# Ternary structures in Hilbert spaces

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This thesis is dedicated to

**my husband, mother and father.**

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*I kiss the cup that me apprise*

*Of the world's turnings and its disguise.*

Hafez-e-Shirazi

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# Abstract

Ternary structures in Hilbert spaces arose in the study of infinite dimensional manifolds in differential geometry. In this thesis, we develop a structure theory of Hilbert ternary algebras and Jordan Hilbert triples which are Hilbert spaces equipped with a ternary product. We obtain several new results on the classification of these structures. Some results have been published in [2].

A Hilbert ternary algebra is a real Hilbert space $(V, \langle\cdot,\cdot\rangle)$ equipped with a ternary product $[\cdot,\cdot,\cdot]$ satisfying $\langle [a,b,x], y\rangle = \langle x, [b,a,y]\rangle$ for $a, b, x$ and $y$ in $V$. A Jordan Hilbert triple is a real Hilbert space in which the ternary product $\{\cdot,\cdot,\cdot\}$ is a Jordan triple product. It is called a $JH$-triple if the identity $\langle \{a,b,x\}, x\rangle = \langle x, \{b,a,x\}\rangle$ holds in $V$. $JH$-triples correspond to a class of Lie algebras which play an important role in symmetric Riemannian manifolds.

We begin by proving new structure results on ideals, centralizers and derivations of Hilbert ternary algebras. We characterize primitive tripotents in Hilbert ternary algebras and use them as coordinates to classify abelian Hilbert ternary algebras. We show that they are direct sums of simple ones, and each simple abelian Hilbert ternary algebra is ternary isomorphic to the algebra $\mathcal{C}_2(H,K)$ of Hilbert-Schmidt operators between real, complex or quaternion Hilbert spaces $H$ and $K$.

Further, we describe completely the ternary isomorphisms and ternary anti-isomorphisms between abelian Hilbert ternary algebras. We show that each ternary isomorphism $\tau$ between simple algebras $\mathcal{C}_2(H,K)$ and $\mathcal{C}_2(H',K')$ is of the form $\tau(x) = Jxj$ where $j : H' \longrightarrow H$ and $J : K \longrightarrow K'$ are isometries. A ternary anti-isomorphism is of the form $\tau(x) = Jx^*j$ where $j : H' \longrightarrow K$ and $J : H \longrightarrow K'$ are isometries.

The structures of Hilbert ternary algebras and $JH$-triples are closely related. Indeed, we show that each $JH$-triple $(V, \{\cdot,\cdot,\cdot\})$ admits a decomposi-

tion $V = V_s \bigoplus V_s^{\perp}$ where $(V_s, \{\cdot,\cdot,\cdot\})$ is a Hilbert ternary algebra which is usually nonabelian and unless $V = V_s$, the orthogonal complement $V_s^{\perp}$ is always a nonabelian Hilbert ternary algebra in the derived ternary product $[a, b, c]_d = \{a, b, c\} - \{b, a, c\}$. Hence $JH$-triples provide important examples of nonabelian Hilbert ternary algebras. We determine exactly when $V_s$ and $V_s^{\perp}$ are Jordan triple ideals of $V$. We show, in each dimension at least 2, there is a $JH$-triple $(V, \{\cdot,\cdot,\cdot\})$ for which $V \neq V_s$, equivalently, $(V, \{\cdot,\cdot,\cdot\})$ is not a Hilbert ternary algebra.

# Contents

# Chapter 1

# Introduction

In what follows, we develop a structure theory of Hilbert ternary algebras and Jordan Hilbert triples which are Hilbert spaces equipped with a ternary product. We obtain several new results on these structures and their classification. Some of the results in this thesis have been published in [2].

We begin in Chapter 2 with some preliminaries needed for later chapters. In particular, we discuss involutions of the quaternions $\mathbb{H}$, octonions $\mathbb{O}$ and split octonions, and prove some basic results which will be used in subsequent development.

Chapter 3 is concerned with the structure theory of Hilbert ternary algebras. The concept of a Hilbert ternary algebra has been introduced in [8]. It originates from the notion of a *ternary algebra*, introduced by Hestenes [16] to develop a spectral theory for non-Hermitian matrices and operators. A *Hilbert ternary algebra* is a *real* Hilbert space $(V, \langle \cdot, \cdot \rangle)$ equipped with a trilinear map $[\cdot, \cdot, \cdot] : V^3 \longrightarrow V$ satisfying

$$\langle [a, b, x], y \rangle = \langle x, [b, a, y] \rangle \qquad (a, b, x, y \in V).$$

Every Hilbert space $(V, \langle \cdot, \cdot \rangle)$ has an inherent ternary algebraic structure defined by

$$[x, y, z] = \frac{1}{2}\langle x, y\rangle z + \frac{1}{2}\langle z, y\rangle x$$

which plays an important role in differential geometry. The class of $JH$-triples introduced recently in [9] provides further interesting examples of Hilbert ternary algebras. They are in one-one correspondence with a class of Riemannian symmetric spaces [9]. These examples, together with the examples of $H^*$-algebras, motivate our investigation of the structures of Hilbert ternary algebras.

We begin by showing that the ternary product in a Hilbert ternary algebra is automatically continuous. Then we prove new structure results on ideals, centralizers and derivations of Hilbert ternary algebras. We introduce the important concept of primitive tripotents in Hilbert ternary algebras and use them as coordinates to classify abelian Hilbert ternary algebras in Section 3.4.

An important feature of a Hilbert ternary algebra $V$ is that if $I$ is an ideal in $V$, then the orthogonal complement $I^\perp$ is also an ideal. This enables one to establish a Wedderburn type theorem for a Hilbert ternary algebra $V$ with zero annihilator, namely, $V$ can be decomposed as a direct sum $\bigoplus_\alpha V_\alpha$ of minimal closed ideals, each of which is a simple Hilbert ternary algebra.

We study the centralizers and derivations of a Hilbert ternary algebra in Section 3.3. We show that the set of all centralizers $\mathcal{Z}(V)$ of a Hilbert ternary algebra $V$ with zero annihilator forms a von Neumann algebra which is in fact, a direct sum $\bigoplus_\alpha \mathcal{C}_\alpha$ where $\mathcal{C}_\alpha \simeq \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Using the complex result in [36] and complexification, we prove that every derivation of a Hilbert ternary algebra with zero annihilator is automatically continuous.

In Section 3.4, we study the structure of abelian Hilbert ternary algebras. A Hilbert ternary algebra $V$ is called *abelian* if

$$[a, b, [x, y, z]] = [[a, b, x], y, z]$$

for all $a, b, x, y, z \in V$. Our objective is to classify abelian Hilbert ternary algebras. Using primitive tripotents as coordinates and adapting the proof in [37] for Complex Hilbert triple systems, we are able to show that every simple abelian Hilbert ternary algebra is ternary isomorphic to the algebra $\mathcal{C}_2(H, K)$ of Hilbert-Schmidt operators between Hilbert spaces $H$ and $K$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Together with the Wedderburn type theorem mentioned earlier, this gives a complete classification of abelian Hilbert ternary algebras. We should note that the above result for simple abelian Hilbert ternary algebras has also been obtained in [6], in the context of *associative $H^*$-triple systems*, using the technique of Peirce subspaces.

The last section of Chapter 3 is devoted to the study of ternary isomorphisms between Hilbert ternary algebras. We describe completely these isomorphisms. A linear bijection $\tau : V \longrightarrow V'$ between Hilbert ternary algebras is called a *ternary isomorphism* if

$$\tau[x, y, z] = [\tau x, \tau y, \tau z]'$$

and, is called a *ternary anti-isomorphism* if instead $\tau[x, y, z] = [\tau z, \tau y, \tau x]'$ for all $x, y, z \in V$. We show that each ternary isomorphism $\tau$ between simple abelian Hilbert ternary algebras $\mathcal{C}_2(H, K)$ and $\mathcal{C}_2(H', K')$ is of the form

$$\tau(x) = Jxj$$

where $j : H' \longrightarrow H$ and $J : K \longrightarrow K'$ are isometries. A ternary anti-isomorphism is of the form

$$\tau(x) = Jx^*j$$

where $j : H' \longrightarrow K$ and $J : H \longrightarrow K'$ are isometries.

Interestingly, nonabelian Hilbert ternary algebras arose in the study of Jordan triple structures in Hilbert spaces which is the subject of Chapter 4. To motivate the concept of a Jordan triple system, we recall in Section 4.1 the one-one correspondence between Jordan triple systems and Tits-Kantor-Koecher Lie algebras.

In Section 4.2, we show that a normed real Jordan triple admits continuous inner derivations if and only if the symmetric part of the corresponding Tits-Kantor-Koecher Lie algebra is a *normed* Lie algebra. This is an improvement of a result in [9] for the case of *non-degenerate* Jordan triples. We study the structures of $JH$-triples in Section 4.3. A *$JH$-triple* is a real Jordan triple $(V, \{\cdot,\cdot,\cdot\})$, with a continuous triple product, which is also a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ satisfying

$$\langle\{a,b,x\},x\rangle = \langle x,\{b,a,x\}\rangle \qquad (a,b,x \in V).$$

If the inner product is associative, that is, if

$$\langle\{a,b,x\},y\rangle = \langle x,\{b,a,y\}\rangle \qquad (a,b,x,y \in V),$$

then $V$ is called a *real $JH^*$-triple*. Real $JH^*$-triples are Hilbert ternary algebras. They are the *real forms* of the complex $JH^*$-triples defined in [23] where the one-one correspondence is shown between the $JH^*$-triples and Hermitian symmetric spaces.

The structures of Hilbert ternary algebras and $JH$-triples are closely related. We show that every $JH$-triple gives rise to a nonabelian Hilbert ternary algebra. Indeed, each $JH$-triple $(V, \{\cdot,\cdot,\cdot\})$ admits an orthogonal decomposition $V = V_s \bigoplus V_s^\perp$ where both $V_s$ and $V_s^\perp$ are Hilbert ternary algebras in the *derived* ternary product

$$[a,b,c]_d = \{a,b,c\} - \{b,a,c\}.$$

Unless $V = V_s$, the orthogonal complement $(V_s^\perp, [\cdot,\cdot,\cdot]_d)$ is always a nonabelain Hilbert ternary algebra. Hence $JH$-triples provide important examples of non-abelian Hilbert ternary algebras.

In the above decomposition, we show that $(V_s, \{\cdot,\cdot,\cdot\})$ is a real $JH^*$-triple and hence its structure is completely known by the classification result of [23, 30]. We determine exactly when $V_s$ and $V_s^{\perp}$ are Jordan triple ideals of $V$. We show, in each dimension of at least 2, there is a $JH$-triple $(V, \{\cdot, \cdot, \cdot\})$ for which $V \neq V_s$. We conclude with further examples of nonabelian Hilbert ternary algebras.

# Chapter 2

# Hilbert spaces

Our objective is to investigate ternary algebraic structures in real Hilbert spaces which would involve the quaternions and octonions. In this chapter, we introduce the notations and prove some basic results needed for later investigation.

## 2.1 Hilbert spaces over real division algebras

We begin with the quaternions and octonions as well as the split octonions.

Let $\mathbb{H}$ denote the *quaternions* which can be described as a 4-dimensional real vector space with basis $\{\mathbf{1}, i, j, k\}$ in which the multiplication is defined by

$$i^2 = j^2 = k^2 = ijk = -\mathbf{1},\ ij = k = -ji.$$

The quaternions $\mathbb{H}$ is a noncommutative division algebra in which the inverse of any nonzero element $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ is given by

$$\frac{\bar{a}}{\|a\|^2}$$

where $\bar{a} = a_0 - a_1 i - a_2 j - a_3 k$ is the conjugate of $a$ in $\mathbb{H}$ and $\|a\|$ is the norm of $a$ defined by

$$\|a\| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.$$

We call $a_0$ the *real part* of $a$ and write $\operatorname{Re} a = a_0$.

We will always equip $\mathbb{H}$ with the real inner product

$$\langle a, b\rangle = \operatorname{Re}(a\bar{b}) \qquad (a, b \in \mathbb{H})$$

so that $\mathbb{H}$ is a real Hilbert space.

The real *octonions* $\mathbb{O}$ can be described as an 8-dimensional real vector space with basis $\{\mathbf{1}, e_1, e_2, \cdots, e_7\}$ in which $e_1 = i$, $e_2 = j, e_3 = k$, the quaternion basis, and $e_5 = ie_4$, $e_6 = je_4$, $e_7 = ke_4$. The multiplication in $\mathbb{O}$ is defined by the following table:

| $\cdot$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $-\mathbf{1}$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ | $e_6$ |
| $e_2$ | $-e_3$ | $-\mathbf{1}$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_2$ | $-e_1$ | $-\mathbf{1}$ | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $-\mathbf{1}$ | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | $-\mathbf{1}$ | $-e_3$ | $e_2$ |
| $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $-e_2$ | $e_3$ | $-\mathbf{1}$ | $-e_1$ |
| $e_7$ | $-e_6$ | $e_5$ | $e_4$ | $-e_3$ | $-e_2$ | $e_1$ | $-\mathbf{1}$ |

The octonions $\mathbb{O}$ is a non-associative division algebra. The inverse of a nonzero element $a = a_0 + a_1e_1 + \cdots + a_7e_7 \in \mathbb{O}$ is given by

$$\frac{\bar{a}}{\|a\|^2}$$

where $\bar{a} = a_0 - a_1e_1 - \cdots - a_7e_7$ is the conjugate of $a$ in $\mathbb{O}$ and $\|a\|$ is the norm:

$$\|a\| = (a_0^2 + a_1^2 + \cdots + a_7^2)^{1/2}.$$

We call $a_0$ the *real part* of $a$ and write $\operatorname{Re} a = a_0$.

Now we define the *split octonions* as pairs of the octonions. Let

$$u_0 = \frac{1}{2}(\mathbf{1}, e_7) \qquad u_2 = \frac{1}{2}(e_2, -e_5) \qquad u_4 = \frac{1}{2}(e_1, e_6) \qquad u_6 = \frac{1}{2}(e_3, -e_4)$$
$$u_1 = \frac{1}{2}(\mathbf{1}, -e_7) \qquad u_3 = \frac{1}{2}(e_2, e_5) \qquad u_5 = \frac{1}{2}(e_1, -e_6) \qquad u_7 = \frac{1}{2}(e_3, e_4)$$

where $\mathbf{1}, e_1, e_2, \cdots, e_7$ are the octonion basis elements. The *real split octonion* $\mathbb{O}_s$ is the 8-dimensional real vector space spanned by the above elements. We define the *complex split octonion* $\mathcal{O}_s$ to be the complex linear span of the above elements. Both $\mathbb{O}_s$ and $\mathcal{O}_s$ are non-associative algebras where the multiplication table of the split octonions can be drawn by multiplying above elements in the usual way, i.e $(a,b)(c,d) = (ac-bd, ad+bc)$ where $a,b,c,d \in \mathbb{O}$, using the octonion multiplication table (see [4]).

Throughout the thesis, $\mathbb{F}$ will always denote the real division algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We are going to study ternary structures of Hilbert spaces over $\mathbb{F}$ where, by a *Hilbert space $H$ over* $\mathbb{H}$, or a *quaternion Hilbert space $H$*, we mean a right $\mathbb{H}$-module $H$, equipped with an $\mathbb{H}$-valued inner product $\langle \cdot, \cdot \rangle : H \times H \longrightarrow \mathbb{H}$ where $a = a_0 + a_1 i + a_2 j + a_3 k \geq 0$ in $\mathbb{H}$ means $a_0 \geq 0$ and $a_1 = a_2 = a_3 = 0$. In the sequel, unless otherwise stated, the inner product of a Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$.

Let $H$ and $K$ be Hilbert spaces over $\mathbb{H}$. By an $\mathbb{H}$-*linear operator*, or simply, a *linear operator* $T : H \longrightarrow K$, we mean

$$T(x\lambda + y\mu) = T(x)\lambda + T(y)\mu \qquad (x, y \in H,\ \lambda, \mu \in \mathbb{H})$$

in which case $T$ is also real linear. We call an operator $T : H \longrightarrow K$ *conjugate linear* if

$$T(x\lambda + y\mu) = T(x)\bar{\lambda} + T(y)\bar{\mu} \qquad (x, y \in H,\ \lambda, \mu \in \mathbb{H})$$

where $\bar{\lambda}, \bar{\mu}$ are the conjugates of $\lambda, \mu$ in $\mathbb{H}$.

Given a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ over $\mathbb{C}$ or $\mathbb{H}$, we consider $H$ as a real Hilbert space by restricting to the real scalar multiplication and taking the real inner product

$$\ll \cdot,\cdot \gg = \mathrm{Re}\,\langle\cdot,\cdot\rangle.$$

We call $(H, \ll\cdot,\cdot\gg)$ the *real restriction* of $(H, \langle\cdot,\cdot\rangle)$. If $(H, \langle\cdot,\cdot\rangle)$ is a complex Hilbert space, then we have

$$\langle x, y\rangle = \ll x, y \gg - \ll ix, y \gg i,$$

and for a quaternion Hilbert space $(H, \langle\cdot,\cdot\rangle)$, we have

$$\langle x, y\rangle = \ll x, y \gg - \ll xi, y \gg i - \ll xj, y \gg j - \ll xk, y \gg k.$$

Besides the usual involutions on $\mathbb{H}$ and $\mathbb{O}$, we need to consider other involutions for the study of classification of Hilbert spaces with a Jordan triple structure. The *usual involution* on $\mathbb{R}$ is the identity map, on $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ is the conjugation.

An *involution of the second kind* on $\mathbb{H}$ is an involution $\delta$ of $\mathbb{H}$ commuting with the usual involution. For instance, $(a_0+a_1i+a_2j+a_3k)^\delta = -a_0+a_1i+a_2j+a_3k$ is such an involution.

The usual involution $'$ on the split octonions $\mathbb{O}_s$ and $\mathcal{O}_s$ is defined by

$$(\alpha_0u_0 + \alpha_1u_1 + \cdots + \alpha_7u_7)' = \alpha_1u_0 + \alpha_0u_1 - \alpha_2u_2 - \alpha_3u_3 - \cdots - \alpha_7u_7$$

where $\alpha_i$, $i = 0, 1, \cdots, 7$ are the scalars. Another involution on $\mathbb{O}_s$ and $\mathcal{O}_s$, needed later, is defined by

$$(\alpha_0u_0+\alpha_1u_1+\cdots+\alpha_7u_7)^\times = \overline{\alpha}_0u_0+\overline{\alpha}_1u_1-\overline{\alpha}_3u_2-\overline{\alpha}_2u_3-\overline{\alpha}_5u_4-\overline{\alpha}_4u_5-\overline{\alpha}_7u_6-\overline{\alpha}_6u_7$$

where $\overline{\alpha}_j = \alpha_j$ if $\alpha_j \in \mathbb{R}$. For more details of these involutions, see [30].

**Definition 2.1.1.** Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space. On the complexification $H_{\mathbb{C}} = H \bigoplus iH$, define

$$\langle x+iy, z+iw\rangle_{\mathbb{C}} = \langle x, z\rangle + \langle y, w\rangle + i\langle y, z\rangle - i\langle x, w\rangle$$

for all $x, y, z, w \in H$. Then $(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ is a complex Hilbert space and we have

$$\|x \pm iy\|^2 = \|x\|^2 + \|y\|^2$$

for all $x, y \in H$. The complex Hilbert space $H_{\mathbb{C}}$ is called the *complexification* of $H$.

Given a real or complex Hilbert space $H$, the set of all bounded linear operators on $H$ is denoted by $\mathcal{B}(H)$. Let $H_{\mathbb{C}}$ be the complexification of a real Hilbert space $H$. Then we have

$$\mathcal{B}(H_{\mathbb{C}}) = \mathcal{B}(H) \bigoplus i\mathcal{B}(H)$$

which is the complexification of $\mathcal{B}(H)$.

Given a family $\{H_\alpha : \alpha \in I\}$ of Hilbert spaces, we define as usual the Hilbert space direct sum $\bigoplus_{\alpha \in \mathrm{I}} H_\alpha$ to be the space consisting of elements $(x_\alpha)_{\alpha \in \mathrm{I}}$ in the Cartesian product $\prod_{\alpha \in I} H_\alpha$ satisfying $\sum_{\alpha \in \mathrm{I}} \|x_\alpha\|^2 < \infty$ (see [18, p.123]). The inner product in $\bigoplus_{\alpha \in \mathrm{I}} H_\alpha$ is given by

$$\langle (x_\alpha)_{\alpha \in \mathrm{I}}, (y_\alpha)_{\alpha \in \mathrm{I}} \rangle = \sum_{\alpha \in \mathrm{I}} \langle x_\alpha, y_\alpha \rangle$$

where the symbol $\langle \cdot, \cdot \rangle$ is interpreted in an obvious way.

Let $H$ and $K$ be Hilbert spaces over $\mathbb{F}$ and choose an orthonormal basis $\{e_\beta\}_{\beta \in \Gamma}$ in $H$. An $\mathbb{F}$-linear operator $a : H \longrightarrow K$ is called a *Hilbert-Schmidt operator* if

$$\sum_{\beta \in \Gamma} \|a(e_\beta)\|^2 < \infty.$$

This definition does not depend on the choice of the basis $\{e_\beta\}_{\beta \in \Gamma}$. The Hilbert-Schmidt operators between two $\mathbb{F}$-Hilbert spaces form a real Hilbert space, denoted by $\mathcal{C}_2(H, K)$, with inner product

$$\langle a, b \rangle = \operatorname{Re} \operatorname{Trace}(a^* b) \qquad (a, b \in \mathcal{C}_2(H, K))$$

where $a^*$ denotes the adjoint of an operator $a \in \mathcal{C}_2(H, K)$. We write $\mathcal{C}_2(H)$ for $\mathcal{C}_2(H, H)$.

Let $k > 0$. Then $\mathcal{C}_2(H, K)$ is also a real Hilbert space with the inner product

$$\langle a, b \rangle_k = k \operatorname{Re} \operatorname{Trace}(a^* b) \qquad (a, b \in \mathcal{C}_2(H, K)).$$

We write $\mathcal{C}_2^k(H, K)$ for $(\mathcal{C}_2(H, K), \langle \cdot, \cdot \rangle_k)$, and $\mathcal{C}_2^1(H, K)$ is just $\mathcal{C}_2(H, K)$.

For two $\mathbb{F}$-Hilbert spaces $H$ and $K$ and $u \in H$, $v \in K$, we define the *rank-one operators* $v \otimes u : H \longrightarrow K$ and $u \otimes v : K \longrightarrow H$ by

$$(v \otimes u)(x) = v\langle x, u \rangle_H, \quad (u \otimes v)(y) = u\langle y, v \rangle_K \qquad (x \in H,\ y \in K)$$

where $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_K$ denote the inner products in $H$ and $K$ respectively. Evidently $v \otimes u \in \mathcal{C}_2(H, K)$. We note that the dual $(v \otimes u)^* \in \mathcal{C}_2(K, H)$ is given by

$$(v \otimes u)^*(y) = u\langle y, v \rangle_K \qquad (y \in K)$$

and hence $(v \otimes u)^* = u \otimes v$.

Given $a \in \mathcal{C}_2(K, H)$, we have

$$a(v \otimes u) = a(v) \otimes u$$

where $u \in H$ and $v \in K$. In fact, for any $h \in H$, we have

$$a(v \otimes u)(h) = a(v\langle h, u \rangle_H) = a(v)\langle h, u \rangle_H = (a(v) \otimes u)(h).$$

Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be an orthonormal basis of $K$. Then we have

$$(f_\alpha \otimes e_\beta)(e_\gamma) = \begin{cases} f_\alpha & \text{if } \gamma = \beta \\ 0 & \text{if } \gamma \neq \beta \end{cases} \tag{2.1}$$

for all $\alpha, \beta$. The rank-one operator $f_\alpha \otimes e_\beta$ can be conveniently represented as a matrix $(m_{\alpha\beta})_{(\alpha,\beta) \in \Lambda \times \Gamma}$ with the $(\alpha, \beta)$-entry 1, but 0 elsewhere.

We note that $\{f_\alpha \otimes e_\beta : \alpha \in \Lambda, \beta \in \Gamma\}$ is an orthogonal set in $\mathcal{C}_2^k(H, K)$ for if $\alpha \neq \mu$ in $\Lambda$ or $\beta \neq \gamma$ in $\Gamma$, then

$$\begin{aligned}
\langle f_\alpha \otimes e_\beta, f_\mu \otimes e_\gamma \rangle_k &= k \operatorname{Re} \operatorname{Trace}\left((f_\alpha \otimes e_\beta)^*(f_\mu \otimes e_\gamma)\right) \\
&= k \operatorname{Re} \langle (f_\alpha \otimes e_\beta)^*(f_\mu), e_\gamma \rangle_H \\
&= k \operatorname{Re} \langle (e_\beta \otimes f_\alpha)(f_\mu), e_\gamma \rangle_H \\
&= k \operatorname{Re} \langle e_\beta \langle f_\mu, f_\alpha \rangle_K, e_\gamma \rangle_H \\
&= k \operatorname{Re} \langle e_\beta, e_\gamma \rangle_H \langle f_\mu, f_\alpha \rangle_K \\
&= 0.
\end{aligned}$$

In fact, $\{f_\alpha \otimes e_\beta : \alpha \in \Lambda, \beta \in \Gamma\}$ is an orthogonal basis of $\mathcal{C}_2^k(H, K)$. For if $a \in \mathcal{C}_2^k(H, K)$ and $\langle a, f_\alpha \otimes e_\beta \rangle_k = 0$ for all $\alpha \in \Lambda$ and $\beta \in \Gamma$, then for all $\alpha \in \Lambda$, we have

$$\begin{aligned}
k \operatorname{Re} \langle a^*(f_\alpha), e_\beta \rangle_H &= k \operatorname{Re} \sum_{\gamma \in \Gamma} \langle a^*(f_\alpha), e_\gamma \rangle_H \langle e_\gamma, e_\beta \rangle_H \\
&= k \operatorname{Re} \sum_{\gamma \in \Gamma} \langle (a^*(f_\alpha) \otimes e_\beta) e_\gamma, e_\gamma \rangle_H \\
&= k \operatorname{Re} \sum_{\gamma \in \Gamma} \langle a^*(f_\alpha \otimes e_\beta) e_\gamma, e_\gamma \rangle_H \\
&= k \operatorname{Re} \operatorname{Trace}(a^*(f_\alpha \otimes e_\beta)) \\
&= \langle a, f_\alpha \otimes e_\beta \rangle_k = 0
\end{aligned}$$

where $k > 0$. Hence $a = 0$.

