_{j,2}|$. Then we can write $r_{j+1} \equiv a^{\varepsilon p}\ x_{j,2}\ z'\ x_{j+1,3}\ x_{j+1,4}\ x_{j,2}^{-1}$, where $x_{j,2}\ z' \equiv x_{j+1,2}$. Since u does not end in a^{ε} , comparing r_j and r_{j+1} shows that $a^{\varepsilon\ell}\ x_{j,2}$ is a piece. Hence $\theta^*(r_j) \leq 5$, contradicting Corollary (3.6). Then $h(Y_j) = 1$. Similarly, when $m = 1$ or 2 , $h(Y_j) \leq m-2$. If we assume either (2) or (3), then by an argument similar to the above (see also the proof of Lemma (3.10)), we get the desired result.

(3.13) <u>Lemma</u> Let $\theta(a^{\varepsilon p}) = 2$. Let $y_j \equiv a^{\varepsilon\ell}\ u$, $\ell < p$, $j \geq 3$, $\varepsilon = \pm 1$. Let $y_{j-1} \equiv a^{\varepsilon p}$. Let $\theta(u) = m$, where $1 \leq m \leq 3$. Suppose that u does not begin with a^{ε} . Further assume that one of the following is true:

(1) $y_{j-1} \equiv a^{\varepsilon p}$, $d(\mu(Y_{j-2}))=3$ and $h(Y_{j-1}) = 0$.

(2) $y_s \equiv y_{s+1} \equiv a^{\varepsilon p}$, $d(\mu(Y_s))=3$ and $h(Y_{s+1}) = 0$, where either

$2 \le s \le j-2$ or $j+1 \le s \le L-1$.

(3) $y_s \equiv y_{s+1} \equiv a^{-\varepsilon p}$, and $h(Y_s) = 0$, where either

$2 \le s \le j-3$ or $j+1 \le s \le L-1$.

Then $h(Y_j) \le m-2$.

Proof: By an argument similar to that in Lemma (3.11).

(3.14) Lemma Let $\theta(a^{\varepsilon p}) = 2$. Let $y_j \equiv a^{\varepsilon \ell} u a^{\eta k}$, where $\ell, k < p$, $\varepsilon = \pm 1$, $\eta = \pm 1$, $j \ge 3$.

Let $y_{j+1} \equiv a^{\eta p}$ and $y_{j-1} \equiv a^{\varepsilon p}$. Let $\theta(u) = m$, where $1 \le m \le 2$.

Suppose that u does not begin with a^{ε} and does not end in a^{ε}.

Further assume that one of the following is true:

(1) $y_{j+2} \equiv a^{\eta p}$, $y_{j-2} \equiv a^{\varepsilon p}$, $d(\mu(Y_j))=3$, $d(\mu(Y_{j-2}))=3$ and $h(Y_{j+1}) = h(Y_{j-1}) = 0$.

(2) $y_s \equiv y_{s+1} \equiv a^{\nu p}$, $\nu = \pm 1$, $d(\mu(Y_j))=3$, $d(\mu(Y_{j-1}))=3$ and $h(Y_s) = h(Y_{s+1}) = 0$,

where either $2 \le s \le j-2$ or $j+2 \le s \le L-1$.

Then $h(Y_j) \le m-2$.

Proof: By an argument similar to that in Lemma (3.12).

Section 4

Let $G = \langle A ; R\rangle$ be a C(6)-group in which every generator is a piece. As explained prior to Convention (3.5), our approach in all cases leads us to consider an R-diagram $\mathfrak{M}$, with T-path $Y_1, Y_2, \ldots, Y_L$ such that $\phi(\delta(\mathfrak{M}))$ is the cyclically reduced word derived from a supposed relation in a given pair of generators.

Let a and b be any two distinct generators in A. We have the following possibilities:

(1) There exist positive integers m and n such that $a^m \in R$ and $b^n \in R$. (Then such m and n are unique, by the C(6) condition.)

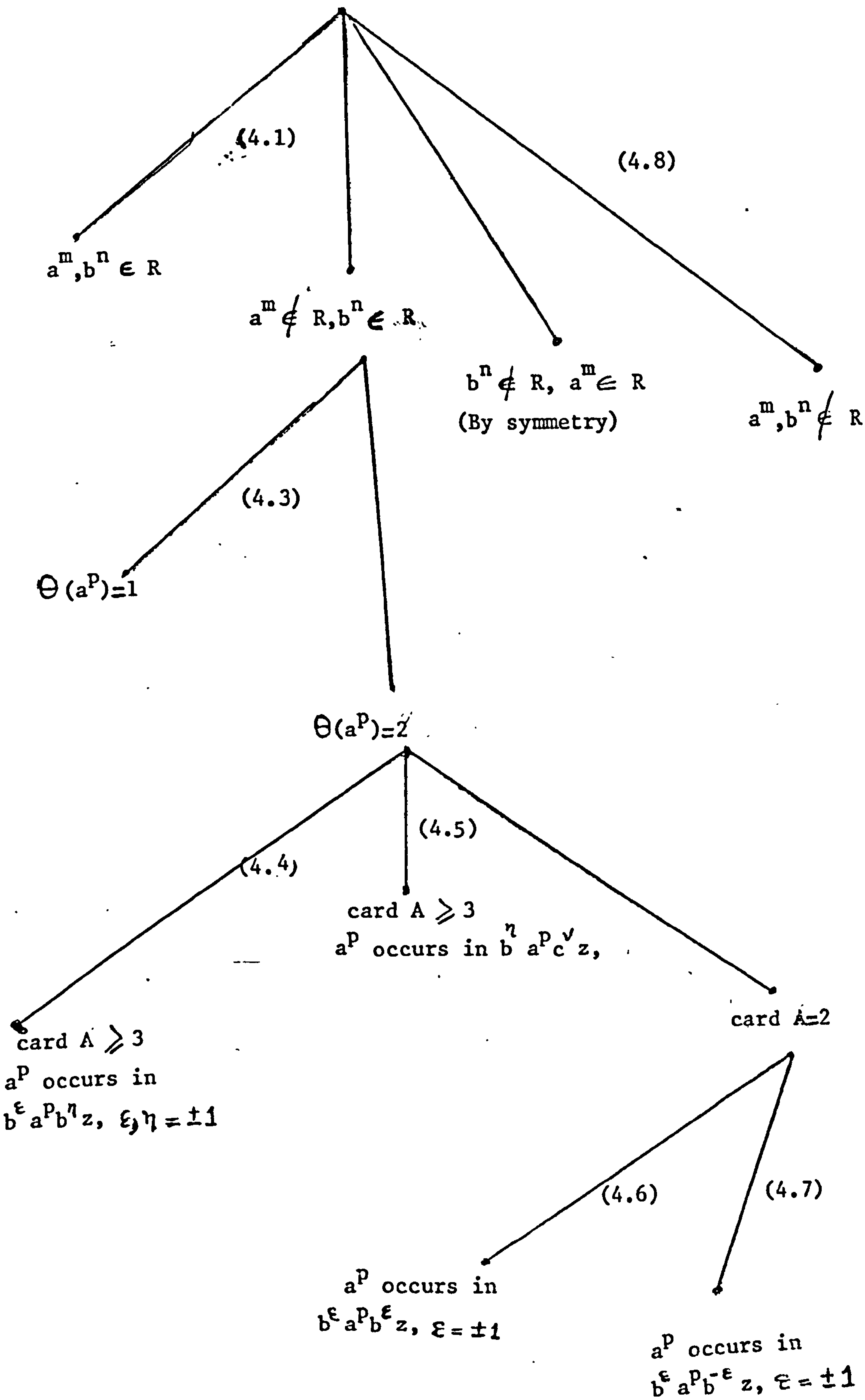
(2)(i) For every integer m, $a^m \notin R$ and there exists an integer n such that $b^n \in R$.

(ii) The dual of (i) with a and b interchanged.

(3) No element of R is of form a^m or b^n.

Recall that our aim is to determine free generators for free subgroups of G. For case 1, we shall use Proposition (4.1). For case 2 we shall use Propositions (4.3)-(4.7) (or appeal to symmetry). Finally for case 3, we shall use Proposition (4.8).

We illustrate the case division in this section in a more detailed way as follows. (For notation see Convention (4.2).)


(4.1)
(4.8)
a^m,b^n ∈ R
a^m ∉ R,b^n ∈ R
b^n ∉ R, a^m ∈ R
(By symmetry)
a^m,b^n ∉ R
(4.3)
Θ(a^p)=1
Θ(a^p)=2
(4.4)
(4.5)
card A ⩾ 3
a^p occurs in b^η a^p c^ν z,
card A ⩾ 3
a^p occurs in
b^ε a^p b^η z, ε,η = ±1
card A=2
(4.6)
(4.7)
a^p occurs in
b^ε a^p b^ε z, ε = ±1
a^p occurs in
b^ε a^p b^-ε z, ε = ±1

At a given vertex we have the case in which we assume all the conditions which occur at vertices in the path between the given vertex and the top vertex. The case occurring at an extremal vertex is solved by the proposition labelling the edge incident to the vertex.

Propositions (4.1) - (4.6) are comparatively easy. Propositions (4.7) and (4.8) are longer and more complicated, partly because we have endeavoured to obtain simple short free generators in these cases. Some reduction in the length of the proofs could have been obtained at the expense of having more complicated generators.

We make some points about that after Proposition (4.8).

(4.1) Proposition Let a and b be any two generators in A such that a^m and b^n are relators in R, where $m, n > 0$. Let a^p be the maximal power of a which appears in the relators of R distinct from $a^{\pm m}$. Let b^q be the maximal power of b which appears in the relators of R distinct from $b^{\pm n}$. Then $a^{p+1}(b^{2q+1}\ a^{2p+1})^2\ b^{2q+1}a^{p+1}$ and $b^{q+1}(a^{2p+1}\ b^{2q+1})^2a^{2p+1}b^{q+1}$ are free generators in G.

Proof. Since a^p and b^q are pieces, $\theta(a^p) = \theta(b^q) = 1$. Also, no larger power of a or b is a piece. It follows that $m \geq 5p+1$ and $n \geq 5q+1$. Let w be a word in $a^{p+1}(b^{2q+1}\ a^{2p+1})^2b^{2q+1}a^{p+1}$ and $b^{q+1}(a^{2p+1}\ b^{2q+1})^2a^{2p+1}b^{q+1}$, and assume w is cyclically reduced as a word in these generators. Then it is clear that w is cyclically reduced in a and b. Suppose that $w = 1$ in G. Then there exists an associated R-diagram $\mathfrak{M}$ for w, by Lemma (3.1). Let E be a boundary edge of $\mathfrak{M}$. By our choice of generators, $\phi(E)$ is a subword of one of:

$$(a^p b^q)^\nu, (a^{2p+1})^\nu, (b^{2q+1})^\nu, (a^{2p+2})^\nu, (b^{2q+2})^\nu, \text{ where } \nu = \pm 1 .$$

In all cases, $\theta(\phi(E)) \le 4$ and so $\mathfrak{M}$ has a T-path $Y_1, Y_2, \ldots, Y_L$, by Lemma (3.4).

Now $\theta(y_1) = \theta(\phi(Y_1)) \ge 3$ and so y_1 must be one of $(a^{2p+1})^{\pm 1}, (b^{2q+1})^{\pm 1}, (a^{2p+2})^{\pm 1}, (b^{2q+2})^{\pm 1}$.

If $y_1 \equiv a^{2p+2}$, then $r_1 \equiv a^m$.

If $p \geqq 2$, then $\theta(y_1) = 3$ and $h(Y_1) = 1$, $d(\mu(Y_1)) = 3$ and D_1 is adjacent to D_2. It follows that $r_2 \not\equiv b^{\pm n}$. Hence $y_2 \equiv b^{\pm k}$, $1 \le k \le q$. Then $\theta(y_2) = 1$, whence $h(Y_2) \le -1$, by Lemma (3.7). Therefore $\sum_{j=1}^{2} h(Y_j) = 0$, a contradiction since $Y_1, \ldots, Y_L$ is a T-path.

If $p = 1$, then $\theta(y_1) = 4$. Hence $h(Y_1) \le 2$, by Lemma (3.7). From the nature of our chosen generators and the maximality of q, $y_2 \equiv b^{k_1}$ and $y_3 \equiv b^{k_2}$, where $1 \le k_1, k_2 \le q$. Hence $\theta(y_2) = \theta(y_3) = 1$. If follows that $h(Y_2)$, $h(Y_3) \le -1$, by Lemma (3.7); and so

$$\sum_{j=1}^{3} h(Y_j) \le 0, \text{ a contradiction.}$$

Assume $y_1 \equiv a^{2p+1}$, then $h(Y_1) = 1$, $x_{1,2}$ begins with a. Since $d(\mu(Y_1)) = 3$, $x_{2,n(2)}$ ends with a^{-1} and so $y_2 \equiv b^k$, $1 \le k \le q$. By Lemma (3.7), $h(Y_2) \le -1$. It follows that $h(Y_1) + h(Y_2) \le 0$, absurd. Therefore, $y_1 \not\equiv a^{2p+1}$.

The other possibilities can be ruled out easily. Therefore $w \neq 1$ in G.

(4.2) Convention If R contains no relator of form a^m, then a^p denotes the maximal power of a occurring in any relator of R. If some $a^m \in R$, then there exists a unique such. In these circumstances a^p denotes the maximal power occurring in relators different from a^m.

(4.3) Proposition Let a and b be any two generators in A. Suppose that no relator in R is of the form a^m. Further, assume that $\theta(a^p) = 1$ and $b^n \in R$. Then $b^{-1} a^{3p}$ and $b a^{-3p}$ are free generators in G.

Proof. Let w be a word in $b^{-1} a^{3p}$ and $b a^{-3p}$, and assume w is cyclically reduced as a word in these generators. Then w is cyclically reduced in a and b; and if $w = 1$ in G, then there is an associated R-diagram $\mathfrak{M}$ for w, by Lemma (3.1). By the nature of our chosen generators, $\theta(\phi(E)) \le 4$ for each edge E in $\beta(\mathfrak{M})$. Then, by Lemma (3.4), there exists a T-path $Y_1, \ldots, Y_L$ in $\mathfrak{M}$. Also, from the nature of our chosen generators $y_1 \equiv a^{\nu\ell_1} b^{\eta k} a^{\varepsilon\ell_2}$ where $1 \le \ell_1, \ell_2 \le p$, $k = 1$ or 2, $\varepsilon, \eta, \nu = \pm 1$. (Otherwise $\theta(y_1) \le 2$, absurd.) Hence $y_2 \equiv a^{\nu\ell_3}$ and $y_3 \equiv a^{\nu\ell_4}$, $\nu = \pm 1$, $1 \le \ell_3, \ell_4 \le p$. It follows that $h(Y_2), h(Y_3) \le -1$, by Lemma (3.7) and since $h(Y_1) \le 2$, $\sum_{j=1}^{3} h(Y_j) \le 0$, which is absurd. Therefore $w \ne 1$ in G.

If $\theta(a^p) = 2$, then a^p is not a piece, and so there exists a unique relator r in R which involves a^p. This is of form $r \equiv (a^p z)^m$, $m \ge 1$, where z is nontrivial and without occurrences of a^p (and a^{-p}).

(4.4) Proposition Let a, b and c be three distinct generators in A. Let $\theta(a^p) = 2$. Suppose that the unique relator r in R

involving a^p has the form $b^\varepsilon a^p b^\eta z$, where $\varepsilon, \eta = \pm 1$, $z \in F(A)$. Then $c a^{4p} c^{-1}$ and $a^{4p} c a^{4p}$ are free generators in G.

<u>Proof</u> Let w be cyclically reduced as a word in $c a^{4p} c^{-1}$ and $a^{4p} c a^{4p}$. Cancelling occurrences of $c^{-1}c$ gives a word w^* cyclically reduced in a and c. If $w = 1$ in G then $w^* = 1$ in G and there is an associated R-diagram $\mathfrak{M}$ for w^*, by Lemma (3.1).

Let E be a boundary edge of $\mathfrak{M}$. Since $a^p c^{\pm 1}$ and $c^{\pm 1} a^p$ do not occur in any relator of R, $\phi(E)$ must be a subword of some $a^{\varepsilon \ell_1} c^\eta a^{\nu \ell_2}$ where $1 \le \ell_1, \ell_2 < p$, $\varepsilon, \eta, \nu = \pm 1$. Then certainly $\theta(\phi(E)) \le 3$ and so $\mathfrak{M}$ has a T-path $Y_1, Y_2, \ldots, Y_L$. Since $\theta(y_1) \ge 3$, $y_1 \equiv a^{\varepsilon \ell_1} c^\eta a^{\nu \ell_2}$ where $1 \le \ell_1, \ell_2 < p$, $\varepsilon, \eta, \nu = \pm 1$; and so $h(Y_1) = 1$. If $y_2 \equiv a^{\eta \ell}$, $\ell < p$ then $h(Y_1) \le -1$ by Lemma (3.7); and so $\sum_{j=1}^{2} h(Y_j) \le 0$, absurd. Therefore $y_2 \equiv a^{\eta p}$ and $h(Y_2) = 0$, whence $y_3 \equiv a^{\eta p}$. It follows that $h(Y_1) = 0$, by Lemma (3.10). This is a contradiction. Therefore $w \ne 1$ in G.

(4.5) <u>Proposition</u> Let a, b and c be three distinct generators in A. Let $\theta(a^p) = 2$ and $b^n \in R$. Suppose that the unique relator in R which involves a^p has the form $c^\nu a^p b^\eta z$ or $b^\eta a^p c^\nu z$, where $\eta, \nu = \pm 1$. Then $b^{-\eta} a^{5p} c^{-\nu}$ and $a^{5p} b^{-\eta} a^{5p}$ are free generators in G.

<u>Proof</u> Let w be a word in $b^{-\eta} a^{5p} c^{-\nu}$ and $a^{5p} b^{-\eta} a^{5p}$, and assume w is cyclically reduced word in these generators. Then w is cyclically reduced in a, b and c. W.l.o.g. we may assume that the unique relator r in R which involves a^p has the form $r \equiv c a^p b z$, (hence w is a word in $b^{-1} a^{5p} c^{-1}$ and $a^{5p} b^{-1} a^{5p}$).

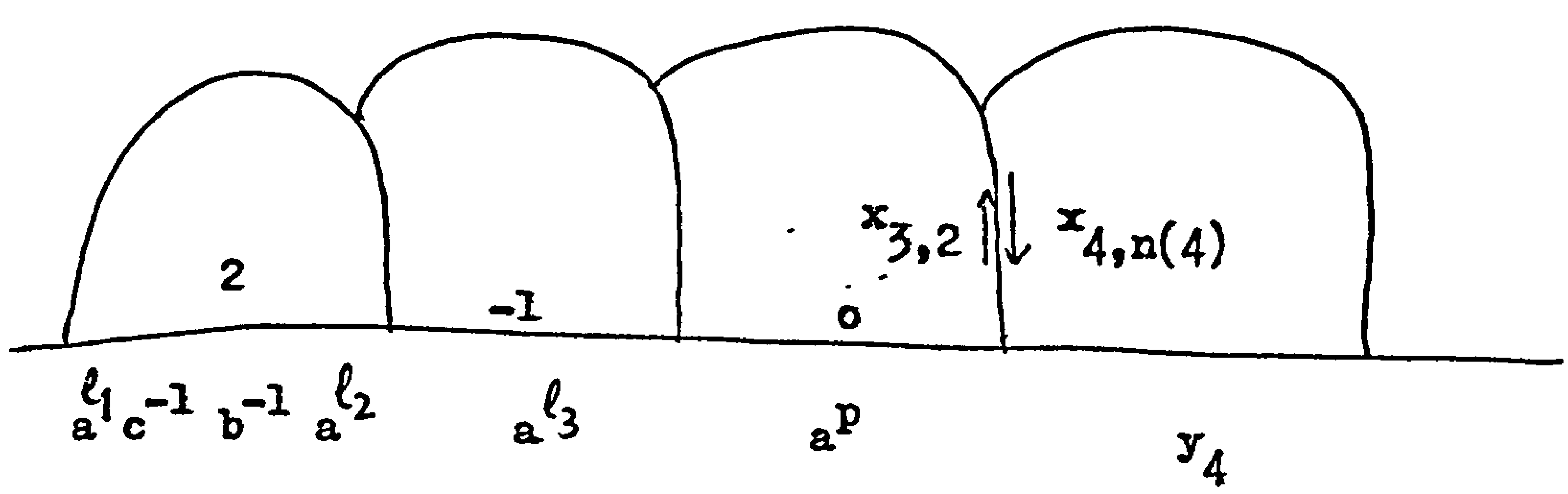
If $w = 1$ in G, then there exists an associated R-diagram $\mathfrak{M}$ for w. From the nature of our generators, the maximality of p and the nature of r, the label on any boundary edge E must be a subword of one of

(I) $(a^{\ell_1} c^{-1} b^{-1} a^{\ell_2})^{\nu}$, (II) $a^{\varepsilon\ell_1} c^{\nu} a^{\eta\ell_2}$, (III) $a^{\varepsilon\ell_1} b^{\nu} a^{\eta\ell_2}$,

where $1 \le \ell_1, \ell_2 < p$, $\varepsilon, \eta, \nu = \pm 1$.

In all cases $\theta(\phi(E)) \le 4$ and so there is a T-path $Y_1, \ldots, Y_L$ in $\mathfrak{M}$. Then y_1 is one of (I), (II) or (III) above.

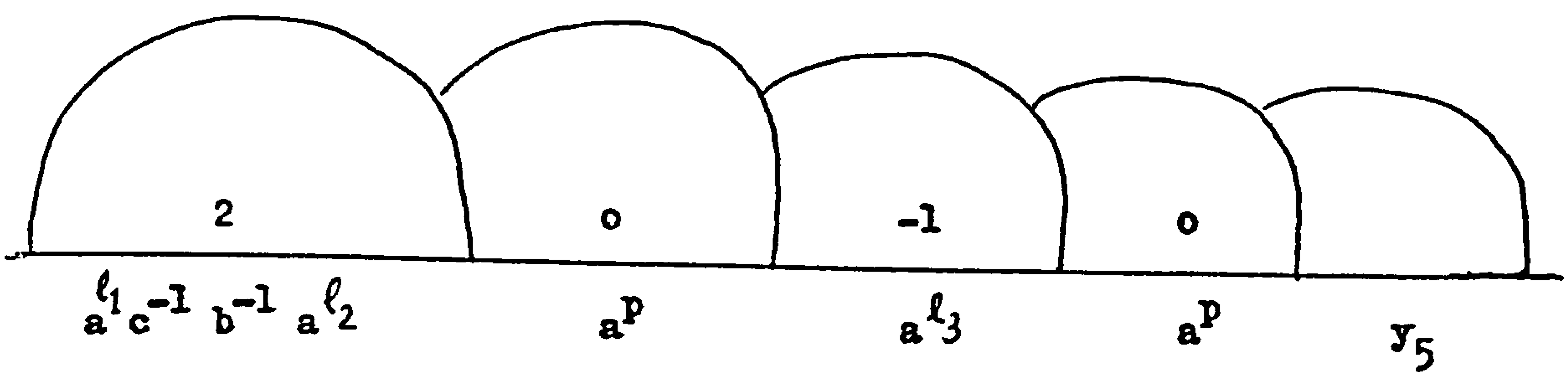
Assume (I) occurs with $y_1 \equiv a^{\ell_1} c^{-1} b^{-1} a^{\ell_2}$. Then $h(Y_1) = 1$ or 2. We shall show that $y_2 \equiv a^p$; suppose $y_2 \equiv a^{\ell_3}$, $1 \le \ell_3 < p$. Then $h(Y_2) \le -1$, by Lemma (3.7), and since $Y_1, Y_2, \ldots, Y_L$ is a T-path, $h(Y_1) = 2$ and $h(Y_2) = -1$. Again since $Y_1, Y_2, \ldots, Y_L$ is a T-path $h(Y_3) \ge 0$ and so $y_3 \equiv a^p$ with $h(Y_3) = 0$, as shown below.



(See the note after the proposition for an explanation of this diagram.)

We compare r_3 and r, noting that a^p is not a piece. Then $x_{3,2}$ begins with b whence $x_{4,n(4)}$ ends in b^{-1}. Now our choice of generators gives $y_4 \equiv a^{\ell_4}$, $1 \le \ell_4 \le p$ and a comparison of r_4 and r shows that $\ell_4 \ne p$. Thus $h(Y_4) \le -1$ and $\sum_{j=1}^{4} h(Y_j) \le 0$, absurd. therefore $y_2 \equiv a^p$.

Now compare r_2 and r, noting that a^p is not a piece. Then $x_{2,1}$ begins with b. Since we have a T-path, $d(\mu(Y_2)) = 3$ and therefore $x_{3,n(3)}$ ends in b^{-1}, and $y_3 \equiv a^{\ell_3}$, $1 \le \ell_3 < p$ (compare r_3 and r to get $\ell_3 \ne p$). To maintain a positive count we have $h(Y_1) = 2$, $h(Y_3) = -1$, $y_4 \equiv a^p$ and $h(Y_4) = 0$ – as shown below.



But now we get $y_5 \equiv a^{\ell_4}$, $1 \le \ell_4 < p$ just as we get $y_3 \equiv a^{\ell_3}$ and $\sum_{j=1}^{5} h(Y_j) \le 0$ which is contradictory. So $y_1 \not\equiv a^{\ell_1} c^{-1} b^{-1} a^{\ell_2}$.

Similar arguments rule out $y_1 \equiv a^{-\ell_2} b c a^{-\ell_1}$ and (II) and (III).

Note: In the diagrams above, the edges $Y_1, Y_2, Y_3, \ldots$ of the T-path occur in left to right order and each edge is directed from left to right. The number above Y_j is the value of $h(Y_j)$ and the word below Y_j is the label $\phi(Y_j)$. The drawing of such diagrams is essential for easy comprehension of our argument. Mostly, we leave the drawing of these diagrams to the reader.

and $b^n \in R$.

(4.6) Proposition Let $\theta(a^p) = 2$ Suppose that Card $A = 2$. Further suppose that the unique relator r in R which involves a^p has the form $r \equiv b^{\varepsilon} a^p b^{\varepsilon} z$, where $\varepsilon = \pm 1$. Then $b^{-\varepsilon} a^{5p} b^{-\varepsilon}$ and $a^{5p} b^{-\varepsilon} a^{5p}$ are free generators in G.

Proof W.l.o.g. we assume that the unique relator r in R which involves a^p has the form $r \equiv b a^p b z$. Let w be a word in $b^{-1} a^{5p} b^{-1}$ and $a^{5p} b^{-1} a^{5p}$, and assume w is cyclically reduced in these generators. Then w is cyclically reduced in a and b. If

$w = 1$ in G, then there exists an associated R-diagram $\mathfrak{M}$ for w. By the nature of our chosen generators, the maximality of p and the nature of r, we will again have a T-path $Y_1, Y_2, \ldots, Y_L$, and y_1 must be one of the following:

(I) $y_1 \equiv (a^{\ell_1} b^{-2} a^{\ell_2})^{\nu}$, (II) $y_1 \equiv (a^p b a^{-\ell_1})^{\nu}$,

(III) $y_1 \equiv (a^{-\ell_1} b a^p)^{\nu}$, (IV) $y_1 \equiv a^{\varepsilon \ell_1} b^{\nu} a^{\eta \ell_2}$, where $1 \le \ell_1, \ell_2 < p$ and $\varepsilon, \eta, \nu = \pm 1$ (v) $y_1 \equiv (a^p b^{\varepsilon})^{\pm 1}$ or $(b^{\varepsilon} a^p)^{\pm 1}$.

We shall firstly show that y_1 does not end with $a^{\nu p}$; suppose not. Compare r_1 and r, noting that a^p is not a piece. Then $x_{1,2}$ must begin with b^{ν}, whence $x_{2,n(2)}$ ends in $b^{-\nu}$ since $d(\mu(Y_1))$ must be 3. From the nature of our chosen generators, $y_2 \equiv a^{\nu \ell_3}$, $1 \le \ell_3 < p$ (compare r_2 and r to get $\ell_3 \ne p$). Then $h(Y_1) = 2$ and $h(Y_2) = -1$ (use Lemma (3.7) and the fact that $Y_1, Y_2, \ldots, Y_L$ is a T-path). It follows that $y_3 \equiv a^{\nu p}$ with $h(Y_3) = 0$. By a similar argument as with y_2 we can show that $y_4 \equiv a^{\nu \ell_4}$, $1 \le \ell_4 < p$ and so $h(Y_4) \le -1$, by Lemma (3.7). Thus $\sum_{j=1}^{4} h(Y_j) \le 0$, absurd. Therefore y_1 cannot end with $a^{\nu p}$.

Suppose that y_1 ends with $a^{\nu \ell'}$, $1 \le \ell' < p$, $\nu = \pm 1$. If $y_2 \equiv a^{\nu \ell_3}$, $1 \le \ell_3 < p$, then by a similar argument as above we can show that $y_3 \equiv a^{\nu p}$ with $h(Y_3) = 0$ and $y_4 \equiv a^{\nu \ell_4}$, $1 \le \ell_4 < p$. Thus $\sum_{j=1}^{4} h(Y_j) \le 0$, absurd. Hence $y_2 \equiv a^{\nu p}$ and $h(Y_2) = 0$ or -1. Compare r_2 and r, noting that a^p is not a piece. Since $d(\mu(Y_2)) = 3$, $y_3 \equiv a^{\nu \ell_3}$, $1 \le \ell_3 < p$. Thus $h(Y_1) = 2$, $h(Y_2) = 0$ and $h(Y_3) = -1$. Since $\sum_{j=1}^{3} h(Y_j) = 1$, $y_4 \equiv a^{\nu p}$ with

$h(Y_4) = 0$. Then it is easy to see that $h(Y_5) = -1$, absurd. Therefore y_1 cannot end with $a^{\nu\ell'}$, $1 \le \ell' < p$.

The remaining possibilities are easily ruled out.

(4.7) Proposition Let $\theta(a^p) = 2$. Let b^n be a relator in R. Let q be the largest integer such that b^q is a subword of some relator r in R, $r \not\equiv b^{\pm n}$. Suppose that Card $A = 2$. Further suppose that the unique relator r in R which involves a^p has the form $r_1 \equiv b^{-\varepsilon} a^p b^\varepsilon z$, where $\varepsilon = \pm 1$.

(1) If $q = 1$, then $(a^{5p} b)^5 a^{5p}$ and $(b a^{5p})^5 b$ are free generators in G.

(2) If $q \ge 2$, then $(a^{5p} b^{2q+1})^2 a^{5p}$ and $b^{q+1} a^{5p} (b^{2q+1} a^{5p})^2 b^{q+1}$ are free generators in G.

Proof W.l.o.g. we may assume that the unique relator r in R which involves a^p has the form $r \equiv b^{-1} a^p b z$.

(1) Let w be a word in $(a^{5p} b)^5 a^{5p}$ and $(b a^{5p})^5 b$, and assume w is cyclically reduced in a and b.

If $w = 1$ in G, it follows that in the associated R-diagram there is a T-path, since the label on any boundary edge is a subword of $b^{2\varepsilon}$, $(a^p b a^\ell)^\varepsilon$, $(a^p b a^{-\ell})^\varepsilon$, or $(a^{-p} b a^\ell)^\varepsilon$, where $1 \le \ell < p$ and $\varepsilon = \pm 1$.

Since $\theta(y_1) \ge 3$, y_1 must be one of the following:

(i) $y_1 \equiv (a^{\ell_1} b a^{\ell_2})^\varepsilon$, (ii) $y_1 \equiv (a^{-\ell_1} b a^{\ell_2})^\varepsilon$,

(iii) $y_1 \equiv (a^p b a^{-\ell})^\varepsilon$, (iv) $y_1 \equiv (a^{-p} b a^\ell)^\varepsilon$,

(v) $y_1 \equiv (a^p b a^\ell)^\varepsilon$, (vi) $y_1 \equiv (a^p b)^\varepsilon$ or (vii) $y_1 \equiv (b^{-1} a^p)^\varepsilon$,

where $1 \le \ell, \ell_1, \ell_2 < p$ and $\varepsilon = \pm 1$.

Assume (i) occurs. Then $\theta(y_1) = 3$ and so $h(Y_1) = 1$. To maintain a positive count, we have $y_2 \equiv y_3 \equiv a^{\varepsilon p}$ with $h(Y_2) = h(Y_3) = 0$; and Lemma (3.12) gives a contradiction.

By a similar argument we can show that (ii) does not occur.

Assume (iii) occurs with $y_1 \equiv a^p\, b\, a^{-\ell}$. We shall firstly show that $h(Y_1) \neq 2$; suppose not. Now, y_2 must begin with a^{-1}. Since $d(\mu(Y_1) = 3$, $x_{2,n(2)}$ does not end in a^{-1}. If $y_2 \equiv a^{-\ell_1}$, $\ell_1 < p$, then $h(Y_2) = -1$; and to maintain a positive count we have $y_3 \equiv y_4 \equiv a^{-p}$ with $h(Y_3) = 0$. By Lemma (3.10), $h(Y_2) \leq -2$, absurd; and so $y_2 \equiv a^{-p}$. By a similar argument involving y_4 and y_5, we get $y_3 \equiv a^{-p}$; and then using Lemma (3.11), we can show that $y_4 \equiv a^{-p}$. Clearly, $h(Y_2) = 0$ or $h(Y_3) = 0$. Then Lemma (3.12) gives a contradiction. Since $Y_1, Y_2, \ldots, Y_L$ is a T-path, $h(Y_1) = 1$.

We now have to go a little deeper to derive a contradiction. If y_j involves only $a^{\pm 1}$, then $h(Y_j) \leq 0$ and so the T-path includes all edges up to the next edge in which $b^{\pm 1}$ occurs. Specifically, let s be the least j, $2 \leq j \leq L$ such that y_j involves $b^{\pm 1}$. Then $s > 4$.

As y_1 involves b and $a^{-\ell}$ which are of opposite sign, our choice of generators ensures that y_s is a negative word. Also, since $h(Y_1) = 1$, we have $y_j \equiv a^{-p}$ and $h(Y_j) = 0$, $2 \leq j \leq s-1$ whence y_s begins with a^{-1}. Since

$\sum_{j=1}^{s-1} h(Y_j) = 1$, $y_s \not\equiv a^{-(p-\ell)}\, b^{-1}$, and thus $y_s \equiv a^{-(p-\ell)}\, b^{-1}\, a^{-\ell_1}$,

where $1 \leq \ell_1 < p$ (compare with r_1 to get $\ell_1 \neq p$). By Lemma (3.13), $h(Y_s) = 0$, and so $y_{s+1} \equiv a^{-p}$ with $h(Y_{s+1}) \neq 0$. By

Lemma (3.14), $h(Y_s) \leq -1$, absurd. Therefore $y_1 \not\equiv a^p\, b\, a^{-\ell}$ and so $y_1 \equiv a^{\ell}\, b^{-1}\, a^{-p}$.

Again if y_j involves only $a^{\pm 1}$, then $h(Y_j) \leq 0$ and so there are y_j's, $2 \leq j \leq L$ which involve $b^{\pm 1}$. Let s be the least such j; then $s > 4$.

As y_1 involves a^{ℓ} and b^{-1}, again y_s is a negative word and using Lemmas (3.10) and (3.11) when $h(Y_1) = 2$, we have for each $2 \leq j \leq s-1$, $y_j \equiv a^{-p}$. Since $y_{s-1} \equiv a^{-p}$, $x_{s-1,2}$ begins with b (for $r \equiv b^{-1}\, a^p\, b\, z$). As $d(\mu(Y_{s-1}))$ must be three, $x_{s,n(s)}$ ends in b^{-1} and so y_s cannot begin with a^{-p}. Since $q = 1$, y_s must be b^{-1}, i.e. $r_s \equiv b^{-n}$ and $h(Y_s) = -1$. Clearly $d(\mu(Y_s)) = 3$. Then $x_{s+1,n(s+1)}$ must end in b, which means that $y_{s+1} \equiv a^{-k}$, $k < q$, (compare with r to get $k \neq p$). By Lemma (3.7) $h(Y_{s+1}) \leq -1$ and so

$\sum_{j=1}^{s+1} h(Y_j) \leq 0$, absurd. Therefore (iii) does not occur.

Assume (iv) occurs with $y_1 \equiv a^{-\ell}\, b^{-1}\, a^p$. This needs our lengthiest argument so far. Braodly we consider edges $Y_1, Y_s, Y_{s'}, Y_{s''}, Y_{s'''}, Y_t$ whose labels contain occurrences of b and which are chosen so that intermediate edges Y_j always have label b-free. Roughly we show that such an intermediate edge Y_j has label a^p with $h(Y_j) = 0$. We shall make numerous uses of Lemmas (3.7)-(3.14). Also note that since y_1 involves b^{-1} and a, our choice of generators ensures that $y_s, y_{s'}, y_{s''}, y_{s'''}$, y_t will be positive words.

We divide the proof of this case into steps to avoid repetition of the same arguments.

Let s be the least j, $2 \leq j \leq L$ such that y_j involves b.

Step (iv) 1 - To show that y_s begins with a^p.

By the maximality of p, $s \geqslant 5$. Since $h(Y_1) = 1$ or 2, $y_j \equiv a^p$, $2 \le j \le s-1$, using Lemmas (3.10) and (3.11) when $h(Y_1) = 2$. In particular, $d(\mu(Y_{s-1})) = 3$ and $x_{s,n(s)}$ ends in b^{-1}. So y_s must begin with a^p.

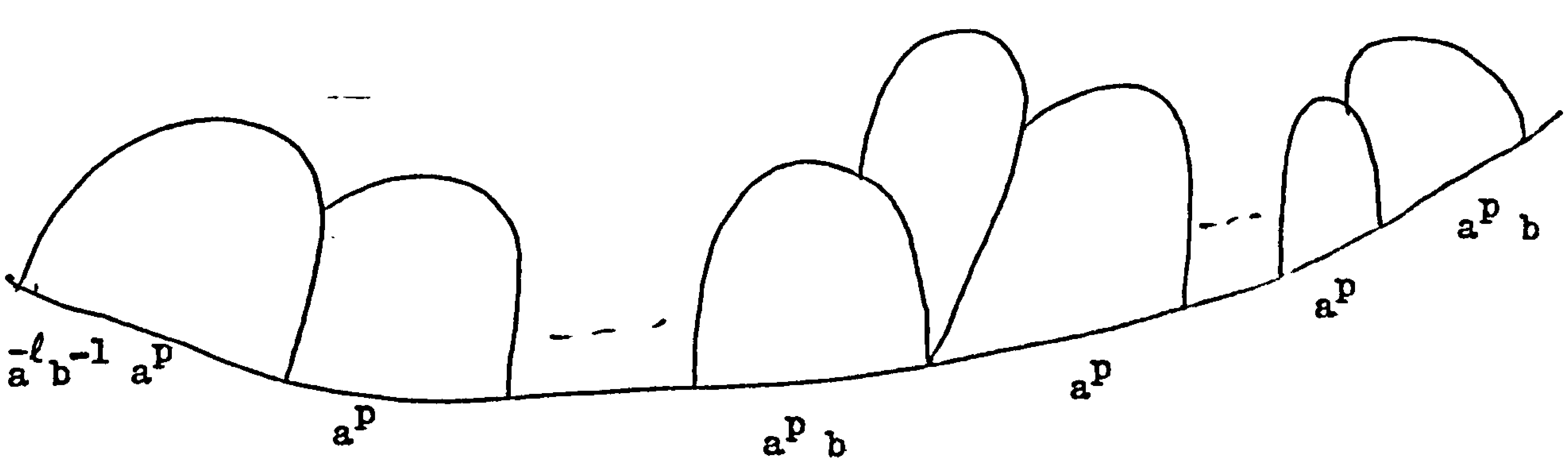
Step (iv) 2 - If $y_s \equiv a^p b$, we shall show that $d(\mu(Y_s)) = 3$.

