Queen Mary University of London

Applications of cosmological perturbation theory

Christopherson, Adam J.

Astronomy Unit
School of Mathematical Sciences
Queen Mary, University of London

Applications of Cosmological Perturbation Theory

Adam J. Christopherson

Submitted in part fulfilment of the requirements for the degree of
Doctor of Philosophy in Mathematical Sciences of the University of London

Declaration

I hereby certify that this thesis, which is approximately 35,000 words in length, has been written by me; that it is the record of the work carried out by me at the Astronomy Unit, Queen Mary, University of London, and that it has not been submitted in any previous application for a higher degree. Parts of this work have been completed in collaboration with Karim A. Malik and David R. Matravers, and are published in the following papers:

- A. J. Christopherson and K. A. Malik:
  Phys. Lett. B **675** (2009) pp. 159-163,
- A. J. Christopherson, K. A. Malik and D. R. Matravers:
  Phys. Rev. D **79** 123523 (2009),
- A. J. Christopherson and K. A. Malik:
  JCAP **11** (2009) 012,
- A. J. Christopherson:
  Phys. Rev. D **82** 083515 (2010),
- A. J. Christopherson, K. A. Malik and D. R. Matravers:
  arXiv:1008.4866 [astro-ph.CO] (submitted).
- A. J. Christopherson and K. A. Malik:
  Invited article, Class. Quant. Grav. (in press),

I have made a major contribution to all the original research presented in this thesis.

Adam J. Christopherson

Abstract

Cosmological perturbation theory is crucial for our understanding of the universe. The linear theory has been well understood for some time, however developing and applying the theory beyond linear order is currently at the forefront of research in theoretical cosmology.

This thesis studies the applications of perturbation theory to cosmology and, specifically, to the early universe. Starting with some background material introducing the well-tested 'standard model' of cosmology, we move on to develop the formalism for perturbation theory up to second order giving evolution equations for all types of scalar, vector and tensor perturbations, both in gauge dependent and gauge invariant form. We then move on to the main result of the thesis, showing that, at second order in perturbation theory, vorticity is sourced by a coupling term quadratic in energy density and entropy perturbations. This source term implies a qualitative difference to linear order. Thus, while at linear order vorticity decays with the expansion of the universe, the same is not true at higher orders. This will have important implications on future measurements of the polarisation of the Cosmic Microwave Background, and could give rise to the generation of a primordial seed magnetic field. Having derived this qualitative result, we then estimate the scale dependence and magnitude of the vorticity power spectrum, finding, for simple power law inputs a small, blue spectrum.

The final part of this thesis concerns higher order perturbation theory, deriving, for the first time, the metric tensor, gauge transformation rules and governing equations for fully general third order perturbations. We close with a discussion of natural extensions to this work and other possible ideas for off-shooting projects in this continually growing field.

Acknowledgements

I would like to thank Karim Malik for all his help, support and guidance, without which this truly would not have been possible. I am very grateful to David Matravers for his valuable input into an enjoyable collaboration. I would like to thank everyone in the cosmology group at Queen Mary, in particular Ian Huston, James Lidsey and Reza Tavakol, and my fellow students of 301, both past and present.

Finally, I would like to thank my parents for their encouragement and Christine for her constant support, helping me remember that there exists a world outside of science!

This work was funded by the Science and Technology Facilities Council (STFC), and by the Astronomy Unit and School of Mathematical Sciences at Queen Mary, University of London.

To the loving memory of Ted and Iris.

Contents

List of Figures

List of Tables

1 Introduction

Over the last few decades cosmology has moved from a mainly theoretical discipline to one in which data is of increasing importance. This is promising, since it means that we are no longer confined to the theorists' playground, but instead have observational data with which to constrain our models.

At present the main observable that we have with which to test our theories is the Cosmic Microwave Background (CMB) radiation. This is radiation that was produced when the universe was around 380,000 years old and had cooled enough to allow electrons and protons to combine to produce Hydrogen atoms. Its first detection in 1964 by Penzias and Wilson was hailed as a firm success of the hot Big Bang cosmological model, and experiments have been performed in the years after in order to obtain more details of this radiation. The Cosmic Background Explorer (COBE) [23] and the Wilkinson Microwave Anisotropy Probe (WMAP) [77] satellites have since probed the anisotropies of the CMB, finding that it is extremely isotropic (up to one part in 100,000), and the Planck [1] satellite is currently taking data to further increase our wealth of data on the CMB. This observation of small anisotropies is very much in agreement with the theory, which states that quantum perturbations in the field driving inflation produce small primordial density fluctuations which are then amplified through gravitational instability to form the structure that exists in the universe today.

In order to study the theoretical framework of the standard cosmological model (see, e.g. Ref. [107], for a particularly lucid review), one uses cosmological perturbation theory, which is the main topic of this thesis. The basic idea is quite simple: we model the universe as a homogeneous 'background' which has inhomogeneous perturbations on top. The perturbations can then be split up order-by-order, with each order being smaller than the one before.

Early studies of the linear order theory were mainly done by the following authors. Lifshitz pioneered the early work on perturbations in Ref. [89], which was later extended in Ref. [90], with Bonnor considering density perturbations in Ref. [25]. This early work by Lifshitz was conducted in the synchronous gauge, which has since been shown to exhibit gauge artefacts if one is not careful [128]. Though the authors used intricate geometrical arguments in order to remove these unphysical modes, the fact that the calculations cannot be done easily was far from ideal. Thus, Hawking [63] and Olson [124] took a different approach to this problem and attempted the first fully covariant study of cosmological perturbations, focussing not on perturbations of the metric tensor, but instead on perturbations of the curvature tensor. However, the most pioneering work in modern cosmological perturbation theory was completed by Bardeen. In Ref. [16], Bardeen presented a systematic method for removing the gauge artefacts by constructing gauge invariant variables. His work focussed on the two metric potentials Ψ and Φ, which correspond to the two gauge invariant scalar metric perturbations in the longitudinal gauge. This work was then followed by the two review articles by Kodama and Sasaki [75] and Mukhanov, Feldman and Brandenberger [120]. These three articles together arguably form the basis of linear metric cosmological perturbation theory.

An alternative approach to metric perturbation theory is often called the covariant approach. The approach defines gauge invariant variables using the Stewart-Walker lemma (a perturbation which vanishes in the background is gauge invariant [145]) and was mostly pioneered by Ellis and collaborators [49–51]. An interesting paper made a first step towards relating the covariant approach to the metric approach which was written by Bruni *et al.* [29].

Cosmological perturbations can be decomposed into scalar, vector and tensor perturbations, as we will show in Chapter 2 and, at linear order, the three types of perturbations decouple from one another. The scalar modes are related to density perturbations, vector modes are vortical or rotational perturbations, and the tensor modes are related to gravitational waves. The CMB radiation is polarised, and the different types of perturbations induce different polarisations. Scalar modes produce only E-mode (or curl-free) polarisation and tensor modes produce only B-mode (divergence-free) polarisation. Vectors produce both, but are usually deemed negligible, since any produced in the early universe will be inflated away, or will decay with the expansion of the universe [75].

However, once we go beyond linear order, different types of perturbation no longer decouple and so, for example, vector and tensor perturbations are sourced by scalar modes. This mathematical difference between linear and higher orders therefore plays an important role in the theory and can result in qualitatively different physics beyond linear order which will, in turn, generate different observational signatures. This is, really, the main reason for extending perturbation theory beyond linear order, which has been studied by many authors in the last few years [3, 18, 30, 41, 105, 110, 119, 121–123, 148] (see Ref. [112] for a recent review and a comprehensive list of references on second order cosmological perturbations). Perturbation theory beyond linear order is the central theme of this thesis.

This thesis is organised as follows. In the remainder of this introductory Chapter we present the standard model of cosmology in a more detailed sense, briefly introducing inflationary cosmology, and restating our notation that will be used for the remainder of this thesis at the end of the Chapter. In Chapter 2 we introduce the theory of non-linear cosmological perturbations up to second order. We present the perturbed metric tensor, and energy momentum tensor for a perfect fluid (i.e. a fluid with no anisotropic stress) and a scalar field. We consider next the transformation behaviour of the different perturbations under a gauge transformation, using these to choose gauges and define gauge invariant variables. Next, we present a discussion of the thermodynamics of a perfect fluid, discussing the pressure and energy density perturbations and the definition of the non-adiabatic pressure perturbation, which will play a central role in the following work. Finally, to close Chapter 2, we briefly consider how non-adiabatic pressure perturbations can arise naturally in multiple fluid or multi-field inflationary models.

In Chapter 3 we continue presenting the foundations of cosmological perturbation theory and present the dynamic and constraint equations up to second order. Starting with the linear order theory we present the governing equations for scalar, vector and tensor perturbations of a perfect fluid without fixing a gauge. We then fix a gauge, giving the equations in terms of gauge invariant variables for three different gauges: the uniform density, uniform curvature and longitudinal gauges, solving the equations for the latter two. We then present the Klein-Gordon equation for a scalar field, and highlight the important difference between the adiabatic sound speed and the speed with which perturbations travel for a scalar field system. Finally, we investigate the perturbations of a system containing both dark energy and dark matter. Having laid the foundations with the linear theory, we move on to the second order theory, presenting the governing equations for a perfect fluid coming form energy momentum conservation and the Einstein equations in gauge dependent form. We then present all equations in the uniform curvature gauge which we will use in Chapter 4, including now only the canonical Klein-Gordon equation. Finally, in order to connect with the literature, we give the equations for scalars in the Poisson gauge.

Having now developed the tools for second order perturbation theory, in Chapter 4 we use the qualitative differences between the linear theory and higher order theory to show that, at second order in perturbation theory, vorticity is sourced by a coupling between first order energy density and entropy perturbations. This is analogous to the case of classical fluid mechanics and generalises Crocco's theorem to an expanding background. To show this, we start by defining the vorticity tensor in general relativity, and then calculate the vorticity tensor at linear and second order using the fluid four velocity and the metric tensor defined in Chapter 2. We then compute the evolution equation for the vorticity, making use of the governing equations in Chapter 3. At linear order vorticity is not sourced, however there exists a non-zero source term at second order when allowing for fluid with a general equation of state that depends upon both the energy and the entropy. Having derived this qualitative result, we then give a first quantitative solution, estimating the magnitude and scale dependence of the induced vorticity using simple input power spectra: the energy density derived in Chapter 3, and using a simple ansatz for the non-adiabatic pressure perturbation.

In Chapter 5 we extend the formalism from the second order theory to third order, presenting the gauge transformation rules and constructing gauge invariant variables. Then, considering perfect fluids and including all types of perturbation, we present the energy and momentum conservation equations and give components of the Einstein tensor up to third order. We also give the Klein-Gordon equation for a scalar field minimally coupled to gravity. Finally, to close, we conclude in Chapter 6 and present possible directions in which one can extend the work presented in this thesis.

1.1 Standard Cosmology

We now introduce some elements of standard cosmology in a more quantitative sense and, in doing so, define our notation. The basic starting point in cosmology is the cosmological principle which states that, on large enough scales, the universe is both isotropic and homogeneous. In general relativity, geometry is encoded in the metric tensor $g_{\mu\nu}$, or the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The general line element for an isotropic and homogeneous spacetime, and thus one which obeys the cosmological principle, takes the form[1] [87]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\Big(d\theta^2 + \sin^2\theta d\chi^2\Big)\right], \tag{1.1}$$

in spherical coordinates $(t; r, \theta, \chi)$, where t denotes the coordinate time, $a(t)$ is a function that depends only on time, and K denotes the global curvature of the

[1]Throughout this thesis we use the so-called 'East coast' metric signature $(- + ++)$ and the positive $(+ + +)$ sign convention in the notation of Misner *et al.* [116].

spatial slices, where $K = 1$ denotes a positively curved, or closed, universe, $K = 0$ a flat universe, and $K = -1$ a negatively curved, or open, universe. This is the Friedmann-Robertson-Walker (FRW) metric. An alternative way of representing the FRW metric is

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \,, \tag{1.2}$$

where γ_{ij} is the metric tensor on spatial hypersurfaces.

Since current observations are consistent with a flat, $K = 0$ universe, which is also in agreement with inflation, we adopt this choice henceforth and so write the line element as

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,, \tag{1.3}$$

where δ_{ij} denotes the Kronecker delta.

We note here the importance of the function $a(t)$, called the scale factor, in an expanding spacetime. We can picture space as a coordinate grid which expands uniformly with the increase of time. The comoving distance, $\boldsymbol{x}$ between two points, which is just measured by the comoving coordinates, remains constant as the universe expands. The physical distance, $\boldsymbol{r}$, is proportional to the scale factor,

$$\boldsymbol{r} = a(t)\boldsymbol{x} \,, \tag{1.4}$$

and so does evolve with time. Thus, an isotropic and homogeneous universe is characterised not only by its geometry, but also by the evolution of the scale factor [45]. In order to quantify the expansion rate, we introduce the Hubble parameter, $H(t)$ defined as

$$H(t) = \frac{da/dt}{a} = \frac{\dot{a}}{a} \,, \tag{1.5}$$

where an overdot denotes a derivative with respect to coordinate time, which measures how rapidly the scale factor changes.

It is often convenient to use, instead of t, the conformal time coordinate η defined through

$$\eta = \int_{\infty}^{t} \frac{dt}{a} \,, \tag{1.6}$$

in terms of which the line element (1.3) becomes

$$ds^2 = a^2(\eta)\Big[-d\eta^2 + \delta_{ij}dx^i dx^j\Big] \,. \tag{1.7}$$

In doing this, we have increased the spatial coordinate grid introduced above to a coordinate grid over the entire spacetime. We can furthermore define the conformal

Hubble parameter

$$\mathcal{H}(\eta) = \frac{da/d\eta}{a} = \frac{a'}{a}\,, \tag{1.8}$$

where we have used a prime to denote a derivative with respect to conformal time. Then, the Hubble parameter in coordinate and conformal time are related to one another by

$$\mathcal{H} = aH\,. \tag{1.9}$$

Having introduced the metric tensor of an FRW universe, we can now go on to discuss the dynamical equations. In general relativity the curvature of a given spacetime is encoded in the Riemann curvature tensor, defined as [33]

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\lambda\beta}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\mu\beta}\,, \tag{1.10}$$

where $\Gamma^{\sigma}_{\delta\lambda}$ are the Christoffel connection coefficients, defined in terms of the metric tensor and its derivatives as

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda})\,, \tag{1.11}$$

where we have introduced the notation $g_{\alpha\beta,\gamma} \equiv \partial_\gamma g_{\alpha\beta}$. There are two contractions of the Riemann tensor that are particularly useful: the Ricci tensor is given by

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}\,, \tag{1.12}$$

and the Ricci scalar, which is the contraction of the Ricci tensor

$$R \equiv g^{\mu\nu}R_{\mu\nu}\,. \tag{1.13}$$

The Riemann tensor obeys the following identity

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0\,, \tag{1.14}$$

where the square brackets denote anti-symmetrisation over the relevant indices. Eq. (1.14) is often called the Bianchi identity. If we introduce the Einstein tensor $G_{\mu\nu}$, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\,, \tag{1.15}$$

then the Bianchi identity implies that the divergence of this tensor vanishes identically:

$$\nabla^\mu G_{\mu\nu} = 0\,. \tag{1.16}$$

The equation of motion in general relativity is the Einstein equation,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}\,, \tag{1.17}$$

where G is Newton's gravitational constant. $T_{\mu\nu}$ is the energy momentum tensor, which describes the energy and momentum of the matter content of the spacetime. We take the perfect fluid energy momentum tensor which has the following form

$$T^\mu{}_\nu = (\rho_0 + P_0) u^\mu_{(0)} u_{(0)\nu} + P_0 \delta^\mu{}_\nu\,, \tag{1.18}$$

where ρ and P are the energy density and pressure, and u^μ the fluid four velocity, satisfying the constraint $u^\mu u_\mu = -1$. Note that the subscript '0' denotes the value of the quantity in the homogeneous and isotropic background: the importance of this notation will become apparent in Chapter 2. In addition to the Einstein equation, there also exist a set of evolution equations for the matter variables. These are obtained through the covariant conservation of the energy momentum tensor,[2]

$$\nabla_\mu T^\mu{}_\nu = 0\,. \tag{1.19}$$

This equation is obtained from the conservation of the Einstein tensor, Eq. (1.16), and by using the Einstein equation, Eq, (1.17).

Now, let us consider the FRW spacetime. Using the fluid four velocity

$$u^\mu_{(0)} = \frac{1}{a}(1,0)\,, \qquad u_{(0)\mu} = -a(1,0)\,. \tag{1.20}$$

and the Christoffel symbols for the (flat) FRW spacetime

$$\Gamma^0_{00} = \mathcal{H}\,, \qquad \Gamma^0_{ij} = \mathcal{H}\delta_{ij}\,, \qquad \Gamma^i_{j0} = \mathcal{H}\delta^i{}_j\,, \tag{1.21}$$

$$\Gamma^0_{0i} = 0\,, \qquad \Gamma^i_{00} = 0\,, \qquad \Gamma^i_{jk} = 0\,, \tag{1.22}$$

[2] It should be noted that this is not the only way one can obtain evolution equations. Instead, varying the action with respect to the matter fields will result in the same evolution equations except in the case where the system cannot be described by an action (e.g., dissipative fluids [2]).

the Einstein equations (1.17) are then

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho_0 a^2\,, \tag{1.23}$$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\rho_0 + 3P_0)a^2\,, \tag{1.24}$$

where the first equation comes from the 0-0 component and the second from the trace of the spatial Einstein equation. Eqs. (1.23) and (1.24) are called the Friedmann and acceleration equations, respectively. The acceleration equation can be rewritten, by introducing the constant equation of state $P_0 = w\rho_0$, as

$$\mathcal{H}' = -\frac{4\pi G}{3}(1 + 3w)\rho_0 a^2\,, \tag{1.25}$$

where w is the equation of state parameter. Energy conservation gives the continuity equation

$$\rho_0' = -3\mathcal{H}(\rho_0 + P_0)\,, \tag{1.26}$$

which can also be rewritten as

$$\rho_0' = -3\mathcal{H}(1 + w)\rho_0\,. \tag{1.27}$$

This can then be integrated to give

$$\rho_0 = \bar{\rho}_0 \left(\frac{a}{\bar{a}}\right)^{-3(1+w)}\,, \tag{1.28}$$

where an overbar denotes the value of a quantity today. The scale factor and Hubble parameter can then be shown to evolve as

$$a = \bar{a}\left(\frac{\eta}{\bar{\eta}}\right)^{2/(1+3w)}\,, \qquad \mathcal{H} = \frac{2}{1+3w}\eta^{-1}\,. \tag{1.29}$$

We now highlight the evolution of the parameters in the different eras of the universe in the hot Big Bang model. The first is the radiation era, where the universe is filled with a fluid of particles moving at (or close to) the speed of light, such as photons.[3] The equation of state parameter for such a fluid is $w = 1/3$, which

[3]We should note that, generically, one would expect the initial radiation era to be violently disordered. However, the existence of the inflationary era guarantees that the radiation era is smooth. We will introduce inflationary cosmology in the next section.

gives

$$\rho_0 \propto a^{-4}\,, \qquad \text{and} \qquad a \propto \eta\,. \tag{1.30}$$

The next epoch is the matter domination era, where the universe is filled with collisionless, non-relativistic particles which better models a universe filled with galaxies. This matter is called dust, and is well modelled by a pressureless fluid with equation of state parameter $w = 0$. In this era the energy density and scalar factor evolve, respectively, as

$$\rho_0 \propto a^{-3}\,, \qquad \text{and} \qquad a \propto \eta^2\,. \tag{1.31}$$

1.2 Inflationary Cosmology

Any summary of modern cosmology would not be complete without a short discussion on the inflationary paradigm[4], which is a crucial part of the standard cosmological model. Historically, it was introduced in an attempt to solve some outstanding problems in the big bang model. These are:

- *the Horizon problem*, that CMB radiation coming from areas of the universe that were never in causal contact are observed to have the same temperature;
- *the Flatness problem*, that in order for the universe to be so close to flat today, it must have started off very close to flat;
- *the Relic problem*, that no topological relics[5] are observed, though they are likely produced in the early universe.

In order to solve these problems, inflation was postulated in the early '80s, simultaneously by Guth [60], Starobinsky [143], Albrecht and Steinhardt [8] and Linde [91, 92]. The precise definition of inflation is simple: it is a period during which the universe undergoes accelerated expansion, i.e.

$$\ddot{a} > 0\,, \tag{1.32}$$

[4]See, e.g., Ref. [87] for a more complete treatment of the classical Big Bang problems

[5]Topological relics, such as magnetic monopoles are generically produced if the symmetry of a Grand Unified Theory is restored in the early universe and then broken spontaneously. Such an abundance of relics is higher than observation allows. See, e.g., Refs. [64, 150] for more details.

where we have switched to coordinate time in the section for clarity. Another definition, equivalent to the first is

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0\,. \tag{1.33}$$

This definition is more intuitive, since $1/aH$ is the Hubble horizon size, and so inflation is defined as a period in which the Hubble size is decreasing. This is precisely the condition required to solve the flatness problem. Finally, a third equivalent definition of inflation is

$$\rho_0 + 3P_0 < 0\,, \tag{1.34}$$

which, since ρ_0 is always positive from the weak energy condition [62], implies a negative pressure during inflation.

In order to be entirely critical of inflation, we should state that inflation does not completely remove the initial flatness problem. To begin an inflationary phase, there must exist a Planck-scale patch of spacetime that is roughly smooth at an early time. It is hoped that the tuning necessary to achieve this is less than that required to suppress the spatial curvature at early times in the absence of an inflationary phase, but this is not necessarily true; it may be much worse. Similarly, one might argue that inflation does not entirely solve the relic problem, depending on the model of inflation, since some hybrid models may produce topological defects as a by-product of their operation. Thus, it is too simplistic to claim that inflation solves all the problems completely, however any further investigation into this is beyond the remit of this thesis.

The most popular type of matter which can drive inflation is a scalar field. A scalar field has a Lagrangian density of the form

$$\mathcal{L} = p(X, \varphi)\,, \tag{1.35}$$

where X is the relativistic kinetic energy, $X \equiv \frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$ which, for a homogeneous field, is then $X = -\frac{1}{2}\dot{\varphi_0}^2$, where $\varphi_0(t)$ is the scalar field. Inflationary models can be classified according to the form of the Lagrangian, and whether they contain a single scalar field or multiple fields. The most simple single field inflationary model has a Lagrangian

$$p(X, \varphi_0) = X - U(\varphi_0)\,, \tag{1.36}$$

where $U(\varphi_0)$ is the potential of the field. This simple model can take very different forms depending on the choice of this potential function. More exotic models have been considered more recently which modify the Lagrangian through changing the dependence on X as well or instead of changing the potential. These models have been of interest since they enable us to evade observational bounds placed on the simple, canonical, single field models.

For the simple, canonical model, then, we can write down a pressure and energy density for the scalar field as

$$\rho_0 = \frac{1}{2}\dot{\varphi_0}^2 + U(\varphi_0)\,, \tag{1.37}$$

$$P_0 = \frac{1}{2}\dot{\varphi_0}^2 - U(\varphi_0)\,. \tag{1.38}$$

Then, using the energy conservation equation (1.26) we obtain the Klein-Gordon equation for the homogeneous field

$$\ddot{\varphi}_0 + 3\mathcal{H}\dot{\varphi}_0 + U_{,\varphi} = 0\,, \tag{1.39}$$

and the Friedmann equation (1.23) becomes

$$H^2 = \frac{8\pi G}{3}\Big(U(\varphi_0) + \frac{1}{2}\dot{\varphi_0}^2\Big)\,. \tag{1.40}$$

A useful approximation exists, which consists of neglecting the first term of Eq. (1.39) and the last term of Eq. (1.40), to give

$$H^2 \simeq \frac{8\pi G}{3}U\,, \tag{1.41}$$

$$3H\dot{\varphi}_0 \simeq -U_{,\varphi}\,, \tag{1.42}$$

which, for this approximation to be true, demands two parameters defined as

$$\varepsilon(\varphi_0) = 4\pi G\left(\frac{U_{,\varphi}}{U}\right)^2\,, \tag{1.43}$$

$$\eta(\varphi_0) = 8\pi G\,\frac{U_{,\varphi\varphi}}{U}\,, \tag{1.44}$$

to be small (i.e. much less than 1). This approximation is called the *slow-roll approximation*, and if it holds guarantees that inflation will occur. In fact, inflation ends when ε becomes 1. We do not discuss specific models of inflation here, instead pointing the interested reader to one of the many textbooks available on the topic [87].

To close this brief section on inflationary cosmology, we consider some observational signatures of the early universe. Since this is somewhat beyond the main aim of this thesis, we simply present some results, and refrain from any derivations. To begin, we use the two point correlator of the linear scalar field perturbation, $\langle\delta\varphi_1\delta\varphi_1\rangle$, defined later on, which one can then relate to the spectrum of the curvature perturbation that will source the anisotropies in the CMB, e.g. the comoving curvature perturbation $\langle\mathcal{R}_1\mathcal{R}_1\rangle$, which is defined in Eq. (2.143). This can then be evolved forwards using a Boltzmann code [86, 138, 154], to give predictions for the anisotropies in the CMB.

In the notation of the WMAP team [77], the primordial spectrum is then taken to be a power law with amplitude $\Delta_{\mathcal{R}}(k_0)^2$ and spectral index $n_{\rm s}$

$$\Delta_{\mathcal{R}}(k)^2 = \Delta_{\mathcal{R}}(k_0)^2\left(\frac{k}{k_0}\right)^{n_{\rm s}-1}, \tag{1.45}$$

where $\Delta_{\mathcal{R}}(k_0)^2 = 2.38\times10^{-9}$ and $n_{\rm s} = 0.969$, at the pivot scale of $k_0 = 0.002{\rm Mpc}^{-1}$. The observations are compatible with the mostly adiabatic and Gaussian inflationary initial condition, but are incompatible with other a priori equally motivated suggestions, such as density perturbations induced by topological defects. Constraints on inflationary model building are discussed in detail in, e.g., Refs. [5, 6]. Similarly, we can write the spectrum for tensor perturbations as

$$\Delta_h(k)^2 = \Delta_h(k_0)^2\left(\frac{k}{k_0}\right)^{n_{\rm T}}, \tag{1.46}$$

and we can then define the tensor-scalar ratio, r, as [77]

$$r \equiv \frac{\Delta_h(k_0)^2}{\Delta_{\mathcal{R}}(k_0)^2}\,. \tag{1.47}$$

Inflationary models can then be tested according to their predictions for these observables. The scalar spectral index is well constrained by WMAP observations, and in fact a generic, successful feature of inflation is its ability to generate a near scale invariant spectrum [87]. However, the tensor-scalar ratio is less constrained, and different inflationary models make varying predictions for r (e.g. Refs. [4, 5, 99, 142] and references therein). It is hoped that we will be able to narrow the error bars on the tensor-scalar ratio by future observations of the polarisation of the CMB from the sky (Refs. [1, 21]) and from the ground (e.g. Ref. [53]) which will in turn enable us to rule out some models of inflation.

1.3 Notation

To close the introduction, we briefly state some of the notational conventions that will be used in this thesis.

- We use the mostly positive metric signature, $(-+++)$, and the $(+++)$ convention in the notation of [116].
- Coordinate time is denoted with t and an overdot denotes a derivative with respect to coordinate time; the Hubble parameter is $H = \dot{a}/a$.
- Conformal time is denoted by η and a prime denotes a derivative with respect to conformal time; the conformal Hubble parameter is $\mathcal{H} = a'/a$.
- Greek indices $\{\mu, \nu, \ldots\}$ cover the entire spacetime and take the range $\{0 \ldots 3\}$.
- Latin indices $\{i, j, \ldots\}$ cover the spatial slice and take the range $\{1 \ldots 3\}$.
- The index 0 (as in u^0) denotes conformal time, and an index t (as in u^t) denotes coordinate time.
- The order in the perturbative expansion is denoted with a subscript, ϕ_1, or with a subscript enclosed in parentheses if the meaning could be ambiguous, as in $u^\mu_{(0)}$.
- A comma denotes a partial derivative so, e.g.,

$$X_{,\mu} \equiv \partial_\mu X \equiv \frac{\partial X}{\partial x^\mu}\,, \qquad \text{or, when } X \equiv X(Y), \qquad X_{,Y} \equiv \frac{\partial X}{\partial Y}\,.$$

- A semicolon denotes a covariant derivative with respect to the full spacetime metric, $g_{\mu\nu}$, i.e.,

$$X_{\nu;\mu} \equiv \nabla_\mu X_\nu\,.$$

- The Lie derivative along a vector field ξ^μ is denoted $\pounds_\xi$ and takes the following forms for a scalar φ, a vector, v_μ, and a tensor, $t_{\mu\nu}$:

$$\pounds_\xi \varphi = \xi^\lambda \varphi_{,\varphi}\,, \tag{1.48}$$

$$\pounds_\xi v_\mu = v_{\mu,\alpha}\xi^\alpha + v_\alpha {\xi^\alpha}_{,\mu}\,, \tag{1.49}$$

$$\pounds_\xi t_{\mu\nu} = t_{\mu\nu,\lambda}\xi^\lambda + t_{\mu\lambda}{\xi^\lambda}_{,\nu} + t_{\lambda\nu}{\xi^\lambda}_{,\mu}\,. \tag{1.50}$$

2 Cosmological Perturbations

As discussed in the introduction, cosmological perturbation theory is an extremely useful, and successful, tool to study the universe in which we live. In this Chapter we introduce cosmological perturbation theory in the metric approach (i.e. where we consider perturbations to the metric á la Bardeen [16]) in a more formal and quantitative manner. We introduce the line element for the most general perturbations to FRW, and then define the perturbed energy momentum tensor for both a perfect fluid and for a scalar field (with both a canonical and non-canonical action). We then derive gauge transformations for the different types of perturbation (scalar, vector and tensor), and use these to define choices of gauge and gauge invariant variables. We focus on gauge choice at linear order and give an illustrative example of how to choose a gauge at second order. We close this Chapter by considering the thermodynamics of a perfect fluid, defining what we mean by non-adiabatic, or entropic, perturbations, and briefly discuss how such perturbations can naturally arise in multi-field or multiple fluid systems.

2.1 Metric Tensor

We consider the most general perturbations to the flat FRW metric, which gives a line element of the form

$$ds^2 = a^2(\eta)\Big[-(1+2\phi)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + 2C_{ij})dx^i dx^j\Big]\,. \tag{2.1}$$

We choose a flat background metric because it agrees with observations and is mathematically easier to work with, but should note that all the techniques used in this Chapter are valid for a background spacetime with non-zero curvature.

The perturbations of the spatial components of the metric can be further decomposed as

$$B_i = B_{,i} - S_i \,, \tag{2.2}$$

$$C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij} \,. \tag{2.3}$$

The perturbations are classified as scalar, vector and tensor perturbations according to their transformation behaviour on spatial three hypersurfaces [112, 120]. The scalar metric perturbations are ϕ, the lapse function, ψ, the curvature perturbation and E and B, which make up the scalar shear. S_i and F_i are divergence free vector perturbations, and h_{ij} is a transverse, traceless tensor perturbation. The perturbations therefore obey the following relations

$$\partial_i S^i = 0 \,, \tag{2.4}$$

$$\partial_i F^i = 0 \,, \tag{2.5}$$

$$\partial_i h^{ij} = 0 = h^i{}_i \,. \tag{2.6}$$

Since the perturbations are inhomogeneous, they depend upon both space and time, e.g.

$$\phi \equiv \phi(x^\mu) = \phi(\eta, x^i) \,. \tag{2.7}$$

The scalar perturbations each contribute one degree of freedom to the perturbed metric tensor, each divergence free vector perturbation has two degrees of freedom, as does the transverse, traceless tensor perturbation, so we see that, in total, there are 10 degrees of freedom – the same as the number of independent components of the perturbed metric tensor. Each perturbation can then be expanded in a series: For example the lapse function is split as

$$\phi = \sum_n \frac{\epsilon^n}{n!}\phi_n \tag{2.8}$$

$$= \epsilon\phi_1 + \frac{1}{2}\epsilon^2\phi_2 + \frac{1}{3!}\epsilon^3\phi_3 + \cdots \,, \tag{2.9}$$

where the subscript denotes the order of the perturbation and ϵ is a fiducial expansion parameter. We will often omit ϵ when not required for brevity and write

$$\phi = \phi_1 + \frac{1}{2}\phi_2 + \frac{1}{3!}\phi_3 + \cdots \,, \tag{2.10}$$

In order to define this expansion uniquely, we choose the first order quantity, ϕ_1, to have Gaussian statistics. The series is then truncated at the required order.

Performing this split then gives us the covariant components of the metric tensor up to second order

$$g_{00} = -a^2\left(1 + 2\phi_1 + \phi_2\right)\,, \tag{2.11}$$

$$g_{0i} = a^2\left(2B_{1i} + B_{2i}\right)\,, \tag{2.12}$$

$$g_{ij} = a^2\left(\delta_{ij} + 2C_{1ij} + C_{2ij}\right)\,. \tag{2.13}$$

The contravariant components of the metric tensor are obtained by imposing the constraint

$$g_{\mu\nu}g^{\nu\lambda} = \delta_\mu{}^\lambda\,, \tag{2.14}$$

to the appropriate order. To second order this gives

$$g^{00} = -\frac{1}{a^2}\Big(1 - 2\phi_1 - \phi_2 + 4{\phi_1}^2 - B_{1k}B_1^k\Big)\,, \tag{2.15}$$

$$g^{0i} = \frac{1}{a^2}\Big(B_1^i + \frac{1}{2}B_2^i - 2\phi_1 B_1^i - 2B_{1k}C_1^{ki}\Big)\,, \tag{2.16}$$

$$g^{ij} = \frac{1}{a^2}\Big(\delta^{ij} - 2C_1^{ij} - C_2^{ij} + 4C_1^{ik}C_{1k}{}^j - B_1^i B_1^j\Big)\,. \tag{2.17}$$

2.2 Energy Momentum Tensor

The matter content of the universe is described by the energy momentum tensor. Since General Relativity links the geometry of spacetime to its matter content, perturbations in the metric tensor invoke perturbations in the energy momentum tensor.[1] In this section, we outline the perturbed energy momentum tensor for a perfect fluid and a scalar field, to second order in perturbation theory.

2.2.1 Perfect Fluid

The energy momentum tensor for a perfect fluid, i.e. in the absence of anisotropic stress, is as presented in the previous section,

$$T^\mu{}_\nu = (\rho + P)u^\mu u_\nu + P\delta^\mu{}_\nu\,, \tag{2.18}$$

where ρ is the energy density, P is the pressure and u^μ is the four velocity of the fluid. Note that, for the purposes of this thesis, we define a 'perfect fluid' to be a

[1]There are, in fact, works where this is not the case and, for example, only the energy momentum tensor is perturbed. However, in order for the work to be consistent, one should perturb both the geometry and the matter content of the spacetime.

fluid with a diagonal energy momentum tensor as in, e.g., Ref. [57].

The energy density and the pressure are expanded up to second order in perturbation theory in the standard way

$$\rho = \rho_0 + \delta\rho_1 + \frac{1}{2}\delta\rho_2\,, \tag{2.19}$$

$$P = P_0 + \delta P_1 + \frac{1}{2}\delta P_2\,. \tag{2.20}$$

The fluid four velocity, which is defined as

$$u^\mu = \frac{dx^\mu}{d\tau}\,, \tag{2.21}$$

where τ is an affine parameter, here the proper time, and is subject to the constraint

$$u^\mu u_\mu = -1\,. \tag{2.22}$$

To second order in perturbation theory the fluid four velocity has the contravariant components

$$u^i = \frac{1}{a}\left(v_1{}^i + \frac{1}{2}v_2{}^i\right)\,, \tag{2.23}$$

$$u^0 = \frac{1}{a}\left(1 - \phi_1 - \frac{1}{2}\phi_2 + \frac{3}{2}\phi_1^2 + \frac{1}{2}v_{1k}v_1{}^k + v_{1k}B_1{}^k\right)\,, \tag{2.24}$$

and the covariant components,

$$u_i = a\left(v_{1i} + B_{1i} + \frac{1}{2}(v_{2i} + B_{2i}) - \phi_1 B_{1i} + 2C_{1ik}v_1{}^k\right)\,, \tag{2.25}$$

$$u_0 = -a\left(1 + \phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\phi_1^2 + \frac{1}{2}v_1{}^k v_{1k}\right)\,, \tag{2.26}$$

where v^i is the spatial three velocity of the fluid. Then, by substituting the components of the four velocity along with the expansions of the energy density and pressure into Eq. (1.18), we obtain the components of the energy momentum tensor, up to second order

$$T^0{}_0 = -(\rho_0 + P_0)(v_1{}^k + B_1{}^k)v_{1k} - (\rho_0 + \delta\rho_1 + \frac{1}{2}\delta\rho_2)\,, \tag{2.27}$$

$$T^0{}_i = (\rho_0 + P_0)\left(v_{1i} + B_{1i} + \frac{1}{2}(v_{2i} + B_{2i}) - \phi_1(v_{1i} + 2B_{1i}) + 2C_{1ik}v_1{}^k\right)$$
$$+ (\delta\rho_1 + \delta P_1)(v_{1i} + B_{1i})\,, \tag{2.28}$$

$$T^i{}_j = \left(P_0 + \delta P_1 + \frac{1}{2}\delta P_2\right)\delta^i{}_j + (\rho_0 + P_0)v_1{}^i(v_{1j} + B_{1j})\,. \tag{2.29}$$

2.2.2 Scalar Field

As shown in Section 1.2, scalar fields play an important role in modern cosmology through the theory of inflation. The energy momentum tensor for a scalar field minimally coupled to gravity is defined through the Lagrangian density. For a canonical scalar field, the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - U(\varphi)\,, \tag{2.30}$$

where $U(\varphi)$ is the potential energy of the scalar field. The variational energy momentum tensor (or Hilbert stress-energy tensor) is then defined as [87]

$$T_{\mu\nu} \equiv -2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}\,, \tag{2.31}$$

and, for a scalar field φ, we obtain

$$T^\mu{}_\nu = g^{\mu\lambda}\varphi_{,\lambda}\varphi_{,\nu} - \delta^\mu{}_\nu\left(U(\varphi) + \frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}\right)\,. \tag{2.32}$$

Expanding the scalar field in the usual way, and using the definition of the metric tensor given in Section 2.1, gives the components of the energy momentum tensor for a scalar field up to second order in perturbation theory (see, e.g., Refs. [3, 105, 123])

$$
\begin{aligned}
T^0{}_0 = & -\frac{\varphi_0'^2}{2a^2}\Big[1-2\phi_1+2\frac{\delta\varphi_1'}{\varphi_0'}+4{\phi_1}^2-B_{1k}B_1^k+2\phi_1\delta\varphi_1'-\left(\frac{\delta\varphi_1'}{\varphi_0'}\right)^2 \\
& +\frac{\delta\varphi_{1,k}\delta\varphi_{1,}^k}{\varphi_0'^2}-\phi_2-\frac{\delta\varphi_2'}{\varphi_0'}\Big]-U(\varphi_0)-U_{,\varphi}\delta\varphi_1-U_{,\varphi\varphi}\delta\varphi_1^2-U_{,\varphi}\delta\varphi_2\,,
\end{aligned}
\tag{2.33}
$$

$$
\begin{aligned}
T^i{}_0 = & \frac{\varphi_0'^2}{a^2}\Big[B_1^i+\frac{\delta{\varphi_{1,}}^j}{\varphi_0'}-2\phi_1B_1^i-2B_{1k}C_1^{ki}+2\frac{B_1^i\delta\varphi_1'}{\varphi_0'}+\frac{\delta\varphi_1'\delta{\varphi_{1,}}^i}{\varphi_0'^2} \\
& -2\frac{\delta\varphi_{1,j}C_1^{ij}}{\varphi_0'}+\frac{1}{2}\Big(B_2^i+\frac{\delta{\varphi_{2,}}^i}{\varphi_0'}\Big)\Big]\,,
\end{aligned}
\tag{2.34}
$$

$$
T^0{}_i = -\frac{\varphi_0'}{a^2}\Big[\delta\varphi_{1,i}+\frac{\delta\varphi_1'\delta\varphi_{1,i}}{\varphi_0'}-2\phi_1\delta\varphi_{1,i}+\frac{1}{2}\delta\varphi_{2,i}\Big]\,,
\tag{2.35}
$$

$$
\begin{aligned}
T^i{}_j = & \frac{\varphi_0'^2}{a^2}\Big[\frac{B_1^i\delta\varphi_{1,j}}{\varphi_0'}+\frac{\delta{\varphi_{1,}}^i\delta\varphi_{1,j}}{\varphi_0'^2}\Big]-\delta^i{}_j\Big\{U(\varphi_0)+U_{,\varphi}\delta\varphi_1+U_{,\varphi\varphi}\delta\varphi_1^2+U_{,\varphi}\delta\varphi_2 \\
& -\frac{\varphi_0'^2}{2a^2}\Big[1-2\phi_1+2\frac{\delta\varphi_1'}{\varphi_0'}+4{\phi_1}^2-B_{1k}B_1^k+4\frac{\phi_1\delta\varphi_1'}{\varphi_0'}-2\frac{B_1^k\delta\varphi_{1,k}}{\varphi_0'} \\
& -\frac{\delta{\varphi_{1,}}^k\delta\varphi_{1,k}}{\varphi_0'^2}-\left(\frac{\delta\varphi_1'}{\varphi_0'}\right)^2-\phi_2-\frac{\delta\varphi_2'}{\varphi_0'}\Big]\Big\}\,.
\end{aligned}
\tag{2.36}
$$

We can write down a pressure and energy density for the scalar field by comparing the components of the scalar field energy momentum tensor to that of a perfect fluid given in Eqs. (2.27)-(2.29). In the background this gives, as shown in Section 1.2,

$$
\rho_0=\frac{1}{2a^2}\varphi_0'^2+U(\varphi_0)\,,\qquad P_0=\frac{1}{2a^2}\varphi_0'^2-U(\varphi_0)\,,
\tag{2.37}
$$

and, to linear order,

$$
\delta\rho_1=\frac{1}{a^2}\Big(\delta\varphi_1'\varphi_0'-\phi_1\varphi_0'^2\Big)+U_{,\varphi}\delta\varphi_1\,,
\tag{2.38}
$$

$$
\delta P_1=\frac{1}{a^2}\Big(\delta\varphi_1'\varphi_0'-\phi_1\varphi_0'^2\Big)-U_{,\varphi}\delta\varphi_1\,.
\tag{2.39}
$$

We can also express the fluid velocity in terms of the field by comparing Eq. (2.28) to Eq. (2.35) to give

$$
V_1=-\frac{\delta\varphi_1}{\varphi_0'}\,,
\tag{2.40}
$$

where $V_1\equiv B_1+v_1$.

There has been a lot of recent interest in scalar fields with non-canonical actions: early work in the realm of the early universe such as k-inflation [12, 56] and more current work including the string theory motivated Dirac–Born–Infeld (DBI) inflation [9, 66, 88, 135, 140] . Scalar fields with non-canonical actions have also recently been considered as dark energy candidates (see, e.g. Ref. [149]). These models all modify the Lagrangian (2.30) to the more general function[2]

$$\mathcal{L} = p(X, \varphi)\,, \tag{2.41}$$

where $X = -\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}$. The Lagrangian for DBI inflation is then given by

$$p(X,\varphi) = -T(\varphi)\sqrt{1-2T^{-1}(\varphi)X} + T(\varphi) - V(\varphi)\,, \tag{2.42}$$

where $T(\varphi)$ and $V(\varphi)$ are functions that further specify the model.

The energy momentum tensor for the general Lagrangian (2.41) is then obtained from Eq. (2.31) as

$$T^{\mu}{}_{\nu} = p_{,X}g^{\mu\lambda}\varphi_{,\lambda}\varphi_{,\nu} + \delta^{\mu}{}_{\nu}p\,. \tag{2.43}$$

This has the components, up to linear order in perturbation theory, and in coordinate time, t,

$$T^{t}{}_{t} = (p_0 - 2p_{,X}X_0) - \left[(p_{,X} + 2X_0 p_{,XX})\,\delta X_1 + 2X_0 p_{,X\varphi}\delta\varphi_1 - p_{,\varphi}\delta\varphi\right]\,, \tag{2.44}$$

$$T^{t}{}_{i} = -p_{,X}\dot{\varphi}_0\delta\varphi_{1,i}\,, \tag{2.45}$$

$$T^{i}{}_{j} = (p_0 + p_{,X}\delta X_1 + p_{,\varphi}\delta\varphi_1)\,\delta^{i}{}_{j}\,, \tag{2.46}$$

where we have defined $\delta X_1 = \dot{\varphi}_0\delta\dot{\varphi}_1 - \phi_1\dot{\varphi}_0{}^2$, and $X_0 = \frac{1}{2}\dot{\varphi}_0{}^2$. As in the canonical case, we can define a pressure and energy density for the scalar field. The background quantities are

$$\rho_0 = 2p_{,X}X_0 - p\,, \qquad P_0 = p\,, \tag{2.47}$$

and the linear order quantities are

$$\delta\rho_1 = (p_{,X} + 2X_0 p_{,XX})\delta X_1 + (2X_0 p_{,X\varphi} - p_{,\varphi})\delta\varphi_1\,, \tag{2.48}$$

$$\delta P_1 = p_{,X}\delta X_1 + p_{,\varphi}\delta\varphi_1\,. \tag{2.49}$$

[2]One could, of course, cook up even more general Lagrangians. However, typically if higher-order derivatives of the scalar field are present then the model will exhibit Ostrogradski instabilities (see Ref. [47] for an outline of Ostrogradski's original argument).