Let $\mathcal{C}_2^k(H, K)^{\mathbb{R}}$ be the real linear span of

$$\{f_\alpha \otimes e_\beta : \alpha \in \Lambda, \beta \in \Gamma\}$$

in $\mathcal{C}_2^k(H, K)$. Then we have the natural identification

$$\mathcal{C}_2^k(H, K) = \mathcal{C}_2^k(H, K)^{\mathbb{R}} \bigotimes_{\mathbb{R}} \mathbb{F}.$$

We choose the following way to embed a Hilbert space $H$ into $\mathcal{C}_2(H,K)$. Let $\{e_\beta\}_{\beta\in\Gamma}$ be an orthonormal basis of $H$ and $\{f_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis of $K$ and fix some $f_{\alpha_0}$ in $\{f_\alpha\}_{\alpha\in\Lambda}$. Then one can embed $H$ into $\mathcal{C}_2(H,K)$ by the linear isometry

$$\psi : x \in H \longmapsto f_{\alpha_0} \otimes x \in \mathcal{C}_2(H,K)$$

where

$$\begin{aligned}
\langle f_{\alpha_0} \otimes x, f_{\alpha_0} \otimes h\rangle &= \operatorname{Re}\operatorname{Trace}((f_{\alpha_0}\otimes x)^*(f_{\alpha_0}\otimes h)) \\
&= \operatorname{Re}\operatorname{Trace}((x\otimes f_{\alpha_0})(f_{\alpha_0}\otimes h)) \\
&= \operatorname{Re}\operatorname{Trace}(x\otimes h) \\
&= \operatorname{Re}\sum_{\gamma\in\Gamma}\langle (x\otimes h)e_\gamma, e_\gamma\rangle_H \\
&= \operatorname{Re}\sum_{\gamma\in\Gamma}\langle x, e_\gamma\rangle_H\langle e_\gamma, h\rangle_H \\
&= \operatorname{Re}\langle x, h\rangle_H.
\end{aligned}$$

for any $h \in H$.

**Definition 2.1.2.** For any element $a$ in a complex Banach algebra $\mathcal{A}$ with identity $e$, the *spectrum* of $a$ is denoted by

$$\sigma(a) = \{\lambda \in \mathbb{C} :\ \lambda e - a \text{ is not invertible in } \mathcal{A}\}.$$

Let $A$ be a real Banach algebra with identity $e$ and let $A_{\mathbb{C}} = A \bigoplus iA$ be the complexification of $A$ where $A_{\mathbb{C}}$ is a Banach algebra in a suitable norm (see [31]). Let $a \in A$. We define its *spectrum*, denoted by $\sigma_A(a)$ or simply $\sigma(a)$, to be the spectrum $\sigma_{A_{\mathbb{C}}}(a)$ of $a$ in the complexification $A_{\mathbb{C}}$. We note that $\overline{\sigma_A(a)} = \sigma_A(a) \subset \mathbb{C}$ where $^-$ denotes the complex conjugation, and that

$$\sigma_A(a) = \left\{\alpha + i\beta :\ \alpha,\beta \in \mathbb{R} \text{ and } (\alpha e - a)^2 + \beta^2 \text{ is not invertible in A}\right\} \text{ [14].}$$

If $\mathcal{A}$ is a complex algebra and $a \in \mathcal{A}$, then we have $\sigma_{\mathcal{A}_{\mathbb{C}}}(a) = \sigma_{\mathcal{A}}(a) \cup \overline{\sigma_{\mathcal{A}}(a)}$ where $\sigma_{\mathcal{A}}(a)$ is the spectrum with respect to the *complex* algebra $\mathcal{A}$.

Let $A$ be a real Banach algebra containing $a$ and $A_{\mathbb{C}}$ be its complexification. For any holomorphic function $f$ on an open set $U$ containing $\sigma(a) = \sigma_A(a)$, such that $\overline{U} = U$, there is a well-defined element $f(a)$ in the complexification $A_{\mathbb{C}}$ via the Cauchy formula [31, 32]. If moreover we have $f(\overline{\lambda}) = \overline{f(\lambda)}$ for $\lambda \in U$, then

$$\begin{aligned} \overline{f(a)} &= \frac{1}{2\pi i}\int_{\rho} \overline{f(\lambda)}(a-\overline{\lambda})^{-1}d\overline{\lambda} \\ &= \frac{1}{2\pi i}\int_{\rho} f(\overline{\lambda})(a-\overline{\lambda})^{-1}d\overline{\lambda} \\ &= f(a) \end{aligned}$$

which implies $f(a) \in A$, where $\rho$ is a smooth simple closed curve in $U$, enclosing $\sigma(a)$.

Let $T$ be a bounded linear operator on a real Banach space $V$. We can consider $T$ as an element in the real Banach algebra $\mathcal{B}(V)$ of all bounded linear operators on $V$. Since

$$\|I\| + \|T\| + \left\|\frac{T^2}{2!}\right\| + \cdots + \left\|\frac{T^n}{n!}\right\| + \cdots \le \sum_{n=0}^{\infty} \frac{\|T\|^n}{n!} < \infty,$$

the operator $\exp(T) = I + T + \dfrac{T^2}{2!} + \cdots + \dfrac{T^n}{n!} + \cdots$ is well-defined in $\mathcal{B}(V)$.

The following two lemmas for real Banach spaces are extensions of results for complex Banach spaces proved in [37]. It will be used to prove Theorem 3.5.2 later.

**Lemma 2.1.3.** *Let $T$ be a bounded linear operator on a real Banach space $V$ and $S = \exp(T)$. If $S(x) = x$ for some $x \in V$ and the spectrum $\sigma(T)$ does not contain $\pm 2k\pi i$ for all $k \in \mathbb{N}$, then $T(x) = 0$.*

*Proof.* The complex mapping

$$f(\lambda) = \frac{e^{\lambda} - 1}{\lambda}$$

is holomorphic on the entire complex plane. Consider $T \in \mathcal{B}(V)$. Since $f$ is holomorphic on any open set $U$ containing $\sigma(T)$, which is symmetric with respect to the real axis, and satisfies $f(\overline{\lambda}) = \overline{f(\lambda)}$, by the above remark we have $f(T) \in \mathcal{B}(V)$, and noting that $\pm 2k\pi i \notin \sigma(T)$, the spectral mapping theorem and functional calculus imply $0 \notin \sigma(f(T))$, that is, $f(T)$ is invertible. From $f(\lambda)\lambda = e^{\lambda} - 1$ we have $f(T)T(x) = S(x) - x = 0$. Hence $T(x) = f(T)^{-1} f(T)T(x) = 0$. □

We recall that a bounded linear operator $T$ on a Hilbert space $H$ is called *positive* if $T = T^*$ and $\sigma(T) \subset [0, \infty)$. If $T$ is positive, then $T^n$ is positive for all $n \in \mathbb{N}$.

**Lemma 2.1.4.** *Let $T$ be a positive operator on a real Hilbert space $H$ such that $\|T\| = 1$. Then the sequence $\big(T^n(x)\big)$ converges for all $x \in H$.*

*Proof.* There exists some positive operator $S$ on $H$ such that $T = S^2$ [14] and $\langle Tx, x\rangle \geq 0$ for all $x \in H$. Fix $x \in H$ and define the sequence

$$\alpha_n = \langle T^n x, x\rangle.$$

We have $\alpha_n \geq 0$ and that $(\alpha_n)$ is a decreasing sequence:

$$\alpha_{n+1} = \langle T^{n+1}x, x\rangle = \langle S^n T S^n x, x\rangle \leq \|T\| \langle T^n x, x\rangle = \alpha_n$$

which implies $\alpha = \lim_{n\to\infty} \alpha_n$ exists. Now, for integers $n$ and $m$, we have

$$\|T^n(x) - T^m(x)\|^2 = \langle T^{2n}x, x\rangle - 2\langle T^{n+m}x, x\rangle + \langle T^{2m}x, x\rangle \longrightarrow \alpha - 2\alpha + \alpha = 0$$

where $n, m \to \infty$. Hence $\big(T^n(x)\big)$ is a Cauchy sequence in $H$ and therefore converges. □

## 2.2 Conjugations in Hilbert spaces

**Definition 2.2.1.** A *conjugation* on a Hilbert space $H$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ is a conjugate linear isometry $J$ of $H$ such that $J^2 = \mathrm{id}$, the identity map on H. A conjugate linear isometry $j$ of $H$ satisfying $j^2 = -\mathrm{id}$ is called an *anticonjugation.* If $H$ is over $\mathbb{R}$, then conjugate linear maps are the same as linear maps.

Bounded conjugate linear operators on Hilbert spaces have adjoints and we have $J^* = J$ for a conjugation $J$ and $j^* = -j$ for an anticonjugation $j$.

For any bounded conjugate linear operator $T$ on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ we have

$$\begin{aligned} \langle Tx, y\rangle &= \ll Tx, y \gg -i \ll iTx, y \gg \\ &= \ll x, T^*y \gg -i \ll -ix, T^*y \gg \\ &= \ll T^*y, x \gg -i \ll T^*y, -ix \gg \\ &= \ll T^*y, x \gg -i \ll iT^*y, x \gg \\ &= \langle T^*y, x\rangle \end{aligned}$$

for all $x, y \in H$.

Given a conjugation $J$ on a complex Hilbert space $H$, let

$$H^J = \{x \in H : \; Jx = x\}.$$

Then $H^J$ is a real Hilbert space. Indeed $\langle x, y\rangle$ is a real number for all $x, y \in H^J$ since

$$\langle x, y\rangle = \langle Jx, y\rangle = \langle J^*y, x\rangle = \langle Jy, x\rangle = \langle y, x\rangle.$$

Therefore $H = H^J \bigoplus iH^J$ is the complexification of $H^J$.

**Lemma 2.2.2.** *Let $(H, \langle\cdot,\cdot\rangle)$ ba a complex Hilbert space. There always exists a conjugation on $H$. Also, there exists an anticonjugation on $H$ if and only if* $\dim H$ *is even or infinite. Moreover, if $J$ and $J'$ are two conjugations or two anticonjugations on $H$, then there is a linear isometry $U$ on $H$ such that $J' = UJU^*$.*

*Proof.* See [15, Lemma 7.5.6]. $\square$

## 2.3 $H^*$-algebras

The theory of complex $H^*$-algebras was developed by Ambrose [1] (see also [33] and [34]) and has been extended to the real case by several authors (see for example, [3], [5], [11] and [21]). A real Hilbert space $(A, \langle \cdot, \cdot \rangle)$ which is also an associative algebra with an involution $^*$ is called an *$H^*$-algebra* if

$$\langle xy, z \rangle = \langle x, zy^* \rangle = \langle y, x^* z \rangle$$

for all $x, y, z \in A$.

The real Hilbert space $\mathcal{C}_2(H)$ of Hilbert-Schmidt operators on a Hilbert space $H$ over $\mathbb{F}$, with operator composition as multiplication, is a typical example of a real $H^*$-algebra.

We recall some definitions concerning $H^*$-algebras for later reference. We define the *left annihilator* of an $H^*$-algebra $A$, denoted by $L_0(A)$, to be

$$L_0(A) = \{a \in A : \, aA = \{0\}\}.$$

A *right ideal* of an $H^*$-algebra $A$ is a closed subspace $R$ of $A$ such that $RA \subset R$. A *minimal right ideal* of an $H^*$-algebra is a proper right ideal which contains no right ideal other than itself and $\{0\}$. An *idempotent* in an $H^*$-algebra $A$ is an element $p \in A$ such that $p^2 = p$. Two nonzero idempotents $p_1$ and $p_2$ are said to be *orthogonal* if $p_1 p_2 = p_2 p_1 = 0$. An element $x \in A$ is called *self-adjoint* if $x^* = x$.

We extend below two results in [1] for complex $H^*$-algebras to real $H^*$-algebras for later applications.

**Lemma 2.3.1.** *For every idempotent $p$ in an $H^*$-algebra $A$ with $L_0(A) = \{0\}$, the right ideal $R = pA$ is minimal if and only if $p$ cannot be decomposed as a sum $p = p_1 + p_2$ of nonzero orthogonal idempotents $p_1$ and $p_2$.*

*Proof.* Since $A$ has zero left annihilator, we have $R \neq \{0\}$. If $R$ is not minimal, then $R$ contains a proper nonzero right ideal $R_1$ and we have $R = R_1 \bigoplus R_1^\perp$ where $R_1^\perp = \{x \in R : \langle x, R_1 \rangle = 0\} \neq \{0\}$. Note that $R_1^\perp$ is also a right ideal of $A$. Let $p = p_1 + p_2$ with $p_1 \in R_1$ and $p_2 \in R_1^\perp$. Then $p_1 \neq 0$ and $p_2 \neq 0$. We have

$$p_1 = pp_1 = (p_1 + p_2)p_1 = p_1 p_1 + p_2 p_1$$

which implies $p_2 p_1 \in R_1 \cap R_1^\perp$. Hence $p_2 p_1 = 0$ and $p_1 = p_1^2$. Similarly $p_2 = p_2^2$.

Conversely, let $p = p_1 + p_2$ where $p_1, p_2$ are nonzero orthogonal idempotents. Then $0 \neq p_1 \in p_1 A \subset pA$ since $pp_1 = (p_1 + p_2)p_1 = p_1 = p_1 p$. Moreover, $p_2 \in pA \setminus p_1 A$ otherwise $p_2 = p_1 p_2 = 0$. Therefore $pA$ is not a minimal ideal. □

**Lemma 2.3.2.** *Every idempotent $p$ of an $H^*$-algebra $A$ with zero left annihilator has a decomposition $p = p_1 + \cdots + p_n$ into mutually orthogonal idempotents such that each $p_i$ admits no further (nontrivial) orthogonal decomposition. Moreover, $p_i$'s are self-adjoint if $p$ is self-adjoint.*

*Proof.* Consider the right ideal $R = pA$. Then $R$ is nonzero since $L_0(A) = \{0\}$. If $R$ is minimal, then assertion follows from Lemma 2.3.1. Otherwise $R = R_1 \bigoplus R_1^\perp$ where $R_1$ and $R_1^\perp$ are nonzero right ideals. We continue this process. Then at each stage there is a decomposition of $R$ into an orthogonal sum of right ideals $R = R_1 \bigoplus \cdots \bigoplus R_n$. Let $p = p_1 + \cdots + p_n$ where $p_i \in R_i$. We have

$$p_i = pp_i = (p_1 + \cdots + p_n)p_i = p_1 p_i + \cdots + p_n p_i \in R_i$$

where $p_j p_i \in R_j$. Since $\langle p_j p_i, p_k p_i \rangle = 0$ for $k \neq j$, we have

$$\|p_j p_i\|^2 = \langle p_j p_i, p_j p_i \rangle = \langle p_j p_i, p_1 p_i + \cdots + p_n p_i \rangle = \langle p_j p_i, pp_i \rangle = \langle p_j p_i, p_i \rangle = 0$$

for $j \neq i$. Therefore $p_i = p_i^2$. We note that for an idempotent $p_i$ in an $H^*$-algebra, $\|p_i\| \geq 1$. Since $\|p\|^2 = \|p_1\|^2 + \cdots + \|p_n\|^2 \geq n$, the decomposition process must end after a finite number of steps.

Now let $R = R_1 \bigoplus \cdots \bigoplus R_n$ where $R_i$ are minimal right ideals. Then $p = p_1 + \cdots + p_n$ in which every $p_i$ cannot be decomposed into two nonzero orthogonal idempotents. We have

$$p_1 + \cdots + p_n = p = p^2 = (p_1 + \cdots + p_n)p = p_1 p + \cdots + p_n p = pp_1 + \cdots + pp_n$$

which implies $p_i = p_i p = pp_i$. Let $p$ be self-adjoint. Then we get $p_i^* = pp_i^* = p_i^* p$. For any $x, y \in A$, we have $p_i x \in R_i$ and $p_j y \in R_j$, and then $\langle p_j^* p_i x, y \rangle = \langle p_i x, p_j y \rangle = 0$ for $j \neq i$. This gives $p_j^* p_i = 0$ when $j \neq i$. Therefore from

$$p_i^* = p_i^* p = p_i^*(p_1 + \cdots + p_n) = p_i^* p_i$$

and

$$p_i = (p_1^* + \cdots + p_n^*)p_i = p_i^* p_i,$$

we have $p_i^* = p_i$. □

# Chapter 3

# Hilbert ternary algebras

It is well known that Jordan ternary structures play an important role in differential geometry. For instance, it has been shown by Kaup [22] that the class of bounded symmetric domains in complex Banach spaces is in one-one correspondence with a class of complex Banach spaces equipped with a Jordan ternary structure, called $JB^*$-triples. Recently, Chu [9] has introduced a class of real Hilbert spaces equipped with a Jordan ternary structure called $JH$-triples and showed that they correspond to a large class of Riemannian symmetric spaces. Each $JH$-triple gives rise to a Hilbert ternary algebra. Conversely, each abelian Hilbert ternary algebra gives rise to a $JH$-triple. Motivated by their close connection to geometry as described above, we develop a structure theory for Hilbert ternary algebras in this chapter.

We study ideals, centralizers and derivations of Hilbert ternary algebras. We prove a Wedderburn type theorem for a Hilbert ternary algebra $V$ with zero annihilator, namely, $V$ can be decomposed as a direct sum $\bigoplus_\alpha V_\alpha$ of minimal closed ideals, each of which is a simple Hilbert ternary algebra. We show that each simple abelian Hilbert ternary algebra is isomorphic to the algebra of Hilbert-Schmidt operators between two real, complex or quaternion Hilbert spaces, up to a positive multiple of the inner product. The latter result has been published in [2]. We describe completely ternary isomorphisms between abelian Hilbert ternary algebras. We give examples of nonabelian Hilbert ternary algebras, but a more detailed study of nonabelian Hilbert ternary algebras in relation with Jordan Hilbert triples will be given in the next chapter.

## 3.1 Basic algebraic structures

**Definition 3.1.1.** A real Hilbert space $(V, \langle\cdot,\cdot\rangle)$ is called a *Hilbert ternary algebra* if there exists a trilinear map $[\cdot,\cdot,\cdot] : V \times V \times V \longrightarrow V$ satisfying

$$\langle [x,y,z], w\rangle = \langle x, [w,z,y]\rangle = \langle z, [y,x,w]\rangle$$

for all $x, y, z, w \in V$.

**Definition 3.1.2.** A complex Hilbert space $(V, \langle\cdot,\cdot\rangle)$ is called a *Hermitian Hilbert ternary algebra* if there exists a ternary multiplication $[\cdot,\cdot,\cdot] : V^3 \longrightarrow V$ which is linear in the first and third variables but conjugate linear in the second one and satisfies

$$\langle [x,y,z], w\rangle = \langle x, [w,z,y]\rangle = \langle z, [y,x,w]\rangle$$

for all $x, y, z, w \in V$.

**Remark 3.1.3.** A Hermitian Hilbert ternary algebra $(V, \langle\cdot,\cdot\rangle)$ can be regarded as a (real) Hilbert ternary algebra by restricting to the real scalars and real inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$.

Hermitian Hilbert ternary algebras are called *Hilbert triple systems* in [37].

Let $V$ be a Hilbert ternary algebra and let $V_{\mathbb{C}} = V \bigoplus iV$ be the Hilbert space complexification of $V$.

We define a ternary product $[\cdot,\cdot,\cdot]_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by

$$\begin{aligned}[a+ib, x+iy, u+iv]_{\mathbb{C}} &= [a,x,u]+[a,y,v]-[b,x,v]+[b,y,u] \\ &\quad +i([a,x,v]-[a,y,u]+[b,x,u]+[b,y,v])\end{aligned}$$

for $a,b,x,y,u,v \in V$. With this ternary product, $V_{\mathbb{C}}$ is a Hermitian Hilbert ternary algebra.

**Example 3.1.4.** Let $V$ be a complex Hilbert space and $j : V \longrightarrow V$ a conjugate linear mapping such that $j^2(x) = -x$ and $\langle jx, jy\rangle = \langle y, x\rangle$ for all $x, y$ in $V$. For example, the conjugate linear map $j : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ defined by

$$j(a,b) = (-\overline{b}, \overline{a})$$

satisfies these conditions. Define a ternary product on $V$ by

$$[x,y,z] = \langle z,y\rangle x - \langle x, jz\rangle jy.$$

Then $V$ is a Hermitian Hilbert ternary algebra.

**Example 3.1.5.** Given an $H^*$-*algebra* $A$, we can define a ternary product $[\cdot,\cdot,\cdot] : A^3 \longrightarrow A$ by

$$[x,y,z] = xy^*z \qquad (x,y,z \in A).$$

Then $(A, \langle\cdot,\cdot\rangle, [\cdot,\cdot,\cdot])$ is a Hilbert ternary algebra.

**Example 3.1.6.** Let $H$ and $K$ be Hilbert spaces over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $k > 0$. Then $\mathcal{C}_2^k(H,K)$ is a Hilbert ternary algebra with the ternary product defined by

$$[a,b,c] = ab^*c \qquad (a,b,c \in \mathcal{C}_2^k(H,K))$$

where $a^*$ is the adjoint operator of $a$.

**Definition 3.1.7.** By a *ternary subalgebra* of a Hilbert ternary algebra $V$, we mean a vector subspace $W \subset V$ satisfying $[x,y,z] \in W$ for all $x,y,z \in W$.

We saw in Chapter 2 that the Hilbert space $H$ can be embedded into $\mathcal{C}_2(H,K)$ by the linear isometry $\psi : x \in H \longmapsto f_{\alpha_0} \otimes x \in \mathcal{C}_2(H,K)$ where $f_{\alpha_0}$ is a fixed element of the basis $\{f_\alpha\}_{\alpha\in\Lambda}$ of $K$. The embedding also satisfies

$$[\psi(x), \psi(y), \psi(z)] = \psi(z)\langle y, x\rangle \qquad (x, y, z \in H).$$

Indeed we have

$$\begin{aligned}
[f_{\alpha_0} \otimes x, f_{\alpha_0} \otimes y, f_{\alpha_0} \otimes z](h) &= (f_{\alpha_0} \otimes x)(f_{\alpha_0} \otimes y)^*(f_{\alpha_0} \otimes z)(h) \\
&= (f_{\alpha_0} \otimes x)(y \otimes f_{\alpha_0})(f_{\alpha_0}\langle h, z\rangle_H) \\
&= (f_{\alpha_0} \otimes x)(y\langle f_{\alpha_0}, f_{\alpha_0}\rangle_K \langle h, z\rangle_H) \\
&= f_{\alpha_0}\langle y, x\rangle_H \langle h, z\rangle_H \\
&= (f_{\alpha_0}\langle y, x\rangle_H \otimes z)(h) \\
&= (f_{\alpha_0} \otimes z)\langle y, x\rangle_H(h)
\end{aligned}$$

for all $h \in H$. Therefore, with the ternary product

$$[x, y, z] = z\langle y, x\rangle,$$

$H$ can be regarded as a ternary subalgebra of $\mathcal{C}_2(H,K)$.

**Definition 3.1.8.** Given $a, b$ in a Hilbert ternary algebra $V$, we define the *box operator* $a \,\square\, b : V \longrightarrow V$ by

$$(a \,\square\, b)(x) = [a, b, x]$$

for all $x \in V$.

Let $a \in V$. The *quadratic operator* $Q_a : V \longrightarrow V$, induced by $a$, is defined by

$$Q_a(x) = [a, x, a] \qquad (x \in V).$$

**Lemma 3.1.9.** *The box operator $a \,\square\, b : V \longrightarrow V$ is continuous for any $a, b$ in a Hilbert ternary algebra $V$. Moreover, the adjoint of $a \,\square\, b$ is the box operator $b \,\square\, a$.*

*Proof.* From the definition of the Hilbert ternary algebra, we have

$$\langle (a \,\square\, b)x, y\rangle = \langle x, (b \,\square\, a)y\rangle \qquad (\forall\, x, y \in V).$$

Hence $a \,\square\, b : V \longrightarrow V$ is weakly continuous, as well as norm continuous [12].

The above equation shows $(a \,\square\, b)^* = b \,\square\, a$. $\square$

The following proposition is an extension of the same result in [37] for Hilbert triple systems to Hilbert ternary algebras.

**Proposition 3.1.10.** *The ternary product of every Hilbert ternary algebra is continuous.*

*Proof.* Let $V$ be a Hilbert ternary algebra. For every $a \in V$, define the maps $\phi_a, \psi_a : V \to \mathcal{B}(V)$ by

$$\phi_a(x) = a \,\square\, x, \quad \psi_a(x) = x \,\square\, a$$

respectively, where $\mathcal{B}(V)$ denotes the real Banach algebra of all bounded linear operators on $V$. Then $\phi_a$ and $\psi_a$ are continuous. Indeed, let $x_n \longrightarrow 0$ in $V$ and $\phi_a(x_n) \longrightarrow T \in \mathcal{B}(V)$ as $n \longrightarrow \infty$. Then for every $x, y \in V$, we have

$$\begin{aligned} \langle Tx, y\rangle &= \lim_{n\to\infty} \langle \phi_a(x_n)x, y\rangle \\ &= \lim_{n\to\infty} \langle (a \,\square\, x_n)x, y\rangle \\ &= \lim_{n\to\infty} \langle [a, x_n, x], y\rangle \\ &= \lim_{n\to\infty} \langle a, [y, x, x_n]\rangle \\ &= \lim_{n\to\infty} \langle [x, y, a], x_n\rangle = 0. \end{aligned}$$

This shows $T = 0$ and therefore by the closed graph theorem, $\phi_a$ is continuous for any $a \in V$. Similarly, one can prove that $\psi_a$ is continuous for all $a \in V$.