If $d(\mu(Y_s)) > 3$, then $d(\mu(Y_s)) = 4$ and $h(Y_s) = -1$ (and hence $h(Y_1) = 2$). Let s' be the least j, $s < j \le L$, such that y_j involves b. Then $s' > s+3$ and since

$\sum_{j=1}^{s} h(Y_j) = 1$, for each $s < j < s'$, $y_j \equiv a^p$ with $h(Y_j) = 0$.

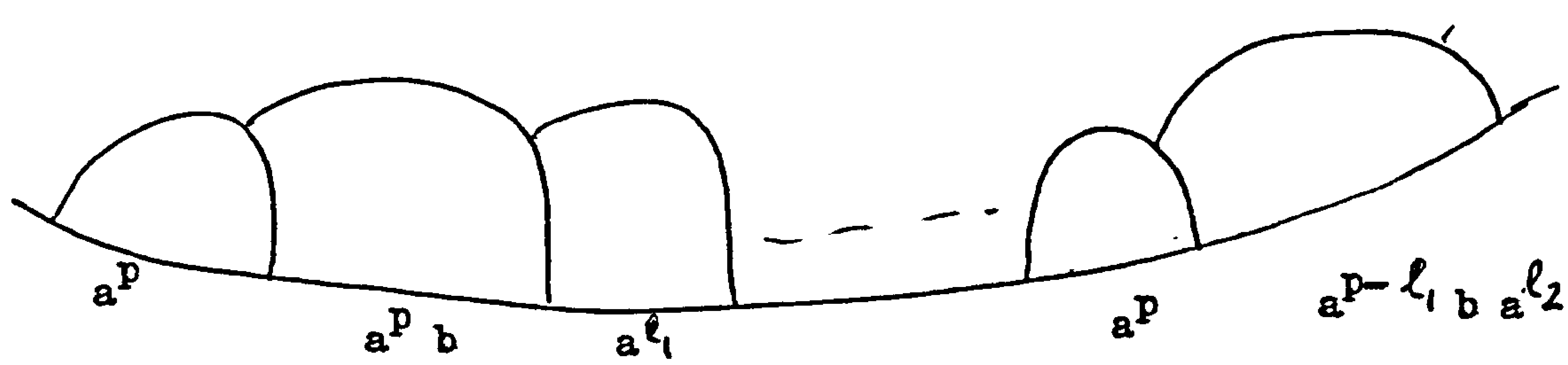
By a similar argument as for y_s (in Step (iv) 1), $y_{s'}$ does not begin with b.

Now, suppose that $y_{s'} \equiv a^p b$. (This situation is illustrated in the diagram below).



Since $\sum_{j=1}^{s-1} h(Y_j) = 1$, $h(Y_{s'}) = 0$ or 1 (and so $d(\mu(Y_{s'})) = 3$). Since $q = 1$, $x_{s',2}$ does not begin with b. Then $y_{s'+1} \not\equiv a^p$ and so $y_{s'+1} \equiv a^{\ell_1}$, $1 \le \ell_1 < p$. Then $h(Y_{s'+1}) = -1$ and $h(Y_{s'}) = 1$. Let s'' be the least j, $s'+2 \le j \le L$ such that

y_j involves b. By the maximality of p, $s'' \geq s'+4$. Since $\sum_{j=1}^{s'+1} h(Y_j) = 1$, $y_k \equiv a^p$ with $h(Y_k) = 0$ for each $s'+2 \leq k \leq s''-1$. Hence $y_{s''}$ must begin with a. By Lemma (3.13), $y_{s''} \not\equiv a^{(p-\ell_1)} b$, (note that $\sum_{j=1}^{s''-1} h(Y_j) = 1$). Hence $y_{s''} \equiv a^{p-\ell_1} b a^{\ell_2}$, with $\ell_2 < p$ (compare with r to get $\ell_2 \neq p$), as illustrated below.



By Lemma (3.13), $h(Y_{s''}) = 0$, and so $y_{s''+1} \equiv a^p$. Hence $h(Y_{s''}) \leq -1$, by Lemma (3.14). This is a contradiction. Therefore $d(\mu(Y_s)) = 3$.

<u>Step (iv) 3</u> - To show that $y_s \equiv a^p b a^{\ell_1}$, where $1 \leq \ell_1 < p$.

Suppose $y_s \equiv a^p b$; then $y_{s+1} \not\equiv a^p$. (If $y_{s+1} \equiv a^p$, then $x_{s+1,n(s+1)}$ ends in b^{-1} and so $x_{s,2}$ begins with b; this is a contradiction for $q = 1$). Thus $y_{s+1} \equiv a^{\ell_1}$ and $h(Y_{s+1}) \leq -1$. Again let $y_{s'}$ involve b. Thus $s+4 < s' \leq L$ and in the usual way we can show that for each $s+2 \leq j \leq s'-1$, $y_j \equiv a^p$ with $h(Y_j) = 0$. Then either $y_{s'} \equiv a^{p-\ell_1} b$ or $y_{s'} \equiv a^{p-\ell_1} b a^{\ell_2}$, where $1 \leq \ell_2 < p$. If $y_{s'} \equiv a^{p-\ell_1} b$, then $h(Y_{s'}) \leq -1$, by Lemma (3.13). Since $\sum_{j=1}^{s'-1} h(Y_j) \leq 2$, $h(Y_{s'}) = -1$. Thus $d(\mu(Y_{s'})) = 3$ and since

$q = 1$, $y_{s'+1} \not\equiv a^p$. Thus $y_{s'+1} \equiv a^{\ell_2}$, $1 \le \ell_2 < p$ and so $\sum_{j=1}^{s'+1} h(Y_j) \le 0$, absurd. Hence $y_{s'} \equiv a^{p-\ell_1} b\, a^{\ell_2}$, where $1 \le \ell_2 < p$ (compare with r to get $\ell_2 \ne p$) and, using Lemma (3.13), $h(Y_{s'}) \le 0$. Let s'' be the least j such that y_j involves b. Then $s'+4 < s'' \le L$ and in the usual way, (using Lemmas (3.10) and (3.11)), for each $s'+1 \le j \le s''-1$, $y_j \equiv a^p$ with $h(Y_j) = 0$. Then by Lemma (3.14), $h(Y_{s'}) = -1$ and $\sum_{j=1}^{s''-1} h(Y_j) = 1$; and so $h(Y_{s''}) \ge 0$. The same argument as for $y_{s'}$ gives $h(Y_{s''}) \le -1$ which is impossible. Therefore $y_s \not\equiv a^p\, b$ and so $y_s \equiv a^p\, b\, a^{\ell_1}$, where $1 \le \ell_1 < p$.

Step (iv) 4 – To show that $h(Y_1) + h(Y_s) \le 3$.

Suppose that $h(Y_1) = h(Y_s) = 2$. Then $\theta(y_1) = \theta(y_s) = 4$ and $\theta(a^{-\ell}\, b^{-1}) = \theta(b^{-1}\, a) = \theta(ab) = \theta(b\, a^{\ell_1}) = 2$. If $\ell_1 \ge \ell$, then we can write $r_1 \equiv a^{-\ell}\, b^{-1}\, a^p\, z_1$ and $r_s^{-1} \equiv a^{-(\ell_1-\ell)}\, a^{-\ell}\, b^{-1}\, a^{-p}\, z_s^{-1}$. Hence $\theta(a^{-\ell}\, b^{-1}) = 1$, absurd. Thus $\ell > \ell_1$, and so we can write $r_1^{-1} \equiv a^{-p}\, b\, a^{\ell_1}\, a^{\ell-\ell_1}\, z_1^{-1}$ and $r_s \equiv a^p\, b\, a^{\ell_1}\, z_s$. Hence $\theta(a\, b^{\ell_1}) = 1$, absurd. Therefore $\sum_{j=1}^{s} h(Y_j) \le 3$.

Step (iv) 5 – To show that $d(\mu(Y_s)) = 3$.

If $d(\mu(Y_s) > 3$, then $d(\mu(Y_s)) = 4$ and so $\sum_{j=1}^{s} h(Y_j) = 1$. Again let s' be the least j, $s < j \le L$ such that y_j involves b. Then $s' > s+3$ and, as usual, for each $s < j < s'$, $y_j \equiv a^p$ with $h(Y_j) = 0$. So $y_{s'}$ must begin with a. Arguing almost exactly as in Step (iv) 3, we get $y_{s'} \equiv a^{p-\ell_1}\, b\, a^{\ell_2}$ and using Lemma (3.13), $h(Y_{s'}) \le 0$. Then $y_{s'+1} \equiv a^p$, whence $h(Y_{s'}) \le -1$

by Lemma (3.14), and so $\sum_{j=1}^{s} h(Y_j) \le 0$, absurd. Therefore $d(\mu(Y_s)) = 3$.

Step (iv) 6 - To show that $y_{s+1} \equiv a^p$.

Suppose $y_{s+1} \equiv a^{\ell_2}$, $1 \le \ell_2 < p$. If $y_{s+2} \equiv a^{\ell_3}$, $1 \le \ell_3 < p$, then $h(Y_{s+1}) = h(Y_{s+2}) = -1$. Then $y_{s+3} \equiv a^p$ with $h(Y_{s+3}) = 0$. By Lemma (3.10) (using the labels of a pair of consecutive edges Y_j, Y_{j+1}, $2 \le j \le s-1$) we get $h(Y_{s+2}) \le -2$, absurd. Hence $y_{s+2} \equiv a^p$ and the same argument gives $h(Y_{s+1}) \le -2$. Thus $h(Y_{s+1}) = -2$ and $\sum_{j=1}^{s+1} h(Y_j) = 1$. Since $d(\mu(Y_{s+1})) = 3$, $x_{s+1,2}$ begins with b (compare r_{s+2} and r, noting that a^p is not a piece. Then $x_{s+2,n(s+2)}$ ends in b^{-1}). Comparing r_{s+1} and r_s shows that ab is a piece and so $h(Y_s) = 1$. Let s' be the least j, $s < j \le L$ such that y_j involves b. Then $s' > s+3$ and for each $s+2 \le j \le s'-1$, $y_j \equiv a^p$ with $h(Y_j) = 0$.

Now, $y_s \equiv a^p\, b\, a^{\ell_1}$ and $y_{s+1} \equiv a^{\ell_2}$. If $\ell_1 + \ell_2 \ne p$, then it follows from Lemma (3.13) that $y_{s'} \equiv a^{\ell_3}\, b\, a^{\ell_4}$, where $\ell_3 = p - (\ell_1+\ell_2)$ or $\ell_3 = 2p - (\ell_1+\ell_2)$ and $1 \le \ell_4 < p$. By Lemma (3.13), $h(Y_{s'}) \le 0$ (or compare $r_{s'}$ with r_s to see that $a^{\ell_3} b$ is a piece). Since $\sum_{j=1}^{s'-1} h(Y_j) = 1$, $h(Y_{s'}) = 0$ and $y_{s'+1} \equiv a^p$. By Lemma (3.14), $h(Y_{s'}) \le -1$, absurd. Then $\ell_1 + \ell_2 = p$ and so $y_{s+1} \equiv a^{p-\ell_1}$, with $h(Y_{s+1}) = -2$. Then $s' > s+3$ and $y_{s'} \equiv a^p\, b\, a^{\ell_3}$, $1 \le \ell_3 < p$, by essentially the argument of Step (iv) 3. Now we noted that ab is a piece (when showing $h(Y_s) = 1$) and so $h(Y_{s'}) \le 1$. As $\sum_{j=1}^{s'-1} h(Y_j) = 1$, $h(Y_{s'}) = 0$ or 1.

Suppose $y_{s''}$ involves b and y_j is b-free, $s' < j < s''$. Then we easily get $y_j \equiv a^p$, $s' < j < s''$ and $y_{s''} \equiv a^{p-\ell_3} b a^{\ell_4}$, $1 \le \ell_4 < p$. As $a^{p-\ell_3} b$ is a piece (compare $r_{s''}$ and r_s), $h(Y_{s''}) \le 0$ and we see easily that $y_{s''+1} \equiv a^p$. By Lemma (3.14), $h(Y_{s''}) \le -1$ and so $h(Y_{s'}) = 1$, $h(Y_{s''}) = -1$. Finally, if $y_{s'''}$ involves b and y_j is b-free, $s'' < j < s'''$, then the same argument as gave $h(Y_{s''}) \le -1$ will show $h(Y_{s'''}) \le -1$ which is impossible. Therefore $y_{s+1} \equiv a^p$.

<u>Step (iv) 7</u> – To show that $y_{s'} \equiv a^{p-\ell_1} b a^{\ell_2}$, $1 \le \ell_2 < p$ (where $y_{s'}$ involves b, and y_j is b-free, $s < j < s'$).

Arguing as in Step (iv) 5, we can show that for $s+1 \le j \le s'-2$, $d(\mu(Y_j)) = 3$. Then arguing as in Step (iv) 6, we can show that $y_j \equiv a^p$, $s+2 \le j \le s'-2$.

We claim that $h(Y_s) + h(Y_{s+1}) \le 0$. Since $y_{s+1} \equiv a^p$, $x_{s,2}$ begins with b and, looking at y_s, we see that ab is a piece. So $h(Y_s) \le 1$. Suppose $h(Y_s) = 1$ and $h(Y_{s+1}) = 0$. Since $y_{s+2} \equiv a^p$ then one of $a^{\ell_1} x_{s,2}$ and $a^{\ell_1} x_{s+1,2}$ is an initial segment of the other. This means that $a^{\ell_1} x_{s,2}$ is a piece (since $a^{\ell_1} x_{s+1,2}$ is not). However, as ab is also a piece, we get $\delta^*(r_s) \le 5$ and this gives $h(Y_s) + h(Y_{s+1}) \le 0$ as desired.

Now we look at $y_{s'}$; suppose $y_{s'}$ begins with b so that $y_{s'-1}$ is $a^{p-\ell_1}$. Then $y_{s'} \not\equiv b$ since otherwise

$h(Y_{s'-1}) + h(Y_{s'}) \le -2$ giving $\sum_{j=1}^{s'} h(Y_j) = 0$. Since $q = 1$, we

have $y_{s'} \equiv b a^{\ell_2}$, $1 \le \ell_2 < p$. Since $\sum_{j=1}^{s'-1} h(Y_j) = 1$, $y_{s'+1} \equiv a^p$

which gives $h(Y_{s'}) \le -1$, absurd. Hence $y_{s'}$ must begin with $a^{p-\ell_1} b$ and then $y_{s'-1} \equiv a^p$, $d(\mu(Y_{s'-1})) = 3$.

Suppose $y_{s'} \equiv a^{p-\ell_1} b$; by Lemma (3.13), $h(Y_{s'}) \le -1$, and so $d(\mu(Y_{s'})) = 3$. Since $q = 1$, $y_{s'+1} \not\equiv a^p$ and so $h(Y_{s'+1}) \le -1$, absurd. Therefore $y_{s'} \equiv a^{p-\ell_1} b a^{\ell_2}$ with $1 \le \ell_2 < p$ since $r \equiv b^{-1} a^p b z$.

<u>Step (iv) 8</u> - To show that $y_{s'+1} \equiv a^p$.

Using $h(Y_s) + h(Y_{s+1}) \le 0$, from Step (iv) 7, we have $\sum_{j=1}^{s'-1} h(Y_j) \le 2$. Comparison of $r_{s'}$ and r_s shows that $a^{p-\ell_1} b$ is a piece and hence $h(Y_{s'}) \le 0$. Then $\sum_{j=1}^{s'} h(Y_j) \le 2$. If $y_{s'+1} \equiv a^{\ell_3}$, $1 \le \ell_3 < p$ then $y_{s'+2} \equiv y_{s'+3} \equiv a^p$ and Lemma (3.10) gives $h(Y_{s'+1}) \le -2$ which is impossible. So $y_{s'+1} \equiv a^p$; and of course by Lemma (3.14), $h(Y_{s'}) \le -1$.

<u>Step (iv) 9</u> - The final contradiction.

We have $\sum_{j=1}^{s'} h(Y_j) \le 1$ and if $y_{s''}$ involves b and y_j is b-free, $s' < j < s''$ then $y_j \equiv a^p$ and $h(Y_j) = 0$, $s' < j < s''$. The argument of Step (iv) 7 gives $y_{s''} \equiv a^{p-\ell_2} b a^{\ell_3}$, $1 \le \ell_3 < p$ and the argument of Step (iv) 8 (the full argument is not in fact required) gives $h(Y_{s''}) \le -1$.

Next we show that $y_1 \not\equiv a^{-p} b a^{\ell}$, where $1 \le \ell < p$; suppose not, and let s be the least j, $2 \le j \le L$ such that y_j involves b. Then $5 \le s \le L$ and for $2 \le j \le s-1$, $y_j \equiv a^p$ with $h(Y_j) = 0$

(using Lemmas (3.10) and (3.11)). Then by Lemma (3.10), $h(Y_1) = 1$ and Lemma (3.13) giving $y_s \not\equiv a^{p-\ell} b$. Thus $y_s \equiv a^{p-\ell} b a^{\ell_1}$, $1 \le \ell_1 < p$ and so $h(Y_s) = 0$ by Lemma (3.13). Hence $y_{s+1} \equiv a^p$ with $h(Y_{s+1}) = 0$. By Lemma (3.14), $h(Y_s) \le -1$, absurd. Therefore (iv) does not occur.

Assume (v) occurs with $y_1 \equiv a^{-\ell} b^{-1} a^{-p}$. We rule this out by an argument similar to that for $y_1 \equiv a^{-\ell} b^{-1} a^p$ establishing the following. In Step 1, y_s begins with $a^{-p} b$; in Steps 2 and 3, $y_s \equiv a^{-p} b a^{\ell_1}$; Step 4, $h(Y_1) + h(Y_s) \le 3$; Step 5, $d(\mu(Y_s)) = 3$; Step 6, $Y_{s+1} \equiv a^p$ (but replace r_s, Y_s by r_1, Y_1 to get ab is a piece); Step 7, $y_{s'} \equiv a^{p-\ell_1} b a^{\ell_2}$, $1 \le \ell_2 < p$. (We show $\sum_{j=1}^{s+1} h(Y_j) \le 2$, rather than $h(Y_s) + h(Y_{s+1}) \le 0$).

By almost same argument as with $y_1 \equiv a^{-p} b a^{\ell}$, we can show that $y_1 \not\equiv a^p b a^{\ell}$ and so (v) does not occur.

Assume (vi) occurs with $y_1 \equiv a^p b$. Since $h(Y_1) = 1$, $y_2 \equiv a^p$ and $h(Y_2) = 0$. So $x_{2,n(2)}$ ends in b^{-1} which means that $x_{1,2}$ must begin with b since $d(\mu(Y_1)) = 3$. Then we can write $r_1 \equiv a^p b^2 z'$ which is impossible for $q = 1$.

Now suppose $y_1 \equiv b^{-1} a^{-p}$. Let s be the least j, $2 \le j < L$ such that y_j involves b. Then $s > 4$ and for each $2 \le j \le s-1$, $y_j \equiv a^{-p}$ with $h(Y_j) = 0$. So $y_s \equiv a^{-p} b a^{\ell}$. As in Step (vi) 5, we can show that $d(\mu(Y_s)) = 3$, and as in Step (vi)6 we can show that $y_{s+1} \equiv a^p$. By Lemma (3.12), $h(Y_s) \le 1$ and so $\sum_{j=1}^{s} h(Y_j) = 2$. Thus we can apply the same argument as in (iv) (when $y_1 \equiv a^{-p} b a^{\ell}$) to show that (vii) does not occur.

(2) Let w be a word in $(a^{5p}\, b^{2q+1})^2\, a^{5p}$ and $b^{q+1}\, a^{5p}\, (b^{2q+1}\, a^{5p})^2\, b^{q+1}$ and assume w is cyclically reduced in these generators (then w is cyclically reduced in a and b).

If $w = 1$ in G, then by the nature of the chosen generators, the associated R-diagram $\mathfrak{M}$ for w has a T-path and y_1 must be one of the following:

(i) $y_1 \equiv (a^p\, b^k)^\nu$, (ii) $y_1 \equiv (b^{-k}\, a^p)^\nu$, (iii) $y_1 \equiv (b^{2q+1})^\nu$ or (iv) $y_1 \equiv (b^{2q+2})^\nu$, where $1 \le k \le q$ and $\nu = \pm 1$.

It is clear that b^q is a piece and so $\theta(b^q) = 1$.

Assume (i) occurs with $y_1 \equiv a^p\, b^k$, $1 \le k \le q$ and so $h(Y_1) = 1$. Since $d(\mu(Y_1)) = 3$, y_2 cannot be b^{k_1}, where $k_1 > q$. It follows that $y_2 \equiv b^{q+1-k}\, a^\ell$, where $1 \le \ell < p$. Hence $y_3 \equiv y_4 \equiv a^p$ with $h(Y_3) = 0$ and so, by Lemma (3.10), $h(Y_2) \le -1$, absurd. Therefore $y_1 \not\equiv a^p\, b^k$, where $1 \le k \le q$.

Now suppose that $y_1 \equiv b^{-k}\, a^{-p}$. Let s be the least j, $2 \le j \le L$ such that y_j involves $b^{\pm 1}$. Then $s > 4$ and y_s does not begin with b. (If not; then $y_i \equiv a^{-p}$, $2 \le i \le s-1$ and $h(Y_i) = 0$ since $\sum_{j=1}^{s-1} h(Y_j) = 1$. So $x_{s-1,2}$ begins with b and hence $x_{s,n(s)}$ ends with b^{-1}, absurd.)

Next we show that y_s does not begin with b^{-1}; suppose not. Then $y_s \equiv b^{-k_1}$, where $q < k_1$, i.e. $r_s \equiv b^{-n}$. Now $h(Y_s) = 0$ or 1. Since $d(\mu(Y_s)) = 3$ and $x_{s,2}$ begins with b^{-1}, $y_{s+1} \equiv a^{\varepsilon\ell}$, where $\varepsilon = \pm 1$, $1 \le \ell < p$ (compare with r to get $\ell \ne p$). Thus $h(Y_s) = 1$, $h(Y_{s+1}) = -1$. Since $\sum_{j=1}^{s+1} h(Y_j) = 1$, $y_{s+2} \equiv a^{-p}$.

By Lemma (3.10), $h(Y_{s+1}) \le -2$, absurd. Therefore y_s does not begin with b^{-1} . Hence $y_s \equiv a^{-p} b^{k_2}$, where $1 \le k_2 \le q$ and $y_{s+1} \equiv b^{k_3}$, $1 \le k_3 \le q$ and so $h(Y_{s+1}) = -1$ and $h(Y_s) = 1$.

Let s' be the least j , $s' < j \le L$ such that y_j involves b . By the same argument as with y_s we can show that $y_{s'} \equiv a^p b^{k_4}$, $1 \le k_4 \le q$ and $y_{s'+1} \equiv b^{k_5}$, $1 \le k_4 \le q$. Then $h(Y_{s'+1}) = -1$ and $h(Y_{s'}) = 1$. Since $\sum_{j=1}^{s'+1} h(Y_j) = 1$, $y_{s'+2} \equiv b^{k_6} a^{\ell_1}$, $1 \le k_6 \le q$ and $1 \le \ell_1 < p$ and $y_{s'+2} \equiv a^p$. So by Lemma (3.12), $h(Y_{s'+2}) \le -1$, absurd. Therefore (i) does not occur.

By using an argument similar to that in (i), we can rule out (ii).

Assume (iii) occurs with $y_1 \equiv b^{2q+1}$. Then $h(Y_1) = 1$. If y_2 begins with b , then $y_2 \equiv b\, a^{\ell_1}$, where $1 \le \ell_1 < p$ and so $y_3 \equiv y_4 \equiv a^p$ with $h(Y_3) = h(Y_4) = 0$. Then by Lemma (3.10), $h(Y_2) \le -1$, absurd.

Now, $y_2 \equiv a^p$. (Otherwise $h(Y_2) \le -1$, absurd.) Let s be the least j , $2 \le j \le L$ such that y_j involves b . Then $y_s \equiv a^p b^k$, $y_{s+1} \equiv b^{k_1}$, where $1 \le k$, $k_1 \le q$; it follows that $h(Y_{s+1}) = -1$ and $h(Y_s) = 1$. So $y_{s+2} \equiv b^{k_2} a^{\ell_1}$, $1 \le \ell_1 < p$, $k_2 = 2q+1 - (k+k_1)$. Since $\sum_{j=1}^{s+2} h(Y_j) = 1$, $y_{s+3} \equiv y_{s+4} \equiv a^p$ with $h(Y_{s+3}) = 0$. By Lemma (3.10), $h(Y_{s+2}) \le -1$, absurd. Therefore $y_1 \not\equiv b^{2q+1}$. Similarly $y_1 \not\equiv b^{-2q-1}$.

Finally by almost the same argument as in (iii) we can rule out (iv).

(4.8) <u>Proposition</u> Let a and b be any two generators in A such that no relator in R has the form a^m or b^n. Let b^q be the maximal power of b which appears in the relators of R, and a^p be defined as in Convention (4.2).

(1) If there is no relator r in R of the form $r \equiv a^{\varepsilon p} b^{\eta q} a^{-\varepsilon \ell} b^{-\eta k}$ or $b^{\eta q} a^{\varepsilon p} b^{-\eta k} a^{-\varepsilon \ell}$, where $1 \le \ell < p$, $1 \le k < q$, $\varepsilon, \eta = \pm 1$, then a^{3p} and b^{3q} are free generators in G.

(2) If there exists r in R such that $r \equiv a^{\varepsilon p} b^{\eta q} a^{-\varepsilon \ell} b^{-\eta k}$ or $r \equiv b^{\eta q} a^{\varepsilon p} b^{-\eta k} a^{-\varepsilon \ell}$, where $1 \le \ell < p$, $1 \le k < q$, $\varepsilon, \eta = \pm 1$, then $a^{p+1} b^{q+1} a^{p+1}$ and $b^{q+1} a^{p+1} b^{q+1}$ are free generators in G.

<u>Proof</u> Note that in (2), $\theta(a^{\varepsilon p}) = \theta(b^{\varepsilon q}) = 2$ and if $r \equiv a^{\varepsilon p} b^{\eta q} a^{-\varepsilon \ell} b^{-\eta k}$ then $b^{\eta q} a^{\varepsilon p} b^{-\eta k} a^{-\varepsilon \ell} \notin R$ since R is a C(6)-set.

(1) Let w be a word in a^{3p} and b^{3q}, and assume that w is cyclically reduced as a word in these generators. Then w is cyclically reduced in a and b.

If $w = 1$ in G, then in the associated R-diagram there is a T-path, and y_1 must be either $(a^{\varepsilon p} b^{\eta k})^{\nu}$ or $(b^{\eta q} a^{\varepsilon \ell})^{\nu}$, where $1 \le k < q$, $1 \le \ell < p$ and $\varepsilon, \eta, \nu = \pm 1$ (see Corollary (3.9)).

If $\theta(a^p) = \theta(b^q) = 1$, then $\theta(y_1) \le 2$. This is a contradiction since $\theta(y_1) \ge 3$.

We shall show that $\theta(a^p) = \theta(b^q) = 2$. Suppose that $\theta(a^p) = 1$ and $\theta(b^q) = 2$. Since $\theta(y_1) \ge 3$, y_1 must contain $b^{\eta q}$. If $y_1 \equiv b^{\eta q} a^{\varepsilon \ell}$, where $1 \le \ell \le p$ and $\varepsilon, \eta = \pm 1$, then $h(Y_1) = 1$. So $y_2 \equiv a^{\varepsilon \ell_1}$, $1 \le \ell_1 \le p$ and $h(Y_2) \le -1$, which is absurd. Hence $y_1 \equiv a^{\varepsilon \ell} b^{\eta q}$, where $1 \le \ell \le p$, $\varepsilon, \eta = \pm 1$, with $h(Y_1) = 1$. So

$y_2 \equiv b^{\eta q}$ with $h(Y_2) = 0$. Let s be the least integer, $3 \le s \le L$, such that y_s involves $a^{\pm 1}$. Then $y_j \equiv b^{\eta q}$ with $h(Y_j) = 0$, $2 \le j \le s-1$. Thus $h(Y_s) = 0$ or 1. Compare r_1 and r_2, noting that b^q is not a piece. Since $d(\mu(Y_1)) = 3$, $x_{1,2}$ must begin with $a^{-\varepsilon}$. Then $y_s \equiv b^{\eta q} a^{-\varepsilon \ell_1}$, $1 \le \ell_1 \le p$, (compare r_s and r_1 to show that $b^{\eta q}$ must be succeeded by $a^{-\varepsilon}$). So $y_{s+1} \equiv a^{-\varepsilon \ell_2}$, $1 \le \ell_2 \le p$, $h(Y_{s+1}) = -1$ and $h(Y_s) = 1$. Since $\sum_{j=1}^{s+1} h(Y_j) = 1$, $s' = s+2$ and $y_{s'} \equiv a^{-\varepsilon \ell_3} b^{\eta k}$, where $1 \le k < q$, $\eta = \pm 1$ (compare $r_{s'}$ and r_1 to show $k \ne q$). Thus $y_{s+3} \equiv b^{\eta q}$ and $h(Y_{s+2}) = h(Y_{s+3}) = 0$. By using a similar argument as in Lemma (3.12), we can show that $h(Y_{s+2}) \le -1$, which is absurd. Therefore $\theta(a^p) = \theta(b^q) = 2$, since the case $\theta(a^p) = 2$, $\theta(b^q) = 1$ is disposed of similarly.

W.l.o.g. we may assume that y_1 is one of the following: (i) $y_1 \equiv a^p b^q$, (ii) $y_1 \equiv b^{-k} a^p$, $1 \le k < q$, (iii) $y_1 \equiv a^p b^k$, $1 \le k < q$. The remaining possibilities can be dealt with by appealing to symmetry or by replacing a by a^{-1} or b by b^{-1} as a basic generating symbol.

We emphasise that a^p and b^q are not pieces.

Assume (i) occurs and so $h(Y_1) = 1$ or 2. In fact we need to consider the sequence of edges $Y_1, Y_s, Y_{s'}, Y_{s''}, Y_{s'''}, Y_t, Y_{t'}, Y_{t''}, Y_{t'''}$ in the T-path, where $1 < s < s' < s'' < s''' < t < t' < t'' < t''' < L$ (in some cases, we need not consider all of them), whose labels satisfy the following:

(1) $y_1 \equiv a^p b^q$;

(2) y_s involves $a^{\pm 1}$ and for each $2 \le j \le s-1$, y_j is $a^{\pm 1}$ free;

(3) $y_{s'}$ involves $b^{\pm 1}$ and for each $s+1 \le j \le s'-1$, y_j is $b^{\pm 1}$-free;

(4) $y_{s''}$ involves $a^{\pm 1}$ and for each $s'+1 \le j \le s''-1$, y_j is $a^{\pm 1}$-free;

(5) $y_{s'''}$ involves $b^{\pm 1}$, and for each $s''+1 \le j \le s'''-1$, y_j is $b^{\pm 1}$-free;

(6) y_t involves $a^{\pm 1}$, and for each $s'''+1 \le j \le t-1$, y_j is $a^{\pm 1}$-free;

(7) $y_{t'}$ involves $b^{\pm 1}$, and for each $t+1 \le j \le t'-1$, y_j is $b^{\pm 1}$-free;

(8) $y_{t''}$ involves $a^{\pm 1}$, and for each $t'+1 \le j \le t''-1$, y_j is $a^{\pm 1}$-free;

(9) $y_{t'''}$ involves $b^{\pm 1}$, and for each $t''+1 \le j \le t'''-1$, y_j is $b^{\pm 1}$-free.

It is clear that Y_1 and Y_s always exist.

We shall show that $h(Y_1) \ne 1$; suppose not. Thus $y_2 \equiv b^q$ and $h(Y_2) = 0$. Compare r_1 and r_2 to show that $x_{1,2}$ must begin with a^{-1} since $d(\mu(Y_1)) = 3$. Clearly, $s \ge 3$ and $h(Y_s) \ge 0$. Since $h(Y_1) = 1$, for each $2 \le j \le s-1$, $y_j \equiv b^q$ with $h(Y_j) = 0$. Compare r_1 and r_{s-1} to show that $x_{s,n(s)}$ ends in a since $d(\mu(Y_{s-1})) = 3$. So y_s does not begin with $a^{\pm 1}$ (otherwise, $y_s \equiv a^{\nu \ell}$, $1 \le \ell < p$, $\nu = \pm 1$, and $h(Y_s) \le -1$ which is absurd). Therefore y_s must begin with b and so $y_s \equiv b^q a^{-\ell}$, $1 \le \ell < p$ (compare r_s and r_1 to show $\ell \ne p$).

If $y_{s+1} \equiv a^{-\ell_1}$, $1 \le \ell_1 < p$, then $y_{s+2} \equiv a^{-p}$, $h(Y_s) = 1$,

$h(Y_{s+1}) = -1$ and $h(Y_{s+2}) = 0$. If y_{s+3} begins with a^{-p} then the argument used in the proof of Lemma (3.10) shows that either $x_{s+1,n(s+1)}$ ends in a^{-1} or one of $a^{-\ell} x_{s+1,2}$, $a^{-\ell_1} x_{s+2,2}$ is a piece. Since neither conclusion can be valid, y_{s+3} must be $a^{-\ell_2} b^{\nu k}$, where $p \neq \ell_2 = 2p - (\ell_1+\ell)$, $\nu = \pm 1$ and, after a moment's observation, $k < q$. Clearly $h(Y_{s+3}) = 0$. Since

$\sum_{j=1}^{s+3} h(Y_j) = 1$, $y_{s+4} = y_{s+5} \equiv b^{\nu q}$ with $h(Y_{s+4}) = h(Y_{s+5}) = 0$; and

Lemma (3.10) gives $h(Y_{s+3}) \leq -1$, which is absurd. Therefore $y_{s+1} \equiv a^{-p}$. We also want $y_{s+2} \equiv a^{-p}$; suppose $y_{s+2} \equiv a^{-\ell_1}$, $1 \leq \ell_1 < p$. If $\ell_1+\ell \neq p$, then we get $y_{s'} \equiv a^{-\ell_2} b^{\nu k}$, $1 \leq k < q$ with $s' = s+3$. Also $y_{s'+1} \equiv b^{\nu q}$ whence $h(Y_{s'}) \leq -1$, which is absurd. Further, we get $y_{s''} \equiv b^{\nu(q-k)} a^{\eta \ell_1}$, $\eta = \pm 1$, $1 \leq \ell_2 < p$, (compare $r_{s''}$, $r_{s'}$ and r_1 to show $\ell_2 \neq p$) and $y_j \equiv b^{\nu q}$, $s'+1 \leq j \leq s''-1$. Using Lemmas (3.12) and (3.13) we can show that $h(Y_{s'}) = 0$ and $h(Y_{s''}) \leq -1$. Thus

$\sum_{j=1}^{s''} h(Y_j) \leq 0$, which is absurd. Therefore $y_{s+2} \equiv a^{-p}$, whence

$h(Y_s) = 0$, by Lemma (3.13).

If $y_{s'}$ begins with $b^{\pm 1}$, then $y_{s'-1} \equiv a^{-(p-\ell)}$ and so

$\sum_{j=1}^{s'-1} h(Y_j) \leq 0$, which is absurd. Hence $y_{s'} \equiv a^{-(p-\ell)} b^{\nu k}$,

$1 \leq k < q$, $\nu = \pm 1$ (compare r_1, r_s and $r_{s'}$ to show $k \neq q$) and

$h(Y_{s'}) = 0$. Since $\sum_{j=1}^{s'} h(Y_j) = 1$, $y_j \equiv a^{-p}$ with $h(Y_j) = 0$,

where $s+1 \leq j \leq s'-1$. By Lemma (3.13), $h(Y_{s'}) \leq -1$, which is absurd. Therefore $h(Y_1) = 2$; so $\theta(a^p b^q) = 4$ and $\theta(ab) = 2$.

Suppose that $y_2 \equiv b^k$, $1 \leq k < q$. Then $h(Y_2)$ must be -1

and $d(\mu(Y\ )) = 3$. So $y_3 \equiv b^q$ with $h(Y_3) = 0$. Then, of course, $x_{2,2}$ does not begin with $b^{\pm 1}$. Now $y_4 \not\equiv b^q$, (using Lemma (3.10)), and so $y_4 \equiv b^{q-k}\, a^{\varepsilon \ell}$, $1 \le \ell \le p$. If $\ell < p$, then $y_5 \equiv y_6 \equiv a^{\ p}$ with $h(Y_5) = h(Y_6) = 0$ whence $h(Y_4) \le -1$ by Lemma (3.13), which is absurd. So $y_4 \equiv b^{q-k}\, a^p$ (compare r_1 and r_4 to show $\varepsilon = 1$).

Compare r_1 and r_4 , noting that a^p is not a piece. Then $x_{5,n(5)}$ ends in b^{-1} since $d(\mu(Y\ )) = 3$. So $y_5 \equiv a^{\ell_1}$, $1 \le \ell_1 < p$ (compare r_4 and r_5 to show $\ell_1 \ne p$). Thus $h(Y_5) = -1$ and $h(Y_4) = 1$. Since

$\sum_{j=1}^{s} h(Y_j) = 1$, $y_6 \equiv a^p$ with $h(Y_j) = 0$. Since $d(\mu(Y_1)) = 3$,

$x_{s,n(2)}$ ends in a and $r_2 \equiv a\, b^k\, z_2$, where z_2 does not begin with b . So $x_{1,2}$ does not begin with a^{-1} since ab is not a piece. Hence $y_7 \equiv a^{p-\ell_1}\, b^{\varepsilon k_1}$, where $1 \le k_1 < q$ and $\varepsilon = \pm 1$; (compare r_7 and r_1 to show $\varepsilon = -1$ and $k_1 \ne q$). Thus $y_8 \equiv y_9 \equiv b^{-q}$ with $h(Y_8) = h(Y_9) = 0$. By Lemma (3.12), $h(Y_7) \le -1$, which is absurd. All our possibilities have led to contradiction and therefore $y_2 \equiv b^q$.

Next we show that $h(Y_2) = 0$. Suppose that $h(Y_2) = -1$. Then $y_s \equiv b^q\, a^{-\ell}$, $1 \le \ell < p$, $h(Y_s) = 0$ or 1 , and $y_j \equiv b^q$ with $2 \le j \le s-1$. If $y_{s+1} \equiv a^{-\ell_1}$, $1 \le \ell_1 < p$, then $h(Y_{s+1}) = -1$, $h(Y_s) = 1$ and $y_{s+2} \equiv a^{-p}$; and if y_{s+3} begins with a^{-p} then $h(Y_{s+1}) \le -1$, which is absurd. Hence $s' = s+3$ with $y_{s'} \equiv a^{-\ell_2}\, b^{\nu k}$, where $p \ne \ell_2 = 2p - (\ell + \ell_1)$, $\nu = \pm 1$, and $k < q$. So $y_{s+4} \equiv y_{s+5} \equiv b^q$ and Lemma (3.10) gives a contradiction. Hence $y_{s+1} \equiv a^{-p}$. If $y_{s+2} \equiv a^{-\ell_1}$, $1 \le \ell_1 < p$, then $s' = s+3$, $y_{s'} \equiv a^{-p}\, b^k$, $1 \le k < q$. It is clear that $x_{s+2,2}$ does not begin with a^{-1}. Since $d(\mu(Y_{s+1})) = 3$, x_{s+1} begins with b. (Compare r_{s+1} with $r_{s'}$, noting that a^{-p} is not a piece.) So $r_{s+2} \equiv b^{-1} a^{-\ell_1} z_{s+2}$ where z_{s+2} does not begin with a^{-1}. But $b^{-1} a^{-1}$ is not a piece. Compare r_{s+1} and r_1 to get $\ell_1 = p$, which is a contradiction since $\ell_1 < p$. Therefore $h(Y_2) = 0$.

Knowing that $y_2 \equiv b^q$, we see that $x_{1,2}$ and $x_{2,2}$ begin with a^{-1}, whence $x_{3,n(3)}$ ends in a. The possibility that $y_3 \equiv b^k$, $1 \le k < q$, in which case $d(\mu(Y_3)) = 3$, is immediately rejected by comparison with r_1. Thus for each $2 \le j \le s-1$, $y_j \equiv b^q$, and the same argument as with y_2 gives $h(Y_j) = 0$, $2 \leqslant j \leqslant s-1$. We shall consider two cases.