2.3 Gauge Transformations at First and Second Order

Gauge transformations play an important role in cosmological perturbation theory. General relativity is a theory of differential manifolds with no preferred coordinate charts, and is therefore required to be covariant under coordinate transformations. However, when we come to consider perturbations in general relativity we must, for consistency, consider perturbations of the spacetime itself. That is, in the language of differential geometry, we consider a one parameter family of four-manifolds M_ϵ, embedded in a five-manifold N [108, 144]. Each M_ϵ represents a spacetime, with the base spacetime, or unperturbed $\epsilon = 0$ manifold, M_0. The problem comes in perturbation theory when attempting to compare two objects which 'live' in different spaces. In order to deal with this, we introduce a point identification map $p_\epsilon : M_0 \to M_\epsilon$ which relates points in the perturbed manifold with those in the background. This correspondence introduces a new vector field, X on N, and points which lie on the same integral curve γ of X are regarded as being the same physical point.

However, the choice of this point identification map and, therefore, the vector field X is not unique. The choice of the correspondence between the points on the M_0 and those on M_ϵ or, equivalently, the choice of the vector field X is called a *choice of gauge*, and X is then the gauge generator. A gauge transformation then tells us how we move from one choice of gauge to another.

There are two approaches to gauge transformations. First, consider a point p on M_0. Two generating vectors X and Y then define a correspondence between this point p and two different points s and t on M_ϵ. Clearly, then, these choices induce a coordinate change (gauge transformation) on M_ϵ. This is known as the passive view. Alternatively, consider a point p on M_ϵ. We then find a point s on M_0 which maps to p under the gauge choice X and a point t, also on M_0 that maps to p under the choice Y. In this case, a gauge transformation is induced on the background manifold, M_0. This is known as the active view. We can think of the active approach as the one in which the transformation of the perturbed quantities is evaluated at the same coordinate point, and the passive approach where the transformation of the perturbed quantities is taken at the same physical point.

In this section we go on to discuss the active and passive approach briefly then, adopting the active approach, we derive the gauge transformation rules for scalars, vectors and tensors up to second order in the perturbations.

2.3.1 Active Approach

In the active approach to gauge transformations, the exponential map is the starting point [108, 119]. Once the generating vector of gauge transformation, ξ^μ, has been specified, we can immediately write down how a general tensor $\mathbf{T}$ transforms. The exponential map is

$$\widetilde{\mathbf{T}} = e^{\pounds_\xi}\mathbf{T}\,, \tag{2.50}$$

where $\pounds_\xi$ denotes the Lie derivative with respect to the generating vector ξ^μ which, up to second order in perturbation theory, is

$$\xi^\mu \equiv \epsilon\xi_1^\mu + \frac{1}{2}\epsilon^2\xi_2^\mu + \cdots\,. \tag{2.51}$$

The exponential map is then

$$\exp(\pounds_\xi) = 1 + \epsilon\pounds_{\xi_1} + \frac{1}{2}\epsilon^2\pounds_{\xi_1}^2 + \frac{1}{2}\epsilon^2\pounds_{\xi_2} + \cdots \tag{2.52}$$

up to second order in perturbation theory. Splitting $\mathbf{T}$ order by order, we find that the tensorial quantities transform at zeroth, first and second order, respectively, as [30, 119]

$$\widetilde{\mathbf{T}_0} = \mathbf{T}_0\,, \tag{2.53}$$

$$\epsilon\widetilde{\delta\mathbf{T}_1} = \epsilon\delta\mathbf{T}_1 + \epsilon\pounds_{\xi_1}\mathbf{T}_0\,, \tag{2.54}$$

$$\epsilon^2\widetilde{\delta\mathbf{T}_2} = \epsilon^2\Big(\delta\mathbf{T}_2 + \pounds_{\xi_2}\mathbf{T}_0 + \pounds_{\xi_1}^2\mathbf{T}_0 + 2\pounds_{\xi_1}\delta\mathbf{T}_1\Big)\,. \tag{2.55}$$

By noting that the Lie derivative acting on a scalar is just the directional derivative, $\pounds_\xi = \xi^\mu(\partial/\partial x^\mu)$, the exponential map can be applied to the coordinates x^μ to obtain the following relationship between coordinates at two points, p and q[3]

$$x^\mu(q) = e^{\xi^\lambda\frac{\partial}{\partial x^\lambda}\big|_p}\,x^\mu(p)\,. \tag{2.56}$$

Expanding this to second order gives

$$x^\mu(q) = x^\mu(p) + \epsilon\xi_1^\mu(p) + \frac{1}{2}\epsilon^2\Big(\xi^\mu_{1,\lambda}(p)\xi_1^\lambda(p) + \xi_2^\mu(p)\Big)\,. \tag{2.57}$$

[3]Note that, in some of the literature, a different sign is taken in the exponent in this equation to obtain correspondence between the active and passive approach at linear order. Since we are not solely working at linear order in this work, there is no advantage gained by making such a choice.

This coordinate relationship is not required to perform calculations in the active approach, but will be useful for the discussion in Section 2.3.2 below.

2.3.2 Passive Approach

A natural starting point for discussing the passive approach to gauge transformations is the coordinate relationship Eq. (2.57) since, in the passive approach, one states the relationship between two coordinate systems and then calculates how variables change when transforming from one coordinate system to the other. However, since in the passive approach quantities are evaluated at the same physical point we need to rewrite Eq. (2.57) [108]. Choosing p and q such that $\widetilde{x^\mu}(q) = x^\mu(p)$, Eq. (2.57) enables us to write

$$\begin{aligned}\widetilde{x^\mu}(q) &= x^\mu(p)\\ &= x^\mu(q) - \epsilon\xi_1^\mu(p) - \frac{1}{2}\epsilon^2\Big[\xi^\mu_{1,\lambda}(p)\xi_1^\lambda(p) + \xi_2^\mu(p)\Big]\,. \end{aligned} \tag{2.58}$$

Using the first terms of Eq. (2.57),

$$x^\mu(q) = x^\mu(p) + \epsilon\xi_1^\mu(p)\,, \tag{2.59}$$

allows us to Taylor expand ξ_1^μ as

$$\begin{aligned}\xi_1^\mu(p) &= \xi_1^\mu\big(x^\mu(q) - \epsilon\xi_1^\mu(p)\big)\\ &= \xi_1^\mu(q) - \epsilon\xi^\mu_{1,\lambda}(q)\xi_1^\lambda(q)\,, \end{aligned} \tag{2.60}$$

which is valid up to second order. Substituting Eq. (2.60) into Eq. (2.58) gives the relationship between the two coordinate systems at the same point, q

$$\widetilde{x^\mu}(q) = x^\mu(q) - \epsilon\xi_1^\mu(q) - \frac{1}{2}\epsilon^2\Big[\xi_2^\mu(q) - \xi^\mu_{1\lambda}(q)\xi_1^\lambda(q)\Big]\,. \tag{2.61}$$

Having highlighted the two approaches to gauge transformations we now focus on the active approach and give some concrete examples.

2.3.3 Four Scalars

We will now look at the transformation of four scalars, choosing the energy density, ρ, which can be expanded as in Eq. (2.19),

$$\rho = \rho_0 + \delta\rho_1 + \frac{1}{2}\delta\rho_2 + \frac{1}{3!}\delta\rho_3 + \cdots \,, \tag{2.62}$$

as an example.

First Order

Before studying the transformation behaviour of perturbations at first order, we split the generating vector ξ_1^μ into a scalar temporal part α_1 and a spatial scalar and divergence free vector part, respectively β_1 and ${\gamma_1}^i$, as

$$\xi_1^\mu = (\alpha_1, {\beta_1}_{,}{}^i + {\gamma_1}^i)\,. \tag{2.63}$$

Since the Lie derivative of a scalar ρ with respect to the vector ξ^μ is simply

$$\pounds_\xi \rho = \xi^\mu \rho_{,\mu}\,, \tag{2.64}$$

from Eq. (2.54), we then find that the energy density transforms, at linear order, as

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1\,. \tag{2.65}$$

We see that, at first order, the gauge transformation is completely determined by the time slicing, α_1.[4]

Second Order

At second order we split the generating vector ξ_2^μ in an analogous way to first order as

$$\xi_2^\mu = (\alpha_2, {\beta_2}_{,}{}^i + {\gamma_2}^i)\,. \tag{2.66}$$

Then, using Eq. (2.55), we find that the second order energy density perturbation transforms as

$$\widetilde{\delta\rho_2} = \delta\rho_2 + \rho_0'\alpha_2 + \alpha_1(\rho_0''\alpha_1 + \rho_0'\alpha_1' + 2\delta\rho_1') + (2\delta\rho_1 + \rho_0'\alpha_1)_{,k}({\beta_1}_{,}{}^k + {\gamma_1}^k)\,. \tag{2.67}$$

[4]Note that α_1 does not generate a foliation of spacetime by spatial hypersurfaces – this is inherited from the foliation already present in the background spacetime. Instead, α_1 labels the time slicing.

Thus, at second order, the gauge transformation is only fully determined once the time slicing is specified at first and second order (α_1 and α_2) and the spatial threading (or spatial gauge perturbation) is specified at first order (β_1 and ${\gamma_1}^i$) [112].

2.3.4 The Metric Tensor

We will now focus on the transformation behaviour of the metric tensor. Again, the starting point is the Lie derivative, which for a the metric tensor, $g_{\mu\nu}$, is given by

$$\pounds_\xi g_{\mu\nu} = g_{\mu\nu,\lambda}\xi^\lambda + g_{\mu\lambda}\xi^\lambda{}_{,\nu} + g_{\lambda\nu}\xi^\lambda{}_{,\mu} \,. \tag{2.68}$$

First Order

At first order, the metric tensor transforms, from Eqs. (2.54) and (2.68) as

$$\widetilde{\delta g^{(1)}_{\mu\nu}} = \delta g^{(1)}_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi_1^\lambda + g^{(0)}_{\mu\lambda}\xi_1{}^\lambda{}_{,\nu} + g^{(0)}_{\lambda\nu}\xi_1{}^\lambda{}_{,\mu} \,. \tag{2.69}$$

We can obtain the transformation behaviour of each particular metric function by extracting it, in turn, from the above general expression using the method outlined in Ref. [112]. From Eq. (2.68) we obtain the following transformation behaviour for C_{1ij}

$$2\widetilde{C}_{1ij} = 2C_{1ij} + 2\mathcal{H}\alpha_1\delta_{ij} + \xi_{1i,j} + \xi_{1j,i} \,. \tag{2.70}$$

From this we can extract the transformation behaviour of the spatial metric functions. Here we do not focus on the details, but instead quote results. We find that the scalar metric perturbations transform as

$$\widetilde{\phi_1} = \phi_1 + \mathcal{H}\alpha_1 + \alpha_1' \,, \tag{2.71}$$

$$\widetilde{\psi_1} = \psi_1 - \mathcal{H}\alpha_1 \,, \tag{2.72}$$

$$\widetilde{B_1} = B_1 - \alpha_1 + \beta_1' \,, \tag{2.73}$$

$$\widetilde{E_1} = E_1 + \beta_1 \,, \tag{2.74}$$

the vectors as

$$\widetilde{{S_1}^i} = {S_1}^i - {\gamma_1}^{i\prime} \,, \tag{2.75}$$

$$\widetilde{{F_1}^i} = {F_1}^i + {\gamma_1}^i \,, \tag{2.76}$$

and, as is well known, the tensor component, h_{1ij}, is gauge invariant. Finally, the scalar shear, which is defined as

$$\sigma_1 \equiv E_1' - B_1 \,, \tag{2.77}$$

transforms as

$$\widetilde{\sigma_1} = \sigma_1 + \alpha_1 \,, \tag{2.78}$$

which will be useful later when we come to define gauges and gauge invariant variables.

Second Order

At second order we obtain the transformation behaviour of the metric tensor from Eqs. (2.55) and (2.68), noting that

$$\pounds_{\xi_1}^2 \mathbf{T} = \pounds_{\xi_1}(\pounds_{\xi_1}\mathbf{T}) \,. \tag{2.79}$$

The metric tensor therefore transforms as

$$\begin{aligned}
\widetilde{\delta g^{(2)}_{\mu\nu}} = \delta g^{(2)}_{\mu\nu} &+ g^{(0)}_{\mu\nu,\lambda}\xi_2^\lambda + g^{(0)}_{\mu\lambda}\xi_2{}^\lambda{}_{,\nu} + g^{(0)}_{\lambda\nu}\xi_2{}^\lambda{}_{,\mu} + 2\Big[\delta g^{(1)}_{\mu\nu,\lambda}\xi_1^\lambda + \delta g^{(1)}_{\mu\lambda}\xi_1{}^\lambda{}_{,\nu} + \delta g^{(1)}_{\lambda\nu}\xi_1{}^\lambda{}_{,\mu}\Big] \\
&+ g^{(0)}_{\mu\nu,\lambda\alpha}\xi_1^\lambda\xi_1^\alpha + g^{(0)}_{\mu\nu,\lambda}\xi_1{}^\lambda{}_{,\alpha}\xi_1^\alpha + 2\Big[g^{(0)}_{\mu\lambda,\alpha}\xi_1^\alpha\xi_1{}^\lambda{}_{,\nu} + g^{(0)}_{\lambda\nu,\alpha}\xi_1^\alpha\xi_1{}^\lambda{}_{,\mu} + g^{(0)}_{\lambda\alpha}\xi_1{}^\lambda{}_{,\mu}\xi_1{}^\alpha{}_{,\nu}\Big] \\
&+ g^{(0)}_{\mu\lambda}\left(\xi_1{}^\lambda{}_{,\nu\alpha}\xi_1^\alpha + \xi_1{}^\lambda{}_{,\alpha}\xi^\alpha_{1,\,\nu}\right) + g^{(0)}_{\lambda\nu}\left(\xi_1{}^\lambda{}_{,\mu\alpha}\xi_1^\alpha + \xi_1{}^\lambda{}_{,\alpha}\xi^\alpha_{1,\,\mu}\right) \,,
\end{aligned} \tag{2.80}$$

from which we can extract, as at first order, the transformation behaviour of individual metric perturbation functions. It is a little trickier to obtain the transformation behaviour of ψ_2 from this expression. However, we note that the expression in Eq. (2.80) gives the transformation of C_{2ij}, namely,[5]

$$2\widetilde{C}_{2ij} = 2C_{2ij} + 2\mathcal{H}\alpha_2\delta_{ij} + \xi_{2i,j} + \xi_{2j,i} + \mathcal{X}_{ij} \,, \tag{2.81}$$

where $\mathcal{X}_{ij}$ contains terms quadratic in the first order perturbations and is defined below in Eq. (2.87), and so we need intermediate methods in order to extract the transformation of a particular component. Again, we we do not go into the unnecessary details here, but instead refer the interested reader to Ref. [112]. After this

[5] In the following and for the rest of this section, we do not split up the spatial part of the gauge transformation generating vector into scalar and vector parts for brevity. We denote the spatial part of ξ^μ by ξ^i.

calculation we obtain

$$\widetilde{\psi}_2 = \psi_2 - \mathcal{H}\alpha_2 - \frac{1}{4}\mathcal{X}^k_{\ k} + \frac{1}{4}\nabla^{-2}\mathcal{X}^{ij}_{\ \ ,ij}\,. \tag{2.82}$$

We find that the other second order scalar metric perturbations transform as [112]

$$\begin{aligned}\widetilde{\phi}_2 &= \phi_2 + \mathcal{H}\alpha_2 + {\alpha_2}' + \alpha_1\left[{\alpha_1}'' + 5\mathcal{H}{\alpha_1}' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\alpha_1 + 4\mathcal{H}\phi_1 + 2\phi_1'\right] \qquad (2.83)\\ &\quad + 2{\alpha_1}'\left({\alpha_1}' + 2\phi_1\right) + \xi_{1k}\left({\alpha_1}' + \mathcal{H}\alpha_1 + 2\phi_1\right)_{,}^{\ k} + \xi_{1k}'\left[\alpha_{1,}^{\ k} - 2B_{1k} - \xi_1^{k\prime}\right],\\ \widetilde{E}_2 &= E_2 + \beta_2 + \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathcal{X}^{ij}_{\ \ ,ij} - \frac{1}{4}\nabla^{-2}\mathcal{X}^k_{\ k}\,, \qquad (2.84)\\ \widetilde{B}_2 &= B_2 - \alpha_2 + \beta_2' + \nabla^{-2}{\mathcal{X}_{\rm B}}^k_{\ ,k}\,, \qquad (2.85)\end{aligned}$$

where $\mathcal{X}_{\mathrm{B}i}$ and $\mathcal{X}_{ij}$ are defined as

$$\begin{aligned}\mathcal{X}_{\mathrm{B}i} \equiv\ & 2\Big[\left(2\mathcal{H}B_{1i} + B_{1i}'\right)\alpha_1 + B_{1i,k}\xi_1^k - 2\phi_1\alpha_{1,i} + B_{1k}\xi^k_{1,\ i} + B_{1i}\alpha_1' + 2C_{1ik}\xi_1^{k\prime}\Big]\\ & + 4\mathcal{H}\alpha_1\left(\xi_{1i}' - \alpha_{1,i}\right) + \alpha_1'\left(\xi_{1i}' - 3\alpha_{1,i}\right) + \alpha_1\left(\xi_{1i}'' - \alpha_{1,i}'\right)\\ & + \xi_1^{k\prime}\left(\xi_{1i,k} + 2\xi_{1k,i}\right) + \xi_1^k\left(\xi_{1i,k}' - \alpha_{1,ik}\right) - \alpha_{1,k}\xi^k_{1,\ i}\,, \end{aligned} \tag{2.86}$$

and

$$\begin{aligned}\mathcal{X}_{ij} \equiv\ & 2\Big[\left(\mathcal{H}^2 + \frac{a''}{a}\right)\alpha_1^2 + \mathcal{H}\left(\alpha_1\alpha_1' + \alpha_{1,k}{\xi_1}^k\right)\Big]\delta_{ij} + 2\left(B_{1i}\alpha_{1,j} + B_{1j}\alpha_{1,i}\right)\\ & + 4\Big[\alpha_1\left(C_{1ij}' + 2\mathcal{H}C_{1ij}\right) + C_{1ij,k}{\xi_1}^k + C_{1ik}{\xi_1}^k_{\ ,j} + C_{1kj}{\xi_1}^k_{\ ,i}\Big]\\ & + 4\mathcal{H}\alpha_1\left(\xi_{1i,j} + \xi_{1j,i}\right) - 2\alpha_{1,i}\alpha_{1,j} + 2\xi_{1k,i}{\xi_1}^k_{\ ,j} + \alpha_1\left(\xi_{1i,j}' + \xi_{1j,i}'\right)\\ & + \xi_{1i,k}{\xi_1}^k_{\ ,j} + \xi_{1j,k}{\xi_1}^k_{\ ,i} + \xi_{1i}'\alpha_{1,j} + \xi_{1j}'\alpha_{1,i} + \left(\xi_{1i,jk} + \xi_{1j,ik}\right){\xi_1}^k\,. \end{aligned} \tag{2.87}$$

Furthermore, the vector perturbations transform as

$$\begin{aligned}\widetilde{S}_{2i} &= S_{2i} - {\gamma_{2i}}' - \mathcal{X}_{\mathrm{B}i} + \nabla^{-2}{\mathcal{X}_{\rm B}}^k_{\ ,ki}\,, \qquad (2.88)\\ \widetilde{F}_{2i} &= F_{2i} + \gamma_{2i} + \nabla^{-2}\mathcal{X}_{ik,}^{\ \ \ k} - \nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}_{\ \ ,kli}\,, \qquad (2.89)\end{aligned}$$

and the tensor perturbation which at second order, unlike at first order, is not gauge invariant, as

$$\begin{aligned}\widetilde{h}_{2ij} =\ & h_{2ij} + \mathcal{X}_{ij} + \frac{1}{2}\left(\nabla^{-2}\mathcal{X}^{kl}_{\ \ ,kl} - \mathcal{X}^k_{\ k}\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}_{\ \ ,klij}\\ & + \frac{1}{2}\nabla^{-2}\mathcal{X}^k_{\ k,ij} - \nabla^{-2}\left(\mathcal{X}_{ik,}^{\ \ \ k}{}_j + \mathcal{X}_{jk,}^{\ \ \ k}{}_i\right)\,. \end{aligned} \tag{2.90}$$

2.3.5 Four Vectors

Finally, we move on to the transformation behaviour of a four vector. The Lie derivative of a vector, W_μ, is given by

$$\pounds_\xi W_\mu = W_{\mu,\alpha}\xi^\alpha + W_\alpha \xi^\alpha{}_{,\mu} \,. \tag{2.91}$$

First Order

A four vector transforms at first order, from Eq. (2.91), as

$$\widetilde{\delta W}_{1\mu} = \delta W_{1\mu} + W'_{(0)\mu}\alpha_1 + W_{(0)\lambda}\xi^\lambda_{1,\mu} \,, \tag{2.92}$$

which gives, for the specific case of the four-velocity, the transformation rule

$$\widetilde{V}_{1i} = V_{1i} - \alpha_{1,i} \,, \tag{2.93}$$

where the quantity V_{1i} is defined as

$$V_{1i} \equiv v_{1i} + B_{1i} \,. \tag{2.94}$$

Then, on splitting the velocity perturbation into a vector part and the gradient of a scalar, as

$$v_{1i} = v^{\rm V}_{1i} + v_{1,i} \,, \tag{2.95}$$

recalling that the metric perturbation B_{1i} can be split up as

$$B_{1i} = B_{1,i} - S_{1i} \,, \tag{2.96}$$

and making use of Eqs. (2.73) and (2.75), we obtain the transformation rules

$$\widetilde{v_1} = v_1 - \beta'_1 \,, \tag{2.97}$$

$$\widetilde{v_1^{{\rm V}i}} = v_1^{{\rm V}i} - \gamma_1{}^{i\prime} \,. \tag{2.98}$$

Second Order

At second order we find that a four vector, $W_{2\mu}$, transforms as

$$\begin{aligned}\widetilde{\delta W}_{2\mu} = \delta W_{2\mu} + W'_{(0)\mu}\alpha_2 + W_{(0)0}\alpha_{2,\mu} + W''_{(0)\mu}\alpha_1^2 + W'_{(0)\mu}\alpha_{1,\lambda}\xi^\lambda_1 \,, \\ + 2W'_{(0)0}\alpha_1\alpha_{1,\mu} + W_{(0)0}\Big(\xi^\lambda_1\alpha_{1,\mu\lambda} + \alpha_{1,\lambda}\xi^\lambda_{1,\mu}\Big) + 2\Big(\delta W_{1\mu,\lambda}\xi^\lambda_1 + \delta W_{1\lambda}\xi^\lambda_{1,\mu}\Big) \,.\end{aligned} \tag{2.99}$$

Focussing again on the fluid four velocity, we obtain from the ith component of Eq. (2.99), the transformation behaviour of V_{2i}:

$$\begin{aligned}\widetilde{V_{2i}} = V_{2i} - \alpha_{2,i} + 4\xi_1^{k\prime}C_{ik} + 2\alpha_1'\left(\xi_1' + B_{1i} - \frac{3}{2}\alpha_{1,i}\right) + \alpha_1(2V_{1i}' - \alpha_{1,i}') \\ + 2(\xi_{1i,k} + \xi_{1k,i})(\xi_1^{k\prime} - v_1^k) + 2\mathcal{H}\alpha_1(2B_{1i} - v_{1i} + 3\xi_{1i}') \\ + \xi_1^k(2V_{1i,k} + \alpha_{1,ik}) + \xi_{1,i}^k(2V_{1k} - \alpha_{1,k}) + 2\phi_1(\xi_1' - 2\alpha_{1,i})\,. \end{aligned} \tag{2.100}$$

Then, using the transformation behaviour of B_{2i}, given by

$$\widetilde{B_{2i}} = B_{2i} + \xi_{2i}' - \alpha_{2,i} + \mathcal{X}_{\mathrm{B}i}\,, \tag{2.101}$$

we find that the second order three-velocity, v_{2i}, transforms as

$$\begin{aligned}\widetilde{v_{2i}} = v_{2i} - \xi_{2i}' + \xi_{1i}'(2\phi_1 + \alpha_1' + 2\mathcal{H}\alpha_1) - \alpha_1\xi_1'' - \xi_1^k\xi_{1i,k}' \\ + \xi_1^{k\prime}\xi_{1i,k} + 2\alpha_1(v_{1i}' - \mathcal{H}v_{1i}) - 2v_1^k\xi_{1i,k} + 2v_{1i,k}\xi_1^k\,. \end{aligned} \tag{2.102}$$

2.4 Gauge Choices and Gauge Invariant Variables

As mentioned previously, a central element to Einstein's theory of general relativity is the covariance of the theory under coordinate reparametrisation. However, a problem arises when undertaking metric cosmological perturbation theory since the process of splitting the spacetime into a background and a perturbation is not a covariant process (see, e.g., Ref. [48] and Section 2.2 of Ref. [108]) Therefore, in doing so, one introduces spurious gauge modes so that variables depend upon the coordinate choice. Observational quantities should not depend upon the choice of coordinate used, and therefore this so-called *gauge problem* of perturbation theory seemingly would introduce confusion and erroneous results. However, as long as one is careful to remove the gauge modes, this will not pose us a problem.

The gauge problem was first 'solved' by Bardeen in a consistent way in Ref. [16], and has been studied in much detail, and extended beyond linear order, in the decades since. The solution lies with the introduction of gauge invariant variables, that is, variables which no longer change under a gauge transformation. Bardeen constructed two such variables for scalar perturbations, which happen to coincide with the lapse function and curvature perturbation in the longitudinal gauge (see Section. 2.4.2). However, the systematic approach can be extended to other gauges: one simply inspects the gauge transformation rules presented in the previous sections, and chooses coordinates such that, e.g., two of the scalar metric perturbations are zero. This enables one to remove the gauge dependencies α_1 and β_1, rendering the other scalar perturbations gauge invariant. Similarly, this can be extended beyond scalar perturbations, and one can inspect the transformation rules for a vector perturbation, setting it to zero, and thus removing the dependency on γ_1^i.

Finally, let us emphasise the 'gauge issue' by considering degrees of freedom of the metric. In four dimensions, the metric starts with 16 degrees of freedom: 6 are lost because of symmetry, 4 more because of coordinate invariance (*gauge* choice) of the metric and 4 to do with the Hamiltonian constraints, which arise when writing down the field equations – also called a *gauge* choice. This leaves 2 degrees of freedom for the two polarisations of the graviton. But this is so far not related to perturbation theory. By perturbing the metric, one introduces further degrees of freedom which are not addressed by the above – the option to change the *gauge*, or map. These are the degrees of freedom addressed by the perturbation *gauge* choice.

This paragraph does well to highlight that 'gauge' is a well used term in theoretical physics, and is often used to mean different (albeit closely related) things. When discussing cosmological perturbation theory, and for the rest of this thesis, we reserve the phrase 'choice of gauge' to mean a specification of the mapping between the background and the perturbed spacetimes.

In this section we concentrate on the linear theory, present the definitions of various gauges commonly used throughout the literature, and define some gauge invariant variables. We then give an example of how the theory works at second order, by describing the uniform curvature gauge. This section is mainly a review, and more details can be found in Ref. [112].

2.4.1 Uniform Curvature Gauge

A possible choice of gauge is one in which the spatial metric is unperturbed. At linear order, this amounts to setting $\widetilde{E_1} = \widetilde{\psi_1} = 0$ and $\widetilde{F_{1i}} = 0$. This specifies the gauge generating vector, ξ_1^μ, using Eqs. (2.72) and (2.74), as

$$\widetilde{\psi_1} = \psi_1 - \mathcal{H}\alpha_1 = 0 \tag{2.103}$$

$$\Rightarrow \alpha_{1\mathrm{flat}} = \frac{\psi_1}{\mathcal{H}}\,, \tag{2.104}$$

$$\widetilde{E_1} = E_1 + \beta_1 = 0 \tag{2.105}$$

$$\Rightarrow \beta_{1\text{flat}} = -E_1\,, \tag{2.106}$$

and

$$\widetilde{F_1}^i = {F_1}^i + {\gamma_1}^i = 0 \tag{2.107}$$

$$\Rightarrow {\gamma_{1\text{flat}}}^i = -{F_1}^i\,. \tag{2.108}$$

The other scalars in this gauge, which are then gauge invariant are, from Eqs. (2.71) and (2.73),

$$\widetilde{\phi_{1\text{flat}}} = \phi_1 + \psi_1 + \left(\frac{\psi_1}{\mathcal{H}}\right)'\,, \tag{2.109}$$

$$\widetilde{B_{1\text{flat}}} = B_1 - \frac{\psi_1}{\mathcal{H}} - {E_1}'\,. \tag{2.110}$$

The gauge invariant vector metric perturbation is, from Eq. (2.75),

$$\widetilde{{S_{1\text{flat}}}}^i = {S_1}^i + {F_1}^i\,. \tag{2.111}$$

Note that the scalar field perturbation on spatially flat hypersurfaces is the gauge invariant Sasaki-Mukhanov variable [118, 133], often denoted by $\mathcal{Q}$,

$$\mathcal{Q} \equiv \widetilde{\delta\varphi_{1\text{flat}}} = \delta\varphi_1 + \varphi_0'\frac{\psi_1}{\mathcal{H}}\,. \tag{2.112}$$

2.4.2 Longitudinal (Poisson) Gauge

The longitudinal gauge is the gauge in which the shear, σ, vanishes. It is also known as the conformal Newtonian or orthogonal zero-shear gauge. Its extension to include vector and tensors is called the Poisson gauge. This gauge is commonly used in the literature, since the remaining gauge invariant scalars in this gauge are the variables introduced by Bardeen [16].

At linear order the temporal part of the gauge generating vector is specified by the choice $\widetilde{\sigma_1} = 0$ as, using Eq. (2.78)

$$\widetilde{\sigma_1} = \sigma_1 + \alpha_1 = 0 \tag{2.113}$$

$$\Rightarrow \alpha_{1\ell} = -\sigma_1\,, \tag{2.114}$$

where the subscript ℓ denotes the value in the longitudinal gauge. The generating vector is fully specified, for scalar perturbations, by making the gauge choice $\widetilde{E_1} = 0$ (and hence $\widetilde{B_1} = 0$) as

$$\widetilde{E_1} = E_1 + \beta_1 = 0 \tag{2.115}$$
$$\Rightarrow \beta_{1\ell} = -E_1 \,. \tag{2.116}$$

The other two scalar metric perturbations in this gauge are then

$$\widetilde{\phi_{1\ell}} = \phi_1 - \mathcal{H}\sigma_1 - \sigma_1' \,, \tag{2.117}$$
$$\widetilde{\psi_{1\ell}} = \psi_1 + \mathcal{H}\sigma_1 \,, \tag{2.118}$$

which, by using the definition of the shear, Eq. (2.77), give

$$\widetilde{\phi_{1\ell}} = \phi_1 - \mathcal{H}({E_1}' - B_1) - ({E_1}' - B_1)' \,, \tag{2.119}$$
$$\widetilde{\psi_{1\ell}} = \psi_1 + \mathcal{H}({E_1}' - B_1) \,. \tag{2.120}$$

These are then identified with the two Bardeen potentials, Φ_1 and Ψ_1, respectively (or, in Bardeen's notation, $\Phi_A Q^{(0)}$ and $-\Phi_H Q^{(0)}$).

In the Poisson gauge, the generalisation of the longitudinal gauge beyond scalar perturbations, the spatial vector component of the gauge transformation generating vector by demanding that $\widetilde{{S_1}^i} = 0$ gives, using Eq. (2.75),

$$\widetilde{{S_1}^i} = {S_1}^i + {\gamma_1}^{i\prime} = 0 \tag{2.121}$$
$$\Rightarrow {\gamma_{1\ell}}^i = \int {S_1}^i d\eta + {\mathcal{F}_1}^i(x^j) \,, \tag{2.122}$$

where ${\mathcal{F}_1}^i$ is an arbitrary constant three-vector. Thus, when having fixed the Poisson gauge, there still exists some residual freedom in this choice of constant vector.

The remaining gauge invariant vector metric perturbation in the Poisson gauge is then

$$\widetilde{{F_{1\ell}}^i} = {F_1}^i + \int {S_1}^i d\eta + {\mathcal{F}_1}^i(x^j) \,. \tag{2.123}$$

2.4.3 Uniform Density Gauge

As an alternative to the gauges above, we can define a gauge with respect to the matter perturbations. One example is the uniform density gauge which is based upon choosing a spacetime foliation such that the density perturbation vanishes.

At first order we can fix α_1 by demanding that $\widetilde{\delta\rho_1} = 0$. Using Eq. (2.65) we obtain

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1 = 0 \tag{2.124}$$

$$\Rightarrow \alpha_{1\delta\rho} = -\frac{\delta\rho_1}{\rho_0'}\,. \tag{2.125}$$

We still have the freedom to choose the spatial scalar part of the gauge transformation generating vector, which can be done unambiguously by choosing, e.g., $\widetilde{E_1} = 0$.

An especially interesting variable is the curvature perturbation in this gauge, ζ_1, which is a gauge invariant variable and defined as

$$-\zeta_1 \equiv \widetilde{\psi_{1\delta\rho}} = \psi_1 + \mathcal{H}\frac{\delta\rho_1}{\rho_0'}\,, \tag{2.126}$$

where the sign is chosen to agree with that in Ref. [17][6]. This quantity will come in useful in later chapters because it is conserved on large scales for an adiabatic system, as will be shown in Section 3.1.1.

2.4.4 Synchronous Gauge

The synchronous gauge was popular in early work on cosmological perturbation theory, and was introduced by Lifshitz in the groundbreaking work of Ref. [89]. It is characterised by $\widetilde{\phi_1} = 0 = \widetilde{B_{1i}}$, so that the g_{00} and g_{0i} components of the metric are left unperturbed and any perturbation away from FRW is confined to the spatial part of the metric. It can be thought of physically as the gauge in which η defines proper time for all comoving observers. This gauge is also used in many modern Boltzmann codes such as CMBFAST [138], and is discussed in detail, and compared to the longitudinal gauge in Ref. [102].

The synchronous gauge condition fixes the temporal gauge function through

$$\widetilde{\phi_1} = \phi_1 + \mathcal{H}\alpha_1 + \alpha_1' = 0 \tag{2.127}$$

$$\Rightarrow a\phi_1 + a'\alpha_{1\mathrm{syn}} + a\alpha_{1\mathrm{syn}}' = a\phi_1 + (a\alpha_{1\mathrm{syn}})' = 0 \tag{2.128}$$

$$\Rightarrow \alpha_{1\mathrm{syn}} = -\frac{1}{a}\Big(\int a\phi_1 d\eta - \mathcal{C}(x^i)\Big)\,, \tag{2.129}$$

[6]See Ref. [151] for a detailed comparison of different sign conventions and notation used for the curvature perturbations in different papers.

and the spatial gauge functions as

$$\beta_{1\mathrm{syn}} = \int (\alpha_{1\mathrm{syn}} - B_1) d\eta + \mathcal{D}(x^i) \,, \tag{2.130}$$

$$\gamma^i_{1\mathrm{syn}} = \int S_1^i d\eta + \mathcal{E}^i(x^k) \,. \tag{2.131}$$

The function $\mathcal{D}_{,i}(x^k) + \mathcal{E}_i(x^k)$ affects the labelling of the initial spatial hypersurface. However, the function $\mathcal{C}(x^k)$ affects the scalar perturbations, and so the synchronous gauge does not determine the time-slicing unambiguously. It is therefore not possible to define gauge invariant variables from the metric in this gauge, since the remaining scalars (for example the curvature perturbation), have spurious gauge dependence:

$$\widetilde{\psi_{1\mathrm{syn}}} = \psi_1 + \frac{\mathcal{H}}{a}\Big(\int a\phi_1 d\eta - \mathcal{C}(x^k) \Big) \,. \tag{2.132}$$

In Lifshitz' original work, the gauge mode was removed using symmetry arguments. Nowadays a systematic approach is used to remove this gauge mode, and a further gauge condition is taken, setting the perturbation in the three velocity of the dark matter fluid to zero. Then,

$$\widetilde{v_{1\mathrm{cdm,\,syn}}} = v_1 - \beta'_{1\mathrm{syn}} = 0 \tag{2.133}$$

$$\Rightarrow a(v_1 + B_1) + \int a\phi_1 d\eta - \mathcal{C}(x^i) = 0 \,. \tag{2.134}$$

However, we then refer to the field equations, Eq. (3.2) which, in the synchronous gauge guarantees that for the cold dark matter perturbation (where $c_{\mathrm{s}}^2 = 0 = \delta P_1$), $V_{1\mathrm{cdm}} = a(v_1 + B_1)|_{\mathrm{cdm}}$ is a constant (in conformal time). Thus, we obtain

$$\mathcal{C}(x^i) = a(v_1 + B_1)\Big|_{\mathrm{cdm}} \,, \tag{2.135}$$

which removes the gauge freedom. Note that we can only choose the dark matter fluid with which to define the gauge since it is pressureless. The same does not hold true for a fluid with non-zero pressure. In order to see this we need to use an equation that we will derive in full detail later from the momentum conservation, Eq. (3.7) which, in coordinate time, is

$$\dot{V}_1 - 3\mathcal{H}c_{\mathrm{s}}^2 V_1 + \frac{c_{\mathrm{s}}^2 \delta\rho_1}{\rho_0 + P_0} + \phi_1 = 0 \,. \tag{2.136}$$

In the synchronous gauge, $B_1 = 0 = \phi_1$, so this becomes

$$a\dot{v}_1 - 3\mathcal{H}c_s^2 a v_1 + \frac{c_s^2 \delta\rho_1}{\rho_0 + P_0} = 0 \,. \tag{2.137}$$

If we then set the three-velocity to zero, in order to specify the threading, Eq. (2.137) becomes

$$\frac{c_s^2 \delta\rho_1}{\rho_0 + P_0} = 0 \,, \tag{2.138}$$

which states that the fluid is pressureless. Therefore, we can only define the synchronous gauge as comoving with respect to a pressureless fluid, e.g. cold dark matter, and not with respect to a fluid with pressure.

2.4.5 Comoving Gauge

Another example of a gauge defined by the gauge transformation of a matter variable is the comoving gauge. This is defined by choosing the gauge such that the three-velocity of the fluid vanishes, $\widetilde{v}_{1i} = 0$. Furthermore, this choice implies that $\widetilde{V}_{1i} = 0$. Then,

$$\widetilde{V}_1 = V_1 - \alpha_1 = 0 \tag{2.139}$$

$$\Rightarrow \alpha_{1\mathrm{com}} = V_1 = v_1 + B_1 \,, \tag{2.140}$$

and

$$\widetilde{v}_1 = v_1 - \beta_1' = 0 \tag{2.141}$$

$$\Rightarrow \beta_{1\mathrm{com}} = \int v_1 d\eta + \mathcal{F}(x^k) \,, \tag{2.142}$$

where $\mathcal{F}(x^k)$ denotes a residual gauge freedom. However, note that scalar perturbations are all independent of $\mathcal{F}(x^k)$. One of the remaining scalars is the curvature peturbation on comoving hypersurfaces, which is quite popular in the literature, and often denoted $\mathcal{R}$:

$$\mathcal{R} \equiv \widetilde{\psi_{1\mathrm{com}}} = \psi_1 - \mathcal{H}V_1 \,. \tag{2.143}$$

2.4.6 Beyond Linear Order

At second order the procedure is very much the same as at linear order, and the various gauges are defined in an analogous way to linear order. Of course, the expressions obtained are much longer, due to the fact that the second order gauge transformations contain many more terms than those at first order. Since we do not intend this work to be a comprehensive summary of gauge choice and gauge invariant variables (this topic has already been covered in full, gory detail in Ref. [112]), we instead sketch how the gauge choice and construction of gauge invariant variables work at second order for one choice of gauge: the uniform curvature gauge.

As at linear order, detailed above in Section 2.4.1, we determine the components of the gauge transformation generating vector through the conditions $\widetilde{\psi_2} = \widetilde{E_2} = 0$ and $\widetilde{F_{2i}} = \mathbf{0}$. The first gives, from Eq. (2.82),

$$\alpha_{2\mathrm{flat}} = \frac{\psi_2}{\mathcal{H}} + \frac{1}{4\mathcal{H}}\left(\nabla^{-2}\mathcal{X}^{ij}_{\mathrm{flat},ij} - \mathcal{X}^{k}_{\mathrm{flat}k}\right), \tag{2.144}$$

where $\mathcal{X}_{\mathrm{flat}ij}$ is defined as Eq. (2.87) with the first order generators given above in Eqs. (2.104), (2.106) and (2.108), and the second condition gives

$$\beta_{2\mathrm{flat}} = -E_2 - \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathcal{X}^{ij}_{\mathrm{flat},ij} + \frac{1}{4}\nabla^{-2}\mathcal{X}^{k}_{\mathrm{flat}k}\,. \tag{2.145}$$

Finally, imposing the condition $\widetilde{F_{2i}} = \mathbf{0}$, gives

$$\gamma_{2\mathrm{flat}i} = -F_{2i} - \nabla^{-2}\mathcal{X}_{\mathrm{flat}ik,}{}^{k} + \nabla^{-2}\nabla^{-2}\mathcal{X}^{kl}_{\mathrm{flat},kli}\,. \tag{2.146}$$

As an example of a gauge invariant variable at second order, we present the lapse function in the uniform curvature gauge, $\phi_{2\mathrm{flat}}$. Using Eq. (2.83) along with Eqs. (2.104) and (2.144), we obtain

$$\begin{aligned}\widetilde{\phi_{2\mathrm{flat}}} = \phi_2 &+ \frac{\mathcal{A}_2}{\mathcal{H}} + \frac{1}{\mathcal{H}}\left[\left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right) + \partial_\eta\right]\left[\nabla^{-2}\mathcal{X}_{\mathrm{flat},kl}^{\;kl} - \mathcal{X}^{k}_{\mathrm{flat}k}\right] \\ &+ \frac{1}{\mathcal{H}^2}({\psi_1}''\psi_1 + 2{\psi_1}^2) + \left(2 - \frac{\mathcal{H}''}{\mathcal{H}^3}\right){\psi_1}^2 + \frac{1}{\mathcal{H}}\left(5 - 6\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\psi_1{\psi_1}' + \frac{2}{\mathcal{H}}{\phi_1}'\psi_1 \\ &+ \frac{4}{\mathcal{H}}\mathcal{A}_1\phi_1 + \frac{1}{\mathcal{H}}\Big[\mathcal{A}_1 + 2\mathcal{H}\phi_1\Big]_{,k}\xi^{k}_{1\mathrm{flat}} + \frac{1}{\mathcal{H}}\Big[\mathcal{A}_{1,k} - 2\mathcal{H}B_{1ik}\Big]{\xi^{k}_{1\mathrm{flat}}}'\,,\end{aligned} \tag{2.147}$$

where we have defined

$$\mathcal{A}_{(n)} = \psi'_{(n)} + \left(\mathcal{H} - \frac{\mathcal{H}'}{\mathcal{H}}\right)\psi_{(n)}\,. \tag{2.148}$$

2.5 Thermodynamics of a Perfect Fluid

Considerable physical insight can be gained by studying the thermodynamic properties of a system [80, 96]. In this section, we study a single perfect fluid system. Such a system is fully characterised by three variables, of which only two are independent. Here we choose the energy density, ρ and the entropy, S, as independent variables, with the pressure being given by the equation of state $P = P(\rho, S)$. The pressure perturbation can then be expanded, at linear order in perturbation theory, as

$$\delta P_1 = \left.\frac{\partial P}{\partial S}\right|_\rho \delta S_1 + \left.\frac{\partial P}{\partial \rho}\right|_S \delta\rho_1 \,. \tag{2.149}$$

This can be cast in the more familiar form

$$\delta P_1 = \delta P_{\mathrm{nad}\,1} + c_{\mathrm{s}}^2 \delta\rho_1 \,, \tag{2.150}$$

by introducing the adiabatic sound speed

$$c_{\mathrm{s}}^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_S \,, \tag{2.151}$$

and by defining the non-adiabatic pressure perturbation, which is proportional to the perturbation in the entropy, as [152]

$$\delta P_{\mathrm{nad}\,1} \equiv \left.\frac{\partial P}{\partial S}\right|_\rho \delta S_1 \,. \tag{2.152}$$

Note that $\delta P_{\mathrm{nad}\,1}$ is gauge invariant. This can be shown by considering the gauge transformation for the energy density perturbation, Eq. (2.65), along with the analogous equation for the pressure perturbation. One can extend Eq. (2.149) to higher order by simply not truncating the expansion at linear order, that is

$$\begin{aligned} \delta P = & \frac{\partial P}{\partial S}\delta S + \frac{\partial P}{\partial \rho}\delta\rho \\ & + \frac{1}{2}\left[\frac{\partial^2 P}{\partial S^2}\delta S^2 + \frac{\partial^2 P}{\partial\rho\partial S}\delta\rho\delta S + \frac{\partial^2 P}{\partial \rho^2}\delta\rho^2\right] + \ldots \,. \end{aligned} \tag{2.153}$$

The entropy, or non-adiabatic pressure perturbation at second order, for example, is then found from Eq. (2.153), as [110]

$$\delta P_{2\mathrm{nad}} = \delta P_2 - c_{\mathrm{s}}^2 \delta\rho_2 - \frac{\partial c_{\mathrm{s}}^2}{\partial \rho}\, \delta\rho_1^2 \,. \tag{2.154}$$

2.5.1 Entropy or Non-Adiabatic Perturbations from Inflation

One way in which a non-adiabatic pressure perturbation can be generated is through the relative entropy perturbation between two or more fluids or scalar fields. For example, the relative entropy or isocurvature perturbation, at first order, between two fluids denoted with subscripts A and B is [113] (dropping here the subscripts denoting the order, for brevity)

$$\mathcal{S}_{AB} = 3\mathcal{H}\left(\frac{\delta\rho_B}{\rho'_{0B}} - \frac{\delta\rho_A}{\rho'_{0A}}\right) . \tag{2.155}$$

In a system consisting of multiple fluids, the non-adiabatic pressure perturbation is split as [75, 112]

$$\delta P_{\rm nad} = \delta P_{\rm intr} + \delta P_{\rm rel} , \tag{2.156}$$

where the first term is the contribution from the intrinsic entropy perturbation of each fluid, and the second term is due to the relative entropy perturbation between each fluid, $\mathcal{S}_{AB}$, and is defined as

$$\delta P_{\rm rel} \equiv \frac{1}{6\mathcal{H}\rho'_0} \sum_{A,B} \rho'_{0A}\rho'_{0B}\left(c_B^2 - c_A^2\right)\mathcal{S}_{AB} , \tag{2.157}$$

where c_A^2 and c_B^2 are the adiabatic sound speed of each fluid. Thus, for a multiple fluid system, even when the intrinsic entropy perturbation is zero for each fluid, there is a non-vanishing overall non-adiabatic pressure perturbation. This can be extended to the case of scalar fields by using standard techniques of treating the fields as fluids. Much recent work has been focussed on the discussion of entropy, or isocurvature perturbations in multi-field inflationary models. See, e.g. Refs. [7, 19, 31, 32, 54, 59, 76, 78, 81, 82, 84, 93, 94, 101, 127, 134] and references therein.