Now let $S = \{\phi_a : \|a\| \leq 1\}$. Then for $\phi_a \in S$, we have

$$\|\phi_a(x)\| = \|a \,\square\, x\| = \|\psi_x(a)\| \leq \|\psi_x\|\|a\| \leq \|\psi_x\|$$

for all $x \in V$. By the uniform boundedness principle, there exist some constant $M > 0$ such that $\|\phi_a\| \leq M$ for all $\phi_a \in S$. Hence $\|\phi_{x/\|x\|}\| \leq M$ yields

$$\|\phi_x\| \leq M\|x\|$$

for all $x \in V$. It follows that

$$\|x \,\square\, y\| = \|\phi_x(y)\| \leq M\|x\|\|y\|$$

and

$$\|[x, y, z]\| = \|(x \,\square\, y)(z)\| \leq \|x \,\square\, y\|\|z\| \leq M\|x\|\|y\|\|z\|$$

for all $x, y, z \in V$. Hence $[\cdot, \cdot, \cdot]$ is continuous. $\square$

**Definition 3.1.11.** A Hilbert ternary algebra $(V, [\cdot, \cdot, \cdot])$ is called *abelian* if

$$[a, b, [x, y, z]] = [[a, b, x], y, z]$$

for all $a, b, x, y, z \in V$.

The Hilbert ternary algebra $\mathcal{C}_2^k(H, K)$ of Hilbert-Schmidt operators between two $\mathbb{F}$-Hilbert spaces $H$ and $K$, with the ternary product $[a, b, c] = ab^*c$, is abelian. On the other hand, $\mathcal{C}_2(H, K)$ with the triple product

$$\{a, b, c\} = \frac{1}{2}ab^*c + \frac{1}{2}cb^*a$$

is a Hilbert ternary algebra which is not abelian. For example, Let $V = \mathcal{C}_2(\mathbb{R}^2, \mathbb{R}^2)$ which is the space $M_2(\mathbb{R})$ of $2 \times 2$ real matrices. Then $V$ is a Hilbert ternary algebra with the inner product $\langle a, b\rangle = \text{Trace}(a^*b)$ and the above triple product $\{\cdot, \cdot, \cdot\}$. Let $a = b = x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\{a, b, \{x, y, z\}\} = \frac{1}{4}y^* \neq \frac{1}{2}y^* = \{\{a, b, x\}, y, z\}$, that is, $V$ is not abelian.

Throughout this chapter, the ternary product of a Hilbert ternary algebra $\mathcal{C}_2^k(H, K)$ is denoted by $[\cdot, \cdot, \cdot]$, unless otherwise stated.

**Remark 3.1.12.** In an abelian Hilbert ternary algebra $V$, we have

$$[a, b, [x, y, z]] = [a, [y, x, b], z]$$

for all $a, b, x, y, z \in V$. Indeed, we have

$$\begin{aligned} \langle [a, b, [x, y, z]], w \rangle &= \langle [x, y, z], [b, a, w] \rangle \\ &= \langle z, [y, x, [b, a, w]] \rangle \\ &= \langle z, [[y, x, b], a, w] \rangle \\ &= \langle [a, [y, x, b], z], w \rangle \end{aligned}$$

for any $w \in V$. We note that in an abelian Hilbert ternary algebra, $[a, b, [x, y, z]]$ need not equal $[a, [b, x, y], z]$. For example, let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be two matrices in $\mathcal{C}_2(\mathbb{R}^2)$ with the ternary product $[a, b, c] = ab^*c$. Then we have $[a, a, [a, b, b]] = a \neq 0 = [a, [a, a, b], b]$.

We give here some examples of nonabelian Hilbert ternary algebras.

Let $V$ be a Hilbert ternary algebra and $0 \neq a \in V$. Then the smallest closed ternary subalgebra $V(a)$ of $V$ containing $a$ is the norm closed real linear span of the odd powers $a^1 = a$, $a^3 = [a, a, a]$, $a^5 = [a^3, a, a]$, $\cdots$. It is a Hilbert ternary algebra with the inherited inner and ternary products.

If $H$ is a real Hilbert space with $\dim H \geq 2$ and is equipped with the ternary product

$$[a, b, c] = \frac{1}{2}\langle a, b \rangle c + \frac{1}{2}\langle c, b \rangle a \qquad (a, b, c \in H),$$

then $H$ is a nonabelian Hilbert ternary algebra, for otherwise there is a nonzero $a \in H$ orthogonal to $z \in H \setminus \{0\}$ and

$$\langle x, y \rangle \langle z, b \rangle a = \langle x, b \rangle \langle a, y \rangle z \qquad (b, x, y \in H)$$

from $[a, b, [x, y, z]] = [[a, b, x], y, z]$. Now let $x = y = a$ and $b = z$. We have

$$\|z\|^2 a = \langle a, z \rangle z = 0$$

which is a contradiction. However the closed ternary subalgebra $H(a)$ of $H$ generated by any nonzero element $a$ in $H$ is an abelian ternary subalgebra of $H$, in fact, $H(a) = \mathbb{R}a$.

We also note that the Hilbert ternary algebra $V$ in Example 3.1.4 is not abelian for $\dim V \geq 3$. Indeed, if it were, then we would have

$$\langle x, jz\rangle\langle jy, b\rangle a + \langle x, jz\rangle\langle a, y\rangle jb = \langle x, b\rangle\langle a, jz\rangle jy - \langle a, jx\rangle\langle z, b\rangle jy$$

for all $a, b, x, y, z \in V$. But this equality does not always hold. For example, if $\dim V \geq 3$, we can choose $y$ orthogonal to $a$ and $ja$, all nonzero, and $a = b = z$, then

$$\langle x, ja\rangle\langle jy, a\rangle a + \langle x, ja\rangle\langle a, y\rangle ja = \langle x, a\rangle\langle a, ja\rangle jy - \langle a, jx\rangle\langle a, a\rangle jy$$

where $\langle a, ja\rangle = 0$ implies

$$\|a\|^2\langle a, jx\rangle jy = 0.$$

Therefore $\langle a, jx\rangle = 0$. But one can choose element $x \in V$ such that $\langle x, ja\rangle \neq 0$.

For another example of a nonabelian Hilbert ternary algebra, consider the *spin factor* which is a real or complex Hilbert space $H$ with the ternary product

$$[a, b, c] = \langle a, b\rangle c + \langle c, b\rangle a - \langle a, \overline{c}\rangle \overline{b} \qquad (a, b, c \in H)$$

where $^{-}$ is a conjugation of $H$. Spin factors of dimension $\geq 3$ are nonabelian Hilbert ternary algebras. Otherwise for $a, b, x, y, z \in H$ such that $0 \neq \overline{a} = a = z$, $x = y$ and $\langle a, b\rangle = \langle a, y\rangle = 0$, the abelian condition $[a, b, [x, y, z]] = [[a, b, x], y, z]$ would imply

$$\|y\|^2 b = \langle b, y\rangle y$$

which is not true in general, when, for example, $b$ and $y$ are nonzero and mutually orthogonal.

It is interesting to note that each Hilbert ternary algebra gives rise to a nonabelian Hilbert ternary algebra. Let $(V, [\cdot,\cdot,\cdot])$ be a Hilbert ternary algebra. Define a new ternary product on the same Hilbert space $V$ by

$$[a,b,c]_1 = \frac{1}{2}[a,b,c] - \frac{1}{2}[c,b,a] \qquad (a,b,c \in V).$$

Then $(V, [\cdot,\cdot,\cdot]_1)$ is a nonabelian Hilbert ternary algebra. In fact, it contains no nonzero tripotents where a *tripotent* is an element $a$ satisfying $a = [a,a,a]$. It will be shown in Proposition 3.4.3 later that every abelian Hilbert ternary algebra admits a nonzero tripotent.

**Definition 3.1.13.** Given a Hilbert ternary algebra $(V, \langle\cdot,\cdot\rangle, [\cdot,\cdot,\cdot])$, we let $V_- = V$ and define

$$[\cdot,\cdot,\cdot]_- := -[\cdot,\cdot,\cdot].$$

Then $(V_-, \langle\cdot,\cdot\rangle, [\cdot,\cdot,\cdot]_-)$ is a Hilbert ternary algebra, called the *dual* of $V$. We will always assume that $V_-$ is equipped with this ternary structure.

Two Hilbert ternary algebras $(V, \langle\cdot,\cdot\rangle, [\cdot,\cdot,\cdot])$ and $(V', \langle\cdot,\cdot\rangle', [\cdot,\cdot,\cdot]')$ are said to be *(ternary) isomorphic* if there is a linear isometry $\tau : V \longrightarrow V'$ which preserves the ternary product, that is

$$\tau[x,y,z] = [\tau x, \tau y, \tau z]'.$$

## 3.2 Ideal structures

We begin with the definition of a ternary ideal of a Hilbert ternary algebra. The aim of this section is to show that every Hilbert ternary algebra is an orthogonal sum of its annihilator and minimal ternary ideals.

**Definition 3.2.1.** Let $V$ be a Hilbert ternary algebra and $I$ a subspace of $V$. Then $I$ is called

(i) a *ternary left ideal* of $V$ if $[V, V, I] \subset I$.

(ii) a *ternary right ideal* of $V$ if $[I, V, V] \subset I$.

(iii) a *ternary ideal* of $V$ if $[I, V, V] + [V, I, V] + [V, V, I] \subset I$.

(iv) a *ternary inner ideal* of $V$ if $[I, V, I] \subset I$.

If confusion is unlikely, we sometimes omit the word *ternary* in the above definition. Note that every left or right ideal is an inner ideal.

**Definition 3.2.2.** Let $V$ be a Hilbert ternary algebra. The subspace

$$V_0 = \{a \in V : [a, V, V] = \{0\}\}$$

is called the *annihilator* of $V$.

We note that

$$V_0 = \{a \in V : [V, a, V] = \{0\}\} = \{a \in V : [V, V, a] = \{0\}\}$$

and $V_0$ is a ternary ideal of $V$.

A Hilbert ternary algebra $V$ is called *simple* if $[V, V, V] \neq \{0\}$ and, $\{0\}$ and $V$ are the only closed ternary ideals of $V$. Note that for a simple Hilbert ternary algebra $V$ we have $V_0 = \{0\}$ because $V_0$ is a closed ternary ideal of $V$.

A closed ternary ideal $I$ of $V$ is called *minimal* if $I \neq \{0\}$ and $I$ does not properly contain a nonzero closed ternary ideal of $V$.

We extend below the result in [37, Theorem 1] for Hermitian Hilbert ternary algebras to Hilbert ternary algebras.

**Lemma 3.2.3.** *Every Hilbert ternary algebra $V$ can be decomposed into an orthogonal sum $V = V_0 \bigoplus V_1$ of two Hilbert ternary algebras, where $V_0$ is the annihilator of $V$ and $V_1$ is a Hilbert ternary algebra with zero annihilator.*

*Proof.* We have $V = V_0 \bigoplus V_1$ where $V_1 = V_0^{\perp}$ is the orthogonal complement of $V_0$. We show that $V_1$ is a ternary subalgebra of $V$ with zero annihilator. Since

$$\langle [V_1, V_1, V_1], V_0 \rangle = \langle V_1, [V_0, V_1, V_1] \rangle = \{0\},$$

we have $[V_1, V_1, V_1] \subset V_0^{\perp} = V_1$.

Let $a$ be in the annihilator of $V_1$. Then we have

$$[a, V, V] = [a, V_0, V_0] + [a, V_0, V_1] + [a, V_1, V_0] + [a, V_1, V_1] = \{0\}$$

which implies $a \in V_0$. Hence $a \in V_0 \cap V_1 = \{0\}$. □

Our objective is to prove that a Hilbert ternary algebra with zero annihilator can be decomposed into an orthogonal sum of simple Hilbert ternary algebras. First we consider some facts about ternary ideals of a Hilbert ternary algebra.

**Lemma 3.2.4.** *Let $V$ be a Hilbert ternary algebra.*

(i) *If $I$ is a ternary left ideal of $V$, then $I^{\perp}$ is also a ternary left ideal of $V$. The same statement is true for ternary right ideals.*

(ii) *If $V_0 = \{0\}$ and $I$ is a closed ternary ideal of $V$, then*

$$\{x \in V : [x, V, I] = \{0\}\} = I^{\perp} = \{x \in V : [I, V, x] = \{0\}\}.$$

(iii) *If $I$ is a closed ternary ideal of $V$ and $K$ a closed ternary ideal of $I$, then $K$ is a closed ternary ideal of $V$.*

(iv) *Let $I$ and $K$ be minimal ternary ideals of $V$, then either $I = K$ or $\langle I, K \rangle = \{0\}$.*

*Proof.* (i). Let $I$ be a left ideal of $V$. From

$$\langle [V, V, I^{\perp}], I \rangle = \langle I^{\perp}, [V, V, I] \rangle \subset \langle I^{\perp}, I \rangle = \{0\},$$

it follows that $I^\perp$ is a left ideal of $V$. Similar argument applies to right ideals.

(ii). Let us write

$$\begin{aligned} L_0(I) &= \{x \in V : [x, V, I] = \{0\}\} \\ R_0(I) &= \{x \in V : [I, V, x] = \{0\}\}. \end{aligned}$$

Note that $I^\perp$ is a left ideal and a right ideal of $V$ since $I$ is an ideal of $V$. Then we have

$$I^\perp \subset L_0(I) \cap R_0(I).$$

To prove the reverse inclusion, take any $x \in L_0(I)$ and write $x = y + z$ with $y \in I$ and $z \in I^\perp$. Pick $a, b \in V$ and write $b = b_1 + b_2$ with $b_1 \in I$, $b_2 \in I^\perp$. Then we have

$$\begin{aligned} [y, a, b] &= [x, a, b] - [z, a, b] \\ &= [x, a, b_1] + [x, a, b_2] - [z, a, b] \\ &= [x, a, b_2] - [z, a, b_1] - [z, a, b_2] \\ &= [x, a, b_2] - [z, a, b_2] \\ &= [y, a, b_2] = 0. \end{aligned}$$

where $[I, V, I^\perp] \subset I \cap I^\perp = \{0\}$ since $I^\perp$ is a left ideal of $V$. Hence $y \in V_0$ and $y = 0$ by assumption. Therefore $x = z \in I^\perp$. This proves $L_0(I) \subset I^\perp$. Similarly, we can prove $R_0(I) \subset I^\perp$.

(iii). We have

$$\begin{aligned} [V, V, K] &= [I, V, K] + [I^\perp, V, K] \\ &= [I, V, K] \\ &= [I, I, K] + [I, I^\perp, K] \\ &= [I, I, K] \subset K \end{aligned}$$

since $\langle [I, I^\perp, K], I\rangle = \langle K, [I^\perp, I, I]\rangle = \{0\}$ implies that $[I, I^\perp, K] \subset I \cap I^\perp = \{0\}$. Similarly, $[K, V, V] \subset K$ and $[V, K, V] \subset K$. Hence $K$ is a ternary ideal of $V$.

(iv). Obviously $I \cap K$ is also a closed ideal of $V$ which is contained in $I$ and $K$. Since $I$ and $K$ are minimal, $I \cap K = I = K$ or $I \cap K = \{0\}$. So, either

$$I = K \qquad \text{or} \qquad [I, V, K] \subset I \cap K = \{0\}$$

which implies $I \subset L_0(K) = K^\perp$ by (ii). □

**Lemma 3.2.5.** A closed subspace $I$ of a Hilbert ternary algebra $V$ is a ternary ideal if and only if it is both a ternary left ideal and a ternary right ideal.

*Proof.* We prove the sufficiency. Let I be a left and a right ideal of $V$. Then by Lemma 3.2.4 (i), $I^\perp$ is also a left and a right ideal of $V$. This implies that $[I^\perp, V, I] \subset I \cap I^\perp = \{0\}$, and hence

$$\langle [V, I, V], I^\perp\rangle = \langle V, [I^\perp, V, I]\rangle = \{0\}$$

which gives $[V, I, V] \subset I^{\perp\perp} = I$. □

**Lemma 3.2.6.** *Let $I$ be a closed ternary inner ideal of a Hilbert ternary algebra $V$. Then*

$$\langle I, [V, I, V]\rangle = \langle [I, V, I], I\rangle.$$

*Proof.* We have

$$\begin{aligned}\langle I, [V, I, V]\rangle &= \langle [I, V, I], V\rangle \\ &= \langle [I, V, I], I\rangle + \langle [I, V, I], I^\perp\rangle \\ &= \langle [I, V, I], I\rangle\end{aligned}$$

where $[I, V, I] \subset I$. □

**Lemma 3.2.7.** *Let $I$ be a closed subspace of a Hilbert ternary algebra $V$. Then $I$ is a ternary inner ideal of $V$ if and only if $[I, I^\perp, I] = \{0\}$.*

*Proof.* Let $I$ be an inner ideal of $V$. Then $[I,V,I] \subset I$ and we have

$$\begin{aligned} \langle [I, I^\perp, I], V\rangle &= \langle I, [I^\perp, I, V]\rangle \\ &= \langle [I,V,I], I^\perp\rangle = \{0\} \end{aligned}$$

which implies $[I, I^\perp, I] = \{0\}$. Conversely, if $[I, I^\perp, I] = \{0\}$, then $[I,V,I] \subset I^{\perp\perp} = I$ by the above computation. $\square$

**Lemma 3.2.8.** *For any closed ternary ideal $I$ of a Hilbert ternary algebra $V$ with zero annihilator, the closed linear span of $[I,I,I]$ is equal to $I$.*

*Proof.* Let $K$ be the linear span of $[I,I,I]$ and $\langle a, K\rangle = \{0\}$. Then by Lemma 3.2.4 (i), $[I,I,I^\perp] + [I^\perp, I, I] + [I^\perp, I, I^\perp] \subset I \cap I^\perp = \{0\}$ and we have

$$\begin{aligned} \langle [a,V,I], V\rangle &= \langle a, [V,I,V]\rangle \\ &= \langle a, [I,I,I] + [I,I,I^\perp] + [I^\perp,I,I] + [I^\perp,I,I^\perp]\rangle \\ &= \langle a, [I,I,I]\rangle \\ &= \{0\} \end{aligned}$$

which implies that $a \in I^\perp$ by Lemma 3.2.4 (ii). This shows that $K^\perp \subset I^\perp$ and therefore $I = I^{\perp\perp} \subset K^{\perp\perp} = \overline{K} \subset I$ which gives $\overline{K} = I$. $\square$

**Lemma 3.2.9.** *If $V$ is an abelian Hilbert ternary algebra, then $[a,V,a]$ is a ternary inner ideal in $V$ for any $a \in V$.*

*Proof.* Indeed, we have

$$[[a,V,a],V,[a,V,a]] = [a,[V,[a,V,a],V],a] \subset [a,V,a]$$

for all $a \in V$. $\square$

The ternary inner ideal $[a,V,a]$ in Lemma 3.2.9 is called the *principal inner ideal* determined by $a \in V$.

We now show the existence of minimal closed ternary ideals. For a Hilbert ternary algebra $V$, a nonzero element $x$ in $V$ is called *minimal*, if for every closed ternary ideal $I$ of $V$, either $x \in I$ or $x \in I^{\perp}$.

Let $x$ be a minimal element of $V$ and $I(x)$ the closed ternary ideal in $V$ generated by $x$. Suppose $K \subset I(x)$ for some closed ternary ideal $K$. Since $x$ is minimal, $x \in K$ or $x \in K^{\perp}$. This implies $I(x) = K$ or $I(x) \subset K^{\perp}$. The latter implies that $K \subset K^{\perp}$ and $K = \{0\}$. It follows that $I(x)$ is a minimal closed ternary ideal.

Therefore the existence of minimal elements in Hilbert ternary algebras implies the existence of minimal closed ternary ideals.

Using arguments similar to those in [37] for Hermitian Hilbert ternary algebras, we prove in the following a Wedderburn type theorem for Hilbert ternary algebras.

**Proposition 3.2.10.** *A Hilbert ternary algebra $V$ with zero annihilator contains a minimal element.*

*Proof.* For each $x \in V$, define $\phi_x : V \longrightarrow \mathcal{B}(V)$ as in the proof of Proposition 3.1.10. Define a new norm on $V$ by $|x| = \|\phi_x\|$ for $x \in V$. It is indeed a norm since $\phi_x = 0$ if, and only if, $x = 0$, as $V_0 = \{0\}$.

By the Krein-Milman theorem, the closed unit ball $B_1$ of the dual space $(V, |\cdot|)^*$ has an extreme point $f_0$ with $|f_0| = 1$. By the proof of Proposition 3.1.10, we have $|x| = \|\phi_x\| \leq M\|x\|$ for some $M > 0$. Hence $f_0 \in (V, \|\cdot\|)^*$ which implies $f_0(\cdot) = \langle \cdot, a\rangle$ for some $a \in V$. We show that $a$ is a minimal element in $V$.

Given a closed ideal $I$ in $(V, \|\cdot\|)$, we have $V = I \bigoplus I^{\perp}$. We show $a \in I$ or $a \in I^{\perp}$.

We first show that $|x + y| = \max\{|x|, |y|\}$ for $x \in I$ and $y \in I^{\perp}$. Let $z, w \in V$.

Then by orthogonality,

$$\begin{aligned}\|\phi_{x+y}(z)(w)\|^2 &= \|[x,z,w]+[y,z,w]\|^2\\ &= \|[x,z,w]\|^2+\|[y,z,w]\|^2\\ &= \|\phi_x(z)(w)\|^2+\|\phi_y(z)(w)\|^2.\end{aligned}$$

This implies

$$\|\phi_{x+y}(z)(w)\| \;\geq\; \|\phi_x(z)(w)\|,\, \|\phi_y(z)(w)\|$$

and

$$|x+y| = \|\phi_{x+y}\| \geq \max\{\|\phi_x\|, \|\phi_y\|\} = \max\{|x|,|y|\}.$$

On the other hand, for $w = u+v \in I \bigoplus I^{\perp}$, we have

$$\begin{aligned}\|\phi_{x+y}(z)(w)\|^2 &= \|[x,z,w]+[y,z,w]\|^2\\ &= \|[x,z,u]\|^2+\|[y,z,v]\|^2\\ &= \|\phi_x(z)(u)\|^2+\|\phi_y(z)(v)\|^2\\ &\leq \|\phi_x(z)\|^2\|u\|^2+\|\phi_y(z)\|^2\|v\|^2\\ &\leq \|\phi_x\|^2\|z\|^2\|u\|^2+\|\phi_y\|^2\|z\|^2\|v\|^2\\ &\leq \max\{|x|,|y|\}^2\|z\|^2(\|u\|^2+\|v\|^2)\\ &= \max\{|x|,|y|\}^2\|z\|^2\|w\|^2.\end{aligned}$$

Hence

$$|x+y| = \|\phi_{x+y}\| \leq \max\{|x|,|y|\}.$$

Let $U = \big\{g \in (V, |\cdot|)^* : g(I^{\perp}) = \{0\}\big\}$ and $W = \{g \in (V, |\cdot|)^* : g(I) = \{0\}\}$. We show that

$$|g+h| = |g| + |h|$$

for $g \in U$ and $h \in W$.

Choose $\varepsilon > 0$ such that $\varepsilon < 2\min\{|g|,|h|\}$, there exist elements $x \in I$ and

$y \in I^{\perp}$ such that $|x| = |y| = 1$, $g(x) > |g| - \varepsilon/2$ and $h(y) > |h| - \varepsilon/2$ which gives

$$\begin{aligned} |g+h| &\geq \frac{|(g+h)(x+y)|}{|x+y|} \\ &= \frac{g(x)+h(y)}{\max\{|x|,|y|\}} \\ &= g(x)+h(y) \\ &> |g|+|h|-\varepsilon. \end{aligned}$$

This proves $|g+h| \geq |g|+|h|$. The reverse inequality is trivial.

We now show that $(V, |\cdot|)^*$ can be decomposed into an $\ell_1$-direct sum $(V, |\cdot|)^* = U \bigoplus W$. Evidently, $U \cap W = \{0\}$.

Let $P$ and $Q$ be projections from $V$ onto $I$ and $I^{\perp}$, respectively. Let $z \in V$ and write $z = x + y$ with $x \in I$ and $y \in I^{\perp}$. The inequality

$$|P(z)| = |x| \leq \max\{|x|, |y|\} = |z|$$

implies that $P$ is $|\cdot|$-continuous. Likewise for $Q$. Given $f \in (V, |\cdot|)^*$, we let $g = f \circ P$ and $h = f \circ Q$, then $g \in U$, $h \in W$ and $f = g + h$. This proves $(V, |\cdot|)^* = U \bigoplus_{\ell_1} W$.

We have $f_0 = g + h \in U \bigoplus_{\ell_1} W$. Then $g = 0$ or $h = 0$. Otherwise

$$f_0 = |g|\frac{g}{|g|} + |h|\frac{h}{|h|} \qquad (|g|+|h| = 1)$$

implies $f_0 = g/|g| = h/|h| = 0$ which is impossible. If $f_0 \in W$, then $\langle x, a\rangle = f_0(x) = 0$ for all $x \in I$, that is, $a \in I^{\perp}$. If $f_0 \in U$, then $\{0\} = f_0(I^{\perp}) = \langle I^{\perp}, a\rangle$ which gives $a \in I^{\perp\perp} = I$. □

We are now ready to prove a Wedderburn type theorem for Hilbert ternary algebras.