<u>Case 1</u>: Suppose that $x_{1,3}$ ends in b^{-1} (i.e. the unique relator r in R which involves a^p can be written as $r \equiv b^{-1} a^p z$).

We shall show that this situation leads to a contradiction in the following manner. We shall construct an infinite sequence $L_o, L_1, L_2, \ldots$ of integers such that for each $f = 0,1,2, \ldots$ the following hold:

(a) $1 \le L_f < L$;

(b) $y_{L_f} \equiv a^p b^q$;

(c) $\sum_{j=1}^{L_f} h(Y_j) = 2$;

(d) $L_f < L_{f+1}$.

Conditions (a) and (b) give the required contradiction.

We take $L_o = 1$. Now, the argument which constructs L_1 from

L_o will depend only on the three facts

(b_o) $y_{L_o} \equiv a^p b^q$,

(c_o) $\sum_{j=1}^{L_o} h(Y_j) = 2$,

$(*)$ $\theta(ab) = 2$ and a^p is preceeded by b^{-1} .

Since $(*)$ is valid, independently of f , the same argument, verbatim, will show how L_{f+1} is constructed from L_f .

We divide Case 1, according to the form of y_s , into two subcases.

Subcase 1: Suppose that y_s begins with $a^{\pm 1}$. Then $s \geq 4$, $y_s \equiv a^{\ell}$, $1 \leq \ell < p$ and $h(Y_s) = -1$. (Compare r_1, r_2 and r_{s-1} to show that $x_{s,n(s)}$ ends in a ; and by the maximality of p , $\ell \neq p$). So $y_{s+1} \equiv y_{s+2} \equiv a^p$ with $h(Y_{s+1}) = h(Y_{s+2}) = 0$. Then $y_{s'} \equiv a^{p-\ell} b^{\varepsilon k}$, $1 \leq k \leq q$, $\varepsilon = \pm 1$. Since ab is not a piece and $x_{s',n(s')}$ does not end in a , $\varepsilon = -1$; and Lemma (3.13) gives that $k = q$. By Lemma (3.13), $h(Y_{s'}) = 0$ and so $\sum_{j=1}^{s'} h(Y_j) = 1$. Thus $y_{s'+1} \equiv b^{-q}$ with $h(Y_{s'+1}) = 0$. Further, we get $y_{s''} \equiv b^{-q} a^{-p}$.

We shall prove that $d(\mu(Y_{s''})) = 3$. If $d(\mu(Y_{s''}) > 3$ then, clearly, $d(\mu(Y_{s''})) = 4$ and so $h(Y_{s''}) = 0$. Since $\sum_{j=1}^{s''} h(Y_j) = 1$, $y_{s''+1} \equiv a^{-p}$ and $h(Y_{s''+1}) = 0$. Moreover, for for each $s''+1 \leq j \leq s'''-1$, $y_j \equiv a^{-p}$ and $h(Y_j) = 0$. Since $d(\mu(Y_{s'''-1}) = 3$, $x_{s'''-1,2}$ must begin with b and hence $x_{s''',n(s''')}$

ends in b^{-1}. By the maximality of q, $y_{s'''} \not\equiv b^{-q}$. Thus $y_{s'''} \equiv a^{-p} b^{k}$, $1 \lesssim k < q$. Suppose that $y_{s'''+1}$ b^{k} , 1 k q. Then $h(Y_{s'''+1}) = -1$ and $y_{s'''+2}$ b^{q}. By Lemma (3.10) $h(Y_{s'''+1}) \leq -2$, which is absurd. Then $y_{s'''+1} \equiv b^{q}$ with $h(Y_{s'''+1}) = 0$. It follows that $y_{t} \equiv b^{q-k} a^{\varepsilon \ell_1}$, $1 \leq \ell_1 < p$; and so Lemma (3.13) gives that $h(Y_{t}) \leq -1$ which is impossible. Therefore $d(\mu(Y_{s''})) = 3$.

Suppose that $h(Y_{s''}) = 2$. We write $r_{s''} \equiv a^{p} b^{q} x_{s'',2} x_{s'',3}$. Since $d(\mu(Y_{s'-1})) = 3$ and $y_{s'-1} \equiv a^{p}$, $x_{s'-1,2}$ begins with b and so $x_{s',n(s')}$ ends in b^{-1}. Now, we can write $r_{s'} \equiv b^{-k'} a^{p-\ell} b^{-q} a^{-p} b^{\ell'} z_{s'}$, where $z_{s'}$ does not begin with b and does not end in b^{-1}, and $1 \leq k'$, $\ell' < q$. Comparing $r_{s''-1}$ and $r_{s''}$ gives that $x_{s'',3} \equiv a^{\ell_1}$, $1 \leq \ell_1 < p$ since $d(\mu(Y_{s''-1})) = 3$. Also comparing $r_{s'}$ and $r_{s''}$ gives that $\ell_1 \leq p-\ell$ and $x_{s'',2} \equiv b^{\ell'} z_{s'} b^{-k'} a^{p-(\ell-\ell_1)}$. Since $d(\mu(Y_{s''})) = 3$, $x_{s''+1,n(s''+1)} \equiv a^{-p+\ell+\ell_1} b^{k'} z_{s'}^{-1} b^{-\ell'}$ and hence $r_{s''+1}$ contains $b^{-1} a^{-1}$ which is not a piece. Compare $r_{s''+}$ and $r_{s''}$, noting that $z_{s'}^{-1}$ does not end in b^{-1} and $1 \leq \ell' < q$. This is impossible since ab is not a piece. Thus $h(Y_{s''}) \neq 2$.

Next we need to show that $y_{s''+1} \equiv a^{-p}$; suppose that $y_{s''+1} \equiv a^{-\ell_1}$, $1 \leq \ell_1 < p$. Compare r_1 and $r_{s''}$ noting that a^{p} is not a piece. Then $x_{s'',2}$ begins with b. Since $d(\mu(Y_{s''})) = 3$, $x_{s''+1,n(s''+1)}$ ends in b^{-1}. Now, $x_{s''+1,2}$ does not begin with a^{-1} since $y_{s''+2}$ begins with a^{-1}; and it follows that $\ell_1 = p$, (since ab is not a piece), which is absurd. Therefore $y_{s''+1} \equiv a^{-p}$. Since $r_{s''+1}$ contains $a^{-p} b$, an easy iterative argument shows that for each $s''+2 \leq j \leq s'''-1$, r_j contains $b^{-1} a^{-1}$ and therefore $y_j \equiv a^{-p}$.

Suppose that $y_{s'''}$ begins with a^{-1}; and so $y_{s'''} \equiv a^{-p} b^{k}$, where $1 \leq k < q$. If $d(\mu(Y_{s'''})) > 3$, then $d(\mu(Y_{s'''})) = 4$; and so $\sum_{j=1}^{s'''} h(Y_j) = 1$. Thus for each $s'''+1 \leq j \leq t-1$, $y_j \equiv b^q$ with $h(Y_j) = 0$. Hence $y_t \equiv b^{q-k} a^{\nu\ell}$, $1 \leq \ell < p$ (compare r_t, r_1 and $r_{s'''}$ to show $\ell \neq p$). By Lemma (3.11), $h(Y_t) \leq -1$, which is absurd. Therefore $d(\mu(Y_{s'''})) = 3$.

Next we need to show that $y_{s'''+1} \equiv b^q$; suppose that $y_{s'''+1} \equiv b^{k_1}$, $1 \leq k_1 < q$. If $y_{s'''+2} \equiv b^{k_2}$, $1 \leq k_2 < q$, then $h(Y_{s'''+1}) = h(Y_{s'''+2}) = -1$ and $h(Y_{s'''}) = 1$. If $y_{s'''+1}$ begins with b^q, then a contradiction can be easily obtained. So $y_{s'''+3} \equiv b^{k_3} a^{\nu\ell_1}$, where $q \neq k = 3q - (k+k_1+k_2)$, $1 \leq \ell_1 < p$ and $\nu = \pm 1$ (compare $r_{s'''+3}$, $r_{s'''}$, r_1 to show $\ell_1 \neq p$). Thus $y_{s'''+4} \equiv a^{\nu p}$ with $h(Y_{s'''+4}) = 0$ and Lemma (3.12) gives a contradiction. Hence $y_{s'''+2} \equiv b^q$ and so $h(Y_{s'''+1}) = -2$. Since $\sum_{j=1}^{s'''+1} h(Y_j) = 1$, $y_j \equiv b^q$ with $h(Y_j) = 0$, where $s'''+2 \leq j \leq t-1$. Now, if $k+k_1 \neq p$ then $y_t \equiv b^{k_2} a^{\nu\ell}$, where $k_2 = 2p - (k_1+k)$ or $k_2 = p - (k_1+k)$, $1 \leq \ell < p$ and $\nu = \pm 1$. So $y_{t+1} \equiv a^{\nu p}$ and Lemma (3.12) gives a contradiction. Thus $k+k_1 = p$ and so $y_t \equiv b^q a^{-\ell_1}$, $1 \leq \ell_1 < p$. By Lemma (3.10), $h(Y_{s'''+2}) = -2$, hence $\sum_{j=1}^{s'''+1} h(Y_j) = 1$. It is easy to see that $y_j \equiv a^{-p}$, $t+1 \leq j \leq t'-1$. Then $y_{t'} \equiv a^{-(p-\ell_1)} b^{\nu k_2}$, $1 \leq k_2 < q$. By Lemma (3.13), $h(Y_{t'}) = -1$ and so $y_{t'+1} \equiv b^{-\nu q}$. By the same arguments as in Lemmas (3.12) and (3.13) we can show that $x_{t',n(t')} a^{-(p-\ell_1)}$ and $b^{\nu k_2} x_{t,2}$ are pieces. Then $h(Y_{t'}) \leq -2$, which is absurd. Therefore $y_{s'''+1} \equiv b^q$ and so $h(Y_{s'''}) = 0$.

Since $r_{s'''+1}$ contains $b^q a^{-1}$, an easy iterative argument shows that for each $s'''+2 \le j \le t-1$, r_j contains ab and therefore $y_j \equiv b^q$. Thus $y_t \equiv b^{q-k} a^{\nu\ell_1}$, $1 \le \ell_1 < p$, $\nu = \pm 1$. So $y_{t+1} \equiv a^{\nu p}$ and by the same argument as in Lemmas (3.12) and (3.13) we can show that $x_{t,n(t)} b^{q-k}$ and $a^{\nu\ell_1} x_{t,2}$ are pieces. Then $h(Y_t) \le -2$, which is absurd. Therefore $y_{s'''}$ does not begin with a^{-1}. Thus $y_{s'''} \equiv b^{-k}$, $1 \le k < q$ (since $x_{s''',n(s''')}$ must end in b^{-1}, $k \ne q$). So $y_j \equiv b^{-q}$ and $h(Y_j) = 0$, $s'''+1 \le j \le t-1$ since $\sum_{j=1}^{s'''} h(Y_j) = 1$. Thus $y_t \equiv b^{-(q-k)} a^p$ and $h(Y_t) = 0$. So $y_j \equiv a^p$ and $h(Y_j) = 0$, $t+1 \le j \le t'-1$. It is easy to see that $y_{t'} \equiv a^p b^q$.

We shall show that $d(\mu(Y_{t'})) = 3$; suppose not. Then $d(\mu(Y_{t'})) = 4$ and $h(Y_{t'}) = 0$. Since $\sum_{j=1}^{t'} h(Y_j) = 1$, $y_i \equiv b^q$ and $h(Y_i) = 0$, $t'+1 \le i \le t''-1$. It follows that $y_{t''} \equiv b^q a^{-\ell_1}$, $1 \le \ell_1 < p$, $y_j \equiv a^{-p}$ for each $t''+1 \le j \le t'''-1$ and so $h(Y_{t''}) = 0$. Thus $h(Y_{t'''}) \equiv a^{-(p-\ell_1)} b^{\nu k_1}$, $1 \le k_1 < q$. By Lemma (3.13), $h(Y_{t'''}) \le -1$, which is absurd. Therefore $d(\mu(Y_{t'})) = 3$.

Since $x_{t',2}$ begins with a^{-1}, $y_{t'+1} \equiv b^{k_1}$, $1 \le k_1 < q$ and so $y_{t'+1} \equiv b^q$. Compare $r_{t'-1}$, $r_{t'}$ and $r_{t'+1}$ noting that a^p and b^q are not pieces. Then $x_{t',2} \equiv a^{-\ell'}$, $1 \le \ell' < p$ and $x_{t',n(t')} \equiv b^{-k'}$, $1 \le k' < q$. If $h(Y_{t'}) = 2$, then $r_{t'}$ can be written as $r_{t'} \equiv a^p b^q a^{-\ell'} b^{-k'}$, where $1 \le \ell' < p$ and $1 \le k' < q$ which is impossible. Therefore $h(Y_{t'}) = 1$.

Since $\sum_{j=1}^{t'} h(Y_j) = 2$, we can take $L_1 = t'$. It is clear that the construction of L_1 did not depend on the fact that $L_o = 1$, but only on the facts that the sum up to (and including) $h(Y_{L_o})$ is 2, that ab is not a piece, and that a^p is preceeded by b^{-1}.

Subcase 2: Suppose that y_s begins with $b^{\pm 1}$. Then $y_s \equiv b^{\nu q}a^{-\ell}$, $1 \leq \ell \leq p$, since for each $2 \leq j \leq s-1$, $y_j \equiv b^q$.

We shall firstly show that $d(\mu(Y_s)) = 3$; suppose not. Then $d(\mu(Y_s)) = 4$ and so for each $s+1 \leq j < s'-1$, $y_j \equiv a^{-p}$ and $h(Y_j) = 0$. Thus $y_{s'} \equiv a^{-(p-\ell)}\, b^{\nu k}$, $1 \leq k < q$ and $\nu = \pm 1$ (compare r_s, r_1 and r_2 to get $k \neq q$) and so Lemma (3.13), gives a contradiction. Therefore $d(\mu(Y_s)) = 3$.

Suppose that $y_{s+1} \equiv a^{-\ell_1}$, $1 \leq \ell_1 < p$. If $y_{s+2} \equiv a^{-\ell_2}$, $1 \leq \ell_2 < p$, then $h(Y_s) = 1$ and $h(Y_{s+1}) = h(Y_{s+2}) = -1$. If $y_{s'}$ begins with $b^{\pm 1}$ then $s' > s+3$ and so $y_{s'-1} \equiv a^{-p}$, whence $y_{s'} \equiv b^{\nu k}$, $1 \leq k < q$, giving $h(Y_{s'}) \leq -1$, which is absurd. So $y_{s'}$ begins with a^{-1}; hence $s' > s+2$. If $y_{s'} \equiv a^{-\ell_3}\, b^{\nu k}$, $1 \leq \ell_3 < p$, then we can see that $k < q$ and so $y_{s'+1} \equiv y_{s'+2}$ $b^{\nu q}$ giving $h(Y_{s'}) \leq -1$, again absurd. So $y_{s'} \equiv a^{-p}\, b^{\nu k}$, $1 \leq k < q$. Then $s' = s+3$ and $\ell+\ell_1+\ell_2 = 2p$ (otherwise, use Lemma (3.10) to give a contradiction). Making the usual comparison with parts of r_2 we can show firstly $y_{s'+1} \equiv b^{\nu q}$ and hence $h(Y_{s'}) = 0$.

If $\nu = -1$, then $y_{s'+1} \equiv b^{-q}$ and $x_{s'+1,n(s'+1)}$ ends in a; and so $x_{s',2}$ begins with a^{-1} since $d(\mu(Y_{s'})) = 3$. Compare $r_{s'}$ and r_1 noting that ab is not a piece. Then $k = q$, which is absurd. Then $\nu = 1$ and so $y_{s''} \equiv b^{q-k}\, a^{\eta\ell_1}$, $1 \leq \ell_1 < p$, $\eta = \pm 1$ (compare $r_{s''}$, $r_{s'}$ and r_1 to show $\ell_1 \neq p$). By Lemma (3.13), $h(Y_{s''}) \leq -1$ and so

$\sum_{j=1}^{s''} h(Y_j) \le 0$, which is absurd. Therefore $y_{s+2} \equiv a^{-p}$ which forces $\ell+\ell_1 = p$ and $y_{s'} \equiv a^{-p} b^k$. In this situation $h(Y_{s+1}) + h(Y_{s+2}) = -2$ and then the same argument as above shows that $h(Y_{s''}) \le -1$. So we can conclude that $y_{s+1} \equiv a^{-p}$.

An iterative method will show that for each $s+2 \le j \le s'-1$, $y_j \equiv a^{-p}$. Suppose that $y_{s'}$ begins with a^{-1}. Then $y_{s'} \equiv a^{-(p-\ell)} b^{\nu k}$, $1 \le k < q$. If $s' > s+3$, then we can apply Lemmas (3.12) and (3.13) to show that $h(Y_s) = 0$ and $h(Y_{s'}) = -1$; and so $\sum_{j=1}^{s'} h(Y_j) = 1$. Hence $y_{s'+1} \equiv b^{\nu q}$, and a similar argument as in Lemmas (3.12) and (3.13) shows that $x_{s',n(s')}\ a^{-(p-\ell)}$ and $b^{\nu k}\ x_{s',2}$ are pieces, and so $h(Y_{s'}) \le -2$, which is absurd. Thus $s' = s+3$.

We claim that $h(Y_{s+1}) + h(Y_{s+2}) = 0$; suppose not. Then $h(Y_{s+1}) + h(Y_{s+2}) = -2$ or -1. If $h(Y_{s+1}) + h(Y_{s+2}) = -2$, then $y_{s'+1} \equiv y_{s'+2}\ b^{\nu q}$ with $h(Y_{s'+1}) = 0$; and so Lemma (3.12) gives that $h(Y_{s'}) \le -1$, which is absurd. If $h(Y_{s+1}) + h(Y_{s+2}) = -1$, then $h(Y_s) + h(Y_{s+1}) + h(Y_{s+2}) + h(Y_{s'}) = -1$ and so $\sum_{j=1}^{s'} h(Y_j) = 1$. Thus $y_{s'+1} \equiv y_{s'+2} \equiv b^{\nu q}$ with $h(Y_{s'+1}) = 0$; and again we have a contradiction. Therefore $h(Y_{s+1}) + h(Y_{s+2}) = 0$.

By Lemmas (3.12) and (3.13), $h(Y_s) = 0$, $h(Y_{s'}) = -1$ and so $\sum_{j=1}^{s'} h(Y_j) = 1$. Thus $y_{s'+1} \equiv b^{\nu q}$ and a contradiction can be easily obtained. Therefore $y_{s'}$ must begin with $b^{\pm 1}$ and $y_{s'-1} \equiv a^{-(p-\ell_1)}$.

If $y_{s'} \equiv b^q$, then $y_{s''} \equiv b^q a^{-\ell_1}$, $1 \le \ell_1 < p$ and so

$y_{s'''-1} \equiv a^{-(p-\ell_1)}$, $h(Y_{s''}) = 0$ and $h(Y_{s'''-1}) \le -1$, which is absurd. Therefore $y_{s'} \equiv b^{-q}$. Thus $h(Y_s) = 0$, $h(Y_{s'-1}) = -1$ and $\sum_{j=1}^{s'} h(Y_j) = 1$. It follows that $y_{s''} \equiv b^{-q} a^{-p}$ and for each $s'+1 \le j \le s''-1$, $y_j \equiv b^{-q}$ with $h(Y_j) = 0$. It is easy to see that $d(\mu(Y_{s''})) = 3$. Then $y_{s''+1} \equiv a^{-p}$ since $x_{s'',2}$ begins with b . If $h(Y_{s''}) = 2$, then $r_{s''}$ can be written as $r_{s''} \equiv b^{-q} a^{-p} b^{k*} a^{\ell*}$, $1 \le k* < q$, $1 \le \ell* < p$ which is impossible. Therefore $h(Y_{s''}) = 1$.

We observed in Subcase 1 that from the point where we established $h(Y_{s''}) = 1$, we used the following facts: $y_1 \equiv a^p b^q$, $y_2 \equiv b^q$ with $h(Y_2) = 0$, y_3 begins with b^q , ab is not a piece and $x_{1,3}$ ends in b^{-1} to construct L_1 . Since all these facts are valid, the argument goes exactly as for Subcase 1.

Case 2 $x_{1,3}$ does not end in b^{-1} (i.e. a^p is not preceeded by b^{-1}).

We shall show that our assumption in this case also leads to a contradiction, by using an analogous method to that in Case 1, i.e. we construct an infinite sequence $L_o, L_1, \ldots,$ of integers such that for each $f = 0,1,2, \ldots,$ the following are valid:

(a') $1 < L_f < L$;

(b') $y_{L_f} \equiv b^{-q} a^{-p}$;

(c') $\sum_{j=1}^{L_f} h(Y_j) = 3$;

(d') $L_f < L_{f+1}$.

Again conditions (a') and (d') give the required contradiction.

We take $L_o = s''$ and so the argument which constructs L_1 from L_o will depend only on the following facts:

(a*) $y_1 \equiv a^p b^q$;

(b*) $y_{L_o} \equiv b^{-q} a^{-p}$;

(c*) $\sum_{j=1}^{L_o} h(Y_j) = 3$;

(d*) $\theta(ab) = 2$ and a^p is not preceeded by b^{-1}.

Then the same argument will show how L_{f+1} is constructed from L_f since (d*) is valid independently of f.

We shall firstly show that $y_s \equiv b^q a^{-\ell}$, $1 \le \ell < p$. Since for each $2 \le j \le s-1$, $y_j \equiv b^q$, $x_{s-1,2}$ begins with a^{-1} and so $x_{s,n(s)}$ ends in a. If y_s begins with $a^{\pm 1}$, then $y_s \equiv a^{\ell}$, $1 \le \ell < p$ and so $h(Y_s) = -1$. Since $\sum_{j=1}^{s} h(Y_j) = 1$, $y_{s+1} \equiv a^p$ with $d(\mu(Y_{s+1})) = 3$. Compare r_{s+1} and r_1 noting that a^p is not a piece. Then $x_{s+1,2}$ begins with b and so $x_{s+2,n(s+2)}$ ends in b^{-1}. Thus $y_{s+2} \equiv a^{-\ell_2}$, $1 \le \ell_1 < p$ and $h(Y_{s+2}) \le -1$, which is absurd. Therefore $y_s \equiv b^q a^{-\ell}$.

Suppose that for each $s+1 \le j \le s'-1$, $y_j \not\equiv a^{-p}$, then $y_{s'} \equiv a^{-\ell'} b^{\nu k}$, $1 \le k < q$, $h(Y_{s'}) = 0$ and $\sum_{j=1}^{s'} h(Y_j) = 1$. So $y_{s'+1} \equiv b^{\nu q}$ and Lemma (3.12) gives a contradiction. Therefore there exists j, $s+1 \le j \le s'-1$, such that $y_j \equiv a^{-p}$.

Next we show that for each $s+1 \leq j \leq L$, there are no two consecutive edges Y_j, Y_{j+1} with y_j ends in $a^{\varepsilon p}$, y_{j+1} begins with $a^{\varepsilon p}$ and $d(\mu(Y_j)) = 3$; suppose not. Compare r_1 and r_{j+1}, noting that a^{-p} is not a piece. Then $x_{j+1,n(j+1)}$ ends in b^{-1} and so $x_{j,2}$ begins with b since $d(\mu(Y_j)) = 3$. This is a contradiction. Therefore if y_j ends in a^{-p} with $d(\mu(Y_j)) = 3$, then $y_{j+1} \not\equiv a^{-p}$.

It is easy to see that $y_{s'} \equiv b^{-q}$ and $\sum_{j=1}^{s'} h(Y_j) = 1$. Then

$y_{s'+1} \equiv b^{-q}$, $y_{s''} \equiv b^{-q}\, a^{-p}$ and $\sum_{j=1}^{s''} h(Y_j) \leq 3$. If

$d(\mu(Y_{s''})) > 3$, then $d(\mu(Y_{s''})) = 4$; and so $y_{s''+1} \equiv a^{-p}$. Hence $y_{s''+2} \equiv a^{-\ell_1}$, $1 \leq \ell_1 < p$ and so $h(Y_{s''+2}) \leq -1$, which is absurd. Therefore $d(\mu(Y_{s''})) = 3$. Thus $y_{s''+1} \equiv a^{-\ell_1}$, $1 \leq \ell_1 < p$; and in the usual way we can show that $y_{s''+2} \equiv a^{-p}$, $y_{s'+3} \equiv a^{-(p-\ell_1)}$, $h(Y_{s''+1}) = h(Y_{s''+3}) = -1$ and $h(Y_{s''+2}) = 0$. It is not difficult to see that $s''' = s''+4$ and $y_{s'''} \equiv b^{-q}$. So $y_{s'''+1} \equiv b^{-q}$ with

$h(Y_{s'''+1}) = 0$. It follows that $y_t \equiv b^{-q}\, a^{-p}$ and $\sum_{j=1}^{t} h(Y_j) = 3$.

We take $L_1 = t$. Hence the construction of y_{L_1} did not depend on the fact that $L_0 = s''$, but only on the fact that the sum up to (and including) $h(Y_1)$ is 3, ab is not a piece, and a^p is not preceeded by b^{-1}. Therefore $y_1 \not\equiv a^p\, b^q$.

Assume (ii) occurs, i.e. $y_1 \equiv b^{-k}\, a^p$, $1 \leq k < q$. Hence $h(Y_1) = 1$; and so for each $2 \leq j \leq s-1$, $y_i \equiv a^p$ where s is the least integer, $3 \leq s \leq L$ such that y_s involves $b^{\pm 1}$. Thus $y_s \equiv a^p\, b^q$.

Again we need to consider the sequence of edges $Y_1, Y_s, Y_{s'}, Y_{s''}, Y_{s'''}, Y_t$; note the variation from before, which is due to the assumption that y_1 ends with b rather than a.

Suppose that ab is a piece, and so $h(Y_s) = 1$ or 0. Then for each $s+1 \le j \le s'-1$, $y_j \equiv b^q$ with $h(Y_j) = 0$. So, in the usual way, $y_{s'} \equiv a^{\ell}$, $1 \le \ell < p$, $y_{s''} \equiv a^{p-\ell} a^{-q}$, $s'' > s'+2$, $h(Y_{s'}) = -1$, $h(Y_{s''}) = 0$ and for each $s'+1 \le j \le s''-1$, $y_j \equiv a^p$ with $h(Y_j) = 0$. Thus $s''' > s''+1$, $y_{s'''} \equiv b^{-q} a^{-p}$ and for each $s''+1 \le j \le s'''-1$, $y_j \equiv b^{-q}$ with $h(Y_j) = 0$, whence $t > s'''+1$, $y_t \equiv a^{-p} b^{k_1}$, $h(Y_t) = 0$, $y_j \equiv a^p$ with $h(Y_j) = 0$ for each $s'''+1 \le j \le t-1$ and $\sum_{j=1}^{t} h(Y_j) \le 2$. Finally, $t' > t+2$, $y_{t'} \equiv b^{q-k} a^{\nu\ell}$, $1 \le \ell < p$ and $\nu = \pm 1$. So for each $t+1 \le j \le t'-1$, $y_j \equiv b^q$ with $h(Y_j) = 0$. Hence $h(Y_{t'}) = -1$ and $\sum_{j=1}^{t'} h(Y_j) = 1$ whence $y_{t'+1} \equiv a^{\nu p}$, $\nu = \pm 1$. Then we can show that $a^{\nu\ell} x_{t',2}$ and $x_{t',n(t')} b^{q-k}$ are pieces and so $h(Y_{t'}) \le -2$, which is absurd. Therefore ab is not a piece.

Suppose that $h(Y_s) = 2$. So we can write $r_s \equiv b^q a^p x_{s,2} x^{-1}_{x-1,2}$. Now, $\sum_{j=1}^{s} h(Y_j) \le 3$. Then in the usual way we can show that $d(\mu(Y_s)) = 3$, $y_{s+1} \equiv b^q$; it follows that we can write r_s as $a^p b^q a^{-\ell_1} b^{-k_1}$, $1 \le \ell_1 < p$ and $1 \le k_1 < q$ which is impossible. Therefore $h(Y_s) \ne 2$.

Suppose that $h(Y_s) = 0$. Then $\sum_{j=1}^{s} h(Y_j) = 1$ and so $y_{s+1} \equiv b^q$. So $s' > s+1$ and $y_{s'} \equiv b^q a^{-\ell}$, $1 \le \ell < p$ whence $s'' > s'+1$, $y_{s''} \equiv a^{-(p-\ell)} b^{\nu k}$, $1 \le k < q$, $\nu = \pm 1$, $h(Y_{s'}) = 0$

and for each $s'+1 \le j \le s''-1$, $y_j \equiv a^{-p}$ with $h(Y_j) = 0$. It follows that $h(Y_{s''}) \le -1$ and so $\sum_{j=1}^{s''} h(Y_j) \le 0$, which is absurd. Therefore $h(Y_s) \ne 0$ and so $h(Y_s) = 1$.

Now, $\sum_{j=1}^{s} h(Y_j) = 2$, $y_s \equiv a^p\, b^q$, ab is not a piece and a^p is preceeded by b^{-1}. So we may use the same argument as in Subcase 1 of Case 1 in (i) to get a contradiction.

Assume (iii) occurs, i.e. $y_1 \equiv a^p\, b^k$, $1 \le k < q$. So $h(Y_1) = 1$, $y_2 \equiv y_3 \equiv b^q$ with $h(Y_2) = h(Y_3) = 0$ and whence Lemma (3.12) gives a contradiction.

(2) W.l.o.g. we may assume that $r \equiv a^p\, b^q\, a^{-\ell}\, b^{-k}$, $1 \le \ell < p$ and $1 \le k < q$. So $\theta(a^p\, b^q) = 4$ and $\theta(ab) = 2$.

Let w be a word in $(a^{2p+1}\, b^{2q+1})^3\, a^{2p+1}$ and $(b^{2q+1}\, a^{2p+1})^3\, b^{2q+1}$ and assume w is cyclically reduced as a word in these generators. Then it is clear that w is cyclically reduced in a and b.

If $w = 1$ in G then, by the nature of our chosen generators in the associated R-diagram $\mathcal{M}$ for w, there is a T-path, and y_1 must be one of the following: (i) $y \quad a^p\, b^q$, (ii) $y_1 \equiv a^p\, b^{\ell}$, $1 \le \ell' < q$, (iii) $y_1 \equiv a^{\ell'}\, b^q$, $1 \le \ell' < p$, (iv) $y_1 \equiv b^{-k'}\, a^p$, $1 \le k' < k$ or (v) $y_1 \equiv b^q\, a^{-\ell'}$, $1 \le \ell' < \ell$.

Assume (i) occurs, i.e. $y_1 \equiv a^p\, b^q$. Then $h(Y_1) = 1$ or 2. Suppose $y_2 \equiv b^{k_1}$, $1 \le k_1 < q$. Then $x_{2,2}$ does not begin with b since y_3 begins with b and $d(\mu(Y_2)) = 3$. It is clear that $x_{1,2}$ begins with a^{-1} and so $x_{2,n(2)}$ ends in a. So we can write $r_2 \equiv a\, b^{k_2} z_2$ where z_2 does not begin with b, which is absurd. Therefore $y_2 \equiv b^q$. Suppose that y_4 begins with b. From the nature of our chosen generators, we can show that $y_3 \equiv y_4 \equiv b^q$ and $y_5 \equiv b^2$.

If $q \geq 3$, then $x_{s,2}$ must begin with b and $y_6 \equiv y_7 \equiv a^p$ with $h(Y_6) = h(Y_7) = 0$. Thus $y_8 \equiv a\, b^k$, $1 \leq k \leq q$ which is a contradiction since $x_{8,n(8)}$ does not end in a. So $q = 2$ and $x_{s,2}$ begins with a^{-1}. Thus $y_6 \equiv a^{\ell_1}$, $1 \leq \ell_1 < p$ and $h(Y_6) = -1$. So $y_7 \equiv a^p$ with $h(Y_7) = 0$. It follows that $y_8 \equiv a^p b^2$ which forces $\ell_1 = 1$, i.e. $y_6 \equiv a$. By the same argument as with y_2, $y_4 \equiv b^2$. Since ab is not a piece, $y_{10} \equiv b$ and $x_{10,2}$ must begin with b. So $y_{11} \equiv y_{12} \equiv a^p$, $y_{13} \equiv a$ and $x_{13,2}$ must begin with a. Thus $h(Y_1) = h(Y_8) = 2$, $h(Y_6) = h(Y_{10}) = h(Y_{13}) = -1$ and $\sum_{j=1}^{13} h(Y_j) = 1$. So $y_{14} \equiv y_{15} \equiv b^q$. It follows that $h(Y_{16}) \leq -1$, which is absurd. Therefore y_4 does not begin with b. Obviously, y_4 does not begin with b^{-1}.

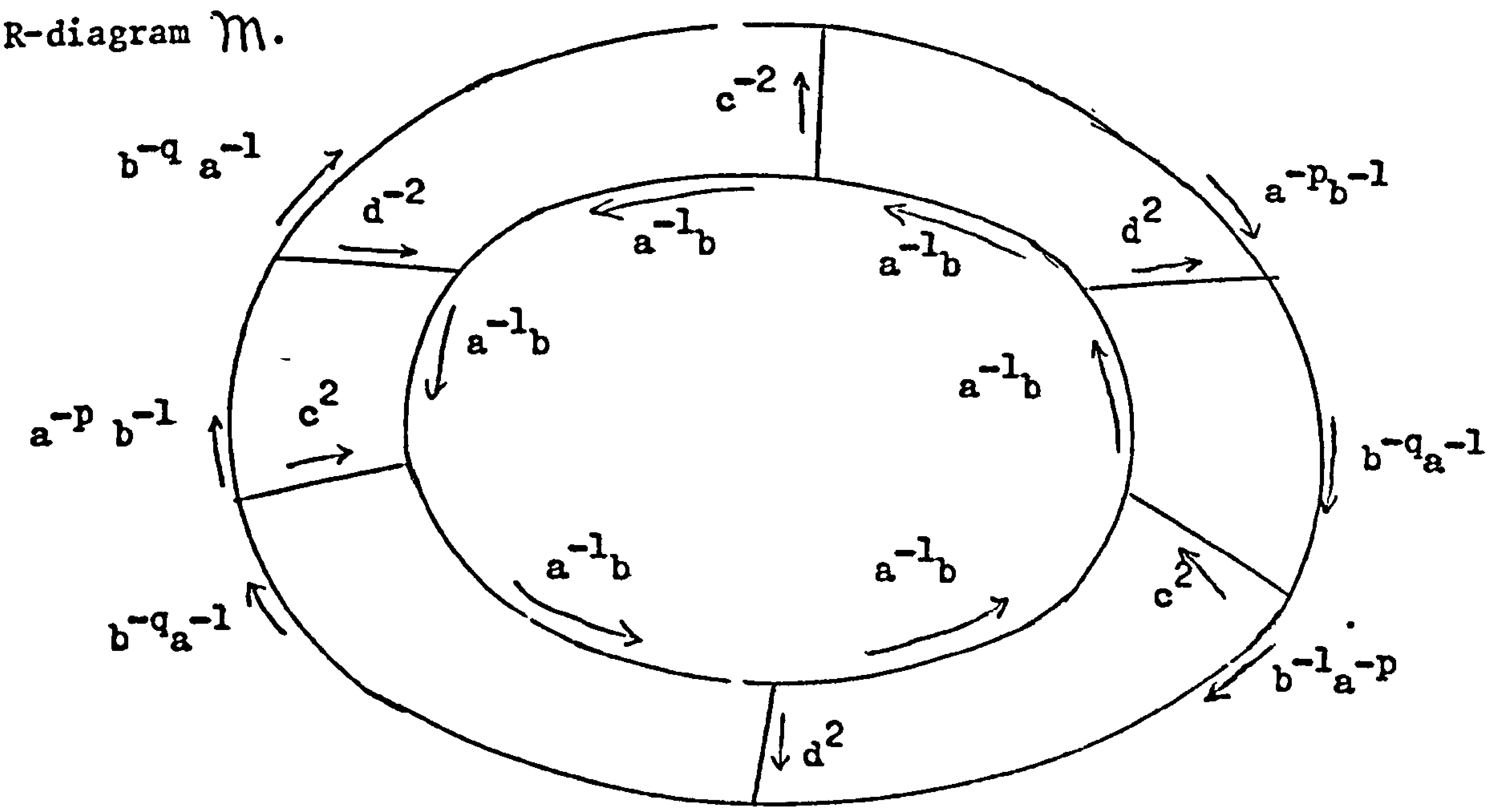
Suppose that y_4 begins with a^{-1}. Then $y_3 \equiv b$, $y_4 \equiv a^{-p}$, $h(Y_1) = 2$, $h(Y_3) = -1$ and $h(Y_4) = 0$. Since $\sum_{j=1}^{4} h(Y_j) = 1$, $y_5 \equiv a^{-p}$ with $h(Y_5) = 0$ whence $y_6 \equiv a^{-1} b^{-k_1}$, $1 \leq k_1 < q$. Compare r_5 and r, noting that a^p is not a piece and $d(\mu(Y_5)) = 3$. Then $x_{6,n(6)}$ ends in b^{-1}, which is absurd since ab is not a piece. Therefore y_4 must begin with a. So $y_4 \equiv y_5 \equiv a^p$ with $h(Y_4) = h(Y_5) = 0$ and $y_6 \equiv a\, b^{\nu k}$, $1 \leq k < q$. Thus $h(Y_6) \leq -1$, which is absurd since $\sum_{j=1}^{5} h(Y_j) = 1$. Therefore

$y_1 \neq a^p b^q$. It follows that $y_1 \neq a^p b^{\ell'}$ and $y_1 \neq a^{\ell'} b^q$ and the other two possibilities can be ruled out easily.//

It turned out that the most difficult case was when $a^m, b^n \notin R$. It is clear that a^p and b^q are not in general free generators (e.g. a relation $(a^p b^q)^3 = 1$). The following example shows that even a^{p+1} and b^{q+1} need not be free generators.

(4.9) Example

Let $G = \langle a,b,c,d \,;\ r_1 = d^2 a^p b c^2 b^{-1} a$,
$r_2 = c^2 b^q a d^{-2} b^{-1} a$, $r_3 = (a^{-1} b)^6\rangle$, where $p,q \geq 2$. It is clear that G is a C(6)-group. Let R be a symmetrized set generated by r_i, $i = 1,2,3$. R is a subset of $F(a,b,c,d)$, the free group on $\{a,b,c,d\}$. Consider the following connected reduced R-diagram $\mathfrak{M}$.



Now $\phi(\delta(\mathfrak{M})) \equiv (a^{p+1} b^{q+1})^{-3} \equiv w$, say, is a word in a^{p+1} and b^{q+1} which is cyclically reduced in a and b. By the Normal Subgroup Lemma (I.2.2), $w \in \langle F(a,b,c,d) < R\rangle$, where $F(a,b,c,d)$ is the free group on $\{a,b,c,d\}$.

We have established that when $a^n, b^m \notin R$, (provided no relator r in R has the form $r \equiv b^{\eta q} a^{\varepsilon p} b^{-\eta \ell} a^{-\varepsilon k}$ or $a^{\varepsilon p} b^{\eta q} a^{-\varepsilon k} b^{-\eta \ell}$, where $1 \le k < p$, $1 \le \ell < q$, $\varepsilon, \eta = \pm 1$), free generators can be chosen as proper powers of a and b (see Proposition (4.8)). It seems very likely that a^{2p} and b^{2q} are free generators, but to prove this we should have to consider many more possibilities than in the proof of Proposition (4.8)(1); on the other hand, the proof may be shortened considerably by using higher powers of a^p and b^q. We conclude this chapter with our main theorem.