3 Dynamics and Constraints

In this Chapter we give the governing equations for perturbations of a FRW universe. The background evolution and constraint equations are presented in Chapter 1, so here we consider the equations at first and second order in perturbation theory. Starting with the linear order theory we present the governing equations for scalar, vector and tensor perturbations for a universe filled with a perfect fluid without fixing a gauge. We then make three choices of gauge, the uniform density, uniform curvature and longitudinal gauges, and solve the evolution equations for scalar perturbations in the latter two case. Next, we present the evolution equation for a scalar field – the Klein-Gordon equation – for a field with both a canonical and non-canonical Lagrangian, highlighting the importance of the difference between the adiabatic sound speed and the phase speed for a scalar field system. We finish our discussion of linear perturbations with an investigation into the perturbations of a system with both a dark matter and a dynamical dark energy component.

Having discussed linear order perturbations, we then move on to the second order theory. We derive the governing equations for a perfect fluid in a perturbed FRW universe from energy momentum conservation and the Einstein equations, without fixing gauge. We go on to present the equations for scalar and vector perturbations in the uniform curvature gauge, which will come in useful in Chapter 4, and then for scalar perturbations only, including now the canonical Klein-Gordon equation at second order. Finally, to connect with other parts of the literature, we give the equations for scalars in the Poisson gauge.

3.1 First Order

In this section we give the evolution and constraint equations at first order in cosmological perturbation theory for a universe filled with a perfect fluid, in a gauge dependent form,[1] and for all scalar, vector and tensor perturbations, neglecting anisotropic stress. We also present the equations in some commonly used gauges,

[1]By gauge dependent, we mean that the equations are presented in a form where the gauge functions α, β and γ^i have not yet been specified

present the evolution equation for a scalar field – the Klein-Gordon equation – for both a canonical and non-canonical field, and solve some of the evolution equations.

First, in gauge dependent form, energy conservation at linear order gives

$$\delta\rho_1{}' + 3\mathcal{H}(\delta\rho_1 + \delta P_1) = (\rho_0 + P_0)(3\psi_1{}' - \nabla^2 E_1{}' - v_{1i,}{}^{i})\,, \tag{3.1}$$

where ∇^2 denotes the spatial Laplacian, $\nabla^2 \equiv \partial_k\partial^k$,[2] while momentum conservation gives

$$V_{1i}' + \mathcal{H}(1 - 3c_{\rm s}^2)V_{1i} + \left[\frac{\delta P_1}{\rho_0 + P_0} + \phi_1\right]_{,i} = 0\,, \tag{3.2}$$

where we have introduced the covariant velocity perturbation as $V_{1i} = v_{1i} + B_{1i}$. Note, Eq. (3.2) is the analogue of the Euler equation in an expanding background (see Chapter 4).

The Einstein equations give the energy constraint equation

$$3\mathcal{H}(\psi_1{}' + \mathcal{H}\phi_1) - \nabla^2(\psi_1 + \mathcal{H}E_1{}') + \mathcal{H}\nabla^2 B_1 = -4\pi G a^2 \delta\rho_1\,, \tag{3.3}$$

and the momentum constraint

$$\psi_{1,i}' + \frac{1}{4}S_{1i} + \mathcal{H}\phi_{1,i} = -4\pi G a^2(\rho_0 + P_0)V_{1i}\,. \tag{3.4}$$

Finally, the (i,j) component gives the equation

$$\begin{aligned}
&E_{1,}{}^{i''}{}_{j} + 2\mathcal{H}E_{1,}{}^{i'}{}_{j} + (\psi_1 - \phi_1)_{,}{}^{i}{}_{j} - \frac{1}{2}(\partial_\eta + 2\mathcal{H})\left(2B_{1,}{}^{i}{}_{j} - S^i_{1,j} - S_{1j,}{}^{i}\right)\\
&+(\partial_\eta^2 + 2\mathcal{H}\partial_\eta - \nabla^2)\left(F_1^{(i}{}_{,j)} + \frac{1}{2}h^i_{1j}\right) + \delta^i{}_j\bigg\{ - 2\phi_1\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big) + 2\psi_1{}''\\
&+\nabla^2(\phi_1 - E_1'' - \psi_1 + B_1') + 2\mathcal{H}\Big(2\psi_1{}' + \nabla^2(B_1 - E_1') + \phi_1{}'\Big)\bigg\}\\
&= 8\pi G a^2(P_0 + \delta P_1)\delta^i{}_j\,.
\end{aligned} \tag{3.5}$$

Simplifying now to scalar perturbations, the energy-momentum conservation equa-

[2]Since we are working with a background whose spatial submanifold is Euclidean, the position of the Latin indices does not have a meaning. However, we preserve the position in order to keep with notational conventions such as the summation convention, and for ease of future generalisation.

tions then become

$$\delta\rho_1{}' + 3\mathcal{H}(\delta\rho_1 + \delta P_1) = (\rho_0 + P_0)(3\psi_1{}' - \nabla^2(E_1' - v_1))\,, \tag{3.6}$$

$$V_1' + \mathcal{H}(1 - 3c_{\rm s}^2)V_1 + \frac{\delta P_1}{\rho_0 + P_0} + \phi_1 = 0\,, \tag{3.7}$$

and the Einstein equations are

$$3\mathcal{H}(\psi_1{}' + \mathcal{H}\phi_1) - \nabla^2(\psi_1 + \mathcal{H}E_1{}') + \mathcal{H}\nabla^2 B_1 = -4\pi G a^2 \delta\rho_1\,, \tag{3.8}$$

$$\psi_1' + \mathcal{H}\phi_1 = -4\pi G a^2(\rho_0 + P_0)V_1\,, \tag{3.9}$$

$$\begin{aligned} &E_{1,}{}^{i\prime\prime}{}_{j} + 2\mathcal{H}E_{1,}{}^{i\prime}{}_{j} + (\psi_1 - \phi_1)_{,}{}^{i}{}_{j} - B_{1,}{}^{i\prime}{}_{j} - 2\mathcal{H}B_{1,}{}^{i}{}_{j} + \delta^i{}_j\Big\{ - 2\phi_1\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big) \\ &+ \nabla^2(\phi_1 - E_1'' - \psi_1 + B_1') + 2\psi_1{}'' + 2\mathcal{H}\Big(2\psi_1{}' + \nabla^2(B_1 - E_1') + \phi_1{}'\Big)\Big\} \\ &= 8\pi G a^2(P_0 + \delta P_1)\delta^i{}_j\,. \end{aligned} \tag{3.10}$$

We can obtain from Eq. (3.10) two scalar equations. Firstly, by applying the operator $\partial_i\partial^j$, using the method outlined in Ref. [109], we obtain

$$\psi_1{}'' + 2\mathcal{H}\psi_1{}' + \mathcal{H}\phi_1{}' + \phi_1\left(\frac{2a''}{a} - \mathcal{H}^2\right) = 4\pi G a^2 \delta P_1\,, \tag{3.11}$$

and then, taking the trace of Eq. (3.10) and using Eq. (3.11), we obtain

$$(B_1 - E_1')' + 2\mathcal{H}(B_1 - E_1') + \phi_1 - \psi_1 = 0\,, \tag{3.12}$$

which can be recast in terms of the shear ($\sigma_1 \equiv E_1' - B_1$) as

$$\sigma_1' + 2\mathcal{H}\sigma_1 - \phi_1 + \psi_1 = 0\,. \tag{3.13}$$

3.1.1 Uniform Density Gauge

We now work in the uniform density gauge which, as we saw in the previous chapter, is specified by $\widetilde{\delta\rho_1} = 0$, and consider only scalar perturbations. Eq. (3.6) evaluated in this gauge is

$$3\mathcal{H}\widetilde{\delta P_{1\delta\rho}} = (\rho_0 + P_0)\Big[3\widetilde{\psi_{1\delta\rho}}{}' - \nabla^2(\widetilde{E_{1\delta\rho}}{}' - \widetilde{v_{1\delta\rho}})\Big]\,. \tag{3.14}$$

Noting that $\widetilde{\psi_{1\delta\rho}} \equiv -\zeta_1$ is the gauge invariant curvature perturbation on uniform density hypersurfaces, introduced in Eq. (2.126), and $\widetilde{\delta P_{1\delta\rho}} = \delta P_{\rm nad1}$, this then becomes

$$3\zeta_1' = -\frac{3\mathcal{H}}{(\rho_0 + P_0)}\delta P_{\rm nad1} - \nabla^2(\widetilde{E'_{1\delta\rho}} - \widetilde{v_{1\delta\rho}})\,. \tag{3.15}$$

Recall, as mentioned in section 2.4.3, that the uniform density gauge condition does not specify the spatial gauge function, β_1. The transformations of E_1 and v_1 do not depend upon the temporal gauge function, so $\widetilde{E_{1\delta\rho}} = \widetilde{E}$, and similarly for v_1. However, the combination $\widetilde{E_1}' - \widetilde{v_1}$ is a gauge invariant variable, the three-velocity in the longitudinal gauge, and so Eq. (3.15) can be written solely in terms of gauge invariant variables as

$$3\zeta_1' = -\frac{3\mathcal{H}}{(\rho_0 + P_0)}\delta P_{\rm nad1} - \nabla^2 v_{1\ell}\,. \tag{3.16}$$

We then recover the known result that, on large scales, the evolution of the curvature perturbation is proportional to the non-adiabatic pressure perturbation:

$$\zeta_1' = -\frac{\mathcal{H}}{(\rho_0 + P_0)}\delta P_{\rm nad1}\,. \tag{3.17}$$

Thus, the curvature perturbation on uniform density hypersurfaces is conserved on large scales for adiabatic perturbations such as those from a single fluid or a single scalar field [152].

Let us now consider whether a general scalar field can support non-adiabatic perturbations. We can calculate the non-adiabatic pressure perturbations for a general scalar field with Lagrangian $\mathcal{L} \equiv \mathcal{L}(X, \varphi)$, Eq. (2.41). Noting that the pressure and energy density are both functions of X and φ, and so their perturbations can be expanded in a series as

$$\delta P_1 = \frac{\partial P}{\partial \varphi}\delta\varphi_1 + \frac{\partial P}{\partial X}\delta X_1\,, \tag{3.18}$$

and

$$\delta \rho_1 = \frac{\partial \rho}{\partial \varphi}\delta\varphi_1 + \frac{\partial \rho}{\partial X}\delta X_1\,, \tag{3.19}$$

the non-adiabatic pressure perturbation can then be obtained by substituting Eqs. (3.18) and (3.19) into Eq. (2.149), giving

$$\delta P_{\rm nad1} = \rho_{,\varphi}\left(\frac{P_{\varphi}}{\rho_{,\varphi}} - c_{\rm s}^2\right)\delta\varphi_1 + \rho_{,X}\left(\frac{P_X}{\rho_{,X}} - c_{\rm s}^2\right)\delta X_1\,. \tag{3.20}$$

In such a system and on large scales, the relationship between $\delta\varphi_1$ and δX_1 is [37]

$$\delta X_1 = \ddot{\varphi}_0 \delta\varphi_1\,, \tag{3.21}$$

which, when substituted into Eq. (3.20), along with the definition for the adiabatic sound speed, $c_{\rm s}^2$, yields

$$\delta P_{\rm nad1} = 0\,. \tag{3.22}$$

Therefore, the curvature perturbation on uniform density hypersurfaces is conserved on large scales in an expanding universe not only for a canonical scalar field, but for any scalar field with Lagrangian (2.41).[3] This result is in agreement with Ref. [83].

3.1.2 Uniform Curvature Gauge

Now, returning to the full equations, we neglect tensor perturbations and work in uniform curvature gauge, where $\widetilde{E_1} = \widetilde{\psi_1} = \widetilde{F_1^i} = 0$. The energy conservation equation then becomes[4]

$$\delta\rho_1{}' + 3\mathcal{H}(\delta\rho_1 + \delta P_1) + (\rho_0 + P_0)v_{1i,}{}^i = 0\,, \tag{3.23}$$

while momentum conservation gives

$$V_{1i}' + \mathcal{H}(1 - 3c_{\rm s}^2)V_{1i} + \left[\frac{\delta P_1}{\rho_0 + P_0} + \phi_1\right]_{,i} = 0\,. \tag{3.24}$$

Let us now consider the dynamics of scalar, linear perturbations in the uniform curvature gauge. The energy-momentum conservation equations then reduce to

$$\delta\rho_1{}' + 3\mathcal{H}(\delta\rho_1 + \delta P_1) = (\rho_0 + P_0)\nabla^2 v_1\,, \tag{3.25}$$

$$V_1' + \mathcal{H}(1 - 3c_{\rm s}^2)V_1 + \frac{\delta P_1}{\rho_0 + P_0} + \phi_1 = 0\,, \tag{3.26}$$

and the Einstein equations give

$$3\mathcal{H}^2\phi_1 + \mathcal{H}\nabla^2 B_1 = -4\pi G a^2 \delta\rho_1\,, \tag{3.27}$$

$$\mathcal{H}\phi_1 = -4\pi G a^2(\rho_0 + P_0)V_1\,, \tag{3.28}$$

$$\mathcal{H}\phi_1{}' + \phi_1\left(\frac{2a''}{a} - \mathcal{H}^2\right) = 4\pi G a^2 \delta P_1\,, \tag{3.29}$$

$$B_1' + 2\mathcal{H}B_1 + \phi_1 = 0\,. \tag{3.30}$$

We will now solve this set of equations for the energy density perturbation of a perfect fluid. Let us first rewrite the equations in terms of the 'new' velocity

[3]Note that, in a contracting universe this is not true (see, e.g., Ref. [73]).
[4]We omit the tildes and gauge subscripts for brevity.

perturbation

$$\mathcal{V}_1 \equiv (\rho_0 + P_0)(v_1 + B_1) \ . \tag{3.31}$$

This enables us to write Eq. (3.28), using Eq. (1.23), the background Friedmann equation, as

$$\phi_1 = -\frac{3}{2}\frac{\mathcal{H}}{\rho_0}\mathcal{V}_1 \,, \tag{3.32}$$

and using this, Eq. (3.27) becomes

$$\nabla^2 B_1 = \frac{9}{2}\frac{\mathcal{H}^2}{\rho_0}\mathcal{V}_1 - \frac{3}{2}\frac{\mathcal{H}}{\rho_0}\delta\rho_1 \,. \tag{3.33}$$

Then the evolution equations, Eqs. (3.25) and (3.26) are

$$\delta\rho_1' + \frac{3}{2}\mathcal{H}(3+w)\,\delta\rho_1 + 3\mathcal{H}\delta P_1 - k^2\mathcal{V}_1 - \frac{9}{2}\mathcal{H}^2(1+w)\,\mathcal{V}_1 = 0 \,, \tag{3.34}$$

$$\mathcal{V}_1{}' + \frac{\mathcal{H}}{2}(5-3w)\,\mathcal{V}_1 + \delta P_1 = 0 \,, \tag{3.35}$$

where $w \equiv P_0/\rho_0$ defines the background equation of state,[5] and we are working in Fourier space, k being the comoving wavenumber. Equations (3.34) and (3.35) make up a system of coupled, linear, ordinary differential equations. Given an equation of state and initial conditions, this system can be solved immediately (for a given k), numerically. One can also obtain a qualitative solution by considering a system of two equations such as this one. However, if one wants to solve the system quantitatively, and analytically, it is easier to rewrite the system as a single second order differential equation, which we do in the following. We solve Eq. (3.34) for $\mathcal{V}_1$ and get

$$\mathcal{V}_1 = \frac{1}{\mathfrak{T}}\left(\delta\rho_1' + \frac{3}{2}\mathcal{H}(3+w)\,\delta\rho_1 + 3\mathcal{H}\delta P_1\right) \,, \tag{3.36}$$

where $\mathfrak{T} \equiv k^2 + \frac{9}{2}\mathcal{H}^2(1+w)$.

After some further algebraic manipulations of Eq. (3.36), and using Eq. (3.35), we arrive at the desired evolution equation

$$\delta\rho_1'' + \left(7\mathcal{H} - \frac{\mathfrak{T}'}{\mathfrak{T}}\right)\delta\rho_1' + \frac{3}{2}\mathcal{H}(3+w)\left[\frac{\mathcal{H}'}{\mathcal{H}} + \frac{w'}{(3+w)} - \frac{\mathfrak{T}'}{\mathfrak{T}} + \frac{1}{2}\mathcal{H}(5-3w)\right]\delta\rho_1 + 3\mathcal{H}\delta P_1' + \left[3\mathcal{H}\left(\frac{\mathcal{H}'}{\mathcal{H}} - \frac{\mathfrak{T}'}{\mathfrak{T}}\right) + \frac{3}{2}\mathcal{H}^2(5-3w) + \mathfrak{T}\right]\delta P_1 = 0 \,. \tag{3.37}$$

[5] We do not demand that w be constant here.

Equation (3.37) is a linear differential equation, of second order in (conformal) time. It is valid on all scales, for a single fluid with any (time dependent) equation of state. Furthermore, it assumes nothing more than a perfect fluid and hence allows for non-zero non-adiabatic pressure perturbations.

Having derived a general governing equation (3.37) valid in any epoch, we now restrict our analysis to radiation domination, where the background equation of state parameter is $w = \frac{1}{3}$ and the adiabatic sound speed is $c_{\rm s}^2 = \frac{1}{3}$. We recall from Eq. (2.150) that the first order pressure perturbation can be expanded as

$$\delta P_1 \equiv c_{\rm s}^2 \delta\rho_1 + \delta P_{\rm nad1}\,,$$

where $\delta P_{\rm nad1}$ is the non-adiabatic pressure perturbation, which is proportional to the perturbation in the entropy, and is defined in Eq. (2.152). Then, the general governing equation, Eq. (3.37), becomes, during radiation domination, where $\mathcal{H} \propto 1/\eta$

$$\delta\rho_1'' + 4\mathcal{H}\left(2 + \frac{3\mathcal{H}^2}{k^2 + 6\mathcal{H}^2}\right)\delta\rho_1' + \left(6\mathcal{H}^2 + \frac{1}{3}(k^2 + 6\mathcal{H}^2) + \frac{72\mathcal{H}^4}{k^2 + 6\mathcal{H}^2}\right)\delta\rho_1$$
$$+ 3\mathcal{H}\delta P_{\rm nad}{}_1' + \left[9\mathcal{H}^2 + 36\frac{\mathcal{H}^4}{k^2 + 6\mathcal{H}^2} + k^2\right]\delta P_{\rm nad1} = 0\,. \tag{3.38}$$

For the case of zero non-adiabatic pressure perturbations the second line in Eq. (3.38) vanishes, and the resulting equation can be solved directly using the Frobenius method, to give [40]

$$\delta\rho_1(\boldsymbol{k},\eta) = C_1(\boldsymbol{k})\eta^{-5}\left[\cos\left(\frac{k\eta}{\sqrt{3}}\right)k\eta - 2\sin\left(\frac{k\eta}{\sqrt{3}}\right)\sqrt{3}\right]$$
$$+ C_2(\boldsymbol{k})\eta^{-5}\left[2\cos\left(\frac{k\eta}{\sqrt{3}}\right)\sqrt{3} + \sin\left(\frac{k\eta}{\sqrt{3}}\right)k\eta\right], \tag{3.39}$$

where C_1 and C_2 are functions of the wave-vector, $\boldsymbol{k}$, and k is the wavenumber, $k \equiv |\boldsymbol{k}|$. For small $k\eta$, the trigonometric functions can be expanded in power series giving, to leading order, the approximation

$$\delta\rho_1(\boldsymbol{k},\eta) \simeq A(\boldsymbol{k})k\eta^{-4} + B(\boldsymbol{k})\eta^{-5}\,, \tag{3.40}$$

for some $A(\boldsymbol{k})$ and $B(\boldsymbol{k})$, determined by the initial conditions.

In order to solve for a non vanishing non-adiabatic pressure, we make the ansatz that the non-adiabatic pressure grows as the decaying branch of the density perturbation in Eq. (3.34), i.e.,

$$\delta P_{\mathrm{nad}\,1}(\boldsymbol{k},\eta) = D(\boldsymbol{k})k^{\lambda}\eta^{-5}\,. \tag{3.41}$$

This assumption is well motivated, since we would expect the non-adiabatic pressure to decay faster than the energy density, and in fact observations are consistent with a very small entropy perturbation today. Furthermore, if one is considering a relative entropy perturbation between more than one fluid or field, as time increases the system will equilibriate and one species will dominate.[6] This gives the solution

$$\begin{aligned}\delta\rho_1(\boldsymbol{k},\eta) = C_1(\boldsymbol{k})\,\eta^{-5}&\left[\cos\left(\frac{k\eta}{\sqrt{3}}\right)k\eta - 2\,\sin\left(\frac{k\eta}{\sqrt{3}}\right)\sqrt{3}\right]\\ &+C_2(\boldsymbol{k})\,\eta^{-5}\left[2\,\cos\left(\frac{k\eta}{\sqrt{3}}\right)\sqrt{3}+\sin\left(\frac{k\eta}{\sqrt{3}}\right)k\eta\right] - 3D(\boldsymbol{k})k^{\lambda}\eta^{-5}\,.\end{aligned} \tag{3.42}$$

3.1.3 Longitudinal Gauge

In this section we consider scalar, linear perturbations in the longitudinal gauge in order to make a connection with the literature. In the longitudinal gauge, $E_1 = B_1 = 0$, and $\psi_1 = \Psi_1$, $\phi_1 = \Phi_1$, and so Eq. (3.12) tells us that, in the absence of anisotropic stress, $\Phi_1 = \Psi_1$. Taking this into account, the energy-momentum conservation equations become

$$\delta\rho_1{}' + 3\mathcal{H}(\delta\rho_1 + \delta P_1) = (\rho_0 + P_0)(3\Phi_1 + \nabla^2 v_1)\,, \tag{3.43}$$

$$v_1' + \mathcal{H}(1-3c_{\mathrm{s}}^2)v_1 + \frac{\delta P_1}{\rho_0 + P_0} + \Phi_1 = 0\,, \tag{3.44}$$

and the Einstein equations are

$$3\mathcal{H}(\Phi_1' + \mathcal{H}\Phi_1) - \nabla^2\Phi_1 = -4\pi G a^2\delta\rho_1\,, \tag{3.45}$$

$$\Phi_1' + \mathcal{H}\Phi_1 = -4\pi G a^2(\rho_0 + P_0)v_1\,, \tag{3.46}$$

$$\Phi_1'' + 3\mathcal{H}\Phi_1' + \Phi_1\left(\frac{2a''}{a} - \mathcal{H}^2\right) = 4\pi G a^2\delta P_1\,, \tag{3.47}$$

Assuming adiabatic perturbations, in which case $\delta P_1 = c_{\mathrm{s}}^2\delta\rho_1$ and $w = c_{\mathrm{s}}^2$, we can

[6] Of course, given a specific model of the early universe this relative entropy perturbation can be calculated.

combine Eqs. (3.45) and (3.47), using Eqs. (1.23) and (1.25) to give

$$\Phi_1'' + 3\mathcal{H}(1 + c_{\rm s}^2)\Phi_1' + c_{\rm s}^2 k^2 \Phi_1 = 0\,, \tag{3.48}$$

which, in radiation domination (where we recall from section 1.1 that $c_{\rm s}^2 = 1/3$ and $\mathcal{H} \propto \eta^{-1}$), then becomes

$$\eta\Phi_1'' + 4\Phi_1' + \frac{1}{3}\eta k^2 \Phi_1 = 0\,. \tag{3.49}$$

This equation can then be solved (see, e.g., Ref. [10] for the general method) to give

$$\begin{aligned}\Phi_1(\boldsymbol{k},\eta) = \frac{\tilde{C}_1(\boldsymbol{k})}{(k\eta)^3}&\left[\cos\left(\frac{k\eta}{\sqrt{3}}\right)k\eta - \sin\left(\frac{k\eta}{\sqrt{3}}\right)\sqrt{3}\right]\\ &+ \frac{\tilde{C}_2(\boldsymbol{k})}{(k\eta)^3}\left[\cos\left(\frac{k\eta}{\sqrt{3}}\right)\sqrt{3} + \sin\left(\frac{k\eta}{\sqrt{3}}\right)k\eta\right].\end{aligned} \tag{3.50}$$

3.1.4 Scalar Field Evolution and Sound Speeds

In this section we consider the dynamics of a scalar field. Many oscillating systems can be described by a wave equation, that is a second order evolution equation of the form

$$\frac{1}{c_{\rm ph}^2}\ddot{\mathfrak{X}} - \nabla^2\mathfrak{X} + F(\mathfrak{X},\dot{\mathfrak{X}}) = 0\,, \tag{3.51}$$

where $\mathfrak{X}$ is, for example, the velocity potential, $F(\mathfrak{X},\dot{\mathfrak{X}})$ is the damping term, and $c_{\rm ph}^2$ is the phase speed, i.e. the speed with which perturbations travel through the system [79, 141]. The situation for a scalar field is not dissimilar, and the evolution equation in this case is the Klein-Gordon equation, where $\mathfrak{X} = \varphi$ is the scalar field. The Klein-Gordon equation is obtained by substituting the expressions for the energy density and the pressure into the energy conservation equation. The background equation is given in Section 1.2 and here we focus on the perturbations.

There is some confusion in the literature on the meaning of adiabatic sound speed and phase speed. Although this seems not to affect the results of previous works, and the adiabatic sound speed is often simply used as a convenient shorthand for $\dot{P}_0/\dot{\rho}_0$, as defined in Eq. (2.151), it is often confusingly used to denote the phase speed. The adiabatic sound speed defined above describes the response of the pressure to a change in density at constant entropy, and is the speed with which pressure perturbations travels through a classical fluid. A more intuitive introduction of the adiabatic sound speed is using the compressibility [79],

$$\beta \equiv \frac{1}{\rho}\frac{\partial \rho}{\partial P}\Big|_S = \frac{1}{\rho c_{\rm s}^2}\,, \tag{3.52}$$

which describes the change in density due to a change in pressure while keeping the entropy constant.

A collection of scalar fields can also be described as a fluid, but the analogy is not exact. Whereas in the fluid case the phase speed $c_{\rm ph}^2$ and the adiabatic sound speed $c_{\rm s}^2$ are equal, this is not true in the scalar field case and the speed with which perturbations travel is given *only* by $c_{\rm ph}^2$. The phase speed, $c_{\rm ph}^2$ can be read off from the perturbed Klein-Gordon equation governing the evolution of the scalar field. This is just a damped wave equation, like Eq. (3.51), and so by comparing the coefficients of the second order temporal, and second order spatial derivatives, we obtain the phase speed.

We first focus on the canonical scalar field with Lagrangian (2.30); the background equation is given in section 1.2. The equation for linear perturbations is, in the uniform curvature gauge,

$$\delta\varphi_1'' + 2\mathcal{H}\delta\varphi_1' + 2a^2 U_{,\varphi}\phi_1 - \varphi_0'\nabla^2 B_1 - \nabla^2\delta\varphi_1 - \varphi_0'\phi_1{}' + a^2 U_{,\varphi\varphi}\delta\varphi_1 = 0\,. \tag{3.53}$$

We can express this in closed form, that is containing only matter perturbations, by replacing the metric perturbations using the appropriate field equations. Doing so gives (e.g. Ref. [106])

$$\begin{aligned}\delta\varphi_1'' &+ 2\mathcal{H}\delta\varphi_1' - \nabla^2\delta\varphi_1 \\ &+ a^2\left[U_{,\varphi\varphi} + \frac{8\pi G}{\mathcal{H}}\left(2\varphi_0' U_{,\varphi} + \varphi_0'^2\frac{8\pi G}{\mathcal{H}}U(\varphi_0)\right)\right]\delta\varphi_1 = 0\,.\end{aligned} \tag{3.54}$$

Similarly, we can calculate the Klein-Gordon equation for a scalar field with the non-canonical Lagrangian, (2.41). We obtain

$$\frac{1}{c_{\rm ph}^2}\ddot{\delta\varphi} + \left[\frac{3H}{c_{\rm ph}^2} + C_1\right]\dot{\delta\varphi} + \left[\frac{k^2}{a^2} + C_2\right]\delta\varphi = 0\,, \tag{3.55}$$

where the coefficients C_1 and C_2, both functions of φ and X, are

$$C_1 = \frac{c_{\rm ph}^2}{p_{,X}^2}\left[p_{,\varphi} - 3Hp_{,X}\dot{\varphi}_0 - p_{,X\varphi}\dot{\varphi_0}^2\right]\left[3p_{,XX}\dot{\varphi}_0 + \dot{\varphi_0}^3 p_{,XXX}\right]$$
$$+ \frac{1}{p_{,X}}\left[\dot{\varphi}_0 p_{,X\varphi} + \dot{\varphi_0}^3 p_{,XX\varphi}\right] , \qquad (3.56)$$

$$C_2 = \frac{c_{\rm ph}^2}{p_{,X}^2}\left[p_{,\varphi} - 3Hp_{,X}\dot{\varphi}_0 - p_{,X\varphi}\dot{\varphi_0}^2\right]$$
$$\times\left[p_{,X\varphi} + \dot{\varphi_0}^2 p_{,XX\varphi} - \frac{4\pi G\dot{\varphi}_0 p_{,X}}{H}\left(5\dot{\varphi_0}^2 p_{,XX} + \dot{\varphi_0}^4 p_{,XXX} + 2p_{,X}\right)\right]$$
$$- \frac{4\pi G\dot{\varphi}_0}{Hc_{\rm ph}^2}\left[3H\dot{\varphi}_0 p_{,X} + p_{,\varphi} - \frac{p_{,X}\dot{\varphi}_0}{H}(3H^2 + 2\dot{H}) + c_{\rm ph}^2(\dot{\varphi_0}^4 p_{,XX\varphi} + p_{,\varphi})\right]$$
$$+ \frac{1}{p_{,X}}\left[\dot{\varphi_0}^2 p_{,X\varphi\varphi} - p_{,\varphi\varphi} + 3H\dot{\varphi}_0 p_{,X\varphi}\right] . \qquad (3.57)$$

The constants in this equation of motion can be interpreted physically: C_1 is an additional damping term, and C_2 affects the frequency of the oscillations. The phase speed is then read off as

$$c_{\rm ph}^2 = \frac{p_{,X}}{p_{,X} + 2X_0 p_{,XX}} , \qquad (3.58)$$

which, using Eq. (2.37) reduces to [56]

$$c_{\rm ph}^2 = \frac{P_{0,X}}{\rho_{0,X}} . \qquad (3.59)$$

The adiabatic sound speed for the Lagrangian (2.41) is given by

$$c_{\rm s}^2 = \frac{p_{,X}\ddot{\varphi}_0 + p_{,\varphi}}{p_{,X}\ddot{\varphi}_0 - p_{,\varphi} + p_{,XX}\dot{\varphi_0}^2\ddot{\varphi}_0 + p_{,X\varphi}\dot{\varphi_0}^2} . \qquad (3.60)$$

Finally, the adiabatic sound speed for the canonical Lagrangian is

$$c_{\rm s}^2 = 1 + \frac{2U_{,\varphi}}{3H{\varphi_0}'} , \qquad (3.61)$$

and becomes $c_{\rm s}^2 = -1$ in slow-roll. The phase speed for a canonical scalar field is $c_{\rm ph}^2 = 1$, as can be read off from Eq. (3.53).

Thus we note that, although in classical fluid systems the adiabatic sound speed and phase speed are the same, they are different in the scalar field models studied here, with only the phase speed describing the speed with which perturbations travel. We emphasise that only the definition for the adiabatic sound speed, Eq. (2.151), enters the definition of the pressure perturbation. Similarly, in the adiabatic case when $\delta P_{\rm nad1} = 0$, Eq. (2.152) reduces to $\delta P_1 = c_{\rm s}^2 \delta\rho_1$. Again, this convenient relation between the pressure and the energy density perturbation is only correct when using the adiabatic sound speed, as defined in Eq. (2.151).

3.1.5 Combined Dark Energy and Dark Matter System

In this section, as an application of gauge choice and the linear equations, we consider perturbations of a combined dark matter and dark energy system. The importance of considering dark energy perturbations in such a system is still under discussion. Recently, in Ref. [126], it was shown that ignoring the dark energy perturbations results in gauge dependent predictions. The authors showed this by first taking the governing equations for the dark matter/dark energy system calculating, using a numerical simulation, the matter power spectrum in three gauges: the comoving gauge, the uniform curvature gauge and the uniform expansion gauge. As expected, the same result was obtained for each gauge. They then ignored perturbations to the dark energy component, and repeated the analysis, this time obtaining different results for each of the three gauges. They then concluded that it was crucial to include the dark energy perturbations when analysing such a system. Thus one cannot use the familiar evolution equation for the dark matter density contrast, $\delta_{\rm c} \equiv \delta\rho_{\rm c}/\rho_{\rm c}$,

$$\ddot{\delta}_{\rm c} + 2H\dot{\delta}_{\rm c} - 4\pi G\rho_{\rm c}\delta_{\rm c} = 0\,, \tag{3.62}$$

which is obtained by ignoring dark energy perturbations once the equations have been put into a gauge comoving with the dark matter.

In this section we consider a similar question, and ask whether we can come to the same result without using numerical simulations, but instead by using the formalism of cosmological perturbation theory. The section contains work published in Ref. [35].

We consider here the linear governing equations for scalars only in coordinate time and without fixing a gauge (dropping the subscript '1' in this section for ease of presentation). Though some of the equations from previous sections are replicated here, since we are using a different time coordinate in previous sections, we give all equations used for completeness. We assume that the dark matter and dark energy are non-interacting, and so energy momentum conservation for each fluid is given by

$$\nabla_\mu T^\mu_{(\alpha)\nu} = 0\,. \tag{3.63}$$

Note that this assumption is for brevity, and to allow us to deal with more manageable equations. The results highlighted in this work will still hold in the case of interacting fluids for which the overall energy-momentum tensor is covariantly conserved, but the components obey

$$\nabla_\mu T^\mu_{(\alpha)\nu} = Q_{(\alpha)\nu}\,, \tag{3.64}$$

where $Q_{(\alpha)\nu}$ is the energy-momentum transfer to the αth fluid [70, 112].

Then, we obtain an evolution equation for each fluid from the energy (temporal) component of Eq. (3.63),

$$\dot{\delta\rho_\alpha} + 3H(\delta\rho_\alpha + \delta P_\alpha) + (\rho_0 + P_0)\frac{\nabla^2}{a^2}(V_\alpha + \sigma) = 3(\rho_\alpha + P_\alpha)\dot{\psi}\,, \tag{3.65}$$

where the covariant velocity potential of each fluid is defined as $V_\alpha = a(v_\alpha + B)$.

Considering the dark matter fluid and the dark energy scalar field, respectively, Eq. (3.65) then gives

$$\dot{\delta_{\rm c}} = 3\dot{\psi} - \frac{\nabla^2}{a^2}(\sigma + V_{\rm c})\,, \tag{3.66}$$

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \left(U_{,\varphi\varphi} - \frac{\nabla^2}{a^2}\right)\delta\varphi = \dot{\varphi}\left(3\dot{\psi} - \frac{\nabla^2}{a^2}\sigma + \dot{\phi}\right) - 2U_{,\varphi}\phi\,, \tag{3.67}$$

where we have used the expressions for the energy density perturbation and pressure perturbation of a scalar field given in Section 2.2, and note that the dark matter is pressureless. We have also used the background evolution equation for the dark energy scalar field

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 + U_{,\varphi} = 0\,, \tag{3.68}$$

the background conservation equation for the dark matter,

$$\dot{\rho}_{\rm c} + 3\mathcal{H}\rho_{\rm c} = 0\,, \tag{3.69}$$

and the expression

$$V_\varphi = -\frac{\delta\varphi}{\dot{\varphi}_0}\,. \tag{3.70}$$

There is also a momentum conservation equation, coming from the spatial component of Eq. (3.63), corresponding to each fluid

$$\dot{V}_\alpha - 3Hc_\alpha^2 V_\alpha + \phi + \frac{\delta P_\alpha}{\rho_\alpha + P_\alpha} = 0\,, \tag{3.71}$$

where $c_\alpha^2 = \dot{P}_\alpha / \dot{\rho}_\alpha$.
The Einstein field equations give (from the previous section or, e.g., Ref. [111])

$$3H(\dot{\psi} + H\phi) - \frac{\nabla^2}{a^2}(\psi + H\sigma) + 4\pi G\delta\rho = 0\,, \tag{3.72}$$

$$\dot{\psi} + H\phi + 4\pi G(\rho_0 + P_0)V = 0\,, \tag{3.73}$$

$$\dot{\sigma} + H\sigma - \phi + \psi = 0\,, \tag{3.74}$$

$$\ddot{\psi} + 3H\dot{\psi} + H\dot{\phi} + (3H^2 + 2\dot{H})\phi - 4\pi G\delta P = 0\,, \tag{3.75}$$

where the total matter quantities are defined as the sum of the quantity for each fluid or field, i.e.

$$\delta\rho = \delta\rho_\varphi + \delta\rho_{\mathrm{c}}\,, \tag{3.76}$$

$$\delta P = \delta P_\varphi\,, \tag{3.77}$$

$$(\rho + P)V = (\rho_\varphi + P_\varphi)V_\varphi + \rho_{\mathrm{c}} V_{\mathrm{c}}\,, \tag{3.78}$$

and we have used the fact that the dark matter is pressureless, i.e. $P_{\mathrm{c}} = 0 = \delta P_{\mathrm{c}}$. Introducing a new variable $\mathcal{Z}$, both for notational convenience and to assist with the following calculations, defined as

$$\mathcal{Z} \equiv 3(\dot{\psi} + H\phi) - \frac{\nabla^2}{a^2}\sigma\,, \tag{3.79}$$

we can rewrite Eq. (3.75), using Eqs. (3.72) and (3.74), as

$$\dot{\mathcal{Z}} + 2H\mathcal{Z} + \left(3\dot{H} + \frac{\nabla^2}{a^2}\right)\phi = 4\pi G(\delta\rho + 3\delta P)\,. \tag{3.80}$$

Then, from Eq. (3.66),

$$\mathcal{Z} = \dot{\delta}_{\mathrm{c}} + 3H\phi + \frac{\nabla^2}{a^2}V_{\mathrm{c}}\,, \tag{3.81}$$

and from Eq. (3.71) for the dark matter fluid (for which $c_{\mathrm{s}}^2 = 0$),

$$\phi = -\dot{V}_{\mathrm{c}}\,. \tag{3.82}$$

Differentiating Eq. (3.81) gives

$$\dot{\mathcal{Z}} = \ddot{\delta}_{\rm c} + 3(H\phi)\dot{} + \frac{\nabla^2}{a^2}\Big(\dot{V}_{\rm c} - 2HV_{\rm c}\Big)\,. \tag{3.83}$$

Substituting this into Eq. (3.80) gives

$$\begin{aligned}\ddot{\delta}_{\rm c} + 2H\dot{\delta}_{\rm c} - 4\pi G\rho_{\rm c}\delta_{\rm c} &= 8\pi G(2\dot{\varphi}_0\dot{\delta\varphi} - U_{,\varphi}\delta\varphi) \\ &\quad + \dot{V}_{\rm c}(6(\dot{H} + H^2) + 16\pi G\dot{\varphi_0}^2) + 3H\ddot{V}_{\rm c}\,. \end{aligned} \tag{3.84}$$

The evolution equation for the field is then obtained solely in terms of matter perturbations from Eq. (3.67) by using Eqs. (3.82) and (3.81):

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \Big(U_{,\varphi\varphi} - \frac{\nabla^2}{a^2}\Big)\delta\varphi = \dot{\varphi}\Big(\dot{\delta}_{\rm c} + \frac{\nabla^2}{a^2}V_{\rm c} + 3H\dot{V}_{\rm c} + 3H\dot{V}_{\rm c} - \ddot{V}_{\rm c}\Big) + 2U_{,\varphi}\dot{V}_{\rm c}\,. \tag{3.85}$$

It is worth restating that we have derived these equations in a general form without fixing a gauge. If we set the dark matter velocity to zero, i.e. $V_{\rm c} = 0$, then they reduce to those presented in, for example, Refs. [34, 70] (for the case of a zero energy-momentum transfer).