**Theorem 3.2.11.** *Let $V$ be a Hilbert ternary algebra with zero annihilator. Then $V$ can be decomposed into an orthogonal sum $V = \bigoplus_{\alpha\in\Lambda} I_\alpha$ of a family $\{I_\alpha\}_{\alpha\in\Lambda}$ of minimal closed ternary ideals, each of which is a simple Hilbert ternary algebra.*

*Proof.* By Proposition 3.2.10, $V$ has a minimal closed ideal. Let $\{I_\alpha : \alpha \in \Lambda\}$ be a maximal family of mutually orthogonal minimal closed ideals of $V$. Let $I = \bigoplus_{\alpha\in\Lambda} I_\alpha$ be the orthogonal sum. Then both $I$ and $I^\perp$ are closed ideals of $V$. Suppose $I^\perp \neq \{0\}$. By Proposition 3.2.10, $I^\perp$ contains a minimal closed ideal $K$. By Lemma 3.2.4 (iii), $K$ is a minimal closed ideal of $V$. Therefore $K \subset I$ and $\langle K, I\rangle = 0$, which is a contradiction. Hence $I^\perp = \{0\}$ and $V = I$.

Further, each $I_\alpha$ is a simple Hilbert ternary algebra since every closed ideal of $I_\alpha$ is $\{0\}$ or $I_\alpha$, by minimality of $I_\alpha$. Also, $[I_\alpha, I_\alpha, I_\alpha] \neq \{0\}$ for otherwise $I_\alpha \subset L_0(I_\alpha) = I_\alpha^\perp$ (cf. the proof of Lemma 3.2.4 (ii)) and $I_\alpha = \{0\}$. □

## 3.3 Centralizers and derivations

In this section, we study the structure of centralizers of Hilbert ternary algebras and show that they form a real von Neumann algebra. We also show continuity of derivations on Hilbert ternary algebras with zero annihilator.

**Definition 3.3.1.** Let $V$ be a Hilbert ternary algebra. A linear map $C : V \longrightarrow V$ is called a *centralizer* of $V$ if

$$C[x, y, z] = [Cx, y, z] = [x, y, Cz]$$

for all $x, y, z \in V$.

Let $\mathcal{Z}(V)$ be the real vector space of centralizers of $V$.

**Lemma 3.3.2.** *Let $V$ be a Hilbert ternary algebra. The following statements are equivalent.*

(i) $a \,\square\, b \in \mathcal{Z}(V)$ *for all* $a, b \in V$.

(ii) $V$ *is abelian and* $[a \,\square\, b, c \,\square\, d] = 0$ *for all* $a, b, c, d \in V$.

*Proof.* (i) $\Rightarrow$ (ii). Suppose $a \,\square\, b \in \mathcal{Z}(V)$ for all $a, b \in V$. Then $a \,\square\, b$ commutes with any box operator $x \,\square\, y$ where $x, y \in V$ since

$$\begin{aligned}(a \,\square\, b)(x \,\square\, y)(z) &= (a \,\square\, b)[x, y, z] \\ &= [x, y, (a \,\square\, b)z] \\ &= (x \,\square\, y)(a \,\square\, b)(z)\end{aligned}$$

for all $z \in V$. We also have $(a \,\square\, b)[x, y, z] = [(a \,\square\, b)x, y, z]$ for each $x, y, z \in V$ which verifies the abelian condition for $V$.

(ii) $\Rightarrow$ (i). Let $V$ be abelian and $a, b \in V$. Then we have

$$\begin{aligned}(a \,\square\, b)[x, y, z] &= [a, b, [x, y, z]] \\ &= [[a, b, x], y, z] \\ &= [(a \,\square\, b)x, y, z]\end{aligned}$$

and also

$$\begin{aligned}(a \,\square\, b)[x, y, z] &= (a \,\square\, b)(x \,\square\, y)z \\ &= (x \,\square\, y)(a \,\square\, b)z \\ &= [x, y, (a \,\square\, b)z]\end{aligned}$$

for all $x, y, z \in V$. Therefore $a \,\square\, b \in \mathcal{Z}(V)$. $\square$

**Lemma 3.3.3.** *Every centralizer of a Hilbert ternary algebra with zero annihilator is a bounded linear operator.*

*Proof.* Let $C : V \longrightarrow V$ be a centralizer of Hilbert ternary algebra $V$. For $a, b, x, y \in V$ and $\lambda \in \mathbb{R}$, we have

$$[\lambda Cx - C(\lambda x), a, b] = \lambda[Cx, a, b] - [\lambda x, a, Cb] = \lambda[Cx, a, b] - \lambda[x, a, Cb] = 0$$

$$[(C(x + y) - Cx - Cy), a, b] = [x + y, a, Cb] - [x, a, Cb] - [y, a, Cb] = 0$$

which implies linearity of $C$.

Let $(x_n)$ be a null sequence in $V$ such that $C(x_n) \longrightarrow y \in V$. Then for arbitrary $a, b \in V$, we have

$$\begin{aligned} [y, a, b] &= \lim_{n\to\infty} [Cx_n, a, b] \\ &= \lim_{n\to\infty} [x_n, a, Cb] = 0 \end{aligned}$$

by Proposition 3.1.10. Since the annihilator of $V$ is zero, $y = 0$ and by the closed graph theorem, $C$ is bounded. □

**Proposition 3.3.4.** *Let $V$ be a Hilbert ternary algebra with zero annihilator. Then $\mathcal{Z}(V)$ is a von Neumann subalgebra of $\mathcal{B}(V)$.*

*Proof.* Let $C^* \in \mathcal{B}(V)$ be the adjoint of $C \in \mathcal{Z}(V)$. Let $x, y, z \in V$. We have

$$\begin{aligned} \langle a, C^*[x, y, z]\rangle &= \langle Ca, [x, y, z]\rangle \\ &= \langle [y, x, Ca], z\rangle \\ &= \langle C[y, x, a], z\rangle \\ &= \langle [y, x, a], C^*z\rangle \\ &= \langle a, [x, y, C^*z]\rangle \end{aligned}$$

for all $a \in V$. Hence $C^*[x, y, z] = [x, y, C^*z]$.

Likewise $C^*[x, y, z] = [C^*x, y, z]$. Therefore $C^* \in \mathcal{Z}(V)$.

Let $\{C_\alpha\}$ be a net in $\mathcal{Z}(V)$, converging to $C$ in the strong operator topology. Then for $x, y, z \in V$, we have

$$\begin{aligned} C[x, y, z] &= \lim_\alpha C_\alpha[x, y, z] \\ &= \lim_\alpha [C_\alpha x, y, z] \\ &= \lim_\alpha [x, y, C_\alpha z] \\ &= [Cx, y, z] = [x, y, Cz] \end{aligned}$$

by continuity of the ternary product (cf. Proposition 3.1.10). Hence $C \in \mathcal{Z}(V)$ and $\mathcal{Z}(V)$ is closed in the strong operator topology. $\square$

**Lemma 3.3.5.** *Let $V$ be a simple Hilbert ternary algebra and let $C : V \longrightarrow V$ be a self-adjoint centralizer. Then $C$ is a scalar multiple of the identity operator.*

*Proof.* By Proposition 3.3.4, $\mathcal{Z}(V)$ is a real von Neumann algebra and therefore its self-adjoint part is the closed linear span of its projections (cf. [10, Proposition 3.2]). Let $P \in \mathcal{Z}(V)$ be a projection. For any $x \in \text{Ker}P$ and $a, b \in V$, we have

$$P[a, b, x] = [a, b, Px] = 0$$

$$P[x, a, b] = [Px, a, b] = 0$$

which implies that $\text{Ker}P$ is a closed ternary ideal of $V$ by Lemma 3.2.5. Since $V$ is simple, $\text{Ker}P = \{0\}$ or $\text{Ker}P = V$. It follows that $P = \text{id}_V$ or $P = 0$ where $\text{id}_V$ is the identity operator of $V$. Hence every hermitian element is a scalar multiple of $\text{id}_V$. $\square$

**Proposition 3.3.6.** *Let $V$ be a simple Hilbert ternary algebra. Then $\mathcal{Z}(V) \simeq \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.*

*Proof.* Let $0 \neq C \in \mathcal{Z}(V)$. By Lemma 3.3.5, we have $CC^* = \lambda\, \text{id}_V$ for some $\lambda > 0$. Since $\|C^*C\| = \|CC^*\| = \lambda$, we have $C^*C = \lambda\, \text{id}_V$ by Proposition 3.3.4. Hence $C$ is invertible and $\mathcal{Z}(V)$ is a real associative division algebra. $\square$

**Corollary 3.3.7.** *Let $V$ be a Hilbert ternary algebra with zero annihilator. Then $\mathcal{Z}(V) = \bigoplus_\alpha \mathcal{C}_\alpha$ where $\mathcal{C}_\alpha \simeq \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.*

*Proof.* By the proof of Lemma 3.2.8, a closed ternary ideal $I$ of $V$ is the closed real linear span of $[I, I, I]$. It follows that $C(I) \subset I$ for any $C \in \mathcal{Z}(V)$ since $C[I, I, I] = [C(I), I, I] \subset I$. Hence the restriction of $C$ to $I$ is a centralizer of $I$.

Now let $V = \bigoplus_{\alpha \in \Lambda} I_\alpha$ where $I_\alpha$ is a minimal closed ternary ideal of $V$, which is also a simple Hilbert ternary algebra by Theorem 3.2.11. Let $\mathcal{C}_\alpha = \mathcal{Z}(I_\alpha)$. Then

$$\mathcal{Z}(V) = \bigoplus_{\alpha \in \Lambda} \mathcal{C}_\alpha$$

by the above remark. By Proposition 3.3.6, each $\mathcal{C}_\alpha$ is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. $\square$

**Lemma 3.3.8.** *Let $V$ be a Hilbert ternary algebra with zero annihilator and $C \in \mathcal{Z}(V)$. Then we have $C[x,y,z] = [x, C^*y, z]$ for $x, y, z \in V$ and hence*

$$C[x,[a,b,c],y] = [x,[a,Cb,c],y] \qquad (a,b,c,x,y \in V).$$

*Proof.* By Proposition 3.3.4, $C^* \in \mathcal{Z}(V)$ and for $x, y, z \in V$, we have

$$\begin{aligned}
\langle C[x,y,z], w\rangle &= \langle [x,y,z], C^*w\rangle \\
&= \langle z, [y,x,C^*w]\rangle \\
&= \langle z, [C^*y,x,w]\rangle \\
&= \langle [x,C^*y,z], w\rangle
\end{aligned}$$

for all $w \in V$.

Hence $C[x,[a,b,c],y] = [x, C^*[a,b,c], y] = [x,[a,Cb,c],y]$. $\square$

**Proposition 3.3.9.** *If $V$ is a simple Hilbert ternary algebra and $C \in \mathcal{Z}(V)$, then there exists $\mu > 0$ such that*

$$C[x,y,z] = \mu[Cx,Cy,Cz] \qquad (x,y,z \in V).$$

*Proof.* Let $C \in \mathcal{Z}(V) \setminus \{0\}$. By the proof of Proposition 3.3.6, $C^*C = CC^* = \mu\, \mathrm{id}_V$ for some $\mu > 0$. Therefore Lemma 3.3.8 implies

$$\begin{aligned}
[Cx,Cy,Cz] &= CC^*C[x,y,z] \\
&= \mu\, C[x,y,z]
\end{aligned}$$

for all $x, y, z \in V$. $\square$

We now discuss derivations of Hilbert ternary algebras.

**Definition 3.3.10.** A linear operator $D$ on a Hilbert ternary algebra $V$ is called a *derivation* if satisfies

$$D[x,y,z] = [Dx,y,z] + [x,Dy,z] + [x,y,Dz]$$

for all $x, y, z \in V$.

A simple example of a derivation is an inner derivation. Let $H$ and $K$ be Hilbert spaces, $a \in \mathcal{C}_2(K)$ and $b \in \mathcal{C}_2(H)$ be skew Hermitian. Then $D : \mathcal{C}_2(H,K) \longrightarrow \mathcal{C}_2(H,K)$ defined by

$$D(x) = ax - xb$$

is a derivation.

For any linear operator $T$ on a Hilbert ternary algebra, its *separating subspace* is defined to be the following closed subspace $V$:

$$S(T) = \left\{ y \in V : \, y = \lim_{n\to\infty} T(x_n) \quad \text{where} \quad \lim_{n\to\infty} x_n = 0 \right\}.$$

By the closed graph theorem, $T$ is continuous if and only if $S(T) = \{0\}$.

The following two propositions are extensions of results in [36] for Hermitian Hilbert ternary algebras.

**Proposition 3.3.11.** *Let $V$ be a Hilbert ternary algebra and $D$ a derivation on $V$. Then the separating subspace $S(D)$ is a closed ternary ideal of $V$.*

*Proof.* Let $x \in S(D)$. Then there exists a sequence $(x_n)$ in $V$ such that $x_n \to 0$ and $x = \lim_{n\to\infty} Dx_n$. For $y, z \in V$, we have

$$\begin{aligned}
\lim_{n\to\infty} D[x_n,y,z] &= \lim_{n\to\infty}[Dx_n,y,z] + \lim_{n\to\infty}[x_n,Dy,z] + \lim_{n\to\infty}[x_n,y,Dz] \\
&= [x,y,z]
\end{aligned}$$

where $[x_n, y, z] \to 0$. Hence $[x, y, z] \in S(D)$ and $S(D)$ is a right ideal of $V$. Similarly $S(D)$ is also a left ideal. Since $S(D)$ is closed subspace, it follows that it is a ternary ideal of $V$ by Lemma 3.2.5. $\square$

**Proposition 3.3.12.** *Let $V$ be a Hilbert ternary algebra with zero annihilator. For any derivation $D$ and closed ternary ideal $I$ of $V$, we have $D(I) \subset I$.*

*Proof.* We note that $I \neq \{0\}$ implies $[I, I, I] \neq \{0\}$. Suppose $[I, I, I] = \{0\}$. Then $[I, V, I] = \{0\}$ since $[I, I^{\perp}, I] = \{0\}$. We have

$$\begin{aligned} [I, V, V] &= [I, V, I] + [I, V, I^{\perp}] \\ &= [I, V, I^{\perp}] \\ &\subset I \cap I^{\perp} = \{0\} \end{aligned}$$

which implies $I \subset V_0$ and hence $I = \{0\}$.

Suppose $I \neq V$ and for an arbitrary $a \in I$, write $D(a) = x + y$ where $x \in I$ and $y \in I^{\perp}$. If $y \neq 0$, we have $[I^{\perp}, I^{\perp}, I^{\perp}] \neq \{0\}$ by the above remark. Hence there exist $y_1, y_2 \in I^{\perp}$ such that $[y, y_1, y_2] \neq 0$ since $V$ has zero annihilator. Since $[x, y_1, y_2] \in I \cap I^{\perp}$, we have

$$\begin{aligned} [Da, y_1, y_2] &= [x, y_1, y_2] + [y, y_1, y_2] \\ &= [y, y_1, y_2] \neq 0. \end{aligned}$$

On the other hand, using, for example, $[a, y_1, y_2] \in I \cap I^{\perp}$, we have

$$[Da, y_1, y_2] = D[a, y_1, y_2] - [a, Dy_1, y_2] - [a, y_1, Dy_2] = 0$$

gives a contradiction. Hence $y = 0$ and $D(a) \in I$. $\square$

**Lemma 3.3.13.** *Let $D : V \longrightarrow V$ be a derivation on a Hilbert ternary algebra $V$ and let $V_{\mathbb{C}}$ be the complexification of $V$. Then $D_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ defined by $D_{\mathbb{C}}(a + ib) = D(a) + iD(b)$ is a derivation of $V_{\mathbb{C}}$.*

*Proof.* Straightforward computation. □

**Theorem 3.3.14.** *Every derivation of a Hilbert ternary algebra with zero annihilator is continuous.*

*Proof.* Let $D$ be a derivation of a Hilbert ternary algebra $V$ with $V_0 = \{0\}$. The complexification $V_{\mathbb{C}} = V \bigoplus iV$ of $V$ is a Hermitian Hilbert ternary algebra with zero annihilator. Let $D_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ be the complexification of $D$ defined above. Then $D_{\mathbb{C}}$ is a derivation on $V_{\mathbb{C}}$. By [36], $D_{\mathbb{C}}$ is continuous on $V_{\mathbb{C}}$. Hence $D$ is continuous. □

## 3.4 Structure theorems

Our goal in this Section is to show that every simple abelian Hilbert ternary algebra is isomorphic to the ternary algebra $\mathcal{C}_2^k(H,K)$ or its dual $\mathcal{C}_2^k(H,K)_-$ for some positive number $k$ and some Hilbert spaces $H$ and $K$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Let $\{e_\beta\}_{\beta\in\Gamma}$ and $\{f_\alpha\}_{\alpha\in\Lambda}$ be orthonormal bases of Hilbert spaces $H$ and $K$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, respectively. Let $\mathcal{C}_2^k(H,K) = \mathcal{C}_2^k(H,K)^{\mathbb{R}} \bigotimes_{\mathbb{R}} \mathbb{F}$ where $\mathcal{C}_2^k(H,K)^{\mathbb{R}}$ is the real linear span of $\{f_\alpha \otimes e_\beta : \alpha \in \Lambda, \beta \in \Gamma\}$ (see Section 2.1). An easy computation shows that:

(i) $[f_\alpha \otimes e_\beta, f_\alpha \otimes e_\beta, f_\alpha \otimes e_\beta] = f_\alpha \otimes e_\beta$;

(ii) $[f_\alpha \otimes e_\beta, \mathcal{C}_2^k(H,K), f_\gamma \otimes e_\eta] = (f_\alpha \otimes e_\eta)\,\mathbb{F}$;

(iii) $[f_\alpha \otimes e_\beta, f_\gamma \otimes e_\beta, f_\gamma \otimes e_\eta] = f_\alpha \otimes e_\eta$;

(iv) $[f_\alpha \otimes e_\beta, f_\gamma \otimes e_\eta, f_\delta \otimes e_\lambda] = 0$, if $\beta \neq \eta$ or $\gamma \neq \delta$.

We recall that $\{f_\alpha \otimes e_\beta : \alpha \in \Lambda, \beta \in \Gamma\}$ is an orthogonal basis of $\mathcal{C}_2^k(H,K)$.

**Lemma 3.4.1.** *Let $H$ and $K$ be Hilbert spaces. Then $\mathcal{C}_2^k(H,K)$ is simple with respect to the ternary product $[\cdot,\cdot,\cdot]$.*

*Proof.* It suffices to consider the case of $k = 1$. Let $I$ be a nonzero closed ternary ideal of $\mathcal{C}_2(H, K)$. Pick $0 \neq x \in I$. Then $x^*x \neq 0$ since

$$\langle x^*x(h), h\rangle = \langle x(h), x(h)\rangle = \|x(h)\|^2$$

for all $h \in H$. We have $v \otimes x^*x(u) = [v \otimes u, x, x]$ for all $u \in H$ and $v \in K$, since

$$\begin{aligned}(v \otimes x^*x(u))(h) &= v\langle h, x^*x(u)\rangle \\ &= v\langle x^*x(h), u\rangle \\ &= (v \otimes u)x^*x(h) \\ &= [v \otimes u, x, x](h)\end{aligned}$$

for all $h \in H$. Hence $v \otimes x^*x(u) \in I$. Also, for any $w \in H$, we have

$$\begin{aligned}[v \otimes x^*x(u), v \otimes x^*x(u), v \otimes w] &= (v \otimes x^*x(u))(x^*x(u) \otimes v)(v \otimes w) \\ &= (v \otimes v)(v \otimes w) \\ &= v \otimes w\end{aligned}$$

which implies $v \otimes w \in I$ for any vector $v \in K$ and $w \in H$. It follows that $I = \mathcal{C}_2(H, K)$. □

**Definition 3.4.2.** Let $V$ be a Hilbert ternary algebra. An element $e \in V$ is called a *tripotent* if

$$[e, e, e] = e$$

and a *negative tripotent* if $[e, e, e] = -e$.

Evidently $f_\alpha \otimes e_\beta$ is a tripotent in $\mathcal{C}_2^k(H, K)$ for $\alpha \in \Lambda$ and $\beta \in \Gamma$.

The following extends a result in [37] for Hermitian Hilbert ternary algebras.

**Proposition 3.4.3.** *Let $V$ be an abelian Hilbert ternary algebra with zero annihilator. For any closed ternary left ideal $L$ of $V$ and closed ternary right ideal $R$ of $V$ satisfying $L \cap R \neq \{0\}$, the set $L \cap R$ contains a tripotent or a negative tripotent.*

*Proof.* First we prove that $[a,a,a] \neq 0$ for any $0 \neq a \in V$. Suppose $[a,a,a]=0$ for nonzero $a \in V$, then we have

$$\begin{aligned} \|[a,a,b]\|^2 &= \langle [a,a,b],[a,a,b]\rangle \\ &= \langle b,[a,a,[a,a,b]]\rangle \\ &= \langle b,[[a,a,a],a,b]\rangle = 0 \end{aligned}$$

for all $b \in V$, which implies $a \square a = 0$. This gives

$$\begin{aligned} \|[c,b,a]\|^2 &= \langle [c,b,a],[c,b,a]\rangle \\ &= \langle c,[[c,b,a],a,b]\rangle \\ &= \langle c,[c,b,[a,a,b]]\rangle = 0 \end{aligned}$$

for all $b,c \in V$. Hence $a \in V_0 = \{0\}$ which is a contradiction. Hence $[a,a,a] \neq 0$ and $a \square a \neq 0$ for any $a \neq 0$.

Let $x \in L \cap R$ and $\|x \square x\| = 1$. The operator $S = x \square x$ is self-adjoint, therefore $T = S^2$ is a positive operator in $\mathcal{B}(V)$ and $\|T\| = \|S^2\| = \|S\|^2 = 1$. By Lemma 2.1.4, we can define

$$u = \lim_{n\to\infty} T^n(x), \quad v = \lim_{n\to\infty} T^n[x,x,x].$$

Both $u$ and $v$ belong to $L \cap R$ since $L$ and $R$ are closed. We show that $u \neq 0$ or $v \neq 0$. Otherwise $u = v = 0$ and hence $\lim_{n\to\infty} S^n(x) = 0$ because $S(x) = [x,x,x]$ and $S^{2n+1}(x) = T^n[x,x,x]$. Since the Hilbert ternary algebra $V$ is abelian, it follows that $S[x,y,z] = [Sx,y,z]$ for $y,z \in V$. By continuity of the ternary product, there exists some $M > 0$ such that

$$\|[x,y,z]\| \leq M\|x\|\|y\|\|z\| \qquad (y,z \in V)$$

Since $\|S\| = 1$, we have $\|S^n\| = \|S\|^n = 1$. It follows that

$$\begin{aligned} 1 &= \|S^{n+1}\| \\ &= \sup\left\{\|S^{n+1}y\| : \|y\| \leq 1\right\} \\ &= \sup\{\|[S^n x, x, y]\| : \|y\| \leq 1\} \\ &\leq \sup\{M\,\|S^n x\|\,\|x\|\} \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$, which is a contradiction.

Now let

$$w_1 = \frac{u+v}{2} \qquad \text{and} \qquad w_2 = \frac{u-v}{2}.$$

Then $w_1 \neq 0$ or $w_2 \neq 0$. In the first case, $w_1$ is a tripotent and in the second case, $w_2$ is a negative tripotent. In fact, since $V$ is abelian, we have $[Tx, Tx, Tx] = T^3[x, x, x]$ and therefore

$$[T^n x, T^n x, T^n x] = T^{3n}[x, x, x].$$

From this it follows that $[u, u, u] = [u, v, v] = [v, u, v] = [v, v, u] = v$. Likewise $[v, v, v] = [v, u, u] = [u, v, u] = [u, u, v] = u$. Therefore a straightforward computation shows that $w_1$ is a tripotent and $w_2$ is a negative tripotent. $\square$

**Corollary 3.4.4.** *Every abelian Hilbert ternary algebra with zero annihilator contains a tripotent or a negative tripotent.*

**Proposition 3.4.5.** *Let $V$ be a simple abelian Hilbert ternary algebra.*

(i) *If $x, y \in V$ are nonzero elements, then $[x, V, y] \neq \{0\}$.*

(ii) *$V$ contains only tripotents or only negative tripotents.*

(iii) *The Hilbert ternary algebra $V$ and its dual $V_-$ are not isomorphic.*

*Proof.* (i). First we define the *left annihilator* of $y \in V \setminus \{0\}$ by

$$L_0(y) = \{x \in V : [x, V, y] = \{0\}\}.$$

Then $L_0(y)$ is a closed linear subspace of $V$ by continuity of the ternary product. We show that $L_0(y)$ is both a left ideal and a right ideal of $V$. Indeed, for any $x \in L_0(y)$, we have

$$[[V, V, x], V, y] = [V, V, [x, V, y]] = \{0\}$$

$$[[x, V, V], V, y] = [x, [V, V, V], y] = \{0\}.$$

Hence by Lemma 3.2.5, $L_0(y)$ is a closed ternary ideal of $V$. Since $V$ is simple, $L_0(y) = V$ or $L_0(y) = \{0\}$. In the case $L_0(y) = V$, we have $[y, y, y] = 0$. But this is impossible as was shown in the proof of Proposition 3.4.3. Therefore $L_0(y) = \{0\}$.

(ii). Suppose $V$ contains both a tripotent $e$ and a negative tripotent $f$. For every $x \in V$, we have

$$\begin{aligned}
\|[e, [f, x, e], f]\|^2 &= \langle [e, [f, x, e], f], [e, [f, x, e], f] \rangle \\
&= \langle [[f, x, e], e, [e, [f, x, e], f]], f \rangle \\
&= \langle [[[f, x, e], e, e], [f, x, e], f], f \rangle \\
&= \langle [[f, x, [e, e, e]], [f, x, e], f], f \rangle \\
&= \langle [[f, x, e], [f, x, e], f], f \rangle \\
&= \langle [f, x, e], [f, f, [f, x, e]] \rangle \\
&= -\langle [f, x, e], [f, x, e] \rangle \\
&= -\|[f, x, e]\|^2
\end{aligned}$$

which yields to $[f, V, e] = \{0\}$. This contradicts (i).

(iii). This follows directly from (ii). $\square$

In the classification of abelian Hilbert ternary algebras, according to the Proposition 3.4.5, we may restrict our attention to those simple Hilbert ternary algebras which contain only tripotents.