(4.10) Theorem Let $G = \langle A ; R \rangle$ be a nontrivial finitely related C(6)-group. Then G contains a free subgroup of rank two unless G is isomorphic to one of the following three groups:

(i) $G_1 = \langle a; \emptyset \rangle$,

(ii) $G_2 = \langle a; a^m, a^{-m} \rangle$, $m \neq 0$,

(iii) $G_3 = \langle a,b; a^2, a^{-2}, b^2, b^{-2} \rangle$.

Proof Use Proposition (1.3) or Proposition (1.4), if there exists a in A which is not a piece. Otherwise, appeal to the tree in the beginning of this section.

CHAPTER III

A THEORY OF DIAGRAMS

In this Chapter we shall use R.C. Lyndon's result (10), about an arbitrary map, (see Lemma (I.1.3)), to introduce a theory of diagrams which is a tool in dealing with embedding problems. In section 1 we define a "suitable" map with three types of regions $\mathcal{G}$, $\mathcal{K}$, $\mathcal{C}$ and show that in such a map there exists a $\mathcal{C}$-region whose artificial degree is positive. In section 2 we show that, if a countable group K, (under certain conditions), is not embeddable in a quotient of a C(6)-group G then, roughly, there exists a suitable diagram (depending on G and K). This is our main result (Proposition (2.17)). Finally, in section 3 we shall establish some results on a given suitable diagram. We shall employ this theory in the next Chapter.

Section 1

Let $\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ be an oriented map. We shall partition $\mathcal{F}$ by $\mathcal{F} = \mathcal{G} \cup \mathcal{K} \cup \mathcal{C}$, where $\mathcal{G}, \mathcal{C} \neq \emptyset$.

Let X and Y be regions in $\mathcal{F}$ (not necessarily distinct). Let $\mathcal{X}, \mathcal{Y} \in \{\mathcal{G}, \mathcal{K}, \mathcal{C}\}$. Let E be an interior edge. We say that E is an (X,Y)-edge if $\rho(E) = X$ and $\sigma(E) = Y$. We say that E is an $(X,\mathcal{Y})$-edge (or $E \in (X,\mathcal{Y})$) if there exists a region $Y' \in \mathcal{Y}$ such that E is an (X,Y')-edge. We say that E is a $(\mathcal{X},Y)$-edge (or $E \in (\mathcal{X},Y)$) if there exists a region $X' \in \mathcal{X}$ such that E is an (X',Y)-edge. We say that E is an $(\mathcal{X},\mathcal{Y})$-edge (or $E \in (\mathcal{X},\mathcal{Y})$) if there exists X',Y', regions in $\mathcal{X}$, $\mathcal{Y}$ respectively, such that E is an (X',Y')-edge. Thus the partition of $\mathcal{F}$ induces a partition of $\mathcal{E}$ via $E \in (\mathcal{X},\mathcal{Y})$ iff $\rho(E) \in \mathcal{X}$ and $\sigma(E) \in \mathcal{Y}$.

Let v be an interior vertex. Let $E_1, E_2, \ldots, E_n$ be edges such the following four conditions are valid

(i) $\lambda(E_i) = v$, where $1 \leq i \leq n$;

(ii) $\rho(E_i) = \sigma(E_{i+1})$, $1 \leq i \leq n-1$, and $\rho(E_n) = \sigma(E_1)$;

(iii) $E_1, E_2, \ldots, E_n$ are distinct ;

(iv) If E is an edge with $\lambda(E) = v$, then there exists i, $1 \leq i \leq n$ such that $E = E_i$.

We shall refer to $\delta(v) = (E_1, E_2, \ldots, E_n)$ as the <u>ordered star</u> of v (note that $\delta(v)$ is defined only up to cyclic permutation). It is clear that $d(v) = n$.

Let each of $\mathcal{X}_1, \ldots, \mathcal{X}_m$, $m \geq 1$, be one of $\mathcal{G}, \mathcal{K}, \mathcal{L}$. Let v be an interior vertex with $\delta(v) = (E_1, E_2, \ldots, E_m)$. We shall say that v is an $(\mathcal{X}_1, \ldots, \mathcal{X}_m)$-vertex if $\rho(E_i) \in \mathcal{X}_i$ for each $1 \leq i \leq m$.

(1.1) <u>Convention</u> From now on we shall use constantly the following notation:

Let X,Y be any regions in $\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ with $\mathcal{F} = \mathcal{G} \cup \mathcal{K} \cup \mathcal{L}$. Let v be an interior vertex. Let each of $\mathcal{X}$ and $\mathcal{Y}$ be one of $\mathcal{G}$, $\mathcal{K}$ or $\mathcal{L}$.

(i) $v^{\#} = \text{card}\,\{D \in \mathcal{F} \mid v \in \beta(D)\}$

(ii) $[\mathcal{X}\,;\,v]^{\#} = \text{card}\,\{D \in \mathcal{X} \mid v \in \beta(D)\}$

(iii) $[(\mathcal{X},\mathcal{Y})\,;\,v]^{\#} = \text{card}\,\{E \in (\mathcal{X},\mathcal{Y}) \mid \lambda(E) = v \text{ or } \mu(E) = v.\}$

(iv) $[(X,\mathcal{Y})\,;\,v]^{\#} = \text{card}\,\{E \in (X,\mathcal{Y}) \mid \lambda(E) = v \text{ or } \mu(E) = v.\}$

(v) $[(\mathcal{X},Y)\,;\,v]^{\#} = \text{card}\,\{E \in (\mathcal{X},Y) \mid \lambda(E) = v \text{ or } \mu(E) = v.\}$

(vi) $(X,Y)^{\#} = \text{card}\ \{E \in \mathcal{E} \mid E \text{ is an } (X,Y)\text{-edge}\}$

(vii) $(\mathcal{X},Y)^{\#} = \text{card}\ \{E \in \mathcal{E} \mid E \text{ is an } (\mathcal{X},Y)\text{-edge}\}$

(viii) $(X,\mathcal{Y})^{\#} = \text{card}\ \{E \in \mathcal{E} \mid E \text{ is an } (X,\mathcal{Y})\text{-edge}\}$

(ix) $(\mathcal{X},\mathcal{Y})^{\#} = \text{card}\ \{E \in \mathcal{E} \mid E \text{ is an } (\mathcal{X},\mathcal{Y})\text{-edge}\}$

(x) $\mathcal{G}^{\#} = \text{card}\ (\mathcal{G})$

(1.2) Definition

Let $\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ be a map. Then $\mathcal{M}$ is called a suitable map if $\mathcal{F}$ can be given a partition $\mathcal{F} = \mathcal{K} \cup \mathcal{G} \cup \mathcal{C}$ such that the following conditions are valid.

(I) $\delta(\mathcal{M})$ is a simple closed path.

(II) A boundary region of $\mathcal{M}$ is a $\mathcal{G}$-region or a $\mathcal{C}$-region.

(III) (i) Any boundary region not semi-interior is a $\mathcal{C}$-region.

(ii) If X is a boundary region which is not semi-interior and $E \in \beta(X)$ is an interior edge of $\mathcal{M}$, then $E \notin (X,\mathcal{K})$.

(iii) If X is a boundary region then $\beta(X)$ has at most one (unoriented) boundary edge of $\mathcal{M}$.

(IV) A semi-interior $\mathcal{C}$-region X has a unique $(X,\mathcal{K})$-edge.

(V) (i) If X is a $\mathcal{G}$-region, then $d(X) \geq 2$.

(ii) If X is a $\mathcal{K}$-region, then $d(X) \geq 6$.

(iii) If X is a $\mathcal{C}$-region then $d(X) \geq 7$.

(VI) (i) There are no $(\mathcal{K},\mathcal{K})$-edges .

(ii) There are no $(\mathcal{G},\mathcal{K})$-edges .

(VII) If X is a $\mathcal{G}$-region and all the edges in $\beta(X)$ are $(\mathcal{G},\mathcal{G})$-edges then $i(X) \geq 6$.

Throughout this section let $\mathcal{M}=\mathcal{V}\cup\mathcal{E}\cup\mathcal{F}$ be a suitable map. Let X be any region of $\mathcal{M}$ and put

$$\alpha(X) = \begin{cases} 6 & \text{if } X \text{ is an interior region} \\ 4 & \text{if } X \text{ is a boundary region} \end{cases}$$

Let $\mathcal{M}^*$ be the dual of $\mathcal{M}$ as defined in (16).
Applying Lemma(I.1.3(2)), with $p = 6$ and $q = 3$, to $\mathcal{M}^*$ and noting that, in the obvious notation,
$((\mathcal{V}^*)^{\#})^{\cdot} \leq ((\mathcal{E}^*)^{\#})^{\cdot}$, we have

(F1) : $$6 \leq (*) + (**) \text{ , where}$$

$$(*) = \sum_{M\in\mathcal{C}} [\alpha(M) - i(M)] + \sum_{D\in\mathcal{G}} [\alpha(D) - i(D)]$$

$$(**) = \sum_{K\in\mathcal{K}} [\alpha(K) - i(K)] + 2 \sum_{v\in\mathcal{V},\ v \text{ interior}} [3 - d(v)]$$

Since $\mathcal{M}$ is suitable, and so $\mathcal{C} \neq \phi$, it is easy to derive the following

(F2) $$\sum_{D\in\mathcal{G}} [\alpha(D) - i(D)] > 6$$

Let $D\in\mathcal{G}$. We shall say that D is a $\mathcal{G}_{\mathcal{C}}$-region (or $D\in\mathcal{G}_{\mathcal{C}}$) if $(D,\mathcal{C})^{\#} > 0$, and D is a $\mathcal{G}_M$-region (or $D\in\mathcal{G}_M$), where $M\in\mathcal{C}$, if $(D,M)^{\#} > 0$. Hence we have

(F3) $$\sum_{D \in \mathcal{G}_{\mathcal{C}}} [\alpha(D) - i(D)] > 6 \quad .$$

This is clear for if $D \notin \mathcal{G}_{\mathcal{C}}$, then $i(D) \geq 6$. (See condition (VII)). However (F3) seems inappropriate for us. We obtain a more useful inquality by defining for each $M \in \mathcal{C}$, its artificial degree $\tilde{d}(M)$ and showing $\sum_{M \in \mathcal{C}} \tilde{d}(M) \geq 6$. Roughly, the sum $\sum_{D \in \mathcal{G}} [\alpha(D) - i(D)] + 2\sum [3 - d(v)]$ is broken up and distributed among the $\mathcal{C}$-regions. The exact way of doing that is made precise by the following sequence of definitions.

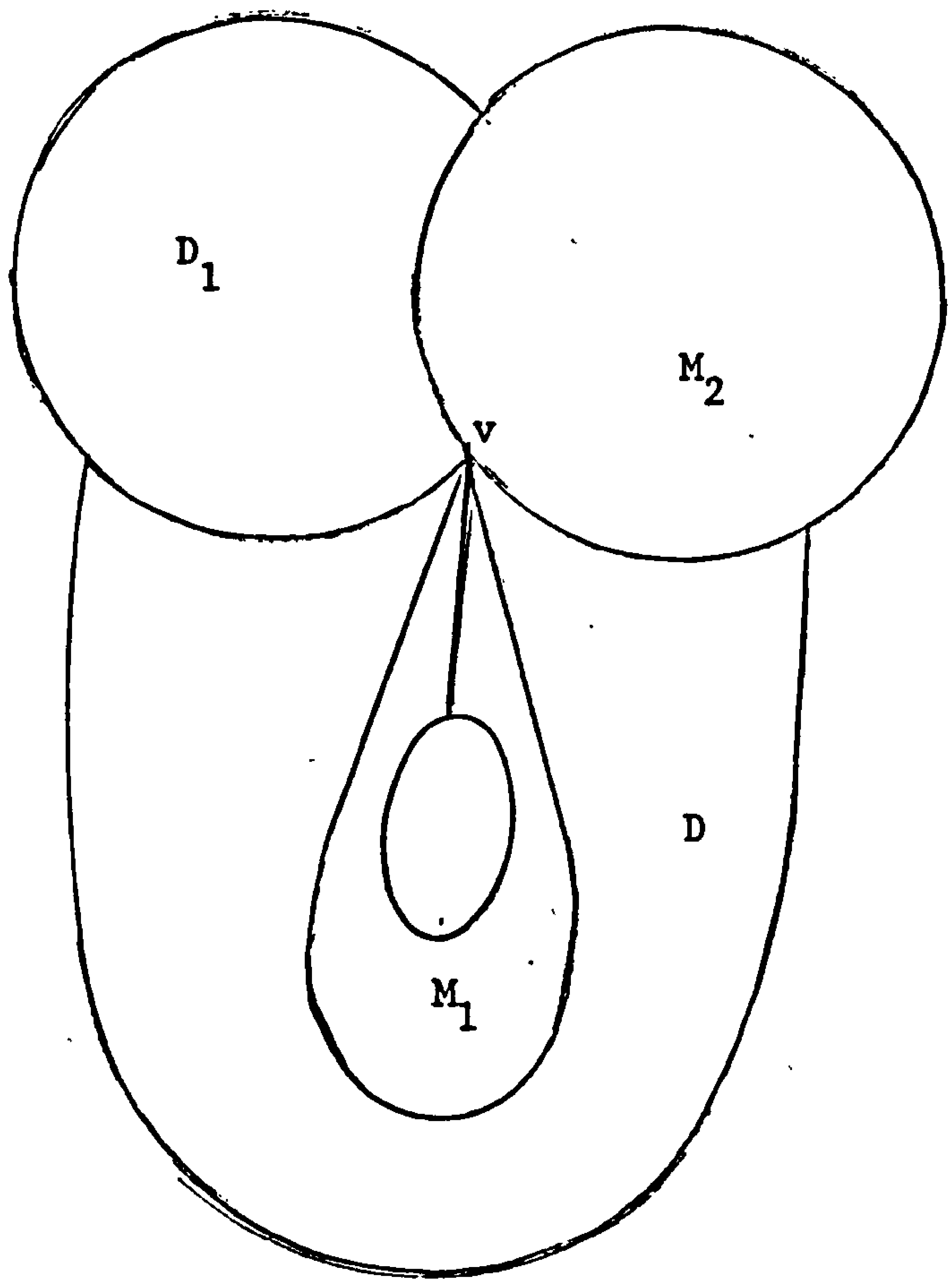
As a first step we share out the value $3 - d(v)$, where v is any interior vertex, among the regions in whose boundary v lies.

(1.3) <u>Definition</u>

Let v be an interior vertex and D a $\mathcal{G}$-region with $v \in \beta(D)$. Define

$$d(v,D) = \begin{cases} \dfrac{3-d(v)}{v^{\#}} & \text{if } [(D,\mathcal{C});v]^{\#} = 0 \\[2ex] \dfrac{3-d(v)}{v^{\#}}\left(\dfrac{1}{[(D,\mathcal{C});v]^{\#}} + \dfrac{[\mathcal{C};v]^{\#}+[\mathcal{K};v]^{\#}}{[(\mathcal{G},\mathcal{C});v]^{\#}}\right) & \text{otherwise} \end{cases}$$

We illustrate the Definition (1.3) in the following example. Let M_1, $M_2 \in \mathcal{C}$ and D, $D_1 \in \mathcal{G}$. Let $v \in \beta(M_1) \cap \beta(M_2) \cap \beta(D) \cap \beta(D_1)$ where v is an interior vertex.



Here $d(v) = 6$, $v^{\#} = 4$, $[\mathcal{C}\,; v]^{\#} = 2$, $[(\mathcal{G},\mathcal{C})\,; v]^{\#} = 3$, $[(D,\mathcal{C});v]^{\#} = 2$, $[(D_1,\mathcal{C});v]^{\#} = 1$, $[\mathcal{K};v]^{\#} = 0$.

(1.4) <u>Definition</u>

Let D be a $\mathcal{G}_{\mathcal{C}}$-region. Define

$$\underset{\sim}{d}(D) = \frac{1}{(D,\mathcal{C})^{\#}}\Big[\alpha(D) - i(D) + 2 \sum_{v\in\beta(D),\ v \text{ interior}} \underline{d}(v,D)\Big]$$

The idea of the definition is the incorporation with $\alpha(D) - i(D)$ of the contributions from the vertices in $\beta(D)$ with the resulting quantity being "averaged" in regard to $\mathcal{C}$-regions.

Finally, if a $\mathcal{G}$-region D has an edge in common with a $\mathcal{C}$-region M then we transfer the value $\underset{\sim}{d}(D)$ to the region M thereby replacing $\alpha(M) - i(M)$ by the new quantity $\tilde{d}(M)$.

(1.5) Definition

Let $M \in \mathcal{C}$. Define

$$\tilde{d}(M) = \alpha(M) - i(M) + \sum_{D \in \mathcal{G}_M} (D,M)^{\#}\, \underset{\sim}{d}(D) \quad .$$

(1.6) Proposition

$$\sum_{M \in \mathcal{C}} \tilde{d}(M) \geq 6 \quad .$$

Proof

From (F1) and conditions VI, VII, we have

$$\sum_{M \in \mathcal{C}} [\alpha(M) - i(M)] + \sum_{D \in \mathcal{G}_{\mathcal{C}}} [\alpha(D) - i(D)] + 2 \sum_{v\ \text{interior}} [3-d(v)] \geq 6 \; .$$

By Definition (1.5),

$$\sum_{M \in \mathcal{C}} \tilde{d}(M) = \sum_{M \in \mathcal{C}} [\alpha(M) - i(M)] + \sum_{M \in \mathcal{C}} \sum_{D \in \mathcal{G}_M} (D,M)^{\#}\, \underset{\sim}{d}(D)$$

Now,

$$\sum_{M \in \mathcal{C}} \sum_{D \in \mathcal{G}_M} (D,M)^{\#}\, \underset{\sim}{d}(D) = \sum_{D \in \mathcal{G}_{\mathcal{C}}} (D,\mathcal{C})^{\#}\, \underset{\sim}{d}(D)$$

$$= \sum_{D \in \mathcal{G}_{\mathcal{C}}} [\alpha(D) - i(D)] + 2 \sum_{D \in \mathcal{G}_{\mathcal{C}}} \sum_{\substack{v \in \beta(D) \\ v\ \text{interior}}} \underset{\sim}{d}(v,D) \; .$$

We claim that $\displaystyle\sum_{D \in \mathcal{G}_{\mathcal{C}}} \sum_{\substack{v \in \beta(D) \\ v\ \text{interior}}} \underset{\sim}{d}(v,D) \geq \sum_{v\ \text{interior}} [3-d(v)]$

Let $\mathcal{G}_{\ell}(v) = \{D \in \mathcal{G}_{\ell} \mid v \in \beta(D)\}$, $[\mathcal{G}_{\ell};v]^{\#} = \text{card}(\mathcal{G}_{\ell}(v))$,

$\mathcal{G}_{\ell}^{+}(v) = \{D \in \mathcal{G}_{\ell}(v) \mid [(D,\ell) ; v]^{\#} > 0\}$ and

$[\mathcal{G};v]^{\natural} = \text{card}(\mathcal{G}_{\ell}^{+}(v))$.

Fix an interior vertex v_o. Then we can write

$$\sum_{D \in \mathcal{G}_{\ell}(v_o)} d^{-}(v_o,D) = (1) + (2) + (3)$$

where

$$(1) = \frac{3-d(v_o)}{v_o^{\#}} \left([\mathcal{G}_{\ell};v_o]^{\#} - [\mathcal{G};v_o]^{\natural}\right) ,$$

$$(2) = \frac{3-d(v_o)}{v_o^{\#}} \left[\sum_{D \in \mathcal{G}_{\ell}^{+}(v_o)} \frac{1}{[(D,\ell);v_o]^{\#}}\right]$$

and

$$(3) = \frac{[\mathcal{G};v_o]^{\natural}}{[(\mathcal{G};\ell);v_o]^{\#}} \left([\mathcal{K};v_o]^{\#} + [\ell;v_o]^{\#}\right) \frac{3-d(v_o)}{v_o^{\#}}$$

Note that

$$\sum_{D \in \mathcal{G}_{\ell}^{+}(v_o)} \frac{1}{[(D,\ell);v_o]^{\#}} \leq [\mathcal{G},v_o]^{\natural} \quad \text{and}$$

$$[\mathcal{G};v_o]^{\natural} \leq [(\mathcal{G},\ell) ; v_o]^{\#}$$

Since $\dfrac{3-d(v_o)}{v_o^{\#}} \leq 0$, we obtain

$$\sum_{D \in \mathcal{G}_{\ell}(v_o)} d^{-}(v_o,D) \geq \frac{3-d(v_o)}{v_o^{\#}} \left([\mathcal{G}_{\ell};v_o]^{\#}+[\mathcal{K};v_o]^{\#}+[\ell;v_o]^{\#}\right)$$

$$\geq 3 - d(v_o)$$

Let $\mathcal{V}_o = \{v \in \mathcal{V} \mid v$ is interior and there exists $D \in \mathcal{G}_{\mathcal{C}}$ with $v \in \beta(D)\}$

Hence $$\sum_{D \in \mathcal{G}_{\mathcal{C}}} \sum_{\substack{v \in \beta(D) \\ v \text{ interior}}} \underset{\sim}{d}(v, D) = \sum_{v \in \mathcal{V}_o} \left[\sum_{D \in \mathcal{G}_{\mathcal{C}}(v)} \underset{\sim}{d}(v,D) \right]$$

$$= \sum_{v \in \mathcal{V}_o} (3-d(v)) \geq \sum_{v \text{ interior}} (3-d(v)) ,$$

since $3-d(v) \leq o$, for any v.

(1.7) <u>Corollary</u> (i) There exists a $\mathcal{C}$-region M in $\mathcal{M}$ with $\tilde{d}$ (M) > o .

(ii) If $\tilde{d}$ (M) > o, then there exists a $\mathcal{G}_M$-region D with $\underset{\sim}{d}(D) > o$.

<u>Proof</u> (i) Obvious

(ii) Use the fact that $i(M) \geq 6$, (condition III(iii)).

(1.8) <u>Definition</u>

Let M be a $\mathcal{C}$-region. For each edge E occurring in $\delta(M)$ we define

$$\tilde{d}(E;M) = \begin{cases} \underset{\sim}{d}(\sigma(E)) - 1 & \text{if } \sigma(E) \in \mathcal{G} \\ -1 & \text{otherwise,} \end{cases}$$

and call this the artificial degree of E relative to M.

(1.9) <u>Proposition</u>

Let M be a $\mathcal{C}$-region. If $\tilde{d}(M) > o$, then $\sum \tilde{d}(E;M) > -6$, where the sum is over all E occurring in $\delta(M)$.

Proof From Definition (1.5) we have

$$0 < \tilde{d}(M) = \alpha(M) - i(M) + \sum_{D \in \mathcal{G}_M} (D,M)^{\#} \underset{\sim}{d}(D) .$$

If M is semi-interior then

$$-\alpha(M) < (D,M)^{\#}[\underset{\sim}{d}(D) - 1] - (M,\mathcal{C})^{\#} - 1, (\text{ since } (M,\mathcal{K}) - 1) .$$

If M is not semi-interior then

$$-\alpha(M) < \sum (D,M)^{\#}[\underset{\sim}{d}(D) - 1] - ((M,\mathcal{C})^{\#} .$$

In both cases $-6 < \sum_{E \text{ in } (M)} \tilde{d}(E,M)$.

(1.10) Remark

Let M be a $\mathcal{C}$-region such that $\tilde{d}(M) > o$. If $(M,\mathcal{C})^{\#} \geq \alpha(M)$, then there exists an $(M,\mathcal{G})$-edge E such that

$$\tilde{d}\,(E;M) > o \quad \text{i.e.} \quad \underset{\sim}{d}\,(\sigma(E)) > 1.$$

Section 2

For any set S, we denote the free group on S by F(S). If S is a finite, say, $S = \{x_1,\ldots, x_n\}$ we may write $F(S) = F(x_1,x_2,\ldots, x_n)$

Let A be a finite set such that card $A \geq 2$. Let R be a finite symmetrical subset of F(A). We shall assume that R satisfies

Hypothesis (H1)

(1) No element of R is of form az, where $a \in A$, $z \in F(A)$ and z is a - free.

(2) If $a \in A$, which is not a piece relative to R, then no element in R is a proper power of a.

(3) R satisfies C(6).

Let $G = \langle A \; ; \; R \rangle = F(A) \, / \, \langle F(A) \langle R \rangle$. Let x and y be any two elements not in A. Let B be a normal subgroup of F(x,y) which satisfies

Hypothesis (H2)

If $u \in B$ with $u \neq 1$ in F(x,y), then $|u| \geq 6$.

We write $K = \langle x,y \; ; \; B \rangle = F(x,y)/B$. Let B_o be the set of all cyclically reduced words of B. Then K also has a presentation $\langle x,y \; ; \; B_o \rangle$ since any element of B is conjugate to an element of B_o.

Let $s_1 \equiv xz_1$, and $s_2 \equiv yz_2$, where $z_1, z_2 \in F(A)$. Then $s_1, s_2 \in F(A,x,y)$ and we can form the set C of all cyclic permutations of $s_1^{\pm 1}$ and $s_2^{\pm 1}$. We shall assume C satisfies

Hypothesis (H3)

z_1 and z_2 are not expressible as a product of fewer than six pieces relative to $R \cup C$.

Let $R' = R \cup B_o \cup C$ and let $N = \langle F(A,x,y) \langle R' \rangle$. Let w be a freely reduced word in N. Then by the Lyndon-Van Kampen Theorem (I.2.1), there exists an associated R'-diagram $\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ such that $\phi(\delta(\mathcal{M})) = w^{-1}$. We partititon $\mathcal{F}$ by $\mathcal{F} = \mathcal{J} \cup \mathcal{K} \cup \mathcal{F}$, where

$$\mathcal{J} = \{D \in \mathcal{F} \mid \quad \phi(\delta(D)) \in R\} \ ,$$
$$\mathcal{K} = \{D \in \mathcal{F} \mid \quad \phi(\delta(D)) \in B_o\} \ , \quad \text{and}$$
$$\mathcal{L} = \{D \in \mathcal{F} \mid \quad \phi(\delta(D)) \in C\} \ .$$

It is clear that $F(A,x,y)/N$ is a factor group of G, since x and y can be expressed in terms of $\mathcal{L}$ elements of A. Also K is embedded in

$$F(A,x,y) \ / \ N \quad \text{iff} \quad N \cap F(x,y) = B.$$

We shall use our theory of diagrams to show that $N \cap F(x,y) = B$. We shall proceed by contradiction and assume

<u>Hypothesis</u> (H4) $\quad N \cap F(x,y) \neq B$

Obviously $B \leq N \cap F(x,y)$. Suppose that $w \in N \cap F(x,y)$ and $w \notin B$. Then any R'-diagram for w must involve regions other than $\mathcal{K}$-regions (otherwise, by the Normal Subgroup Lemma (I.2.2), $w \in B$).

Now, let Δ be the set of all cyclically reduced words in $(N \cap F(x,y)) - B$. (It follows from (H4) that Δ is non-empty). We shall not distinguish between two elements of Δ if one is a cyclic permutation of the other.

Among all R'-diagrams whose boundary labels are elements of Δ we select a diagram $\mathcal{M}(\Delta)$ with the minimum number of regions. Thus $\mathcal{M}(\Delta)$ is R'-reduced. Our main result in this Chapter is to show that $\mathcal{M}(\Delta)$ is a suitable map. A series of Lemmas is required. For emphasis we repeat that hypothesis (H1) - (H4) are assumed.

(2.1) Lemma

Let $m = \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$ (the number of regions in $\mathcal{M}(\Delta)$) and let $u \equiv \phi(\delta(\mathcal{M}(\Delta)))$. If

$$u = \prod_{1=1}^{n} w_i r_i^* w_o^{-1}, \quad r_i^* \in R'$$

is a representation of u as a product of conjugates of elements of R', then $m \leq n$.

Proof Suppose not. By the Lyndon-Van Kampen Theorem (I.2.1) we can construct an R'-diagram $\mathcal{M}^*$ for u with fewer regions than $\mathcal{M}(\Delta)$. This is a contradiction.

(2.2) Lemma

Let $m = \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$. Let $\mathcal{N}$ be a connected, simply connected diagram such that for any region D, $\phi(\delta(D)) \in R \cup B \cup C$. If $\phi(\delta(\mathcal{N})) \notin B$, then $\mathcal{F}^{\#}_{\mathcal{N}} \geq m$.

Proof Clearly $\phi(\delta(\mathcal{N})) \neq 1$. Let $\mathcal{F}^{\#}_{\mathcal{N}} = n$.

Suppose that $n < m$. By the Normal Subgroup Lemma (I.2.2),

$$\phi(\delta(\mathcal{N})) = \prod_{i=1}^{n} w_i r_i w_i^{-1}, \quad r_i \in R \cup B \cup C$$

$$= \prod_{i=1}^{k} w_i' r_i' w_i'^{-1}, \quad r_i' \in R' \text{ and } k \leq n$$

(replacing r_i by a suitable conjugate if $r_i \in B - B_o$).

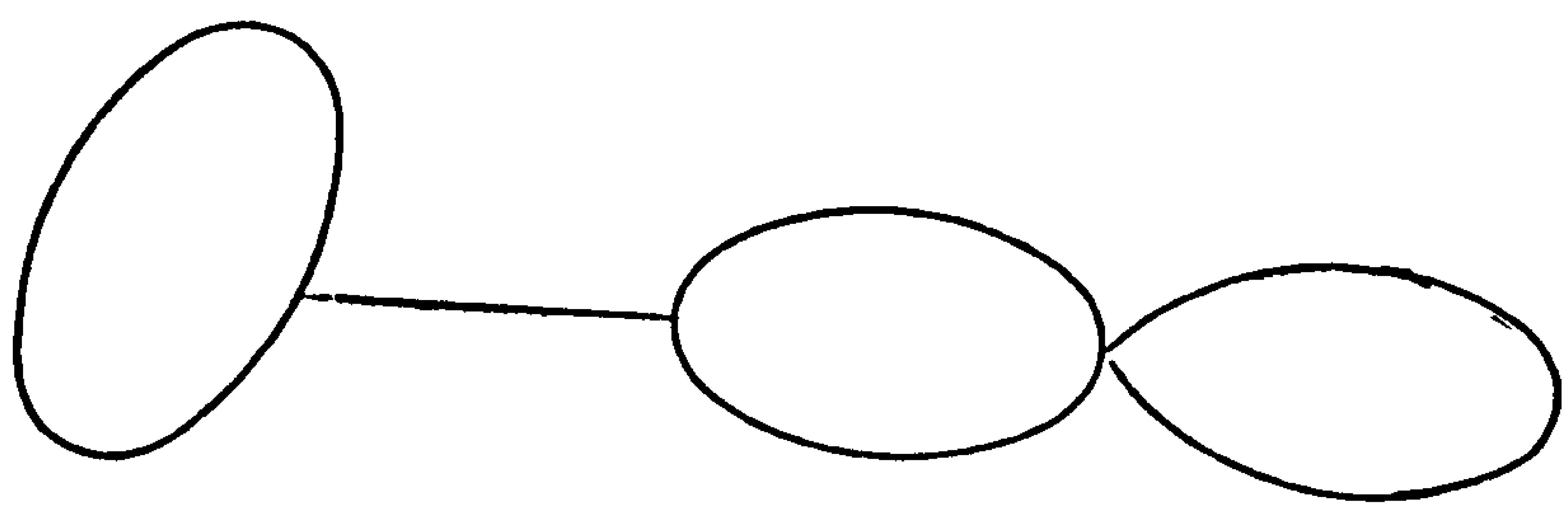
Now, by the Lyndon-Van Kampen Theorem (I.2.1), there exists an R'-diagram $\mathcal{N}'$ with

$$\phi(\delta(\mathcal{N}')) = \left[\prod_{i=1}^{k} w_i' \, r_i' \, w_i'^{-1}\right]^{-1} \notin B.$$

Finally there exists a reduced R'-diagram $\mathcal{N}^*$ with $\phi(\delta(\mathcal{N}^*)) \in \Delta$ and $\mathcal{F}^{\#}_{\mathcal{N}^*} < \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$ which is impossible.

(2.3) <u>Lemma</u> $\delta(\mathcal{M}(\Delta))$ is a simple closed path.

<u>Proof</u> Suppose not. (See the figure below)



If there exists a disc $\mathcal{L}$ in $\mathcal{M}(\Delta)$ such that $\phi(\delta(L)) \notin B$, then by Lemma (2.2), $\mathcal{F}^{\#}_{\mathcal{L}} \geq \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$ since for each region D in $\phi(\delta(D)) \in R \cup B \cup C$. This is a contradiction. Thus for each disc in $\mathcal{M}(\Delta)$, $\phi(\delta(\mathcal{L})) \in B$.

Let $\mathcal{M}_1$ be the diagram obtained from $\mathcal{M}(\Delta)$ by deleting all interior edges and interior vertices in $\mathcal{M}(\Delta)$.
Then $\phi(\delta(\mathcal{M}_1)) = \phi(\delta(\mathcal{M}(\Delta))$. By the Normal Subgroup Lemma (I.2.2), $\phi(\delta(\mathcal{M}_1)) \in B$ which is impossible. (Note. In $\mathcal{M}_1$ the interior of each disc of $\mathcal{M}$ is to be regardeda as a region.)

(2.4) <u>Lemma</u>

Let L be a $\mathcal{K}$-region in $\mathcal{M}(\Delta)$. Then $\delta(L)$ is a simple closed path.

Proof

Any closed path has a simple closed subpath. Moreover if $\delta(L)$ is not simple closed, then there must exist a simple closed subpath γ with L exterior to γ since every region in $\mathcal{M}(\Delta)$ is homeomorphic to the open unit disc.

Let $\gamma = (E_1, \ldots, E_k)$, $k \geqslant 1$, be a simple closed subpath of $\delta(L)$ with L exterior to γ. Let $\mathcal{N}$ be the subdiagram of $\mathcal{M}(\Delta)$ consisting of γ and all vertices, edges and regions interior to γ. Then

$$\mathcal{F}^{\#}_{\mathcal{M}(\Delta)} \geq \mathcal{F}^{\#}_{\mathcal{N}} + 1.$$

Case 1 $\phi(\gamma) \notin B$. Since for each region D in $\mathcal{N}$, $\phi(\delta(D)) \in B \cup R \cup C$, Lemma (2.2) shows that $\mathcal{F}^{\#}_{\mathcal{N}} \geq \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$ which is impossible.

Case 2 $\phi(\gamma) \in B$. We write $\delta(L) = (E_1, \ldots, E_k, E_{k+1}, \ldots, E_\ell)$, where $(E_1, \ldots, E_k) = \gamma$, $\ell = d(L)$. Consider the path $\gamma' = (E_{k+1}, \ldots, E_\ell)$. Here $\mu(E_\ell) = \lambda(E_1) = \mu(E_k) = \lambda(E_{k+1})$. Then γ' is a closed path. Let $\mathcal{L}$ be the subdiagram of $\mathcal{M}(\Delta)$ consisting of γ', all vertices, edges and regions interior to γ' - in effect $\mathcal{L} = \mathcal{N} \cup \bar{L}$.

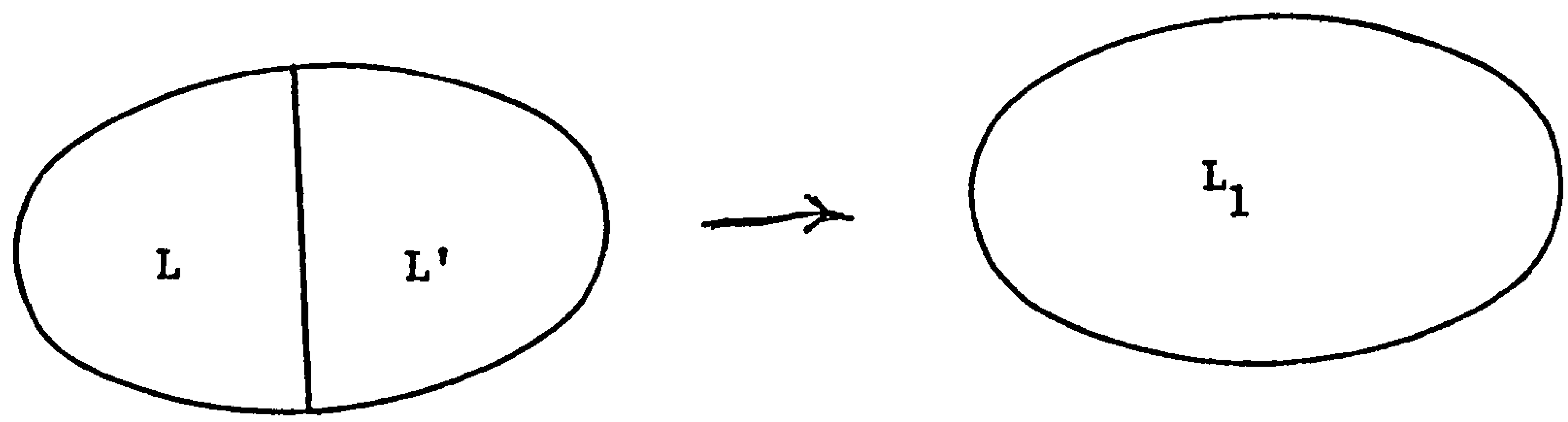
Now, $\phi(\delta(L)) = \phi(\gamma)\,\phi(\gamma') \in B$ whence $\phi(\gamma') \in B$. Let $\mathcal{M}_1$ be the diagram obtained from $\mathcal{M}(\Delta)$ by deleting all the interior vertices, edges and regions of $\mathcal{L}$ and regarding $\mathcal{L}$ as a single region. Clearly $\phi(\delta(\mathcal{M}(\Delta)) \equiv \phi(\delta(\mathcal{M}_1))$ and $\mathcal{F}^{\#}_{\mathcal{M}_1} < \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$. Since for each region D in $\mathcal{M}_1$, $\phi(\delta(D)) \in B \cup R \cup C$, $\mathcal{F}^{\#}_{\mathcal{M}(\Delta)} \leqslant \mathcal{F}^{\#}_{\mathcal{M}_1}$, by Lemma (2.2), which is impossible.

(2.5) <u>Lemma</u>

Let L and L' be any two distinct $\mathfrak{k}$-regions in $\mathfrak{M}(\Delta)$. Then $\beta(L) \cap \beta(L') = \emptyset$.

<u>Proof</u> Suppose not; then we consider two cases.

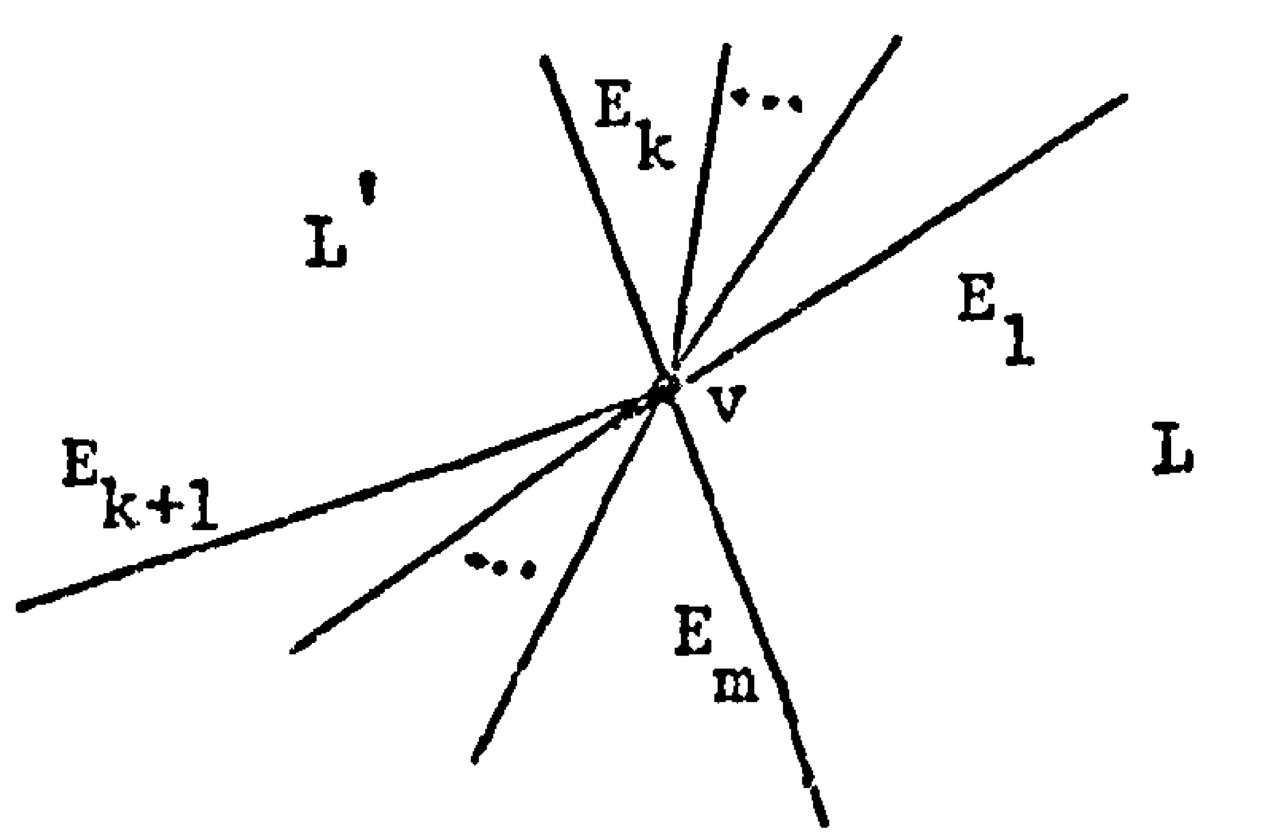
<u>Case 1</u> - When $\beta(L) \cap \beta(L')$ contains an edge E. Let $\mathfrak{M}_1$ be a diagram obtained from $\mathfrak{M}(\Delta)$ by deleting E and replacing L and L' by a single region L_1.