We now want to consider fixing the gauge freedoms in this set of governing equations. In order to do so, we need to consider the gauge transformations of the variables under the transformations

$$\widetilde{t} = t - \alpha\,, \tag{3.86}$$

$$\widetilde{x}^i = x^i - \beta_{,}{}^{i}\,, \tag{3.87}$$

where the generating vector is then defined as

$$\xi^\mu = (\alpha, \beta_{,}{}^{i})\,. \tag{3.88}$$

Note that, since the choice of time coordinate is different to that used in the previous chapter, the following gauge transformations will differ from those presented in, e.g., Section 2.3.4 due to the different definition of ξ^μ. Then, scalar quantities such as the field perturbation transform as

$$\widetilde{\delta\varphi} = \delta\varphi + \dot{\varphi}\alpha\,, \tag{3.89}$$

the density contrast transforms as

$$\widetilde{\delta_{\rm c}} = \delta_{\rm c} + \frac{\dot{\rho}_{\rm c}}{\rho_{\rm c}}\alpha\,, \tag{3.90}$$

and the components of the velocity potential as

$$\widetilde{v_\alpha} = v_\alpha - a\dot{\beta}\,. \tag{3.91}$$

Furthermore, by considering the transformation behaviour of the metric tensor we obtain the following transformation rules for the scalar metric perturbations

$$\widetilde{\phi} = \phi + \dot{\alpha}\,, \tag{3.92}$$
$$\widetilde{\psi} = \psi - H\alpha\,, \tag{3.93}$$
$$\widetilde{B} = B - \frac{1}{a}\alpha + a\dot{\beta}\,, \tag{3.94}$$
$$\widetilde{E} = E + \beta\,, \tag{3.95}$$

and so the scalar shear transforms as

$$\widetilde{\sigma} = \sigma + \alpha\,. \tag{3.96}$$

Finally Eq. (3.91) gives the transformation behaviour of the components of the covariant velocity potential

$$\widetilde{V_\alpha} = V_\alpha - \alpha\,. \tag{3.97}$$

Turning now to the case at hand, choosing the gauge in which the perturbation in the dark energy field is zero, $\widetilde{\delta\varphi} = 0$, fixes α as

$$\alpha = -\frac{\delta\varphi}{\dot{\varphi}}\,. \tag{3.98}$$

Since none of the gauge transformations of the quantities involved in the governing equations depend upon the threading β, we do not need to consider fixing it explicitly here. (Of course, one can rigorously fix β by choosing a suitable gauge condition, as outlined in Section 2.4.) The governing equations in this gauge are then

$$\ddot{\widehat{\delta_{\rm c}}} + 2H\dot{\widehat{\delta_{\rm c}}} - 4\pi G\rho_{\rm c}\widehat{\delta_{\rm c}} = \dot{\widehat{V_{\rm c}}}(6(\dot{H}+H^2) + 16\pi G\dot{\varphi}^2) + 3H\ddot{\widehat{V_{\rm c}}}\,, \tag{3.99}$$
$$\ddot{\widehat{V_{\rm c}}} - \frac{2U_{,\varphi}}{\dot{\varphi}_0}\dot{\widehat{V_{\rm c}}} - \frac{\nabla^2}{a^2}\widehat{V_{\rm c}} = -\dot{\widehat{\delta_{\rm c}}}\,, \tag{3.100}$$

where the hat denotes that the variables are evaluated in the uniform field fluctuation gauge. That is, in this gauge, $\widehat{\delta_{\rm c}}$ and $\widehat{V_{\rm c}}$ are gauge invariant variables defined as

$$\widehat{\delta_{\rm c}} = \delta_{\rm c} - \frac{\dot{\rho}_{\rm c}}{\rho_{\rm c}\dot{\varphi}_0}\delta\varphi\,, \qquad \widehat{V_{\rm c}} = V_{\rm c} + \frac{\delta\varphi}{\dot{\varphi}_0}\,. \tag{3.101}$$

Alternatively, choosing a gauge comoving with the dark matter, in which $\widetilde{V_{\rm c}} = 0$ fixes the generating vector as

$$\alpha = V_{\rm c}\,, \tag{3.102}$$

and reduces the governing equations to

$$\ddot{\bar{\delta_{\rm c}}} + 2H\dot{\bar{\delta_{\rm c}}} - 4\pi G\rho_{\rm c}\bar{\delta_{\rm c}} = 8\pi G(2\dot{\varphi}_0\dot{\bar{\delta\varphi}} - U_{,\varphi}\bar{\delta\varphi})\,, \tag{3.103}$$

$$\ddot{\bar{\delta\varphi}} + 3H\dot{\bar{\delta\varphi}} + \left(U_{,\varphi\varphi} - \frac{\nabla^2}{a^2}\right)\bar{\delta\varphi} = \dot{\varphi}_0\dot{\bar{\delta_{\rm c}}}\,, \tag{3.104}$$

where the bar denotes variables in the comoving gauge and we have

$$\bar{\delta_{\rm c}} = \delta_{\rm c} + \frac{\rho_{\rm c}}{\dot{\rho}_{\rm c}}V_{\rm c}\,, \qquad \bar{\delta\varphi} = \delta\varphi + \dot{\varphi}_0 V_{\rm c}\,. \tag{3.105}$$

By studying the above systems of equations, it is evident that choosing the dark energy field perturbation to be zero is a well defined choice of gauge, reducing the governing equations to Eqs. (3.99) and (3.100). Then, having done so, we are no longer allowed the freedom to make another choice of gauge. Alternatively, choosing a gauge comoving with the dark matter uses up the gauge freedom, and so we are not permitted to neglect the perturbation in the dark energy field. In fact, doing so will result in erroneous gauge dependent results. It is clearest to see why this is the case by considering the set of governing equations. By making our choice of gauge we are left with a set of equations which is gauge invariant: that is, performing a gauge transformation will leave the set of equations unchanged. However, by neglecting the perturbation in the dark energy field after having chosen the gauge comoving with the dark matter amounts to setting the right hand side of Eq. (3.103) to zero. This resulting equation will, in general, then no longer be gauge invariant.

Thus, we conclude that the dark energy perturbation must be considered in a system containing a mixture of dark matter and dark energy. Our result is consistent with that of Ref. [126], though we have shown this by simply using the formalism of cosmological perturbation theory instead of relying on more involved numerical calculations.

3.2 Second Order

We can extend the governing equations presented in the previous section to beyond linear order by simply not truncating the expansion of each variable after the first term. Doing so, we obtain equations with similar structure to those at linear order, however with new couplings between different type of perturbation. In fact, these couplings will turn out to be the reason for the qualitative difference between the linear and higher order theories. In this section, we will present the full second order equations for scalar, vector and tensor perturbations in a gauge dependent format. These equations have been derived previously, for example in Ref. [123], though are derived in a slightly different way, using the ADM split, and so are written in terms of the extrinsic curvature. The energy conservation equations have also been derived in Ref. [112].

Conservation of energy momentum again gives us two equations: an energy conservation equation

$$\begin{aligned}
&\delta\rho_2{}' + 3\mathcal{H}(\delta\rho_2 + \delta P_2) + (\rho_0 + P_0)(C^{i\prime}_{2i} + v_{2i,}{}^{i}) + 2(\delta\rho_1 + \delta P_1)_{,i}v^i_1 \\
&+ 2(\delta\rho_1 + \delta P_1)(C^{i\prime}_{1i} + v_{1i,}{}^{i}) + 2(\rho_0 + P_0)\Big[(V^i_1 + v^i_1)V'_{1i} + v^i_{1,i}\phi_1 \\
&- 2C'_{1ij}C^{ij}_1 + v^i_1(C^j_{1j,i} + 2\phi_{1,i}) + 4\mathcal{H}v^i_1(V_{1i} + v_{1i})\Big] = 0\,,
\end{aligned} \tag{3.106}$$

and a momentum conservation equation

$$\begin{aligned}
&\Big[(\rho_0 + P_0)V_{2i}\Big]' + (\rho_0 + P_0)(\phi_{2,i} + 4\mathcal{H}V_{2i}) + \delta P_{2,i} + 2\Big[(\delta\rho_1 + \delta P_1)V_{1i}\Big]' \\
&+ 2(\delta\rho_1 + \delta P_1)(\phi_{1,i} + 4\mathcal{H}V_{1i}) - 2(\rho_0 + P_0)'\Big[(B_{1i} + V_{1i})\phi_1 - 2C_{1ij}v^j_1\Big] \\
&+ 2(\rho_0 + P_0)\Big[V_{1i}(C^{j\prime}_{1j} + v^j_{1,j}) - B_{1i}(\phi_1{}' + 8\mathcal{H}\phi_1) + (2C_{1ij}v^j_1)' \\
&+ v^j_1(V_{1i,j} - B_{1j,i} + 8\mathcal{H}C_{1ij}) - \phi_1(V'_{1i} + B'_{1i} + 2\phi_{1,i} + 4\mathcal{H}v_{1i})\Big] = 0\,,
\end{aligned} \tag{3.107}$$

where we have not expanded the terms in C_{ij} in order to keep the equations compact, but recall from Eq. (2.3), that

$$C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}\,. \tag{3.108}$$

The Einstein equations give the $(0-0)$ component

$$
\begin{aligned}
&\nabla^2 C^j_{2j} - C_{2ij,}{}^{ij} + 2\mathcal{H}(-C^{i'}_{2\,i} + B^i_{2,i} + 3\mathcal{H}\phi_2) + 2C^j_{1j,i}(\frac{1}{2}C^k_{1\,k,}{}^i - 2C^{ik}_1{}_{,k})\\
&+ 2B^i_1\Big[C^{j'}_{1\,j,i} - C'_{1ij,}{}^j + \frac{1}{2}\left(\nabla^2 B_{1i} - B_{1j,i}{}^j\right) + 2\mathcal{H}\left(C^{1j}{}_{j,i} - 2C_{1ij,}{}^j - \phi_{1,i}\right)\Big]\\
&+ 4C^{ij}_1\Big[2C_{1jk,i}{}^k - C^k_{1\,k,ij} - \nabla^2 C_{1ij} + 2\mathcal{H}(C'_{1ij} - B_{1i,j})\Big] + 2C_{1jk,i}(C^{ik}_1{}_{,}{}^j - \frac{3}{2}C^{jk}_1{}_{,}{}^i)\\
&+ 2C^{i'}_{1\,i}(B_{1j,}{}^j - \frac{1}{2}C^{j'}_{1\,j} + 4\mathcal{H}\phi) + 4C^{ij}_1{}_{,i}C_{1jk,}{}^k + 2C'_{1ij}(\frac{1}{2}C^{ij'}_1 - B^j_{1,}{}^i)\\
&+ \frac{1}{2}B_{1j,i}(B^i_{1,}{}^j + B^j_{1,}{}^i) - 6\mathcal{H}^2(4\phi_1^2 - B_{1i}B^i_1) - B^i_{1,i}B_{1j,}{}^j - 8\mathcal{H}B^i_{1,i}\phi_1\\
&= -8\pi G a^2\Big[2(\rho_0 + P_0)V^k_1 v_{1k} + \delta\rho_2\Big]\,,
\end{aligned}
\tag{3.109}
$$

and the $(0-i)$ component

$$
\begin{aligned}
&C^{k'}_{2\,k,i} - C'_{2ik,}{}^k - \frac{1}{2}\left(B_{2k,i}{}^k - \nabla^2 B_{2i}\right) - 2\mathcal{H}\phi_{2,i} + 16\mathcal{H}\phi_{1,i}\phi_1 - 2C^{j'}_{1\,j}\phi_{1,i}\\
&+ 2C'_{1ij}\left(2C^{kj}_1{}_{,k} - C^k_{1\,k,}{}^j + \phi_{1,}{}^j\right) + 4C^{kj}_1\left[C'_{1ik,j} - C'_{1jk,i} + \frac{1}{2}\left(B_{1k,ij} - B_{1i,kj}\right)\right]\\
&+ 2B^j_1\left(C_{1kj,i}{}^k - C^k_{1\,k,ij} + C_{1ik,k}{}^j - \nabla^2 C_{1ij} - 2\mathcal{H}B_{1j,i}\right) - \Big(B_{1i,j} + B_{1j,i}\Big)\phi_{1,}{}^j\\
&+ 2\left(B_{1i,j} - B_{1j,i}\right)\left(\frac{1}{2}C^k_{1\,k,}{}^j - C^{jk}_1{}_{,k}\right) - 2C_{1ik,j}\left(B^k_{1,}{}^j - B^j_{1,}{}^k\right) + 2B^j_{1,j}\phi_{1,i}\\
&+ 2\phi_1\left[B_{1j,i}{}^j - \nabla^2 B_{1i} + 2\left(C'_{1ij,}{}^j - C^{1j'}{}_{j,i}\right)\right] - 2C^{kj'}_1 C_{kj,i}\\
&= 16\pi G\Big[\frac{1}{2}V_{2i} - \phi_1(V_{1i} + B_{1i}) + 2C_{1ik}v^k_1 + (\delta\rho_1 + \delta P_1)V_{1i}\Big]\,.
\end{aligned}
\tag{3.110}
$$

We present the full $(i-j)$ component in Appendix A.

The equations for scalar perturbations only in a gauge dependent form are then obtained by substituting $C_{ij} = -\psi\delta_{ij} + E_{,ij}$ and $B_i = B_{,i}$, at both first and second order, into the above. The energy conservation equation then becomes

$$
\begin{aligned}
&\delta\rho_2{}' + 3\mathcal{H}(\delta\rho_2 + \delta P_2) + (\rho_0 + P_0)\Big(\nabla^2(E_2{}' + v_2) - 3\psi_2{}'\Big) + 2(\delta\rho_1 + \delta P_1)_{,i}v_1{}^i\\
&+ 2(\delta\rho_1 + \delta P_1)\Big(\nabla^2(E_1{}' + v_1) - 3\psi_1{}'\Big) + 2(\rho_0 + P_0)\Big[(V'_{1,i} + 4\mathcal{H}v_{1,i})(V_{1,i} + v_{1,i})\\
&+ 3\psi_1\psi_1{}' + \nabla^2 v_1\phi_1 - (\psi_1\nabla^2 E)' + E'_{1,ij}E_{1,}{}^{ij} + v_{1,}{}^i(2\phi_{1,i} - 3\psi_{1,i} + \nabla^2 E_{1,i})\Big] = 0\,,
\end{aligned}
\tag{3.111}
$$

while the momentum conservation equation is

$$
\begin{aligned}
&\Big[(\rho_0+P_0)V_{1,i}\Big]' + (\rho_0+P_0)\Big(\phi_2+4\mathcal{H}V_2\Big)_{,i} + \delta P_{2,i} + 2\Big[V_{1,i}(\delta\rho_1+\delta P_1)\Big]' \\
&+2(\delta\rho_1+\delta P_1)\Big(\phi_1+4\mathcal{H}V_1\Big)_{,i} - 2(\rho_0+P_0)'\Big[(V_1+B_1)_{,i}\phi_1 - 2(E_{1,ij}v_{1,}{}^{j}-\psi_1 v_{1,i})\Big] \\
&+2(\rho_0+P_0)\Big[V_{1,i}\Big(\nabla^2(E_1{}'+v_1)-3\psi_1{}'\Big) - B_{1,i}(\phi_1{}'+8\mathcal{H}\phi_1) + v_{1,}{}^{j}(v_{1,ij}+8\mathcal{H}E_{1,ij}) \\
&+2\Big(v_{1,}{}^{j}E_{1,ij}-\psi_1 v_{1,i}\Big)' - \phi_1\Big((V_1+B_1)'+2\phi_1+4\mathcal{H}v_1\Big)_{,1} - 8\mathcal{H}\psi_1 v_{1,i}\Big] = 0\,.
\end{aligned}
\tag{3.112}
$$

Turning now to the Einstein equations, the energy constraint is

$$
\begin{aligned}
&3\mathcal{H}(\psi_2{}'+\mathcal{H}\phi_2) + \nabla^2\Big(\mathcal{H}(B_2-E_2{}')-\psi_2\Big) + \nabla^2 B_1\Big(\nabla^2(E_1{}'-\frac{1}{2}B_1)-2\psi_1{}'\Big) \\
&+B_{1,i}\Big(\mathcal{H}(3\mathcal{H}B_{1,}{}^{i}-2\nabla^2E_{1,}{}^{i}-2(\psi_1+\phi_1)_{,}{}^{i})-2\psi_{1,}{}^{i\prime}\Big) + 2E_{1,}{}^{ij}(\psi_1-2\mathcal{H}B_1)_{,ij} \\
&+4\mathcal{H}(\psi_1-\phi_1)\Big(3\psi_1{}'-\nabla^2(E_1{}'-B_1)\Big) + E_{1,}{}^{ij\prime}\Big(4\mathcal{H}E_1+\frac{1}{2}E_1{}'-B_1\Big)_{,ij} \\
&+\psi_1{}'\Big(2\nabla^2(E_1{}'-2\mathcal{H}E_1)-3\psi_1{}')\Big) + \psi_{1,}{}^{i}(2\nabla^2E_1-3\psi_1)_{,i} + 2\nabla^2\psi_1(\nabla^2E_1-4\psi_1) \\
&-12\mathcal{H}^2\phi_1{}^2 + \frac{1}{2}\Big(B_{1,ij}B_{1,}{}^{ij} + \nabla^2E_{1,j}\nabla^2E_{1,}{}^{j} - E_{1,ijk}E_{1,}{}^{ijk} - \nabla^2E_1{}'\nabla^2E_1{}'\Big) \\
&= -4\pi Ga^2\Big(2(\rho_0+P_0)V_{1,}{}^{k}v_{1,k}+\delta\rho_2\Big)\,,
\end{aligned}
\tag{3.113}
$$

and the momentum constraint

$$
\begin{aligned}
&\psi_{2,i}' + \mathcal{H}\phi_{2,i} - E_{1,ij}'(\psi_1+\phi_1+\nabla^2E_1)_{,}{}^{j} + B_{1,ij}(2\mathcal{H}B_1+\phi_1)_{,}{}^{j} \\
&-\Big[\psi_{1,i}(\nabla^2E_1-4\psi_1)\Big]' - \phi_{1,i}\Big(8\mathcal{H}\phi_1+2\psi_1{}'+\nabla^2(E_1{}'-B_1)\Big) \\
&-B_{1,j}\psi_{1,i}{}^{j} + 2\psi_{1,}{}^{j\prime}E_{1,ij} + E_{1,jk}{}'E_{1,i}{}^{jk} - \psi_{1,i}'(\nabla^2E_1+4\phi_1) - \nabla^2\psi_1 B_{1,i} \\
&= -4\pi Ga^2\Big[(\rho_0+P_0)\Big(V_{2,i}-2\phi_1(V_1+B_1)_{,i}-4(\psi_1 v_{,i}-E_{1,ik}v_{1,}{}^{k})\Big) \\
&\qquad\qquad + 2(\delta\rho_1+\delta P_1)V_{1,i}\Big]\,,
\end{aligned}
\tag{3.114}
$$

while, from the trace of the $i-j$ component, we obtain

$$
\begin{aligned}
&3\mathcal{H}(2\psi_2+\phi_2)'+\nabla^2(E_2''+2E_2'+2\psi_2-B_2'-\phi_2+2\mathcal{H}B_2)-3\phi_2\Big(\mathcal{H}^2-2\frac{a''}{a}\Big)+3\psi_2''\\
&+(\psi_1-\phi_1)\Big(12(\psi_1''+2\mathcal{H}\psi_1')+4\nabla^2(\phi_1+(B_1-E_1)')+8\mathcal{H}\nabla^2(B_1-E_1')\Big)\\
&+E_{1,}{}^{ij}\Big(8\mathcal{H}(E_1'-B_1)_{,ij}+2\psi_{1,ij}-4\phi_{1,ij}-4B_{1,ij}'\Big)+E_{1,}{}^{ij}\Big(\frac{5}{2}E_{1,ij}'-B_{1,ij}\Big)\\
&+2\nabla^2E_1'\Big(4\phi_1'-\nabla^2(E_1'-2B_1)\Big)+\nabla^2E_{1,}{}^{i}\Big(\nabla^2E_{1,i}+2\phi_{1,i}-4\mathcal{H}B_{1,i}-2B_{1,i}'\Big)\\
&+\psi_{1,}{}^{i}\Big(2\nabla^2E_{1,i}-4\mathcal{H}B_{1,i}-2(\psi_1+\phi_1)_{,i}-2B_{1,i}'\Big)-2\phi_1'(\nabla^2B_1+12\mathcal{H}\phi_1)\\
&-2\phi_{1,i}\phi_{1,}{}^{i}+2\nabla^2\psi_1(\nabla^2E_1-4\psi_1)+\psi_1'\Big(3\psi_1'-6\phi_1'-8\mathcal{H}\nabla^2E_1-2\nabla^2(E_1'+B_1)\Big)\\
&+2B_{1,}{}^{i}\Big(\mathcal{H}(3B_1'-2\phi_1)-3\psi_1'\Big)_{,i}+\frac{1}{2}\Big(B_{1,ij}B_{1,}{}^{ij}-E_{1,ijk}E_{1,}{}^{ijk}-\nabla^2B_1\nabla^2B_1\Big)\\
&+4(E_{1,}{}^{ij}E_{1,ij}''-\psi_1''\nabla^2E_1)+3\Big(\mathcal{H}^2-2\frac{a''}{a}\Big)\Big(4\phi_1{}^2-B_{1,i}B_{1,}{}^{i}\Big)\\
&=4\pi Ga^2\Big(3\delta P_2+2(\rho_0+P_0)v_{1,}{}^{i}V_{1,i}\Big)\,. \qquad (3.115)
\end{aligned}
$$

We can obtain, from the $i-j$ component a fourth Einstein equation, as at linear order, by applying the operator $\partial_i\partial^j$. However, since we do not require this equation for the work in this thesis, as one can always use the energy-momentum conservation equations in place of the $i-j$ equations, we omit that equation here.

3.2.1 Uniform Curvature Gauge

We now present the second order equations in the uniform curvature gauge, where $\widetilde{\psi}=\widetilde{E}=\widetilde{F}_i=0$, at both first and second orders. We also consider only scalar and vector perturbations in this section, i.e. we choose to neglect tensor perturbations, and so $C_{ij}=0$, at both first and second order. We obtain the energy conservation equation

$$
\begin{aligned}
&\delta\rho_2'+3\mathcal{H}(\delta\rho_2+\delta P_2)+(\rho_0+P_0)v_{2i,}{}^{i}+2(\delta\rho_1+\delta P_1)_{,i}v_1^i+2(\delta\rho_1+\delta P_1)v_{1i,}{}^{i}\\
&+2(\rho_0+P_0)\Big[(V_1^i+v_1^i)V_{1i}'+v_{1,i}^i\phi_1+2v_1^i\phi_{1,i}+4\mathcal{H}v_1^i(V_{1i}+v_{1i})\Big]=0\,, \qquad (3.116)
\end{aligned}
$$

and momentum conservation equation

$$\Big[(\rho_0+P_0)V_{2i}\Big]'+(\rho_0+P_0)(\phi_{2,i}+4\mathcal{H}V_{2i})+\delta P_{2,i}+2\Big[(\delta\rho_1+\delta P_1)V_{1i}\Big]'$$
$$+2(\delta\rho_1+\delta P_1)(\phi_{1,i}+4\mathcal{H}V_{1i})-2(\rho_0+P_0)'(B_{1i}+V_{1i})\phi_1$$
$$+2(\rho_0+P_0)\Big[V_{1i}v^j_{1,j}-B_{1i}({\phi_1}'+8\mathcal{H}\phi_1)+v^j_1(V_{1i,j}-B_{1j,i})$$
$$-\phi_1(V'_{1i}+B'_{1i}+2\phi_{1,i}+4\mathcal{H}v_{1i})\Big]=0\,. \tag{3.117}$$

The Einstein equations give us an energy constraint

$$2\mathcal{H}(B^i_{2,i}+3\mathcal{H}\phi_2)+2B^i_1\Big[\frac{1}{2}\left(\nabla^2B_{1i}-B_{1j,i}{}^{j}\right)-2\mathcal{H}\phi_{1,i}\Big]$$
$$+\frac{1}{2}B_{1j,i}(B^i_{1,}{}^{j}+B^j_{1,}{}^{i})-6\mathcal{H}^2(4\phi_1^2-B_{1i}B^i_1)-B^i_{1,i}B_{1j,}{}^{j}-8\mathcal{H}B^i_{1,i}\phi_1$$
$$=-8\pi Ga^2\Big[2(\rho_0+P_0)V^k_1v_{1k}+\delta\rho_2\Big]\,, \tag{3.118}$$

a momentum constraint

$$\frac{1}{2}\left(\nabla^2B_{2i}-B_{2k,i}{}^{k}\right)-2\mathcal{H}\phi_{2,i}+16\mathcal{H}\phi_{1,i}\phi_1-4\mathcal{H}B^j_1B_{1j,i}$$
$$-\Big(B_{1i,j}+B_{1j,i}\Big)\phi_{1,}{}^{j}+2B^j_{1,j}\phi_{1,i}+2\phi_1(B_{1j,i}{}^{j}-\nabla^2B_{1i})$$
$$=16\pi G\Big[\frac{1}{2}V_{2i}-\phi_1(V_{1i}+B_{1i})+(\delta\rho_1+\delta P_1)V_{1i}\Big]\,, \tag{3.119}$$

and a third equation from the $(i-j)$ component

$$-\frac{1}{2}(B^{i\prime}_{2\ ,j}+B_{2j,}{}^{i\prime})-\phi_{2,}{}^{i}{}_{j}-\mathcal{H}(B^i_{2,j}+B_{2j,}{}^{i})$$
$$+\delta^i{}_j\Big\{2\phi_2\Big(\frac{2a''}{a}-\mathcal{H}^2\Big)+2\mathcal{H}(B^k_{2,k}+\phi'_2)+B^{k\prime}_{2\ k}+\nabla^2\phi_2\Big\}+2\phi_{1,}{}^{i}\phi_{1,j}$$
$$+B^k_1(B_{1j,}{}^{i}{}_{k}+B^i_{1,jk}-2B_{1k,}{}^{i}{}_{j})+(B^i_{1,j}+B_{1j,}{}^{i})({\phi_1}'+B^k_{1,k})-B_{1k,}{}^{i}B^k_{1,j}-B_{1j,}{}^{k}B^i_{1,k}$$
$$+B^i_1(B_{1k,j}{}^{k}-\nabla^2B_{1j}+4\mathcal{H}\phi_{1,j})+2\phi_1[B_{1j,}{}^{i\prime}+B_1{}^{i\prime}{}_{,j}+2\phi_{1,}{}^{i}{}_{j}+2\mathcal{H}(B_{1j,}{}^{i}+B_1{}^{i}{}_{,j})]$$
$$+2\delta^i{}_j\Big\{\Big(\mathcal{H}^2-\frac{2a''}{a}\Big)(4\phi_1^2-B_{1k}B_1{}^{k})-2\phi_1[B^{k\prime}_{1\ ,k}-\nabla^2\phi_1+2\mathcal{H}(2\phi'_1+B^k_{1,k})]$$
$$B^k_1[\nabla^2B_{1k}-B_{1l,k}{}^{l}+2\mathcal{H}(B'_{1k}-\phi_{1,k})]-\frac{1}{4}(2B^k_{1,k}B_{1l,}{}^{l}-B_{1l,k}B^{k}_{1,}{}^{l}-3B^l_{1,k}B_{1l,}{}^{k})$$
$$-\phi'_1B^k_{1,k}-\phi_{1,k}\phi_{1,}{}^{k}\Big\}=8\pi G\Big[2(\rho_0+P_0)v^i_1V_{1j}+\delta^i{}_j\delta P_2\Big]\,. \tag{3.120}$$

Considering scalar perturbations only, energy conservation gives

$$
\begin{aligned}
&\delta\rho_2{}' + 3\mathcal{H}(\delta\rho_2 + \delta P_2) + (\rho_0 + P_0)\nabla^2 v_2 + 2(\delta\rho_1 + \delta P_1)_{,i} v_1{}^i \\
&2(\delta\rho_1 + \delta P_1)\nabla^2 v + 2(\rho_0 + P_0)\Big[(V_{1,i}' + 4\mathcal{H}v_{1,i})(V_{1,i} + v_{1,i}) \\
&+ \nabla^2 v_1 \phi_1 + 2v_{1,}{}^i \phi_{1,i}\Big] = 0\,,
\end{aligned}
\tag{3.121}
$$

and momentum conservation gives

$$
\begin{aligned}
&\Big[(\rho_0 + P_0)V_{1,i}\Big]' + (\rho_0 + P_0)\Big(\phi_2 + 4\mathcal{H}V_2\Big)_{,i} + \delta P_{2,i} + 2\Big[V_{1,i}(\delta\rho_1 + \delta P_1)\Big]' \\
&+ 2(\delta\rho_1 + \delta P_1)\Big(\phi_1 + 4\mathcal{H}V_1\Big)_{,i} - 2(\rho_0 + P_0)'(V_1 + B_1)_{,i}\phi_1 \\
&+ 2(\rho_0 + P_0)\Big[V_{1,i}\nabla^2 v_1 - B_{1,i}(\phi_1{}' + 8\mathcal{H}\phi_1) + v_{1,}{}^j v_{1,ij} \\
&- \phi_1\Big((V_1 + B_1)' + 2\phi_1 + 4\mathcal{H}v_1\Big)_{,1}\Big] = 0\,.
\end{aligned}
\tag{3.122}
$$

Finally, the Einstein equations for scalars only in the uniform curvature gauge are

$$
\begin{aligned}
&3\mathcal{H}^2\phi_2 + \nabla^2\mathcal{H}B_2 - \psi_2 - \frac{1}{2}(\nabla^2 B_1)^2 + \mathcal{H}B_{1,i}(3\mathcal{H}B_{1,}{}^i - 2\phi_{1,}{}^i) + 4\mathcal{H}\phi_1 B_1 \\
&- 12\mathcal{H}^2\phi_1{}^2 + \frac{1}{2}B_{1,ij}B_{1,}{}^{ij} = -4\pi G a^2\Big(2(\rho_0 + P_0)V_{1,}{}^k v_{1,k} + \delta\rho_2\Big)\,,
\end{aligned}
\tag{3.123}
$$

$$
\begin{aligned}
&\mathcal{H}\phi_{2,i} + B_{1,ij}(2\mathcal{H}B_1 + \phi_1)_{,}{}^j - \phi_{1,i}\Big(8\mathcal{H}\phi_1 - \nabla^2 B_1\Big) \\
&= -4\pi G a^2\Big[(\rho_0 + P_0)\Big(V_{2,i} - 2\phi_1(V_1 + B_1)_{,i}\Big) + 2(\delta\rho_1 + \delta P_1)V_{1,i}\Big]\,,
\end{aligned}
\tag{3.124}
$$

and

$$
\begin{aligned}
&3\mathcal{H}\phi_2{}' + \nabla^2(2\mathcal{H}B_2 - B_2{}' - \phi_2) - \phi_1\Big(4\nabla^2(\phi_1 + B_1{}') + 8\mathcal{H}\nabla^2 B_1\Big) \\
&- 2\phi_1{}'(\nabla^2 B_1 + 12\mathcal{H}\phi_1) - 2\phi_{1,i}\phi_{1,}{}^i + 2\mathcal{H}B_{1,}{}^i\Big(3B_1{}' - 2\phi_1\Big)_{,i} \\
&+ \frac{1}{2}\Big(B_{1,ij}B_{1,}{}^{ij} - \nabla^2 B_1 \nabla^2 B_1\Big) + 3\Big(\mathcal{H}^2 - 2\frac{a''}{a}\Big)\Big(4\phi_1{}^2 - B_{1,i}B_{1,}{}^i - \phi_2\Big) \\
&= 4\pi G a^2\Big(3\delta P_2 + 2(\rho_0 + P_0)v_{1,}{}^i V_{1,i}\Big)\,.
\end{aligned}
\tag{3.125}
$$

The Klein-Gordon equation can be obtained at second order by using the same technique as at first order: comparing the energy-momentum tensor for the scalar field to that of a perfect fluid and using energy conservation. The equation for a canonical scalar field is

$$\delta\varphi_2'' + 2\mathcal{H}\delta\varphi_2' - \nabla^2\delta\varphi_2 + a^2 U_{,\varphi\varphi}\delta\varphi_2 + a^2 U_{,\varphi\varphi\varphi}\delta\varphi_1^2 + 2a^2 U_{,\varphi}\phi_2 - \varphi_0'(\nabla^2 B_2 + {\phi_2}')$$
$$+ 4\varphi_0' B_{1,k}{\phi_{1,}}^k + 2(2\mathcal{H}\varphi_0' + a^2 U_{,\varphi})B_{1,k}{B_{1,}}^k + 4\phi_1(a^2 U_{,\varphi\varphi}\delta\varphi_1 - \nabla^2\delta\varphi_1) + 4\varphi_0'\phi_1{\phi_1}'$$
$$- 2\delta\varphi_1'(\nabla^2 B_1 + {\phi_1}') - 4\delta\varphi_{1,k}'{B_{1,}}^k = 0\,, \tag{3.126}$$

where we are yet to use the field equations to remove the metric perturbations. See Refs.[65, 67, 68, 106] for the closed form of the Klein-Gordon equation at second order and for detailed work on the second order Klein-Gordon equation.

3.2.2 Poisson Gauge

In this section we present the second order equations in the Poisson gauge, in order to connect with the literature which often uses this gauge (for example, Ref. [3] presents the Einstein equations with scalar field matter in this gauge). The gauge is defined by $\widetilde{E} = 0 = \widetilde{B}$, and then $\widetilde{\phi} = \Phi$ and $\widetilde{\psi} = \Psi$. In the absence of anisotropic stress, as is the case for this work, $\Psi_1 = \Phi_1$ (though note that this does not hold true for the second order variables Φ_2 and Ψ_2). Note also that, in this gauge, $V = v$.

Energy conservation then becomes

$$\delta{\rho_2}' + 3\mathcal{H}(\delta\rho_2 + \delta P_2) + (\rho_0 + P_0)\Big(\nabla^2 v_2 - 3\Psi_2'\Big) + 2(\delta\rho_1 + \delta P_1)_{,i}{v_1}^i$$
$$+ 2(\delta\rho_1 + \delta P_1)\Big(\nabla^2 v_1 - 3\Phi_1'\Big) + 2(\rho_0 + P_0)\Big[2(v_{1,i}' + 4\mathcal{H}v_{1,i}){v_{1,}}^i$$
$$+ 3\Phi_1\Phi_1' + \nabla^2 v_1\Phi_1 - {v_{1,}}^i\Phi_{1,i}\Big] = 0\,, \tag{3.127}$$

while the momentum conservation equation is

$$\Big[(\rho_0 + P_0)v_{2,i}\Big]' + (\rho_0 + P_0)\Big(\Phi_2 + 4\mathcal{H}v_2\Big)_{,i} + \delta P_{2,i} + 2\Big[v_{1,i}(\delta\rho_1 + \delta P_1)\Big]'$$
$$+ 2(\delta\rho_1 + \delta P_1)\Big(\Phi_1 + 4\mathcal{H}v_1\Big)_{,i} - 6(\rho_0 + P_0)'\Phi_1 v_{1,i}$$
$$+ 2(\rho_0 + P_0)\Big[v_{1,i}\Big(\nabla^2 v_1 - 3\Phi_1'\Big) + {v_{1,}}^j v_{1,ij} - \Phi_1\Big(v_1' + 2\Phi_1 + 4\mathcal{H}v_1\Big)_{,i}$$
$$- 2\Big(\Phi_1 v_{1,i}\Big)' - 8\mathcal{H}\Phi_1 v_{1,i}\Big] = 0\,. \tag{3.128}$$

Then, the Einstein equations (where we here do not decompose the velocity into a scalar and divergenceless vector) are

$$3\mathcal{H}(\Psi_2' + \mathcal{H}\Phi_2) - \nabla^2\Psi_2 - 3\Phi_1'\Phi_1' - 3\Phi_{1,}{}^{i}\Phi_{1,i} - 8\nabla^2\Phi_1\Phi_1 - 12\mathcal{H}^2\Phi_1^2$$
$$= -4\pi G a^2\Big(2(\rho_0 + P_0)v_1{}^{k}v_{1k} + \delta\rho_2\Big)\,, \tag{3.129}$$

$$\Psi'_{2,i} + \mathcal{H}\Phi_{2,i} + 4(\Phi_{1,i}\Phi)' - \Phi_{1,i}(8\mathcal{H}\Phi_1 + 2\Phi_1') - 4\Phi'_{1,i}\Phi_1$$
$$= -4\pi G a^2\Big[(\rho_0 + P_0)(v_{2i} - 6\Phi_1 v_{1i}) + 2(\delta\rho_1 + \delta P_1)v_{1i}\Big]\,, \tag{3.130}$$

and

$$\Psi_2'' + \mathcal{H}(2\Psi_2 + \Phi_2)' + \frac{1}{3}\nabla^2(\Phi_2 - \Psi_2) + \Big(\frac{2a''}{a} - \mathcal{H}^2\Big)\Phi_2$$
$$+ 4\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big) - 2\Phi_{1,}{}^{i}\Phi_{1,i} - 8\mathcal{H}\Phi_1\Phi_1' - \frac{8}{3}\nabla^2\Phi_1\Phi_1 - 3(\Phi_1')^2$$
$$= 4\pi G a^2\Big(\delta P_2 + \frac{2}{3}(\rho_0 + P_0)v_1{}^{i}v_{1i}\Big)\,. \tag{3.131}$$

For completeness, we present the fourth field equation, obtained by applying the operator $\partial_i\partial^j$ to the $i-j$ component of the Einstein equations, Eq. (A.1):

$$\Psi_2'' + \mathcal{H}(2\Psi_2' + \Phi_2') + \Big(\frac{2a''}{a} - \mathcal{H}^2\Big)\Phi_2 = 4\pi G a^2\delta P_2 + 8\pi G a^2(\rho_0 + P_0)\nabla^{-2}\partial_i\partial^j(v_1{}^{i}v_{1j})$$
$$- \nabla^{-2}\Big\{2\Phi_{1,k}\Phi_{1,}{}^{k\prime} + 4\Phi_{1,}{}^{i}{}_{j}\Phi_{1,i}{}^{j} - \nabla^2\Big[\Phi_1 + \Phi_1'' + 2\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big)\Big]$$
$$+ \Phi_1'\Big(4\nabla^4\Phi_1' - 3\nabla^2\Phi_1' + 2\mathcal{H}\nabla^2\Phi_1\Big)\Big\}\,, \tag{3.132}$$

where ∇^{-2} is the inverse Laplacian operator. Finally, combining Eqs. (3.131) and (3.132), we obtain

$$\nabla^2(\Psi_2 - \Phi_2) = 24\pi G a^2(\rho_0 + P_0)\Big[v_1^i v_{1i} - \nabla^{-2}\big(\partial_i\partial^j(v^i v_j)\big)\Big] + 12\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big)$$
$$- 18\mathcal{H}\Phi_1\Phi_1' - 3\nabla^{-2}\Big\{2\Phi_{1,k}{}'\Phi_{1,}{}^{k\prime} + 4\Phi_{1,}{}^{i}{}_{j}\Phi_{1,i}{}^{j} + \Phi_1'(4\nabla^4\Phi_1' - 3\nabla^2\Phi_1' + 2\mathcal{H}\nabla^2\Phi_1)\Big\}$$
$$- 6\Phi_{1,i}\Phi_{1,}{}^{i} + \Phi_1\Phi_1'' + (\Phi_1')^2 + 2\Phi_1^2\Big(\mathcal{H}^2 - \frac{2a''}{a}\Big)\,. \tag{3.133}$$

which is the second order analogue of the equation which, at first order, tells us that the two Newtonian potentials are identical in the absence of anisotropic stress.

3.3 Discussion

In this chapter we have presented the dynamical equations for general scalar, vector and tensor perturbations of a flat FRW spacetime at both linear and second order. The case of linear perturbations is relatively simple since the different types of perturbation decouple from one another. However, as one moves beyond linear order, this is no longer the case and so things necessarily become more involved. As we have seen, at second order the energy-momentum conservation equations, for example, not only depend upon the true second order perturbations, but also involve terms quadratic in first order perturbations. So, while in this chapter we have managed to solve the linear equations analytically in, for example, the uniform curvature gauge, doing this at second order would be far more complicated. But with this complication comes great reward, since this coupling between lower order perturbations provides a source which can result in qualitatively new results, and thus new observational phenomena. In the next chapter we will discuss one such example: vorticity generation at second order in perturbation theory.

4 Vorticity

Vorticity is a common phenomenon in situations involving fluids in the 'real world' (see e.g. Refs. [2, 79]). There has also been some interest recently in studying vorticity in astrophysical scenarios, including the inter galactic medium [44, 155], but relatively little attention has been paid to the role that vorticity plays in cosmology and the early universe.

In this chapter we will consider vorticity in early universe cosmology. Starting with a summary of vorticity in classical fluid dynamics as motivation, we show that vorticity is generated by gradients in energy density and entropy. We then consider vorticity in cosmology, for which we need to use general relativity and cosmological perturbation theory. At linear order, there is no source term present in the evolution equation, and any vorticity present in the early universe will decay with the universe expansion. At second order, however, vorticity is induced by linear order perturbations. As mentioned in Chapter 1, while at first order perturbations of different types decouple, this is no longer true at higher orders. Recent work in the area of second order gravitational waves has exploited this fact [11, 13–15, 20, 114, 117, 125, 132, 148], as have recent studies of induced vector perturbations [97, 98]. Though Ref. [98] assumed the restrictive condition of adiabaticity which therefore could not source vorticity at any order, we show that, in analogy with the classical case, vorticity is sourced at second order in perturbation theory by a term quadratic in energy density and entropy (or non-adiabatic pressure) perturbations. Finally, we present a first estimate of the magnitude and scale dependence of this induced vorticity, using the expression for the energy density derived in Section 3.1.2 as an input power spectrum along with a sensible ansatz for that of the non-adiabatic pressure perturbation. We close the chapter with a discussion of the results, and highlight some potential observational consequences. The results in the chapter have been published in Refs. [38–40].

4.1 Introduction

In classical fluid dynamics the evolution of an inviscid fluid in the absence of body forces is governed by the Euler, or momentum, equation [79]

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P\,, \tag{4.1}$$

where $\boldsymbol{v}$ is the velocity vector, ρ the energy density and P the pressure of the fluid. The vorticity, $\boldsymbol{\omega}$, is a vector field and is defined as

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \boldsymbol{v}\,, \tag{4.2}$$

and can be thought of as the circulation per unit area at a point in the fluid flow.[1] An evolution equation for the vorticity can be obtained by taking the curl of Eq. (4.1), which gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) + \frac{1}{\rho^2}\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P\,. \tag{4.3}$$

The second term on the right hand side of Eq. (4.3), often called the baroclinic term in the literature, then acts as a source for the vorticity. Evidently, this term vanishes if lines of constant energy and pressure are parallel, or if the energy density or pressure are constant. A special class of fluid for which the former is true is a barotropic fluid, defined such that the equation of state is a function of the energy density only, i.e. $P \equiv P(\rho)$, and so $(1/\rho^2)\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P = 0$.

For a barotropic fluid, the vorticity evolution equation, Eq. (4.3), can then be written, by using vector calculus identities, as

$$\frac{D\boldsymbol{\omega}}{Dt} \equiv \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\omega}(\boldsymbol{\nabla} \cdot \boldsymbol{v})\,, \tag{4.4}$$

where D/Dt denotes the convective, or material derivative, which is commonly used in fluid dynamics. From Eq. (4.4) it is clear that the vorticity vector has no source, in this case, and so $\boldsymbol{\omega} = \boldsymbol{0}$ is a solution.

[1] The circulation, Γ, is defined as

$$\Gamma = \oint_C \boldsymbol{v} \cdot \boldsymbol{dl}\,,$$

where C is the boundary of the surface S. Then, using Stokes theorem, this becomes

$$\Gamma = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \boldsymbol{dS}\,,$$

which gives the result that the vorticity is the circulation per unit area at a point in the fluid flow. □

Allowing for a more general perfect fluid with an equation of state depending not only on the energy density, but of the form $P \equiv P(S, \rho)$ will mean that, in general, the baroclinic term is no longer vanishing, and so acts as a source for the evolution of the vorticity. This is Crocco's theorem [43] which states that vorticity generation is sourced by gradients in entropy in classical fluid dynamics.

4.2 Vorticity in Cosmology

In General Relativity, the vorticity tensor is defined as the projected anti symmetrised covariant derivative of the fluid four velocity, that is [75]

$$\omega_{\mu\nu} = \mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta u_{[\alpha;\beta]} \,, \tag{4.5}$$

where the projection tensor $\mathcal{P}_{\mu\nu}$ into the instantaneous fluid rest space is given by

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu . \tag{4.6}$$

Note that, in analogy with the classical case, it is possible to define a vorticity vector as $\omega_\mu = \frac{1}{2}\varepsilon_{\mu\nu\gamma}\omega^{\nu\gamma}$, where $\varepsilon_{\mu\nu\gamma} \equiv u^\delta \varepsilon_{\delta\mu\nu\gamma}$ is the covariant permutation tensor in the fluid rest space (see Refs. [62, 98]). However, since this is less general we choose to work with the vorticity tensor when deriving the equations, and only switch to using the vorticity vector when solving the equation in Section 4.4.

The vorticity tensor can then be decomposed in the usual way, up to second order in perturbation theory, as $\omega_{ij} \equiv \omega_{1ij} + \frac{1}{2}\omega_{2ij}$. Working in the uniform curvature gauge, and considering only scalar and vector perturbations, we can obtain the components of the vorticity tensor by substituting the expressions for the fluid four velocity, Eq. (2.23), along with the metric tensor into Eq. (4.5). At first order this gives us

$$\omega_{1ij} = aV_{1[i,j]} \,, \tag{4.7}$$

and at second order

$$\omega_{2ij} = aV_{2[i,j]} + 2a \left[V'_{1[i}V_{1j]} + \phi_{1,[i}\left(V_1 + B_1\right)_{j]} - \phi_1 B_{1[i,j]} \right] . \tag{4.8}$$

The first order vorticity is gauge invariant. In order to see this we recall, from Eq. (2.94), that V_{1i} transforms under a gauge transformation as

$$\widetilde{V_{1i}} = V_{1i} - \alpha_{1,i} \,, \tag{4.9}$$

so the first order vorticity transforms as

$$\widetilde{\omega_{1ij}} = a\widetilde{V_{1[i,j]}} = a(V_{1i,j} - \alpha_{1,ij} - V_{1j,i} + \alpha_{1,ji}) = \omega_{1ij}\,, \tag{4.10}$$

and is therefore gauge invariant.