In the sequel, we assume all abelian Hilbert ternary algebras contain *only* tripotents.

Let $V$ be an abelian Hilbert ternary algebra and let $e \in V$ be a tripotent. For $x, y \in V$, we define the product

$$x \circ_e y = [x, e, y].$$

Then $(V, \circ_e)$ is an associative algebra. Indeed, we have

$$(x \circ_e y) \circ_e z = [[x, e, y], e, z] = [x, e, [y, e, z]] = x \circ_e (y \circ_e z)$$

for $x, y, z \in V$.

Let $V_e = [e, V, e]$ be the principal inner ideal determined by $e$, which is a ternary subalgebra of $V$.

**Lemma 3.4.6.** *Let $V$ be an abelian Hilbert ternary algebra and let $e \in V$ be a tripotent. Define an involution $^{*_e}$ on $V_e$ by*

$$[e, x, e]^{*_e} = [e, [e, x, e], e] \qquad (x \in V).$$

*Then $(V_e, \circ_e, ^{*_e})$ is an $H^*$-algebra with identity $e$, and zero left annihilator.*

*Proof.* We note that $V_e$ is a subalgebra of $(V, \circ_e)$ since

$$\begin{aligned} [e, x, e] \circ_e [e, y, e] &= [[e, x, e], e, [e, y, e]] \\ &= [[[e, x, e], e, e], y, e] \\ &= [[e, x, [e, e, e]], y, e] \\ &= [e, [y, e, x], e] \end{aligned}$$

for all $x, y \in V$. Plainly $e \in V_e$ is an identity of $V_e$.

Since $V$ is abelian, we have

$$V_e = \{x \in V : x = [e, e, x] = [x, e, e]\}.$$

Indeed, if $x$ belongs to the set on the right hand side above, then we have $x = [e, e, [x, e, e]] = [e, [e, x, e], e] \in V_e$. Also $V_e$ is closed in $V$ by continuity of the ternary product and hence a Hilbert space.

Next we show that ${}^{*_e}$ is an algebra involution on $V$. We have

$$\begin{aligned} [e, x, e]^{*_e *_e} &= [e, [e, x, e], e]^{*_e} \\ &= [e, [e, [e, x, e], e], e] \\ &= [[e, e, e], x, [e, e, e]] \\ &= [e, x, e] \end{aligned}$$

and also

$$\begin{aligned} [e, x, e]^{*_e} \circ_e [e, y, e]^{*_e} &= [e, [e, x, e], e] \circ_e [e, [e, y, e], e] \\ &= [[e, [e, x, e], e], e, [e, [e, y, e], e]] \\ &= [e, [e, x, e], [e, e, [e, [e, y, e], e]]] \\ &= [e, [e, x, e], [[e, e, e], [e, y, e], e]] \\ &= [e, [e, x, e], [e, [e, y, e], e]] \\ &= [e, [[e, y, e], e, [e, x, e]], e] \\ &= [e, [e, y, e] \circ_e [e, x, e], e] \\ &= ([e, y, e] \circ_e [e, x, e])^{*_e} \end{aligned}$$

for all $x, y \in V$.

Finally we prove that $(V_e, \circ_e, {}^{*_e}, \langle \cdot, \cdot \rangle)$ is an $H^*$-algebra with zero left annihilator. For $x, y, z \in V$, we have

$$\begin{aligned}
\langle [e,y,e],[e,x,e]^{*_e} \circ_e [e,z,e]\rangle &= \langle [e,y,e],[e,[e,x,e],e] \circ_e [e,z,e]\rangle \\
&= \langle [e,y,e],[[e,[e,x,e],e],e,[e,z,e]]\rangle \\
&= \langle [e,y,e],[e,[e,e,[e,x,e]],[e,z,e]]\rangle \\
&= \langle [e,y,e],[e,[e,x,e],[e,z,e]]\rangle \\
&= \langle [[e,x,e],e,[e,y,e]],[e,z,e]\rangle \\
&= \langle [e,x,e] \circ_e [e,y,e],[e,z,e]\rangle
\end{aligned}$$

and likewise

$$\langle [e,x,e],[e,z,e] \circ_e [e,y,e]^{*_e}\rangle = \langle [e,x,e] \circ_e [e,y,e],[e,z,e]\rangle.$$

Let $x \in V_e$ be such that $x \circ_e V_e = \{0\}$. Then

$$\langle x^{*_e}, y\rangle = \langle e, x \circ_e y\rangle = 0 \qquad (y \in V_e)$$

implies $x^{*_e} = 0$ and hence $x = 0$. $\square$

By Ingelstam's theorem [17], $(V_e, \circ_e)$ is a division algebra if

$$\|e\| = 1 \qquad \text{and} \qquad \|x \circ_e y\| \leqslant \|x\|\|y\|$$

for all $x, y \in V_e$. These restrictive norm conditions may not be satisfied in $V_e$. Nevertheless, we give below algebraic criteria for $(V_e, \circ_e)$ to be a division algebra.

**Definition 3.4.7.** Two tripotents $e$ and $f$ in a Hilbert ternary algebra $V$ are said to be *ternary orthogonal* to each other if

$$[e,e,f] = 0 = [f,f,e] \qquad \text{and} \qquad [e,f,f] = 0 = [f,e,e].$$

The basis elements $\{f_\alpha \otimes e_\beta : \alpha \in \Lambda, \beta \in \Gamma\}$ of $\mathcal{C}_2^k(H,K)$ are mutually ternary orthogonal. We note that two ternary orthogonal tripotents in a Hilbert ternary algebra are orthogonal with respect to the inner product.

An important concept in Hilbert ternary algebras is that of a primitive tripotent. These tripotents provide the basic building blocks of an abelian Hilbert ternary algebra.

**Definition 3.4.8.** A nonzero tripotent $e$ in a Hilbert ternary algebra $V$ is called *primitive* if it does not admit a decomposition

$$e = e_1 + e_2$$

into the sum of two nonzero ternary orthogonal tripotents $e_1$ and $e_2$.

The following lemma is crucial in the classification of abelian Hilbert ternary algebras.

**Lemma 3.4.9.** *Let $e$ be a tripotent in an abelian Hilbert ternary algebra $V$. The following conditions are equivalent.*

(i) *$e$ is primitive.*

(ii) *$(V_e, \circ_e)$ is a real division algebra.*

(iii) $\{x \in V : x = [e, x, e]\} = \mathbb{R}e$.

*Proof.* (i) $\Rightarrow$ (ii). Let $e$ be primitive. We show that the $H^*$-algebra $V_e$ is a division algebra. For every $0 \neq a \in V_e$ we have $a = a \circ_e e \in a \circ_e V$ and $a \circ_e V$ is a nonzero ternary right ideal of $V$. Also, $a \circ_e V \subset e \circ_e V$ since

$$a \circ_e x = (e \circ_e a) \circ_e x = e \circ_e (a \circ_e x)$$

for any $x \in V$. We show that $e \circ_e V$ is a minimal right ideal of $V$. Suppose, otherwise, $e \circ_e V = I \bigoplus \left(I^\perp \cap (e \circ_e V)\right)$ for some proper right ideal $I$ of $V$. Then we have $e = e_1 + e_2$ with $e_1 \neq 0 \neq e_2$, $e_1 \in I$ and $e_2 \in I^\perp \cap (e \circ_e V)$. Since

$$\langle [[e, e_1, e], e, e_2], x \rangle = \langle e_2, [e, [e, e_1, e], x] \rangle = \langle e_2, [[e, e, e_1], e, x] \rangle = \langle e_2, [e_1, e, x] \rangle = 0$$

for all $x \in V$, we have $[[e, e_1, e], e, e_2] = 0$ and

$$\begin{aligned} e_2 &= e \circ_e e_2 \\ &= [[e, e, e], e, e_2] \\ &= [[e, e_1, e], e, e_2] + [[e, e_2, e], e, e_2] \\ &= [e, [e, e, e_2], e_2] \\ &= [e, e_2, e_2]. \end{aligned}$$

Therefore $[e, e_2, e] = [e, e_2, e_1] + [e, e_2, e_2] = e_2$ as $\langle [e, e_2, e_1], x\rangle = \langle e_1, [e_2, e, x]\rangle = 0$ for all $x \in V$. Similarly $[e, e_1, e] = e_1$. It follows that $e_2 \circ_e e_1 = 0$ since for any $x \in V$, we have

$$\begin{aligned} \langle e_2 \circ_e e_1, x\rangle &= \langle [e_2, e, e_1], x\rangle \\ &= \langle e_1, [e, [e, e_2, e], x]\rangle \\ &= \langle e_1, [[e, e, e_2], e, x]\rangle \\ &= \langle e_1, [e_2, e, x]\rangle = 0. \end{aligned}$$

Therefore

$$\begin{aligned} e_1 &= e \circ_e e_1 \\ &= e_1 \circ_e e_1 + e_2 \circ_e e_1 \\ &= [e_1, e, e_1] \\ &= [e_1, e_1, e_1] + [e_1, e_2, e_1] \\ &= [e_1, e_1, e_1] \end{aligned}$$

where $[e_1, e_2, e_1] = [e_1, [e, e_2, e], e_1] = [e_1, e, [e_2, e, e_1]] = 0$. Likewise $e_2 = [e_2, e_2, e_2]$ is a tripotent. Moreover, $e_1$ and $e_2$ are ternary orthogonal. Indeed,

$$[e_2, e_2, e_1] = [e_2, [e, e_2, e], e_1] = [e_2, e, [e_2, e, e_1]] = 0$$

and then $[e_2, e_1, e_1] = [e_2, e, e_1] - [e_2, e_2, e_1] = 0$. Similarly one can show that $[e_1, e_1, e_2] = [e_1, e_2, e_2] = 0$. This contradicts primitivity of $e$ and therefore $e \circ_e V$ is a minimal right ideal in $V$.

It follows that $a \circ_e V = e \circ_e V$ and there exists an element $b \in V$ such that $a \circ_e b = e$. Let $c = [[e, e, b], e, e] \in V_e$. Then

$$a \circ_e c = [a, e, [[e, e, b], e, e]] = [[a, e, b], e, e] = e$$

and $a$ is right invertible in $V_e$. Similarly $a$ is left invertible in $V_e$. This proves that $V_e$ is a division algebra.

(ii) $\Rightarrow$ (iii). Since the unital $H^*$-algebra $(V_e, \circ_e)$ is a real division algebra, $V_e$ is unital $^*$-isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (cf. [21, Lemma 3]), we can choose a basis $\{v_1, \cdots, v_n\}$ in $V_e$ with $v_1 = e$, $n = 1, 2$ or $4$ and $[e, v_n, e] = v_n^{*_e} = -v_n$ when $n \geq 2$. Let $x = [e, x, e]$ with $x = \sum_{j=1}^n \lambda_j v_j \, (\lambda_j \in \mathbb{R})$. Then

$$\begin{aligned} x &= \lambda_1 e + \sum_{j=2}^n \lambda_j v_j \\ &= \lambda_1 e + \sum_{j=2}^n \lambda_j [e, v_j, e] \\ &= \lambda_1 e - \sum_{j=2}^n \lambda_j v_j \end{aligned}$$

which implies $\lambda_2 = \cdots = \lambda_n = 0$ and $x = \lambda_1 e \in \mathbb{R}e$.

(iii) $\Rightarrow$ (i). Suppose $e = e_1 + e_2$ for two ternary orthogonal tripotents $e_1$ and $e_2$. We have

$$[e_2, e_1, e_2] = [e_2, [e_1, e_1, e_1], e_2] = [e_2, e_1, [e_1, e_1, e_2]] = 0$$

and

$$[e_1, e_1, e] = [e_1, e_1, e_1] + [e_1, e_1, e_2] = e_1$$

by ternary orthogonality of $e_1$ and $e_2$. Hence

$$\begin{aligned}[e,e_1,e]-e_1 &= [e,e_1,e]-[e_1,e_1,e] \\ &= [e-e_1,e_1,e] \\ &= [e_2,e_1,e] \\ &= [e_2,e_1,e_1]+[e_2,e_1,e_2]=0\end{aligned}$$

which implies $e_1=\lambda e$ for some $\lambda\in\mathbb{R}$ by condition (iii). Likewise $e_2=\lambda' e$ for some $\lambda'\in\mathbb{R}$. Therefore $0=[e_2,e_1,e]=[\lambda' e,\lambda e,e]=\lambda\lambda' e$. Hence $\lambda\lambda'=0$ which gives $e_1=0$ or $e_2=0$. This proves that $e$ is primitive. $\square$

**Lemma 3.4.10.** *The primitive tripotents in $\mathcal{C}_2^k(H,K)$ with respect to the ternary product $[\cdot,\cdot,\cdot]$ are exactly the rank-one operators $v\otimes u$ for unit vectors $u\in H$ and $v\in K$.*

*Proof.* Let $u\in H$ and $v\in K$ be unit vectors. Then

$$[v\otimes u, v\otimes u, v\otimes u]=(v\otimes u)\|u\|^2\|v\|^2=v\otimes u$$

implies that $v\otimes u$ is a tripotent. Now we show that $v\otimes u$ is primitive. Let $v\otimes u=e+f$ say, $e,f$ are nonzero ternary orthogonal tripotents. We have

$$(v\otimes u)(x)=v\langle x,u\rangle_H=e(x)+f(x) \qquad (x\in H)$$

where $\langle e(x),f(x)\rangle_K=\langle e(x),ff^*f(x)\rangle_K=\langle ff^*e(x),f(x)\rangle_K=0$ by ternary orthogonality of $e$ and $f$. Let $[u]=u\mathbb{F}$ be the one-dimensional subspace in $H$ generated by $u$ and consider

$$H=[u]\bigoplus[u]^\perp$$

Let $x\in[u]^\perp$. Then $\langle x,u\rangle_H=0$ and $e(x)+f(x)=0$. Hence $\|e(x)\|^2+\|f(x)\|^2=\|e(x)+f(x)\|^2=0$ and $e(x)=f(x)=0$. Therefore we have $e|_{[u]^\perp}=f|_{[u]^\perp}=0$.

We must have

$$e=e(u)\otimes u.$$

Indeed, any $h \in H$ can be written as $h = u\alpha \oplus x \in [u] \bigoplus [u]^{\perp}$ for some $\alpha \in \mathbb{F}$. Hence $e(h) = e(u)\alpha$ and we have

$$(e(u) \otimes u)(h) = e(u)\langle h, u\rangle_H = e(u)\langle u\alpha, u\rangle_H = e(u)\alpha = e(h).$$

Since $ee^*e = e$ implies

$$e(u) \otimes u = (e(u) \otimes u)(u \otimes e(u))(e(u) \otimes u) = (e(u) \otimes u)\|e(u)\|^2\|u\|^2,$$

we have $\|e(u)\| = 1$. Likewise $\|f(u)\| = 1$. But we have

$$v = (v \otimes u)(u) = e(u) + f(u)$$

and $e(u) \perp f(u)$ in $K$. Therefore

$$1 = \|v\|^2 = \|e(u)\|^2 + \|f(u)\|^2 = 2$$

which is a contradiction. Hence $v \otimes u$ can not be split into two nonzero ternary orthogonal tripotents.

Conversely, let $e \in \mathcal{C}_2^k(H, K)$ be a primitive tripotent. We show $e = v \otimes u$ for some unit vectors $u \in H$ and $v \in K$. Since $ee^*e = e$, $e$ is a partial isometry and the range of $e(H)$ is closed in $K$. Let

$$K = e(H) \bigoplus e(H)^{\perp}.$$

Likewise, let

$$H = e^*(K) \bigoplus e^*(K)^{\perp}.$$

Note that $e(e^*(K)^{\perp}) = \{0\}$. Indeed, $\langle e(e^*(K)^{\perp}), y\rangle_K = \langle e^*(K)^{\perp}, e^*(y)\rangle_H = \{0\}$ for all $y \in K$. Hence $e(H) = e(e^*(K))$ and we can pick $u \in e^*(K)$, $\|u\| = 1$ and $e(u) \neq 0$. Note that $e^*e(u) = u$ since $u = e^*(k_0)$ for some $k_0 \in K$ and

$$e^*e(u) = e^*ee^*(k_0) = e^*(k_0) = u.$$

Let $x \in H$ with $x = e^*(k) \oplus h \in e^*(K) \bigoplus e^*(K)^\perp$. Then $e(x) = ee^*(k)$. Hence, writing $v = e(u)$, then $\|v\| = \|e(u)\| = \|u\| = 1$ since $e$ is an isometry on $e^*(K)$. Moreover,

$$\begin{aligned}
[e, v \otimes u, e](x) &= e(u \otimes v)e(x) \\
&= e(u)\langle e(x), v\rangle_K \\
&= v\langle ee^*(k), e(u)\rangle_K \\
&= v\langle e^*(k), e^*e(u)\rangle_H \\
&= v\langle e^*(k), u\rangle_H + v\langle h, u\rangle_H \\
&= v\langle x, u\rangle_H \\
&= (v \otimes u)(x)
\end{aligned}$$

Therefore by Lemma 3.4.9, we have

$$v \otimes u = \lambda e \qquad (\lambda \in \mathbb{R}).$$

It follows that $|\lambda| = 1$ and $\lambda = \pm 1$. Therefore we get

$$e = v \otimes u \qquad \text{or} \qquad e = (-v) \otimes u.$$

$\square$

We extend below the result in [37] for Hermitian Hilbert ternary algebras to Hilbert ternary algebras using the new concept of primitive tripotents.

**Proposition 3.4.11.** *Let $V$ be a simple abelian Hilbert ternary algebra. Let $L$ be a ternary left ideal of $V$ and $R$ a ternary right ideal of $V$. Then the following assertions hold.*

(i) *If $L \cap R = \{0\}$, then $L = \{0\}$ or $R = \{0\}$.*

(ii) *If $L$ and $R$ are closed and nonzero, then $L \cap R$ contains a primitive tripotent.*

*Proof.* (i). Suppose that $L \cap R = \{0\}$. Then $[R, V, L] \subset R \cap L = \{0\}$ and by Proposition 3.4.5 (i), we have $L = \{0\}$ or $R = \{0\}$.

(ii). From (i), we have $L \cap R \neq \{0\}$ which implies that $L \cap R$ contains a tripotent $e$ by Proposition 3.4.3. Form the $H^*$-algebra $(V_e = [e, V, e], \circ_e)$ as defined in Lemma 3.4.6. Then $e$ is an idempotent in $V_e$. By Lemma 2.3.2, $e$ can be expressed as a finite sum of idempotents $e = p_1 + \cdots + p_n$ in which $p_i \circ_e p_j = 0$ for $i \neq j$ and every $p_i$ cannot be decomposed any more. This gives $p_1 = [p_1, e, e] = [e, e, p_1] \in L \cap R$, where

$$[p_1, e, e] = [p_1, e, p_1 + \cdots + p_n] = p_1 \circ_e p_1 + \cdots + p_1 \circ_e p_n = p_1.$$

Since $e$ is self-adjoint in $V_e$, Lemma 2.3.2 implies that $p_i$ is self-adjoint for every $i = 1, 2, \cdots, n$ and we get

$$\begin{aligned} [p_1, p_1, p_1] &= [p_1, p_1^{*_e}, p_1] \\ &= [p_1, [e, p_1, e], p_1] \\ &= [[p_1, e, p_1], e, p_1] \\ &= [p_1 \circ_e p_1, e, p_1] \\ &= [p_1, e, p_1] \\ &= p_1. \end{aligned}$$

Hence $p_1$ is a tripotent in $L \cap R$.

We want to show that $p_1$ is primitive. By Lemma 2.3.1, $p_1 \circ_e V_e$ is a minimal right ideal of the $H^*$-algebra $(V_e, \circ_e)$, and as in the proof of Lemma 3.4.9, one can show that $p_1 \circ_e V_e \circ_e p_1$ is a real division algebra with identity $p_1$. Indeed, let $0 \neq a \in p_1 \circ_e V_e \circ_e p_1$. Let $a = p_1 \circ_e y \circ_e p_1$ for some $y \in V_e$. For any $x \in V_e$, we have

$$a \circ_e x = (p_1 \circ_e y \circ_e p_1) \circ_e x = p_1 \circ_e (y \circ_e p_1 \circ_e x)$$

which implies $a \circ_e V_e \subset p_1 \circ_e V_e$. Therefore $a \circ_e V_e = p_1 \circ_e V_e$ by minimality of $p_1 \circ_e V_e$. Thus there exists $b \in V_e$ such that $a \circ_e b = p_1$. Let $c = p_1 \circ_e b \circ_e p_1$. Then

$$a \circ_e c = a \circ_e (p_1 \circ_e b \circ_e p_1) = (a \circ_e b) \circ_e p_1 = p_1 \circ_e p_1 = p_1.$$

This proves that $a$ is right invertible. Likewise one can prove that $a$ is left invertible in $p_1 \circ_e V_e \circ_e p_1$.

We note that

$$V_{p_1} = [p_1, V, p_1] = [[p_1, e, e], V, [e, e, p_1]] = [[p_1, e, V_e], e, p_1] = p_1 \circ_e V_e \circ_e p_1 \subset V_e$$

and for $a, b \in V_{p_1}$, we have $a \circ_{p_1} b = a \circ_e b$. Indeed,

$$\begin{aligned} a \circ_{p_1} b &= [a, p_1, b] \\ &= [a, [p_1, [p_1, e, p_1], p_1], b] \\ &= [[[a, p_1, p_1], e, p_1], p_1, b] \\ &= [a, e, [p_1, p_1, b]] \\ &= [a, e, b] \\ &= a \circ_e b. \end{aligned}$$

Hence $(V_{p_1}, \circ_{p_1})$ is a division algebra and $p_1$ is a primitive tripotent by Lemma 3.4.9. □

Now we introduce two concepts which will be needed later. Two primitive tripotents $e$ and $f$ are said to be *horizontally connected* if

$$\langle V_e, f \rangle = \{0\}, \quad [e, V, f] \subset V_e \quad \text{and} \quad [f, V, e] \subset V_f$$

where $V_e$ and $V_f$ are principal inner ideals of $V$ determined by $e$ and $f$, respectively. The primitive tripotents $e$ and $f$ are called *vertically connected* if

$$\langle V_e, f \rangle = \{0\}, \quad [e, V, f] \subset V_f \quad \text{and} \quad [f, V, e] \subset V_e.$$

According to this definition, $e$ is not vertically connected or horizontally connected to itself.

We note that, if $e$ and $f$ are horizontally connected, then for any $y \in V_f$,

$$\langle e, y \rangle = \langle e, [f, f, y] \rangle = \langle [e, y, f], f \rangle = 0$$

since $\langle V_e, f \rangle = \{0\}$. Hence $\langle e, V_f \rangle = \{0\}$. The same is true when $e$ and $f$ are vertically connected.

Two following propositions are extensions of similar results in [37] for Hermitian Hilbert ternary algebras, but with a different definition for horizontal and vertical connectedness. We also use the new concept of primitive tripotents.

**Proposition 3.4.12.** *Let $V$ be a simple abelian Hilbert ternary algebra and let $e, f \in V$ be horizontally connected primitive tripotents. Then we have*

(i) $[e, e, f] = [f, f, e] = 0$.

(ii) $[e, f, e] = [f, e, f] = 0$.

(iii) $[e, f, f] = e, [f, e, e] = f$.

(iv) $\langle e, [f, V, V] \rangle = \langle f, [e, V, V] \rangle = \{0\}$.

(v) $\|e\| = \|f\|$.

*Proof.* (i). Using the horizontal connectedness, we have $[e, e, f] \in V_e$ and

$$\begin{aligned} \|[e, e, f]\|^2 &= \langle [e, e, f], [e, e, f] \rangle \\ &= \langle [e, e, [e, e, f]], f \rangle \\ &= \langle [[e, e, e], e, f], f \rangle \\ &= \langle [e, e, f], f \rangle = 0 \end{aligned}$$

since $[e,e,f] \in V_e$. Similarly $[f,f,e] = 0$.

(ii). We have

$$\begin{aligned}
\|[e,f,e]\|^2 &= \langle [e,f,e],[e,f,e] \rangle \\
&= \langle e, [[e,f,e],e,f] \rangle \\
&= \langle e, [e,f,[e,e,f]] \rangle = 0
\end{aligned}$$

which yields $[e,f,e] = 0$. Likewise $[f,e,f] = 0$.

(iii). By the horizontal connectedness of $e$ and $f$, we have $[f,e,e] \in V_f$, $[e,f,f] \in V_e$ and

$$\begin{aligned}
[e,f,f] \circ_e [e,f,f] &= [[e,f,f],e,[e,f,f]] \\
&= [e,f,[[f,e,e],f,f]] \\
&= [e,f,[f,e,e]] \\
&= [[e,f,f],e,e] \\
&= [e,f,f]
\end{aligned}$$

which implies $[e,f,f] = 0$ or $[e,f,f] = e$ in $V_e$. In the first case,

$$[f,x,e] = [[f,x,e],f,f] = [f,x,[e,f,f]] = 0$$

for all $x \in V$ by horizontal connectedness. But this is impossible by Proposition 3.4.5 (i) and therefore $[e,f,f] = e$. Similarly $[f,e,e] = f$.

(iv). For arbitrary $x, y \in V$, we have

$$\begin{aligned}
\langle e, [f,x,y] \rangle &= \langle e, [[f,f,f],x,y] \rangle \\
&= \langle e, [f,f,[f,x,y]] \rangle \\
&= \langle [f,f,e],[f,x,y] \rangle = 0
\end{aligned}$$

since $[f,f,e] = 0$ for horizontally connected primitive tripotents $e$ and $f$. Likewise $\langle f, [e,V,V] \rangle = \{0\}$.

(v). We have

$$\begin{aligned} \|e\|^2 &= \langle e, e\rangle = \langle [e, f, f], e\rangle = \langle f, [f, e, e]\rangle \\ &= \langle f, f\rangle = \|f\|^2. \end{aligned}$$

□

We note that two horizontally connected primitive tripotents $e$ and $f$ are not ternary orthogonal since however $[e, e, f] = 0 = [f, f, e]$ but $[e, f, f] = e \neq 0$ and $[f, e, e] = f \neq 0$.