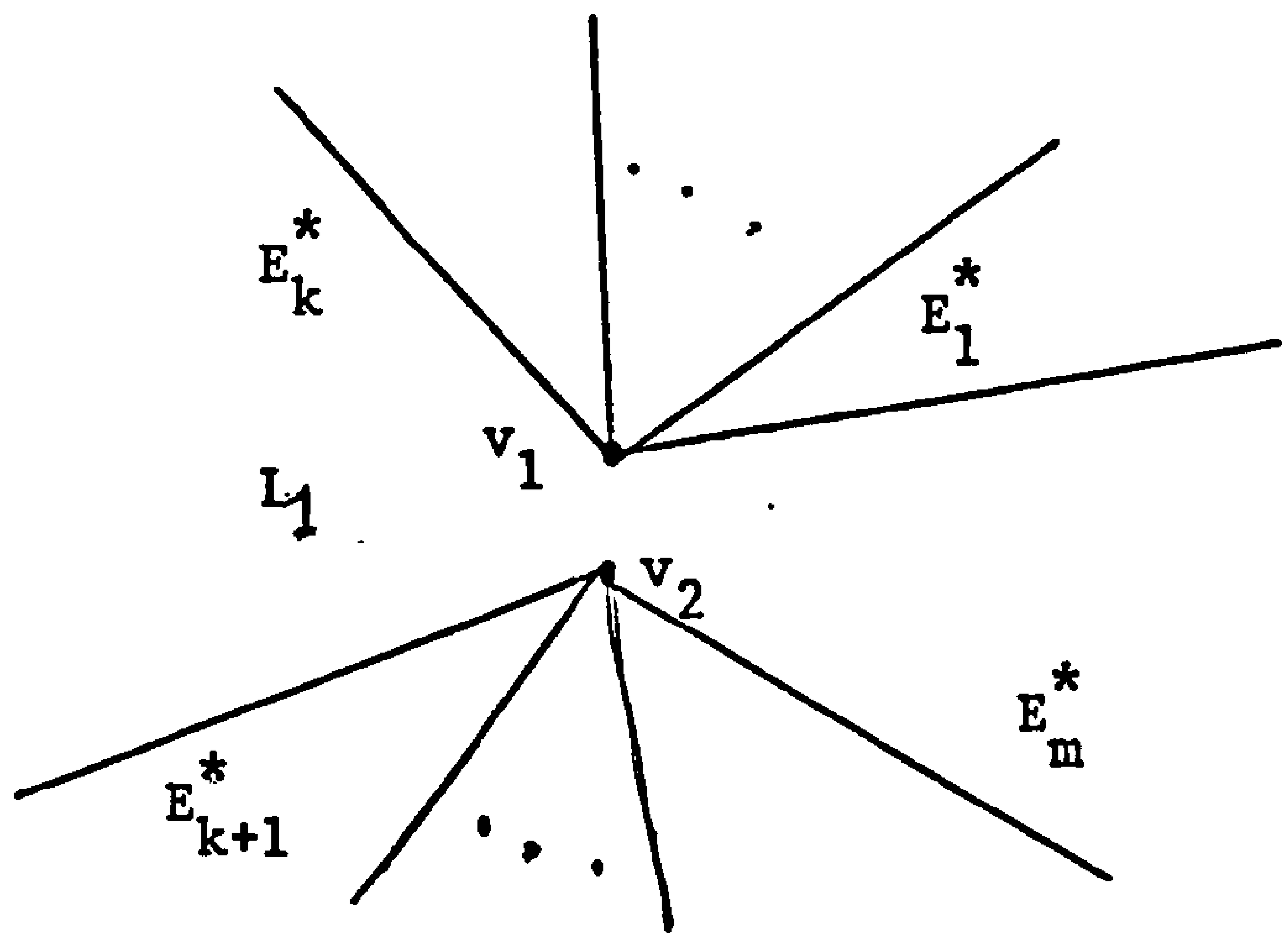
Then $\phi(\delta(\mathfrak{M}_1)) = \phi(\delta(\mathfrak{M}(\Delta))$. Apply Lemma (2.2) to $\mathfrak{M}_1$ to show a contradiction.

<u>Case 2</u> - When $\beta(L) \cap \beta(L')$ contains a vertex and no edges.

Let v be an interior vertex in $\beta(L) \cap \beta(L')$; then by case 1, $d(v) \geq 4$. Let $\delta(v) = (E_1, E_2, \ldots, E_m)$ where $E_1, E_m \in \beta(L)$ and $E_k, E_{k+1} \in \beta(L')$, $2 \leq k \leq m-2$



We construct a new diagram $\mathcal{M}_1$ by "tearing $\mathcal{M}(\Delta)$ apart" at v in such a way that v is replaced by two new vertices v_1 and v_2 , and also L and L' are replaced by a single region L_1.



Formally v, L, L' and $E_1, E_2, \ldots, E_m$ are all deleted and are replaced by v_1, v_2, L_1 and $E^*_1, E^*_2, \ldots, E^*_m$ where $\lambda(E^*_j) = v_1$, $1 \leqslant j \leqslant k$, $\lambda(E^*_j) = v_2$, $k+1 \leqslant j \leqslant m$ and $\mu(E^*_j) = \mu(E_j)$, $1 \leqslant j \leqslant m$. Also we put $\phi(E^*_j) = \phi(E_j)$, $1 \leqslant j \leqslant m$.

Then $\phi(\delta(\mathcal{M}_1)) = \phi(\delta(\mathcal{M}(\Delta)))$ and $\mathcal{F}^{\#}_{\mathcal{M}_1} < \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$. Since $\phi(L_1) = \phi(L)\ \phi(L')$, $\phi(L_1) \in B$ and so Lemma (2.2) gives a contradiction.

If v is a boundary vertex, then the <u>star sequence</u> of edges at v can be written as $\delta^*(v) = (E_1, E_2, \ldots, E_m)$, where E_1 and E_2 are boundary edges since $\delta(\mathcal{M})$ is simple closed path. And (almost) the same argument as with v interior can be applied to give a contradiction.

(2.6) <u>Corollary</u> There are no $(\mathcal{K},\mathcal{K})$-edges in $\mathcal{M}(\Delta)$.

(2.7) <u>Lemma</u>

Let L be a $\mathcal{K}$-region in $\mathcal{M}(\Delta)$. Then $\beta(L) \cap \beta(\mathcal{M}(\Delta)) = \phi$.

<u>Proof</u>

<u>Case 1</u> when L is an sbc-region. We write $\delta(\mathcal{M}(\Delta)) = (E_1, E_2, \ldots, E_k)$ where $E_1 \in \beta(L)$. So we can write $\delta(L) = (E_1', E_2', \ldots, E_\ell')$, where $E_1' = E_1^{-1}$. Now, $\mu(E_\ell') = \lambda(E_1') = \mu(E_1) = \lambda(E_2)$ and $\mu(E_k) = \lambda(E_1) = \mu(E_1') = \lambda(E_2')$. Then we have $\gamma = (E_2, \ldots, E_k, E_2', \ldots, E_\ell')$ a closed path (in fact γ is a simple closed path) and let $\mathcal{L}$ be the subdiagram of $\mathcal{M}(\Delta)$ consisting of γ all vertices, edges and regions interior to γ. Hence $\mathcal{F}_{\mathcal{L}}^{\#} < \mathcal{F}_{\mathcal{M}(\Delta)}^{\#}$, and for each region D of $\mathcal{L}$, $\phi(D) \in R'$. If $\phi(\gamma) \notin B$, then by Lemma (2.2), $\mathcal{F}_{\mathcal{L}}^{\#} \geq \mathcal{F}_{\mathcal{M}(\Delta)}^{\#}$ which is absurd. Thus $\phi(\gamma) \in B$.

Let $\mathcal{M}_1$ be the diagram obtained from $\mathcal{M}(\Delta)$ by removing all the interior vertices, edges and regions of $\mathcal{L}$ and regarding $\mathcal{L}$ as a single region. Then $\phi(\delta(\mathcal{M}_1)) \equiv \phi(\delta(\mathcal{M}(\Delta))$. By the normal subgroup Lemma (I.2.2) $\phi(\delta(\mathcal{M}_1)) \in B$ which is impossible.

<u>Case 2</u> . When L is a wbc-region . Let $v_0 \in \beta(L) \cap \beta(\mathcal{M})$.
Let $\mathcal{L} = \mathcal{M}(\Delta) - L$; then $\mathcal{L}$ has boundary cycle $(E_1, \ldots, E_k, E'_1, \ldots, E'_\ell)$ where $\delta(\mathcal{M}) - (E_1, \ldots, E_k)$ and $\delta(L) - (E'_1, \ldots, E'_k)$. Tearing apart at v_0 we obtain a connected simply connected map $\mathcal{L}^*$ with $\varphi(\delta(\mathcal{L}^*)) \in (F(x,y) - B)$. As $\mathcal{F}_{\mathcal{L}^*}^{\#} < \mathcal{F}_{\mathcal{M}(\Delta)}^{\#}$ we have a contradiction .

<u>Case 3</u> - when L is a bdc-region. Let $\delta(\mathcal{M}) = (E_1, E_2, \ldots, E_m)$. We consider the subsequence $(E_{k_1}, E_{k_2}, \ldots, E_{k_t})$ of $\delta(\mathcal{M})$ where

(1) $k_{j-1} < k_j$, $1 < j \leq t$,

(2) $E_{k_j} \notin \beta(L)$, $1 \leq j \leq t$,

(3) $\lambda(E_{k_j}) \in \beta(L)$

(4) $\lambda(E_i) \notin \beta(L)$ for $k_{j-1} < i \leq k_j - 2$ and $\lambda(E_{k_j-1}) \in \beta(L)$ if and only if $E_{k_j-1} \in \beta(L)$. Such a sequence must exist since $\delta(\mathcal{M})$ and $\delta(L)$ are simple closed.

Let $\gamma_j - (E_{k_{j-1}}, \ldots, E_{k_j-2}, (E_{k_j-1}))$ where E_{k_j-1} is icluded if and if $E_{k_j-1} \notin \beta(L)$. Now there is a subpath $\gamma_j' = (E'_{\ell_{j-1}}, \ldots, E'_{\ell_j})$ of $\delta(L)$ with the property that (i) $\lambda(E' \ ) = \begin{cases} \mu(E_{k_j-1}) & \text{if } E_{k_j-1} \notin \beta(L) \\ \mu(E_{k_j-2}) & \text{otherwise} \end{cases}$ and, (ii) $\lambda(E'_{\ell_j}) = E_k$. (Such a path always exsist.) Thus (γ_j, γ_j') is a simple closed path.

For each $1 \leq j \leq t$, let $\mathcal{L}_j$ be the subdiagram of $\mathcal{M}(\Delta)$ consisting of (γ_j, γ_j') and all vertices, edges and regions interior to (γ_j, γ_j'). (In effect $\mathcal{L}_j$, $1 \leq j \leq t$, are the submaps whose supports are the closure of the components of $\mathcal{M} - \bar{L}$).

Now, Lemma (2.2) shows that for each $1 \leq j \leq t$, $\phi(\delta(\mathcal{L}_j)) \in B$. Let $\mathcal{M}_1$ be the diagram obtained from $\mathcal{M}(\Delta)$ by deleting all vertices, edges and regions interior to the $\mathcal{L}_j$'s and regarding each $\mathcal{L}_j$ as a single region. By the Normal Subgroup Lemma $\phi(\delta(\mathcal{M}_1)) \in B$. This is a contradiction since $\phi(\delta(\mathcal{M}_1)) \equiv \phi(\delta(\mathcal{M}(\Delta)))$.

We observe, from the labels of $\mathcal{C}$-regions and using Lemma (2.7) the following

(2.8) Remarks

(1) Let D and D' be any two $\mathcal{C}$-regions in $\mathcal{M}(\Delta)$. Let E be an edge such that $\phi(E) = x^{\pm 1}$ or $y^{\pm 1}$. Assume that $E \in \beta(D) \cap \beta(D')$. We write $\delta(D) = (E, E_1, \ldots, E_\ell)$ and $\delta(D') = (E^{-1}, E_1', \ldots, E_k')$. Then $\phi(\delta(D)) \equiv (\phi(\delta(D')))^{-1}$.

(2) Let D be a boundary region in $\mathcal{M}(\Delta)$ which is not semi-interior. Let E be a boundary edge occurring in $\delta(D)$. Then

(i) D is a $\mathcal{C}$-region.

(ii) E is the unique boundary edge which occurs in $\delta(D)$.

(iii) $\phi(E) = x^{\pm 1}$ or $y^{\pm 1}$.

(iv) If E' is an interior edge occurring in $\delta(D)$, then $E' \notin (D, \mathcal{K})$.

(3) If D is a semi-interior $\mathcal{C}$-region in $\mathcal{M}(\Delta)$, then there is a unique $(D, \mathcal{K})$-edge which occurs in $\delta(D)$.

In Chapter II, Section 3, we defined $\theta_R(w)$ and $\theta_R^*(r)$. From now on we shall use them.

(2.9) Lemma (i) Let E be a $(\mathcal{G}, \mathcal{G})$-edge. Then $\theta_R(\phi(E)) = 1$.

(ii) for each $D \in \mathcal{G}$, $\theta_R^*(\phi(\delta(D)) \geq 6$, provided that $\phi(\delta(D))$ is a product of pieces relative to R.

Proof (i) The label on a $(\mathcal{G},\mathcal{G})$-edge is a piece relative to R.

(ii) This is true since R satisfies C(6).

(2.10) Remark There are no $(\mathcal{G},\mathcal{K})$-edges in $\mathcal{M}(\Delta)$.

Next we assume the following hypothesis is valid on $\mathcal{M}(\Delta)$.

Hypothesis (H5)

Let E be a $(\mathcal{G},\mathcal{L})$-edge.

(1) If $\phi(E)$ is a product of pieces relative to R, then $\theta_R(\phi(E)) \leq 3$.

(2) If $\phi(E)$ involves a generator a in A which is not a piece relative to R, then $\phi(E) \equiv u_1 \, a \, u_2$ where u_1 and u_2 are a - free.

(2.11) Lemma

Let D be a region of $\mathcal{M}(\Delta)$.

(1) If D is a $\mathcal{G}$-region then $d(D) \geq 2$.

(2) If D is a $\mathcal{K}$-region then $d(D) \geq 6$.

(3) If D is a $\mathcal{L}$-region then $d(D) \geq 7$.

Proof

(1) Let $E \in \beta(D)$. If $\phi(E)$ involves a generator which is not a piece relative to R, then using hypothesis (H5) and (H1) it is easy to show that $d(D) \geq 2$.

Now, if for each $E \in \beta(D)$, $\phi(E)$ is a product of pieces then, using hypothesis (H5) and Lemma (2.9) (ii), again $d(D) \geq 2$.

(2) By Remark (2.10) and Corollary (2.6), neither $(\mathcal{K},\mathcal{G})$-edges nor $(\mathcal{K},\mathcal{K})$-edges can exist in $\mathcal{M}(\Delta)$. Since the label on a $(\mathcal{K},\mathcal{L})$-edge is either $x^{\pm 1}$ or $y^{\pm 1}$, $d(D) \geq 6$ by hypothesis (H2).

(3) This follows easily from hypothesis (H3).

(2.12) <u>Lemma</u> (i) $\mathcal{L} \neq \emptyset$, and

(ii) $\mathcal{G} \neq \emptyset$.

<u>Proof</u> (i) Use Lemma (2.7) and the fact that $\phi(\delta(\mathcal{M}(\Delta))) \in \Delta$.

(ii) Using Lemmas (2.11) and (I.3.1), we can show that $\mathcal{G} \neq \emptyset$.

Next, we want to show that every countable group can be embedded in a countable group $K = \langle x,y \; ; B \rangle$ which satisfies (H2). We shall use an argument similar to that in the proof of Theorem (I.3.1).

Let F^* be a free product of nontrivial groups G_j. Then each non-identity element w of F^* has a unique representation in normal form $w=z_1 \ldots z_n$ where each of the letters z_i is a nontrivial element of the factors G_i and where no adjacent z_i, z_{i+1} come from the same factor. The integer n is the syllable length of w, written $|w|_{F^*}$. Let $u,v \in F^*$, $u = x_1 \ldots x_k$ and $v = y_1 \ldots y_s$ in normal form. A word w has <u>semi-reduced</u> form uv if w = uv and there is no cancellation between u and v. If y_1 and x_k are in different factors of F^*, then w = uv has <u>reduced form</u>. An element w in F^* with normal form $w = z_1 \ldots z_n$ is said to be <u>weakly cyclically reduced</u> if $z_n \neq z_1^{-1}$ or $|w|_{F^*} \leq 1$. An element w in F^* with normal form $w = z_1 \ldots z_n$ is said to be <u>cyclically reduced</u> if $|w|_{F^*} \leq 1$ or z_1 and z_n are in different factors of F.

A subset R^* of F^* is called symmetrized if every $r \in R^*$ is weakly cyclically reduced and every weakly cyclically reduced conjugate of r and r^{-1} is also in R^*.

A word b is called a <u>piece</u> if R^* contains distinct elements r_1 and r_2 with semi-reduced forms $r_1 \equiv bc_1$ and $r_2 \equiv bc_2$. Note that the last letter of b does not have to be a letter of the normal form of r_1 or r_2.

<u>Condition</u> $C'_{F^*}(\xi)$

If $r \in R^*$, $r = bc$ in semi-reduced form and b is a piece, then $|b|_{F^*} < \xi |r|_{F^*}$, and always $|r|_{F^*} > \frac{1}{\xi}$.

(2.15) <u>Theorem</u> (Lyndon (10))

Let F^* be a free product. Let R^* be a symmetrized subset of F^* which satisfies $C'_{F^*}(\xi)$ for $\xi \leq \frac{1}{6}$. Let N^* be the normal closure of R^* in F^*. If w is a nontrivial element of N^* then w has a reduced factorization $w = usv$ in reduced form, where there is a cyclically reduced $r \in R^*$ with $r = st$ in reduced form and $|s|_{F^*} > (1-3\xi)\ |r|_{F^*}$

(2.16) <u>Lemma</u> Every countable group can be embedded in a group K with a presentation $K = \langle x,y\ ;\ B\rangle$ that satisfies (H2).

<u>Proof</u>

Let $K' = \langle k_1, \ldots\ ;\ B'\rangle$ be any countable group. Let $F^* = K' * \langle x\rangle * \langle y\rangle$ be a nontrivial free product, where $\langle x\rangle$ and $\langle y\rangle$ are the infinite cyclic groups generated by x and y respectively.

$$\text{Let } r_i = k_i^{-1} \prod_{\ell=1}^{80} [(xy)^{\ell \cdot 10^{2i}} y] \quad , \quad i = 1, 2, \ldots$$

The symmetrized set R^* generated by the $r_i (i \geq 1)$ satisfies $C'_{F^*}(\frac{1}{10})$, since $[(xy)^{\ell \cdot 10^{2i}} y\, (xy)^{(\ell+1)10^{2i}} y]$ cannot be a subword of a piece. Let $N^* = \langle F^* \langle R^* \rangle$ and put $K = F^*/N^*$. By Theorem (2.15), if $w \neq 1$ in F^* and w belongs to N^*, then w contains at least $\frac{7}{10}$ of an element of R^*; and so K embeds all the factors of F^*. By the nature of our chosen relators r_i, K is generated by x and y. In fact we can eliminate the relations R^* by Tietze transformation, rewriting the relations B' in terms of x and y to give the presentation $\langle x,y ; B\rangle$ for K.

Now, if $w = 1$ in K, (hence $w \in N^*$), then $|w|_{F^*} \geq 6$, since $\frac{7}{10}$ of an element of K^* has syllable length much larger than 6. Thus $|w| \geq 6$.

Now, before we give the main proposition, we summarise what we have done in this section. We begin with a finitely presented group $G = \langle A ; R\rangle$ and $K = F(x,y)/B = \langle x,y ; B_0\rangle$ where B is a normal subgroup of $F(x,y)$ and B_0 is the set of all cyclically reduced words in B. We choose C to be the symmetrized subset of $F(A,x,y)$ generated by s_1 and s_2, where $s_1 \equiv x z_1$ and $s_2 \equiv y z_2$, and put $R' = R \cup B_0 \cup C$. Then $\mathcal{C}$ is the set of all cyclically reduced words in $(F(x,y) \cap N) - B$, where $N = \langle F(A,x,y) \langle R \rangle$. We assume the following:

(H1) R satisfies C(6), no element in R has the form az where $a \in A$ and z is a - free and if $a \in A$ which is not a piece relative to R, then no element r in R has the form $r \equiv a^m$, any m.

(H2) If $u \in B$, $u \neq 1$, then $|u| \geq 6$.

(H3) $\theta_{R \cup C}(z_1)$, $\theta_{R \cup C}(z_2) \geq 6$

(H4) $N \cap F(x,y) \neq B$.

Since (H4) is given, $\Delta \neq \phi$, and we can select an R'-diagram $\mathcal{M}(\Delta)$ with $\phi(\delta(\mathcal{M}(\Delta))) \in \Delta$ and for each R'-diagram $\mathcal{N}$, if $\phi(\delta(\mathcal{N})) \in \Delta$, then

$$\mathcal{F}^{\#}_{\mathcal{N}} \geq \mathcal{F}^{\#}_{\mathcal{M}(\Delta)}$$

Further, we assume

(H5) If E is an edge in $\mathcal{M}(\Delta)$ with $\sigma(E)$ or $\rho(E) \in \mathcal{G}$, then $\theta_R(\phi(E)) \leq 3$ provided that $\phi(E)$ is a product of pieces; otherwise $\phi(E) \equiv z_1 a z_2$ where a is not a piece relative to R, z_1 and z_2 are a - free.

Finally, we conclude this section with our main result.

(2.17) Proposition

Given, G,A,R,K,B,B_o,C,R',N,Δ and $\mathcal{M}(\Delta)$ as above with (H1) - (H5) valid. Then $\mathcal{M}(\Delta)$ is a suitable R'-diagram.

Proof

$\mathcal{M}(\Delta)$ is a suitable R'-diagram if $\mathcal{M}(\Delta)$ satisfies the conditions I - VIII of a suitable map. Now, condition I follows from Lemma (2.3); condition II follows from Lemma (2.7); conditions III and IV follow from Remark (2.8); condition V follows from Lemma (2.11); condition VI (i) follows from Corollary (2.6); condition VI (ii) follows from Remark (2.10); condition VII follows from Lemma (2.9)

Section 3

Throughout this section, let $\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ be a suitable map with $\mathcal{F} = \mathcal{G} \cup \mathcal{K} \cup \mathcal{L}$. In this section we establish some results which we may need in the next Chapter .

(3.1) Lemma Let D be a $\mathcal{G}$-region in $\mathcal{M}$ with $1 \leqslant (D,\mathcal{E})^{\#} \leqslant 3$. Assume that $i(D) \leqslant 5$. Let v be an interior vertex in $\beta(D)$. (For notation see convention (1.1)).

(1) If $[(D,\mathcal{E})\ ;\ v]^{\#} = o$, then $\underset{\sim}{d}(v,D) \leqslant \frac{3}{d(v)} - 1.$

(2) If $[(D,\mathcal{E})\ ;\ v]^{\#} > o$, then we have the following

(i) if $d(v) = 3$, then $\underset{\sim}{d}(v,D) = o$;

(ii) if $d(v) = 4$ and $[(D,\mathcal{E})\ ;\ v]^{\#} = 1$, then $\underset{\sim}{d}(v,D) \leqslant -\frac{3}{8}$;

(iii) if $d(v) = 4$ and $[(D,\mathcal{E})\ ;\ v]^{\#} = 2$, then $\underset{\sim}{d}(v,D) \leqslant -\frac{1}{4}$;

(iv) if $d(v) = 5$ and $[(D,\mathcal{E})\ ;\ v]^{\#} = 1$, then $\underset{\sim}{d}(v,D) \leqslant -\frac{3}{5}$;

(v) if $d(v) = 5$ and $[(D,\mathcal{E})\ ;\ v]^{\#} = 2$, then $\underset{\sim}{d}(v,D) \leqslant -\frac{2}{5}$;

(vi) if $d(v) = 5$ and $[(D,\mathcal{E})\ ;\ v]^{\#} = 3$, then $\underset{\sim}{d}(v,D) = -\frac{5}{12}$;

(vii) if $d(v) = 6$, then

$$\underset{\sim}{d}(v,D) \leqslant \frac{3 - d(v)}{d(v)} \quad \frac{1}{(D,\mathcal{E})^{\#}} \leqslant \frac{1}{6}.$$

Proof

(1) Since $1 \leq v^{\#} \leq d(v)$ and $3 - d(v) \leq 0$,

$$\underset{\sim}{d}(v,D) = \frac{3-d(v)}{v^{\#}} \leq \frac{3-d(v)}{d(v)} = \frac{3}{d(v)} - 1$$

(2) (i) This is obvious since $3 - d(v) \leq 0$.

(ii) Here v is either a $(\mathcal{G}, \mathcal{B}, \mathcal{X}, \mathcal{G})$-vertex or a $(\mathcal{G}, \mathcal{G}, \mathcal{X}, \mathcal{B})$-vertex, where $\mathcal{X}$ is either $\mathcal{B}$ or $\mathcal{G}$, since there are no $(\mathcal{K}, \mathcal{G})$-edges in $\mathcal{M}$.

$$\underset{\sim}{d}(v,D) = \frac{3-d(v)}{v^{\#}} \left| \frac{1}{[(D,\mathcal{B});v]^{\#}} + \frac{[\mathcal{B};v]^{\#}}{[(\mathcal{G},\mathcal{B});v]^{\#}} \right|$$

$$\leq \frac{3-4}{v^{\#}} [\frac{1}{1} + \frac{1}{2}] = -\frac{3}{v^{\#}.2} \leq -\frac{3}{8} \text{ since}$$

$$v^{\#} \leq 4 .$$

(iii) Here v is a $(\mathcal{G}, \mathcal{X}, \mathcal{Y}, \mathcal{B})$-vertex where

$\mathcal{X} \in \{\mathcal{G}, \mathcal{B}\}$ and $\mathcal{Y} \in \{\mathcal{B}, \mathcal{G}, \mathcal{K}\}$. Let $\mathcal{X} = \mathcal{B}$.

Case 1 - when $\mathcal{Y} = \mathcal{G}$; $\underset{\sim}{d}(v,D) = \frac{3-4}{v^{\#}} [\frac{1}{2} + \frac{[\mathcal{B};v]^{\#}}{4}] \leq -\frac{1}{4}$

Case 2 - when $\mathcal{Y} = \mathcal{B}$; $\underset{\sim}{d}(v,D) = \frac{3-4}{v^{\#}} [\frac{1}{2} + \frac{[\mathcal{B};v]^{\#}}{2}] \leq -\frac{1}{3}$

Case 3 - when $\mathcal{Y} = \mathcal{K}$; $\underset{\sim}{d}(v,D) = -\frac{1}{4} [\frac{1}{2} + \frac{3}{2}] = -\frac{1}{2}$

If $\mathcal{X} = \mathcal{G}$, then it is easy to show that $v^{\#} \leq 3$ and $\underset{\sim}{d}(v,D) \leq -(1/3)$.

(iv) Here v is either a $(\mathcal{G}, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \mathcal{G})$-vertex or a $(\mathcal{G}, \mathcal{G}, \mathcal{X}, \mathcal{Y}, \mathcal{B})$-vertex, where $\mathcal{X}, \mathcal{Y} \in \{\mathcal{G}, \mathcal{K}, \mathcal{B}\}$.

In all cases $\underset{\sim}{d}(v,D) =$

$$\frac{3-5}{v^{\#}}\left(\frac{1}{1} + \frac{[\mathcal{C};v]^{\#}+[\mathcal{K};v]^{\#}}{[(\mathcal{G};\mathcal{C});v]^{\#}}\right) \leq -\frac{3}{5} .$$

(v) Follows by a similar argument as that in the previous cases .

(vi) Note that $i(D) \leq 5$. Then $[\mathcal{C};v]^{\#} = 2$ and $[(\mathcal{G},\mathcal{C});v]^{\#} = 4$ (Use Jordan curve theorem). So

$$\underset{\sim}{d}(v,D) = -\frac{2}{4}\left[\frac{1}{3}+\frac{2}{4}\right] = -\frac{5}{12} .$$

(vii)

By definition, if an edge E contributes 1 to $[(D,\mathcal{C});v]^{\#}$ then $E \in (D,\mathcal{C})$. Thus $[(D,\mathcal{C});v]^{\#} \leq (D,\mathcal{C})^{\#} \leq 3$. When $d(v) \geq 6$, $\frac{3-d(v)}{d(v)}$ attains its maximum with $d(v) = 6$.

Let M be a $\mathcal{C}$-region. Let E be an edge in $\delta(M)$. In section 1, (Definition (1.8), we introduce the artificial degree of E relative to M and in Remark (1.10), we observed that if $(M,\mathcal{C})^{\#} \geq \alpha(M)$, then there exists an edge E with $\tilde{d}(E;M) > 0$.

We shall call an edge E a <u>positive</u> edge iff $\tilde{d}(E;M) > 0$, a <u>negative</u> edge iff $\tilde{d}(E;M) < 0$, and a <u>neutral</u> edge iff $\tilde{d}(E;M) = 0$. So, obviously, if $(M,\mathcal{C})^{\#} \geq \alpha(M)$, then there must exist a positive edge. Also if an edge E is not a negative edge then E must be a $(\mathcal{C},\mathcal{G})$-edge.

Let $w \equiv x_{j_1}^{\varepsilon_1} \ldots x_{j_m}^{\varepsilon_m}$ be a reduced word with $\varepsilon_i = \pm 1$, $i = 1, \ldots m$.

w is called a <u>same sign</u> word if $\varepsilon_1 = \ldots = \varepsilon_m$. Hence we have the following

(3.2) <u>Lemma</u> Assume that for each $M \in \mathcal{C}$, $\phi(\delta(M))$ is a same sign word. Then for each $M \in \mathcal{C}$ there are no (M,M)-edges in $\mathcal{M}$.

(3.3) <u>Lemma</u> Assume that for each $M \in \mathcal{C}$, $\phi(\delta(M))$ is a same sign word. Let E be an edge occurring in $\delta(M)$ with $\sigma(E) \in \mathcal{G}$, where M is a $\mathcal{C}$-region. If $i(\sigma(E)) = (\sigma(E), \mathcal{C})^{\#} = 3$, then E is a negative edge.

<u>Proof</u> Suppose not; and so $d(v) = 3$, all $v \in \beta(D) \cap \mathcal{V}$. Then examination of signs of symbols on the successor and predecessor edges gives a contradiction, (using the fact that $\phi(\delta(M))$ is a same sign word for each $M \in \mathcal{C}$).

Now, let $\mathcal{M} = \mathcal{M}(\Delta)$ be the suitable R'-diagram constructed in section 2, assuming that (H1) - (H5) are valid.

(3.4) <u>Lemma</u> Let D be a $\mathcal{G}$-region in $\mathcal{M}(\Delta)$ such that $i(D) = 2$. Then the edges which occur in $\delta(D)$ are $(D, \mathcal{C})$-edges.

<u>Proof</u> Use Lemma (2.9), condition VII, and hypotheses (H5) and (H1).

Let E be a $(\mathcal{C}, \mathcal{G})$-edge in $\mathcal{M}(\Delta)$. Then E is called a <u>stable</u> edge (or an S-edge) if $\phi(E)$ contains exactly one kind of letter, and E is called a <u>transition</u> (or a T-edge) otherwise.

(3.5) Lemma Let E be a $(\mathcal{C},\mathcal{G})$-edge in $\mathcal{M}(\Delta)$. E is a positive edge iff $\rho(E)$ is a region D such that $2 \leq i(D) \leq 4$ and

(i) if $i(D) = 2$, then $\sum_{v \in \beta(D),\ v \text{ interior}} \underset{\sim}{d}(v,D) > -1$,

(ii) if $i(D) = 3$, then $(D,\mathcal{C})^{\#} = 2$ and

$$\sum_{v \in \beta(D),\ v \text{ interior}} \underset{\sim}{d}(v,D) > -\frac{1}{2}$$

(iii) if $i(D) = 4$, then $(D,\mathcal{C})^{\#} = 1$ with

$$\sum_{v \in \beta(D),\ v \text{ interior}} \underset{\sim}{d}(v,D) > -\frac{1}{2}$$

Proof Easy argument from the definitions ,(H1) and (H5).

(3.6) Corollary Let M be a $\mathcal{C}$-region in $\mathcal{M}(D)$. Let E be an edge occurring in $\delta(M)$. Then $\tilde{d}(E;M) \leq 1$.

(3.7) Let M be a $\mathcal{C}$-region. Let E be an edge occuring in $\delta(M)$ with $\sigma(E) \in \mathcal{G}$ and $\dfrac{\alpha(\sigma(E))-i(\sigma(E))}{(\sigma(E),\mathcal{C})^{\#}} \leq 1$. If E is a negative edge then $\tilde{d}(E,M) \leq -\frac{1}{6}$.

Proof Use Lemma (3.1) and the defintions.

Chapter IV

THE SQ-UNIVERSALITY OF C(6)-GROUPS

This chapter is devoted to study the SQ-universality of C(6)-groups. It has been proved, (Theorem (II.4.10)), that C(6)-groups, with few exceptions, contain non-abelian free subgroups. In general, the property of having non-abelian free subgroups does not imply SQ-universality, (for example, $PSL(n,\mathbb{Z})$, $n \geq 3$). However, we shall see that for C(6)-groups this property characterizes SQ-universality. In particular, the aim of this chapter is to prove the following result.

MAIN THEOREM

Let $G = \langle A; R \rangle$ be a nontrivial finitely related C(6)-group. Then G is an SQ-universal group unless G is isomorphic to one of the following three groups:

(i) $G_1 = \langle a; \phi \rangle$,

(ii) $G_2 = \langle a; \; a^m, a^{-m} \rangle$, $m \neq 0$,

(iii) $G_3 = \langle a,b; \; a^2, a^{-2}, b^2, b^{-2} \rangle$.

Note that if A is infinite, then there are some generators in A which do not occur in R, and hence $G = \langle A;R \rangle$ is a nontrivial free product which is SQ-univeral, by Theorem (I.3.1). Therefore, w.l.o.g., we consider $G = \langle A;R \rangle$ as a finitely presented C(6)-group.

Now, $G = \langle A;R \rangle \cong G^* = \langle A^*,R^* \rangle$ if and only if G can be transformed into G^* by a finite sequence of Tietze transformations. From now on, we will restrict our attention to a "suitable" presentation of G by repeated applications of Tietze transformation; so we can assume that there is no element in R of the form az, where $a \in A$, $z \in F(A)$ and z is a-free. If an element $a \in A$, is not a piece relative to R with $a^m \in R$, $m > 0$, and G is neither $G_2 = \langle a;a^m,a^{-m} \rangle$ nor $G_3 = \langle a,b;\ a^2,a^{-2},b^2,b^{-2} \rangle$, then G is cyclic or a nontrivial free product which is SQ-universal by Theorem (I.3.1). Hence the only non-SQ-universal C(6)-groups which do not satisfy (H1) are the groups which are isomorphic to one of the following groups

$$G_1 = \langle a; \emptyset \rangle,\quad G_2 = \langle a;\ a^m,a^{-m} \rangle \quad \text{and}$$

$$G_3 = \langle a,b;\ a^2,a^{-2},b^2,b^{-2} \rangle$$

Therefore to prove the Main Theorem, it is sufficient to establish the following proposition.

(0.1) Proposition

Every finitely presented group $G = \langle A;R \rangle$ which satisfies (H1) is SQ-universal.

Note that if $G = \langle A;R \rangle$ satisfies (H1), then card $A \geq 2$. The proof of Proposition (0.1) will be divided into cases according to various assumptions.

<u>case 1</u> - There exists $a \in A$ which is not a piece relative to R such that the unique relator r in R which involves a has the form $r \equiv (az)^n$, $n \geq 2$, z is a-free and $z \not\equiv 1$.

<u>case 2</u> - when every generator in A is a piece relative to R and there exist two distinct generators a and b in A such that $a^m \in R$ and $b^n \in R$, $n,m > 0$.

<u>case 3</u> - when every generator in A is a piece relative to R and there exists $a \in A$ with

(i) $a^m \notin R$, for any $m \neq 0$,

(ii) $\theta_R(a^p) = 1$.

<u>case 4</u> - when every generator in A is a piece relative to R and there exists $a \in A$ with

(i) $a^m \notin R$, for any $m \neq 0$

(ii) $\theta_R(a^p) = 2$,

(iii) for each $b \in A$, $b \neq a$, the unique relator r involving a^p is not of the form $r \equiv (a^p b^{-\varepsilon} z b^{\varepsilon})^n$, $n \geq 1$, $\varepsilon = \pm 1$ where z is a^p-free.

<u>case 5</u> - when every generator in A is a piece relative to R and there exists $a \in A$ with

(i) $a^m \notin R$, for any $m \neq 0$.

(ii) $\theta_R(a^p) = 2$,

(iii) card $A \geq 3$,

(iv) there exists $c \in A$, $c \neq a$, such that the unique relator r involving a^p is of the form $r \equiv (a^p c^{-\varepsilon} z c^{\varepsilon})^m$, $m \geq 1$, $\varepsilon = \pm 1$ where z is a^p-free.

case 6 – when $A = \{a,b\}$, where a and b are pieces relative to R such that

(i) $a^m \notin R$, for any $m \neq 0$,

(ii) $\theta_R(a^p) = 2$,

(iii) the unique relator r in R involving a^p is of the form $r \equiv (a^p b^{\varepsilon} z b^{-\varepsilon})^m$, $\varepsilon = \pm 1$, $m \geq 1$, where z is a^p-free.

Case i will be done in section i, $i = 1,2,\ldots,6$.

Now, every countable group can be embedded in a group $K = F(x,y)/B$ which satisfies (H2). If K is not embeddable in a quotient of G, we choose suitable words u and w in $F(A,x,y)$ and then put

$$s_1 \equiv x\, w \prod_{j=2}^{L+1} (u\, w^j) \quad \text{and}$$

$$s_2 \equiv y\, u \prod_{j=L+2}^{2L} (w^j\, u),$$

where

(1) $L = 2k$, and $k \geq 1000$,

(2) if u begins with a letter a and ends with a letter a',

(a and a' are any two letters in $A \cup A^{-1}$, possibly $a = a'$), then w begins with a letter distinct from a' and ends with a letter distinct from a,

(3) u is not a subword of w^i, $1 \leq i \leq 2L$.