4.3 Vorticity Evolution

In order to obtain an evolution equation for the vorticity at first order, we take the time derivative of Eq. (4.7) to get

$$\omega'_{1ij} = a'V_{1[i,j]} + aV'_{1[i,j]}\,. \tag{4.11}$$

Noting that, from Eq. (3.24),

$$V'_{1[i,j]} + \mathcal{H}(1-3c_{\rm s}^2)V_{1[i,j]} + \left[\frac{\delta P_1}{\rho_0 + P_0} + \phi_1\right]_{,[ij]} = 0\,, \tag{4.12}$$

which gives

$$V'_{1[i,j]} = -\mathcal{H}(1-3c_{\rm s}^2)V_{1[i,j]} = -\frac{1}{a}\mathcal{H}(1-3c_{\rm s}^2)\omega_{1ij}\,, \tag{4.13}$$

and so, from Eq. (4.11),

$$\omega'_{1ij} - 3\mathcal{H}c_{\rm s}^2\omega_{1ij} = 0\,. \tag{4.14}$$

This reproduces the well known result that, during radiation domination, $|\omega_{1ij}\omega_1^{ij}| \propto a^{-2}$ in the absence of an anisotropic stress term [75].

At second order things get somewhat more complicated. We now take the time derivative of Eq. (4.8), to give

$$\begin{aligned}\omega'_{2ij} = a'V_{2[i,j]} + aV_{2[i,j]} + 2a'\Big[V'_{1[i}V_{1j]} + \phi_{1,[i}\left(V_1 + B_1\right)_{j]} - \phi_1 B_{1[i,j]}\Big]\\ + 2a\Big[V''_{1[i}V_{1j]} + V'_{1[i}V'_{1j]} + \phi'_{1,[i}\left(V_1 + B_1\right)_{j]} + \phi_{1,[i}\left(V_1 + B_1\right)'_{j]}\\ - \phi'_1 B_{1[i,j]} - \phi_1 B'_{1[i,j]}\Big]\,.\end{aligned} \tag{4.15}$$

Therefore we now must use the first order conservation and field equations to eliminate the first order metric perturbations as well as the second order conservation equations in order to eliminate the second order metric perturbation variables. This process involves simple algebra, but is very tedious and so we omit the intermediate steps and instead quote the result. We arrive at the evolution equation for the second order vorticity

$$
\begin{aligned}
&\omega_{2ij}' - 3\mathcal{H}c_{\rm s}^2\omega_{2ij} + 2\left[\left(\frac{\delta P_1 + \delta\rho_1}{\rho_0 + P_0}\right)' + V_{1,k}^k - X_{1,k}^k\right]\omega_{1ij} \\
&+ 2\left(V_1^k - X_1^k\right)\omega_{1ij,k} - 2\left(X_{1,j}^k - V_{1,j}^k\right)\omega_{1ik} + 2\left(X_{1,i}^k - V_{1,i}^k\right)\omega_{1jk} \\
&= \frac{a}{\rho_0 + P_0}\Big\{3\mathcal{H}\left(V_{1i}\delta P_{{\rm nad}1,j} - V_{1j}\delta P_{{\rm nad}1,i}\right) \\
&\qquad\qquad + \frac{1}{\rho_0 + P_0}\left(\delta\rho_{1,j}\delta P_{{\rm nad}1,i} - \delta\rho_{1,i}\delta P_{{\rm nad}1,j}\right)\Big\}\,,
\end{aligned}
\tag{4.16}
$$

where X_{1i} is given entirely in terms of matter perturbations as

$$
X_{1i} = \nabla^{-2}\left[\frac{4\pi G a^2}{\mathcal{H}}\left(\delta\rho_{1,i} - \mathcal{H}(\rho_0 + P_0)V_{1i}\right)\right]\,. \tag{4.17}
$$

Eq. (4.16) then shows that the second order vorticity is sourced by terms quadratic in linear order perturbations.

In fact, even assuming zero first order vorticity, i.e. $\omega_{1ij} = 0$, the second order vorticity evolves as

$$
\omega_{2ij}' - 3\mathcal{H}c_{\rm s}^2\omega_{2ij} = \frac{2a}{\rho_0 + P_0}\left\{3\mathcal{H}V_{1[i}\delta P_{{\rm nad}1,j]} + \frac{\delta\rho_{1,[j}\delta P_{{\rm nad}1,i]}}{\rho_0 + P_0}\right\}\,, \tag{4.18}
$$

and so we see that there is a non zero source term for the vorticity at second order in perturbation theory which is, in analogy with classical fluid dynamics, made up of gradients in entropy and density perturbations. Note that, in the absence of a non-adiabatic pressure perturbation, we recover the result of Ref. [98] that there is no vorticity generation.

4.4 Solving the Vorticity Evolution Equation

Having derived an evolution equation for the second order vorticity in the previous section, we now seek an analytic solution to this equation (or, more precisely, the power spectrum of the vorticity governed by this evolution equation), using suitable, realistic approximations for the input power spectra.

4.4.1 The Vorticity Power Spectrum

In order to keep our results conservative and our calculation as simple as possible and hence analytically tractable, we assume that the source term in Eq. (4.18) is dominated by the second term.[2] Then, choosing the radiation era as our background in which $c_{\mathrm{s}}^2 = 1/3$, the evolution equation simplifies to

$$\omega_{ij}' - \mathcal{H}\omega_{ij} = \frac{9a}{8\rho_0^2}\delta\rho_{,[j}\delta P_{\mathrm{nad},i]}\,, \tag{4.19}$$

where we note that, for the remainder of this Chapter, we omit the subscripts denoting the order of the perturbation in order to avoid notational ambiguities, and to keep the expressions as compact and clear as possible: the vorticity is a second order quantity and the energy density and non-adiabatic pressure perturbations are first order quantities. We define the right hand side of Eq. (4.19) to be the source term,

$$S_{ij}(\boldsymbol{x},\eta) \equiv \frac{9a(\eta)}{8\rho_0(\eta)^2}\delta\rho_{,[j}\delta P_{\mathrm{nad},i]}\,, \tag{4.20}$$

which on defining the function $f(\eta)$ as

$$f(\eta) = \frac{9a}{16\rho_0^2}\,, \tag{4.21}$$

is written, for convenience, as

$$S_{ij}(\boldsymbol{x},\eta) \equiv 2f(\eta)\delta\rho_{,[j}\delta P_{\mathrm{nad},i]}\,. \tag{4.22}$$

Since we want to solve the evolution equation we move to Fourier space, in which the source term becomes the convolution integral

$$S_{ij}(\boldsymbol{k},\eta) = -\frac{f(\eta)}{(2\pi)^{3/2}}\int d^3\tilde{k}(\tilde{k}_i k_j - \tilde{k}_j k_i)\delta P_{\mathrm{nad}}(\tilde{\boldsymbol{k}},\eta)\delta\rho_1(\boldsymbol{k}-\tilde{\boldsymbol{k}})\,, \tag{4.23}$$

where $\boldsymbol{k}$ is the wavevector, as usual. Instead of considering the vorticity tensor ω_{ij}, it is easier, and more natural in this case, to work, in analogy with the classical case,

[2] This is a reasonable assumptions since we are working on small scales and the first term has a prefactor $1/k^2$.

with the vorticity vector. This is defined as[3]

$$\omega_i(\boldsymbol{x},\eta) = \epsilon_{ijk}\omega^{jk}(\boldsymbol{x},\eta)\,, \tag{4.24}$$

where ϵ_{ijk} is the totally antisymmetric tensor, and we can define a source vector in an analogous way:

$$S_i(\boldsymbol{x},\eta) = \epsilon_{ijk}S^{jk}(\boldsymbol{x},\eta)\,. \tag{4.25}$$

Note that the vorticity is an axial vector (that is, it arises from the generalisation of the cross product), and so both ω_i and S_i are pseudovectors. Thus, under the transformation of the argument $\boldsymbol{x} \to -\boldsymbol{x}$ the vector vorticity transforms as $\omega_i \to -\omega_i$, and similarly for S_i. The source vector is then Fourier transformed as

$$S_i(\boldsymbol{x},\eta) = \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}\, S_i(\boldsymbol{k},\eta)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \tag{4.26}$$

with the source vector in Fourier space then being split up as

$$S_i(\boldsymbol{k},\eta) = \mathcal{S}_A(\boldsymbol{k},\eta)e_i^A\,, \tag{4.27}$$

where $\mathcal{S}_A$ are the amplitudes, with $A \in \{1,2,3\}$, and the basis vectors are

$$\left\{e_i^1, e_i^2, e_i^3\right\} = \left\{e_i(\boldsymbol{k}), \bar{e}_i(\boldsymbol{k}), \frac{k_i}{|\boldsymbol{k}|} \equiv \hat{k}_i\right\}\,, \tag{4.28}$$

In order to keep a right handed orthonormal basis under the sign reversal of $\boldsymbol{k}$ ($\boldsymbol{k} \to -\boldsymbol{k}$), the basis vectors must obey

$$e_i(-\boldsymbol{k}) = e_i(\boldsymbol{k})\,, \tag{4.29}$$

$$\bar{e}_i(-\boldsymbol{k}) = -\bar{e}_i(\boldsymbol{k})\,; \tag{4.30}$$

the basis vectors are also cyclic:

$$\epsilon_{ijk}\, e_1^j e_2^k = e_{3i}\,. \tag{4.31}$$

Given these definitions, the evolution equation, Eq. (4.19), can then be written as

$$\omega_A'(\boldsymbol{k},\eta) - \mathcal{H}\omega_A(\boldsymbol{k},\eta) = \mathcal{S}_A(\boldsymbol{k},\eta)\,, \tag{4.32}$$

[3]Though strictly this is a covector not a vector, since we are working in a flat background the two are equivalent up to raising or lowering of indices by the Kronecker delta. Therefore, we are slightly loose with terminology here, and do not differentiate between the two.

for each basis state, A (though we omit the subscript in the next few lines, since the evolution equations is the same for each polarisation). The left hand side of Eq. (4.32) can be expressed as an exact derivative, giving

$$a\left(\frac{\omega(\boldsymbol{k},\eta)}{a}\right)' = \mathcal{S}(\boldsymbol{k},\eta)\,, \tag{4.33}$$

which, in radiation domination when $a = \eta$, becomes

$$\left(\frac{\omega(\boldsymbol{k},\eta)}{\eta}\right)' = \eta^{-1}\mathcal{S}(\boldsymbol{k},\eta)\,. \tag{4.34}$$

This can then be integrated to give

$$\omega(\boldsymbol{k},\eta) = \eta\int_{\eta_0}^{\eta}\tilde{\eta}^{-1}\mathcal{S}(\boldsymbol{k},\tilde{\eta})d\tilde{\eta}\,, \tag{4.35}$$

for some initial time η_0. Having solved the temporal evolution of the vorticity, we now move on to considering the power spectrum. In analogy with the standard case for scalar perturbations, we define the power spectrum of the vorticity $\mathcal{P}_\omega$ as

$$\langle\omega^*(\boldsymbol{k}_1,\eta)\omega(\boldsymbol{k}_2,\eta)\rangle = \frac{2\pi}{k^3}\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)\mathcal{P}_\omega(k,\eta)\,, \tag{4.36}$$

where here the star denotes the complex conjugate, and $k = |\boldsymbol{k}|$ is the wavenumber, as usual. On substituting Eq. (4.35) into Eq. (4.36), we can write the correlator for the vorticity as

$$\langle\omega^*(\boldsymbol{k}_1,\eta)\omega(\boldsymbol{k}_2,\eta)\rangle = \eta^2\int_{\eta_0}^{\eta}\tilde{\eta}_1^{-1}\int_{\eta_0}^{\eta}\tilde{\eta}_2^{-1}\left\langle\mathcal{S}^*(\boldsymbol{k}_1,\tilde{\eta}_1)\mathcal{S}(\boldsymbol{k}_2,\tilde{\eta}_2)\right\rangle d\tilde{\eta}_1 d\tilde{\eta}_2\,, \tag{4.37}$$

and so, in order to obtain the vorticity power spectrum, we must calculate the correlator of the source term. Thus, we need to consider how the Fourier amplitudes, $\mathcal{S}_A(\boldsymbol{k},\eta)$, behave under complex conjugation. From Eqs. (4.26) and (4.27) we can write

$$S_i(\boldsymbol{x},\eta) = \frac{1}{(2\pi)^{3/2}}\int d^3k\Big[\mathcal{S}_1(\boldsymbol{k},\eta)\,e_i(\boldsymbol{k}) + \mathcal{S}_2(\boldsymbol{k},\eta)\bar{e}_i(\boldsymbol{k}) + \mathcal{S}_3(\boldsymbol{k},\eta)\hat{k}_i\Big]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \tag{4.38}$$

whose complex conjugate is then

$$S_i(\boldsymbol{x},\eta) = \frac{1}{(2\pi)^{3/2}} \int d^3k \Big[\mathcal{S}_1^*(\boldsymbol{k},\eta)\, e_i(\boldsymbol{k}) + \mathcal{S}_2^*(\boldsymbol{k},\eta) \bar{e}_i(\boldsymbol{k}) + \mathcal{S}_3^*(\boldsymbol{k},\eta) \hat{k}_i \Big] \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \,. \tag{4.39}$$

Alternatively, under the change $\boldsymbol{k} \to -\boldsymbol{k}$, on which $S_i \to -S_i$, as mentioned above, Eq. (4.38) becomes

$$\begin{aligned} S_i(\boldsymbol{x},\eta) = \frac{1}{(2\pi)^{3/2}} \int d^3k \Big[&- \mathcal{S}_1(-\boldsymbol{k},\eta)\, e_i(-\boldsymbol{k}) - \mathcal{S}_2(-\boldsymbol{k},\eta) \bar{e}_i(-\boldsymbol{k}) \\ &- \mathcal{S}_3(-\boldsymbol{k},\eta) \cdot (-\hat{k}_i) \Big] \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \,. \end{aligned} \tag{4.40}$$

Comparing Eqs. (4.39) and (4.40), we can then read off the relationship between the Fourier amplitudes and their conjugates:

$$\mathcal{S}_1^*(\boldsymbol{k},\eta) = -\mathcal{S}_1(-\boldsymbol{k},\eta) \,, \tag{4.41}$$

$$\mathcal{S}_2^*(\boldsymbol{k},\eta) = \mathcal{S}_2(-\boldsymbol{k},\eta) \,, \tag{4.42}$$

$$\mathcal{S}_3^*(\boldsymbol{k},\eta) = \mathcal{S}_3(-\boldsymbol{k},\eta) \,. \tag{4.43}$$

Then, from the definition of the vector source term, we obtain the Fourier amplitudes

$$\mathcal{S}_1(\boldsymbol{k},\eta) = -\frac{f(\eta)}{(2\pi)^{3/2}} \int d^3\tilde{\boldsymbol{k}} \cdot 2k \bar{e}_i \tilde{k}^i \delta P_{\mathrm{nad}}(\boldsymbol{k},\eta) \delta\rho_1(\boldsymbol{k}-\tilde{\boldsymbol{k}},\eta) \,, \tag{4.44}$$

$$\mathcal{S}_2(\boldsymbol{k},\eta) = \frac{f(\eta)}{(2\pi)^{3/2}} \int d^3\tilde{\boldsymbol{k}} \cdot 2k e_i \tilde{k}^i \delta P_{\mathrm{nad}}(\boldsymbol{k},\eta) \delta\rho_1(\boldsymbol{k}-\tilde{\boldsymbol{k}},\eta) \,, \tag{4.45}$$

$$\mathcal{S}_3(\boldsymbol{k},\eta) = 0 \,. \tag{4.46}$$

This last equation tells us that the projection of Eq. (4.38) onto the basis vector k_i gives zero (since contracting Eq. (4.38) with k_i contracts two copies of k with the permutation symbol, automatically giving zero). The complex conjugates are

$$\mathcal{S}_1^*(\boldsymbol{k},\eta) = -\frac{f(\eta)}{(2\pi)^{3/2}} \int d^3\tilde{\boldsymbol{k}} 2k \bar{e}_i \tilde{k}^i \delta P_{\mathrm{nad}}(\boldsymbol{k},\eta) \delta\rho(-(\boldsymbol{k}+\tilde{\boldsymbol{k}}),\eta) \,, \tag{4.47}$$

$$\mathcal{S}_2^*(\boldsymbol{k},\eta) = \frac{f(\eta)}{(2\pi)^{3/2}} \int d^3\tilde{\boldsymbol{k}} 2k e_i \tilde{k}^i \delta P_{\mathrm{nad}}(\boldsymbol{k},\eta) \delta\rho(-(\boldsymbol{k}+\tilde{\boldsymbol{k}}),\eta) \,. \tag{4.48}$$

Having now obtained the required amplitudes and their conjugates, we can compute the correlator of the source terms for the $A = 1$ mode in Eq. (4.37).[4] Assuming that the fluctuations $\delta\rho$ and $\delta P_{\rm nad}$ are Gaussian, we can put the directional dependence into Gaussian random variables $\hat{E}(\boldsymbol{k})$, which obey the relationships

$$\left\langle \hat{E}^*(\boldsymbol{k}_1)\hat{E}(\boldsymbol{k}_2)\right\rangle = \delta^3(\boldsymbol{k}_1 - \boldsymbol{k}_2)\,, \qquad \left\langle \hat{E}(\boldsymbol{k}_1)\hat{E}(\boldsymbol{k}_2)\right\rangle = \delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2)\,, \tag{4.49}$$

and write, for example,[5]

$$\delta\rho(\boldsymbol{k},\eta) = \delta\rho(k,\eta)\hat{E}(\boldsymbol{k})\,. \tag{4.50}$$

The correlator then becomes

$$\begin{aligned}\langle \mathcal{S}^*(\boldsymbol{k}_1,\tilde{\eta}_1)\mathcal{S}(\boldsymbol{k}_2,\tilde{\eta}_2)\rangle = \frac{f_{\tilde{1}}f_{\tilde{2}}}{(2\pi)^3}&\int d^3\tilde{k}_1 2k_1\tilde{k}_{1i}\bar{e}^i_1\delta P_{\rm nad}(\tilde{k}_1,\tilde{\eta}_1)\delta\rho(|\boldsymbol{k}_1+\tilde{\boldsymbol{k}}_1|,\tilde{\eta}_1)\\ &\times\int d^3\tilde{k}_2 2k_2\tilde{k}_{2i}\bar{e}^i_2\delta P_{\rm nad}(\tilde{k}_2,\tilde{\eta}_2)\delta\rho(|\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2|,\tilde{\eta}_2)\\ &\times\left\langle \hat{E}(-\tilde{\boldsymbol{k}}_1)\hat{E}(-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1)\hat{E}(\tilde{\boldsymbol{k}}_2)\hat{E}(\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2)\right\rangle\,,\end{aligned} \tag{4.51}$$

where we have introduced the notation $f_{\tilde{1}} \equiv f(\tilde{\eta}_1)$. Wick's theorem (see, e.g., Ref. [46]) then allows us to express the correlator in the above in terms of delta functions as

$$\begin{aligned}\left\langle \hat{E}(\tilde{\boldsymbol{k}}_1)\hat{E}(-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1)\hat{E}(\tilde{\boldsymbol{k}}_2)\hat{E}(\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2)\right\rangle &= \left\langle \hat{E}(\tilde{\boldsymbol{k}}_1)\hat{E}(-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1)\right\rangle\left\langle \hat{E}(\tilde{\boldsymbol{k}}_2)\hat{E}(\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2)\right\rangle\\ &+\left\langle \hat{E}(\tilde{\boldsymbol{k}}_1)\hat{E}(\tilde{\boldsymbol{k}}_2)\right\rangle\left\langle \hat{E}(-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1)\hat{E}(\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2)\right\rangle\\ &+\left\langle \hat{E}(\tilde{\boldsymbol{k}}_1)\hat{E}(\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2)\right\rangle\left\langle \hat{E}(-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1)\hat{E}(\tilde{\boldsymbol{k}}_2)\right\rangle\\ = \delta^3(\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{k}}_2)\delta^3(-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1+\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2) &+ \delta^3(\tilde{\boldsymbol{k}}_1+\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2)\delta^3(-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{k}}_2)\,,\end{aligned} \tag{4.52}$$

[4] Note that we need only consider one orthogonal component of the source vector, since we can make an appropriate choice such that its component in one direction is zero.

[5] In making this choice, we are assuming that $\delta\rho$ and $\delta P_{\rm nad}$ are completely correlated variables. This is perhaps not the most physically motivated assumption, since one might expect some level of decorrelation between the two variables. However, this assumption will likely give the largest signal (the partially decorrelated case will, in its simplest form, require a new parameter less than one, which characterises how decorrelated the two variables are – this parameter can be determined from the specific model for the production of the non-adiabatic pressure perturbation), so is suitable as a first approximation. We leave the case where the two variables are decorrelated for future work.

which gives

$$\langle \mathcal{S}^*(\boldsymbol{k}_1,\tilde{\eta}_1)\mathcal{S}(\boldsymbol{k}_2,\tilde{\eta}_2)\rangle = \frac{f_{\tilde{1}}f_{\tilde{2}}}{2\pi^3}\int d^3\tilde{k}_1\int d^3\tilde{k}_2\Big\{k_1\bar{e}_i\tilde{k}^i\delta P_{\rm nad}(\tilde{k}_1,\tilde{\eta}_1)\delta\rho(|\boldsymbol{k}_1+\tilde{\boldsymbol{k}}_1|,\tilde{\eta}_1)$$
$$\times\, k_2\bar{e}_i\tilde{k}_2^i\delta P_{\rm nad}(\tilde{k}_2,\tilde{\eta}_2)\delta\rho(|\boldsymbol{k}_2-\tilde{\boldsymbol{k}}_2|,\tilde{\eta}_2)\Big\} \tag{4.53}$$
$$\times\left\{\delta^3(\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{k}}_2)\delta^3(\boldsymbol{k}_2-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1-\tilde{\boldsymbol{k}}_2)+\delta^3(\tilde{\boldsymbol{k}}_1-\tilde{\boldsymbol{k}}_2+\boldsymbol{k}_2)\delta^3(\tilde{\boldsymbol{k}}_2-\boldsymbol{k}_1-\tilde{\boldsymbol{k}}_1)\right\}\,. \tag{4.54}$$

By integrating over the delta functions, and collecting terms, we arrive at

$$\langle \mathcal{S}^*(\boldsymbol{k}_1,\tilde{\eta}_1)\mathcal{S}(\boldsymbol{k}_2,\tilde{\eta}_2)\rangle = \frac{f_{\tilde{1}}f_{\tilde{2}}}{2\pi^3}\delta^3(\boldsymbol{k}_2-\boldsymbol{k}_1)k^2\int d^3\tilde{k}(\bar{e}_i\tilde{k}^i)^2\delta P_{\rm nad}(\tilde{k},\tilde{\eta}_1)\delta\rho_1(|\boldsymbol{k}+\tilde{\boldsymbol{k}}|,\tilde{\eta}_1)$$
$$\times\left[\delta P_{\rm nad}(|\boldsymbol{k}+\tilde{\boldsymbol{k}}|,\tilde{\eta}_2)\delta\rho_1(\tilde{k},\tilde{\eta}_2)-\delta P_{\rm nad}(\tilde{k},\tilde{\eta}_2)\delta\rho_1(|\boldsymbol{k}+\tilde{\boldsymbol{k}}|,\tilde{\eta}_2)\right]\,, \tag{4.55}$$

from which we can read off the power spectrum:

$$\mathcal{P}_\omega(k,\eta) = \frac{k^5\eta^2}{4\pi^4}\int\int f_{\tilde{1}}f_{\tilde{2}}\tilde{\eta}_1^{-1}\tilde{\eta}_2^{-1}d\tilde{\eta}_1 d\tilde{\eta}_2\int d^3\tilde{k}(\bar{e}_i\tilde{k}^i)^2\delta P_{\rm nad}(\tilde{k},\tilde{\eta}_1)\delta\rho_1(|\boldsymbol{k}+\tilde{\boldsymbol{k}}|,\tilde{\eta}_1)$$
$$\times\left[\delta P_{\rm nad}(|\boldsymbol{k}+\tilde{\boldsymbol{k}}|,\tilde{\eta}_2)\delta\rho_1(\tilde{k},\tilde{\eta}_2)-\delta P_{\rm nad}(\tilde{k},\tilde{\eta}_2)\delta\rho_1(|\boldsymbol{k}+\tilde{\boldsymbol{k}}|,\tilde{\eta}_2)\right]\,. \tag{4.56}$$

Now, as a first approximation for the source term, we can expand Eq. (3.42) to lowest order in $k\eta$ to give

$$\delta\rho(k,\eta) = \bar{A}k^\beta\,\eta^{-4}\,, \tag{4.57}$$
$$\delta P_{\rm nad}(k,\eta) = \bar{D}k^\alpha\,\eta^{-5}\,, \tag{4.58}$$

where $\bar{A}$ and $\bar{D}$ are yet unspecified amplitudes, and α and β undetermined powers. Using these approximations gives

$$\mathcal{P}_\omega(k,\eta) = \frac{81}{256}\frac{k^5\eta^2}{4\pi^4}(AD)^2\left[\ln\left(\frac{\eta}{\eta_0}\right)\right]^2$$
$$\times\int d^3\tilde{k}(\bar{e}_i\tilde{k}^i)^2\tilde{k}^\alpha|\boldsymbol{k}+\tilde{\boldsymbol{k}}|^\beta\Big(|\boldsymbol{k}+\tilde{\boldsymbol{k}}|^\alpha\tilde{k}^\beta-\tilde{k}^\alpha|\boldsymbol{k}+\tilde{\boldsymbol{k}}|^\beta\Big) \tag{4.59}$$

where we have performed the temporal integral by noting that as mentioned above, $a\propto\eta$ during radiation domination, and thus $\rho_0\propto\eta^{-4}$. To perform the k-space integral, we first move to spherical coordinates oriented with the axis in the direction of $\boldsymbol{k}$. Then, denoting the angle between $\boldsymbol{k}$ and $\tilde{\boldsymbol{k}}$ as θ, the integral can be transformed as

$$\int d^3\tilde{k} \to 2\pi \int_0^{k_c} \tilde{k}^2 d\tilde{k} \int_0^{\pi} \sin\theta d\theta \,, \tag{4.60}$$

where the prefactor comes from the fact that the integrand has no dependence on the azimuthal angle, and k_c denotes a cut-off on small scales. This cut-off is chosen to be smaller than the typical separation of galaxies, and therefore much smaller than the continuum limit, solely for the purpose of studying this toy model. Noting that, in this coordinate system $\tilde{k}_i \bar{e}^i = \tilde{k}\sin\theta$ the integral in Eq. (4.59) becomes

$$\begin{aligned} I(k) = 2\pi \int_0^{k_c} \int_0^{\pi} & \tilde{k}^{4+\alpha} \sin\theta \, d\theta \, d\tilde{k} \, \sin^2\theta k^{\beta} \Big[1 + (\tilde{k}/k)^2 + 2(\tilde{k}/k)\cos\theta\Big]^{\beta/2} \\ & \times \Big(k^{\alpha}(1 + (\tilde{k}/k)^2 + 2(\tilde{k}/k)\cos\theta)^{\alpha/2}\tilde{k}^{\beta} - \tilde{k}^{\alpha}k^{\beta}(1 + (\tilde{k}/k)^2 + 2(\tilde{k}/k)\cos\theta)^{\beta/2}\Big) \end{aligned} \tag{4.61}$$

Finally, we change variables again to dimensionless u and v defined as [11] (or similarly [27])

$$v = \frac{\tilde{k}}{k} \,, \; u = \sqrt{1 + (\tilde{k}/k)^2 + 2(\tilde{k}/k)\cos\theta} = \sqrt{1 + v^2 + 2v\cos\theta} \,, \tag{4.62}$$

for which the integral (4.61) becomes

$$I(k) = k^{2(\alpha+\beta)+5} \int_0^{k_c/k} \int_{|v-1|}^{v+1} u \, du \, v^3 \, dv \, u^{\beta} v^{\alpha} \Big(1 - \frac{1}{4v^2}(u^2 - v^2 - 1)^2\Big) \Big[u^{\alpha}v^{\beta} - v^{\alpha}u^{\beta}\Big] \,. \tag{4.63}$$

4.4.2 Evaluating the Vorticity Power Spectrum

In order to perform the integral Eq. (4.63) derived above, we need to specify the exponents for the power spectra of the energy density and the non-adiabatic pressure α and β. The energy density perturbation on slices of uniform curvature can be related to the curvature perturbation on uniform density hypersurfaces, ζ, during radiation domination through [112]

$$\delta\rho = -\frac{\rho_0'}{\mathcal{H}}\zeta = 4\rho_0\zeta \,, \tag{4.64}$$

and hence the initial power spectra can be related as $\langle\delta\rho_{\rm ini}\delta\rho_{\rm ini}\rangle = 16\rho_{0\gamma{\rm ini}}^2\langle\zeta_{\rm ini}\zeta_{\rm ini}\rangle$, and we get the power spectrum of the initial density perturbation

$$\delta\rho_{\rm ini} \propto \zeta_{\rm ini} \propto \left(\frac{k}{k_0}\right)^{\frac{1}{2}(n_{\rm s}-1)}, \tag{4.65}$$

where k_0 is the WMAP pivot scale and $n_{\rm s}$ the spectral index of the primordial curvature perturbation [77]. This allows us to relate our ansatz for the density perturbation, to the WMAP-data which gives

$$\delta\rho = \delta\rho_{\rm ini}\left(\frac{k}{k_0}\right)\left(\frac{\eta}{\eta_0}\right)^{-4} = A_{\rm ini}\rho_{0\rm ini}\left(\frac{k}{k_0}\right)^{\frac{1}{2}(n_{\rm s}+1)}\left(\frac{\eta}{\eta_0}\right)^{-4}. \tag{4.66}$$

From this, we can read off that $\beta = \frac{1}{2}(n_{\rm s}+1) \simeq 1$ and the amplitude $A = A_{\rm ini}\rho_{0\rm ini}$. We have some freedom in choosing α, however would expect the non-adiabatic pressure to have a blue spectrum, though the calculation demands $\alpha \neq \beta$. Using the notation of Ref. [77] we get

$$A^2 = \rho_{0\rm ini}^2 P_{\mathcal{R}}(k_0)^2 = \rho_{0\rm ini}^2 k_0^{-6}\Delta_{\mathcal{R}}^4\,, \qquad D^2 = \rho_{0\rm ini}^2 P_{\mathcal{S}}(k_0)^2 = \rho_{0\rm ini}^2 k_0^{-6}\Delta_{\mathcal{S}}^4 \tag{4.67}$$

where we also have the ratio[6]

$$\frac{\Delta_{\mathcal{S}}^2}{\Delta_{\mathcal{R}}^2} = \frac{\alpha(k_0)}{1-\alpha(k_0)}\,, \tag{4.68}$$

and therefore,

$$(AD)^2 = \frac{\alpha(k_0)}{1-\alpha(k_0)}\Delta_{\mathcal{R}}^8\rho_{0\rm ini}^4 k_0^{-12}\,. \tag{4.69}$$

We can then substitute in numerical values for $\Delta_{\mathcal{R}}^2$ and $\alpha(k_0)$ from Ref. [77] later on.

Then, making the choice $\alpha = 2$, the input power spectra are

$$\delta\rho(k,\eta) = A\left(\frac{k}{k_0}\right)\left(\frac{\eta}{\eta_0}\right)^{-4}, \qquad \delta P_{\rm nad}(k,\eta) = D\left(\frac{k}{k_0}\right)^2\left(\frac{\eta}{\eta_0}\right)^{-5}, \tag{4.70}$$

for which the integral Eq. (4.63) becomes

$$I(k) = k^{11}\int_0^{k_c/k}\int_{|v-1|}^{v+1} u^2\,du\,v^5\,dv\left(1-\frac{1}{4v^2}(u^2-v^2-1)^2\right)\left[u^2v - v^2u\right]. \tag{4.71}$$

[6]The parameter $\alpha(k_0)$ is introduced in Refs. [22, 77] in order to quantify the ratio of $\Delta_{\mathcal{S}}^2$ to $\Delta_{\mathcal{R}}^2$.

We can then integrate this analytically to give

$$I(k) = \frac{16}{135} k_{\rm c}^9 k^2 + \frac{12}{245} k_{\rm c}^7 k^4 - \frac{4}{1575} k_{\rm c}^5 k^6 \,, \tag{4.72}$$

which clearly depends upon the small scale cut-off, as expected.[7] For illustrative purposes, we choose $k_{\rm c} = 10{\rm Mpc}^{-1}$ and plot the solution $I(k)$. Fig. 4.1 shows that the amplitude of the integral grows as the wavenumber increases. Fig. 4.2 shows a turn around and a decrease in power at some wavenumber (in fact, for a non-specific cutoff, this point is at $3.7375 k_{\rm c}$). However, we note that this value is greater than our cutoff, and therefore not physical. As long as we consider values of k less than the cutoff, our approximation will still be valid.

Then, using the above, and noting that the input to the temporal integrals are

$$a \propto \left(\frac{\eta}{\eta_0}\right) \,, \qquad \rho_0 = \rho_{0{\rm ini}} \left(\frac{\eta}{\eta_0}\right)^{-4} \,, \tag{4.73}$$

we obtain the power spectrum for the vorticity, for general $k_{\rm c}$ as

$$\mathcal{P}_\omega(k,\eta) = \frac{81}{256} k_0^5 \frac{\eta^2}{4\pi^4} \frac{1}{\rho_{0{\rm ini}}^4} \ln^2\left(\frac{\eta}{\eta_0}\right) \left(\frac{\alpha(k_0)}{1-\alpha(k_0)}\right)^2 k_0^{-12} \Delta_{\mathcal{R}}^8 \rho_{0{\rm ini}}^4 k_{\rm c}^5 \times \left[\frac{16}{135}\frac{k_{\rm c}^4}{k_0^4}\left(\frac{k}{k_0}\right)^7 + \frac{12}{245}\frac{k_{\rm c}^2}{k_0^2}\left(\frac{k}{k_0}\right)^9 - \frac{4}{1575}\left(\frac{k}{k_0}\right)^{11}\right] . \tag{4.74}$$

Substituting in values of parameters from Ref. [77], as presented in Table 4.1 taking a conservative estimate for $\alpha(k_0)$, being 10% of the upper bound as reported by WMAP7, we obtain the vorticity power spectrum

$$\mathcal{P}_\omega(k,\eta) = \eta^2 \ln^2\left(\frac{\eta}{\eta_0}\right) \left[0.87 \times 10^{-12} k_{\rm c}^9 \left(\frac{k}{k_0}\right)^7 {\rm Mpc}^{11} + 3.73 \times 10^{-18} k_{\rm c}^7 \left(\frac{k}{k_0}\right)^9 {\rm Mpc}^9 - 7.71 \times 10^{-25} k_{\rm c}^5 \left(\frac{k}{k_0}\right)^{11} {\rm Mpc}^7\right] , \tag{4.75}$$

[7] It should be noted that if one were to reduce the assumption of 100% correlation between the energy density and entropy perturbation, this could soften this dependence. However, this is left for future investigation.

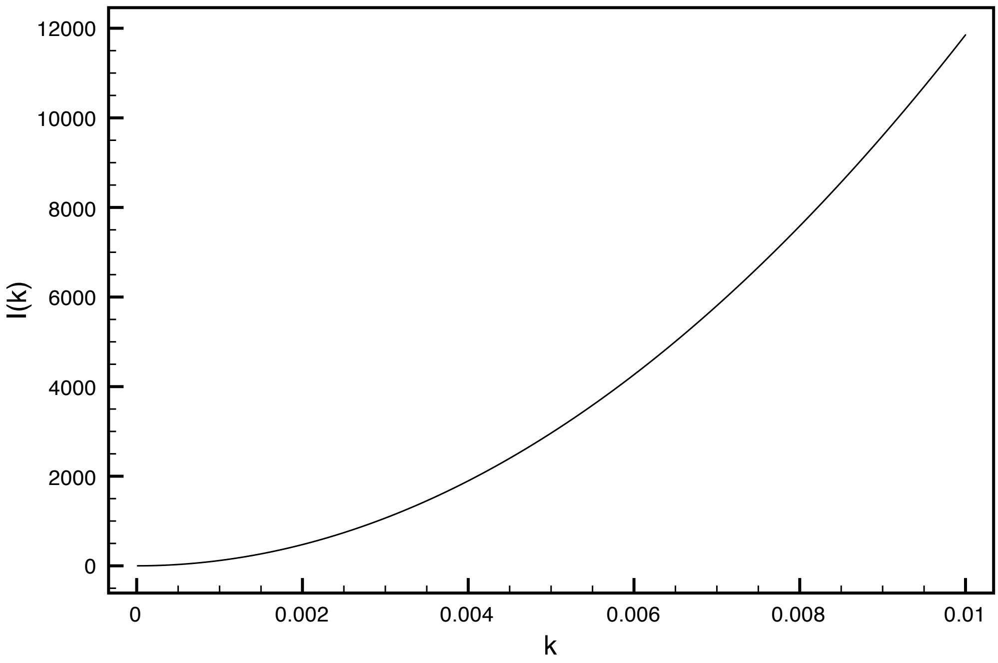

Figure 4.1: Plot of $I(k)$, Eq. (4.71), for the illustrative choice of $k_{\mathrm{c}} = 10\mathrm{Mpc}^{-1}$; small range of $k < k_{\mathrm{c}}$.

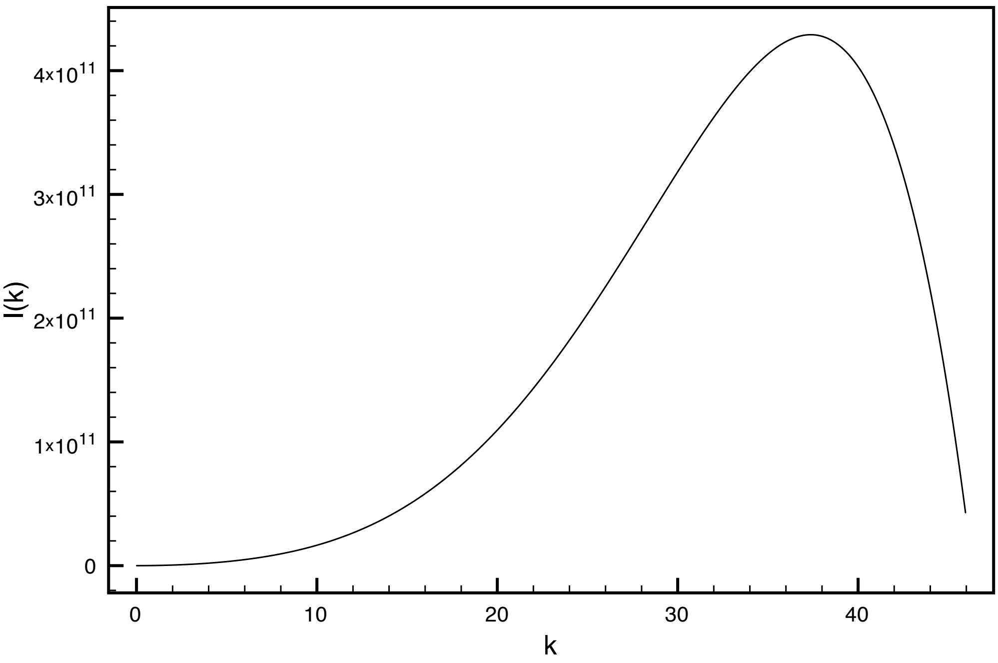

Figure 4.2: Plot of $I(k)$ for the illustrative choice of $k_{\mathrm{c}} = 10\mathrm{Mpc}^{-1}$; wide range of k including $k > k_{\mathrm{c}}$.

Parameter	WMAP7 value
k_0	$0.002\,\mathrm{Mpc}^{-1}$
$\Delta^2_{\mathcal{R}}(k_0)$	2.38×10^{-9}
$\alpha(k_0)$	0.13 (95% CL)

Table 4.1: Parameter values from the WMAP seven year data [77].

and for our above choice of $k_{\rm c} = 10\mathrm{Mpc}^{-1}$,

$$\mathcal{P}_\omega(k,\eta) = \eta^2 \ln^2\left(\frac{\eta}{\eta_0}\right)\left[0.87 \times 10^{-3}\left(\frac{k}{k_0}\right)^7 + 3.73 \times 10^{-11}\left(\frac{k}{k_0}\right)^9 - 7.71 \times 10^{-20}\left(\frac{k}{k_0}\right)^{11}\right]\mathrm{Mpc}^2\,. \tag{4.76}$$

This shows that, under our approximations, the vorticity spectrum has a non-negligible amplitude, with a huge amplification of power on small scales. We plot this, for illustrative purposes, in Figs. 4.3 and 4.4, where we have ignored the time dependence and focused only on the scale dependence of the spectrum. We should emphasise that we are studying the generation of vorticity in the wavenumber region $k_0 < k < k_c$ and do not expect the dynamics to be dominated by an "inverse cascade" (i.e. a feedback of power from smaller scales to larger scales). Therefore, the physics around the cut-off wavenumber cannot influence the vorticity generation. To study this phenomenon in detail, a much more detailed calculation including backreaction effects would have to be performed, beyond the scope of this thesis.

4.5 Discussion

In this Chapter we have studied the generation of vorticity in the early universe, showing that second order in cosmological perturbation theory vorticity is sourced by first order scalar and vector perturbations for a perfect fluid. This is an extension of Crocco's theorem to an expanding, dynamical background, namely, a FRW universe. Whereas previous works assumed barotropic flows, allowing for entropy gives a qualitatively novel result. This implies that the description of the cosmic fluid as a potential flow, which works exceptionally well at first order in the perturbations, will break down at second order for non-barotropic flows. Similarly, in barotropic flow Kelvin's theorem guarantees conservation of vorticity. This is no longer true if

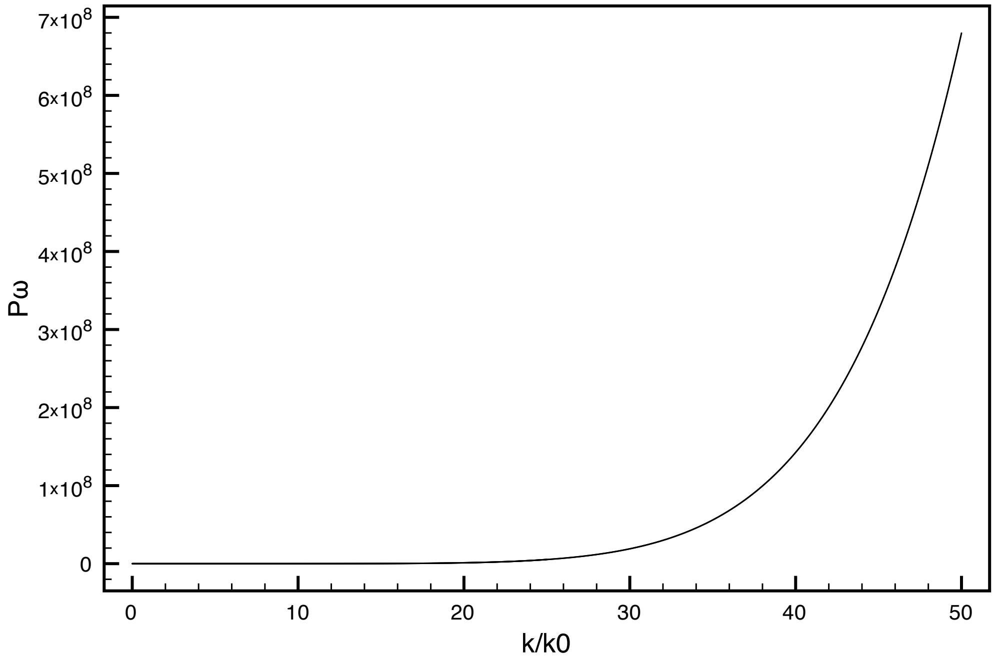

Figure 4.3: Plot of $\mathcal{P}_\omega$(k), i.e. the scale dependence of the vorticity power spectrum.