**Remark 3.4.13.** One can see that if $e$ and $f$ are horizontally connected primitive tripotents in a simple abelian Hilbert ternary algebra $V$, then

$$[e, V, f] = V_e \quad \text{and} \quad [f, V, e] = V_f$$

by Proposition 3.4.12 (iii). In fact, $V_e \subset [e, V, f]$ since

$$[e, x, e] = [[e, f, f], x, [e, f, f]] = [e, [x, f, f], [e, f, f]] = [e, [f, e, [x, f, f]], f]$$

for any $x \in V$. Likewise, $V_f \subset [f, V, e]$.

**Proposition 3.4.14.** *Let $V$ be a simple abelian Hilbert ternary algebra and let $e, f \in V$ be vertically connected primitive tripotents. Then we have*

(i) $[e, f, f] = [f, e, e] = 0.$

(ii) $[e, f, e] = [f, e, f] = 0.$

(iii) $[e, e, f] = f, [f, f, e] = e.$

(iv) $\langle e, [V, V, f]\rangle = \langle f, [V, V, e]\rangle = \{0\}.$

(v) $\|e\| = \|f\|.$

*Proof.* Each statement can be proved in the same way as in Proposition 3.4.12. □

Analogous to Remark 3.4.13, for two vertically connected primitive tripotents $e$ and $f$ in a simple abelian Hilbert ternary algebra $V$, we have $[e, V, f] = V_f$ and $[f, V, e] = V_e$. Also, one can see that two vertically connected primitive tripotents $e$ and $f$ are not ternary orthogonal.

Now we establish the following classification theorem for abelian Hilbert ternary algebras. A similar result in the context of *associative $H^*$-triple systems* is proved in [6].

**Theorem 3.4.15.** *Let $V$ be a simple abelian Hilbert ternary algebra. Then $V$ is isomorphic to the ternary algebra $\mathcal{C}_2^k(H, K)$ or its dual $\mathcal{C}_2^k(H, K)_-$ for some Hilbert spaces $H$ and $K$ over $\mathbb{F}$ and some $k > 0$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.*

*Proof.* Let $V$ contain only tripotents (cf. Proposition 3.4.5). Let $u_{00}$ be a primitive tripotent in $V$. Let

$$\{u_{00}\} \cup \{u_{\alpha 0} : \alpha \in \Lambda\}$$

be a maximal family of pairwise horizontally connected primitive tripotents and

$$\{u_{00}\} \cup \{u_{0\beta} : \beta \in \Gamma\}$$

a maximal family of vertically connected primitive tripotents. We may assume that $0 \notin \Lambda$ and $0 \notin \Gamma$.

For any $\alpha \in \Lambda$ and $\beta \in \Gamma$, define

$$u_{\alpha\beta} = [u_{\alpha 0}, u_{00}, u_{0\beta}].$$

We show that $u_{\alpha\beta}$ is a primitive tripotent with the same norm as $u_{00}$. Also, we prove that $u_{\alpha\beta}$ is orthogonal to $u_{00}$, horizontally connected with $u_{0\beta}$ and vertically connected with $u_{\alpha 0}$ for all $\alpha \in \Lambda$ and $\beta \in \Gamma$.

By Proposition 3.4.12 and Proposition 3.4.14, we have

$$\begin{aligned}
[u_{\alpha\beta}, u_{\alpha\beta}, u_{\alpha\beta}] &= [[u_{\alpha 0}, u_{00}, u_{0\beta}], [u_{\alpha 0}, u_{00}, u_{0\beta}], [u_{\alpha 0}, u_{00}, u_{0\beta}]] \\
&= [u_{\alpha 0}, u_{00}, [u_{0\beta}, [u_{\alpha 0}, u_{00}, u_{0\beta}], [u_{\alpha 0}, u_{00}, u_{0\beta}]]] \\
&= [u_{\alpha 0}, u_{00}, [[u_{0\beta}, u_{0\beta}, u_{00}], u_{\alpha 0}, [u_{\alpha 0}, u_{00}, u_{0\beta}]]] \\
&= [u_{\alpha 0}, u_{00}, [u_{00}, u_{\alpha 0}, [u_{\alpha 0}, u_{00}, u_{0\beta}]]] \\
&= [u_{\alpha 0}, u_{00}, [[u_{00}, u_{\alpha 0}, u_{\alpha 0}], u_{00}, u_{0\beta}]] \\
&= [u_{\alpha 0}, u_{00}, [u_{00}, u_{00}, u_{0\beta}]] \\
&= [u_{\alpha 0}, u_{00}, u_{0\beta}] \\
&= u_{\alpha\beta}
\end{aligned}$$

and hence $u_{\alpha\beta}$ is a tripotent.

Now we show that $u_{\alpha\beta}$ is primitive, using the condition (iii) of Lemma 3.4.9. For this suppose $v = [u_{\alpha\beta}, v, u_{\alpha\beta}]$ for $v \in V$. Then we have

$$\begin{aligned}
[u_{0\beta}, v, u_{\alpha 0}] &= [u_{0\beta}, [u_{\alpha\beta}, v, u_{\alpha\beta}], u_{\alpha 0}] \\
&= [[u_{0\beta}, u_{\alpha\beta}, v], u_{\alpha\beta}, u_{\alpha 0}] \\
&= [[u_{0\beta}, [u_{\alpha 0}, u_{00}, u_{0\beta}], v], [u_{\alpha 0}, u_{00}, u_{0\beta}], u_{\alpha 0}] \\
&= [[[u_{0\beta}, u_{0\beta}, u_{00}], u_{\alpha 0}, v], u_{0\beta}, [u_{00}, u_{\alpha 0}, u_{\alpha 0}]] \\
&= [u_{00}, [u_{0\beta}, v, u_{\alpha 0}], u_{00}]
\end{aligned}$$

and since $u_{00}$ is primitive, $[u_{0\beta}, v, u_{\alpha 0}] = \lambda u_{00}$ for some $\lambda \in \mathbb{R}$ by Lemma 3.4.9. It follows that

$$\begin{aligned}
\lambda u_{\alpha\beta} &= [u_{\alpha 0}, [u_{0\beta}, v, u_{\alpha 0}], u_{0\beta}] \\
&= [[u_{\alpha 0}, u_{\alpha 0}, v], u_{0\beta}, u_{0\beta}] \\
&= [[u_{\alpha 0}, u_{\alpha 0}, [u_{\alpha\beta}, v, u_{\alpha\beta}]], u_{0\beta}, u_{0\beta}] \\
&= [[[u_{\alpha 0}, u_{\alpha 0}, u_{\alpha\beta}], v, u_{\alpha\beta}], u_{0\beta}, u_{0\beta}] \\
&= [[u_{\alpha 0}, u_{\alpha 0}, [u_{\alpha 0}, u_{00}, u_{0\beta}]], v, [[u_{\alpha 0}, u_{00}, u_{0\beta}], u_{0\beta}, u_{0\beta}]] \\
&= [[[u_{\alpha 0}, u_{\alpha 0}, u_{\alpha 0}], u_{00}, u_{0\beta}], v, [u_{\alpha 0}, u_{00}, [u_{0\beta}, u_{0\beta}, u_{0\beta}]]] \\
&= [u_{\alpha\beta}, v, u_{\alpha\beta}] \\
&= v.
\end{aligned}$$

Then $v \in \mathbb{R}u_{\alpha\beta}$ and by Lemma 3.4.9, $u_{\alpha\beta}$ is primitive.

We now show that $V_{u_{\alpha\beta}} = [u_{\alpha 0}, V_{u_{00}}, u_{0\beta}]$. Pick $x \in V_{u_{\alpha\beta}} \setminus \{0\}$. There exists $a \in V_{u_{00}}$ such that $x = [u_{\alpha 0}, a, u_{0\beta}]$ because

$$\begin{aligned}
[u_{\alpha\beta}, V, u_{\alpha\beta}] &= [[u_{\alpha 0}, u_{00}, u_{0\beta}], V, [u_{\alpha 0}, u_{00}, u_{0\beta}]] \\
&= [u_{\alpha 0}, [V, u_{0\beta}, u_{00}], [u_{\alpha 0}, u_{00}, u_{0\beta}]] \\
&= [u_{\alpha 0}, [u_{00}, u_{\alpha 0}, [V, u_{0\beta}, u_{00}]], u_{0\beta}] \\
&= [u_{\alpha 0}, [u_{00}, [u_{0\beta}, V, u_{\alpha 0}], u_{00}], u_{0\beta}].
\end{aligned}$$

Hence $V_{u_{\alpha\beta}} \subset [u_{\alpha 0}, V_{u_{00}}, u_{0\beta}]$. On the other hand, for any $z \in V_{u_{00}}$ we have

$$\begin{aligned}
[u_{\alpha 0}, z, u_{0\beta}] &= [[u_{\alpha 0}, u_{00}, u_{00}], z, [u_{00}, u_{00}, u_{0\beta}]] \\
&= [[u_{\alpha 0}, u_{00}, [u_{0\beta}, u_{0\beta}, u_{00}]], z, [[u_{00}, u_{\alpha 0}, u_{\alpha 0}], u_{00}, u_{0\beta}]] \\
&= [[[u_{\alpha 0}, u_{00}, u_{0\beta}], u_{0\beta}, u_{00}], z, [u_{00}, u_{\alpha 0}, [u_{\alpha 0}, u_{00}, u_{0\beta}]]] \\
&= [u_{\alpha\beta}, [u_{\alpha 0}, [u_{00}, z, u_{00}], u_{0\beta}], u_{\alpha\beta}]
\end{aligned}$$

which implies $[u_{\alpha 0}, V_{u_{00}}, u_{0\beta}] \subset V_{u_{\alpha\beta}}$. Therefore $V_{u_{\alpha\beta}} = [u_{\alpha 0}, V_{u_{00}}, u_{0\beta}]$. We note that $\dim V_{u_{\alpha\beta}} = \dim V_{u_{00}}$. For if $\{v_1, \cdots, v_n\}$ is an orthogonal basis of $V_{u_{00}}$, then the elements $[u_{\alpha 0}, v_1, u_{0\beta}], \cdots, [u_{\alpha 0}, v_n, u_{0\beta}]$ form an orthogonal basis in $V_{u_{\alpha\beta}} = [u_{\alpha 0}, V_{u_{00}}, u_{0\beta}]$. Indeed, we have

$$\begin{aligned}
\langle [u_{\alpha 0}, v_i, u_{0\beta}], [u_{\alpha 0}, v_j, u_{0\beta}] \rangle &= \langle u_{\alpha 0}, [[u_{\alpha 0}, v_j, u_{0\beta}], u_{0\beta}, v_i] \rangle \\
&= \langle [u_{0\beta}, [u_{\alpha 0}, v_j, u_{0\beta}], u_{\alpha 0}], v_i \rangle \\
&= \langle v_j, v_i \rangle = 0
\end{aligned}$$

where

$$\begin{aligned}
[u_{0\beta}, [u_{\alpha 0}, v_j, u_{0\beta}], u_{\alpha 0}] &= [[u_{0\beta}, u_{0\beta}, v_j], u_{\alpha 0}, u_{\alpha 0}] \\
&= [[u_{0\beta}, [u_{00}, u_{00}, u_{0\beta}], v_j], [u_{\alpha 0}, u_{00}, u_{00}], u_{\alpha 0}] \\
&= [[[u_{0\beta}, u_{0\beta}, u_{00}], u_{00}, v_j], u_{00}, [u_{00}, u_{\alpha 0}, u_{\alpha 0}]] \\
&= [[u_{00}, u_{00}, v_j], u_{00}, u_{00}] \\
&= v_j^{*u_{00} * u_{00}} \\
&= v_j
\end{aligned}$$

by vertical connectedness of $u_{0\beta}$ and $u_{00}$, and horizontal connectedness of $u_{\alpha 0}$ and $u_{00}$. Hence all $V_{u_{00}}$, $V_{u_{\alpha 0}}$, $V_{u_{0\beta}}$ and $V_{u_{\alpha\beta}}$ are isomorphic to the same division algebra $\mathbb{F}$ where $\alpha \in \Lambda$ and $\beta \in \Gamma$, and we can identify

$$V_{u_{\alpha\beta}} = u_{\alpha\beta}\mathbb{F}. \tag{3.1}$$

Next, we show that $u_{0\beta}, u_{\alpha\beta}$ are horizontally connected, and $u_{\alpha 0}, u_{\alpha\beta}$ are vertically connected. For $v \in V$, we have

$$\begin{aligned}
[u_{0\beta}, v, u_{\alpha\beta}] &= [[u_{00}, u_{00}, u_{0\beta}], v, [u_{\alpha 0}, u_{00}, u_{0\beta}]] \\
&= [u_{00}, [u_{00}, [u_{0\beta}, v, u_{\alpha 0}], u_{00}], u_{0\beta}]
\end{aligned}$$

which means $[u_{0\beta}, V, u_{\alpha\beta}] \subset V_{u_{0\beta}}$ since $u_{00}$ and $u_{0\beta}$ are vertically connected. By vertical connectedness of $u_{0\beta}$ and $u_{00}$, we have

$$\begin{aligned}
[u_{\alpha\beta}, V, u_{0\beta}] &= [[u_{\alpha 0}, u_{00}, u_{0\beta}], V, [u_{00}, u_{00}, u_{0\beta}]] \\
&= [u_{\alpha 0}, [u_{00}, [u_{0\beta}, V, u_{00}], u_{00}], u_{0\beta}] \\
&\subset [u_{\alpha 0}, V_{u_{00}}, u_{0\beta}] \\
&= V_{u_{\alpha\beta}}.
\end{aligned}$$

In a similar way, one can show that $[u_{\alpha\beta}, V, u_{\alpha 0}] \subset V_{u_{\alpha 0}}$ and $[u_{\alpha 0}, V, u_{\alpha\beta}] = V_{u_{\alpha\beta}}$. Also, we have

$$\langle V_{u_{\alpha\beta}}, u_{00}\rangle = \langle V_{u_{\alpha\beta}}, u_{\alpha 0}\rangle = \langle V_{u_{\alpha\beta}}, u_{0\beta}\rangle = \{0\}.$$

Indeed, for instance,

$$\langle V_{u_{\alpha\beta}}, u_{00}\rangle = \langle u_{\alpha\beta}, [V, u_{\alpha\beta}, u_{00}]\rangle = \langle u_{\alpha\beta}, [V, u_{0\beta}, [u_{00}, u_{\alpha 0}, u_{00}]]\rangle = \{0\}$$

by Proposition 3.4.12 (ii). This proves horizontal connectedness of $u_{0\beta}$ and $u_{\alpha\beta}$, and vertical connectedness of $u_{\alpha 0}$ and $u_{\alpha\beta}$. By Proposition 3.4.12 (v) and Proposition 3.4.14 (v), we have $\|u_{\alpha\beta}\|^2 = \|u_{\alpha 0}\|^2 = \|u_{0\beta}\|^2 = \|u_{00}\|^2 = k$, say.

Now pick two pairs $(\alpha, \beta) \neq (\gamma, \eta) \in \Lambda \times \Gamma$ such that $\alpha \neq \gamma$. Then we get $\langle u_{\alpha 0}, u_{\gamma 0}\rangle = 0$. By horizontal connectedness of $u_{00}, u_{\alpha 0}$ and $u_{\gamma 0}$, we have

$$\begin{aligned}
\langle u_{\alpha\beta}, u_{\gamma\eta}\rangle &= \langle [u_{\alpha 0}, u_{00}, u_{0\beta}], [u_{\gamma 0}, u_{00}, u_{0\eta}]\rangle \\
&= \langle u_{\alpha 0}, [[u_{\gamma 0}, u_{00}, u_{0\eta}], u_{0\beta}, u_{00}]\rangle \\
&= \langle u_{\alpha 0}, [u_{\gamma 0}, [u_{0\beta}, u_{0\eta}, u_{00}], u_{00}]\rangle \\
&= 0
\end{aligned}$$

since $[u_{\gamma 0}, [u_{0\beta}, u_{0\eta}, u_{00}], u_{00}] \in V_{u_{\gamma 0}}$ by horizontal connectedness of $u_{\gamma 0}$ and $u_{00}$. Similarly, $\langle u_{\alpha\beta}, u_{\gamma\eta}\rangle = 0$ for the case $\beta \neq \eta$. Therefore the family

$$S = \{u_{\alpha\beta} : \alpha \in \Lambda \cup \{0\}, \beta \in \Gamma \cup \{0\}\}$$

of tripotents are mutually orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle$ of $V$.

Let $\Lambda_0 = \Lambda \cup \{0\}$ and $\Gamma_0 = \Gamma \cup \{0\}$. Define the orthogonal sum

$$L = \bigoplus_{\beta\in\Gamma_0} [V, V, u_{0\beta}]$$

where the sum is orthogonal since for every $\beta \neq \eta$ and any $x \in V$, we have

$$\begin{aligned}
\|[u_{0\beta}, u_{0\eta}, x]\|^2 &= \langle [u_{0\beta}, u_{0\eta}, x], [u_{0\beta}, u_{0\eta}, x]\rangle \\
&= \langle u_{0\beta}, [[u_{0\beta}, u_{0\eta}, x], x, u_{0\eta}]\rangle \\
&= \langle u_{0\beta}, [u_{0\beta}, [x, x, u_{0\eta}], u_{0\eta}]\rangle = 0
\end{aligned}$$

by vertical connectedness of $u_{0\beta}$ and $u_{0\eta}$, and therefore

$$\begin{aligned}
\langle [V, V, u_{0\eta}], [V, V, u_{0\beta}]\rangle &= \langle V, [[V, V, u_{0\beta}], u_{0\eta}, V]\rangle \\
&= \langle V, [V, V, [u_{0\beta}, u_{0\eta}, V]]\rangle = \{0\}.
\end{aligned}$$

Since $L$ is a ternary left ideal, $L^\perp$ is a (closed) ternary left ideal by Lemma 3.2.4 (i). We want to prove that $V = L$. We define a closed ternary right ideal of $V$ by $R = [u_{00}, V, V]$. In fact, since the ternary product is abelian,

$$R = \{a \in V : a = [u_{00}, u_{00}, a]\} \neq \{0\}.$$

We show $L^\perp = \{0\}$. Suppose otherwise. Then $L^\perp \cap R$ contains a primitive tripotent $e$ by Proposition 3.4.11 (ii). By Proposition 3.4.12 and Proposition 3.4.14, $S \subset L$ and therefore $\langle e, S\rangle = \{0\}$. Also, from $V_{u_{0\beta}} \subset L$, we get $\langle V_{u_{0\beta}}, e\rangle = \{0\}$. Furthermore, $e = [u_{00}, a, b]$ for some $a, b \in V$. This gives

$$[e, V, u_{0\beta}] = [[u_{00}, a, b], V, u_{0\beta}] = [u_{00}, [V, b, a], u_{0\beta}]$$

which implies $[e, V, u_{0\beta}] \subset V_{u_{0\beta}}$ by vertical connectedness of $u_{00}$ and $u_{0\beta}$. Since $e = [e, e, e]$, we have

$$[u_{00}, a, b] = [e, e, [u_{00}, a, b]] = [[e, e, u_{00}], a, b].$$

Hence $u_{00} = [e, e, u_{00}]$ because $V$ is simple and $V_0 = \{0\}$. Using the latter equality, we have

$$\begin{aligned}
[u_{0\beta}, V, e] &= [[u_{00}, u_{00}, u_{0\beta}], V, e] \\
&= [u_{00}, [V, u_{0\beta}, u_{00}], e] \\
&= [[e, e, u_{00}], [V, u_{0\beta}, u_{00}], e] \\
&= [e, [[V, u_{0\beta}, u_{00}], u_{00}, e], e] \\
&= [e, [V, u_{0\beta}, e], e]
\end{aligned}$$

which implies $[u_{0\beta}, V, e] \subset V_e$. It follows that $e$ is vertically connected with all $u_{0\beta}$ where $\beta \in \Gamma_0$, contradicting the maximality of the family $\{u_{00}\} \cup \{u_{0\beta}\}$. Hence $L^\perp \cap R = \{0\}$ and $L^\perp = \{0\}$ by Proposition 3.4.11 (i). Now we have

$$V = \bigoplus_{\beta \in \Gamma_0} [V, V, u_{0\beta}].$$

In a similar way, one proves that that $V = \bigoplus_{\alpha \in \Lambda_0} [u_{\alpha 0}, V, V]$.

Pick an element $[u_{\alpha 0}, x, y] \in [u_{\alpha 0}, V, V]$ and write $y = \sum_\beta [a_\beta, b_\beta, u_{0\beta}]$ with $a_\beta, b_\beta \in V$ and $\beta \in \Gamma_0$. This gives, for each $x \in V$,

$$\begin{aligned}
[u_{\alpha 0}, x, y] &= [u_{\alpha 0}, x, \sum_{\beta \in \Gamma_0} [a_\beta, b_\beta, u_{0\beta}]] \\
&= \sum_{\beta \in \Gamma_0} [u_{\alpha 0}, [b_\beta, a_\beta, x], u_{0\beta}] \\
&= \sum_{\beta \in \Gamma_0} x_{\alpha\beta}
\end{aligned}$$

where $x_{\alpha\beta} \in V_{u_{\alpha\beta}}$, because $[u_{\alpha 0}, V, u_{0\beta}] = V_{u_{\alpha\beta}}$. This implies, by (3.1),

$$\begin{aligned}
V &= \bigoplus_{\alpha \in \Lambda_0, \beta \in \Gamma_0} V_{u_{\alpha\beta}} \\
&\simeq \bigoplus_{\alpha \in \Lambda_0, \beta \in \Gamma_0} u_{\alpha\beta}\, \mathbb{F}.
\end{aligned}$$

We can now define a ternary isomorphism $\tau : \bigoplus_{\alpha,\beta} u_{\alpha\beta}\,\mathbb{F} \longrightarrow \mathcal{C}_2^k(H,K)$ by

$$\sum_{\alpha,\beta} u_{\alpha\beta}\lambda_{\alpha\beta} \longmapsto \sum_{\alpha,\beta}(f_\alpha \otimes e_\beta)\lambda_{\alpha\beta}$$

with $\lambda_{\alpha\beta} \in \mathbb{F}$, where $H$ and $K$ are $\mathbb{F}$-Hilbert spaces with $\dim H = |\Gamma_0|$ and $\dim K = |\Lambda_0|$. Hence $V \simeq \mathcal{C}_2^k(H,K)$.

If $V$ contains negative tripotents, then we have $V \simeq \mathcal{C}_2^k(H,K)_-$ for some $\mathbb{F}$-Hilbert spaces $H$ and $K$. $\square$

## 3.5 Ternary isomorphisms and isometries

In this section, we characterize completely the ternary isomorphisms and ternary anti-isomorphisms between Hilbert ternary algebras. An immediate consequence is that the ternary isomorphisms and ternary anti-isomorphisms on abelian Hilbert ternary algebras are isometries. The converse is false.

**Definition 3.5.1.** Let $(V, [\cdot,\cdot,\cdot])$ and $(W, [\cdot,\cdot,\cdot]')$ be two Hilbert ternary algebras. A linear bijection $\tau : V \longrightarrow W$ is called a *ternary isomorphism* if

$$\tau[x,y,z] = [\tau x, \tau y, \tau z]'$$

for all $x, y, z \in V$.

A ternary isomorphism from a Hilbert ternary algebra onto itself is called a *ternary automorphism.* Also, a linear bijection $\tau : V \longrightarrow W$, where $(V, [\cdot,\cdot,\cdot])$ and $(W, [\cdot,\cdot,\cdot]')$ are Hilbert ternary algebras, is called a *ternary anti-isomorphism* if

$$\tau[x,y,z] = [\tau z, \tau y, \tau x]'$$

for all $x, y, z \in V$.

We begin by showing that ternary automorphisms arise from derivations of Hilbert ternary algebras in a canonical way. Let $V$ be a Hilbert ternary algebra with zero annihilator. We show that a ternary automorphism of $V$ with positive spectrum is the exponential of a derivation.

We will make use of the fact that the family $\mathcal{F}$ of all continuous trilinear mappings $f : V \times V \times V \longrightarrow V$ on a Hilbert ternary algebra $V$ with the supremum norm

$$\|f\| = \sup\{\|f(x,y,z)\| : \|x\|, \|y\|, \|z\| \leq 1, \, x, y, z \in V\}$$

forms a Banach space.

The following result is a real extension of the arguments in [37, Proposition 6] for Hermitian Hilbert ternary algebras.