Let C be the set of all cyclic permutations of $s_1^{\pm 1}$ and $s_2^{\pm 1}$. Then C satisfies (H3) since, by (2) and (3), a piece relative to $R \cup C$ must be a subword of $(w^i u w^{i+1})$, some i. Put $R' = C \cup R \cup B_o$, where B_o is the set of all cyclically reduced words in B. Let $N = \langle F(A,x,y) \langle R' \rangle$. Since K is not embeddable in $F(A,x,y)/N$, $N \cap F(x,y) \neq B$. i.e. condition (H4) is valid. Hence there exists an R'-diagram $\mathcal{M}(\Delta) = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$, where $\mathcal{F} = \mathcal{G} \cup \mathcal{K} \cup \mathcal{L}$ (see the remarks preceding Lemma (III.2.1)). If $\mathcal{M}(\Delta)$ satisfies (H5), then $\mathcal{M}(\Delta)$ is a suitable R'-diagram, by Proposition (III.2.17). We shall show that for each $\mathcal{L}$-region M of $\mathcal{M}(\Delta)$,

$$\sum_{E \text{ occurs in } \delta(M)} \gamma(E;M) < -6$$

which contradicts Proposition (III.1.9).

Throughout this chapter we shall constantly use the following notation. Let M be a $\mathcal{L}$-region in $\mathcal{M}(\Delta)$.

(1) We write $\delta(M) = (E_1, E_2, \ldots, E_n)$, where $\varphi(E_1) = x^{\pm 1}$ or $y^{\pm 1}$.

(2) For each $2 \leq k \leq n$, if E_k is an $(M, \mathcal{G})$ - edge, then we write (i) $D_k = \sigma(E_k)$, and

(ii) $\delta(D_k) = (E_{k,1}, E_{k,2}, \ldots, E_{k,n(k)})$, where $E_{k,1} = E_k^{-1}$.

Evidently, if E_{k+1} is an $(M,\mathcal{S})$-edge and $d(\mu(E_k)) = 3$, then $E_{k+1,2} = E_{k,n(k)}^{-1}$.

(iii) $r_k = \varphi(\delta(D_k)) \equiv e_{k,1}\, e_{k,2} \cdots e_{k,n(k)}$, where $e_{k,j} \equiv \varphi(E_{k,j})$, $1 \leq j \leq n(k)$.

(3) We recall that a piece relative to $R \cup C$ is a subword of $(w^i\, u\, w^{i+1})^{\pm 1}$. Hence for any $\mathcal{L}$-region M, it is possible to define subpaths γ_j, $1 \leq j \leq L/2$ (or $(L/2)+1 \leq j \leq L$), of $\delta(M)$ with the following properties.

(i) If $\varphi(\delta(M)) \equiv s_1^{\varepsilon}$, $\varepsilon = \pm 1$, then $\delta(M) = (E_1, \gamma_1, \gamma_2, \ldots, \gamma_{L/2})$ where $\varphi(E_1) \equiv x^{\pm 1}$.

(ii) If $\varphi(\delta(M)) \equiv s_2^{\varepsilon}$, $\varepsilon = \pm 1$, then $\delta(M) = (E_1, \gamma_{(L/2)+1}, \ldots, \gamma_L)$ where $\varphi(E_1) \equiv y^{\pm 1}$.

(iii) $(u\, w^{2j}\, u)^{\varepsilon}$ is a subword of $\varphi(\gamma_j)$.

(iv) $\varphi(\gamma_j)$ is a subword of $(w^{2j-1}\, u\, w^{2j}\, u\, w^{2j+1})^{\varepsilon}$.

(v) For each j, $1 \leq j \leq (L/2) - 1$ (or $(L/2)+1 \leq j \leq L-1$), the label on the last edge in γ_j must involve a subword of u^{ε}.

Note that for each $1 \leq j \leq L$, $\varphi(\gamma_j)$ must involve w^{2j} and exactly two u's.

Section 1

Let $G = \langle A; R \rangle$ be a finitely presented group which satisfies (H1). Thus card $A \geq 2$. Let $a \in A$ be any generator which is not a piece relative to R. Since G satisfies (H1), the unique relator which involves a has the form $(az)^m$, where $m \geq 2$, z is a-free and $z \not\equiv 1$, (see the introduction).

Now z begins with c and ends with c' (possibly $c = c'$), where $c \not\equiv a^{\varepsilon}$ and $c' \not\equiv a^{\varepsilon}$, $\varepsilon = \pm 1$. Put $w \equiv a^{L/2}$ and $u \equiv b$, where b is a letter in $A \cup A^{-1}$ such that z neither begins nor ends with b. (Such b must exist for, at worst, when card $A = 2$ with $A = \{a,b\}$ must begin and end with b^{η}, $\eta = \pm 1$, and if $\eta = 1$, replace b by b^{-1} as a basic generator). Hence (H3) is valid and so there exists an R'-diagram $\mathcal{M}(\Delta)$. (See the introduction).

<u>(1.1) Lemma</u>

Let E be a $(\mathcal{L},\mathcal{S})$-edge in $\mathcal{M}(\Delta)$. Then E is an S-edge with $\varphi(E) = a^{\pm 1}$ or $b^{\pm 1}$.

<u>Proof</u> Since z neither begins nor ends with b, E must be an S-edge. If $\varphi(E) \equiv a^{\pm n}$, then $n = 1$, by condition (H1) and the fact that a is not a piece relative to R. So $\varphi(E) \equiv a^{\pm 1}$ or $b^{\pm 1}$ as required.

<u>(1.2) Corollary</u> $\mathcal{M}(\Delta)$ is a suitable R'-diagram.

<u>Proof</u>

If $\varphi(E) \equiv b^{\varepsilon}$, $\varepsilon = \pm 1$ and b is a piece relative to R, then $\theta_R(\varphi(E)) = 1$. Thus (H5) is satisfied. As (H1) - (H4) are valid, $\mathcal{M}(\Delta)$ is a suitable R'-diagram by Proposition (III.2.17).

We recall that $\delta(M) = (E_1, E_2, \ldots, E_n)$, where $\varphi(E_1) \equiv x^{\pm 1}$ or $y^{\pm 1}$, (for notation see the introduction).

<u>(1.3) Lemma</u> Let E_m, $2 \leq m \leq n$, be an $(M, \mathcal{G})$-edge that occurs in $\delta(M)$.

(1) If $(D_m, \mathcal{L})^{\#} = 1$, then

(i) $\varphi(E_m) \equiv b^{\varepsilon}$, $\varepsilon = \pm 1$,

(ii) b is a piece relative to R,

(iii) τ_m does not involve $a^{\pm 1}$, and

(iv) $i(D_m) \geq 6$.

(2) If $(D_m, \mathcal{L})^{\#} = 2$ or 3, the $i(D_m) \geq 4$.

<u>Proof</u> (1) (i) - (iii) follow from the fact that G satisfies (H1). Since $\Theta_R(\varphi(E_m)) = 1$, then $i(D_m) \geq 6$ by Lemma (III.2.9).

(2) Since z neither ends nor begins with b and using (H1), $i(D_m) \geq 4$.

<u>(1.4) Corollary</u> For each edge E which occurs in $\delta(M)$, $\tilde{d}(E;M) \leq 0$.

<u>Proof</u>

If E is an $(M, \mathcal{G})$-edge, then $\tilde{d}(E;M) \leq 0$, by using Lemma (1.3). If $\sigma(E) \notin \mathcal{G}$, then by the definition of artical degree of E relative M, $\tilde{d}(E;M) = -1$.

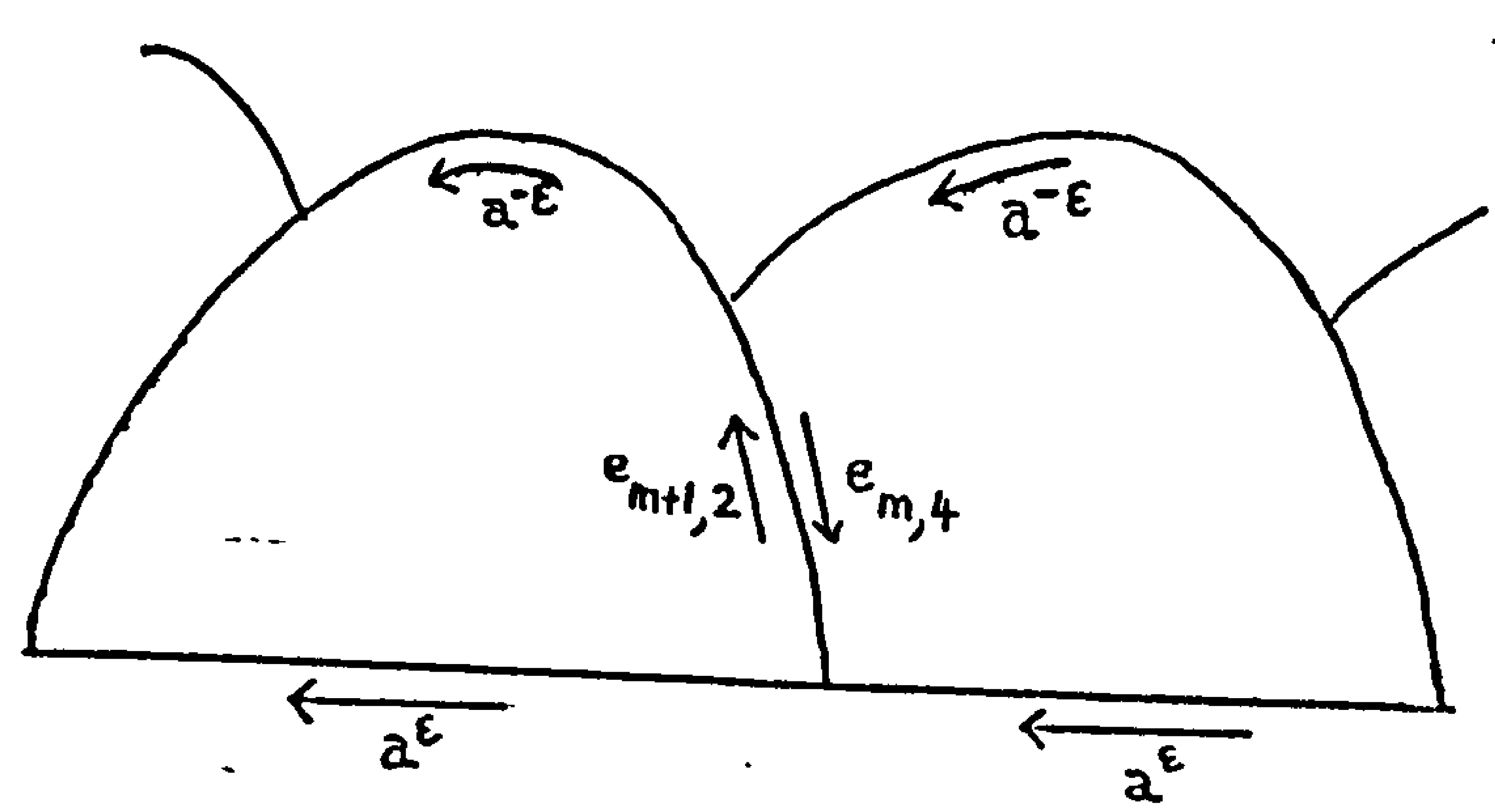
(1.5) Lemma

Let E_m, E_{m+1} be any two $(M,\mathcal{G})$-edges which occur in $\delta(M)$, $2 \leq m \leq n-1$. If $i(D_m) = i(D_{m+1}) = 4$, then $d(\mu(E_m)) \neq 3$.

Proof Suppose that $d(\mu(E_m)) = 3$. By Lemma(1.3), $e_m \equiv e_{m-1} \equiv a^{\varepsilon}$. Let $\delta(D_m) = (E_{m,1}, E_{m,2}, E_{m,3}, E_{m,4})$, and

$\delta(D_{m+1}) = (E_{m+1,1}, E_{m+1,2}, E_{m+1,3}, E_{m+1,4})$. Then

$E_{m,4} = E^{-1}_{m+1,2}$. So $e_{m,1} \equiv e_{m,3} \equiv e_{m+1,1} \equiv e_{m+1,3} \equiv a^{-\varepsilon}$, $\varepsilon = \pm 1$, because $\varphi(E_m) \equiv a^{\varepsilon}$, where $r_m \equiv a^{-\varepsilon} e_{m,2}\, a^{-\varepsilon} e_{m,4}$ and $r_{m+1} \equiv a^{-\varepsilon} e_{m+1,2}\, a^{-\varepsilon} e_{m+1,4}$. Since a is not a piece relative to R, $e_{m,2} \equiv e_{m,4} \equiv e^{-1}_{m+1,2}$. So we can write



$r_m \equiv a^{-\varepsilon} e^{-1}_{m+1,2}\, a^{-\varepsilon} e_{m,4}$ which is absurd since a is not a piece relative to R. (Compare r_m and r_{m+1} to show a contradiction).

(1.6) Proposition Let $G = \langle A;R \rangle$ be a finitely presented group which satisfies (H1). Assume that there exists $a \in A$ which is not a piece relative to R. Then G is SQ-universal.

<u>Proof</u> Suppose not; then we shall have a contradiction by using the method suggested in the introduction.

Now, the unique relator in R which involves a has the form $(a\ z)^m$, where $m \geq 2$, z is a-free and $z \not\equiv 1$. Assume that z neither ends nor begins in b and put $w \equiv a^{L/2}$ and $u \equiv b$. By corollary (1.2), $\mathfrak{M}(\Delta)$ is a suitable R'-diagram. Then there exists a $\mathcal{L}$-region M such that

$$\sum_{E \text{ occurs in } \delta(M)} \tilde{d}(E;M) \geq -6,$$

by Proposition (III.1.9).

Let $\delta(M) = (E_1, E_2, \ldots, E_n)$, where $\delta(E_1) = x^{\pm 1}$ or $y^{\pm 1}$. We write $\delta(M) = (E_1, \gamma_1, \ldots, \gamma_{L/2})$, when $\varphi(\delta(M)) \equiv s_1^{\pm 1}$ and $\delta(M) = (E_1, \gamma_{L/2+1}, \gamma_{L/2+2}, \ldots, \gamma_L)$ when $\varphi(\delta(M)) \equiv s_2^{\pm 1}$. (see the introduction). Hence for each $1 \leq j \leq L$, $\varphi(\gamma_j)$ must involve exactly two u's and w^{2j}.

Suppose that $\delta(M) = (E_1, \gamma_1, \gamma_2, \ldots, \gamma_{L/2})$. If there exists k, $1 \leq k \leq L/2$, such that any edge E occuring in γ_k is an $(M, \mathcal{G})$-edge, then there is a subpath γ_k^* of γ_k with $\varphi(\gamma_k^*) \equiv w^{2k}$, (by using Lemma (1.1)). Then γ_k^* has exactly Lk edges. Let $\gamma_k^* = (E_\ell, \ldots, E_t)$, $2 < \ell < t < n$, and $t - \ell = L k - 1$. By applying Lemma (1.5) inductively, we can show that for each $\ell \leq i \leq t-1$, $d(\mu(E_i)) \geq 4$; and so $\tilde{d}(E;M) \leq -\frac{1}{4}$ by Lemma (III.3.1) and (IV.1.3). Hence

$$\sum_{E \text{ occurs in } \delta(M)} \tilde{d}(E;M) \leq \sum_{E \text{ occurs in } \gamma_k} \tilde{d}(E;M) \leq -\frac{L k}{4} < -6,$$

which is absurd. Thus for each $1 \leq j \leq L/2$, there is an $(M, \mathcal{L})$-edge occuring in γ_j; and so $\sum_{E \text{ occurs in } \gamma_j} \tilde{d}(E;M) \leq -1.$

Therefore $\sum_{E \text{ occurs in } \delta(M)} \tilde{d}(E;M) \leq -L/2$ which is impossible.

By the same argument as above we can show that

$$\sum_{E \text{ occurs in } \delta(M)} \tilde{d}(E;M) < -6, \quad \text{when } \delta(M) = (E_1, \gamma_{L/2+1}, \ldots, \gamma_L),$$

which is absurd.

Section 2

Let $G = \langle A;R \rangle$ be a finitely presented group which satisfies (H1). Throughout this section we assume that every generator in A is a piece relative to R and there exist two distinct generators a and b in A such that $a^{m'} \in R$ and $b^{n'} \in R$, $m',n' > 0$.

Let R_o be the subset of R involving all the elements of R which are a proper power of a generator in A. Let a^p be the maximal power of a occuring in any relator of $R - R_o$. (see Convention (II.4.2)). Let b^q be the maximal power of b occuring in any relator of $R - R_o$. Thus $m' \geq 5p+1$ and $n' \geq 5q+1$ (by using the C(6) condition).

We shall consider five cases.

case 1 Either $p \geq 2$ or $q \geq 2$.

Without loss of generality we assume that $p \geq 2$, and put $w \equiv (a^{2p+1} b^{q+1})^{L/2}$ and $u \equiv a^{2p+1} b^{q+1} a^{2p} b^{q+1}$. Hence (H3) is valid, and there exists an R'-diagram $\mathfrak{M}(\Delta)$ (see the introduction). Then for each edge $E \in (M, \mathcal{S})$, where M is a $\mathcal{L}$-region in $\mathfrak{M}(\Delta)$, $\theta_R(\varphi(E)) \leq 3$. Therefore (H5) is valid; and hence $\mathfrak{M}(\Delta)$ is a suitable R'-diagram. Let M be a $\mathcal{L}$-region in $\mathfrak{M}(\Delta)$ with $\delta(M) = (E_1, E_2, \ldots, E_n)$, where $\varphi(E_{1,1}) \equiv e_{1,1} \equiv x^{\pm 1}$ or $y^{\pm 1}$.

The following Lemma is generally valid and will be applied in particular to case I.

(2.1) Lemma

Let $\mathfrak{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ with $\mathcal{F} = \mathcal{L} \cup \mathcal{K} \cup \mathcal{S}$ be a suitable map. Let E be an $(\mathcal{L}, \mathcal{S})$-edge in $\mathfrak{M}$. Let $\sigma(E) = D$ and $\rho(E) = M$. Assume that $\theta_R(\varphi(E')) \leq 2$, all $E' \in \beta(D)$. Then (i) $\tilde{d}(E,M) \leq 0$;

(ii) if $\theta_R(\varphi(E)) = 1$, then $\tilde{d}(E;M) \leq -1/3$; and

(iii) if there exists a vertex $v \in \beta(D)$ with $d(v) \geq 4$, then $\tilde{d}(E;M) \leq -1/6$.

Proof (i) Clearly, $i(D) \neq 2$; and if $i(D) = 3$, then $(D, \mathcal{L})^{\#} = 3$ whence $\tilde{d}(E;M) = 2 \sum_{v \in \beta(D)} \underset{\sim}{d}(v, D) \leq 0$.

For $i(D) = 4$, $(D, \mathcal{L})^{\#} \geq 2$; and so

$$\tilde{d}\,(E;M) \leq \frac{2}{(D,\mathcal{C})^{\#}} \sum_{v' \in \beta(D)} \underset{\sim}{d}\,(v',D) \leq 0.$$

Obviously, $\tilde{d}\,(E;M) \leq 0$ for $i(D) \geq 5$.

(ii) Since $\theta_R(\varphi(E)) = 1$, $i(D) \geq 4$. For $i(D) = 4$, $(D,\ell)^{\#} \geq 3$ whence

$$\tilde{d}(E;M) = -\frac{1}{3} + \frac{2}{(D,\ell)^{\#}} \sum_{v \in \beta(D)} \underset{\sim}{d}(v,D) \leq -\frac{1}{3}.$$

For $i(D) = 5$, $(D,\ell)^{\#} \geq 2$ whence

$$\tilde{d}\,(E;M) = -\frac{1}{2} + 2 \sum_{v \in \beta(D)} \underset{\sim}{d}(v,D) \leq -\frac{1}{2}.$$

For $i(D) \geq 6$, clearly, $\tilde{d}\,(E;M) \leq -1$.

(iii)

If $[(D,\ell);v]^{\#} \leqslant 2$ then $\underset{\sim}{d}(v,D) \leqslant -\frac{1}{4}$ and the result is clear.

Suppose that $[(D,\ell);v]^{\#} \geqslant 3$. If $i(D) = 3$ then, of course, $(D,\ell)^{\#} = 3 = [(D,\ell);v]^{\#}$ whence $d(v) \geqslant 12$, (there are no (M,M)-edges), and $d(v,D) \leqslant -\frac{1}{4}$ again. If $i(D) \geqslant 4$ then $d(E;M) \leqslant -\frac{1}{3}$.

(2.2) Lemma

Let E_m be an $(M,\mathcal{S})$-edge in $\mathcal{M}(\Delta)$, where $2 \leq m \leq n-1$. Let $\sigma(E_m) = D_m$ and $\varphi(\delta(D_m)) \equiv r_m$. If

$r_m \in R - R_o$, where R_o is the subset of R involving all elements of R which are a proper power of a generator in A, then $\theta_R(\varphi(E)) \leq 2$, all $E \in \beta(D_m)$.

<u>Proof</u> $\theta_R(\varphi(E)) = 3$ if and only if $\varphi(E) \equiv a^{\pm(2p+1)}$.

<u>(2.3) Lemma</u>

Let E_m be an $(M, \mathcal{G})$-edge, where $2 \leq m \leq n-1$. Suppose that $i(D_m) = 4$ and $(D_m, \mathcal{L})^{\#} = 1$. If $d(\mu(E_m)) \geq 4$, then $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{8}$.

<u>Proof</u> Suppose not.

We have $\tilde{d}(E_m;M) = 1 + 2 \sum_{v \in \beta(D)} \underset{\sim}{d}(v, D_m)$

and so $\tilde{d}(E_m;M) \leq 1 + 2\, d(\mu(E_m), D_m)$. By using Lemma (III.3.1), we have the following possibilities:

(i) $d(\mu(E_m)) = 4$ and $\tilde{d}(E_m;M) \leq \frac{1}{4}$

(ii) $d(\mu(E_m)) = 5$ and $\tilde{d}(E_m;M) \leq -\frac{1}{5}$

(iii) $d(\mu(E_m)) \geq 6$ and $\tilde{d}(E_m;M) \leq -\frac{1}{6}$

Since $\theta_R^*(r_m) \geq 6$, our choice of w and u ensures that $\varphi(E_m) \equiv a^{\varepsilon(2p+1)}$ whence

$\varphi(E_{m+1}) \equiv b^{\varepsilon k}$, $1 \leq k \leq q+1$. This certainly means that $\theta_R(\varphi(E)) \leq 2$, all $E \in \beta(D_{m+1})$; and so Lemma (2.1) (iii), gives the required result for the two cases (ii)and(iii).

Suppose (i) occurs; then $E_{m+1,2} \in (\mathcal{G}, \mathcal{G})$ and so we can assume that $i(D_{m+1}) = 4$ or 5. If $i(D_{m+1}) = 4$, then $(D_{m+1}, \mathcal{L})^{\#} = 2$ or 3; and we have either

$$\tilde{d}\,(E_{m+1};M) \leq \underset{\sim}{d}(\mu(E_m), D_{m+1}) \quad \text{or}$$

$$\tilde{d}\,(E_{m+1}; M) \leq \frac{2}{3}\ \underset{\sim}{d}\,(\mu(E_m), D_{m+1}) - \frac{1}{3}$$

Since $[(D_{m+1}; \mathcal{L}), \mu(E_m)]^{\#} = 1$, we are done by Lemma III.3.1.

If $i(D_{m+1}) = 5$, then $\tilde{d}(E_{m+1}; M) \leq 2\ \underset{\sim}{d}(\mu(E_m), D_{m+1}) \leq -\frac{3}{4}$, by Lemma III.3.1 and we are done.

(2.4) Lemma

Let E_m be an $(M, \mathcal{G})$-edge, where $2 \leq m \leq n-1$. Suppose that $i(D_m) = 3$ and $(D_m, \mathcal{L})^{\#} = 2$. Let $\delta(D_m) = (E_{m,1}, E_{m,2}, E_{m,3})$ where $E_{m,3}$ is a $(D_m, \mathcal{G})$-edge. If $d(\mu(E_m)) \geq 4$, then $\tilde{d}\,(E_m; M) + \tilde{d}\,(E_{m+1};M) \leq -\frac{1}{8}$.

Proof By an argument similar to that in Lemma (2.3).

(2.5) Lemma

(i) Let E_m be either a neutral or a positive $(M, \mathcal{G})$-edge, where $2 \leq m \leq n-3$.

(1) If E_m is a T-edge, then it is a neutral edge. Moreover, there exists an integer t, $m+1 \leq t \leq m+5$, such that

$$\sum_{j=m+1}^{t} \tilde{d}\,(E_j;M) \leq -\frac{1}{3}$$

(2) If E_m is an S-edge, then there exists an integer t, $m+1 \leq t \leq m+5$ such that

$$\sum_{j=}^{t} \tilde{d}\,(E_j;M) \leq -\frac{1}{12}$$

(ii) If E_m is a negative edge with $\varphi(E_m)$ involves b^{η}, $\eta = \pm 1$ then $\tilde{d}\,(E_m;M) \leq -\frac{1}{6}$.

<u>Proof</u> (1) As E_m is a T-edge, $e_{m,1}$ is a subword of $(a^p b^q)^{\varepsilon}$ or $(b^q a^p)^{\eta}$ $\varepsilon, \eta = \pm 1$. So $\theta_R(e_{m,1}) \leq 2$; and by Lemma (2.2), $\theta_R(\varphi(E)) \leq 2$, all $E \in \beta(D_m)$ since r_m involves more than one generator.

Therefore, by Lemma (2.1), $\tilde{d}\,(E_m;M) \leq 0$; and so $\tilde{d}\,(E_m;M) = 0$. Clearly, $i(D_m) \neq 2$. If $i(D_m) = 3$, then $(D,\ell)^{\#} = 3$ and so by Remark (III.3.7), $\tilde{d}\,(E\,;M) \leq -\frac{1}{6}$ which is absurd. If $i(D_m) \geq 6$, then $\tilde{d}\,(E_m;M) \leq -1$, which is impossible. Therefore $i(D_m) = 4$ or 5.

Let $i(D_m) = 4$. Then $(D_m,\ell)^{\#} = 2$ and $d(v) = 3$, all $v \in \beta(D_m)$, since $\tilde{d}(E_m;M) = 0$. (Lemma (2.1) (iii)). If E_{m+1} is an (M,ℓ)-edge, then $\tilde{d}\,(E_{m+1};M) = -1$; and so $\tilde{d}\,(E_m;M) + \tilde{d}\,(E_{m+1};M) \leq -1$. Thus $t = m+1$.

Let E_{m+1} be an $(M,\mathcal{S})$-edge, Now, $r_{m+1} \in R - R_o$ for if $r_{m+1} \in R_o$, then there is a cancellation in r_m which is impossible

(see the introduction of this section for the definition of R_o).
Thus if $e_{m+1,1} \equiv a^{\varepsilon\ell}$ or $e_{m+1,1} \equiv b^{\varepsilon k}$, $\varepsilon = \pm 1$, then $1 \leq \ell \leq p$ and $1 \leq k \leq q$. By Lemma (2.1) (ii), $\tilde{d}(E_{m+1};M) \leq -\frac{1}{3}$, since $\theta_R(e_{m+1,1}) = 1$. Hence $t = m + 1$.

Now, $e_{m+1,1} \equiv a^{\varepsilon\ell_1} b^{\varepsilon k_1}$ or $b^{\varepsilon k_1} a^{\varepsilon\ell_1}$, $1 \leq k_1 \leq q$, $1 \leq \ell_1 \leq p$, $\varepsilon = \pm 1$. If E_{m+2} is an (M, ℓ)-edge, then $t = m+2$. Let E_{m+2} be an $(M, \mathcal{g})$-edge and let $e_{m+1,1} \equiv b^{\varepsilon k_1} a^{\varepsilon\ell_1}$. If $e_{m+2,1} \equiv b^{\varepsilon k_2}$, $1 \leq k_2 \leq q$, then $r_{m+2} \not\equiv b^{n'}$, and hence $t = m + 2$. Thus $e_{m+2,1} \equiv a^{\varepsilon\ell_2} b^{\varepsilon(q+1-k_1)}$, $1 \leq \ell_2 \leq p$. If $e_{m+3,1} \equiv a^{\varepsilon\ell_3}$, then $r_{m+3} \not\equiv a^{m'}$, $(a^{m'} \in R)$, and hence $t = m + 3$. Thus $e_{m+3,1} \equiv b^{\varepsilon k_2} a^{\varepsilon p}$ and $\ell_2 = p$, where $1 \leq k_2 \leq q$. If $e_{m+4,1} \equiv b^{\varepsilon k_3}$, then $r_{m+4} \not\equiv b^{n'}$, $(b^{n'} \in R)$, and hence $t = m + 4$. Thus $e_{m+4,1} \equiv a^{\varepsilon\ell_4} b^{\varepsilon(q+1-k_3)}$, $1 \leq \ell_4 \leq p$. Finally, $e_{m+5,1}$ must be $a^{\varepsilon\ell_5}$, and $r_{m+5} \not\equiv a^{n'}$ hence $t = m + 5$.

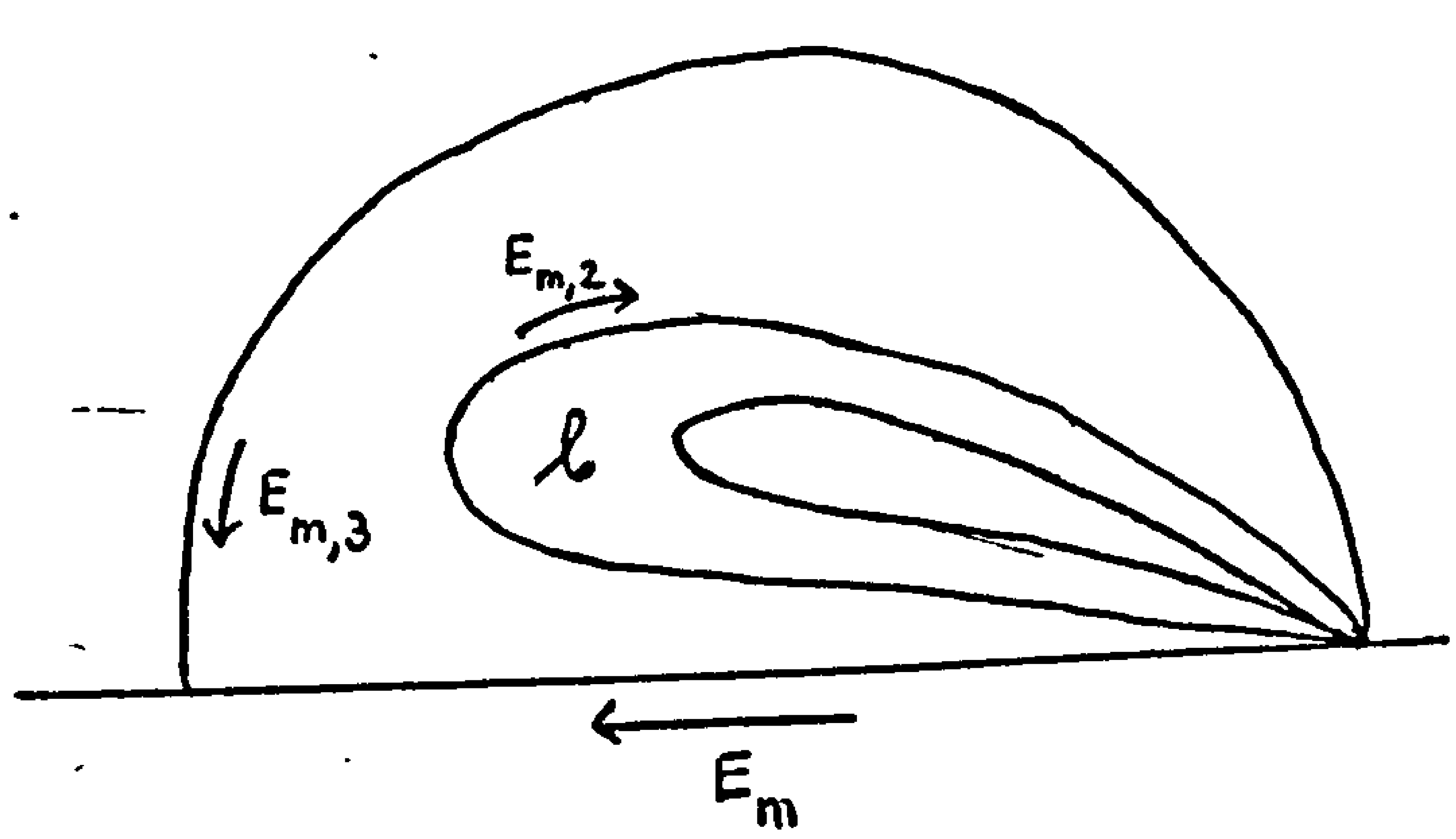
For $e_{m+1} \equiv a^{\varepsilon\ell_1} b^{\varepsilon k_1}$, $1 \leq \ell_1 \leq p$, $1 \leq k_1 \leq q$ and $\varepsilon = \pm 1$, it is not hard to see that $t = m + 4$.

(2) Now E_m is an S-edge. Let $i(D_m) = 3$, then $(D_m, \ell)^{\#} = 2$ since $\tilde{d}(E_m;M) \geq 0$; and so $e_{m,1}$ must be either $a^{\varepsilon 2p}$ or $a^{\varepsilon(2p+1)}$. We recall that $\delta(D_m) = (E_{m,1}, E_{m,2}, E_{m,3})$. Let $E_{m,3}$ be the $(D_m, \mathcal{g})$-edge. If $d(\mu(E_m)) \geq 4$, then by Lemma (2.4), $t = m+1$.

Let $d(\mu(E_m)) = 3$. Then $E_{m,3} = E^{-1}_{m+1,2}$ and so $e_{m+1,2}$ must begin with a^{-1}. It follows that $e_{m+1,1} \equiv b^{\varepsilon k}$,

$1 \leq k \leq q$ and $\varepsilon = \pm 1$ since r_{m+1} involves more than one letter. It is easy to see that $d(\lambda(E_m)) \geq 4$. Thus $\tilde{d}(E_m;M) \leq \frac{1}{4}$. Now, if $d(\mu(E_{m,2})) = 3$, then $e_{m+1,3} \equiv b^{k_1}$, $1 \leq k_1 \leq q$. Since $r_{m+1} \not\equiv b^{n'}$, $(b^{n'} \in R)$, $\theta_R(\varphi(E)) \leq 2$, for all $E \in \beta(D_{m+1})$. Hence $i(D_{m+1}) \geq 5$ and if $i(D_{m+1}) = 5$, then $(D_{m+1}, \ell)^{\#} \geq 3$ whence $\tilde{d}(E_{m+1};M) \leq -\frac{2}{3}$. Therefore $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{6}$ and $t = m+1$.

Let $d(\mu(E_{m,2})) \geq 4$. If $\mu(E_{m,2}) \neq \lambda(E_{m,2})$, then using Lemma (III.3.1), we can show that $\tilde{d}(E_m;M) \leq -\frac{1}{8}$ which is impossible since E_m is either neutral or positive. Then $\lambda(E_{m,2}) = \mu(E_{m,2})$ and so, using the Jordan curve theorem, the fact that $d(v) \geq 3$, all $v \in \mathcal{V}$, and that there is no (M',M')-edge where M' is a ℓ-region in $\mathcal{M}(\Delta)$, we have $d(\mu(E_{m,2})) \geq 7$.



$$\underset{\sim}{d}(\mu(E_{m,2}),D_m) \leq ((3 - d(v)) / (d(v) - 2))\,[1/2 + (2/(d(v) - 1))].$$

Since $(d(v) - 2)(d(v) - 1) < (d(v) - 3)(d(v) - 1)$, $\underset{\sim}{d}(\mu(E_{m,2}),D_m) < -(1/2)$.

Now, let $E_{m,2}$ be the $(D_m, \mathcal{G})$-edge. Then $E_{m,3}$ is a (D_m, ℓ)-edge. and $d(\mu(E_{m,3})) \geq 4$. Let $d(\mu(E_{m,3})) = 4$ and $\sigma(E_{m,3}) = M'$, (possibly $M' = M$.) Then $E_{m+1,2}$ is a (D_{m+1},M')-edge.

Hence either $e_{m,1} \equiv a^{\varepsilon(2p+1)}$ or $e_{m,3} \equiv a^{\varepsilon(2p+1)}$, $\varepsilon = \pm 1$ (by using the condition C(6) and the fact that $\theta_R(a^p) = 1$). If $e_{m,1} \equiv a^{\varepsilon(2p+1)}$, then $e_{m+1,1} \equiv b^{\varepsilon k}$, $1 \leq k \leq q+1$, while if $e_{m,3} \equiv a^{\varepsilon(2p+1)}$, then $e_{m+1,2} \equiv b^{\varepsilon k_1}$, $1 \leq k_1 \leq q+1$.

If $r_m \neq b^{n'}$, $(b^{n'} \in R)$, then either $k \neq q+1$ or $k_1 \neq q+1$ whence either $\theta_R(e_{m+1,1}) = 1$ or $\theta_R(e_{m+1,2}) = 1$. Thus $i(D_{m+1}) \geq 4$. If $i(D_{m+1}) = 4$, then $(D_{m+1}, \mathcal{L})^{\#} \geq 3$ whence $\tilde{d}(E;M) \leq -\frac{1}{3}$. Thus $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$ and so $t = m + 1$.

If $i(D_{m+1}) = 5$, then $(D_{m+1}, \mathcal{L})^{\#} \geq 2$ whence $\tilde{d}(E_{m+1}; M) \leq -\frac{1}{2}$ and so $t = m+1$.

If $i(D_{m+1}) \geq 6$, then $\tilde{d}(E_{m+1};M) \leq -1$; and so $t = m + 1$.

Let $r_{m+1} \equiv b^{\varepsilon n'}$, $(b^{n'} \in R)$. Then $i(D_m) \geq 3$. If $i(D_{m+1}) = 3$, then $(D_{m+1}, \mathcal{L})^{\#} = 3$ since $\theta_R(b^{q+1}) = 2$. Since $d(\mu(E_{m,2})) = 4$, $\mu(E_{m+1}) \neq \lambda(E_{m+1})$.

So $d(E_{m+1};M) \leq \frac{2}{3} \left[\underset{\sim}{d}(\mu(E_{m+1}); D_{m+1}) + \underset{\sim}{d}(\lambda(E_{m+1}); D_{m+1}) \right]$

$\leq \frac{2}{3} \left[-\frac{1}{4} - \frac{1}{4} \right] = -\frac{1}{3}$ (by using

Lemma (III.3.1). Thus $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$ and so that $t = m+1$.

Let $i(D_{m+1}) = 4$. We note that since $d(\mu(E_{m,3})) = 4$, $\tilde{d}(E_m;M) \leqslant \frac{1}{4}$; and using Lemma(III.3.1) we get $\tilde{d}(E_{m+1};M) \leqslant -\frac{1}{4}$. We can take $t = m+1$ unless $(D_{m-1},\mathcal{C})^{\#} = 2$ and $d(v) = 3$, all $v \in \beta(D_{m+1}), v \neq \mu(E_{m,3})$. If $r_{m+1} \equiv b^{n'}$ it is easy to see that we can take $t = m+2$ (using Lemma(2.1)). Otherwise $e_{m+1,1} \equiv b^{\varepsilon \ell} a^{\varepsilon}$. Next we can take $t = m+2$ unless $i(D_{m+2}) = 5$, $(D_{m+2},\mathcal{C}) = 1$, $d(v) = 3$, all $v \in \beta(D_{m-2})$, and $e_{m+2,1} \equiv a^{\varepsilon k_1} b^{\varepsilon(q-1-\ell)}$. Unless $a^{\varepsilon k_1}$ occurs within $a^{\varepsilon 2p}$ (rather than $a^{\varepsilon(2p-1)}$) we can take $t = m+3$. If $a^{\varepsilon k_1}$ occurs within $a^{\varepsilon 2p}$ we need to go on two more steps, in a similar way, and take $t = m+5$.