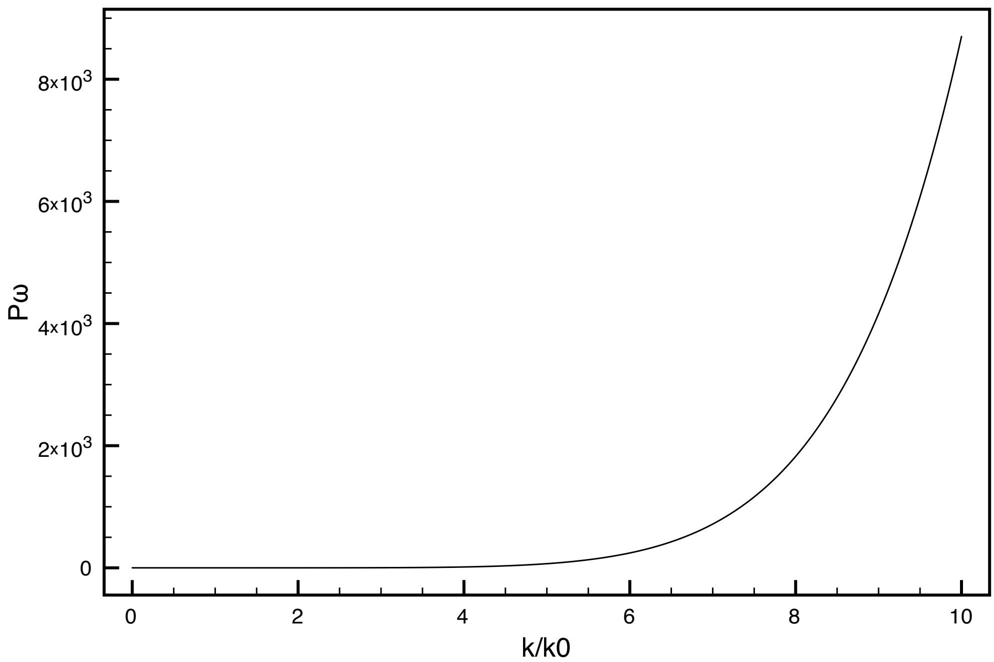

Figure 4.4: Plot of $\mathcal{P}_\omega$(k), for a narrower range of k values than Figure 4.3.

the flow is non-barotropic.

Having derived the qualitative result, we then obtained the first realistic calculation of the amount of vorticity generated at second order. As an input spectrum for the linear energy density, we used the solution obtained in Section 3.1.2 approximated for small $k\eta$ and normalised to WMAP7. Then, making the ansatz that the non-adiabatic pressure perturbation has a bluer spectrum than that of the energy density in order to keep the non-adiabatic pressure sub-dominant on all scales, we obtained an analytical result for the vorticity. Our results show that the vorticity power spectrum has a non-negligible magnitude which depends on the cutoff, $k_{\rm c}$, and the chosen parameters. As this is a second order effect, the magnitude is somewhat surprising. We have also shown that the result has a dependence of the wavenumber to the power of at least seven for the choice $\alpha = 2$, where α is the exponent of the wavenumber for the non-adiabatic pressure input spectrum. Therefore the amplification due to the large power of k is huge, rendering the vorticity not only possibly observable, but also important for the general understanding of the physical processes taking place in the early universe.

The consequences of this significant power are not immediately clear as the model under consideration is only a toy model. Although vorticity is not generated in standard cosmology (at linear order), the vorticity generated at second order will not invalidate the standard predictions, as it is, on large scales, very small and can only be of significant size on small scales. However, any possible observational consequences will depend on the wavenumber at which the power spectrum peaks (to be determined by an actual model).

One prospect for observing early universe vorticity is in the B-mode polarisation of the CMB. Both vector and tensor perturbations produce B-mode polarisation, but at linear order such vector modes decay with the expansion of the universe. However, vector modes produced by gradients in energy density and entropy perturbations, such as those discussed in this chapter, will source B-mode polarisation at second order. Furthermore, it has recently been noted that vector perturbations in fact generate a stronger B-mode polarisation than tensor modes with the same amplitude [55]. Therefore, it is feasible for vorticity to be observed by future surveys such as the space-based CMBPol [21], which is currently in the planning stage, or the ground based experiment Polarbear [53], which is due to start making observations in 2011.

Finally, a non-zero vorticity at second order in perturbation theory has important consequences for the generation of magnetic fields, as it has been long known that vorticity and magnetic fields are closely related (see Refs. [24, 61]). Previous works either used momentum exchange between multiple fluids to generate vorticity, as in Refs. [58, 72, 74, 103, 115, 139, 146], or used intermediate steps to first generate vorticity for example by using shock fronts as in Ref. [129]. However, we do not require such additional steps. Therefore, an important extension to the work presented in this chapter is to consider the magnetic fields generated by our mechanism which could be an important step in answering the question regarding the origin of the primordial magnetic field. We will discuss more future prospects in the concluding chapter of this thesis.

5 Third Order Perturbations

In the preceding chapters we have developed cosmological perturbation theory at both linear and second order. However, one does not need to stop there: it is worthwhile and feasible to consider perturbation theory even beyond second order. In this chapter we explore aspects of cosmological perturbation theory at third order.

We have already seen that extending perturbation theory to second order reveals new phenomena which arise due to quadratic source terms in general, and the coupling between different types of perturbation which is not present at linear order. At third order a new coupling occurs in the energy conservation equation, namely the coupling of scalar perturbations to tensor perturbations. This will allow for the calculation of yet another different observational signature, highlighting another aspect of the underlying full theory.

There has already been some work on third order theory. For example, Refs. [69, 71] considered third order perturbations of pressureless irrotational fluids as "pure" general relativistic correction terms to second order quantities. The calculations focused on the temporal comoving gauge, allowing the authors to consider only second order geometric and energy-momentum components, and neglected vector perturbations. Ref. [85] includes a study of third order perturbations with application to the trispectrum in the two-field ekpyrotic scenario in the large scale limit. There has also been reference in the literature of the need to extend perturbation theory beyond second order. For example, in Ref. [42] UV divergences in the Raychaudhuri equation are found when considering backreaction from averaging perturbations to second order. The authors state that these divergences may be removed by extending perturbation theory to third, or higher, orders.

In this chapter, we develop the essential tools for third order perturbation theory, such as the gauge transformation rules for different types of perturbation, and construct gauge invariant quantities at third order. We consider perfect fluids with non-zero pressure, including all types of perturbation, namely, scalar, vector and tensor perturbations. In particular allowing for vector perturbations is crucial for realistic higher order studies, since vorticity is generated at second order in all models employing non-barotropic fluids as shown in Chapter 4. Hence studying irrotational fluids at higher order will only give partial insight into the underlying physics. We present the energy and momentum conservation equations for such a fluid, and also give the components of the perturbed Einstein tensor, up to third order. All equations are given without fixing a gauge. We also give the Klein-Gordon equation for a scalar field minimally coupled to gravity at third order in cosmological perturbation theory. This work is published in Ref. [36].

5.1 Definitions

As in Chapter 2 we take the perturbed metric tensor with covariant components

$$g_{00} = -a^2(1+2\phi)\,, \qquad g_{0i} = a^2 B_i\,, \qquad g_{ij} = a^2(\delta_{ij} + 2C_{ij})\,, \tag{5.1}$$

In order to obtain the contravariant metric components we impose the constraint Eq. (2.14), $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu{}^\lambda$, up to third order. This gives

$$g^{00} = -\frac{1}{a^2}\Big(1 - 2\phi + 4\phi^2 - 8\phi^3 - B_kB^k + 4\phi B_kB^k + 2B^iB^jC_{ij}\Big)\,, \tag{5.2}$$

$$g^{0i} = \frac{1}{a^2}\Big(B^i - 2\phi B^i - 2B_kC^{ki} + 4\phi^2 B^i + 4B_kC^{ki}\phi + 4C^{kj}C_j{}^iB_k - B^kB_kB^i\Big)\,, \tag{5.3}$$

$$g^{ij} = \frac{1}{a^2}\Big(\delta^{ij} - 2C^{ij} + 4C^{ik}C_k{}^j - B^iB^j + 2\phi B^iB^j - 8C^{ik}C^{jl}C_{kl} + 2B^iC^{kj}B_k + 2B_kB^jC^{ik}\Big)\,. \tag{5.4}$$

Note that in this chapter we do not explicitly split terms up into first, second and third order parts unless where necessary, since doing so would dramatically increase the size of the equations presented. For example, expanding the $0-0$ component fully order by order gives

$$g^{00} = -\frac{1}{a^2}\Big(1 - 2\phi_1 - \phi_2 - \frac{1}{3}\phi_3 + \phi_1^2 + 8\phi_1\phi_2 - 8\phi_1^3 - B_{1k}B_1{}^k - B_{2k}B_1{}^k + 4\phi_1B_1{}^kB_{1k} + 2B_1{}^iB_1{}^jC_{1ij}\Big)\,, \tag{5.5}$$

which when compared to Eq. (5.2) illustrates the increase in number of terms, and thus why we refrain from splitting perturbations up.

Furthermore, to third order in perturbation theory, the fluid four velocity, defined in Eq. (2.21) and satisfying the constraint

$$u^\mu u_\mu = -1\,,$$

has components

$$u^i = \frac{1}{a} v^i\,, \tag{5.6}$$

$$u^0 = \frac{1}{a}\left(1 - \phi + \frac{3}{2}\phi^2 - \frac{5}{2}\phi^3 + \frac{1}{2}v_k v^k + v_k B^k + C_{kj}v^k v^j - 2\phi v^k B_k - \phi v^k v_k\right)\,, \tag{5.7}$$

$$u_i = a\left(v_i + B_i - \phi B_i + 2C_{ik}v^k + \frac{3}{2}B_i\phi^2 + \frac{1}{2}B_i v^k v_k + B_i v^k B_k\right)\,, \tag{5.8}$$

$$u_0 = -a\left(1 + \phi - \frac{1}{2}\phi^2 + \frac{1}{2}\phi^3 + \frac{1}{2}v^k v_k + \phi v_k v^k + C_{kj}v^k v^j\right)\,. \tag{5.9}$$

5.2 Gauge Transformations

We firstly need to extend the gauge transformations derived earlier to third order in cosmological perturbation theory. Expanding the exponential map, (2.50)

$$\widetilde{\mathbf{T}} = e^{\pounds_\xi}\mathbf{T}\,, \tag{5.10}$$

to third order gives

$$\exp(\pounds_\xi) = 1 + \epsilon\pounds_{\xi_1} + \frac{1}{2}\epsilon^2\pounds_{\xi_1}^2 + \frac{1}{2}\epsilon^2\pounds_{\xi_2} + \frac{1}{6}\epsilon^3\pounds_{\xi_3} + \frac{1}{6}\epsilon^3\pounds_{\xi_1}^3 + \frac{1}{4}\epsilon^3\pounds_{\xi_1}\pounds_{\xi_2} + \frac{1}{4}\epsilon^3\pounds_{\xi_2}\pounds_{\xi_1} + \ldots \tag{5.11}$$

Splitting the tensor $\mathbf{T}$ up to third order and collecting terms of like order in ϵ we find that tensorial quantities transform at third order as

$$\begin{aligned}\widetilde{\delta\mathbf{T}}_3 = \delta\mathbf{T}_3 + \Big(\pounds_{\xi_3} + \pounds_{\xi_1}^3 + \frac{3}{2}\pounds_{\xi_1}\pounds_{\xi_2} + \frac{3}{2}\pounds_{\xi_2}\pounds_{\xi_1}\Big)\mathbf{T}_0 \\ + 3\left(\pounds_{\xi_1}^2 + \pounds_{\xi_2}\right)\delta\mathbf{T}_1 + 3\pounds_{\xi_1}\delta\mathbf{T}_2\,.\end{aligned} \tag{5.12}$$

Then, by expanding the coordinate transformation, Eq. (2.56) to third order we obtain the relationship

$$x^\mu(q) = x^\mu(p) + \epsilon\xi_1^\mu(p) + \frac{1}{2}\epsilon^2\left(\xi^\mu_{1,\nu}(p){\xi_1}^\nu(p) + \xi_2^\mu(p)\right)$$
$$+\frac{1}{6}\epsilon^3\left[\xi_3^\mu(p) + \left(\xi^\mu_{1,\lambda\beta}\xi_1^\beta + \xi^\mu_{1,\beta}\xi^\beta_{1,\lambda}\right){\xi_1}^\lambda(p)\right] + \frac{1}{4}\epsilon^3\left(\xi^\mu_{2,\lambda}(p)\xi_1^\lambda(p) + \xi^\mu_{1,\lambda}(p)\xi_2^\lambda(p)\right)\,, \tag{5.13}$$

relating the coordinates of the points p and q.

In the rest of this section, we will derive the gauge transformations at third order in an analogous way to those derived for first and second order in section 2.3.

5.2.1 Four Scalars

At third order we also split the generating vector ξ_3^μ into a scalar temporal and scalar and vector spatial part, as

$$\xi_3^\mu = \left(\alpha_2, {\beta_{2,}}^i + {\gamma_3}^i\right)\,, \tag{5.14}$$

where the vector part is again divergence-free ($\partial_k{\gamma_3}^k = 0$). We then find from Eqs. (5.12) that the energy density transforms as

$$\widetilde{\delta\rho_3} = \delta\rho_3 + \left(\pounds_{\xi_3} + \pounds^3_{\xi_1} + \frac{3}{2}\pounds_{\xi_1}\pounds_{\xi_2} + \frac{3}{2}\pounds_{\xi_2}\pounds_{\xi_1}\right)\rho_0$$
$$+ 3\left(\pounds^2_{\xi_1} + \pounds_{\xi_2}\right)\delta\rho_1 + 3\pounds_{\xi_1}\delta\rho_2\,, \tag{5.15}$$

which gives

$$\widetilde{\delta\rho_3} = \delta\rho_3 + \rho_0'\alpha_3 + \rho_0'''\alpha_1^3 + 3\rho_0''\alpha_1\alpha_{1,\lambda}\xi_1^\lambda + \rho_0'\left(\alpha_{1,\lambda\beta}\xi_1^\lambda + \alpha_{1,\lambda}\xi^\lambda_{1,\ \beta}\right)\xi_1^\beta$$
$$+ 3\rho_0''\alpha_1\alpha_2 + \rho_0'\frac{3}{2}\left(\alpha_{2,\lambda}\xi_1^\lambda + \alpha_{1,\lambda}\xi_2^\lambda\right) + 3\left(\delta\rho_{1,\lambda\beta}\xi_1^\lambda + \delta\rho_{1,\lambda}\xi^\lambda_{1,\ \beta}\right)\xi_1^\beta$$
$$+ 3\delta\rho_{1,\lambda}\xi_2^\lambda + 3\delta\rho_{2,\lambda}\xi_1^\lambda\,. \tag{5.16}$$

Similar to the second order case, we need to specify the time slicings (at all orders), and also the spatial gauge or threading at first and second order, in order to render the third order density perturbation gauge-invariant.

5.2.2 The Metric Tensor

We now give the transformation behaviour of the metric tensor at third order. The starting point is again the Lie derivative, which for a covariant tensor is given by

Eq. (2.68).

As above in the case of the transformation behaviour of a four scalar at third order, the change under a gauge transformation of a two-tensor can be found applying the same methods as at second order. We therefore find that the metric tensor transforms at third order, from Eqs. (5.12) and (2.68), as

$$
\begin{aligned}
\widetilde{\delta g^{(3)}_{\mu\nu}} &= \delta g^{(3)}_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi_3^\lambda + g^{(0)}_{\mu\lambda}\xi_3^\lambda{}_{,\nu} + g^{(0)}_{\lambda\nu}\xi_3^\lambda{}_{,\mu} \\
&+ 3\Big[\delta g^{(1)}_{\mu\nu,\lambda}\xi_2^\lambda + \delta g^{(1)}_{\mu\lambda}\xi_2^\lambda{}_{,\nu} + \delta g^{(1)}_{\lambda\nu}\xi_2^\lambda{}_{,\mu} + \delta g^{(2)}_{\mu\nu,\lambda}\xi_1^\lambda + \delta g^{(2)}_{\mu\lambda}\xi_1^\lambda{}_{,\nu} + \delta g^{(2)}_{\lambda\nu}\xi_1^\lambda{}_{,\mu} \\
&+ \delta g^{(1)}_{\mu\nu,\lambda\alpha}\xi_1^\lambda\xi_1^\alpha + \delta g^{(1)}_{\mu\nu,\lambda}\xi_1^\lambda{}_{,\alpha}\xi_1^\alpha + 2\Big[\delta g^{(1)}_{\mu\lambda,\alpha}\xi_1^\alpha\xi_1^\lambda{}_{,\nu} + \delta g^{(1)}_{\lambda\nu,\alpha}\xi_1^\alpha\xi_1^\lambda{}_{,\mu} + \delta g^{(1)}_{\lambda\alpha}\xi_1^\lambda{}_{,\mu}\xi_1^\alpha{}_{,\nu}\Big] \\
&+ \delta g^{(1)}_{\mu\lambda}\left(\xi_1^\lambda{}_{,\nu\alpha}\xi_1^\alpha + \xi_1^\lambda{}_{,\alpha}\xi^\alpha_{1,\,\nu}\right) + \delta g^{(1)}_{\lambda\nu}\left(\xi_1^\lambda{}_{,\mu\alpha}\xi_1^\alpha + \xi_1^\lambda{}_{,\alpha}\xi^\alpha_{1,\,\mu}\right)\Big] \\
&+ \frac{3}{2}\Big[2g^{(0)}_{\mu\nu,\lambda\beta}\xi_2^\lambda\xi_1^\beta + g^{(0)}_{\mu\nu,\lambda}\left(\xi^\lambda_{2,\beta}\xi_1^\beta + \xi^\lambda_{1,\beta}\xi_2^\beta\right) + 2g^{(0)}_{\lambda\beta}\left(\xi^\lambda_{2,\mu}\xi^\beta_{1,\nu} + \xi^\lambda_{2,\nu}\xi^\beta_{1,\mu}\right) \\
&+ \left(g^{(0)}_{\mu\lambda,\beta}\xi^\lambda_{2,\nu} + g^{(0)}_{\lambda\nu,\beta}\xi^\lambda_{2,\mu} + g^{(0)}_{\mu\lambda}\xi^\lambda_{2,\beta\nu} + g^{(0)}_{\lambda\nu}\xi^\lambda_{2,\beta\mu}\right)\xi_1^\beta + \left(g^{(0)}_{\beta\nu,\lambda}\xi_2^\lambda + g^{(0)}_{\lambda\nu}\xi^\lambda_{2,\beta}\right)\xi^\beta_{1,\nu} \\
&+ \left(g^{(0)}_{\beta\nu,\lambda}\xi_2^\lambda + g^{(0)}_{\lambda\nu}\xi^\lambda_{2,\beta}\right)\xi^\beta_{1,\mu} + \left(g^{(0)}_{\mu\lambda,\beta}\xi^\lambda_{1,\nu} + g^{(0)}_{\lambda\nu,\beta}\xi^\lambda_{1,\mu} + g^{(0)}_{\mu\lambda}\xi^\lambda_{1,\beta\nu} + g^{(0)}_{\lambda\nu}\xi^\lambda_{1,\beta\mu}\right)\xi_2^\beta \\
&+ \left(g^{(0)}_{\beta\nu,\lambda}\xi_1^\lambda + g^{(0)}_{\lambda\nu}\xi^\lambda_{1,\beta}\right)\xi^\beta_{2,\nu} + \left(g^{(0)}_{\beta\nu,\lambda}\xi_1^\lambda + g^{(0)}_{\lambda\nu}\xi^\lambda_{1,\beta}\right)\xi^\beta_{2,\mu}\Big] \\
&+ \Big\{g^{(0)}_{\mu\nu,\alpha\beta\lambda}\xi_1^\alpha\xi_1^\beta\xi_1^\lambda + 3g^{(0)}_{\mu\nu,\lambda\beta}\xi_1^\lambda\xi_1^\alpha\xi^\beta_{1,\alpha} + g^{(0)}_{\mu\nu,\lambda}\xi^\lambda_{1,\alpha\beta}\xi_1^\alpha\xi_1^\beta + g^{(0)}_{\mu\nu,\lambda}\xi^\lambda_{1,\beta}\xi^\beta_{1,\alpha}\xi_1^\alpha \\
&+ 3g^{(0)}_{\mu\beta,\lambda}\left(\xi_1^\lambda\xi^\beta_{1,\alpha}\xi^\alpha_{1,\nu} + \xi_1^\alpha\xi_1^\lambda\xi^\beta_{1,\alpha\nu}\right) + 3g^{(0)}_{\mu\alpha,\lambda}\xi_1^\beta\xi^\lambda_{1,\beta}\xi^\alpha_{1,\nu} + 3g^{(0)}_{\mu\alpha,\beta\lambda}\xi_1^\beta\xi_1^\lambda\xi^\alpha_{1,\nu} \\
&+ g^{(0)}_{\mu\lambda}\Big[\xi_1^\alpha\xi_1^\beta\xi^\lambda_{1,\alpha\beta\nu} + \xi^\beta_{1,\alpha}\xi^\lambda_{1,\beta}\xi^\alpha_{1,\nu} + \xi_1^\alpha\xi^\beta_{1,\alpha}\xi^\lambda_{1,\beta\nu} + 2\xi_1^\beta\xi^\lambda_{1,\alpha\beta}\xi^\alpha_{1,\nu} + \xi_1^\beta\xi^\lambda_{1,\alpha}\xi^\alpha_{1,\beta\nu}\Big] \\
&+ 3g^{(0)}_{\nu\beta,\lambda}\left(\xi_1^\lambda\xi^\beta_{1,\alpha}\xi^\alpha_{1,\mu} + \xi_1^\alpha\xi_1^\lambda\xi^\beta_{1,\alpha\mu}\right) + 3g^{(0)}_{\nu\alpha,\lambda}\xi_1^\beta\xi^\lambda_{1,\beta}\xi^\alpha_{1,\mu} + 3g^{(0)}_{\nu\alpha,\beta\lambda}\xi_1^\beta\xi_1^\lambda\xi^\alpha_{1,\mu} \\
&+ g^{(0)}_{\nu\lambda}\Big[\xi_1^\alpha\xi_1^\beta\xi^\lambda_{1,\alpha\beta\mu} + \xi^\beta_{1,\alpha}\xi^\lambda_{1,\beta}\xi^\alpha_{1,\mu} + \xi_1^\alpha\xi^\beta_{1,\alpha}\xi^\lambda_{1,\beta\mu} + 2\xi_1^\beta\xi^\lambda_{1,\alpha\beta}\xi^\alpha_{1,\mu} + \xi_1^\beta\xi^\lambda_{1,\alpha}\xi^\alpha_{1,\beta\mu}\Big] \\
&+ 6g^{(0)}_{\alpha\beta,\lambda}\xi_1^\lambda\xi^\beta_{1,\nu} + 3g^{(0)}_{\alpha\lambda}\Big[\xi^\lambda_{1,\beta}\left(\xi^\alpha_{1,\mu}\xi^\beta_{1,\nu} + \xi^\alpha_{1,\nu}\xi^\beta_{1,\mu}\right) + \xi_1^\beta\left(\xi^\alpha_{1,\mu}\xi^\lambda_{1,\beta\nu} + \xi^\alpha_{1,\nu}\xi^\lambda_{1,\beta\mu}\right)\Big]\Big\}\,.
\end{aligned}
\tag{5.17}
$$

However, in this case it becomes even more obvious than in Section 5.1 that the expressions at third order are of not inconsiderable size. This will also be clear from the Einstein tensor components and the evolution equations given below in

Section 5.4.

Now, following along the same lines as at second order, Eq. (5.17) gives the transformation for the spatial part of the metric at third order,

$$2\widetilde{C}_{3ij} = 2C_{3ij} + 2\mathcal{H}\alpha_3\delta_{ij} + 2\xi_{3(i,j)} + \mathcal{X}_{3ij}\,, \tag{5.18}$$

where $\mathcal{X}_{3ij}$ contains terms cubic in the first order perturbations. Extracting the curvature perturbation gives

$$\widetilde{\psi_3} = \psi_3 - \mathcal{H}\alpha_3 - \frac{1}{4}\mathcal{X}_3{}^k{}_k + \frac{1}{4}\nabla^{-2}\mathcal{X}_3{}^{ij}{}_{,ij}\,. \tag{5.19}$$

This expression is general, including scalar, vector, and tensor perturbations and is valid on all scales. However, we shall detail here only the expression valid for scalar perturbations and large scales and find that $\mathcal{X}_{3ij}$ takes then the simple form

$$\begin{aligned}\mathcal{X}_{3ij} \equiv 2a^2\delta_{ij}\Bigg\{ & -3\Big[\alpha_2\psi_1' + \frac{1}{2}\alpha_1\psi_2' + \alpha_1\alpha_1'\left(\psi_1' + 2\mathcal{H}\psi_1\right) + \alpha_1^2\left(\psi_1'' + 4\mathcal{H}\psi_1'\right) \\
& + 2\mathcal{H}\alpha_2\psi_1 + \mathcal{H}\alpha_1\psi_2 + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\psi_1\Big] + \left(\frac{a'''}{a} + 3\mathcal{H}\frac{a''}{a}\right)\alpha_1^3 \\
& + 3\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\alpha_1' + \mathcal{H}\alpha_1\left(\alpha_1''\alpha_1 + \alpha_1'^2\right) \\
& + \frac{3}{2}\left[\mathcal{H}\left(\alpha_1\alpha_2' + \alpha_1'\alpha_2\right) + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1\alpha_2\right]\Bigg\}\,.\end{aligned} \tag{5.20}$$

Hence we finally get for the transformation of ψ_3

$$\begin{aligned}-\widetilde{\psi_3} = & -\psi_3 + \mathcal{H}\alpha_3 + \left(\frac{a'''}{a} + 3\mathcal{H}\frac{a''}{a}\right)\alpha_1^3 + 3\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\alpha_1' \\
& + \mathcal{H}\alpha_1\left(\alpha_1''\alpha_1 + \alpha_1'^2\right) + \frac{3}{2}\left[\mathcal{H}\left(\alpha_1\alpha_2' + \alpha_1'\alpha_2\right) + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1\alpha_2\right] \\
& - 3\Big[\alpha_2\psi_1' + \frac{1}{2}\alpha_1\psi_2' + \alpha_1\alpha_1'\left(\psi_1' + 2\mathcal{H}\psi_1\right) + \alpha_1^2\left(\psi_1'' + 4\mathcal{H}\psi_1'\right) + 2\mathcal{H}\alpha_2\psi_1 \\
& + \mathcal{H}\alpha_1\psi_2 + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\alpha_1^2\psi_1\Big]\,.\end{aligned} \tag{5.21}$$

5.3 Gauge Invariant Variables

In the previous section we have described how perturbations transform under a gauge shift. We can now use these results to construct gauge-invariant quantities, in particular the curvature perturbation on uniform density hypersurfaces, ζ. In this section, as before, we consider only scalar perturbations, and restrict ourselves to the large scale limit.

We first define hypersurfaces in different gauges as in Section 2.4. From Eq. (5.16) we find the time slicing defining uniform density hypersurfaces at third order in the large scale limit as

$$\alpha_{3\delta\rho} = -\frac{\delta\rho_3}{\rho_0'} + \frac{1}{2{\rho_0'}^2}\Big[3\left(\delta\rho_1\delta\rho_2' + \delta\rho_1'\delta\rho_2\right) - \frac{\delta\rho_1''{\delta\rho_1}^2}{\rho_0'} - 4{\delta\rho_1'}^2\frac{\delta\rho_1}{\rho_0'} + \rho_0''\delta\rho_1'\left(\frac{\delta\rho_1}{\rho_0'}\right)^2\Big]. \tag{5.22}$$

Similarly, the temporal gauge transformation on uniform curvature hypersurfaces is defined by evaluating (5.21) and gives, at third order,

$$\begin{aligned}\alpha_{3\mathrm{flat}} = \frac{\psi_3}{\mathcal{H}} + \frac{1}{2\mathcal{H}^2}\Big[3\psi_1'\psi_2 + \frac{\psi_1^2\psi_1''}{\mathcal{H}} + 6\mathcal{H}\psi_1\psi_2 + \frac{4\psi_1\psi_1'^2}{\mathcal{H}}\Big] \\ - \frac{\psi_1^2\psi_1'}{\mathcal{H}^4}\Big(\frac{a''}{a} - \frac{37}{2}\mathcal{H}^2\Big) + \frac{8\psi_1^3}{\mathcal{H}}.\end{aligned} \tag{5.23}$$

We can now combine the results found so far to get gauge invariant quantities, and as before choose the curvature perturbation on uniform density slices as well as the density perturbation on uniform curvature hypersurfaces as examples.

One gauge invariant matter quantity of interest is the perturbation to the energy density on uniform curvature hypersurfaces. This is obtained by substituting the temporal gauge transformation components in the uniform density gauge into the appropriate transformation equation, Eq. (5.16). This gives, at third order,

$$\begin{aligned}\widetilde{\delta\rho_{3\mathrm{flat}}} &= \delta\rho_3 + \rho_0'\frac{\psi_3}{\mathcal{H}} + \frac{3\rho_0'}{2\mathcal{H}^2}(2\psi_2\psi_1' + \psi_2'\psi_1) + 3\frac{\psi_1^2}{\mathcal{H}^3}\Big[2(\rho_0'\psi_1 + \psi_1'\rho_0'') + \psi_1''\Big] \\ &+ 3\frac{\psi_2\psi_1}{\mathcal{H}^2}\Big[\rho_0'' + 2\rho_0'\mathcal{H} - \rho_0'\frac{a''}{a}\Big] - 9\rho_0'\frac{\psi_1^2\psi_1'}{\mathcal{H}^3}\Big(\frac{a''}{a} - \mathcal{H}\Big) \\ &+ \frac{\psi_1^3}{\mathcal{H}^3}\Big[\rho_0''' - 3\rho_0''\Big(\frac{a''}{a} + 3\mathcal{H}\Big) + 3\rho_0'\mathcal{H}\Big(3\frac{a''}{a} - \mathcal{H}\Big) + \rho_0'\Big(\Big(\frac{a''}{a}\Big)^2 - \frac{a'''}{a}\Big)\Big].\end{aligned} \tag{5.24}$$

The curvature perturbation on uniform density hypersurfaces, ζ, as introduced in Eq. (2.126), is defined as

$$-\zeta \equiv \widetilde{\psi_{\delta\rho}}. \tag{5.25}$$

This is obtained by substituting the temporal gauge transformation components in the uniform curvature gauge into the appropriate transformation equation, Eq. (5.21).

Evaluating this on spatially flat hypersurfaces then gives, at third order

$$\zeta_3 = -\mathcal{H}\frac{\delta\rho_3}{\rho_0'} + \frac{3\mathcal{H}}{\rho_0'^2}(\delta\rho_2{}'\delta\rho_1 + \delta\rho_1{}'\delta\rho_2) - \frac{3}{\rho_0'^2}\delta\rho_2\delta\rho_1\Big[\frac{\mathcal{H}\rho_0''}{\rho_0'} - \Big(\frac{a''}{a} + \mathcal{H}^2\Big)\Big]$$
$$- \frac{3\mathcal{H}}{\rho_0'^3}\delta\rho_1{}^2\delta\rho_1{}'' - \frac{6\mathcal{H}}{\rho_0'^3}\delta\rho_1{}'^2\delta\rho_1 - \frac{3}{\rho_0'^3}\delta\rho_1{}^2\delta\rho_1{}'\Big[2\Big(\frac{a''}{a} + \mathcal{H}^2\Big) - 3\mathcal{H}\frac{\rho_0''}{\rho_0'}\Big]$$
$$- \frac{\delta\rho_1{}^3}{\rho_0'^3}\Big[3\mathcal{H}\left(\frac{\rho_0''}{\rho_0'}\right)^2 - \mathcal{H}\frac{\rho_0'''}{\rho_0'} + \frac{a'''}{a} + 3\mathcal{H}\frac{a''}{a} - 3\frac{\rho_0''}{\rho_0'}\Big(\frac{a''}{a} + \mathcal{H}^2\Big)\Big]\,. \tag{5.26}$$

There are different definitions of the curvature perturbation present in the literature, depending on different decompositions of the spatial part of the metric tensor. A different definition to the one above, as discussed in e.g. Ref. [112], was used by Maldacena in Ref. [104] to calculate the non-gaussianity from single field inflation, and was introduced by Salopek *et al.* in Refs. [130, 131]. They define the local scale factor $\tilde{a} \equiv e^{\alpha}$, then

$$e^{2\alpha} = a^2(\eta)e^{2\zeta} = a^2(\eta)(1 + 2\zeta_{\rm SB} + 2\zeta_{\rm SB}^2 + \frac{4}{3}\zeta_{\rm SB}^3)\,. \tag{5.27}$$

Comparing to the expansion from perturbation theory

$$e^{2\alpha} = a^2(\eta)(1 + 2\zeta)\,, \tag{5.28}$$

one can obtain the relationship

$$\zeta = \zeta_{\rm SB} + \zeta_{\rm SB}^2 + \frac{2}{3}\zeta_{\rm SB}^3\,. \tag{5.29}$$

Splitting this up order by order gives, at second order

$$\zeta_{\rm 2SB} = \zeta_2 - 2(\zeta_1)^2\,, \tag{5.30}$$

and, at third order,

$$\zeta_{\rm 3SB} = \zeta_3 - 6\zeta_2\zeta_1 + 8(\zeta_1)^3\,. \tag{5.31}$$

Note that it is this definition, (5.31), of the curvature perturbation which occurs in Ref. [85], though with different pre-factors since their perturbative expansion is defined differently. It is perhaps worth mentioning that $\zeta_{\rm SB}$, the variable first introduced by Salopek and Bond and then employed for non-gaussianity calculations by Maldacena, is extremely Gaussian after slow-roll inflation, as opposed to other ζ variables which exhibit non-gaussianity, as can be seen e.g. from Eq. (5.30).

5.4 Governing Equations

Having constructed gauge invariant quantities up to third order in the previous section, we now turn to the evolution and the field equations. The equations presented in this section in full generality are new; Refs. [69, 71] previously considered some governing equations at third order in perturbation theory, however they focussed on pressureless, irrotational fluids.

5.4.1 Fluid Conservation Equation

In this section, we give the energy momentum conservation equations for a fluid with non-zero pressure and in the presence of scalar, vector and tensor perturbations. The latter generalisation is important since at orders above linear order, all types of perturbation are coupled.

As at linear and second order, presented in the previous chapters, energy momentum conservation

$$\nabla_\mu T^\mu{}_\nu = 0, \tag{5.32}$$

gives us evolution equations. Substituting the definition for the energy momentum tensor, Eq. (2.18) expanded to third order, into Eq. (5.32) gives energy conservation (the 0-component)

$$\begin{aligned}
&\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + (\rho_0 + P_0)(C^{i\prime}_{\ i} + v_{i,}{}^{i}) + (\delta\rho + \delta P)(C^{i\prime}_{\ i} + v_{i,}{}^{i}) + (\delta\rho + \delta P)_{,i} v^i \\
&+ (\rho_0 + P_0)\Big[(B^i + 2v^i)(v_i' + B_i') + v^i{}_{,i}\phi - 2C_{ij}' C^{ij} + v^i(C^j{}_{j,i} + 2\phi_{,i}) + 4\mathcal{H} v^i(B_i + 2v_i)\Big] \\
&+ (\delta\rho + \delta P)\Big[(B^i + 2v^i)(v_i' + B_i') - 2C_{ij}' C^{ij} + v^i(C^j{}_{j,i} + 2\phi_{,i} + 4\mathcal{H}(v_i + B_i)) + v^i{}_{,i}\phi\Big] \\
&+ (\delta\rho' + \delta P') v^i (B_i + v_i) + (\delta\rho + \delta P)_{,i}\phi v^i + (\rho_0 + P_0)(2C^{ij} v_i v_j - B_i v^i \phi + v^i v_i \phi) \\
&- (\rho_0 + P_0)\Big[2C^{ij}(C_{ij,k} v^k - 2v_i' v_j + B_i B_j' - 2C_i{}^k C_{jk}') + \frac{1}{2} v^i{}_{,i}(\phi^2 - v^j v_j) \\
&+ (v^i + B^i)(2B_i'\phi - C^{j\prime}_{\ j} v_i) - v^j\Big\{B^i B_{i,j} + v^i v_{i,j} + \phi(v_j + 2v_j' + 3\mathcal{H} v_j - 2\phi_{,j} + C^i{}_{i,j}) \\
&- 2\phi' B_j + 3C_{ij}' v^i\Big\} + B^j(B^i C_{ij}' + B_j\phi' + v_j'\phi)\Big] = 0\,,
\end{aligned} \tag{5.33}$$

and momentum conservation (the $i-$component)

$$
\begin{aligned}
&\Big[(\rho_0+P_0)(v_i+B_i)\Big]'+(\rho_0+P_0)(\phi_{,i}+4\mathcal{H}(v_i+B_i))+\delta P_{,i}+\Big[(\delta\rho+\delta P)(v_i+B_i)\Big]'\\
&+(\delta\rho+\delta P)(\phi_{,i}+4\mathcal{H}(v_i+B_i))-(\rho_0'+P_0')\Big[(2B_i+v_i)\phi-2C_{ij}v^j\Big]\\
&+(\rho_0+P_0)\Big[(v_i+B_i)({C^j}'_j+{v^j}_{,j})-B_i(\phi'+8\mathcal{H}\phi)+v^j(B_{i,j}-B_{j,i}+v_{i,j}+8\mathcal{H}C_{ij})\\
&+(2C_{ij}v^j)'-\phi(v_i'+2B_i'+2\phi_{,i}+4\mathcal{H}v_i)\Big]+(\rho_0'+P_0')\Big[v^j(B_iv_j+B_jv_i+2B_iB_j+\frac{1}{2}v_iv_j\\
&-2C_{ij}\phi)+(\frac{3}{2}v_i+4B_i)\phi^2\Big]+(\rho_0+P_0)\Big[\phi^2\left(4B_i'+\frac{3}{2}v_i'+2\mathcal{H}(8B_i+3v_i)+4\phi_{,i}\right)\\
&+(v_i+2B_i)(v_j'B^j-{C^j}'_j\phi)+(v_i+B_i)(B_j'B^j-C_{jk}'C^{jk})+v^j\Big\{2B_i(B_j'+v_j')\\
&+B_j(2B_i'+2\phi_{,i}+v_i')+2\mathcal{H}(v_i+2B_i)(2B_j+v_j)+2C_{ij}({C^k}'_k+{v^k}_{,k}-4\mathcal{H}\phi)\\
&+{C^k}_{k,j}(B_i+v_i)+v_i(B_j'+\phi_{,j})+v_j(B_i'+\phi_{,i})+v^k(2C_{ik,j}-C_{jk,i})\\
&+(B_{j,i}-B_{i,j}-2C_{ij}')\phi+2C_{ik}{v^k}_{,j}\Big\}+\frac{1}{2}(v_iv_jv^j)'-2C_{ij}{v^j}'\phi+B_i(4\phi'\phi-{v_{j,}}^j\phi)\Big]=0\,.
\end{aligned}
\tag{5.34}
$$

As emphasised earlier, Eq. (5.33) highlights the coupling between tensor and scalar perturbations which occurs only at third order (and higher) in perturbation theory. At both linear and second order, no such coupling exists, since the only terms coupling the spatial metric perturbation, C_{ij}, to scalar perturbations contain either the trace or the divergence of C_{ij} and the tensor perturbation, h_{ij} is, by definition, transverse and trace-free. However, at third order, terms like $\delta\rho C_{ij}'C^{ij}$ occur in the energy conservation equation which, on splitting up order by order and decomposing C_{ij} becomes $\delta\rho_1 h_{1ij}'{h_1}^{ij}$. It is clear that this term only shows up at third order and beyond. Thus, as mentioned earlier, third order is the lowest order at which all the different types of perturbations couple to one another in the evolution equations, which will produce another physical signature of the full theory.