**Theorem 3.5.2.** *Let $(V, [\cdot, \cdot, \cdot])$ be a Hilbert ternary algebra with zero annihilator and $A : V \longrightarrow V$ be a ternary automorphism with spectrum $\sigma(A) \subset (0, \infty)$. Then $A = \exp(D)$ for some derivation $D$ of $V$.*

*Proof.* Since $\sigma(A) \subset (0, \infty)$, we can define a continuous linear map $D = \log(A)$ on $V$ by functional calculus [25]. It suffices to show that $D$ is a derivation. By the spectral mapping theorem, $D$ has a real spectrum. We can define continuous linear operators $D_1, D_2, D_3, D_4 : \mathcal{F} \longrightarrow \mathcal{F}$ by

$$\begin{aligned}
(D_1 f)(x,y,z) &= D(f(x,y,z)) \\
(D_2 f)(x,y,z) &= f(Dx,y,z) \\
(D_3 f)(x,y,z) &= f(x,Dy,z) \\
(D_4 f)(x,y,z) &= f(x,y,Dz) \qquad (f \in \mathcal{F}, \, (x,y,z) \in V^3).
\end{aligned}$$

Let $T = D_1 - D_2 - D_3 - D_4$ and let $A_1, A_2, A_3, A_4$ be mappings from $\mathcal{F}$ to $\mathcal{F}$

defined by

$$\begin{aligned}
(A_1 f)(x,y,z) &= A(f(x,y,z)) \\
(A_2 f)(x,y,z) &= f(A^{-1}x,y,z) \\
(A_3 f)(x,y,z) &= f(x,A^{-1}y,z) \\
(A_4 f)(x,y,z) &= f(x,y,A^{-1}z) \qquad (f \in \mathcal{F},\ (x,y,z) \in V^3).
\end{aligned}$$

Let $S = A_1A_2A_3A_4$. We have $A_1 = \exp(D_1)$ and $A_i = \exp(-D_i)$ for $i = 2,3,4$. We note that $D_1, D_2, D_3$ and $D_4$ are mutually commuting and therefore

$$\begin{aligned}
\exp(T) &= \exp(D_1 - D_2 - D_3 - D_4) \\
&= \exp(D_1)\exp(-D_2)\exp(-D_3)\exp(-D_4) \\
&= A_1A_2A_3A_4 \\
&= S.
\end{aligned}$$

Let $f_0 = [\cdot,\cdot,\cdot]$ be the given ternary product in $V$. Then $S(f_0) = f_0$ since $A$ is a ternary automorphism. We have $\sigma(T) \subset \sigma(D_1) - \sigma(D_2) - \sigma(D_3) - \sigma(D_4)$. From the definition of $D_i$'s, it is not difficult to verify that $\sigma(D_i) \subset \sigma(D)$. Hence $T$ has a real spectrum. We have

$$D_1f_0 - D_2f_0 - D_3f_0 - D_4f_0 = T(f_0) = 0$$

by Lemma 2.1.3 which yields

$$D[x,y,z] = [Dx,y,z] + [x,Dy,z] + [x,y,Dz].$$

Hence $D$ is a derivation of $V$. $\square$

Now we describe the structure of ternary isomorphisms between two simple abelian Hilbert ternary algebras which are algebras $\mathcal{C}_2^k(H,K)$ of Hilbert-Schmidt class operators. We prove two lemmas first.

**Lemma 3.5.3.** *Let $u \in H$ and $v \in K$. Then for each $x \in \mathcal{C}_2^k(K,H)$, we have* $(v \otimes u)(x^*x(v \otimes u))^*(v \otimes u) = (v \otimes u)\|xv\|^2$.

*Proof.* Let $h \in H$. Then we have

$$\begin{aligned}
(v \otimes v)x^*x(v \otimes u)(h) &= (v \otimes v)(x^*xv)\langle h, u\rangle_H \\
&= v\langle x^*xv, v\rangle_K \langle h, u\rangle_H \\
&= v\langle xv, xv\rangle_H \langle h, u\rangle_H \\
&= (v\|xv\|^2 \otimes u)(h) \\
&= (v \otimes u)\|xv\|^2(h).
\end{aligned}$$

Hence

$$\begin{aligned}
(v \otimes u)(x^*x(v \otimes u))^*(v \otimes u) &= [v \otimes u, [x^*, x^*, v \otimes u], v \otimes u] \\
&= [[v \otimes u, v \otimes u, x^*], x^*, v \otimes u] \\
&= [(v \otimes u)(u \otimes v)x^*, x^*, v \otimes u] \\
&= [(v \otimes v)x^*, x^*, v \otimes u] \\
&= (v \otimes v)x^*x(v \otimes u) \\
&= (v \otimes u)\|xv\|^2.
\end{aligned}$$

□

**Lemma 3.5.4.** *Let $u \in H$ and $v \in K$. Then for each $z \in \mathcal{C}_2^k(H,K)$, we have* $(v \otimes u)((v \otimes u)z^*z)^*(v \otimes u) = (v \otimes u)\|zu\|^2$.

*Proof.* Let $h \in H$. Then we have

$$\begin{aligned}
(v \otimes u)z^*z(u \otimes u)(h) &= (v \otimes u)(z^*zu)\langle h, u\rangle_H \\
&= v\langle z^*zu, u\rangle_H \langle h, u\rangle_H \\
&= v\langle zu, zu\rangle_K \langle h, u\rangle_H \\
&= (v\|zu\|^2 \otimes u)(h) \\
&= (v \otimes u)\|zu\|^2(h)
\end{aligned}$$

and therefore

$$\begin{aligned}(v\otimes u)((v\otimes u)z^*z)^*(v\otimes u) &= [v\otimes u,[v\otimes u,z,z],v\otimes u]\\ &= [v\otimes u,z,[z,v\otimes u,v\otimes u]]\\ &= [v\otimes u,z,z(u\otimes v)(v\otimes u)]\\ &= [v\otimes u,z,z(u\otimes u)]\\ &= (v\otimes u)z^*z(u\otimes u)\\ &= (v\otimes u)\|zu\|^2.\end{aligned}$$

□

**Theorem 3.5.5.** *Let $\tau : \mathcal{C}_2^k(H,K) \longrightarrow \mathcal{C}_2^{k'}(H',K')$ be a bounded linear map, where $H,K,H'$ and $K'$ are Hilbert spaces over $\mathbb{F}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$. Then $\tau$ is a ternary isomorphism with respect to the ternary product $[\cdot,\cdot,\cdot]$ if and only if there exist isometries $j : H' \longrightarrow H$ and $J : K \longrightarrow K'$ such that $\tau(x) = Jxj$ for $x \in \mathcal{C}_2^k(H,K)$.*

*Proof.* If $\tau(x) = Jxj$ for isomtries $j : H' \longrightarrow H$ and $J : K \longrightarrow K'$, then it is straightforward to verify that $\tau$ is a ternary isomorphism.

Conversely, let $\tau$ be a ternary isomorphism with respect to $[\cdot,\cdot,\cdot]$. Then $\tau$ preserves ternary orthogonality and therefore sends primitive tripotents to primitive tripotents. Fix unit vectors $v \in K$ and $u \in H$ so that $v\otimes u$ is a primitive tripotent in $\mathcal{C}_2^k(H,K)$ (see Lemma 3.4.10). We have

$$\tau(v\otimes u) = \xi\otimes\eta$$

for some unit vectors $\xi \in K'$ and $\eta \in H'$. We now construct two maps $j^* : H \longrightarrow H'$ and $J : K \longrightarrow K'$ such that $j^*(u) = \eta$ and $J(v) = \xi$.

Each $h \in H$ is of the form $xv$ for some $x \in \mathcal{C}_2^k(K,H)$. Indeed, we have $h = (h\otimes v)(v)$ with $(h\otimes v) \in \mathcal{C}_2^k(K,H)$. Hence we can define $j^* : H \longrightarrow H'$ by

$$j^*(xv) = \tau(x^*)^*\xi.$$

The following shows that $j^*$ is well-defined and an isometry.

$$\begin{aligned}
\|j^*(xv)\|^2 &= \langle j^*(xv), j^*(xv)\rangle_{H'} \\
&= \langle \tau(x^*)^*\xi, \tau(x^*)^*\xi\rangle_{H'} \\
&= \langle \tau(x^*)\tau(x^*)^*\tau(v\otimes u)\eta, \xi\rangle_{K'} \\
&= \langle \tau(x^*x(v\otimes u))\eta, \xi\rangle_{K'} \\
&= \langle \tau(x^*x(v\otimes u))\tau(v\otimes u)^*\xi, \tau(v\otimes u)\eta\rangle_{K'} \\
&= \langle (\tau(v\otimes u)\tau(x^*x(v\otimes u))^*\tau(v\otimes u))^*\xi, \eta\rangle_{H'} \\
&= \langle \tau((v\otimes u)(x^*x(v\otimes u))^*(v\otimes u))^*\xi, \eta\rangle_{H'} \\
&= \langle \tau((v\otimes u)\|xv\|^2)^*\xi, \eta\rangle_{H'} \qquad (3.2) \\
&= \langle (\xi\otimes\eta)^*\xi, \eta\rangle_{H'}\|xv\|^2 \\
&= \langle \eta, \eta\rangle_{H'}\|xv\|^2 \\
&= \|xv\|^2
\end{aligned}$$

where, the equality (3.2) follows from Lemma 3.5.3. Moreover, $j^*$ is surjective since for any $y \in H'$ we have $y = (\xi \otimes y)^*(\xi) = \tau(a)^*\xi = j^*(a^*v)$ for some $a \in \mathcal{C}_2^k(H,K)$ by surjectivity of $\tau$. Also, we have

$$j^*(u) = j^*((u\otimes v)v) = \tau((u\otimes v)^*)^*\xi = (\xi\otimes\eta)^*\xi = \eta.$$

Similarly, each $k \in K$ is of the form $zu$ for some $z \in \mathcal{C}_2^k(H,K)$ since $k = (k\otimes u)u$ where $(k\otimes u) \in \mathcal{C}_2^k(H,K)$. We define $J : K \longrightarrow K'$ by

$$J(zu) = \tau(z)\eta.$$

The following shows that $J$ is well-defined and an isometry.

$$\begin{aligned}
\|J(zu)\|^2 &= \langle J(zu), J(zu)\rangle_{K'} \\
&= \langle \tau(z)\eta, \tau(z)\eta\rangle_{K'} \\
&= \langle \tau(z)^*\tau(z)\tau(v\otimes u)^*\xi, \tau(v\otimes u)^*\xi\rangle_{H'} \\
&= \langle \tau(v\otimes u)(\tau(v\otimes u)\tau(z)^*\tau(z))^*\xi, \xi\rangle_{K'} \\
&= \langle \tau(v\otimes u)\tau((v\otimes u)z^*z)^*\tau(v\otimes u)\eta, \xi\rangle_{K'} \\
&= \langle \tau((v\otimes u)((v\otimes u)z^*z)^*(v\otimes u))\eta, \xi\rangle_{K'} \\
&= \langle \tau((v\otimes u)\|zu\|^2)\eta, \xi\rangle_{K'} \qquad (3.3) \\
&= \langle (\xi\otimes\eta)\eta, \xi\rangle_{K'}\|zu\|^2 \\
&= \langle \xi, \xi\rangle_{K'}\|zu\|^2 \\
&= \|zu\|^2
\end{aligned}$$

where, the equality (3.3) follows from Lemma 3.5.4. For any $w \in K'$, we have $w = (w\otimes\eta)\eta = \tau(z)\eta = J(zu)$ for some $z \in \mathcal{C}_2^k(H,K)$ by surjectivity of $\tau$. Hence $J$ is surjective. Observe that

$$J(v) = J((v\otimes u)u) = \tau(v\otimes u)\eta = (\xi\otimes\eta)\eta = \xi.$$

Now pick any $y \in \mathcal{C}_2^k(K,H)$, we have

$$\begin{aligned}
(Jxj)(\tau(y^*)^*\xi) &= (Jxj)j^*(yv) \\
&= J((xy(v\otimes u))u) \\
&= \tau(xy(v\otimes u))\eta \\
&= \tau(x)\tau(y^*)^*\tau(v\otimes u)\eta \\
&= \tau(x)\tau(y^*)^*(\xi\otimes\eta)\eta \\
&= \tau(x)\tau(y^*)^*\xi
\end{aligned}$$

which implies $\tau(x) = Jxj$ for all $x \in \mathcal{C}_2^k(H,K)$ where $j : H' \longrightarrow H$, being the dual of $j^*$, is also an isometry. $\square$

We have a complete description of ternary isomorphisms between abelian Hilbert ternary algebras.

**Theorem 3.5.6.** *Let $\tau : V \longrightarrow W$ be a ternary isomorphism between abelian Hilbert ternary algebras with zero annihilator. Then $\tau$ is of the form $\tau = \bigoplus_\alpha \tau_\alpha$ where each $\tau_\alpha(x) = J_\alpha x j_\alpha$ for some isometries $J_\alpha$ and $j_\alpha$ between Hilbert spaces.*

*Proof.* Since $\tau$ is a ternary isomorphism, it sends simple ideals onto simple ideals.

By Theorems 3.2.11 and 3.4.15, we have $V = \bigoplus_\alpha I_\alpha$ where each simple ideal $I_\alpha$ is of the form $\mathcal{C}_2^k(H_\alpha, K_\alpha)$ for some Hilbert spaces $H_\alpha$ and $K_\alpha$ and $k > 0$. Let

$$\tau_\alpha = \tau|_{I_\alpha} : I_\alpha \longrightarrow \tau(I_\alpha).$$

Then $\tau(I_\alpha)$ is also a simple abelian Hilbert ternary algebra and $\tau(I_\alpha) = \mathcal{C}_2^{k'}(H'_\alpha, K'_\alpha)$ for some Hilbert spaces $H'_\alpha$ and $K'_\alpha$ and $k' > 0$. Hence each $\tau_\alpha$ is of the form $\tau_\alpha(x) = J_\alpha x j_\alpha$ where $j : H'_\alpha \longrightarrow H_\alpha$ and $J : K_\alpha \longrightarrow K'_\alpha$ are isometries by Theorem 3.5.5. □

Analogously, we derive the following description of ternary anti-isomorphisms between two simple abelian Hilbert ternary algebras $\mathcal{C}_2^k(H, K)$ and $\mathcal{C}_2^{k'}(H', K')$.

**Theorem 3.5.7.** *Let $\tau : \mathcal{C}_2^k(H, K) \longrightarrow \mathcal{C}_2^{k'}(H', K')$ be a bounded linear map, where $H, K, H'$ and $K'$ are Hilbert spaces over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then $\tau$ is a ternary anti-isomorphism with respect to the ternary product $[\cdot,\cdot,\cdot]$ if and only if there exist isometries $j : H' \longrightarrow K$ and $J : H \longrightarrow K'$ such that $\tau(x) = Jx^*j$ for $x \in \mathcal{C}_2^k(H, K)$.*

*Proof.* Given $\tau(x) = Jx^*j$, where $j : H' \longrightarrow K$ and $J : H \longrightarrow K'$ are isometries, a straightforward computation shows that $\tau$ is a ternary anti-isomorphism.

Now, suppose that $\tau$ is a ternary anti-isomorphism. As in the proof of the Theorem 3.5.5, for unit vectors $v \in K$ and $u \in H$ we have

$$\tau(v \otimes u) = \xi \otimes \eta$$

for some unit vectors $\xi \in K'$ and $\eta \in H'$ by Lemma 3.4.10. We construct the isometries $J : H \longrightarrow K'$ and $j^* : K \longrightarrow H'$ such that $J(u) = \xi$ and $j^*(v) = \eta$.

Analogues to the proof of Theorem 3.5.5, each $h \in H$ is of the form $xv$ for some $x \in \mathcal{C}_2^k(K, H)$ and we can define $J : H \longrightarrow K'$ by

$$J(xv) = \tau(x^*)\eta$$

where $J$ is well-defined and an isometry. Indeed,

$$\begin{aligned}
\|J(xv)\|^2 &= \langle J(xv), J(xv)\rangle_{K'} \\
&= \langle \tau(x^*)\eta, \tau(x^*)\eta\rangle_{K'} \\
&= \langle \tau(x^*)^*\tau(x^*)\tau(v \otimes u)^*\xi, \tau(v \otimes u)^*\xi\rangle_{H'} \\
&= \langle \tau(v \otimes u)(\tau(v \otimes u)\tau(x^*)^*\tau(x^*))^*\xi, \xi\rangle_{K'} \\
&= \langle \tau(v \otimes u)\tau(x^*x(v \otimes u))^*\tau(v \otimes u)\eta, \xi\rangle_{K'} \\
&= \langle \tau((v \otimes u)(x^*x(v \otimes u))^*(v \otimes u))\eta, \xi\rangle_{K'} \\
&= \langle \tau((v \otimes u)\|xv\|^2)\eta, \xi\rangle_{K'} \\
&= \langle (\xi \otimes \eta)\eta, \xi\rangle_{K'}\|xv\|^2 \\
&= \|xv\|^2
\end{aligned}$$

where, by Lemma 3.5.3, we have

$$(v \otimes u)(x^*x(v \otimes u))^*(v \otimes u) = (v \otimes u)\|xv\|^2.$$

Moreover, $J$ is surjective since for any $y \in K'$ we have $y = (y \otimes \eta)(\eta) = \tau(a)\eta = J(a^*v)$ for some $a \in \mathcal{C}_2^k(H, K)$ by surjectivity of $\tau$. Also, we have

$$J(u) = J((u \otimes v)v) = \tau((u \otimes v)^*)\eta = (\xi \otimes \eta)\eta = \xi.$$

Each $k \in K$ is of the form $zu$ for some $z \in \mathcal{C}_2^k(H, K)$, we can define $j^* : K \longrightarrow H'$ by

$$j^*(zu) = \tau(z)^*\xi.$$

As in the proof of Theorem 3.5.5, one can use Lemma 3.5.4 to show that

$$\|j^*(zu)\|^2 = \|zu\|^2$$

and therefore $j^*$ is well-defined and an isometry. Hence the dual map $j : H' \longrightarrow K$ is also an isometry. For any $w \in H'$, we have $w = (\xi \otimes w)^*\xi = \tau(z)^*\xi = j^*(zu)$ for some $z \in \mathcal{C}_2^k(H, K)$ by surjectivity of $\tau$. Hence $j^*$ is surjective. Note that

$$j^*(v) = j^*((v \otimes u)u) = \tau(v \otimes u)^*\xi = (\xi \otimes \eta)^*\xi = \eta.$$

Now Pick any $y \in \mathcal{C}_2^k(H, K)$, we have

$$\begin{aligned}
(Jx^*j)(\tau(y)^*\xi) &= (Jx^*j)j^*(yu) \\
&= J((x^*y(u \otimes v))v) \\
&= \tau((x^*y(u \otimes v))^*)\eta \\
&= \tau((v \otimes u)y^*x)\eta \\
&= \tau(x)\tau(y)^*\tau(v \otimes u)\eta \\
&= \tau(x)\tau(y)^*(\xi \otimes \eta)\eta \\
&= \tau(x)\tau(y)^*\xi
\end{aligned}$$

which implies $\tau(x) = Jx^*j$ for all $x \in \mathcal{C}_2^k(H, K)$. $\square$

A linear map $\tau : \mathcal{C}_2(H) \longrightarrow \mathcal{C}_2(H)$ is called a $^*$-*map* if

$$\tau(x^*) = \tau(x)^*$$

for all $x \in \mathcal{C}_2(H)$.

**Corollary 3.5.8.** *Let $\tau : \mathcal{C}_2(H) \longrightarrow \mathcal{C}_2(H)$ be a linear bijection. The following conditions are equivalent.*

(i) *$\tau : \mathcal{C}_2(H) \longrightarrow \mathcal{C}_2(H)$ is a $^*$-antiautomorphism.*

(ii) *$\tau$ is of the form $\tau(x) = jx^*j^{-1}$ for some linear isometry $j : H \longrightarrow H$.*

*Further, $\tau$ has period 2, i.e., $\tau^2$ is the identity map* id*, if and only if $j$ is a conjugation or an anticonjugation.*

*Proof.* We need only prove (i) $\Rightarrow$ (ii). By Theorem 3.5.7, $\tau$ is of the form

$$\tau(x) = Jx^*j$$

where $j, J : H \longrightarrow H$ are isometries. But $\tau(x^*) = \tau(x)^*$ implies

$$Jxj = j^*xJ^*$$

and hence $(jJ)x = x(jJ)^*$ for all $x \in \mathcal{C}_2(H)$. Since $\mathcal{C}_2(H)$ is weakly dense in $\mathcal{B}(H)$, we have $jJ = (jJ)^*$. It follows that $(jJ)\mathcal{B}(H) = \mathcal{B}(H)(jJ)$ and so $jJ = \lambda\,\mathrm{id}$.

If $\tau^2 = \mathrm{id}$, then we have $x = \tau^2(x) = j^2xj^{-2}$ and $j^2x = xj^2$ for all $x \in \mathcal{C}_2(H)$. Hence $j^2$ is a scalar multiple of the identity operator. Since $j^2$ is an isometry, we must have $j^2 = \pm\mathrm{id}$. □

Ternary automorphisms and ternary antiautomorphisms of $\mathcal{C}_2(H)$ are isometries. However, the converse is false. This is in contrast to the case of $JB^*$-triples where a linear isometry of a $JB^*$-triple is necessarily a triple isomorphism.

**Example 3.5.9.** Let $u \in \mathcal{B}(H)$ be a unitary. Define $\varphi : \mathcal{C}_2(H) \longrightarrow \mathcal{C}_2(H)$ by

$$\varphi(x) = ux \qquad (x \in \mathcal{C}_2(H)).$$

Then

$$\|\varphi(x)\|_2^2 = \mathrm{Trace}(\varphi(x)^*\varphi(x)) = \mathrm{Trace}(x^*u^*ux) = \mathrm{Trace}(x^*x) = \|x\|_2^2.$$

Hence $\varphi$ is an isometry.

If $\varphi$ is of the form $jxj^{-1}$ for some linear isometry $j : H \longrightarrow H$, then we have

$$ux^* = \varphi(x^*) = jx^*j^{-1} = (jxj^{-1})^* = \varphi(x)^* = x^*u^*$$

for all $x \in \mathcal{C}_2(H)$ which cannot be true in general. Indeed, let $H = \mathbb{C}^2$ and let $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $u = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$ux^* = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix} = x^*u^*.$$

We conclude by discussing the case of arbitrary, but not necessarily abelian, Hilbert ternary algebras.

**Lemma 3.5.10.** *Let $\tau : V \longrightarrow W$ be a ternary isomorphism between simple Hilbert ternary algebras. Then there exists some $\lambda > 0$ such that $\tau\tau^* = \lambda \,\mathrm{id}_V$.*

*Proof.* For any $a, b, c \in V$ and $z \in W$, we have

$$\langle \tau[a, b, c], z\rangle = \langle [a, b, c], \tau^* z\rangle = \langle c, [b, a, \tau^* z]\rangle$$

and

$$\langle [\tau a, \tau b, \tau c], z\rangle = \langle \tau c, [\tau b, \tau a, z]\rangle = \langle c, \tau^*[\tau b, \tau a, z]\rangle.$$

These identities imply $\tau^*[\tau b, \tau a, z] = [b, a, \tau^* z]$. Given $x, y \in W$, there exist $a, b \in V$ such that $x = \tau a$ and $y = \tau b$. Hence

$$\begin{aligned} \tau\tau^*[x, y, z] &= \tau\tau^*[\tau a, \tau b, z] \\ &= \tau[a, b, \tau^* z] \\ &= [\tau a, \tau b, \tau\tau^* z] \\ &= [x, y, \tau\tau^* z]. \end{aligned}$$

Similarly one can show that

$$\tau\tau^*[x, y, z] = [\tau\tau^* x, y, z]$$

for all $x, y, z \in W$. Therefore $\tau\tau^* \in \mathcal{Z}(V)$. It follows from Lemma 3.3.5 that $\tau\tau^* = \lambda \,\mathrm{id}_V$ for some $\lambda > 0$. □

**Proposition 3.5.11.** *Every ternary isomorphism between simple Hilbert ternary algebras is a scalar multiple of an isometry.*

*Proof.* Let $\tau : V \longrightarrow W$ be a ternary isomorphism. By Lemma 3.5.10, $\tau\tau^* = \lambda\,\mathrm{id}_V$ for some $\lambda > 0$. Since $\tau$ is invertible, we also have $\tau^*\tau = \lambda\,\mathrm{id}_V$. It follows that, for $x \in V$,

$$\langle \tau x, \tau x \rangle = \langle \tau^*\tau x, x \rangle = \langle \lambda x, x \rangle$$

which gives $\|\tau x\| = \sqrt{\lambda}\|x\|$. Hence $\frac{1}{\sqrt{\lambda}}\tau$ is an isometry. $\square$

# Chapter 4

# Jordan structures in Hilbert spaces

## 4.1 Jordan triple systems and Lie algebras

To motivate the concept of a Jordan triple system, we first recall the one-one correspondence between Jordan triple systems and a class of Lie algebras, called the *TKK Lie algebras*. Our construction of the correspondence is more general than the one given in [9], we do not require the Jordan triple systems to be non-degenerate.

Jordan triple systems were introduced by Meyberg [29] to extend the correspondence between Jordan algebras and 3-graded Lie algebras, discovered independently by Tits [35], Kantor [19, 20] and Koecher [24] to whom the name TKK Lie algebra is due (see also [28]).

**Definition 4.1.1.** A real vector space $V$ is called a *Jordan triple system* or simply, a *Jordan triple*, if there exists a trilinear map $\{\cdot,\cdot,\cdot\} : V^3 \longrightarrow V$ satisfying

(i) $\{x, y, z\} = \{z, y, x\}$

(ii) (*Jordan triple identity*)

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

for all $a, b, x, y, z \in V$.

A complex vector space $V$ with a ternary product $\{\cdot, \cdot, \cdot\}$ which is linear in the first and third variables, but conjugate linear in the second variable, and satisfies conditions (i) and (ii) above, is called a *Hermitian Jordan triple*. Restricting to the real scalar field, a Hirmitian Jordan triple can be regarded as a (real) Jordan triple.

Given a Jordan triple system $V$ with $a, b \in V$. We define the box operator $a \square b : V \longrightarrow V$ by left multiplication:

$$(a \square b)(x) = \{a, b, x\} \qquad (x \in V).$$

The Jordan triple identity can be written in the form

$$\{a, b, x\} \square y - x \square \{b, a, y\} = a \square \{y, x, b\} - \{x, y, a\} \square b$$

or $\{a, b, x\} \square y - x \square \{b, a, y\} = [a \square b, x \square y] := (a \square b)(x \square y) - (x \square y)(a \square b)$.

**Definition 4.1.2.** Let $V$ be a Jordan triple system and $a \in V$. As in the case of Hilbert ternary algebras, we define the *quadratic operator* $Q_a : V \longrightarrow V$ by

$$Q_a(x) = \{a, x, a\} \qquad (x \in V).$$

A Jordan triple $V$ is called *degenerate* if there exists an element $a \in V \setminus \{0\}$ such that the quadratic operator $Q_a = 0$.