When $i(D_{m+1}) \geqslant 5$, since $(D_{m+1},\mathcal{C})^{\#} \geqslant 2$ the result is clear.

Let $d(\mu(E_{m,3})) = 5$. Then $d(E_m;M) \leqslant \frac{1}{10}$ whence $t = m+1$ since $\tilde{d}(E_{m+1};M) \leqslant -\frac{4}{15}$. Let $d(\mu(E_{m,3})) \geqslant 6$. Then $\tilde{d}(E_m,M) \leqslant \frac{1}{4}$ since $[(D_m,\mathcal{C})^{\#},\mu(E_{m,3})]^{\#} = 2$. Thus argue similarly to the case when $d(\mu(E_{m,3})) = 4$.

Let $i(D_m) = 4$ and $(D,\mathcal{C}) = 1$. Then $e_{m,1} \equiv a^{\varepsilon(2p-1)}$. If $d(\mu(E_m)) \geqslant 4$, then $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{8}$, by Lemma(2.3). So $t = m+1$. Now let $d(\mu(E_m)) = 3$ and w.l.o.g. we assume that $\varepsilon = 1$. Since $r_m \equiv a^{m'}$, $m' \neq 0$, $e_{m,4}$ must end in a (and hence $e_{m+1,2}$ begins with a^{-1}) and so $e_{m+1,1} \equiv b^k$, $1 \leq k \leq q$ and $r_{m+1} \equiv b^{n'}$, $n' \neq 0$ $(b^{n'} \in R)$.

Suppose that $d(\mu(E_{m,3})) \geqslant 4$. If $d(\mu(E_{m,3})) \geqslant 5$, then $d(\mu(E_{m,3})) = 5$ or 6 with $\tilde{d}(E_m;M) = \frac{1}{5}$. It is not difficult to see that $\tilde{d}(E_{m+1};M) \leqslant \frac{1}{5} - \frac{1}{3}$

$= -\frac{2}{15}$. Then $t = m+1$. Let $d(\mu(E_{m,3})) = 4$. Then $\tilde{d}(E_m;M) \leqslant \frac{1}{2}$, $\sigma(E_{m+1,3}) \in \mathcal{G}$ and $i(D_{m+1}) \geqslant 5$. If $i(D_{m+1}) \geqslant 6$, then $\tilde{d}(E_{m+1};M) \leq -1$ whence $d(E_m;M) + d(E_{m-1};M) \leq -\frac{1}{2}$ and so $t = m+1$. Now let $i(D_{m+1}) = 5$. Then we can take $t = m+1$ unless $(D_{m+1},\mathcal{C})^{\#} = 2$ and $d(v) = 3$, all $v \in \beta(D_{m+1})$, $v \neq \mu(E_{m-1,3})$. (Remember that $e_{m,1} \equiv a^{\varepsilon(2p-1)}$, $e_{m+1} \equiv b^{\varepsilon k_1}$, $1 \leq k_1 \leq q$ since $d(\mu(E_m)) = 3$.) Hence $e_{m+2,1} \equiv b^{\varepsilon(q+1-k_1)} a^{\varepsilon \ell_2}$, $1 \leq \ell_2 \leq p$ and we can take $t = m+2$ unless $i(D_{m+2}) = 2$, $(D_{m+2},\mathcal{C})^{\#} = 2$, $\sigma(E_{m+2,3}) \in \mathcal{C}$ and $d(v) = 3$, all $v \in \beta(D_{m-2})$. It is not difficult to see that $i(D_{m+3}) \geq 4$ with $\theta_R(e_{m+3}) = 1$. Thus $t = m+3$.

Let $d(\mu(E_{m,3})) = 3$. Now, $e_{m+1,1} \equiv b^k$, $1 \leq k \leq q$. It is clear that $E_{m+1,3}$ is a $(D_{m+1},\mathcal{G})$-edge and $\tau_{m+1} \not\equiv b^{n'}$, $n' \neq 0$. Thus $i(D_{m+1}) \geq 5$ and if $i(D_{m+1}) = 5$, then $(D_{m+1},\mathcal{L})^{\#} \geq 2$.

If $d(\mu(E_{m+1})) \geq 4$, then $\tilde{d}(E_{m+1},M) \leq -\frac{3}{4}$; and so $e_{m+2} \equiv b^{k_1}$ or $a^{\ell_1} b^{k_2}$, $1 \leq k_1, k_2 \leq q$ and $1 \leq \ell \leq p$.

If $e_{m+2} \equiv b^{k_1}$, then $\sum_{j=m}^{m+2} \tilde{d}(E_j,M) \leq 1 - \frac{3}{4} - \frac{1}{3} = -\frac{1}{12}$
and so $t = m+2$.

If $e_{m+2} \equiv a^{\ell_1} b^{k_1}$, then $\tilde{d}(E_{m+2};M) \leq -\frac{1}{4}$; and so $e_{m+3} \equiv a^{\ell_2}$, $1 \leq \ell_2 \leq p$. Hence $\tilde{d}(E_{m+3};M) \leq -\frac{1}{3}$ by Lemmas (2.2) and (2.1) since $\tau_m \not\equiv a^{m'}$, $m' \neq 0$. Thus $t = m+3$.

Suppose that $d(\mu(E_{m+1})) = 3$. Hence $\tilde{d}(E_{m+1};M) \leq -\frac{1}{2}$.
If $e_{m+2,1} \equiv b^{k_2}$, $1 \leq k_2 \leq q$, then $\tilde{d}(E_{m+2};M) \leq -\frac{3}{4}$ and so $t = m+2$. And for $e_{m+2,1} \equiv a^{\ell_1} b^{k_2}$, then put $t = m+3$.
Finally, let $i(D_m) = 4$ and $(D_m, \ell)^{\#} = 2$. If $e_{m,1} \equiv (a^{\ell_2} b^{k_2})^{\eta}$, $\eta = \pm 1$, $1 \leq k_2 \leq q$, $1 \leq \ell_2 \leq p$. Then it is easy to see that $m+1 \leq t \leq m+2$; otherwise $t = m+1$.

(ii) An easy argument,(see case(i)),give the required result.

The following result gives the cotradiction we required.

<u>(2.6) Corollary</u>

For any ℓ-region M in $\mathfrak{M}(\Delta)$,

$$\sum_{E \text{ in } \delta(M)} \tilde{d}(E;M) \leq -6 .$$

<u>Proof</u>

Suppose not; then there exists a ℓ-region M in $\mathfrak{M}(\Delta)$ such that

$$\sum_{i=1}^{n} d(E_i;M) > -6, \quad \text{where}$$

$\delta(M) = (E_1, E_2, \ldots, E_n)$ and $\varphi(E_1) \equiv x^{\pm 1}$ or $\varphi(E_1) \equiv y^{\pm 1}$.

If each E_i is a negative-edge, $2 \leq i \leq n$, Lemma (2.5) and the nature of the choice of the words w and u give a contradiction. Otherwise let E_{m_1} be the first edge following E_1 which is neutral or positive. By Lemma (2.5), there exists t, $m_1+1 \leq t \leq m_1+5$ such that

$$\sum_{j=m_1+1}^{t} \tilde{d}\,(E_j;M) \leq -\frac{1}{12}$$

If each E_i is a negative-edge, $m_1+t+1 \leq i \leq n$, then using the same argument as above we get a contradiction. Otherwise let E_{m_2} be the first edge following $E_{m_1+t_1}$ which is neutral or positive. Apply Lemma (2.5) again to find t_2, $m_2+1 \leq t_2 \leq m_{2+5}$ such that

$$\sum_{j=m_2+1}^{t_2} \tilde{d}\,(E_j;M) \leq -\frac{1}{12}$$

Since L is chosen to be large enough, we eventually get

$$\sum_{i=1}^{n} \tilde{d}\,(E_i;M) < -6,$$ which is a contradiction.

The following Lemma is generally valid and will apply to cases II - IV.

(2.7) Lemma

Let M be a k-region in $\mathcal{M}(\Delta)$. Let $\delta(M) = (E_1, E_2, \ldots, E_{L/2})$ or $\delta(M) = (E_1, \gamma_{(L/2)+1}, \ldots, \gamma_L)$ as explained in the introduction. If for each, $1 \leq i \leq L/2$ (or $(L/2)+1 \leq i \leq L$) there is a negative edge E in γ_i, then

$$\sum_{E \text{ in } \delta(M)} \tilde{d}\,(E;M) \leq -6 .$$

Proof A careful examination of all possibilities shows that if E is a negative edge, then $\tilde{d}(E;M) \leq -(1/70)$.

Hence
$$\sum_{E \text{ in } \delta(M)} \tilde{d}\,(E;M) < -(L/12) < -6 .$$

Case II when (i) $p = q = 1$,

(ii) either $\theta_R(ab) = 1$ or $\theta_R(ba) = 1$.

Put $w \equiv (a^2b^2)^{L/2}$ and $u \equiv a^2b^{-2}a^2b^2$. Then (H3) is valid and so there exists an R'-diagram $\mathfrak{M}(\Delta)$. For each $(\mathcal{L},\mathfrak{J})$-edge E in $\mathfrak{M}(\Delta)$, $\varphi(E)$ is a subword of $a^{\varepsilon}\,b^{\eta}$ or $b^{\eta}\,a^{\varepsilon}$, where $\varepsilon, \eta = \pm 1$. Thus $\theta_R(\varphi(E)) \leq 2$. Then certainly (H5) is valid and $\mathfrak{M}(\Delta)$ is a suitable R'-diagram.

(2.8) Lemma

--For any $\mathcal{L}$-region M in $\mathfrak{M}(\Delta)$,

$$\sum_{E \text{ in } \delta(M)} \tilde{d}\,(E;M) \leq -6 .$$

Proof

We can write $\delta(M) = (E_1, \gamma_1, \gamma_2, \ldots, \gamma_{(L/2)})$ or $\delta(M) = (E_1, \gamma_{(L/2)+1}, \ldots, \gamma_L)$, when $\varphi(E_1) \equiv x_1^{\pm 1}$ or $\varphi(E_1) \equiv y^{\pm 1}$. If for each i, $1 \leq i \leq L/2$ (or $(L/2) + 1 \leq i \leq L$) there is an $(M,\mathcal{L})$-edge occuring in γ_i, then by Lemma (2.7),

$$\sum_{E \text{ occur in } \delta(M)} \tilde{d}\,(E;M) \leq -6 \quad \text{as required.}$$

Now, suppose that there exists an integer j such that every edge in γ_j is an $(M,\mathcal{S})$-edge. If an edge E in γ_j is an S-edge then $\varphi(E) \equiv a^{\varepsilon}$ or $\varphi(E) \equiv b^{\eta}$, $\varepsilon, \eta = \pm 1$ and so $\tilde{d}\,(E;M) \leq -\frac{1}{3}$ by Lemma (2.1). Hence

$$\sum_{E \text{ in } \delta(M)} \tilde{d}\,(E;M) \leq -6 \quad \text{as required.}$$

Suppose that for each edge E in γ_j, E is a T-edge. By the nature of the chosen words w and u there must exist edges whose labels are $(ab)^{\eta}$ and $(ba)^{\varepsilon}$ respectively. By (ii), either $\theta_R(ab) = 1$ or $\theta_R(ba) = 1$ and Lemmas (2.1) and (2.7) give the required result.

<u>Case III</u> - when (i) $p = q = 1$,

(ii) $\theta_R(ab) = \theta_R(ba) = 2$,

(iii) The unique relator involving ab is not of the form $r \equiv (babz)^m$, $m \geq 1$.

Put $w \equiv (ab^2)^{L/2}$ and $u \equiv ab^2a^2b^2$. Then again $\mathcal{M}(\Delta)$ is a suitable R'-diagram. And by an almost identical argument as in case II, we show the contradiction required.

<u>Case IV</u> - when (i) $p = q = 1$,

(ii) $\theta_R(ab) = \theta_R(ba) = 2$,

and (iii) The unique relator r in R involving ab is not of the form $r \equiv (abaz)^m$, $m \geq 1$.

Put $w \equiv (a^2b)^{L/2}$ and $u \equiv a^2b^{-2}a^2b^2$ and argue exactly as in Case III.

<u>Case V</u> - when (i) $p = q = 1$,

(ii) $\theta_R(ab) = \theta_R(ba) = 2$,

(iii) The unique relator r in R involving ab is of the form $r \equiv (babaz)^m$.

Then put $w \equiv (a^2b^2)^{L/2}$ and $u \equiv a^2b^{-2}a^2b^2$ and by a similar way as in case II, we show the contradiction required.

Section 3

Let $G = \langle A;R \rangle$ be a finitely presented group which satisfies (H1). Throughout this section we assume that every generator in A is a piece relative to R and there exists a in A with

(i) $a^m \notin R$, for any $m \neq 0$,

(ii) $\theta_R(a^p) = 1$.

We shall consider four cases.

<u>Case I</u> - when there exists $b \in A$, $b \neq a$ with $\theta_R(b^q) = 1$, (see section 2 for the definition of b^q).

Put $w \equiv (a^p)^{L/2}$ and $u \equiv b^{q+1}$. Then (H3) is valid. Now, for each $(\mathcal{L},\mathcal{G})$-edge E in $\mathfrak{M}(\Delta)$, $\varphi(E)$ is a subword of

$(a^pb^q)^{\varepsilon}$, $(b^qa^p)^{\eta}$ or $(b^{q+1})^{\nu}$, $\varepsilon,\eta,\nu = \pm 1$ and so $\theta_R(\varphi(E)) \leq 2$. Then (H5) is valid and $\mathcal{M}(\Delta)$ is a suitable R'-diagram.

Let M be any $\mathcal{L}$-region in $\mathcal{M}(\Delta)$ with $\delta(M) = (E_1, E_2, \ldots, E_n)$, where $\varphi(E_1) = x^{\pm 1}$ or $y^{\pm 1}$.

(3.1) Lemma

Let E_m, $2 \leq m \leq n$ be an $(M,\mathcal{G})$-edge. Then $\tilde{d}(E_m;M) \leq 0$. Moreover, if $e_{m,1} \equiv a^{\varepsilon\ell}$ or $e_{m,1} \equiv b^{\eta k}$, $1 \leq \ell \leq p$, $1 \leq k \leq q$, $\varepsilon,\eta = 1$.

Then $\tilde{d}(E_m;M) \leq -\frac{1}{3}$.

Proof

By Lemma (2.1), $\tilde{d}(E_m;M) \leq 0$. If $e_{m,1} \equiv a^{\varepsilon\ell}$, say, $1 \leq \ell \leq p$ then $\theta_R(e_{m,1}) = 1$ and so $i(D_m) \geq 4$.

If $i(D_m) = 4$, then $(D_m,\mathcal{L})^{\#} \geq 3$ and $\tilde{d}(E_m;M) \leq -\frac{1}{3}$ as required. If $i(D_m) = 5$, then $(D_m,\mathcal{L})^{\#} \geq 2$ and hence $\tilde{d}(E_m;M) \leq -\frac{1}{2}$.

The following Lemma gives the contradiction we require.

(3.2) Lemma

For any $\mathcal{L}$-region M in $\mathcal{M}(\Delta)$,

$$\sum_{E \text{ in } \delta(M)} \tilde{d}(E;M) \leq -6 .$$

Proof

Suppose not; and let M be a $\mathcal{L}$-region with

$$\sum_{E \text{ in } \delta(M)} \tilde{d}(E;M) > -6 .$$

We can write $\delta(M) = (E_1, \gamma_1, \gamma_2, \ldots, \gamma_{L/2})$ or $\delta(M) = (E_1, \gamma_{(L/2)+1}, \ldots, \gamma_L)$ as explained in the introduction. If for each i, $1 \le i \le L/2$ (or $(L/2) + 1 \le i \le L$) there is an $(M,\mathcal{L})$-edge occuring in γ_i then

$$\sum_{E \text{ in } \gamma_i} \tilde{d}(E;M) \le -1 \quad \text{(by Lemma (3.1))}.$$

Thus $$\sum_{E \text{ in } \delta(M)} \tilde{d}(E;M) \le -(L/2),$$ which is absurd.

Hence there exists an integer j such that every edge occuring in γ_j is an $(M,\mathcal{G})$-edge. By the nature of w and u, at most four T-edges and at least $(L-2)$ S-edges occur in γ_j. These S-edges have labels which are subwords of $a^{\varepsilon p}$, $\varepsilon = \pm 1$. Then Lemma (3.1) easily gives a contradiction.

Case II - when (i) card $A \ge 3$,

(ii) for every $b \in A$, $b \ne a$, $\theta_R(b^q) = 2$,

(iii) there exists $b \in A$, $b \ne a$ such that the unique relator r involving b^q is not of the form $r \equiv (b^q a^{-\varepsilon} z a^{\varepsilon})^m$, $m \ge 1$, $\varepsilon = \pm 1$, where z is a b^q-free.

Possibly replacing a by a^{-1}, we can assume that if b is a generator satisfying (iii) then neither $b^q a$ nor ab^q occurs in r. Then we put $w \equiv (a^p)^{L/2}$ and $u \equiv b^{q+1}$; and so (H3) is valid.

Now, $\theta_R(\varphi(E)) \leq 2$, for any $(\mathcal{L},\mathcal{S})$-edge in $\mathcal{M}(\Delta)$ and clearly (H5) is valid so that $\mathcal{M}(\Delta)$ is a suitable R'-diagram.

(3.3) Lemma

For any $\mathcal{L}$-region M in $\mathcal{M}(\Delta)$

$$\sum_{E \text{ in } \mathcal{S}(M)} \tilde{d}\,(E;M) \leq -6 .$$

Proof

As in Lemma (3.2), it suffices to consider a path γ_j in which all edges are $(M,\mathcal{S})$-edges. In such a path there are at most four T-edges, and at least (L - 2) S-edges. Then the labels on the S-edges are subwords of $a^{\varepsilon p}$ and the argument of Lemma (3.1) remains valid to give $\tilde{d}\,(E,M) \leq -\frac{1}{3}$ for each such E.

Case III - when (i) card $A \geq 3$,

(ii) for every $b \in A$, $b \neq a$, $\theta_R(b^q) = 2$,

(iii) for every $b \in A$, $b \neq a$, the unique relator r involving b^q is of the form $r \equiv (b^q a^{-\varepsilon} z a^{\varepsilon})^m$, $m \geq 1$, $\varepsilon = \pm 1$, where z is b^q-free.

Let $c \in A$, $c \neq b$ and $c \neq a$. Let $c^{q'}$ be the maximal power of c occuring in any relator of R. Thus the unique r' in R involving $c^{q'}$ is of the form $r' \equiv (c^{q'} a^{-\eta} z' a^{\eta})^n$, $n \geq 1$, $\eta = \pm 1$, where z' is $c^{q'}$-free.

Put $w \equiv (b^q)^{L/2}$ and $u \equiv c^{10q'+1}$. Then (H3) is valid and so there exists an R'-diagram $\mathcal{M}(\Delta)$. As $\theta_R(\varphi(E)) \leq 2$, all $E \in (\mathcal{L},\mathcal{S})$, (H5) is valid so that $\mathfrak{M}(\Delta)$ is a suitable R'-diagram. The following remarks can be observed easily.

(3.4) Remarks

Let M be a $\mathcal{L}$-region in $\mathfrak{M}(\Delta)$ with $\delta(M) = (E_1, E_2, \ldots, E_n)$ and $\varphi(E_1) = x^{\pm 1}$ or $\varphi(E_1) = y^{\pm 1}$.

(1) $\tilde{d}(E;M) \leq 0$, all $E \in \beta(M)$

(2) If E is an S-edge with $\theta_R(\varphi(E)) = 1$, then $\tilde{d}(E;M) \leq -\frac{1}{3}$.

(3) From (2), $\sum_{E \text{ in } \delta(M)} d(E;M) < -6$, if for edge $E \in \beta(M) \cap (M,\mathcal{S})$, E is an S-edge.

(4) A T-edge does not involve $b^{\eta q}$ or $c^{\varepsilon q'}$, $\varepsilon,\eta = \pm 1$

(5) Let $\sigma(E) = D$, where $E \in (\mathcal{L},\mathcal{S})$. If $\tilde{d}(E;M) > -\frac{1}{6}$, then either $i(D) = 4$ or $i(D) = 5$.

(i) If $i(D) = 4$, then $d(v) = 3$, all $v \in \mathcal{V} \cap \beta(D)$ and $(D,\mathcal{L})^{\#} = 2$.

(ii) If $i(D) = 5$, then $d(v) = 3$, all $v \in \mathcal{V} \cap \beta(D)$ and $(D,\mathcal{L})^{\#} = 1$.

(6) If E is a negative edge then $\tilde{d}(E;M) \leq -\frac{1}{6}$.

(7) If there exists a vertex $v \in \beta(D)$ with $d(v) \geq 4$, where $D = \sigma(E)$ and $E \in (\mathcal{L},\mathcal{S})$, then $\tilde{d}(E;M) \leq -\frac{1}{6}$.

(8) If E is a neutral edge and $\sigma(E) = D \in \mathcal{S}$, then $\theta_R^*(\varphi(\delta(D))) = 6$.

(3.5) Lemma

Let E_m, $10 \leq m \leq n-10$, be a T-edge. Then

$$\sum_{j=m}^{m+2} \tilde{d}(E_j;M) \leq -\frac{1}{6}.$$

Proof

By Remarks (3.4), it is sufficient to consider the case when $\tilde{d}(E_m;M) = 0$,

Now, either $e_{m,1} \equiv b^{\varepsilon \ell} c^{\varepsilon k}$ or $e_{m,1} \equiv c^{\varepsilon k} b^{\varepsilon \ell}$, $1 \leq \ell < q$, $1 \leq k < q'$ and $\varepsilon = \pm 1$. W.l.o.g. we assume that $e_{m,1} \equiv b^{\varepsilon \ell} c^{\varepsilon k}$. By Remarks (3.4), $i(D_m) = 4$ or $i(D_m) = 5$ and in both cases, $\theta_R^*(r_m) = 6$.

Suppose that $i(D_m) = 4$. It is not hard to see that $E_{m,3}$ is the other (D_m, ℓ)-edge, (otherwise there exists a vertex $v \in \beta(D_m)$ such that $d(v) \geq 4$). It is clear that $e_{m+1,1} \equiv b^{\varepsilon q}$ and $e_{m+2,1} \equiv b^{\varepsilon q}$. Applying the same argument as that in Lemma (II.3.12), we can show that $b^{\varepsilon k} e_{m,2}$ is a piece relative to R. Hence $\theta_R^*(r_m) \leq 5$, which is absurd. By the same argument as above we can show that $i(D_m) \neq 5$, which is a contradiction. Therefore there exists $t, m+1 \leq t \leq m+2$, such that $\tilde{d}(E_t;M) < 0$ whence $d(E_t;M) \leq -\frac{1}{6}$ (by Lemma(2.1)).

(3.6) Lemma

For any $\mathcal{L}$-region M in $\mathfrak{M}(\Delta)$,

$$\sum_{E \text{ in } \delta(M)} \tilde{d}(E;M) \leq -\alpha(M).$$

Proof

Suppose that there exists a path γ_j, $1 \leq j \leq L/2$ (or $(L/2) + 1 \leq j \leq L$) such that

$$\sum_{E \text{ in } \gamma_j} \tilde{d}(E;M) = 0.$$

(see the introduction for the definition of γ_j). Then for each E in γ_j, $\tilde{d}(E;M) = 0$. Obviously, there is no $(M,\mathcal{L})$-edge in γ_j. By Lemma (3.5), no E in γ_j can be a T-edge; and by Remark (3.4)(3) either $\varphi(E) = b^{\eta q}$ or $\varphi(E) \equiv$, $\eta, \varepsilon = \pm 1$. This is impossible by the nature of the chosen word u. Hence for each j, $1 \leq j \leq L/2$ (or $(L/2) + 1 \leq j \leq L$), there is a negative edge in γ_j and Lemma (2.7) gives the required result.

Case IV - when (i) $= A = \{a,b\}$

(ii) $\theta_R(b^q) = 2$,

(iii) the unique relator r involving b^q is of the form $r \equiv (b^q a^{\varepsilon} z a^{\varepsilon})^m$, $m \geq 1$ and z is b^q-free.

W.l.o.g. we can assume that $\varepsilon = -1$ and then argue exactly as in case II.

<u>Case V</u> - when (i) $A = \{a,b\}$,

(ii) $\theta_R(b^q) = 2$,

(iii) the unique relator r in R involving b^q is of the form $r \equiv (b^q a^{\varepsilon} z a^{-\varepsilon})^m$, $m \geq 1$, $\epsilon = \pm 1$, where z is b^q-free.

We can take $\varepsilon = 1$. Again put $w \equiv (a^p)^{L/2}$ and $u \equiv b^{q+1}$. Then again $\mathfrak{M}(\Delta)$ is easily seen to be suitable.

<u>(3.7) Lemma</u>

Let E_m, $2 < m < n$ be an $(M,\mathcal{G})$-edge. Suppose that $e_{m,1} \equiv a^{\varepsilon\ell}$, $1 \leq \ell \leq p$ and $\tilde{d}(E_m;M) = 0$. Then either $\tilde{d}(E_{m+1};M) \leq -\frac{1}{3}$ or $d(E_{m-1};M) \leq -\frac{1}{3}$.

<u>Proof</u>

Since $\tilde{d}(E_m;M) = 0$, we must have $i(D) = 4$, $(D_m,\mathcal{L})^{\#} = 2$ and $d(v) = 3$ for every $v \in \beta(D_m)$. (Remember that $\theta_R(a^p) = 1$).

Let $\delta(D_m) = (E_{m,1}, \ldots, E_{m,4})$. If $E_{m,2} \in (D_m,\mathcal{L})$ then $\tilde{d}(E_{m-1};M) = -1$ (since $d(\mu(E_{m-1}) = 3)$ and if $E_{m,4} \in (D_m,\mathcal{L})$ then $\tilde{d}(E_{m+1},M) = -1$ (since $d(\lambda'E_{m+1})) = 3)$.

So suppose that $E_{m,3} \in (D_m,\mathcal{L})$; then $E_{m,2}$ and $E_{m,4}$ are $(D_m,\mathcal{G})$-edges. Thus $e_{m,3} \equiv (b^q a^{\ell_1})^{\eta}$, $\eta = \pm 1$, $1 \leq \ell_1 \leq p$, with $\theta_R(e_{m,3}) = 3$. Let $M' = \sigma(E_{m,3})$ (possibly $M' = M$).

We take $\eta = 1$, when $\eta = -1$ the argument is similar.

Since $d(\mu(E_{m,2})) = 3$, $E_{m+1,3} \in (D_{m+1}, M')$ and $e_{m+1,3} \equiv a^{\ell_2}$, $1 \leq \ell_2 \leq p$. Similarly $e_{m-1,\ n(m-1)-1} \equiv a^{\ell_3} b$, $0 \leq \ell_3 \leq p$. Then $\theta_R(e_{m+1,3}) = 1$ and $\theta_R(e_{m-1,\ n(m-1)-1}) \leq 2$.

If $\theta_R(e_{m+1}) \leq 2$, then we have $i(D_{m+1}) \geq 4$. If $i(D_{m+1}) = 4$, then $(D_{m+1}, \mathcal{L})^{\#} \geq 3$ giving $\tilde{d}(E_{m+1}, M) \leq -\frac{1}{3}$ while if $i(D_{m+1}) = 5$, then $(D_{m+1},\ ) \geq 2$ and $\tilde{d}(E_{m+1}, M) \leq -\frac{1}{2}$.

So suppose that $\theta_R(e_{m+1,1}) = 3$. Then ℓ must be 1 and $e_{m+1,1} \equiv b^q a^{\ell_4}$. So $e_{m-1,1} \equiv a^{\ell_5}$, $1 \leq \ell_5 \leq p$. and $\theta_R(e_{m-1,1}) = 1$. As $\theta_R(e_{m-1,n(m-1)-1}) \leq 2$ $i(D_{m-1}) \geq 4$. Again if $i(D_{m-1}) = 4$, then $(D_{m-1}, \mathcal{L})^{\#} \geq 3$ and $\tilde{d}(E_{m-1}; M) \leq -\frac{1}{3}$ while if $i(D_{m-1}) = 5$, then $(D_{m-1}, \mathcal{L})^{\#} \geq 2$ and $\tilde{d}(E_{m-1}; M) \leq -\frac{1}{2}$.

<u>(3.8) Lemma</u>

Let E_m, $2 < m < n$ be an $(M, \mathcal{G})$-edge and suppose that $e_{m,1} \equiv a^{-\ell} b^{-q}$, $1 \leq \ell \leq p$. If $\tilde{d}(E_m; M) = 1$, then $\tilde{d}(E_{m+1}; M) \leq -\frac{1}{2}$.

<u>Proof</u>

Since $\tilde{d}(E_m; M) = 1$, we must have $(D_m, \mathcal{L})^{\#} = 1$ and $i(D_m) = 4$, $d(v) = 3$, all $v \in \beta(D_m)$. As $E_{m,3} \in (D_m, \mathcal{G})$ and $d(\mu(E_{m,3})) = 3$, $E_{m+1,3} \in (D_{m+1}, \mathcal{G})$ and $\theta_R(e_{m+1,3}) = 1$.

Since $d(\mu(E_{m,4})) = 3$, $\sigma(E_{m+1}) \in \mathcal{G}$ and $e_{m+1,1} \equiv a^{-\ell_1}$, $1 \leq \ell_1 \leq p$.

If $i(D_{m+1}) = 4$ then $\theta_R(e_{m+1,4}) = 3$ and so $e_{m+1,4} \equiv (b^q a^{\ell_2})^{\varepsilon}$, $1 \leq \ell_2 \leq p$, $\varepsilon = \pm 1$. But $\varepsilon = 1$ gives cancellation in r_{m+1} and $\varepsilon = -1$ means that r_{m+1} contains $a^{-1}b^{-q}a^{-1}$ as a subword which is at variance with the fact that $r \equiv (b^q aza^{-1})^n$, $n \geq 1$. Hence $i(D_{m+1}) \neq 4$.

If $i(D_{m+1}) = 5$ then $(D_{m+1}, \ell)^{\#} \geq 2$ and $\tilde{d}(E_{m+1};M) \leq -\frac{1}{2}$ as required.

<u>(3.9) Lemma</u>

Let E_m, $2 < m < n$ be an $(M, \mathcal{G})$-edge and suppose that $e_{m,1} \equiv b^q a^{\ell}$, $1 \leq \ell \leq p$. If $\tilde{d}(E_m;M) = 1$, then $\tilde{d}(E_{m+1};M) \leq -\frac{1}{4}$.

<u>Proof</u>

By the same argument as that in Lemma(3.8).

(3.10) Lemma

Let E_m, $2 \leq m \leq n$, be an S-edge. If E_m is a negative edge then $\tilde{d}(E_m;M) \leq -\frac{1}{6}$.

Proof

As E_m is an S-edge, $i(D_m) \geq 3$. If $i(D_m) = 3$, then $(D,\mathcal{L})^{\#} = 3$. And by Remark (III.3.7), $\tilde{d}(E_m;M) \leq -\frac{1}{6}$.

Let $i(D_m) = 4$. Then $(D_m,\mathcal{L})^{\#} \geq 2$. If $(D_m,\mathcal{L})^{\#} \geq 3$, then $\tilde{d}(E_m;M) \leq -\frac{1}{3}$. Otherwise $(D_m,\mathcal{L})^{\#} = 2$, and so there exists a vertex v in $\beta(D_m)$ such that $d(v) \geq 4$. Hence $\tilde{d}(E_m;M) \leq -\frac{1}{4}$. Similarly for $i(D_m) = 5$ and $(D,\mathcal{L})^{\#} = 1$, there exists a vertex v in $\beta(D_m)$ with $d(v) \geq 4$ whence $\tilde{d}(E_m;M) \leq -\frac{3}{4}$. Otherwise $\tilde{d}(E_m;M) \leq -\frac{1}{2}$. Clearly, for $i(D_m) \geq 6$.

(3.11) Lemma

Let E_m, $3 \leq m \leq n-1$, be a T-edge. If $\tilde{d}(E_m;M) < 1$, then $\tilde{d}(E_m;M) \leq \frac{3}{4}$.

Proof

It is sufficient to consider the case when $i(D_m) = 4$ and

$(D_m, \mathcal{L})^{\#} = 1$. Since $\tilde{d}(E_m;M) < 1$, then there exists a vertex v in $\beta(D_m)$ with $d(v) \geq 4$, (otherwise $\tilde{d}(E_m;M) = 1$). Hence, by Lemma (III.3.1), $d\ (E_m,M) \leq \frac{3}{4}$ as required.

(3.12) Remarks

Now, from Lemmas (3.4) - (3.8), we conclude the following: Let E_m be an $(M,\mathcal{S})$-edge in $\delta(M)$.

(1) If E_m is an S-edge with $\tilde{d}(E_m;M) = 0$, then either $\tilde{d}\ (E_{m+1};M) \leq -\frac{1}{3}$ or $\tilde{d}\ (E_{m-1};M) \leq -\frac{1}{3}$. (Lemma 3.7).

(2) If E_m is a negative S-edge, then $\tilde{d}\ (E_m;M) \leq -\frac{1}{6}$. (Lemma (3.10)).

(3). If E_m is a T-edge with $\tilde{d}\ (E_m;M) = 1$, then $\tilde{d}\ (E_m;M) + \tilde{d}\ (E_{m+1};M) \leq \frac{3}{4}$. (Lemmas (3.8) and (3.9)).

(4) If E_m is a T-edge with $\tilde{d}(E_m;M) < 1$, then $d\ (E_m;M) \leq \frac{3}{4}$. (Lemma (3.11)).

(5) If E_m is a nagative edge, then $d(E_m;M) \leq -\frac{1}{70}$.
(See the proof of Lemma(2.7)).

A subpath $E_{i_1},\ldots,E_{i_k}$, $2 \leq i_1 \leq i_k \leq n$, of $\mathcal{S}(M)$ is called a b-path in $\mathcal{S}(M)$ if it has the following properties.

(i) For each $i_1 \leq j \leq i_k$, e_j ivolves $b^{\pm 1}$.

(ii) If e_{i_1} begins with a^{ε}, $\varepsilon = \pm 1$, and $e_{i_1 - 1}$ ends in b^{ε} then

$E_{i_1 - 1}$ must be an (M,ζ)-edge ,otherwise $e_{i_1 - 1}$ is a $b^{\pm 1}$-free .

(iii) If e_{i_k} ends in a^{ε}, $\varepsilon=\pm 1$, and $e_{i_k + 1}$ begins with b^{ε}, then $E_{i_k + 1}$ (M,ζ) ,otherwise $e_{i_k - 1}$ is a $b^{\pm 1}$-free.

(Note that for $i_k = n$, condition(iii) is still valid.)

(6) If Γ is a b-path in $\mathcal{S}(M)$, then $\sum \tilde{d}(E;M) \leq 1$, where E in Γ .

(3.13) Lemma For any ζ-region in $\mathcal{M}(\Delta)$,

$\sum \tilde{d}(E;M) \leq -6$, where E in $\mathcal{S}(M)$.

Proof Suppose not . Then there exists a ζ-region M such that $\sum d(E;M) > -6$, where E in $\mathcal{S}(M)$. We write $\mathcal{S}(M) = (E_1,E_2,\ldots,E_n)$, where $\phi(E_1) \equiv x^{-1}$ or y^{-1} .Hence $\tilde{d}(E_1;M) = -1$.

Let $\Gamma_1 = (E_{m_{k_1}},\ldots,E_{m_{\ell_1}})$ be the first b-path in $\mathcal{S}(M)$ that follows E_1. Since $\sum_{j = m_{k_1}}^{m_{\ell_1}} \tilde{d}(E_j;M) \leq 1$, $\sum_{j = 1}^{m_{\ell_1}} \tilde{d}(E_j;M) \leq 0$, (See Remark(3.12)(6)). If $\sum_{j = 1}^{m_{\ell_1}} \tilde{d}(E_j;M) > -(1/4)$, then it is not hard to show that $\sigma(E_{m_{\ell_1} + 1}) \in \mathcal{G}$ with $\tilde{d}(E_{m_{\ell_1} + 1};M) \leq -(1/4)$ whence $\sum_{j = 1}^{m_{\ell_1} - 1} \tilde{d}(E_j;M)$ $\leq -(1/4)$.

Let $\Gamma_2 = (E_{m_{k_2}},\ldots,E_{m_{\ell_2}})$ be the first b-path in $\mathcal{S}(M)$ that follows Γ_1 . If $\sum_{j = m_{k_2}}^{m_{\ell_2}} \tilde{d}(E_j;M) \leq -(1/4)$, then $\sum_{j = 1}^{m_{\ell_2}} \tilde{d}(E_j;M) \leq -(1/2)$.

Suppose that $\sum_{j = m_{k_2}}^{m_{\ell_2}} \tilde{d}(E_j;M) > -(1/4)$. If for each i,

$m_{\ell_1}+1 \leq i \leq m_{\ell_2}-1$, E_i is an S-edge then there are sufficiently large number of them with $\sum_{j=m_{\ell_1}+2}^{m_{\ell_1}-1} \tilde{d}(E_j;M)$ -(5/4) whence $\sum_{j=1}^{m_{\ell_2}} \tilde{d}(F_j;M) \leq -(1/2)$.

Now, soppose that there exists an $(M,\mathcal{G})$-edge E_t, $m_{\ell_1}+1 \leq t \leq m_{\ell_2}-1$. If $\sum_{j=t}^{m_{k_2}} \tilde{d}(E_j;M) > -(1/4)$, then $\sigma(E_{m_{k_2}+1}) \in \mathcal{G}$ with $\tilde{d}(E_{m_{k_2}+1};M) \leq -(1/4)$; thus $\sum_{j=1}^{m_{k_2}-1} \tilde{d}(E_j;M) \leq -(1/2)$. Since L is chosen to be large enough, we can repeat the argument to show that $\sum_{j=1}^{n} d(E_j;M) < -6$, which is absurd.

Section 4

Let $G = \langle A;R \rangle$ be a finitely presented group which satisfies (H1). Throughout this section, we assume that every generator is a piece relative to R and there exists $a \in A$ with

(i) $a^m \notin R$, for any $m \neq 0$,

(ii) $\theta_R(a^p) = 2$,

(iii) for each $b \in A$, $b \neq a$, the unique relator r involving a^p is not of the form $r \equiv (a^p b^{-\epsilon} z b^{\epsilon})^n$, $n \geq 1$, $\epsilon = \pm 1$, where z is a^p-free.

We shall consider three cases.

Case I - when (i) card $A \geq 3$,

(ii) the unique relator involving a^p is of the form $r \equiv (a^p b^{\epsilon} z c^{\eta})^m$ where $b \neq c$, $m \geq 1$ and $\epsilon, \eta = \pm 1$.

W.l.o.g. we assume that $r \equiv (a^p b^{-1} zc)^n$, $n \geq 1$,. Put $w \equiv (a^p)^{36L}$ and $u \equiv b$. Then (H3) is valid; and for each $(\mathcal{C}, \mathcal{G})$-edge E in $\mathcal{M}(\Delta)$, $\Theta_R(\varphi(E)) \leq 3$. Hence (H5) is valid and $\mathcal{M}(\Delta)$ is a suitable R'-diagram.

Let M be any $\mathcal{C}$-region in $\mathcal{M}(\Delta)$ with $\mathcal{S}(M) = (E_1, E_2, \ldots, E_n)$, where either $\varphi(E_1) \equiv x^{\pm 1}$ or $\varphi(E_1) \equiv y^{\pm 1}$. We observe the following remarks.

(4.1) Remarks

(i) $\tilde{d}(E;M) \leq 1$, all $E \in \beta(M)$.

(ii) If $\Theta_R(\varphi(E)) = 3$, then $\varphi(E_m) \equiv (a^{\ell_1} b a^{\ell_2})$, where $1 \leq \ell_1, \ell_2 < p$ and

(iii) A T-edge cannot involve $a^{\varepsilon p}$, $\varepsilon = \pm 1$.