Scalars only

It will be useful to have energy and momentum conservation equations for only scalar perturbations. These equations are obtained by making the appropriate substitutions $C_{ij}=-\delta_{ij}\psi+E_{,ij}$, $v_i=v_{,i}$, and $B_i=B_{,i}$ into the above expressions. On doing so, we obtain the energy conservation equation

$$
\begin{aligned}
&\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + (\rho_0 + P_0)(\nabla^2 v + \nabla^2 E - 3\psi') + (\delta\rho + \delta P)(\nabla^2 v + \nabla^2 E - 3\psi') \\
&+ (\delta\rho + \delta P)_{,i} v_{,}{}^{i} + (\rho_0 + P_0)\Big[(B^i + 2v_{,}{}^{i})(v'_{,i} + B'_{,i}) + \nabla^2 v\phi - 2\Big(\psi'(3\psi - \nabla^2 E) \\
&- \psi\nabla^2 E' + E'_{,ij}E_{,}{}^{ij}\Big) + v_{,}{}^{i}(2\phi_{,i} + \nabla^2 E_{,i} - 3\psi_{,i}) + 4\mathcal{H}v_{,}{}^{i}(B_{,i} + 2v_{,i})\Big] \\
&+ (\delta\rho + \delta P)\Big[(B_{,}{}^{i} + 2v_{,}{}^{i})(v'_{,i} + B'_{,i}) - 2\Big(\psi'(3\psi - \nabla^2 E) - \psi\nabla^2 E' + E'_{,ij}E_{,}{}^{ij}\Big) \\
&+ v_{,}{}^{i}(2\phi_{,i} - 3\psi_{,i} + \nabla^2 E + 4\mathcal{H}(v_{,i} + B_{,i})) + \nabla^2 v\phi\Big] + (\delta\rho' + \delta P')v_{,}{}^{i}(B_{,i} + v_{,i}) \\
&+ (\delta\rho + \delta P)_{,i}\phi v_{,}{}^{i} + (\rho_0 + P_0)(v_{,}{}^{i}v_{,i}\phi - B_{,i}v_{,}{}^{i}\phi - 6\psi v_{,}{}^{j}v_{,j} + E_{,}{}^{ij}v_{,i}v_{,j}) \\
&- (\rho_0 + P_0)\Big[2E_{,}{}^{ij}(v_{,}{}^{k}E_{,ijk} - 2v'_{,i}v_{,j} + B_{,i}B'_{,j} - 2E_{,i}{}^{k}E_{,kj}) - 2\nabla^2 E\psi_{,k}v_{,}{}^{k} \\
&- 2\psi\Big(v_{,}{}^{k}(\nabla^2 E_{,k} - 3\psi_{,k}) - 2v'_{,j}v_{,}{}^{j} + B'_{,j}B_{,}{}^{j} + 6\psi\psi'\Big) + \frac{1}{2}\nabla^2 v(\phi^2 - v_{,}{}^{j}v_{,j}) \\
&+ (v_{,}{}^{i} + \nabla^2 E' + B_{,}{}^{i})(2B'_{,i}\phi + 3\psi' v_{,i}) - v_{,}{}^{j}\Big\{B_{,}{}^{i}B_{,ij} + v_{,}{}^{i}v_{,ij} - 2\phi' B_{,j} - 3\psi' v_{,j} \\
&+ \phi(v_{,j} + 2v'_{,j} + 3\mathcal{H}v_{,j} - 2\phi_{,j} - 3\psi_{,j} + \nabla^2 E_{,j}) + E'_{,ij}v_{,}{}^{i}\Big\} \\
&+ B_{,}{}^{j}(B_{,j}\psi' + E'_{,ij}v_{,}{}^{i} + B_{,j}\phi' + v'_{,j}\phi)\Big] = 0\,,
\end{aligned} \tag{5.35}
$$

and the momentum conservation equation

$$
\begin{aligned}
&\Big[(\rho_0+P_0)(v_{,i}+B_{,i})\Big]'+(\rho_0+P_0)(\phi_{,i}+4\mathcal{H}(v_{,i}+B_{,i}))+\delta P_{,i}\\
&+\Big[(\delta\rho+\delta P)(v_{,i}+B_{,i})\Big]'+(\delta\rho+\delta P)(\phi_{,i}+4\mathcal{H}(v_{,i}+B_{,i}))\\
&-(\rho_0'+P_0')\Big[(2B_{,i}+v_{,i})\phi+2\psi v_{,i}-2E_{,ij}v_{,}{}^{j}\Big]\\
&+(\rho_0+P_0)\Big[(v_{,i}+B_{,i})(\nabla^2v-3\psi'+\nabla^2E')-B_{,i}(\phi'+8\mathcal{H}\phi)\\
&+v_{,}{}^{j}\Big(v_{,ij}-8\mathcal{H}(\psi\delta_{ij}-E_{,ij})\Big)-2(\psi v_{,i}+E_{,ij}v_{,}{}^{j})'-\phi(v_{,i}'+2B_{,i}'+2\phi_{,i}+4\mathcal{H}v_{,i})\Big]\\
&+(\rho_0'+P_0')\Big[v_{,}{}^{j}(B_{,i}v_{,j}+B_{,j}v_{,i}+2B_{,i}B_{,j}+\frac{1}{2}v_{,i}v_{,j}+2\phi(\psi\delta_{ij}-E_{,ij}))\\
&+(\frac{3}{2}v_{,i}+4B_{,i})\phi^2\Big]+(\rho_0+P_0)\Big[(v_{,i}+2B_{,i})\Big(v_{,j}'B_{,}{}^{j}-(3\psi'-\nabla^2E')\phi\Big)\\
&+\phi^2\left(4B_{,i}'+\frac{3}{2}v_{,i}'+2\mathcal{H}(8B_{,i}+3v_{,i})+4\phi_{,i}\right)+(v_{,i}+B_{,i})\Big(B_{,j}'B_{,}{}^{j}-3\psi'\psi\\
&+\psi'\nabla^2E+\psi\nabla^2E'-E_{,}{}^{jk}E_{,jk}'\Big)+v_{,}{}^{j}\Big\{2B_{,i}(B_{,j}'+v_{,j}')+B_{,j}(2B_{,i}'+2\phi_{,i}+v_{,i}')\\
&+2\mathcal{H}(v_{,i}+2B_{,i})(2B_{,j}+v_{,j})-(3\psi_{,j}-\nabla^2E_{,j})(B_{,i}+v_{,i})\\
&-2(\psi\delta_{ij}-E_{,ij})(\nabla^2v+\nabla^2E'-3\psi'-4\mathcal{H}\phi)+v_{,i}(B_{,j}'+\phi_{,j})\\
&+v_{,j}(B_{,i}'+\phi_{,i})-2\psi_{,j}v_{,i}+E_{,ikj}v_{,}{}^{k}+\psi_{,i}v_{,j}+2(\psi'\delta_{ij}-E_{,ij}))\phi-2\psi v_{,ij}\\
&+2E_{,ik}v_{,}{}^{k}{}_{j}\Big\}+\frac{1}{2}(v_{,i}v_{,j}v_{,}{}^{j})'+2\psi v_{,i}'\phi-2E_{,ij}v_{,}{}^{j\prime}\phi+B_i(4\phi'\phi-\nabla^2v\phi)\Big]=0\,. \qquad (5.36)
\end{aligned}
$$

Considering the large scale limit, in which spatial gradients vanish, the energy conservation equation becomes

$$
\begin{aligned}
&\delta\rho'+3\mathcal{H}(\delta\rho+\delta P)-3\psi'(\rho_0+P_0)-3\psi'(\delta\rho+\delta P)\\
&-6\psi\psi'(\rho_0+P_0)-6\psi\psi'\ (\delta\rho+\delta P)+12\psi^2\psi'(\rho_0+P_0)=0\,. \qquad (5.37)
\end{aligned}
$$

Splitting up perturbations order by order, this becomes

$$
\begin{aligned}
&\delta\rho_3{}'+3\mathcal{H}(\delta\rho_3+\delta P_3)-3\psi_3'(\rho_0+P_0)-9\psi_2'(\delta\rho_1+\delta P_1)-9\psi_1'(\delta\rho_2+\delta P_2)\\
&-18(\rho_0+P_0)(\psi_2\psi_1'+\psi_1\psi_2')+72\psi_1^2\psi_1'(\rho_0+P_0)=0\,. \qquad (5.38)
\end{aligned}
$$

In the uniform curvature gauge, where $\psi=0$, this is

$$
\delta\rho_{3\text{flat}}'+3\mathcal{H}(\delta\rho_{3\text{flat}}+\delta P_{3\text{flat}})=0\,, \qquad (5.39)
$$

and in the uniform density gauge, where $\delta\rho = 0$,

$$
\begin{aligned}
&3\mathcal{H}\delta P_{3\delta\rho} + 3\zeta_3'(\rho_0 + P_0) + 9\zeta_2'\delta P_{1\delta\rho} + 9\zeta_1'\delta P_{2\delta\rho} \\
&- 18(\rho_0 + P_0)(\zeta_2\zeta_1' + \zeta_1\zeta_2') - 72\zeta_1^2\zeta_1'(\rho_0 + P_0) = 0\,,
\end{aligned}
\tag{5.40}
$$

with ζ as defined above. This can be recast in the more familiar form by introducing the (gauge invariant) non-adiabatic pressure perturbation. At linear order the pressure perturbation can be expanded as, from Eq. (2.149),

$$
\delta P_1 = \frac{\partial P}{\partial S}\delta S_1 + \frac{\partial P}{\partial \rho}\delta\rho_1 \equiv \delta P_{\mathrm{nad}1} + c_{\mathrm{s}}^2\delta\rho_1\,. \tag{5.41}
$$

This can be extended to second order [37] and higher by simply not truncating the Taylor series:

$$
\delta P_{\mathrm{nad}2} = \delta P_2 - c_{\mathrm{s}}^2\delta\rho_2 - \frac{\partial c_{\mathrm{s}}^2}{\partial\rho}\delta{\rho_1}^2\,, \tag{5.42}
$$

$$
\delta P_{\mathrm{nad}3} = \delta P_3 - c_{\mathrm{s}}^2\delta\rho_3 - 3\frac{\partial c_{\mathrm{s}}^2}{\partial\rho}\delta\rho_2\delta\rho_1 - \frac{\partial^2 c_{\mathrm{s}}^2}{\partial\rho^2}\delta{\rho_1}^3\,. \tag{5.43}
$$

Thus, in the uniform density gauge, the pressure perturbation is equal to the non-adiabatic pressure perturbation at all orders. Then, Eq. (5.40) becomes

$$
\zeta_3' + \frac{\mathcal{H}}{\rho_0 + P_0}\delta P_{\mathrm{nad}3} = 6(\zeta_2\zeta_1' + \zeta_1\zeta_2') + 24\zeta_1^2\zeta_1' - \frac{3}{\rho_0 + P_0}(\zeta_2'\delta P_{\mathrm{nad}1} + \zeta_1'\delta P_{\mathrm{nad}2})\,. \tag{5.44}
$$

In the case of a vanishing non-adiabatic pressure perturbation, ζ_1' and ζ_2' are zero and hence we see that ζ_3 is also conserved, on large scales. This was also found in Ref. [85], and previously in Ref. [52].

5.4.2 Klein-Gordon Equation

The energy momentum tensor for a canonical scalar field minimally coupled to gravity is easily obtained by treating the scalar field as a perfect fluid with energy-momentum tensor (c.f. Chapter 2)

$$
T^\mu{}_\nu = g^{\mu\lambda}\varphi_{,\lambda}\varphi_{,\nu} - \delta^\mu{}_\nu\left(\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + U(\varphi)\right)\,, \tag{5.45}
$$

where the scalar field φ is split to third order as

$$
\varphi(\eta, x^i) = \varphi_0(\eta) + \delta\varphi_1(\eta, x^i) + \frac{1}{2}\delta\varphi_2(\eta, x^i) + \frac{1}{3!}\delta\varphi_3(\eta, x^i)\,, \tag{5.46}
$$

and the potential U similarly as

$$U(\varphi) = U_0 + \delta U_1 + \frac{1}{2}\delta U_2 + \frac{1}{3!}\delta U_3\,, \tag{5.47}$$

where we define

$$\begin{aligned} \delta U_1 = U_{,\varphi}\delta\varphi_1\,, \qquad \delta U_2 = U_{,\varphi\varphi}\delta\varphi_1^2 + U_{,\varphi}\delta\varphi_2\,, \\ \delta U_3 = U_{,\varphi\varphi\varphi}\delta\varphi_1^3 + 2U_{,\varphi\varphi}\delta\varphi_1\delta\varphi_2 + U_{,\varphi}\delta\varphi_3\,, \end{aligned} \tag{5.48}$$

and making use of the shorthand notation $U_{,\varphi} \equiv \frac{\partial U}{\partial\varphi}$. Then, Eq. (5.32) gives the Klein-Gordon equation

$$\begin{aligned}
&\delta\varphi_3''-\nabla^2\delta\varphi_3+4\mathcal{H}\delta\varphi_3'+\frac{\varphi_0''}{\varphi_0'}\delta\varphi_3'-\frac{3\delta\varphi_2''}{\varphi_0'}\Big(2\varphi_0'\phi-\delta\varphi_1'\Big)-\frac{3}{\varphi_0'}\Big(\nabla^2\delta\varphi_2\delta\varphi_1'+\delta\varphi_2'\nabla^2\delta\varphi_1\Big)\\
&-\frac{6\delta\varphi_2'}{\varphi_0'}\Big(\phi\varphi_0''-2\mathcal{H}\delta\varphi_1'+\varphi_0\phi'+\varphi_0'B^i{}_{,i}-C^{i\prime}{}_i\varphi_0'+4\mathcal{H}\varphi_0'\phi\Big)-\frac{6(\delta\varphi_1')^2}{\varphi_0'}\Big(\phi'+B^i{}_{,i}+4\mathcal{H}\phi-C^{i\prime}{}_i\Big)\\
&-\frac{6\delta\varphi_1''}{\varphi_0'}\Big(2\phi\delta\varphi_1'-4\varphi_0'\phi^2+2\varphi_0'\phi-\tfrac{1}{2}\delta\varphi_2'+B^iB_i\varphi_0'\Big)-\frac{6\delta\varphi_1'}{\varphi_0'}\Big[\varphi_0''(2\phi-4\phi^2+B^iB_i)\\
&+8\phi\varphi_0'(\mathcal{H}-2\mathcal{H}\phi-2\phi')+B^i(\varphi_0'B_i'+2\delta\varphi_{1,i}'+4\mathcal{H}B_i)+2\varphi_0'\Big\{B^i{}_{,i}+\phi'-C^{i\prime}{}_i(1-2\phi)\\
&-2C^{ij}(B_{j,i}-C_{ij}')+B^iC^j{}_{i,j}-2C^{ij}{}_{,j}B_i-B^i\phi_{,i}-2B^i{}_{,i}\phi\Big\}-2C^{ij}\delta\varphi_{1,ij}\\
&+\delta\varphi_{1,i}(B^{i\prime}+2\mathcal{H}B^i+2C^{ij}{}_{,j}+\phi,^i)\Big]-3\delta\varphi_{2,i}\Big(B^{i\prime}+C^j{}_{j,}{}^i+\phi,^i-2C^{ij}{}_{,j}+2\mathcal{H}B^i\Big)\\
&-6\delta\varphi_{1,i}\Big[B^{i\prime}+C^j{}_{j,}{}^i-2C^{ij}{}_{,j}+\phi,^i+\frac{C^j{}_{j,}{}^i}{\varphi_0'}+2\mathcal{H}B^i-2B_jC^{ij\prime}+B^iC^{j\prime}{}_j+B^jB_{j,}{}^i-B^jB^i{}_{,j}\\
&-B^iB^j{}_{,j}-2C^{ij}B_j'-2C^{ij}C^k{}_{k,j}+4C^{ij}C_{jk,}{}^k-2C^{ij}\phi_{,j}-2B^{i\prime}\phi-B^i\phi'-2\phi,^i\phi-4\mathcal{H}B_jC^{ij}\\
&-4\mathcal{H}B^i\phi+4C^{kj}C_j{}^i{}_{,k}-2C^{kj}C_k{}^i{}_{,j}\Big]-12B^i\Big(\delta\varphi_{1,i}'+\tfrac{1}{2}\delta\varphi_{2,i}'\Big)+24\delta\varphi_{1,i}'(B^i\phi+C^{ij}B_j)\\
&+6\delta\varphi_{2,ij}C^{ij}-6\delta\varphi_{1,ij}(4C^{kj}C_k{}^i-B^iB^j)-24\mathcal{H}\varphi_0'\phi(1-2\phi+4\phi^2-\phi'+3\phi'\phi)\\
&-6\varphi_0''\Bigg(2\phi(1-2\phi+4\phi^2)+B^iB_i-4B^iB_i\phi-2B^iB^jC_{ij}\Bigg)-6C^{ij}{}_{,j}\varphi_0'(B_i\phi-2B_i)\\
&+6C^{i\prime}{}_i\varphi_0'(1-2\phi+4\phi^2-B^jB_j)-6C^j{}_{j,i}\varphi_0'(B^i-2C^{ik}B_k-2B^i\phi)-6C^{ij\prime}\varphi_0'(2C_{ij}-B_iB_j\\
&-4C_{ij}\phi-4C_{kj}C^k{}_i)-12\varphi_0'C^{ij}\Big[2B_iC_{jk,}{}^k+2B^kC_{ki,j}-B_{j,i}-B_iB_j'+B_i\phi_{,j}-B_kC_{ji,}{}^k\\
&+2C_{ik}B^k{}_{,j}+2B_{j,i}\phi-4\mathcal{H}B_iB_j\Big]-6\varphi_0'\Big[\phi+B^i{}_{,i}+B^iB_i'-B^iB_iB^j{}_{,j}-B^iB_jB^j{}_{,i}-4B^iB_i'\phi\\
&+2\mathcal{H}B^iB_i-2B^iB_i\phi'-8\mathcal{H}B^iB_i\phi+4B^i\phi_{,i}-2B^i{}_{,i}\phi+4B^i{}_{,i}\phi^2-B^i\phi_{,i}\Big]+6U_{,\varphi}a^2=0\,.
\end{aligned} \tag{5.49}$$

One can again see the coupling between first order tensor and scalar perturbations. For example, the $\delta\varphi_{1,i}C^{ij}\phi_{,j}$ contains a term that looks like $\delta\varphi_{1,i}{h_1}^{ij}\phi_{1,j}$, which occurs only at third order and beyond.

Again, we refrain from splitting up the perturbations order by order for ease of presentation. Once split up, one can then replace the metric perturbations by using the appropriate order field equations. We present the Einstein tensor at third order in the next section. Note also that Eq. (5.49) implicitly contains the Klein-Gordon equations at first and second order. We refer the reader to, for example, Ref. [106], for a detailed exposition of the second order Klein-Gordon equation.

5.4.3 Einstein Tensor

The Einstein tensor, which describes the geometry of the universe, is defined (as shown in Section 1.1) as

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R\,, \tag{5.50}$$

where $R^\mu{}_\nu$ is the Ricci curvature tensor and R is the Ricci scalar. Here, we give the components of the Einstein tensor up to third order:

$$\begin{aligned} a^2 G^0{}_0 = & -3\mathcal{H}^2 + \nabla^2 C^j{}_j - C_{ij,}{}^{ij} + 2\mathcal{H}(-C^{i\prime}{}_i + B^i{}_{,i} + 3\mathcal{H}\phi) + C^j{}_{j,i}(\tfrac{1}{2}C^k{}_{k,}{}^i - 2C^{ik}{}_{,k}) + C'_{ij}(\tfrac{1}{2}C^{ij\prime} - B^j{}_{,}{}^i) \\ & + B^i\Big[C^{j\prime}{}_{j,i} - C'_{ij,}{}^j + \tfrac{1}{2}\big(\nabla^2 B_i - B_{j,i}{}^j\big) + 2\mathcal{H}\big(C^j{}_{j,i} - 2C_{ij,}{}^j - \phi_{,i}\big)\Big] + 2C^{ij}\Big[2C_{jk,i}{}^k - C^k{}_{k,ij} - \nabla^2 C_{ij} \\ & + 2\mathcal{H}(C'_{ij} - B_{i,j})\Big] + C_{jk,i}(C^{ik}{}_{,}{}^j - \tfrac{3}{2}C^{jk}{}_{,}{}^i) + C^{i\prime}{}_i(B_{j,}{}^j - \tfrac{1}{2}C^{j\prime}{}_j + 4\mathcal{H}\phi) + 2C^{ij}{}_{,i}C_{jk,}{}^k \\ & + \tfrac{1}{4}B_{j,i}(B^i{}_{,}{}^j + B^j{}_{,}{}^i) - 3\mathcal{H}^2(4\phi^2 - B_i B^i) - \tfrac{1}{2}B^i{}_{,i}B_{j,}{}^j - 4\mathcal{H}B^i{}_{,i}\phi + \mathbb{G}^0{}_0\,, \end{aligned} \tag{5.51}$$

$$\begin{aligned} a^2 G^0{}_i = & C^{k\prime}{}_{k,i} - C'_{ik,}{}^k - \tfrac{1}{2}\big(B_{k,i}{}^k - \nabla^2 B_i\big) - 2\mathcal{H}\phi_{,i} + 8\mathcal{H}\phi_{,i}\phi + C'_{ij}\big(2C^{kj}{}_{,k} - C^k{}_{k,}{}^j + \phi_{,}{}^j\big) - C^{j\prime}{}_j\phi_{,i} \\ & + 2C^{kj}\Big[C'_{ik,j} - C'_{jk,i} + \tfrac{1}{2}\big(B_{k,ij} - B_{i,kj}\big)\Big] + B^j\big(C_{kj,i}{}^k - C^k{}_{k,ij} + C_{ik,k}{}^j - \nabla^2 C_{ij} - 2\mathcal{H}B_{j,i}\big) \\ & - \tfrac{1}{2}\Bigg(B_{i,j} + B_{j,i}\Bigg)\phi_{,}{}^j + (B_{i,j} - B_{j,i})\big(\tfrac{1}{2}C^k{}_{k,}{}^j - C^{jk}{}_{,k}\big) - C_{ik,j}\big(B^k{}_{,}{}^j - B^j{}_{,}{}^k\big) + B^j{}_{,j}\phi_{,i} \\ & + \phi\Big[B_{j,i}{}^j - \nabla^2 B_i + 2\Big(C'_{ij,}{}^j - C^{j\prime}{}_{j,i}\Big)\Big] - C^{kj\prime}C_{kj,i} + \mathbb{G}^0{}_i\,, \end{aligned} \tag{5.52}$$

$$
\begin{aligned}
a^2G^i{}_j=&C^{i''}{}_j+2\mathcal{H}C^{i'}_{\ j}-\tfrac{1}{2}(B^{i'}{}_{,j}+B_{j,}{}^{i'})-C^l{}_{l,j}{}^i+C^i{}_{l,j}{}^l-\phi_,{}^i{}_j-\nabla^2C^i{}_j+C_{jl,}{}^{il}-\mathcal{H}\Big(B^i{}_{,j}+B_{j,}{}^i\Big)\\
&+\delta^i{}_j\Big\{\Big(\mathcal{H}^2-\tfrac{2a''}{a}\Big)(1-2\phi)+2\mathcal{H}\Big(B^k{}_{,k}-C^{k'}{}_k+\phi'\Big)+B^{k'}{}_k-C^{kl}{}_{,kl}-C^{k''}{}_k+\nabla^2\Big(\phi+C^l{}_l\Big)\Big\}\\
&+B^k\Big[C_{jk,}{}^{i'}+C^{i'}{}_{k,j}-2C^{i'}{}_{j,k}+2\mathcal{H}(C_{jk,}{}^i+C^i{}_{k,j}-C^i{}_{j,k})+\tfrac{1}{2}\Big(B_{j,}{}^i{}_k+B^i{}_{,jk}-2B_{k,}{}^i{}_j\Big)\Big]\\
&+(C^{k'}_{\ k}-\phi'-B^k{}_{,k})(C^{i'}_{\ j}-\tfrac{1}{2}\Big(B^i{}_{,j}+B_{j,}{}^i\Big))+C^{ik'}\Big(B_{j,k}-2C'_{kj}\Big)+C'_{kj}B^i{}_,{}^k+\phi_,{}^i\phi_{,j}\\
&+(B^{k'}-2C^{kl}{}_{,l}+C^l{}_{l,}{}^k+\phi_,{}^k)(C_{jk,}{}^i+C^i{}_{k,j}-C^i{}_{j,k})+\tfrac{1}{2}B^i(B_{k,j}{}^k-\nabla^2B_j+4\mathcal{H}\phi_{,j}-2C^{k'}{}_{k,j}+2C'_{kj,}{}^k)\\
&+2C^{ik}\Big[\tfrac{1}{2}\Big(B'_{j,k}+B'_{k,j}\Big)-C''_{kj}+\phi_{,jk}-C_{kl,j}{}^l-C_{jl,k}{}^l+\nabla^2C_{kj}+C^l{}_{l,jk}+\mathcal{H}\Big(B_{j,k}+B_{k,j}-2C'_{kj}\Big)\Big]\\
&-\tfrac{1}{2}\Big(B_{k,}{}^iB^k{}_{,j}+B_{j,}{}^kB^i{}_{,k}\Big)+\phi\Big[(B_{j,}{}^{i'}+B^{i'}_{\ ,j}+2\phi_,{}^i{}_j+2\mathcal{H}(B_{j,}{}^i+B^i{}_{,j})-2C^{i''}_{\ j}-4\mathcal{H}C^{i'}_{\ j}\Big]\\
&+2\Big(C^i{}_{k,l}C^k{}_{j,}{}^l-C^l{}_{j,}{}^kC^i{}_{k,l}+C^{kl}{}_{,j}C_{kl,}{}^i\Big)+2C^{kl}\Big[C_{kl,j}{}^i-C_{jl,}{}^i{}_k-C^i{}_{l,jk}+C^i{}_{j,kl}\Big]+\mathbb{G}^i_{oj}\\
&+\delta^i{}_j\Big\{\Big(\mathcal{H}^2-\tfrac{2a''}{a}\Big)(4\phi^2-B_kB^k)+2\phi\Big[C^{k''}{}_k-B^{k'}{}_k-\nabla^2\phi+2\mathcal{H}(C^{k'}{}_k-2\phi'-B^k{}_{,k})\Big]\\
&\quad+B^k\Big[2C^{l'}{}_{l,k}-2C'_{kl,}{}^l+\nabla^2B_k-B_{l,k}{}^l+2\mathcal{H}(B'_k-\phi_{,k}-2C^l{}_{k,l}+C^l{}_{l,k})\Big]+C^{kl'}\Big(\tfrac{3}{2}C'_{kl}-B_{l,k}\Big)\\
&\quad+2C^{kl}\Big[C''_{kl}-\nabla^2C_{kl}+2\mathcal{H}C'_{kl}+2C_{lm,k}{}^m-C^m{}_{m,kl}-2\mathcal{H}B_{l,k}-B'_{l,k}-\phi_{,kl}\Big]+2B^{k'}(C^l{}_{l,k}-C_{kl,}{}^l)\\
&\quad+C^{k'}{}_k\Big(B^l{}_{,l}-\tfrac{1}{2}C^{l'}{}_l\Big)+2C^{kl}{}_{,k}C_{lm,}{}^m+C_{lm,k}\Big(C^{km}{}_,{}^l-\tfrac{3}{2}C^{lm}{}_,{}^k\Big)-C^l{}_{l,k}\Big(2C^k{}_{m,}{}^m-\tfrac{1}{2}C^m{}_{m,}{}^k\Big)\\
&\quad+\phi'\Big(C^{k'}{}_k-B^k{}_{,k}\Big)-\tfrac{1}{4}\Big(2B^k{}_{,k}B_{l,}{}^l-B_{l,k}B^k{}_,{}^l-3B^l{}_{,k}B_{l,}{}^k\Big)+\phi_{,k}\Big(C^l{}_{l,}{}^k-2C^{lk}{}_{,l}-\phi_,{}^k\Big)+\mathbb{G}^i_{dj}\Big\}\,, \qquad (5.53)
\end{aligned}
$$

where $\mathbb{G}^0{}_0, \mathbb{G}^0{}_i, \mathbb{G}^i{}_j$ are the third order corrections (the latter split into a diagonal part $\mathbb{G}^i_{oj}$, and an off diagonal part $\mathbb{G}^i_{dj}$) which we give in the appendix as Eqs. (B.1), (B.2), (B.3) and (B.4), respectively. Note that, in calculating the third order components given above, we have implicitly obtained the full second order Einstein tensor components for fully general perturbations (i.e. including all scalar, vector and tensor perturbations).

5.5 Discussion

In this chapter we have developed the essential tools for cosmological perturbation theory at third order. Starting with the definition of the active gauge transformation we have extended the work presented in Section 2.3 to third order, and derived gauge invariant variables, namely the curvature perturbation on uniform density hypersurfaces, ζ_3, and the density perturbation on uniform curvature hypersurfaces. We also relate the curvature perturbation ζ_3, obtained using the spatial metric split of Ref. [112] to that introduced by Salopek and Bond [130], which is also popular at higher order.

We have then calculated the energy and momentum conservation equations for a general perfect fluid at third order, including all scalar, vector and tensor perturbations. The Klein-Gordon equation for a canonical scalar field minimally coupled to gravity is also presented. We highlight the coupling in these conservation equations between scalar and tensor perturbations which only occurs at third order and above. Finally, we have presented the Einstein tensor components to third order. No large scale approximation is employed for the tensor components or the conservation equations. All equations are given without specifying a particular gauge, and can therefore immediately be rewritten in whatever choice of gauge is desired. However, as examples to illustrate possible gauge choices, we give the energy conservation equation on large scales (and only allowing for scalar perturbations) in the flat and the uniform density gauge. This gives an evolution equation for the curvature perturbation ζ_3, Eq. (5.44). As might be expected from fully non-linear calculations [100] and second order perturbative calculations [110], the curvature perturbation is also conserved at third order on large scales in the adiabatic case. It is worth noting that higher order perturbation theory, as discussed in this chapter, has the advantage of being valid on all scales whereas fully non-linear methods, such as separate universe approaches are gradient expansions (in powers of k/aH), and so are only valid on superhorizon scales.

Another application of our third order variables and equations, in particular the Klein-Gordon equation (5.49), is the calculation of the trispectrum by means of the field equations. Whereas calculations of the trispectrum so far derive the trispectrum from the fourth order action, it should also be possible to use the third order field equations instead. The equivalence of the two approaches for calculating the bispectrum, using the third order action or the second order field equations, has been shown in Ref. [136]. Having included tensor as well as scalar perturbations it will be in particular interesting to see and be an important consistency check for the theory whether we arrive at the same result as Ref. [137].

A final advantage of extending perturbation theory to third order is that, in doing so, one obtains a deeper insight into the second order theory. Also second order perturbation theory, despite remaining challenging, becomes less daunting having explored some of the third order theory.

6 Discussion and Conclusions

6.1 Summary

The main focus of this thesis has been the study of cosmological perturbations beyond linear order. In Section 1.1 we introduced the standard cosmological model, giving the background evolution and constraint equations and briefly discussing inflationary cosmology. In Chapter 2 we introduced the theory of cosmological perturbations up to second order, presenting the perturbed metric tensor and energy momentum tensor for both a perfect fluid, including all types of scalar, vector and tensor perturbations, and a scalar field. We then considered the behaviour of the perturbations under a gauge transformation in the active approach, using this behaviour to define gauges and construct gauge invariant variables. Next, we discussed the thermodynamics of a perfect fluid and defined the non-adiabatic pressure perturbation, closing the chapter by considering how non-adiabatic pressure perturbations can arise naturally in multiple-component systems.

In Chapter 3 we continued the discussion of the foundations of cosmological perturbation theory by presenting the dynamic and constraint equations, from energy momentum conservation and the Einstein field equations, up to second order in the perturbations. Starting with the linear theory, we gave the governing equations for scalar, vector and tensor perturbations of a perfect fluid in a gauge dependent form, i.e. without fixing a gauge. We then presented the gauge invariant form of the equations for three different gauges: the uniform density, the uniform curvature, and the longitudinal gauges, solving the equations for the latter two in the case of scalar perturbations. We then presented the Klein-Gordon equation for a scalar field, to linear order for both a canonical and non-canonical action, and highlighted the important difference between the adiabatic sound speed and the speed with which perturbations travel in a scalar field system (the phase speed). Finally, we investigated the perturbations of a system containing both dark energy and dark matter.

Having laid the foundations of the linear order theory we then discussed the second order theory, presenting the governing equations for a perfect fluid derived, as at first order, from energy momentum conservation and the Einstein field equations. We then presented all equations in the uniform curvature gauge, which we then used in Chapter 4, and for scalars only in the Poisson gauge, in order to connect with the literature.

In Chapter 4 we used the tools developed in the previous chapters to investigate non-linear vector perturbations in the early universe. Using the qualitative difference between the linear theory and the higher order theory we showed that, at second order in perturbation theory, vorticity is sourced by a coupling quadratic in linear energy density and entropy perturbations, extending Crocco's theorem from a classical framework to an expanding, cosmological background. In order to show this, we first defined the vorticity tensor in General Relativity and calculated the vorticity tensor at first and second order in cosmological perturbation theory using the metric tensor and fluid four velocity presented earlier. We then computed the evolution of the vorticity tensor by taking the time derivative, and using the governing equations from Chapter 3 to simplify the expressions and replace the metric perturbation variables. We found that at linear order the vorticity is not sourced in the absence of anisotropic stress, in agreement with the previous results known in the literature. However, at second order we obtained the novel result that there exists a non-zero source term for a fluid with a general equation of state (depending on both the energy density and entropy) which is quadratic in linear energy density and non-adiabatic pressure perturbations.

Having derived this qualitative result we then gave the first quantitative solution, estimating the magnitude and scale dependence of the induced vorticity using simple input power spectra: the energy density derived in Chapter 3 approximated for small $k\eta$, and an ansatz for the non-adiabatic pressure perturbation. We found that the resulting spectrum has a surprisingly large magnitude, given that it is a second order effect, and a dependence on the wavenumber to the power of at least seven, given our assumptions. Thus, this spectrum is hugely amplified on small scales, rendering the vorticity not only possibly observable, but also important for the general understanding of the physical processes taking place in the early universe.

In Chapter 5 we extended the formalism of cosmological perturbation theory from the second order theory to third order, starting with the gauge transformation rules and defining gauge invariant variables. Then, considering perfect fluids and scalar, vector and tensor perturbations we presented the energy and momentum conservation equations and the Klein-Gordon equation for a scalar field without fixing a gauge. Finally, we gave the components of the Einstein tensor at third order also in a gauge dependent form.

6.2 Future Directions

The work presented in this thesis can naturally be extended in several directions. One clear extension is to study the vorticity generation in specific models, moving beyond the simple ansatz for the non-adiabatic pressure perturbation used in Chapter 4. As discussed in Section 2.5.1, non-adiabatic pressure perturbations naturally arise in any system consisting of more than one component, such as a multiple fluid, or multiple scalar fields model. Even in the case of zero intrinsic non-adiabatic pressure perturbation, there exists a relative non-adiabatic pressure perturbation between the different components of the system which is proportional to the relative entropy perturbation, i.e.

$$\delta P_{\rm nad} \propto \mathcal{S}_{IJ} \equiv 3\mathcal{H}\left(\frac{\delta\rho_J}{\rho'_{0J}} - \frac{\delta\rho_I}{\rho'_{0I}}\right) .$$

This can then be written in terms of field variables using the definitions in Chapter 2 in order to obtain an expression for the relative entropy perturbation in multi-field inflation models.

Another way in which this work can be extended is to exploit the potential for this mechanism to generate magnetic fields. As mentioned in Chapter 4, magnetic fields and vorticity are intimately related. This relationship has been studied in some detail in classical fluid mechanics and in astrophysical situations (see, e.g., Ref. [153] for a review), though there is still much work to be done on incorporating magnetic fields into cosmological perturbation theory. Since the energy density is related to the magnetic field, b^i, through the expression

$$\rho_b \sim b^2 , \qquad (6.1)$$

when considering linear perturbations of the energy density one often considers perturbations of the magnetic field to 'half order'. That is, one assumes that the magnetic energy density as shown above to be of order b^2, is formally the same order as the scalar density perturbation. ensuring that the perturbed version of Eq. (6.1) holds at linear order (see, e.g., Ref. [26]). However, one does not need to use this technique and, in fact, when considering higher order perturbations it is not immediately clear how this will work. Instead, one can consider and develop cosmological perturbation theory to consistently include magnetic fields at integer order. Having done that, it will be possible to obtain estimates of the primordial magnetic field produced by the vorticity generated using cosmological perturbations as shown in this thesis by using the simple ansatz considered in Chapter 4 [28].

A further extension to this work will be to consider the primordial magnetic field generated from the relative non-adiabatic pressure perturbation in a system with multiple components, such as a hybrid inflation model.

6.3 Outlook

Cosmological perturbation theory has matured over the last few decades and has been incredibly successful in making predictions that agree with observations. However, with the data sets available to us continually growing in their size and quality it is now a realistic aim to use perturbation theory even beyond linear order to make predictions which are observationally testable.

The main observable with which we can constrain our cosmological models is the CMB, of which we have, to date, collected much information, predominantly from the successful WMAP experiment. The PLANCK satellite [1] will greatly improve the temperature measurements of the CMB and together with the proposed CMBPOL satellite [21], will measure the polarisation of the background radiation. Since the CMB is not affected by the astrophysics of the late universe, usually one prefers the use of CMB data over other techniques for the study of higher order observables.

However, with the recent technological advances, Large Scale Structure (LSS) surveys, such as the Sloan Digital Sky Survey (SDSS) [147] and the proposed 21cm anisotropy maps are attracting more attention as a way to probe the evolution of the universe at different epochs of its history. The 21cm signal, generated by neutral Hydrogen left over after the Big Bang, can probe the era after decoupling but before galaxy formation, i.e. between redshift 200 and 30, while LSS surveys probe out to around redshift 1. The 21cm anisotropy maps contain much more data than the CMB [95], though it should be noted that it is still not clear whether the foregrounds can be removed with enough accuracy to enable reliable results.

This is but one area where calculations at higher order can be tested against observations. Thus, the future study of cosmological perturbation theory will greatly increase our understanding, serving to broaden and deepen our knowledge of the universe in which we live.

A Second Order $i-j$ Einstein Equation

In this appendix we give the second order $(i-j)$ component of the Einstein Equation omitted from Section 3.2:

$$
\begin{aligned}
&C_2^{i''}{}_j+2\mathcal{H}C_{2j}^{i'}-\tfrac{1}{2}(B_2^{i'}{}_{,j}+B_{2j,}{}^{i'})-C_{2l,j}^{l}{}^{i}+C_{2l,j}^{i}{}^{l}-\nabla^2C_{2j}^{i}+C_{2jl,}{}^{il}-\phi_{2,}{}^{i}{}_{j}-\mathcal{H}\Big(B_{2,j}^{i}+B_{2j,}{}^{i}\Big)\\
&+\delta^i{}_j\Big\{2\Big(\tfrac{2a''}{a}-\mathcal{H}^2\Big)\phi_2+2\mathcal{H}\Big(B_{2}^{k}{}_{,k}-C_2^{k'}{}_k+\phi_2'\Big)+B_2^{k'}{}_k-C_2^{kl}{}_{,kl}-C_2^{k''}{}_k+\nabla^2\Big(\phi_2+C_{2l}^{l}\Big)\Big\}\\
&+B_1^k\Big[C_{1jk,}{}^{i'}+C_1^{i'}{}_{k,j}-2C_1^{i'}{}_{j,k}+2\mathcal{H}(C_{1jk,}{}^{i}+C_{1k,j}^{i}-C_{1j,k}^{i})+\tfrac{1}{2}\Big(B_{1j,}{}^{i}{}_{k}+B_{1,jk}^{i}-2B_{1k,}{}^{i}{}_{j}\Big)\Big]\\
&+(C_1^{k'}{}_k-\phi_1'-B_1^k{}_{,k})(C_{1j}^{i'}-\tfrac{1}{2}\Big(B_{1,j}^{i}+B_{1j,}{}^{i}\Big))+C_1^{ik'}\Big(B_{1j,k}-2C_{1kj}'\Big)+C_{1kj}'B_{1,}^{i}{}^{k}+\phi_{1,}{}^{i}\phi_{1,j}\\
&+(B_1^{k'}-2C_1^{kl}{}_{,l}+C_{1l,}^{l}{}^{k}+\phi_{1,}{}^{k})(C_{1jk,}{}^{i}+C_{1k,j}^{i}-C_{1j,k}^{i})+\tfrac{1}{2}B_1^i(B_{1k,j}{}^{k}-\nabla^2B_{1j}+4\mathcal{H}\phi_{1,j}-2C_1^{k'}{}_{k,j}+2C_{1kj,}'{}^{k})\\
&+2C_1^{ik}\Big[\tfrac{1}{2}\Big(B_{1j,k}'+B_{1k,j}'\Big)-C_{1kj}''+\phi_{1,jk}-C_{1kl,j}{}^{l}-C_{1jl,k}{}^{l}+\nabla^2C_{1kj}+C_{1l,jk}^{l}+\mathcal{H}\Big(B_{1j,k}+B_{1k,j}-2C_{1kj}'\Big)\Big]\\
&-\tfrac{1}{2}\Big(B_{1k,}{}^{i}B_{1,j}^{k}+B_{1j,}{}^{k}B_{1,k}^{i}\Big)+\phi_1\Big[(B_{1j,}{}^{i'}+B_{1,j}^{i'}+2\phi_{1,}{}^{i}{}_{j}+2\mathcal{H}(B_{1j,}{}^{i}+B_{1,j}^{i})-2C_{1j}^{i''}-4\mathcal{H}C_{1j}^{i'}\Big]\\
&+2\Big(C_{1k,l}^{i}C_{1j,}^{k}{}^{l}-C_{1j,}^{l}{}^{k}C_{1k,l}^{i}+C_1^{kl}{}_{,j}C_{1kl,}{}^{i}\Big)+2C_1^{kl}\Big[C_{1kl,j}{}^{i}-C_{1jl,}{}^{i}{}_{k}-C_{1l,jk}^{i}+C_{1j,kl}^{i}\Big]\\
&+\delta^i{}_j\Big\{\Big(\mathcal{H}^2-\tfrac{2a''}{a}\Big)(4\phi_1^2-B_{1k}B_1^k)+2\phi_1\Big[C_1^{k''}{}_k-B_1^{k'}{}_k-\nabla^2\phi_1+2\mathcal{H}(C_1^{k'}{}_k-2\phi_1{}'-B_{1,k}^{k})\Big]\\
&\quad+B_1^k\Big[2C_1^{l'}{}_{l,k}-2C_{1kl,}'{}^{l}+\nabla^2B_{1k}-B_{1l,k}{}^{l}+2\mathcal{H}(B_{1k}'-\phi_{1,k}-2C_{1k,l}^{l}+C_{1l,k}^{l})\Big]+C_1^{kl'}\Big(\tfrac{3}{2}C_{1kl}'-B_{1l,k}\Big)\\
&\quad+2C_1^{kl}\Big[C_{1kl}''-\nabla^2C_{1kl}+2\mathcal{H}C_{1kl}'+2C_{1lm,k}{}^{m}-C_1^m{}_{m,kl}-2\mathcal{H}B_{1l,k}-B_{1l,k}'-\phi_{1,kl}\Big]+2B_1^{k'}(C_{1l,k}^{l}-C_{1kl,}{}^{l})\\
&\quad+C_1{}^{k'}{}_k\Big(B_{1,l}^{l}-\tfrac{1}{2}C_1^{l'}{}_l\Big)+2C_1^{kl}{}_{,k}C_{1lm,}{}^{m}+C_{1lm,k}\Big(C_1^{km}{}_{,}{}^{l}-\tfrac{3}{2}C_1^{lm}{}_{,}{}^{k}\Big)-C_{1l,k}^{l}\Big(2C_1^k{}_{m,}{}^{m}-\tfrac{1}{2}C_1^m{}_{m,}{}^{k}\Big)\\
&\quad+\phi_1{}'\Big(C_1{}^{k'}{}_k-B_{1,k}^{k}\Big)-\tfrac{1}{4}\Big(2B_{1,k}^{k}B_{1l,}{}^{l}-B_{1l,k}B_{1,}^{k}{}^{l}-3B_{1,k}^{l}B_{1l,}{}^{k}\Big)+\phi_{1,k}\Big(C_1{}^{l}{}_{l,}{}^{k}-2C_1{}^{lk}{}_{,l}-\phi_{1,}{}^{k}\Big)\Big\}\\
&=8\pi Ga^2\Big\{\delta P_2\delta^i{}_j+2(\rho_0+P_0)v_1{}^{i}(v_{1j}+B_{1j})\Big\}\,. \qquad \text{(A.1)}
\end{aligned}
$$

B Third Order Einstein Tensor

Here, we give the third order corrections to the components of the Einstein tensor. We do not split up perturbations order by order.

$$
\begin{aligned}
\mathbb{G}^0{}_0 ={}& 2C^{ij}\Bigg[2C_{ik}(C^l{}_{l,j}{}^k - 2C^{kl}{}_{,jl} + \nabla^2 C_j{}^k - 2\mathcal{H}C'_j{}^k) + (2C_{jk,i} - C_{ij,k})(C^l{}_{l,}{}^k - 2C^k{}_{l,}{}^l) + C_{il,k}(3C_j{}^l{}_,{}^k - C_j{}^k{}_{,l}) \\
&+ C^k{}_{k,i}(2C^j{}_{l,}{}^l - \tfrac{1}{2}C^l{}_{l,j}) + B^i\Big\{C'_{jk,}{}^k - C^{k\prime}{}_{k,j} + \tfrac{1}{2}(B_{k,j}{}^k - \nabla^2 B_j) + \mathcal{H}(4C_{jk,}{}^k + 2\phi_{,j} - 3\mathcal{H}B_j - 2C^k{}_{k,j})\Big\} \\
&+ C'_{ik}(B^k{}_{,j} + B_{j,}{}^k) - 2C_{ik,}{}^k C_{jl,}{}^l - (B^k{}_{,k} + \phi)(C'_{ij} - B_{j,i}) - \tfrac{1}{4}B_{k,i}(B^k{}_{,j} + 2B_{j,}{}^k) - \tfrac{1}{4}B_{i,k}B_{j,}{}^k - C^{k\prime}{}_k B_{j,i} \\
&+ 8\mathcal{H}C_{ik}B^k{}_{,j}\Bigg] + 2C^{kj}B^i\Big[C'_{ik,j} - C'_{jk,i} + \tfrac{1}{2}(B_{k,ij} - B_{i,jk}) + 2\mathcal{H}(2C_{ik,j} - C_{kj,i})\Big] + C^{ij\prime}\Big[2B^k(C_{ki,j} - C_{ji,k}) \\
&+ B_i(2C_{jk,}{}^k - C^k{}_{k,j}) + B_i\phi_{,j} + (2B_{j,i} - C'_{ij})\phi\Big] + 8\mathcal{H}\phi^2(B^i{}_{,i} - C^{i\prime}_i) + (B_{i,j} - B_{j,i})(\tfrac{1}{2}B^iC^k{}_{k,}{}^j - B^iC^{jk}{}_{,k}) \\
&- \tfrac{1}{2}(B_{i,j} + B_{j,i})(\phi B^j{}_,{}^i + B^i\phi_,{}^j) - 2\mathcal{H}B^iB^jB_{j,i} + B^j{}_{,j}\Big[B^i(2C_{ik,}{}^k - C^k{}_{k,i} + \phi_{,i} - 2\mathcal{H}B_i) + B^i{}_{,i}\phi - 2C^{i\prime}_i\phi\Big] \\
&+ B_{k,j}B^i(C^{kj}{}_{,i} - 2C_i{}^k{}_,{}^j) + 2\mathcal{H}B^i\Big[C^{j\prime}_jB_i + 2\phi(2C_{ij,}{}^j - C^j{}_{j,i} + 2\phi_{,i} - 3\mathcal{H}B_i)\Big] + C^{i\prime}_iC^{j\prime}_j\phi \\
&+ B^iB^j(2C_{jk,i}{}^k - C^k{}_{k,ij} - \nabla^2 C_{ij}) + B^iC^{j\prime}_j(C^k{}_{k,i} - 2C_{ik,}{}^k - \phi_{,i}) + B^i\phi(B^j{}_{,ij} + 2C'_{ij,}{}^j - 2C^{j\prime}{}_{j,i} - \nabla^2 B_i)\,,
\end{aligned} \tag{B.1}
$$

$$
\begin{aligned}
\mathbb{G}^0{}_i ={}& C_{kj}\Big[2C^{jl}\Big(C^k{}_{l,i}{}' - C_{il,}{}^{k\prime}\Big) + C_i{}^j\Big(C^l{}_{l,}{}^k - 2C^{kl}{}_{,l} - 2\phi_,{}^k\Big) + C'_{il}\Big(C^{jk}{}_,{}^l - 2C^{kl}{}_,{}^j\Big) + 2C^{jl\prime}C^k{}_{l,i}\Big] \\
&+ C^{kj}\Big[(B_{j,i} - B_{i,j})(C^l{}_{l,k} - C_{kl,}{}^l) + (B_{l,i} - B_{i,l})(C_{jk,}{}^l - 2C_k{}^l{}_{,j}) + 2(B_{j,l} - B_{l,j})(C^l_{ik,} - C_i{}^l{}_{,k}) + (B_{j,i} + B_{i,j})\phi_{,k} \\
&+ 2(C'_{jk} - B_{k,j})\phi_{,i} + 2\phi(2C'_{jk,i} - 2C'_{ik,j} + B_{i,kj} - B_{k,ij})\Big] + 2B_j\Big[C^{kj}(C^l{}_{l,ik} - C_{kl,i}{}^l - C_{il,k}{}^l + \nabla^2 C_{ik}) \\
&+ C^{kl}(C_{kl,i}{}^j - C^j{}_{l,ik} - C_{il,k}{}^j + C_i{}^j{}_{,kl}) + (C^l{}_{l,}{}^k - C^{kl}{}_{,l})(C^j{}_{k,i} + C_{ik,}{}^j - C^j{}_{i,k}) + C_{il,k}(C^{jl}{}_,{}^k - C^{jk}{}_,{}^l) \\
&+ \tfrac{1}{2}C_{kl,i}C^{kl}{}_,{}^j\Big] + 2C'_{ij}\phi(C^k{}_{k,}{}^j - 2C^{kj}{}_{,k}) + B^j\Big[B_j(C'_{ik,}{}^k - C^{k\prime}{}_{k,i} + \tfrac{1}{2}(B_{k,i}{}^k - \nabla^2 B_i)) + C'_{ik}B_{j,}{}^k \\
&+ B_{j,i}(B^k{}_{,k} - C^{k\prime}_k) + \tfrac{1}{2}B_{j,k}(2C_i^{\prime k} - B^k{}_{,i}B_{i,}{}^k)\Big] + 2\phi\Big[(B_{i,j} - B_{j,i})(C^{kj}{}_{,k} - \tfrac{1}{2}C^k{}_{k,}{}^j) + \phi_,{}^j(B_{i,j} + B_{j,i}) \\
&+ C'_{kj}C^{kj}{}_{,i} + 2C^{j\prime}_j\phi_{,i} - 2C'_{ij}\phi_,{}^j - B_{k,j}(C_i{}^j{}_,{}^k - C_i{}^k{}_,{}^j) - 2B_{j,}{}^j\phi_{,i}\Big] + 2C_{kj}C^{jl}(B_{i,l}{}^k - B_{l,i}{}^k) \\
&+ B^j\Big[\phi_{,i}(C^k{}_{k,j} - 2C_{jk,}{}^k + 4\mathcal{H}B_j) + 2\phi(C^k{}_{k,ij} - C_{jk,}{}^k{}_i - C_{ik,j}{}^k + \nabla^2 C_{ij} + 4\mathcal{H}B_{j,i}) + 2\mathcal{H}(B^kC_{jk,i} + 2C_{kj}B^k{}_{,i}) \\
&+ \phi_,{}^k(C_{ij,k} - C_{ik,j} + C_{jk,i})\Big] + 2\phi^2(2C^{j\prime}{}_{j,i} - 2C'_{ij,}{}^j - B_{j,i}{}^j + \nabla^2 B_i - 12\mathcal{H}\phi_{,i})\,,
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
\mathbb{G}^{i}_{\mathrm{o}j}=&\,2C^{ik}C_{kl}\bigg[2C^{l\prime\prime}_{j}-B^{l\prime}{}_{,j}-B_{j,}{}^{l\prime}-2C^{m}{}_{m,j}{}^{l}+2C^{l}{}_{m,j}{}^{m}+2C_{jm,}{}^{lm}-2\nabla^{2}C^{l}{}_{j}-2\phi_{,j}{}^{l}\bigg]\\
&+4C^{ik}C^{lm}(C_{km,jl}-C_{lm,kj}+C_{jm,kl}-C_{jk,lm})+4C^{km}C_{kl}(C_{jm,}{}^{il}-C^{l}{}_{m,}{}^{i}{}_{j}+C^{i}{}_{m,j}{}^{l}-C^{i}{}_{j,}{}^{l}{}_{m})\\
&+C^{ik}\bigg[2(2C^{lm}{}_{,m}-C^{m}{}_{m,}{}^{l}-\phi_{,}{}^{l}-B^{l\prime})(C_{kl,j}+C_{jl,k}-C_{jk,l})+4C_{jm,l}(C_{k}{}^{l}{}_{,}{}^{m}-C_{k}{}^{m}{}_{,}{}^{l})-2C_{lm,j}C^{lm}{}_{,k}\\
&-2C'_{jk}(C^{l\prime}_{l}-B^{l}{}_{,l}-\phi')+2C'_{jl}(2C^{l\prime}_{k}-B_{k,}{}^{l})+C^{l\prime}_{l}(B_{k,j}+B_{j,k})-2C'_{kl}B_{j,}{}^{l}-(\phi'+B^{l}{}_{,l})(B_{k,j}+B_{j,k})\\
&+B_{l,j}B^{l}{}_{,k}+B_{j,l}B_{k,}{}^{l}-2\phi_{,k}\phi_{,j}-2\phi(B'_{k,j}+B'_{j,k}+\phi_{,jk}+4\mathcal{H}C_{kl}(2C^{l\prime}_{j}-B^{l}{}_{,j}-B_{j,}{}^{l})-4\mathcal{H}B_{k}\phi_{,j}\bigg]\\
&+C^{kl}\bigg[2(C^{m}{}_{m,l}-2C_{lm,}{}^{m})(C^{i}{}_{j,k}-C^{i}{}_{k,j}-C_{jk,}{}^{i})+2(2C^{m}{}_{l,k}-C_{kl,}{}^{m})(C_{jm,}{}^{i}+C^{i}{}_{m,j}-C^{i}{}_{j,m})-4C_{km,}{}^{i}C^{m}{}_{l,j}\\
&+4(C_{jl,m}-C_{jm,l})(C^{im}_{,k}-C^{i}{}_{k,}{}^{m})-2C^{i\prime}_{j}C'_{kl}+4C^{i\prime}_{k}C'_{jl}+(B^{i}{}_{,j}+B_{j,}{}^{i})(C'_{kl}-B_{l,k})-2\phi_{,}{}^{l}(C_{jk,}{}^{i}+C^{i}{}_{k,j}-C^{i}{}_{j,k})\\
&+2C^{i\prime}_{j}B_{l,k}-2C^{i\prime}_{k}B_{j,l}-2C'_{jk}B^{i}{}_{,l}+B_{k,}{}^{i}B_{l,j}-2B'_{k}(C_{jl,}{}^{i}+C^{i}{}_{l,j}-C^{i}{}_{j,l})\bigg]-B^{k}B_{k}\bigg[C^{i\prime\prime}_{j}-\tfrac{1}{2}(B^{i\prime}{}_{,j}+B_{j,}{}^{i\prime})\\
&-\phi_{,}{}^{i}{}_{j}-\mathcal{H}(B^{i}{}_{,j}+B_{j,}{}^{i}-2C^{i\prime}_{j})\bigg]+B^{i}B^{k\prime}\bigg[C^{l}{}_{l,jk}-C^{l}{}_{k,jl}-C^{l}{}_{j,kl}+\nabla^{2}C_{jk}+2\mathcal{H}B_{k,j}\bigg]\\
&+B^{k}B^{l}\bigg[C_{kl,}{}^{i}{}_{j}-C_{jl,}{}^{i}{}_{k}-C^{i}{}_{l,jk}+C^{i}{}_{j,kl}\bigg]+B^{i}\bigg[C'_{jk}(C^{l}{}_{l,}{}^{k}-2C^{kl}{}_{,l}-\phi_{,}{}^{k})+C'_{kl}C^{kl}{}_{,j}+\tfrac{1}{2}\phi_{,}{}^{k}(B_{k,j}+B_{j,k})\\
&+\Big(\tfrac{1}{2}C^{l}{}_{l,}{}^{k}-C^{kl}{}_{,l}\Big)(B_{k,j}-B_{j,k})+B_{l,k}(C^{l}{}_{j,}{}^{k}-C^{k}{}_{j,}{}^{l})-\phi_{,j}(B^{k}{}_{,k}-C^{k\prime}_{k})\bigg]+4\mathcal{H}\phi^{2}(2C^{i\prime}_{j}-B^{i}{}_{,j}-B_{j,}{}^{i})\\
&+B^{k}\bigg[C^{i}{}_{k}(2C^{l\prime}_{l,j}-2C_{jl,}{}^{l\prime}-B^{l}{}_{,jl}+\nabla^{2}B_{j})+C^{il}(4C'_{jl,k}-2C'_{kl,j}-C'_{jk,l}+2B_{k,jl}-B_{l,kj}-B_{j,kl})\\
&+C^{l}{}_{k}(4C^{i\prime}_{j,l}-2C^{i\prime}_{l,j}-C_{jl,}{}^{i\prime}+2B_{l,j}{}^{i}-B_{j,l}{}^{i}-B^{i}{}_{,jl})+C^{i\prime}_{j}(2C_{kl,}{}^{l}-B'_{k}-C^{l}{}_{l,k}+\phi_{,k})+2C^{il\prime}(C_{jl,k}-C_{jk,l})\\
&+2C'_{jl}(C^{il}{}_{,k}-C^{i}{}_{k,}{}^{l})+\tfrac{1}{2}(B'_{k}+C^{l}{}_{l,k}-2C_{kl,}{}^{l})\Big(B_{j,}{}^{i}+B^{i}{}_{,j}\Big)+(C^{l}{}_{j,}{}^{i}+C^{il}{}_{,j}-C^{i}{}_{j,}{}^{l})(B_{k,l}-B_{l,k}-2C'_{kl}-4\mathcal{H}C_{kl})\\
&+(C_{jk,}{}^{i}+C^{i}{}_{k,j}-C^{i}{}_{j,k})(C^{l\prime}_{l}-\phi'-B^{l}{}_{,l})+(B^{i}{}_{,}{}^{l}-4\mathcal{H}C^{il})(C_{jk,l}-C_{jl,k})+(B^{l}{}_{,}{}^{i}-4\mathcal{H}C^{il})C_{kl,j}+B^{l}{}_{,j}C_{kl,}{}^{i}\\
&-\tfrac{1}{2}\phi_{,k}(B^{i}{}_{,j}+B_{j,}{}^{i})+B_{j,l}(C^{i}{}_{k,}{}^{l}-C^{il}{}_{,k})+B_{k,}{}^{i}\phi_{,j}+B_{k,j}\phi_{,}{}^{i}\bigg]+2\phi\bigg[B^{i}\bigg(C^{k\prime}_{k,j}-C_{jk,}{}^{k\prime}+\tfrac{1}{2}(\nabla^{2}B_{j}-B^{k}{}_{,kj})\bigg)\\
&+B^{k}\bigg(2C^{i\prime}_{j,k}-C^{i\prime}_{k,j}-C_{jk,}{}^{i\prime}+B_{k,j}{}^{i}-\tfrac{1}{2}(B_{j,k}{}^{i}+B^{i}{}_{,jk})\bigg)+(B^{k\prime}+\phi_{,}{}^{k}+4\mathcal{H}B^{k})(C^{i}{}_{j,k}-C^{i}{}_{k,j}-C_{jk,}{}^{i})\\
&+C^{i\prime}_{k}(2C^{k\prime}_{j}-B_{j,}{}^{k})+2C^{ik}C''_{jk}-C'_{jk}B^{i}{}_{,}{}^{k}+\tfrac{1}{2}(B^{i}{}_{,j}+B_{j,}{}^{i})(C^{k\prime}_{k}-2\phi'-B^{k}{}_{,k})+C^{i\prime}_{j}(B^{k}{}_{,k}-C^{k\prime}_{k}+2\phi')\\
&+4\phi_{,j}(2\mathcal{H}B^{i}-\phi_{,}{}^{i})-4\mathcal{H}C^{ik}(2C'_{kj}+B_{k,j}+B_{j,k})+\tfrac{1}{2}\Big(B_{k,}{}^{i}B^{k}{}_{,j}+B^{i}{}_{,k}B_{j,}{}^{k}\Big)\bigg]\,,
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\mathbb{G}^{i}_{\mathrm{d}j}={}&4C^{kl}C_{km}\Bigg[\nabla^2C_l{}^m-C^{m}{}''_{l}+C^n{}_{n,l}{}^m-2C^{mn}{}_{,nl}+B^{m}{}'_{,l}+\phi_{,l}{}^m+2\mathcal{H}(B^m{}_{,l}-C^{m}{}'_{l})\Bigg]\\
&+C^{kl}\Bigg[4C_{mn}(C^{mn}{}_{,kl}-2C_l{}^n{}_{,k}{}^m)+2C'_{km}(B^m{}_{,l}+B_{l,}{}^m-3C^{m}{}'_{l})+2(C^n{}_{n,}{}^m-2C^{mn}{}_{,n})(2C_{lm,k}-C_{kl,m})\\
&+C^{kl}\Bigg[+B^{m\prime}(4C_{ml,k}-C_{kl,m})+2B'_k(2C_{lm,}{}^m-C^m{}_{m,l})+C^m{}_{m,k}(4C_{ln,}{}^n-C^n{}_{n,l})+C_{mn,k}(3C^{mn}{}_{,l}-4C^n{}_{l,}{}^m)\\
&+C_{kn,m}(6C_l{}^n{}_{,}{}^m-2C_l{}^m{}_{,}{}^n)+4C_{lm,k}\phi_{,}{}^m-2C^m{}_{m,k}\phi_{,l}-2C_{kl,m}\phi_{,}{}^m+2\phi_{,k}\phi_{,l}+4C_{km,}{}^m(\phi_{,l}-C_{ln,}{}^n)\\
&+2(B_{l,k}-C'_{kl})(B^m{}_{,m}+\phi'+4\mathcal{H}\phi)+4\phi(\phi_{,kl}+B'_{l,k}-C''_{kl})+B_{m,k}(B_{l,}{}^m-\tfrac{3}{2}B^m{}_{,l})\\
&+2B_k\bigg(2C_{lm,}{}^k-2C^{m}{}'_{m,l}+B^m{}_{,lm}-\nabla^2B_l-2\mathcal{H}(B'_l+C^m{}_{m,l}-2C_{lm,}{}^m-\phi_{,l})\bigg)-\tfrac{3}{2}B_{k,m}B_{l,}{}^m\Bigg]\\
&+C^{k}{}'_{k}\Bigg[2C^{ml}(C'_{ml}-B_{l,m})+\phi C^{l}{}'_{l}-8\mathcal{H}\phi^2-4\phi\phi'-2\phi B^l{}_{,l}+B^l(B'_l+C^m{}_{m,l}-2C_{lm,}{}^m-\phi_{,l}+2\mathcal{H}B_l)\Bigg]\\
&+B^k\Bigg[B_k\bigg(C^{l}{}''_{l}-B^{l}{}'_{,l}-\nabla^2\phi-2\mathcal{H}(2\phi'+B^l{}_{,l})\bigg)+B^l\bigg(2C_{lm,k}{}^m-C^m{}_{m,kl}-\nabla^2C_{kl}-2\mathcal{H}(C'_{kl}+B_{l,k})\bigg)\\
&-\Big(\tfrac{2a''}{a}-\mathcal{H}^2\Big)(2C_{kl}+4\phi)+2C^{lm}\bigg(2C'_{km,l}-2C'_{lm,k}+B_{m,lk}-B_{k,ml}-2\mathcal{H}(C_{lm,k}-2C_{km,l}\bigg)\Bigg]\\
&-4\phi^2\Bigg[C^{k}{}''_{k}-B^{k}{}'_{,k}-\nabla^2\phi+2\mathcal{H}(\mathcal{H}\phi-3\phi'-B^k{}_{,k})\Bigg]+C^{kl\prime}\phi\bigg(2B_{l,k}-3C'_{kl}\bigg)+\tfrac{1}{2}B_{l,k}\phi(B^k{}_{,}{}^l-3B^l{}_{,}{}^k)\\
&+B^k{}_{,k}\phi(4\phi'+B^l{}_{,l})+2\phi(C_{kl,}{}^k\phi_{,}{}^l-C^l{}_{l,k}\phi_{,}{}^k+2\phi_{,k}\phi_{,}{}^k)+2B^{k\prime}(2C_{kl,}{}^l\phi-\phi C^l{}_{l,k})\\
&+B^k\Bigg[C^m{}_{m,}{}^l(B_{k,l}-B_{l,k}-2C'_{kl})+C^{lm}{}_{,m}(4C'_{kl}-2B_{k,l})+B^l{}_{,l}(2C_{km,}{}^m-C^m{}_{m,k}-B'_k+\phi_{,k})\\
&+B_{m,l}(C^l{}_{k,}{}^m-3C^m{}_{k,}{}^l+C^{lm}{}_{,k})+C'_{lm}(C_k{}^m{}_{,}{}^l-C^{lm}{}_{,k})-\phi'(C^l{}_{l,k}-2C_{kl,}{}^l)+B^l{}_{,k}C_{ml,}{}^m\\
&+2\phi\bigg(2C_{kl,}{}^{l\prime}-2C^{l}{}'_{l,k}+B_{l,k}{}^l-\nabla^2B_k-4\mathcal{H}(2B'_k+C^l{}_{l,k}-2C_{kl,}{}^l-2\phi_{,k})\bigg)\Bigg].
\end{aligned}
\tag{B.4}
$$

Bibliography

[1] **Planck** Collaboration, "Planck: The scientific programme," `arXiv:astro-ph/0604069`.

[2] Acheson D.J., *Elementary Fluid Dynamics.* Oxford applied mathematics and computing science series. Oxford University Press, 1990.

[3] Acquaviva V., Bartolo N., Matarrese S., and Riotto A., "Second-order cosmological perturbations from inflation," *Nucl. Phys.* **B667** (2003) 119–148, `arXiv:astro-ph/0209156`.

[4] Alabidi L., "The tensor desert," *JCAP* **0702** (2007) 012, `arXiv:astro-ph/0608287`.

[5] Alabidi L. and Huston I., "An update on single field models of inflation in light of WMAP7," *JCAP* **1008** (2010) 037, `arXiv:1004.4794 [astro-ph.CO]`.

[6] Alabidi L. and Lyth D.H., "Inflation models after WMAP year three," *JCAP* **0608** (2006) 013, `arXiv:astro-ph/0603539 [astro-ph]`. * Brief entry *.

[7] Alabidi L., Malik K.A., Byrnes C.T., and Choi K.Y., "How the curvaton scenario, modulated reheating and an inhomogeneous end of inflation are related," `arXiv:1002.1700 [astro-ph.CO]`.

[8] Albrecht A. and Steinhardt P.J., "Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking," *Phys.Rev.Lett.* **48** (1982) 1220–1223.

[9] Alishahiha M., Silverstein E., and Tong D., "DBI in the sky," *Phys. Rev.* **D70** (2004) 123505, `arXiv:hep-th/0404084`.

[10] Ananda K.N., *Exact and Perturbative Cosmological Dynamics.* University of Portsmouth, 2006. Ph.D. Thesis.

[11] Ananda K.N., Clarkson C., and Wands D., "The Cosmological gravitational wave background from primordial density perturbations," *Phys.Rev.* **D75** (2007) 123518, `arXiv:gr-qc/0612013 [gr-qc]`.

[12] Armendariz-Picon C., Damour T., and Mukhanov V.F., "k-Inflation," *Phys. Lett.* **B458** (1999) 209–218, `arXiv:hep-th/9904075`.

[13] Arroja F., Assadullahi H., Koyama K., and Wands D., "Cosmological matching conditions for gravitational waves at second order," *Phys. Rev.* **D80** (2009) 123526, arXiv:0907.3618 [astro-ph.CO].

[14] Assadullahi H. and Wands D., "Gravitational waves from an early matter era," *Phys. Rev.* **D79** (2009) 083511, arXiv:0901.0989 [astro-ph.CO].

[15] Assadullahi H. and Wands D., "Constraints on primordial density perturbations from induced gravitational waves," *Phys. Rev.* **D81** (2010) 023527, arXiv:0907.4073 [astro-ph.CO].

[16] Bardeen J.M., "Gauge Invariant Cosmological Perturbations," *Phys. Rev.* **D22** (1980) 1882–1905.

[17] Bardeen J.M., Steinhardt P.J., and Turner M.S., "Spontaneous Creation of Almost Scale - Free Density Perturbations in an Inflationary Universe," *Phys. Rev.* **D28** (1983) 679.

[18] Bartolo N., Komatsu E., Matarrese S., and Riotto A., "Non-Gaussianity from inflation: Theory and observations," *Phys. Rept.* **402** (2004) 103–266, arXiv:astro-ph/0406398.

[19] Bartolo N., Matarrese S., and Riotto A., "Adiabatic and isocurvature perturbations from inflation: Power spectra and consistency relations," *Phys. Rev.* **D64** (2001) 123504, arXiv:astro-ph/0107502.

[20] Baumann D., Steinhardt P.J., Takahashi K., and Ichiki K., "Gravitational Wave Spectrum Induced by Primordial Scalar Perturbations," *Phys. Rev.* **D76** (2007) 084019, arXiv:hep-th/0703290.

[21] **CMBPol Study Team** Collaboration, Baumann D. *et al.*, "CMBPol Mission Concept Study: Probing Inflation with CMB Polarization," *AIP Conf. Proc.* **1141** (2009) 10–120, arXiv:0811.3919 [astro-ph].

[22] Bean R., Dunkley J., and Pierpaoli E., "Constraining Isocurvature Initial Conditions with WMAP 3-year data," *Phys.Rev.* **D74** (2006) 063503, arXiv:astro-ph/0606685 [astro-ph].

[23] Bennett C.L. *et al.*, "Cosmic temperature fluctuations from two years of COBE differential microwave radiometers observations," *Astrophys. J.* **436** (1994) 423–442, arXiv:astro-ph/9401012.

[24] Biermann L., "Ueber den ursprung der magnetfelder auf sternen und im interstellaren raum (miteinem anhang von a. schlueter)," *Z. Naturforsch. Teil* **A 5** (1950) 65.

[25] Bonnor W.B., "Jeans' formula for gravitational instability," *Mon. Not. Roy. Astron. Soc.* **117** (1957) 104.

[26] Brown I.A., *Primordial Magnetic Fields in Cosmology.* University of Portsmouth, 2006. arXiv:0812.1781 [astro-ph]. Ph.D. Thesis.

[27] Brown I.A., "Concerning the statistics of cosmic magnetism," arXiv:arXiv:1005.2982 [astro-ph.CO].

[28] Brown I.A., Christopherson A.J., Malik K.A., and Matravers D.R. *in preparation* .

[29] Bruni M., Dunsby P.K., and Ellis G.F., "Cosmological perturbations and the physical meaning of gauge invariant variables," *Astrophys.J.* **395** (1992) 34–53.

[30] Bruni M., Matarrese S., Mollerach S., and Sonego S., "Perturbations of spacetime: Gauge transformations and gauge invariance at second order and beyond," *Class. Quant. Grav.* **14** (1997) 2585–2606, arXiv:gr-qc/9609040.

[31] Byrnes C.T. and Choi K.Y., "Review of local non-Gaussianity from multi-field inflation," *Adv. Astron.* **2010** (2010) 724525, arXiv:1002.3110 [astro-ph.CO].

[32] Byrnes C.T. and Wands D., "Curvature and isocurvature perturbations from two-field inflation in a slow-roll expansion," *Phys.Rev.* **D74** (2006) 043529, arXiv:astro-ph/0605679 [astro-ph].

[33] Carroll S., *Spacetime and Geometry: An Introduction to General Relativity.* Benjamin Cummings, 2003.

[34] Chongchitnan S., "Cosmological Perturbations in Models of Coupled Dark Energy," *Phys. Rev.* **D79** (2009) 043522, arXiv:0810.5411 [astro-ph].

[35] Christopherson A.J., "Gauge conditions in combined dark energy and dark matter systems," *Phys. Rev.* **D82** (2010) 083515, arXiv:1008.0811 [astro-ph.CO].

[36] Christopherson A.J. and Malik K.A., "Practical tools for third order cosmological perturbations," *JCAP* **0911** (2009) 012, arXiv:0909.0942 [astro-ph.CO].

[37] Christopherson A.J. and Malik K.A., "The non-adiabatic pressure in general scalar field systems," *Phys. Lett.* **B675** (2009) 159–163, arXiv:0809.3518 [astro-ph].

[38] Christopherson A.J. and Malik K.A., "Can cosmological perturbations produce early universe vorticity?," arXiv:1010.4885 [gr-qc].

[39] Christopherson A.J., Malik K.A., and Matravers D.R., "Vorticity generation at second order in cosmological perturbation theory," *Phys. Rev.* **D79** (2009) 123523, arXiv:0904.0940 [astro-ph.CO].

[40] Christopherson A.J., Malik K.A., and Matravers D.R., "Estimating the amount of vorticity generated by cosmological perturbations in the early universe," arXiv:1008.4866 [astro-ph.CO].

[41] Christopherson A.J., Malik K.A., Matravers D.R., and Nakamura K., "Comparing different formulations of non-linear cosmological perturbation theory," arXiv:1101.3525 [astro-ph.CO].

[42] Clarkson C., Ananda K., and Larena J., "The influence of structure formation on the cosmic expansion," *Phys. Rev.* **D80** (2009) 083525, arXiv:0907.3377 [astro-ph.CO].

[43] Crocco L., "Eine neue stromfunktion fuer die erforschung der bewegung der gase mit rotation," *Z. Angew. Math. Mech* **17** (1937) 1.

[44] Del Sordo F. and Brandenburg A., "Vorticity production through rotation, shear and baroclinicity," arXiv:1008.5281 [astro-ph.GA].

[45] Dodelson S., *Modern cosmology.* Amsterdam, Netherlands: Academic Pr. (2003) 440 p.

[46] Durrer R., *The Cosmic Microwave Background.* Cambridge Univ. Press, Cambridge, 2008.

[47] Durrer R. and Maartens R., "Dark Energy and Dark Gravity," *Gen.Rel.Grav.* **40** (2008) 301–328, arXiv:0711.0077 [astro-ph].

[48] Ellis G. and Matravers D.R., "General covariance in general relativity?," *Gen.Rel.Grav.* **27** (1995) 7.

[49] Ellis G.F.R. and Bruni M., "Covariant and gauge invariant approach to cosmological density fluctuations," *Phys. Rev.* **D40** (1989) 1804–1818.

[50] Ellis G.F.R., Bruni M., and Hwang J., "Density gradient - vorticity relation in perfect fluid Robertson-Walker perturbations," *Phys. Rev.* **D42** (1990) 1035–1046.

[51] Ellis G.F.R., Hwang J., and Bruni M., "Covariant and gauge independent perfect fluid Robertson-Walker perturbations," *Phys. Rev.* **D40** (1989) 1819–1826.

[52] Enqvist K., Hogdahl J., Nurmi S., and Vernizzi F., "Covariant generalization of cosmological perturbation theory," *Phys. Rev.* **D75** (2007) 023515, arXiv:gr-qc/0611020.

[53] Errard J., "The new generation CMB B-mode polarization experiment: POLARBEAR," arXiv:1011.0763 [astro-ph.IM].

[54] Gao X., "On Cross-correlations between Curvature and Isocurvature Perturbations during Inflation," *JCAP* **1002** (2010) 019, arXiv:0908.4035 [hep-th].

[55] Garcia-Bellido J., Durrer R., Fenu E., Figueroa D.G., and Kunz M., "The local B-polarization of the CMB: a very sensitive probe of cosmic defects," arXiv:1003.0299 [astro-ph.CO].

[56] Garriga J. and Mukhanov V.F., "Perturbations in k-inflation," *Phys. Lett.* **B458** (1999) 219–225, arXiv:hep-th/9904176.

[57] Giovannini M., "A primer on the physics of the cosmic microwave background,". Singapore, Singapore: World Scientific (2008) 488 p.

[58] Gopal R. and Sethi S., "Generation of Magnetic Field in the Pre-recombination Era," *Mon. Not. Roy. Astron. Soc.* **363** (2005) 521–528, arXiv:astro-ph/0411170.

[59] Gordon C., Wands D., Bassett B.A., and Maartens R., "Adiabatic and entropy perturbations from inflation," *Phys. Rev.* **D63** (2001) 023506, arXiv:astro-ph/0009131.

[60] Guth A.H., "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," *Phys.Rev.* **D23** (1981) 347–356.

[61] Harrison E.R., "Generation of magnetic fields in the radiation era," *Mon. Not. R. atr. Soc.* **147** (1970) 279.

[62] Hawking S. and Ellis G., *The Large Scale Structure of Space-Time*. Cambridge University Press, Cambridge, 1973.

[63] Hawking S.W., "Perturbations of an expanding universe," *Astrophys. J.* **145** (1966) 544–554.

[64] Hindmarsh M.B. and Kibble T.W.B., "Cosmic strings," *Rept. Prog. Phys.* **58** (1995) 477–562, arXiv:hep-ph/9411342.

[65] Huston I., "Constraining Inflationary Scenarios with Braneworld Models and Second Order Cosmological Perturbations," arXiv:1006.5321 [astro-ph.CO].

[66] Huston I., Lidsey J.E., Thomas S., and Ward J., "Gravitational Wave Constraints on Multi-Brane Inflation," *JCAP* **0805** (2008) 016, arXiv:0802.0398 [hep-th].

[67] Huston I. and Malik K.A., "Numerical calculation of second order perturbations," *JCAP* **0909** (2009) 019, arXiv:0907.2917 [astro-ph.CO].

[68] Huston I. and Malik K.A., "Second Order Perturbations During Inflation Beyond Slow-roll," arXiv:1103.0912 [astro-ph.CO].

[69] Hwang J. and Noh H., "Third-order cosmological perturbations of zero-pressure multi-component fluids: Pure general relativistic nonlinear effects," *JCAP* **0712** (2007) 003, arXiv:0704.2086 [astro-ph].

[70] Hwang J.c. and Noh H., "Cosmological perturbations with multiple fluids and fields," *Class. Quant. Grav.* **19** (2002) 527–550, `arXiv:astro-ph/0103244`.

[71] Hwang J.c. and Noh H., "Third order perturbations of a zero-pressure cosmological medium: Pure general relativistic nonlinear effects," *Phys. Rev.* **D72** (2005) 044012, `arXiv:gr-qc/0412129`.

[72] Ichiki K., Takahashi K., Ohno H., Hanayama H., and Sugiyama N., "Cosmological Magnetic Field: a fossil of density perturbations in the early universe," *Science.* **311** (2006) 827–829, `arXiv:astro-ph/0603631`.

[73] Khoury J. and Piazza F., "Rapidly-Varying Speed of Sound, Scale Invariance and Non-Gaussian Signatures," *JCAP* **0907** (2009) 026, `arXiv:0811.3633 [hep-th]`.

[74] Kobayashi T., Maartens R., Shiromizu T., and Takahashi K., "Cosmological magnetic fields from nonlinear effects," *Phys. Rev.* **D75** (2007) 103501, `arXiv:astro-ph/0701596`.

[75] Kodama H. and Sasaki M., "Cosmological Perturbation Theory," *Prog. Theor. Phys. Suppl.* **78** (1984) 1–166.

[76] Kofman L.A., "What initial perturbations may be generated in inflationary cosmological models," *Phys. Lett.* **B173** (1986) 400.

[77] Komatsu E. *et al.*, "Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation," `arXiv:1001.4538 [astro-ph.CO]`.

[78] Lalak Z., Langlois D., Pokorski S., and Turzynski K., "Curvature and isocurvature perturbations in two-field inflation," *JCAP* **0707** (2007) 014, `arXiv:0704.0212 [hep-th]`.

[79] Landau L.D. and Lifshitz E.M., *Fluid mechanics / by L.D. Landau and E.M. Lifshitz ; translated from the Russian by J.B. Sykes and W.H. Reid.* Pergamon Press ; Addison-Wesley, London : Reading, Mass. :, 1959.

[80] Landau L.D. and Lifshitz E.M., *Statistical Physics (Part 1).* Pergamon Press, Oxford, third ed., August, 1980. `http://www.amazon.com/exec/obidos/redirect?tag=citeulike07-20&path=ASIN/0750633727`.

[81] Langlois D., "Correlated adiabatic and isocurvature perturbations from double inflation," *Phys. Rev.* **D59** (1999) 123512, `arXiv:astro-ph/9906080`.

[82] Langlois D. and Lepidi A., "General treatment of isocurvature perturbations and non- Gaussianities," `arXiv:1007.5498 [astro-ph.CO]`.

[83] Langlois D. and Renaux-Petel S., "Perturbations in generalized multi-field inflation," *JCAP* **0804** (2008) 017, `arXiv:0801.1085 [hep-th]`.

[84] Langlois D., Vernizzi F., and Wands D., "Non-linear isocurvature perturbations and non- Gaussianities," *JCAP* **0812** (2008) 004, arXiv:0809.4646 [astro-ph].

[85] Lehners J.L. and Renaux-Petel S., "Multifield Cosmological Perturbations at Third Order and the Ekpyrotic Trispectrum," *Phys. Rev.* **D80** (2009) 063503, arXiv:0906.0530 [hep-th].

[86] Lewis A., Challinor A., and Lasenby A., "Efficient computation of CMB anisotropies in closed FRW models," *Astrophys. J.* **538** (2000) 473–476, astro-ph/9911177.

[87] Liddle A.R. and Lyth D.H., *Cosmological inflation and large-scale structure.* Cambridge University Press, Cambridge, 2000.

[88] Lidsey J.E. and Huston I., "Gravitational wave constraints on Dirac-Born-Infeld inflation," *JCAP* **0707** (2007) 002, arXiv:0705.0240 [hep-th].

[89] Lifshitz E., "On the Gravitational stability of the expanding universe," *J. Phys. (USSR)* **10** (1946) 116.

[90] Lifshitz E.M. and Khalatnikov I.M., "Investigations in relativistic cosmology," *Adv. Phys.* **12** (1963) 185–249.

[91] Linde A.D., "A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems," *Phys.Lett.* **B108** (1982) 389–393.

[92] Linde A.D., "Chaotic Inflation," *Phys.Lett.* **B129** (1983) 177–181.

[93] Linde A.D., "Generation of Isothermal Density Perturbations in the Inflationary Universe," *Phys. Lett.* **B158** (1985) 375–380.

[94] Linde A.D. and Mukhanov V.F., "Nongaussian isocurvature perturbations from inflation," *Phys. Rev.* **D56** (1997) 535–539, arXiv:astro-ph/9610219.

[95] Loeb A. and Zaldarriaga M., "Measuring the small-scale power spectrum of cosmic density fluctuations through 21-cm tomography prior to the epoch of structure formation," *Phys. Rev. Lett.* **92** (2004) 211301, arXiv:astro-ph/0312134.

[96] Lopez-Monsalvo C.S. and Andersson N., "Thermal Dynamics in General Relativity," arXiv:1006.2978 [gr-qc].

[97] Lu T.H.C., Ananda K., and Clarkson C., "Vector modes generated by primordial density fluctuations," *Phys. Rev.* **D77** (2008) 043523, arXiv:0709.1619 [astro-ph].

[98] Lu T.H.C., Ananda K., Clarkson C., and Maartens R., "The cosmological background of vector modes," *JCAP* **0902** (2009) 023, arXiv:0812.1349 [astro-ph].

[99] Lyth D.H., "What would we learn by detecting a gravitational wave signal in the cosmic microwave background anisotropy?," *Phys. Rev. Lett.* **78** (1997) 1861–1863, arXiv:hep-ph/9606387.

[100] Lyth D.H., Malik K.A., and Sasaki M., "A general proof of the conservation of the curvature perturbation," *JCAP* **0505** (2005) 004, arXiv:astro-ph/0411220.

[101] Lyth D.H. and Wands D., "Generating the curvature perturbation without an inflaton," *Phys. Lett.* **B524** (2002) 5–14, arXiv:hep-ph/0110002.

[102] Ma C.P. and Bertschinger E., "Cosmological perturbation theory in the synchronous and conformal Newtonian gauges," *Astrophys. J.* **455** (1995) 7–25, arXiv:astro-ph/9506072.

[103] Maeda S., Kitagawa S., Kobayashi T., and Shiromizu T., "Primordial magnetic fields from second-order cosmological perturbations:Tight coupling approximation," *Class. Quant. Grav.* **26** (2009) 135014, arXiv:0805.0169 [astro-ph].

[104] Maldacena J.M., "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP* **05** (2003) 013, arXiv:astro-ph/0210603.

[105] Malik K.A., "Gauge-invariant perturbations at second order: multiple scalar fields on large scales," *JCAP* **0511** (2005) 005, arXiv:astro-ph/0506532.

[106] Malik K.A., "A not so short note on the Klein-Gordon equation at second order," *JCAP* **0703** (2007) 004, arXiv:astro-ph/0610864.

[107] Malik K.A., "The Quantum Origin of Cosmic Structure," arXiv:1011.2667 [astro-ph.CO].

[108] Malik K.A. and Matravers D.R., "A Concise Introduction to Perturbation Theory in Cosmology," *Class. Quant. Grav.* **25** (2008) 193001, arXiv:0804.3276 [astro-ph].

[109] Malik K.A., Seery D., and Ananda K.N., "Different approaches to the second order Klein-Gordon equation," *Class. Quant. Grav.* **25** (2008) 175008, arXiv:0712.1787 [astro-ph].

[110] Malik K.A. and Wands D., "Evolution of second-order cosmological perturbations," *Class. Quant. Grav.* **21** (2004) L65–L72, arXiv:astro-ph/0307055.

[111] Malik K.A. and Wands D., "Adiabatic and entropy perturbations with interacting fluids and fields," *JCAP* **0502** (2005) 007, `arXiv:astro-ph/0411703`.

[112] Malik K.A. and Wands D., "Cosmological perturbations," *Phys. Rept.* **475** (2009) 1–51, `arXiv:0809.4944 [astro-ph]`.

[113] Malik K.A., Wands D., and Ungarelli C., "Large-scale curvature and entropy perturbations for multiple interacting fluids," *Phys. Rev.* **D67** (2003) 063516, `arXiv:astro-ph/0211602`.

[114] Matarrese S., Mollerach S., and Bruni M., "Second-order perturbations of the Einstein-de Sitter universe," *Phys. Rev.* **D58** (1998) 043504, `arXiv:astro-ph/9707278`.

[115] Matarrese S., Mollerach S., Notari A., and Riotto A., "Large-scale magnetic fields from density perturbations," *Phys. Rev.* **D71** (2005) 043502, `arXiv:astro-ph/0410687`.

[116] Misner C.W., Thorne K.S., and Wheeler J.A., "Gravitation,". San Francisco 1973, 1279p.

[117] Mollerach S., Harari D., and Matarrese S., "CMB polarization from secondary vector and tensor modes," *Phys. Rev.* **D69** (2004) 063002, `arXiv:astro-ph/0310711`.

[118] Mukhanov V.F., "Quantum Theory of Gauge Invariant Cosmological Perturbations," *Sov. Phys. JETP* **67** (1988) 1297–1302.

[119] Mukhanov V.F., Abramo L.R.W., and Brandenberger R.H., "On the back reaction problem for gravitational perturbations," *Phys. Rev. Lett.* **78** (1997) 1624–1627, `arXiv:gr-qc/9609026`.

[120] Mukhanov V.F., Feldman H.A., and Brandenberger R.H., "Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions," *Phys. Rept.* **215** (1992) 203–333.

[121] Nakamura K., "Gauge invariant variables in two-parameter nonlinear perturbations," *Prog. Theor. Phys.* **110** (2003) 723–755, `arXiv:gr-qc/0303090`.

[122] Nakamura K., "Second-order Gauge-invariant Cosmological Perturbation Theory: Current Status," `arXiv:1001.2621 [gr-qc]`.

[123] Noh H. and Hwang J.c., "Second-order perturbations of the Friedmann world model," *Phys. Rev.* **D69** (2004) 104011.

[124] Olson D.W., "Density Perturbations on Cosmological Models," *Phys. Rev.* **D14** (1976) 327–331.

[125] Osano B., Pitrou C., Dunsby P., Uzan J.P., and Clarkson C., "Gravitational waves generated by second order effects during inflation," *JCAP* **0704** (2007) 003, arXiv:gr-qc/0612108.

[126] Park C.G., Hwang J.c., Lee J.h., and Noh H., "Roles of dark energy perturbations in the dynamical dark energy models: Can we ignore them?," *Phys. Rev. Lett.* **103** (2009) 151303, arXiv:0904.4007 [astro-ph.CO].

[127] Polarski D. and Starobinsky A.A., "Isocurvature perturbations in multiple inflationary models," *Phys. Rev.* **D50** (1994) 6123–6129, arXiv:astro-ph/9404061.

[128] Press W.H. and Vishniac E.T., "Tenacious myths about cosmological perturbations larger than the horizon size," *Astrophys. J.* **239** (1980) 1–11.

[129] Ryu D., Kang H., Cho J., and Das S., "Turbulence and Magnetic Fields in the Large Scale Structure of the Universe," arXiv:0805.2466 [astro-ph].

[130] Salopek D.S. and Bond J.R., "Nonlinear evolution of long wavelength metric fluctuations in inflationary models," *Phys. Rev.* **D42** (1990) 3936–3962.

[131] Salopek D.S., Bond J.R., and Bardeen J.M., "Designing Density Fluctuation Spectra in Inflation," *Phys. Rev.* **D40** (1989) 1753.

[132] Sarkar D., Serra P., Cooray A., Ichiki K., and Baumann D., "Cosmic Shear from Scalar-Induced Gravitational Waves," *Phys. Rev.* **D77** (2008) 103515, arXiv:0803.1490 [astro-ph].

[133] Sasaki M., "Large Scale Quantum Fluctuations in the Inflationary Universe," *Prog. Theor. Phys.* **76** (1986) 1036.

[134] Sasaki M., "Multi-brid inflation and non-Gaussianity," *Prog. Theor. Phys.* **120** (2008) 159–174, arXiv:0805.0974 [astro-ph].

[135] Seery D. and Lidsey J.E., "Primordial non-gaussianities in single field inflation," *JCAP* **0506** (2005) 003, arXiv:astro-ph/0503692.

[136] Seery D., Malik K.A., and Lyth D.H., "Non-gaussianity of inflationary field perturbations from the field equation," *JCAP* **0803** (2008) 014, arXiv:0802.0588 [astro-ph].

[137] Seery D., Sloth M.S., and Vernizzi F., "Inflationary trispectrum from graviton exchange," *JCAP* **0903** (2009) 018, arXiv:0811.3934 [astro-ph].

[138] Seljak U. and Zaldarriaga M., "A Line of sight integration approach to cosmic microwave background anisotropies," *Astrophys.J.* **469** (1996) 437–444, arXiv:astro-ph/9603033 [astro-ph].

[139] Siegel E.R. and Fry J.N., "Cosmological Structure Formation Creates Large-Scale Magnetic Fields," *Astrophys. J.* **651** (2006) 627–635, arXiv:astro-ph/0604526.

[140] Silverstein E. and Tong D., "Scalar Speed Limits and Cosmology: Acceleration from D- cceleration," *Phys. Rev.* **D70** (2004) 103505, `arXiv:hep-th/0310221`.

[141] Sommerfeld A., *Partial differential equations in physics / by Arnold Sommerfeld ; translated by Ernest G. Straws.* Academic Press, New York :, 1964.

[142] Starobinsky A., "Spectrum of relict gravitational radiation and the early state of the universe," *JETP Lett.* **30(11)** ((1979)) 682–685.

[143] Starobinsky A., "A new type of isotropic cosmological models without singularity," *Phys. Lett.* **B91** (1980) 99–102.

[144] Stewart J.M., "Perturbations of friedmann-robertson-walker cosmological models," *Classical and Quantum Gravity* **7** (1990) no. 7, 1169. `http://stacks.iop.org/0264-9381/7/i=7/a=013`.

[145] Stewart J.M. and Walker M., "Perturbations of spacetimes in general relativity," *Proc. Roy. Soc. Lond.* **A341** (1974) 49–74.

[146] Takahashi K., Ichiki K., Ohno H., and Hanayama H., "Magnetic field generation from cosmological perturbations," *Phys. Rev. Lett.* **95** (2005) 121301, `arXiv:astro-ph/0502283`.

[147] **SDSS** Collaboration, Tegmark M. *et al.*, "Cosmological Constraints from the SDSS Luminous Red Galaxies," *Phys. Rev.* **D74** (2006) 123507, `arXiv:astro-ph/0608632`.

[148] Tomita K., "Non-linear theory of gravitational instability in the expanding universe," *Progress of Theoretical Physics* **37** (1967) no. 5, 831–846. `http://ptp.ipap.jp/link?PTP/37/831/`.

[149] Unnikrishnan S., "Can cosmological observations uniquely determine the nature of dark energy ?," *Phys. Rev.* **D78** (2008) 063007, `arXiv:0805.0578 [astro-ph]`.

[150] Vilenkin A. and Shellard E., *Cosmic Strings and Other Toplogical Defects.* Cambridge University Press, 1994.

[151] Wands D., "Local non-Gaussianity from inflation," *Class. Quant. Grav.* **27** (2010) 124002, `arXiv:1004.0818 [astro-ph.CO]`.

[152] Wands D., Malik K.A., Lyth D.H., and Liddle A.R., "A new approach to the evolution of cosmological perturbations on large scales," *Phys. Rev.* **D62** (2000) 043527, `arXiv:astro-ph/0003278`.

[153] Widrow L.M., "Origin of Galactic and Extragalactic Magnetic Fields," *Rev. Mod. Phys.* **74** (2002) 775–823, `arXiv:astro-ph/0207240`.

[154] Zaldarriaga M. and Seljak U., "CMBFAST for spatially closed universes," *Astrophys. J. Suppl.* **129** (2000) 431–434, `arXiv:astro-ph/9911219`.

[155] Zhu W., Feng L.L., and Fang L.Z., "Vorticity of IGM Velocity Field on Large Scales," *Astrophys. J.* **712** (2010) 1–13, `arXiv:1001.4127 [astro-ph.CO]`.