(iv) There are at most $(L/2 - 1)$ T-edges in $\mathcal{S}(M)$.

(v) Let E_m, $2 \leq m \leq n$, be an $(M, \mathcal{G})$-edge.

If E_m is an S-edge and $\tilde{d}(E_m;M) = 0$, then $i(D_m) = 4$ or $i(D_m) = 5$, and; (i) for $i(D_m) = 4$, we have $(D, \mathcal{C})^{\#} = 2$ and $d(v) = 3$ all $v \in \beta(D_m) \cap \vartheta$,

(ii) for $i(D_m) = 5$, we have $(D, \ ) = 1$ and $d(v) = 3$ all $v \in \beta(D_m) \cap \vartheta$.

(4.2) Lemma

Let E_m be any $(M, \mathcal{G})$-edge, where $2 \leq m \leq n-1$. Suppose that E_{m+1} is an $(M, \mathcal{G})$-edge. Further assume that $\varphi(E_m) \equiv \varphi(E_{m+1}) \equiv a^{\varepsilon p}$. Then $d(\mu(E_m)) \neq 3$.

Proof

W.l.o.g. we assume that $s = -1$. If $d(\mu(E_m)) = 3$, then $E_{m,n(m)} = E^{-1}_{m+1,2}$. Recall that the unique relator r involving a^p has the form $r \equiv (a^p b^{-1} zc)^{n'}$, $n' \geq 1$. Compare r_{m+1} and r, noting that a^p is not a piece relative to R. Hence $e_{m,n(m)}$ ends in b which is absurd. Therefore $d((E_m)) \neq 3$.

(4.3) Lemma

Let E_m be an $(M,\mathcal{J})$-edge in $\mathcal{M}(\Delta)$.

(1) Suppose that E_m is an S-edge. Then

(i) $\Theta_R(\varphi(E_m)) \leq 2$,

(ii) E_m cannot be a positive edge,

(iii) If E_{m+1}, E_{m+2}, E_{m+3} are S-edges and E_m is a neutral edge then there exists an integer t, $m+1 \leq t \leq m+3$ such that

$$\sum_{j=m}^{t} \tilde{d}(E_j,M) \leq -\frac{1}{4},$$

(iv) If E_m is a negative edge, then $\tilde{d}(E_m;M) \leq -\frac{1}{6}$.

(2) Suppose that E_m is a T-edge. Then

(i) if E_m is a positive edge with $\tilde{d}(E_m;M) \neq 1$, then $\tilde{d}(E_m;M) \_ \frac{3}{4}$;

(ii) if E_m is a positive edge with $\tilde{d}(E_m;M) = 1$, then there exists an integer t, $m+1 \leq t \leq m$ such that

$$\sum_{j=m}^{t} d\,(E_j;M) \leq \frac{3}{4}.$$

Proof

(i) Given an S-edge E_m, then $\varphi(E_m) \equiv a^{\varepsilon \ell}$ or b^{η}, where $1 \leq \ell \leq p$ and $\varepsilon, \eta = 1$. Hence, clearly, $\theta_R(\varphi(E_m)) \leq 2$; and so (i) is valid. (ii) is valid by using (i) and Lemma (III.3.5).

(iii) Let $\varphi(E_m) \equiv b^{-\eta}$, $\eta = \pm 1$. Then $i(D_m) = 4$ or 5. since E_m is a neutral edge. If $i(D_m) = 5$, then $(D, \mathcal{C})^{\#} \geq 2$ whence $\tilde{d}(E_m;M) \leq -\frac{1}{2}$ which is absurd. Therefore $i(D_m) = 4$ and $(D, \mathcal{C})^{\#} = 2$. Let $\delta(D_m) = (E_{m;1}, E_{m,2}, E_{m,3}, E_{m,4})$. Since E_m is a neutral edge, $d(v) = 3$, for all vertices $v \in \beta(D_m)$. If $E_{m,2}$ is the second $(D_m, \mathcal{C})$-edge, then $e_{m,2} \equiv (a^{\ell_1} b a^{\ell_2})^{-\eta}$, where $1 \leq \ell_1, \ell_2 < p$. So, $E_{m,3}$ and $E_{m,4}$ are $(D_m, \mathcal{G})$-edges. Clearly $E_{m,4} \equiv E^{-1}_{m+1,2}$. Let $\sigma(E_{m,3}) = D' \in \mathcal{G}$. Then $E_{m+1,3}$ is a (D_{m+1}, D')-edge.

Now, $e_{m+1,1} \equiv a^{\eta \ell_3}$, $1 \leq \ell_3 \leq p$. By the condition C(6), $i(D_{m+1}) \neq 3$. If $i(D_{m+1}) = 4$, then $E_{m+1,4}$ is a $(D_{m+1}, \mathcal{C})$-edge and $d(v) = 3$, all vertices $v \in \beta(D_{m+1})$. Thus E_{m+2} ia an $(M, \mathcal{C})$-edge whence $\tilde{d}(E_{m+2};M) = -1$. Thus $t = m+2$.

If $(D_{m+1}) = 5$, then $E_{m+1,4}$ and E_{m+5} must be $(D_{m+1}, \mathcal{G})$-edges with $d(v) = 3$, all vertices $v \in \beta(D_{m+1})$ (otherwise put

$t = m+1$). Thus $e_{m+1,1} \equiv a^{\eta p}$. Let $\sigma(E_{m+1,4}) = D''$. Then $E_{m+2,3}$ is a (D_{m+2}, D'')-edge. By Lemma (4.2), $e_{m+2,1} \equiv a^{\eta \ell_5}$, $1 \leq \ell_5 < p$. If $i(D_{m+2}) = 4$, then $E_{m+2,4}$ must be a T-edge with $e_{m+2,4} \equiv (a^{\ell_6} ba^{\ell_7})^{\eta}$, and so $r_{m+2} \equiv (a^{\ell_6} ba^{\ell_7})^{\eta} a^{\eta \ell_5} e_{m-2,2} e_{m+2,3}$. Thus $\theta_R((a^{\ell_6} ba^{\ell_7})^{\eta} a^{\eta \ell_5}) \leq 3$ so that $_R{*}(r_{m+2}) \leq 5$ which is absurd. Thus $i(D_{m+2}) \geq 5$, and for $i(D_{m+2}) = 5$, $(D_m, \mathcal{C})^{\#} \geq 2$. Hence $\tilde{d}(E_{m+2};M) \leq -\frac{1}{2}$ and so that $t = m+2$.

It is not hard to show the required result for $\varphi(E_m) \equiv a^{\eta \ell}$, $1 \leq \ell \leq p$. (Note that $E_{m+1}, E_{m+2}, E_{m+3}$ are assumed to be S-edges.).

(2) (i) If $\tilde{d}(E_m;M) < 1$, and E_m is a positive edge then either $i(D_m) = 4$ with $(D, \mathcal{C})^{\#} = 1$ or $i(D_m) = 3$ with $(D, \mathcal{C})^{\#} = 2$. For the first case, there exists a vertex $v \in \partial(D_m) \cap \vartheta$ such that $d(v) \geq 4$ whence $\tilde{d}(E_m;M) \leq \frac{3}{4}$ as required. In the second case, clearly $\tilde{d}(E_m;M) \leq \frac{1}{2}$.

(ii) By a similar way as in (1) we can have the required result.

<u>(4.4) Lemma</u>

Let E_m and $E_{m'}$ be any two distinct T-edges in $\mathcal{S}(M)$. If for each $m+1 \leq j \leq m'-1$, E_j is an S-edge, then

$$\sum_{E \text{ in } \delta(M)} \tilde{d}(E;M) \leq -6 .$$

Proof

Suppose not; and let $\sum_{E \text{ in } (M)} \tilde{d}(E;M) > -6$.

By Remark (4.1), $\tilde{d}(E;M) \leq 1$ and T-edges are the only one which can make a positive contribution to the sum $\sum_{E \text{ in} \delta(M)} \tilde{d}(E;M)$.

Since there are at mot L/2 T-edges in $\delta(M)$,

$$-\alpha(M) < \sum_{E \text{ in } (M)} \tilde{d}(E;M) < L/2 ,$$ (see Remark (4.1)).

Now, from the nature of the chosen words w and u, there are at least (36 L - 1) S-edges between E_m and $E_{m'}$. Apply Lemma (4.3), to show that

$$\sum_{j=m}^{m'} \tilde{d}(E_j;M) < -6L ,$$ which is absurd.

(4.5) Corollary

Let E_m and $E_{m'}$ be any two distinct T-edges in $\delta(M)$. If $\sum_{E \text{ in} \delta(M)} \tilde{d}(E;M) > -6$, then there must exist an $(M,\mathcal{C})$-edge between E_m and $E_{m'}$.

Let M be a $\mathcal{C}$-region in $\mathcal{M}(\Delta)$. Let E be an edge that occurs in $\delta(M)$. We shall call E a b-edge. if and only if $\varphi(E)$ involves b^{η}, $\eta = \pm 1$. Hence for each b-edge E in

$\mathcal{S}(M)$, $\varphi(E)$ involves exactly one b^{η}, $\eta = \pm 1$. If the b-edge E is an $(M,\mathcal{C})$-edge, then $\tilde{d}(E;M) = -1$.

<u>(4.6) Lemma</u>

For any -region M in $\mathfrak{M}(\Delta)$,

$$\sum_{E \text{ in } \mathcal{S}(M)} \tilde{d}(E;M) \leq -6 .$$

<u>Proof</u>

Suppose not. Then there exists a $\mathcal{C}$-region M in $\mathfrak{M}(\Delta)$. with

$$\sum_{E \text{ in } (M)} \tilde{d}(E;M) > -6 .$$

If each b-edge is either an $(M,\mathcal{C})$-edge or an S-edge, then clearly we have a contradiction (using Lemma (4.3)). Hence there exists a b-edge in $\mathcal{S}(M)$ which is a T-edge. Apply corollary (4.5) and Lemma (4.3) to show a contradiction.

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<u>Case II</u> - when (i) card $A \geq 3$,

(ii) there exists $c \in A$, $c \neq a$ such that the unique relator in R involving a^p is of the form

$r \equiv (a^p c^{\eta} z' c^{\eta})^m$, $m \geq 1$, $\eta = \pm 1$ and z' is a^p-free.

Put $w \equiv (a^p)^{36L}$ and $u \equiv b$, $b \neq c$. This follows by an almost same way as in case I.

<u>Case III</u> - when (i) $A = \{a,b\}$,

(ii) The unique relator involving a^p is not of the form $r \equiv (a^{p} b^{\varepsilon} z b^{-\varepsilon})^{m}$, $m \geq 1$, z is a^{p}-free, Put $w \equiv (a^{p})^{36L}$ and $u \equiv b$. This again follows by an almost similar way as in case I.

Section 5

Let $G = \langle A;R \rangle$ be a finitely presented group which satisfies (H1). Throughout this section we assume that every generator in A is a piece relative to R and that there exists a in A with

(i) $a^{m} \notin R$, for any $m \neq 0$,

(ii) $\theta_{R}(a^{p}) = 2$,

(iii) card $(A) \geq 3$,

(iv) there exists $c \in A$, $c \neq a$, such that the unique relator r involving a^{p} is of the form $r \equiv (a^{p} c^{\varepsilon} z c^{-\varepsilon})^{m}$, $m \geq 1$ and $\varepsilon = \pm 1$.

Put $w \equiv (a^{p})^{36L}$ and $u \equiv b$, where $b \neq c$ and $b \neq a$. Then for each $(\mathcal{C}, \mathcal{G})$-edge E in $\mathcal{M}(\Delta)$, $\varphi(E)$ is a subword of $a^{\varepsilon p}$ or $a^{\varepsilon l_1} b^{\varepsilon} a^{\varepsilon l_2}$, where $1 \leq l_1, l_2 < p$ and $\varepsilon = \pm 1$.. So that $\theta_{R}(\varphi(E)) \leq 3$ and certainly, in the usual way, $\mathcal{M}(\Delta)$ is suitable R'-diagram.

Let M be any $\mathcal{C}$-region in $\mathcal{M}(\Delta)$ with $\delta(M) = (E_1, E_2, \ldots, E_n)$ and either $\varphi(E_1) \equiv x^{\pm 1}$ or $\varphi(E_1) \equiv y^{\pm 1}$.

(5.1) Remarks

Let E_m, $2 \leq m \leq n$ be an $(M, \mathcal{G})$-edge.

(1) If E_m is an S-edge, then $\theta_R(\varphi(E_m)) \leq 2$.

(2) A T-edge cannot involve $a^{\eta p}$, $\eta = \pm 1$.

(3) $\theta_R(e_m) = 3$ if and only if $e \equiv a^{\eta \ell_1} b^{\eta} a^{\eta \ell_2}$, where

$1 \leq \ell_1, \ell_2 < p$ and $\varepsilon = \pm 1$.

(4) If E_m is an S-edge, then $i(D_m) \geq 4$. Moreover, if $\tilde{d}(E_m;M) = 0$, then $i(D_m) = 4$ or 5 and;

(i) for $i(D_m) = 4$, $(D, \mathcal{C})^{\#} = 2$ and $d(v) = 3$ all $v \in \beta(D_m) \cap \vartheta$,

(ii) for $i(D_m) = 5$, $(D, \mathcal{C})^{\#} = 1$ and $d(v) = 3$, all $v \in \beta(D_m) \cap \vartheta$.

(5) If E_m is an S-edge and there exists a vertex v in $\beta(D_m)$ with $d(v) \geq 4$, then $\tilde{d}(E_m;M) \leq -\frac{1}{6}$.

(6) $i(D_m) \neq 2$.

(7) If $i(D_m) = 3$, then $(D_m, \mathcal{C})^{\#} \geq 2$. Moreover, if $i(D_m) = 3$ and $(D_m, \mathcal{C})^{\#} = 2$, then $(D_m, \mathcal{C})$-edges in $\beta(D_m)$ are T-edges.

(8) Suppose that $i(D_m) = 3$ with $(D, \mathcal{C})^{\#} = 2$ and $\delta(D_m) = (E_{m,1}, E_{m,2}, E_{m,3})$.

(i) If $E_{m,2}$ is a $(D_m, \mathcal{G})$-edge and either $d(\lambda(E_{m,2})) \geq 5$ or $d(\mu(E_{m,2})) \geq 5$, then $\tilde{d}(E_m;M) \leq -\frac{1}{10}$.

(ii) If $E_{m,2}$ is a $(D_m, \mathcal{G})$-edge and either $d(\lambda(E_{m,2})) = 4$ or $d(\mu(E_{m,2})) = 4$, then $\tilde{d}(E_m;M) + \tilde{d}(E_{m-1};M) \leq -\frac{1}{4}$.

(iii) If $E_{m,3}$ is a $(D_m, \mathcal{G})$-edge and either

$d(\lambda(E_{m,3})) \geq 5$ or $d(\mu(E_{m,3})) \geq 5$, then $\tilde{d}(E_m;M) \leq -\frac{1}{10}$.

(iv) If $E_{m,3}$ is a $(D_m,\mathcal{G})$-edge and either $d(\lambda(E_{m,3})) = 4$ or $d(\mu(E_{m,3})) = 4$, then $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{4}$.

(9) If $i(D_m) = 3$ and $(D_m,\mathcal{E})^{\#} = 3$, then $\tilde{d}(E_m;M) \leq -\frac{1}{6}$.

(10) Suppose that $i(D_m) = 4$ and $\delta(D_m) = (E_{m,1},\dots,E_{m,4})$

(i) If $(D_m,\mathcal{E})^{\#} = 1$ and either $\lambda(E_m) \geq 5$ or $d(\mu(E_m)) \geq 5$, then $\tilde{d}(E_m;M) \leq -\frac{1}{5}$.

(ii) If $(D_m,\mathcal{E})^{\#} = 1$ and $d(\lambda(E_m)) = 4$, then $\tilde{d}(E_m;M) + \tilde{d}(E_{m-1};M) \leq -\frac{1}{8}$.

(iii) If $(D_m,\mathcal{E})^{\#} = 1$ and $d(\mu(E_m)) = 4$, then $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{8}$.

(iv) If $(D_m,\mathcal{E}) = 2$ and there exists a vertex $v \in \beta(D_m)$, $d(v) \geq 4$, then $\tilde{d}(E_m;M) \leq -\frac{1}{4}$.

(v) If $(D_m,\mathcal{E}) \geq 3$, then $\tilde{d}(E_m;M) \leq -\frac{1}{3}$.

(11) Suppose that $i(D_m) = 5$ and $\delta(D_m) = (E_{m,1},\dots,E_{m,5})$.

(i) If $(D_m,\mathcal{E})^{\#} = 1$ and either $d(\mu(E_m)) \geq 4$ or $d(\lambda(E_m)) \geq 4$, then $\tilde{d}(E_m;M) \leq -\frac{3}{4}$.

(ii) If $(D_m,\mathcal{E})^{\#} = 2$, then $\tilde{d}(E_m;M) \leq -\frac{1}{2}$.

(iii) If $(D_m,\mathcal{E})^{\#} \geq 3$, then $\tilde{d}(E_m;M) \leq -\frac{2}{3}$.

(12) Suppose that $i(D_m) = 6$, then $\tilde{d}(E_m;M) \leq -1$.

(13) Suppose that $i(D_m) \geq 7$, then $\tilde{d}(E_m;M) \leq -\frac{8}{7}$.

(14) If $i(D_m) = 4$ and $(D_m, \mathcal{C})^{\#} = 1$, then $\theta_R^*(r_m) = 6$. If $i(D_m) = 5$ and $(D_m, \mathcal{C})^{\#} = 1$, then $\theta_R^*(r_m) = 6$ and $\sigma(E_{m,3}) \in \mathcal{G}$.

(15) Let $i(D_m) = 3$, $(D, \mathcal{C})^{\#} = 2\lambda$. Then e_m must end in a^{ε}, $\varepsilon = \pm 1$. Suppose that $d(\mu(E_m)) = 3$ and $\sigma(E_{m+1}) \in \mathcal{G}$. If $e_{m+1} \equiv a^{\varepsilon P}$, $d(\mu(E_{m+1})) = 3$ and $\sigma(E_{m+2}) \in \mathcal{G}$, then $e_{m+2} \not\equiv a^{\varepsilon P}$; Moreover there is no pair of $(M, \mathcal{G})$-edges $E_j; E_{j+1}$, $2 \leq j \leq n-1$ and $j \neq m+1$ such that $e_j \equiv e_{j+1} \equiv a^{\varepsilon P}$.

(16) Let $i(D_m) = 4$ and $(D, \mathcal{C})^{\#} = 1$. Then $e_m \equiv a^{\varepsilon \ell_1} b^{\varepsilon} a^{\varepsilon \ell_2}$, $1 \leq \ell_1, \ell_2 < p$.

If either $e_{m+1} \equiv a^{\varepsilon P}$ or $e_{m-1} \equiv a^{\varepsilon P}$, then there is no pair of $(M, \mathcal{G})$-edges E_j, E_{j+1}, $2 \leq j \leq m-2$ (or $m+2 \leq j \leq n$) such that $e_j \equiv e_{j-1} \equiv a^{\varepsilon p}$

<u>(5.2) Lemma</u>

Let E_m, $2 \leq m \leq n$ be an $(M, \mathcal{G})$-edge. If $\varphi(E_m) \equiv a^{\eta P}$, $\eta = \pm 1$, then $E_{m,2}$ and $E_{m,n(m)}$ must be $(D_m, \mathcal{G})$-edges.

<u>Proof</u>

Compare r_m and r, noting that a^P is not a piece relative to R. It follows that $E_{m,2}$ must begin with c^{ν} and $E_{m,n(m)}$ must end with c^{ν}, $\nu = \pm 1$. Hence both $E_{m,2}$ and $E_{m,n(m)}$ are $(M, \mathcal{G})$-edges.

(5.3) Lemma

Let E_m, $2 \leq m \leq n-2$ be an $(M, \mathcal{J})$-edge. Suppose that E_m is an S-edge with the property that $i(D_m) = 4$, $\tilde{d}(E;M) > -\frac{1}{6}$ and $e_m \equiv a^{\varepsilon \ell}$, $1 \leq \ell \leq p$. Further assume that E_{m+1} and E_{m+2} are S-edges.

Then $\sum_{j=m}^{m+2} \tilde{d}(E_j;M) \leq -\frac{1}{4}$.

Proof

Without loss of generality we can assume $\varepsilon = -1$, i.e. $e_m \equiv a^{-\ell}$, $1 \leq \ell \leq p$.

Suppose that $e_{m,1} \equiv a^{\ell_1}$, $1 \leq \ell_1 < p$. Then $(D, \mathcal{E})^{\#} = 2$ and $d(v) = 3$ all $v \in \beta(D_m) \cap \vartheta$, since $\theta_R(a^{\ell_1}) = 1$ and $\tilde{d}(E_m;M) > -\frac{1}{6}$. (see Remarks 5.1). Moreover the other $(D_m, \mathcal{E})$-edge must be $E_{m,3}$ with $e_{m,3} \equiv (a^{\ell_2} ba^{\ell_3})^{\eta}$, $1 \leq \ell_2, \ell_3 < p$ and $\eta = \pm 1$. Thus $E_{m+1,3}$ is a $(D_{m+1}, \mathcal{E})$-edge and $e_{m+1,3}$ must be a subword of $a^{\eta p}$ since $d(\mu(E_{m,3})) = 3$. Note that it is sufficient to show that either $\tilde{d}(E_{m+1};M) \leq -\frac{1}{4}$ or $\tilde{d}(E_{m+2};M) \leq -\frac{1}{4}$.

If $e_{m+1,1} \equiv b$, then it is not hard to see that $\tilde{d}(E_{m+1};M) \leq -\frac{1}{2}$; and by a similar argument we can show that if $e_{m+1,1} \equiv a^{\ell_3}$, $1 \leq \ell_3 \leq p$, then $\tilde{d}(E_{m+1};M) \leq -\frac{1}{2}$.

Now, let $e_{m+1,1} \equiv a^p$ and $\tilde{d}(E_{m+1};M) = 0$. Then $e_{m+1,3} \equiv a^p$ and $i(D_{m+1}) = 4$ such that $E_{m+1,4} \in (D_{m+1}, \mathcal{G})$ and $d(v) = 3$, all $v \in \beta\,(D_m)$. We write $r_{m+1} \equiv a^p\, e_{m+1,2}\, a^p\, e_{m+1,4}$. If $e_{m+2,1} \equiv a^p$, then we can write $r_{m+2} \equiv a^p e^{-1}_{+1,4}\, z'$ which is absurd, since a^p is not a piece. Thus either $e_{m+2,1} \equiv b$ or $e_{m+2,1} \equiv a^{\ell_5}$, $1 \leq \ell_5 < p$ and in both cases $\tilde{d}(E_m;M) \leq -\frac{1}{4}$.

(5.4) Lemma

Let E_m, $2 \leq m \leq n-2$ be an $(M, \mathcal{G})$-edge. Suppose that $i(D_m) = 3$, $(D_m, \mathcal{E})^{\#} = 2$ and $\tilde{d}(E_m;M) > -\frac{1}{10}$. Then there exists t, $m+1 \leq t \leq m+2$ such that

$$\sum_{j=m+1}^{t} \tilde{d}(E_j;M) \leq -\frac{1}{10}.$$

Proof

Suppose that $E_{m,2}$ is a $(D_m, \mathcal{G})$-edge. Hence $E_{m,3}$ must be in $(D_m, \mathcal{E})$. If $d(\mu(E_m)) \geq 5$, then it is not hard to see that $\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{3}{10}$ and so $t = m+1$.

Let $d(\mu(E_m)) = 4$. Clearly, $e_{m+1,1} \equiv a^{\varepsilon \ell_1}$, $1 \leq \ell_1 < p$ and $e_{m+1,2} \equiv a^{\varepsilon \ell_2}$, $1 \leq \ell_2 < p$; and so $\Theta_R(e_{m+1,1}\, e_{m+1,2}) \leq 2$. If $\Theta_R(e_{m+1,1}\, e_{m+1,2}) = 2$, then $e_{m+1,1}\, e_{m+1,2} \equiv a^{\varepsilon p}$, $\varepsilon = \pm 1$. Hence $E_{m+1,3}$ and $E_{m+1,n(m+1)}$ are $(D_{m+1}, \mathcal{G})$-edges, by Lemma (5.2). So that $i(D_{m+1}) \geq 5$ and for $i(D_{m+1}) = 5$, $(D, \mathcal{E})^{\#} = 3$ whence

$\tilde{d}(E_{m+1};M) \leq -\frac{2}{3}$. Thus $t = m+1$.

Let $\theta_R(e_{m+1,1}\ e_{m+1,2}) = 1$. Since $e_{m+1,2}$ ends in a^{ε}, $\theta_R(\prod_{j=1}^{3} e_{m+1,j}) \leq 3$. If $\theta_R(\prod_{j=1}^{3} e_{m+1,j}) < 3$, then (it is not difficult to see that $i(D_{m+1}) \geq 5$ and), $t = m+1$. If $\theta_R(\prod_{j=1}^{3} e_{m+1,j}) = 3$, then $E_{m+1,3} \in (D_{m+1}, \mathcal{E})$ and $e_{m+1,3}$ must end in a^{ε}, $\varepsilon = \pm 1$. Hence $\theta_R(\prod_{j=1}^{4} e_{m+1},j) \leq 5$ and $i(D_{m+1}) \neq 4$. Thus $i(D_m) \geq 5$. If $i(D_{m+1}) = 5$, then $(D_{m+1}, \mathcal{E})^{\#} \geq 4$. and so $\tilde{d}(E_{m+1};M) \leq -\frac{3}{4}$; and again $t = m+1$.

If $d(\mu(E_m)) = 3$, then $E_{m+1} \in (M, \mathcal{E})$ whence $t = m+1$. Let $E_{m,2}$ be a $(D_m, \mathcal{E})$-edge. Hence $E_{m,3} \in (D_m, \mathcal{G})$. If $d(\mu(E_m)) \geq 4$, then $d(\mu(E_m)) = 4$ and $t = m+1$. (Using Remark 8(iii) and (iv).

Now, we consider the case when $d(\mu(E_m)) = 3$. Let $e_{m+1,1} \equiv a^{\eta\rho}$, $\eta = \pm 1$. If $d(\mu(E_{m+1})) \geq 5$, then

$\tilde{d}(E_m;M) + \tilde{d}(E_{m+1};M) \leq -\frac{1}{10}$ and $t = m+1$. If $d(\mu(E_{m+1})) = 4$, then $t = m+2$. Let $d(\mu(E_{m+1})) = 3$. If $e_{m+1,1} \equiv a^{\eta\rho}$, then $E_{m+1,\,n(m+1)} \in (D_{m+1}, \mathcal{G})$. It is not hard to see that $d(\mu(E_{m,2})) =$ 3 or 4; and for $d(\mu(E_{m,2})) = 4$, $t = m+1$.

Let $d(\mu(E_{m,2})) = 3$, then $E_{m+1,3}$ is a $(D_{m+1}, \sigma(E_{m,2}))$-edge. (Remember that $\sigma(E_{m,2}) \in \mathcal{E}$). Thus $e_{m+1,3} \equiv a^{\varepsilon \ell_2}$, $1 \leq \ell_2 \leq p$. Let $e_{m+1,3} \equiv a^{\varepsilon\rho}$. Then $\varepsilon = \eta$. If $i(D_{m+1}) = 4$, then we write $r_{m+1} \equiv (a^{\eta\rho} e_{m+1,4})^2$. If either $d(\mu(E_{m+1,4})) \geq 5$ or $d(\lambda(E_{m+1,4})) \geq 5$, then $t = m+1$. Also, if either $d(\mu(E_{m+1,4})) = 4$ or $d(\lambda(E_{m+1,4})) = 4$, then $t = m+2$.

Let $d(\mu(E_{m+1,4})) = d(\lambda(E_{m+1,4})) = 3$. Thus $e_{m+2,1} \equiv a^{\eta \ell_3}$ and $e_{m+2,3} \equiv a^{\eta\ell_4}$, $1 \leq \ell_3, \ell_4 < p$. Then $i(D_{m+2}) \geq 5$ and if $i(D_{m+2}) = 5$ then $(D, \mathcal{L})^{\#} \geq 3$; and so $t = m+2$.

Let $e_{m+1,3} \equiv a^{\varepsilon \ell_5}$, $1 \leq \ell_5 < p$. Then $\tilde{d}(E_{m+1};M) \leq -\frac{1}{2}$. Thus if $t \neq m-1$ then $t = m-2$.

Let $e_{m+1,1} \equiv a^{\eta\ell_6}$, $1 \leq \ell_6 < p$. By the usual way we can show that t is either $m+1$ or $m+2$.

By almost the same way we can establish the following results:

<u>(5.5) Lemma</u>

Let E_m, $2 \leq m \leq n-20$ be an $(M, \mathcal{G})$-edge. Suppose that $i(D_m) = 4$, $(D, \mathcal{L}) = 1$ and $\tilde{d}(E_m;M) > -\frac{1}{8}$. Then E_{m-1} and E_{m+1} must be S-edges.and there exists t, $m-1 \leq t \leq m+20$ such that $\sum_{j=m-1}^{t} d(E_j;M) \leq -\frac{1}{8}$, provided that for each $m \leq j \leq t-1$, $E_j \in (M, \mathcal{G})$.

<u>(5.6) Lemma</u>

Let E_m, $2 \leq m < n$ be an $(M, \mathcal{G})$-edge. Suppose $i(D_m) = 4$, $(D, \mathcal{L})^{\#} = 2$ and $d(E_m;M) > -\frac{1}{4}$. Further suppose that $\sigma(E_{m-1}) \in \mathcal{G}$. Then there exists $m-1 \leq t \leq m+2$ such that $\sum_{j=m-1}^{m+2} d(E_j;M) \leq -\frac{1}{4}$.

(5.7) Lemma

Let E_m, $2 \leq m < n$ be an $(M,\mathcal{G})$-edge. Let $i(D_m) = 4$, $(D,\mathcal{C})^{\#} = 2$, $\tilde{d}(E;M) = 0$ and $\sigma(E_{m-1}) \in \mathcal{C}$. Suppose that there exists a b-edge following E_m (see section 4 for the definition of the b-edge) and let $E_{m'}$, $2 < m' \leq n$, be the first b-edge following E_m in $\delta(M)$. Further assume that there does not exist an $(M,\mathcal{C})$-edge between E_m and $E_{m'}$ and that $\sum_{j=m}^{m'-2} \tilde{d}(E_j;M) = 0$. Then $\sum_{j=m}^{t} d(E_j;M) \leq -\frac{1}{10}$, where $m'-1 \leqslant t \leqslant m'-5$.

(5.8) Lemma

Let E_m, $2 \leq m \leq n$, be a T-edge. Suppose that $i(D_m) = 5$, $(D,\mathcal{C})^{\#} = 1$ and $d(E_m;M) > -\frac{1}{4}$. Then there exists t, $m-1 \leq t \leq m+2$ such that $\sum_{j=m-1}^{t} d(E_j;M) \leq -\frac{1}{4}$.

(5.9) Lemma

For any $\mathcal{C}$-region M in $\mathcal{M}(\Delta)$,

$$\sum_{E \text{ in } \delta(M)} \tilde{d}(E;M) \leq -6.$$

Proof

Suppose, by way of contradiction, we have

$$\sum_{E \text{ in } \mathcal{S}(M)} \tilde{d}(E;M) > -6 .$$

First, $\tilde{d}(E_1;M) = -1$. If each E_j, $2 \leq j \leq n$, is an $(M,\mathcal{C})$-edge then there is nothing to prove. Moreover, if there are more than or equavelent to L/8 b-edges in $(M,\mathcal{G})$, then a contradiction can be obtain easily . Otherwise let $E_{m_1}, E_{m_2}, \ldots, E_{m_t}$, $2 \leq t \leq n$, be a subsequence of $(E_1,\ldots,E_n)$ consisting of the b-edges in $(M,\mathcal{G})$. Thus $t \geq 7L/8$. If for each $1 \leq j \leq t$, $\tilde{d}(E_{m_j};M) \leq -(1/10)$, then again we have a contradiction. Otherwise there exists i, $1 \leq i \leq t$, such that $d(E_{m_i};M) > -\frac{1}{10}$. Then $3 \leq i(D_{m_i}) \leq 5$.

Since L is chosen to be large enough, in effect, we have 36L b-edges in $\mathcal{S}(M)$. We can apply one of the Lemmas(5.4)-(5.9), for any b-edge whose artificial degree $> -\frac{1}{10}$ and show that $\sum_{j=1}^{n} d(E_j;M) < -6$, which is impossible .

Section 6

Let $G = \langle a,b; R \rangle$ be a finitely presented group which satisfies (H1). Throughout this section, we assume that

(i) a and b are pieces relative to R,

(ii) $a^m \notin R$, any $m \neq 0$,

(iii) $\theta_R(a^p) = 2$,

(iv) the unique relator r involving a^p is of the form

$r \equiv (a^p b^{\varepsilon} z a^{-\varepsilon})^m, \quad m \geq 1.$

We write $r \equiv (a^p b^{k_1} z' a^{\eta} b^{-k_2})^m$, $m \geq 1$, taking $\varepsilon = 1$.

We shall divide this section into ten cases. In each case we give suitable words w and u and leave the proofs to the reader.

Case I - when (i) $q = 1$,

(ii) $\theta_R(ab) = 1$,

Put $w \equiv (a^p)^{36L}$ and $u \equiv b^2$. Obse

(See section 2).

Case II - when (i) $q = 1$,

(ii) $\theta_R(ab) = 2$.

Put $w \equiv (ab^2)^{L/2}$ and $u \equiv a\, b a^{-\eta} b^2$. (we recall that the unique relator r involving a^p has the form $r \equiv (a^p b z a^{\eta} b^{-1})^m$, $m \geq 1$). (Apply Lemma (2.7)).

Case III - when (i) $q \geq 3$,

(ii) $\theta_R(ab) = 1$,

and (iii) $k_1 = 1$.

Put $w \equiv (a^p)^{36L}$ and $u \equiv b$. (see section 5).

<u>Case IV</u> - when (i) $q \geq 3$,

(ii) $\theta_R(ab) = 1$,

and (iii) $k_1 \geq 2$.

Put $s_1 \equiv xwa \prod_{j=2}^{L+1} (u\ a\ w^j)$, and

$s_2 \equiv y\ u \prod_{j=L+2}^{2L} (a\ w^j\ u)$, where $w \equiv (a^p)^{L/2}$ and

$u \equiv b$. (see section 5).

<u>Case V</u> - when (i) $\theta_R(b^2) = 1$,

(ii) $k_1 = 1$,

(iii) $\theta_R(ab) = 2$.

Put $w \equiv (ab^2)^{L/2}$ and $u \equiv a\ b\ a^{-\eta}\ b^2$. (see section 2).

<u>Case VI</u> - when (i) $\theta_R(b^2) = 1$,

(ii) $\theta_R(ab) = 2$,

and (iii) $k_1 \geq 2$.

Put $w \equiv (ab)^{L/2}$ and $u \equiv a\ b^2\ a^{-\eta}\ b\ a\ b$ (see section 2).

<u>Case VII</u> - when (i) $\theta_R(a^2) = 1$. i.e. $p \geq 3$.

(By symmetry).

<u>Case VIII</u> - when (i) $\theta_R(a^2) = \theta_R(b^2) = 2$, i.e. $p = q = 2$.

and (ii) $r \equiv (a^2b^2a^{-1}zb^{-1})^m$, $m \geq 1$.

(1) If $\theta_R(ab) = 1$, then argue exactly as with case IV.

(2) If $\theta_R(ab) = 2$, then put $w \equiv (ab)^{L/2}$ and $u \equiv ab^{-1}ab$.
(see section 2).

<u>Case IX</u> - when (i) $\theta_R(a^2) \equiv \theta_R(b^2) = 2$,

(ii) $r \equiv (a^2b\, a^{\eta}\, z\, b^{-1})^m, m \geq 1$,

and (iii) $\theta_R(ab) = 1$.

(1) If $\eta = -1$, then argue similarly as with case III.

(2) If $\eta = 1$, then put $w \equiv a^{2L}$ and $u \equiv b^{2L}$.
(see section 2)

<u>Case X</u> - when (i) $\theta_R(a^2) = \theta_R(b^2) = 2$,

(ii) $r \equiv (a^2b\, a^{\eta}\, z\, b^{-1})^m$, $m \geq 1$,

and (iii) $\theta_R(ab) = 2$.

Put $w \equiv (ab)^{L/2}$ and $u \equiv a\, b\, a^{-\eta}\, b$. (see section 2).

REFERENCES

(1) J. L. Britton, Solution of the word problem for certain types of groups, I, II, Proc. Glasgow Math. Assoc. 3 (1956) 45-54(1958) 68-90.

(2) D. J. Collins, Free subgroups of small cancellation groups, Proc. London Math. Soc. Vol. 26 (1973) 193-206.

(3) M. Dehn, Über unendliche diskontinuierliche Gruppen, Math. Ann. 71 (1911) 116-144.

(4) ———, Transformation der Kurven auf zweiseitigen Flächen, Math. Ann. 72 (1912) 413-421.

(5) M. Greendlinger, On Dehn's algorithm for the word problem, Comm. Pure Appl. Math. 13 (1960) 67-83.

(6) ———, On Dehn's algorithm for the word and conjugacy problems with applications, Comm. Pure Appl. Math. 13 (1960) 641-677.

(7) ———, On the word problem and the conjugacy problem, Izv. Akad. Nauk. SSSR Ser. Mat. 29 (1965) 245-268. These results were announced in Doklady Akad. Nauk. SSSR 154 (1964) 507-509.

(8) G. Higman, A finitely generated infinite simple group, J. London Math. Soc. 26 (1951) 61-64.

(9) S. Lipschutz, Elements in S-groups with trivial centralizers, Comm. Pure Appl. Math. 13 (1960) 679-683.

(10) R. C. Lyndon, On Dehn's algorithm, Math. Ann. 166 (1966) 208-228.

(11) W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory: Presentations of groups in terms of generators and defining relations. (Interscience Publishers, New York, London, Sydney, 1966.)

(12) C. F. Miller and P. Schupp, The Geometry of HNN extensions, Comm. Pure Appl. Math. 26 (1973) 787-802.

(13) P. Neumann, The SQ-universality of some finitely presented groups, J. Australian Math. Soc. 16 (1973) 1-6.

(14) K. Reidemeister, Einführung in die Kombinatorische Topologie, (Braunschweig, 1932).

(15) G. Sacerdote and P. Schupp, SQ-universality in HNN groups and one relator groups, J. London Math. Soc. (2) (1974) 733-740.

(16) H. Schiek, Ähnlichkeitsanalyse von Guppenrelationen, Acta Math. 96 (1956) 157-251.

(17) P. E. Schupp, On Dehn's algorithm and the conjugacy problem, Math. Ann. 178 (1968) 119-130.

(18) ——, On Greendlinger's Lemma, Comm. Pure Appl. Math. 23 (1970) 233-240.

(19) ——, A survey of small cancellation theory, in Word Problems. (North-Holland, Amsterdam, London, 1973.)

(20) ——, A survey of SQ-universality, in Conference on Group Theory. (Springer-Verlag, Berlin, Heidelberg, New York, Vol. 319, 1973.)

(21) V. A. Tartakovskii, The sieve method in group theory, Mat. Sbornik (N.S.) 25 (1949) 3-50.

(22) V. A. Tartakovskii, Application of the sieve method to the solution of the word problem for certain types of groups, Math. Sbornik (N.S.) 25 (1949) 251-274.

(23) ——— , Solution of the word problem for groups with a k-reduced basis for k > 6, Izv. Akad. Nauk. SSSR, Ser. Mat. 13 (1949) 483-494.

(24) E. R. Van Kampen, On some lemmas in the theory of groups, Amer. J. Math. 55 (1933) 268-273.

(25) C. M. Weinbaum, Visualizing the word problem, with an application to sixth groups, Pacific J. Math. 16 (1966) 557-578.