In what follows, a *Lie algebra* is a real vector space $\mathfrak{g}$ of any dimension, with a bilinear multiplication, called the *Lie product*,

$$(X, Y) \in \mathfrak{g} \times \mathfrak{g} \longmapsto [X, Y] \in \mathfrak{g}$$

satisfying $[X, X] = 0$ and the *Jacobi identity*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. A linear subspace $\mathfrak{k}$ of $\mathfrak{g}$ is called a *Lie subalgebra* of $\mathfrak{g}$ if $[X, Y] \in \mathfrak{k}$ for all $X, Y \in \mathfrak{k}$.

We define the *Jacobian* of $X, Y, Z \in \mathfrak{g}$ to be

$$J(X, Y, Z) = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y].$$

The Jacobi identity can be written as $J(X, Y, Z) = 0$.

Given two subspaces $\mathfrak{k}$ and $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, we denote by $[\mathfrak{k}, \mathfrak{h}]$ the linear span of

$$\{[X, Y] : X \in \mathfrak{k}; Y \in \mathfrak{h}\}$$

which is a Lie subalgebra of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{g}_{-1} \bigoplus \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$ be a direct sum of subspaces. We say that $\mathfrak{g}$ is *graded* if

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

for $\alpha, \beta = 0, \pm 1$, and $\mathfrak{g}_{\alpha+\beta} = \{0\}$ if $\alpha + \beta \neq 0, \pm 1$.

A linear map $\theta$ on $\mathfrak{g}$ such that

$$\theta[X, Y] = [\theta X, \theta Y] \qquad (X, Y \in \mathfrak{g})$$

is called a *Lie algebra homomorphism*. An *automorphism* of $\mathfrak{g}$ is a bijective Lie algebra homomorphism $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$. It is called an *involutive automorphism* or *involution* if $\theta^2$ is the identity map $\mathrm{id} : \mathfrak{g} \longrightarrow \mathfrak{g}$. By an *involutive Lie algebra* $(\mathfrak{g}, \theta)$, we mean a Lie algebra $\mathfrak{g}$ equipped with an involutive automorphism $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$. Since $\theta^2 = \mathrm{id}$, the automorphism $\theta$ has eigenvalues $\pm 1$. Let $\mathfrak{k}$ be the 1-eignespace of $\theta$ and $\mathfrak{p}$ be the $(-1)$-eignespace of $\theta$. Then we have the *canonical decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Following [9], we call a Lie algebra $\mathfrak{g}$ *orthogonal* if there is a positive definite quadratic form

$$q : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathbb{R}$$

satisfying

$$q([Z, X], X) = 0 \qquad (Z \in \mathfrak{k} \text{ and } X \in \mathfrak{p}),$$

where we define $q$ to be *positive definite* if $q(X, X) > 0$ for $X \in \mathfrak{p} \setminus \{0\}$.

**Example 4.1.3.** On the von Neumann algebra $\mathcal{B}(H)$ of bounded linear operators on a real Hilbert space $H$, there is a natural Lie product

$$[S, T] = ST - TS \qquad (S, T \in \mathcal{B}(H))$$

which turns $\mathcal{B}(H)$ into a Lie algebra. We will assume this Lie structure of $\mathcal{B}(H)$. In $\mathcal{B}(H)$, the Hilbert-Schmidt operators $\mathcal{C}_2(H)$ form a Lie subalgebra. Further, both $\mathcal{B}(H)$ and $\mathcal{C}_2(H)$ are involutive Lie algebras with involution $\theta(T) = T^*$.

**Example 4.1.4.** Let $V$ be a (real) Jordan triple system. Let $V \square V$ be the real linear span of $\{x \square y : x, y \in V\}$ in $\mathcal{L}(V)$, the real vector space of linear self-maps on $V$. Using the Jordan triple identity

$$[x \square y, u \square v] = \{x, y, u\} \square v - u \square \{v, x, y\},$$

we can define a Lie product on $V \square V$ by

$$[h, k] = hk - kh \in V \square V \qquad (h, k \in V \square V).$$

With this product, $V \square V$ is a Lie algebra.

The Cartesian product $(V \square V)^2 = (V \square V) \times (V \square V)$ is a Lie algebra in the coordinatewise Lie product.

**Definition 4.1.5.** A *Tits-Kantor-Koecher Lie algebra* or simpliy, a *TKK Lie algebra*, is a graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \bigoplus \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$ with an involution $\theta$ such that

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$$

for $\alpha = 0, 1, -1$. We call $\mathfrak{g}$ *canonical* if $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.

For completeness and later reference, we now show the correspondence between Jordan triple systems and TKK Lie algebras.

**Theorem 4.1.6.** *Let $V$ be a Jordan triple system. Then there exists a canonical Tits-Kantor-Koecher Lie algebra $\mathfrak{g}(V)$ with grading*

$$\mathfrak{g}(V) = \mathfrak{g}(V)_{-1} \bigoplus \mathfrak{g}(V)_0 \bigoplus \mathfrak{g}(V)_1$$

*and an involution $\theta$ such that $\mathfrak{g}(V)_{-1} = V = \mathfrak{g}(V)_1$ and*

$$\{x, y, z\} = [[x, \theta y], z]$$

*for all $x, y, z \in \mathfrak{g}(V)_{-1}$.*

*Proof.* Let $V \square V$ be the Lie algebra in Example 4.1.4. Let $\mathrm{Ind}(V, V)$ be the real linear span of the set

$$S = \{(x \square y, -y \square x) : x, y \in V\} \subset (V \square V)^2$$

and write $d_{xy} = (x \square y, -y \square x)$ for $x, y \in V$. We call $d_{xy}$ an *inner derivation pair*. Then $\mathrm{Ind}(V, V)$ is a Lie subalgebra of $(V \square V)^2$. Indeed for any $(h^+, h^-) =$

$(h_1 \square h_2, -h_2 \square h_1)$ and $(k^+, k^-) = (k_1 \square k_2, -k_2 \square k_1)$ in $S$, we have

$$\begin{aligned}
&[(h^+, h^-), (k^+, k^-)] \\
&= ([h^+, k^+], [h^-, k^-]) \\
&= ([h_1 \square h_2, k_1 \square k_2], [-h_2 \square h_1, -k_2 \square k_1]) \\
&= ((h_1 \square h_2)(k_1 \square k_2) - (k_1 \square k_2)(h_1 \square h_2), (h_2 \square h_1)(k_2 \square k_1) - (k_2 \square k_1)(h_2 \square h_1)) \\
&= (\{h_1, h_2, k_1\} \square k_2 - k_1 \square \{h_2, h_1, k_2\}, \{h_2, h_1, k_2\} \square k_1 - k_2 \square \{h_1, h_2, k_1\}) \\
&= (\{h_1, h_2, k_1\} \square k_2, -k_2 \square \{h_1, h_2, k_1\}) - (k_1 \square \{h_2, h_1, k_2\}, -\{h_2, h_1, k_2\} \square k_1) \\
&\in S - S \subset \mathrm{Ind}(V, V).
\end{aligned}$$

Form the direct sum

$$\mathfrak{g}(V) = V_{-1} \bigoplus \mathrm{Ind}(V, V) \bigoplus V_1$$

where $V_{-1} = V_1 = V$. We write $x \oplus (h^+, h^-) \oplus y$ for $(x, (h^+, h^-), y)$ and also,

$$\begin{array}{rcl}
x & \text{for} & (x, 0, 0) \\
\bar{y} & \text{for} & (0, 0, y) \\
(h^+, h^-) & \text{for} & (0, (h^+, h^-), 0)
\end{array}$$

if no confusion is likely.

Define the following product on $\mathfrak{g}(V)$ by

$$\begin{aligned}
& [x \oplus (h^+, h^-) \oplus y, u \oplus (k^+, k^-) \oplus v] \\
= \;& (h^+(u) - k^+(x),\, [(h^+, h^-), (k^+, k^-)] + d_{xv} - d_{uy},\, h^-(v) - k^-(y)). \qquad (4.1)
\end{aligned}$$

We show that $\mathfrak{g}(V)$ is a Lie algebra in the above product. Plainly

$$[x \oplus (h^+, h^-) \oplus y, x \oplus (h^+, h^-) \oplus y] = 0$$

and the Jacobi identity $J((x,0,0),(y,0,0),(z,0,0)) = 0$ is satisfied. Note that $[V_{-1}, V_{-1}] = [V_1, V_1] = 0$. For convenience, we write $V_0 = \mathrm{Ind}(V,V)$. We have

$$\begin{aligned} J(V_{-1}, V_{-1}, V_{-1}) &= J(V_{-1}, V_{-1}, V_0) = J(V_{-1}, V_{-1}, V_1) \\ = J(V_{-1}, V_1, V_1) &= J(V_0, V_1, V_1) = J(V_1, V_1, V_1) = 0. \end{aligned}$$

We also have $J(V_0, V_0, V_0) = 0$ since $V_0$ is a Lie algebra. Hence the Jacobi identity is satisfied if and only if

$$J(V_{-1}, V_0, V_1) + J(V_{-1}, V_0, V_0) + J(V_0, V_0, V_1) = 0.$$

Pick $(h^+, h^-), (k^+, k^-) \in V_0$, $u \in V_{-1}$ and $v \in V_1$. Then we have $J(u, (h^+, h^-), v) = 0$ if and only if

$$(-h^+(u) \square v, v \square h^+(u)) + (-u \square h^-(v), h^-(v) \square u) + ([-u \square v, h^+], [v \square u, h^-]) = 0$$

or

$$\begin{aligned} h^+(u) \square v + u \square h^-(v) &= -(u \square v)h^+ + h^+(u \square v) \\ v \square h^+(u) + h^-(v) \square u &= -(v \square u)h^- + h^-(v \square u) \end{aligned}$$

which holds since, for $h^+ = x \square y$, these identities are just the Jordan triple identities

$$\begin{aligned} \{x,y,u\} \square v - u \square \{y,x,v\} &= -(u \square v)(x \square y) + (x \square y)(u \square v) \\ v \square \{x,y,u\} - \{y,x,v\} \square u &= (v \square u)(y \square x) - (y \square x)(v \square u). \end{aligned}$$

Hence $J(V_{-1}, V_0, V_1) = 0$.

We also have

$$\begin{aligned} & J(u, (h^+, h^-), (k^+, k^-)) \\ = \; & [-h^+(u), (k^+, k^-)] + [([h^+, k^+], [h^-, k^-]), u] + [k^+(u), (h^+, h^-)] \\ = \; & k^+h^+(u) + (h^+k^+ - k^+h^+)(u) - h^+k^+(u) \\ = \; & 0. \end{aligned}$$

Hence $J(V_{-1}, V_0, V_0) = 0$.

Similarly we have $J(V_0, V_0, V_1) = 0$. Hence $\mathfrak{g}(V)$ is a Lie algebra. Moreover, it is canonical. For if $x \in V_{-1}$ and $y \in V_1$, we have

$$[x, \bar{y}] = [(x, 0, 0), (0, 0, y)] = (0, d_{xy}, 0)$$

which gives $[V_{-1}, V_1] = \mathrm{Ind}(V, V)$.

Now define a linear map $\theta : \mathfrak{g}(V) \longrightarrow \mathfrak{g}(V)$ by

$$\theta(x \oplus (h^+, h^-) \oplus y) = y \oplus (h^-, h^+) \oplus x \tag{4.2}$$

for $x \oplus (h^+, h^-) \oplus y \in V_{-1} \bigoplus \mathrm{Ind}(V, V) \bigoplus V_1$. Then $\theta$ is an involution. By identifying $V_\alpha$ as subspaces of $\mathfrak{g}(V)$, we see that

$$\theta(V_\alpha) = V_{-\alpha} \qquad \text{for} \qquad \alpha = 0, \pm 1.$$

For $x, y, z \in V_{-1}$ with $d_{xy} = (x \,\square\, y, -y \,\square\, x)$, we have

$$\begin{aligned} [[x, \theta y], z] &= [[x, \bar{y}], z] \\ &= [(0, d_{xy}, 0), (z, 0, 0)] \\ &= (x \,\square\, y)(z) \\ &= \{x, y, z\}. \end{aligned}$$

This proves that $\mathfrak{g}(V) = \mathfrak{g}(V)_{-1} \bigoplus \mathfrak{g}(V)_0 \bigoplus \mathfrak{g}(V)_1$ is a canonical TKK Lie algebra with $\mathfrak{g}(V)_{\pm 1} = V$ and $\mathfrak{g}(V)_0 = V_0$. $\square$

Conversely, given a TKK Lie algebra, one can construct a corresponding Jordan triple system as described below. Without loss of generality, we may assume $\mathfrak{g}_{-1} = \mathfrak{g}_1$ in a TKK Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \bigoplus \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$ since the involution $\theta$ identifies $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$.

**Theorem 4.1.7.** *Let $\mathfrak{g} = \mathfrak{g}_{-1} \bigoplus \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$ be a canonical Tits-Kantor-Koecher Lie algebra with involution $\theta$ and $\mathfrak{g}_1 = \mathfrak{g}_{-1}$. Then there exists a Jordan triple system $V(\mathfrak{g})$ with the triple product defined by*

$$\{x, y, z\} = [[x, \theta y], z]$$

*for all $x, y, z \in V(\mathfrak{g})$ where $[\cdot, \cdot]$ is the Lie product of $\mathfrak{g}$.*

*Proof.* Let $V(\mathfrak{g}) = \mathfrak{g}_{-1}$. We show that $V(\mathfrak{g})$ with the above triple product is a Jordan triple system. From $J(x, \theta y, z) = 0$ for all $x, y, z \in \mathfrak{g}_{-1} = V(\mathfrak{g})$, we get

$$[[x, \theta y], z] + [[\theta y, z], x] = 0$$

since $\mathfrak{g}$ is graded and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$. Hence

$$\begin{aligned}
\{x, y, z\} &= [[x, \theta y], z] \\
&= -[[\theta y, z], x] \\
&= [[z, \theta y], x] \\
&= \{z, y, x\}.
\end{aligned}$$

It follows from the Jacobi identity that

$$\begin{aligned}
&[[a, \theta b], [[x, \theta y], z]] - [[x, \theta y], [[a, \theta b], z]] \\
&= [[a, \theta b], [[x, \theta y], z]] + [[x, \theta y], [z, [a, \theta b]]] \\
&= -[z, [[a, \theta b], [x, \theta y]]]
\end{aligned}$$

and

$$\begin{aligned}
&[[[[a, \theta b], x], \theta y], z] - [[x, \theta[[b, \theta a], y]], z] \\
&= -[[\theta y, [x, [\theta b, a]]], z] - [[x, [[\theta b, a], \theta y]], z] \\
&= [[[\theta b, a], [\theta y, x]], z]
\end{aligned}$$

for all $a, b, x, y, z \in \mathfrak{g}_{-1} = V(\mathfrak{g})$, which implies the Jordan triple identity

$$\{a, b, \{x, y, z\}\} - \{x, y, \{a, b, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\}$$

for all $a, b, x, y, z \in V(\mathfrak{g})$. $\square$

**Remark 4.1.8.** Let $\mathfrak{g} = \mathfrak{g}_{-1} \bigoplus \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$ be a canonical TKK Lie algebra such that $\mathfrak{g}_1 = \mathfrak{g}_{-1}$. Then $\mathfrak{g}$ induces a Jordan triple system $V(\mathfrak{g}) = \mathfrak{g}_{-1}$ by Theorem 4.1.7. Consider the TKK Lie algebra

$$V(\mathfrak{g}) \bigoplus \mathrm{Ind}(V(\mathfrak{g}), V(\mathfrak{g})) \bigoplus V(\mathfrak{g}) = \mathfrak{g}_{-1} \bigoplus \mathrm{Ind}(V(\mathfrak{g}), V(\mathfrak{g})) \bigoplus \mathfrak{g}_1$$

constructed from $\mathfrak{g}$ in Theorem 4.1.6, where $V(\mathfrak{g}) = \mathfrak{g}_{-1}$. Since $\mathfrak{g}$ and $V(\mathfrak{g})$ are canonical, we have $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathrm{Ind}(V(\mathfrak{g}), V(\mathfrak{g}))$. In other words, $\mathfrak{g}(V(\mathfrak{g})) = \mathfrak{g}$ in our notation.

**Example 4.1.9.** The Lie algebra $\mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} y & a \\ b & -y \end{pmatrix} : a, b, y \in \mathbb{R} \right\}$ with the usual brackets $[A, B] = AB - BA$ has grading

$$\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R})_{-1} \bigoplus \mathfrak{sl}(2, \mathbb{R})_0 \bigoplus \mathfrak{sl}(2, \mathbb{R})_1$$

where

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{R})_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in \mathbb{R} \right\} \\ \mathfrak{sl}(2, \mathbb{R})_0 &= \left\{ \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} : y \in \mathbb{R} \right\} \\ \mathfrak{sl}(2, \mathbb{R})_1 &= \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}, \end{aligned}$$

where we have $[\mathfrak{sl}(2, \mathbb{R})_{-1}, \mathfrak{sl}(2, \mathbb{R})_1] = \mathfrak{sl}(2, \mathbb{R})_0$. Define the involution $\theta$ on $\mathfrak{sl}(2, \mathbb{R})$ by

$$\theta \begin{pmatrix} y & a \\ b & -y \end{pmatrix} = \begin{pmatrix} -y & b \\ a & y \end{pmatrix}.$$

We see that $(\mathfrak{sl}(2,\mathbb{R}),\theta)$ is a canonical TKK Lie algebra. The corresponding Jordan triple system is $V(\mathfrak{sl}(2,\mathbb{R})) = \mathfrak{sl}(2,\mathbb{R})_{-1} = \left\{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : b \in \mathbb{R}\right\}$, with triple product

$$\begin{aligned} & \left\{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right\} \\ = & \left[\left[\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \theta\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right] \\ = & \begin{pmatrix} 0 & 0 \\ 2abc & 0 \end{pmatrix}, \end{aligned}$$

which identifies with $\mathbb{R}$ with triple product $\{a,b,c\} = 2abc$.

**Remark 4.1.10.** A Jordan triple system $V$ is called *non-degenerate* if for each $a \in V$, we have $a = 0$ whenever $\{a,x,a\} = 0$ for all $x \in V$. The construction above of the TKK Lie algebras does not require non-degeneracy of the Jordan triple system $V$. However, for a non-degenerate Jordan triple system $V$, the construction is the same as [9] where the TKK Lie algebra is defined by $\mathfrak{g}(V) = V \bigoplus V \square V \bigoplus V$. In fact, if $V$ is non-degenerate, we can identify $\mathrm{Ind}(V,V)$ naturally with $V \square V$ by the mapping

$$\sum_i a_i \square b_i \in V \square V \longmapsto (\sum_{i=1}^n a_i \square b_i, -\sum_{i=1}^n b_i \square a_i) \in \mathrm{Ind}(V,V)$$

which is well-defined by non-degeneracy [9]. In this case, the involution $\theta$ defined in (4.2) identifies with the involution $\theta' : V \bigoplus V \square V \bigoplus V \longrightarrow V \bigoplus V \square V \bigoplus V$ defined by $\theta'(x \oplus \sum a_i \square b_i \oplus y) = y \oplus -\sum b_i \square a_i \oplus x$.

For later use, we recall some basic identities in Jordan triple systems and refer to [26, 27] for a derivation of the following ones:

$$\begin{aligned} \{\{x,y,x\},y,z\} &= \{x,\{y,x,y\},z\} \\ \{x,y,\{x,z,x\}\} &= \{x,\{y,x,z\},x\} \\ \{\{x,y,x\},z,\{x,y,x\}\} &= \{x,\{y,\{x,z,x\},y,\}x\}. \end{aligned} \tag{4.3}$$

We will make use of the *polarization formula*

$$2\{a,x,b\} = \{a+b,x,a+b\} - \{a,x,a\} - \{b,x,b\}$$

in a Jordan triple system $V$.

A *subtriple* of a Jordan triple system $V$ is a subspace $W$ of $V$ such that $\{x,y,z\} \in W$ whenever $x,y,z \in W$. To show that a subspace $W$ of a Jordan triple system $V$ is a subtriple of $V$, it is sufficient to show $\{a,x,a\} \in W$ for all $a,x \in W$ by the polarization formula.

A subspace $I$ of $V$ is called a *triple ideal*, or an *ideal* when there is no confusion with ternary ideal defined before, if

$$\{I,V,V\} + \{V,I,V\} \subset I.$$

Evidently, every triple ideal is a subtriple of $V$.

Let $Q_0 = \{a \in V : Q_a = 0\}$. Then $Q_0$ is a triple ideal of $V$. Indeed, the identities

$$\{\{x,a,y\},z,\{x,a,y\}\} = \{x,\{a,\{y,z,y\},a\},x\} = 0$$

$$\{\{a,x,y\},z,\{a,x,y\}\} = \{a,\{x,\{y,z,y\},x\},a\} = 0$$

with $a \in Q_0$ and $x,y,z \in V$ imply that $\{x,a,y\} \in Q_0$ and $\{a,x,y\} \in Q_0$ for all $x,y \in V$.

We now discuss some basic structures of Jordan triple systems for later use.

**Definition 4.1.11.** A Jordan triple system $V$ is called *abelian* if

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\}$$

holds for all $a, b, x, y, z \in V$.

We note that an abelian Jordan triple system $V$ also satisfies the identity

$$\{a, b, \{x, y, z\}\} = \{a, \{y, x, b\}, z\} \qquad (a, b, x, y, z \in V)$$

by the Jordan triple identity.

**Definition 4.1.12.** A Jordan triple system $V$ is called *flat* if

$$a \,\square\, b = b \,\square\, a$$

for all $a, b \in V$.

**Definition 4.1.13.** Let $V$ be a Jordan triple system. An element $a \in V$ is called a *tripotent* if

$$\{e, e, e\} = e,$$

and a *negative tripotent* if $\{e, e, e\} = -e$.

**Definition 4.1.14.** Two tripotents $e$ and $f$ in a Jordan triple system $V$ are *(triple) orthogonal* if

$$e \,\square\, f = 0$$

in which case, we also have $f \,\square\, e = 0$ (see [8, 13]).

As in the case of Hilbert ternary algebras in Definition 3.3.1, a linear operator $C$ on a Jordan triple system $V$ is called a *centralizer* if

$$C\{x, y, z\} = \{Cx, y, z\}$$

for all $x, y, z \in V$. We denote by $\mathcal{Z}(V)$ the algebra of centralizers on $V$ where as usual, the product is the composition of operators.

**Lemma 4.1.15.** *Let $V$ be a Jordan tripe system. Then for each $C \in \mathcal{Z}(V)$, we have*

$$C\{x, \{a, b, c\}, z\} = \{x, \{a, Cb, c\}, z\}$$

*for all $a, b, c, x, z \in V$.*

*Proof.* The Jordan triple identity can be written as

$$\{x, \{b, a, y\}, z\} = -\{a, b, \{x, y, z\}\} + \{\{a, b, x\}, y, z\} + \{x, y, \{a, b, z\}\}$$

for $a, b, x, y, z \in V$. Now replace $a$ by $Ca$, we have

$$\begin{aligned} & \{x, \{b, Ca, y\}, z\} \\ = \ & -\{Ca, b, \{x, y, z\}\} + \{\{Ca, b, x\}, y, z\} + \{x, y, \{Ca, b, z\}\} \\ = \ & C(-\{a, b, \{x, y, z\}\} + \{\{a, b, x\}, y, z\} + \{x, y, \{a, b, z\}\}) \\ = \ & C\{x, \{b, a, y\}, z\}. \end{aligned}$$

□

**Lemma 4.1.16.** *For any Jordan triple system $V$, the following statements are equivalent.*

(i) *$V$ is abelian.*

(ii) *$a \square b \in \mathcal{Z}(V)$ for all $a, b \in V$.*

*Proof.* (i) $\Rightarrow$ (ii). Let $a, b \in V$. Since $V$ is abelian, for $x, y, z \in V$, we have

$$\begin{aligned} (a \square b)\{x, y, z\} &= \{a, b, \{x, y, z\}\} \\ &= \{\{a, b, x\}, y, z\} \\ &= \{(a \square b)x, y, z\} \end{aligned}$$

and

$$\begin{aligned}(a \square b)\{x, y, z\} &= (a \square b)\{z, y, x\} \\ &= \{(a \square b)z, y, x\} \\ &= \{x, y, (a \square b)z\}\end{aligned}$$

which imply $a \square b \in \mathcal{Z}(V)$ for all $a, b \in V$.

(ii) $\Rightarrow$ (i). For all $a, b, x, y, z \in V$, we get

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\}$$

since $a \square b$ is a centralizer. $\square$

**Definition 4.1.17.** A *triple isomorphism* between Jordan triple systems $(V, \{\cdot, \cdot, \cdot\})$ and $(W, \{\cdot, \cdot, \cdot\}')$ is a linear bijection $\tau : V \longrightarrow W$ satisfying

$$\tau\{x, y, z\} = \{\tau x, \tau y, \tau z\}'$$

for all $x, y, z \in V$. A triple isomorphism from $V$ onto itself is called a *triple automorphism.*

## 4.2 Inner derivations on Jordan triples

To discuss analysis in Jordan triples, we need to consider their topological structures and continuity of various mappings on them. It has been shown in [9] that a normed and non-degenerate Jordan triple $V$ admits continuous inner derivations if and only if the symmetric part $\mathfrak{s}(V)$ of its TKK Lie algebra $\mathfrak{g}(V)$ is a normed Lie algebra. In this section, we improve this result by removing the non-degenerate condition on $V$.

Let $V$ be a Jordan triple system equipped with a norm and let $\mathcal{B}(V)$ be the normed space of continuous linear operators on $V$. We call $V$ (topologically) *simple* if $\{V, V, V\} \neq 0$ and, $\{0\}$ and $V$ are the only closed triple ideals of $V$.

**Definition 4.2.1.** A *normed Lie algebra* is a Lie algebra $\mathfrak{g}$ which is a normed linear space such that the Lie product is continuous, that is, there is some $C > 0$ such that

$$\|[X,Y]\| \leq C\|X\|\|Y\| \qquad (X, Y \in \mathfrak{g}).$$

Let $V$ be a Jordan triple system and

$$\mathfrak{g}(V) = V_{-1} \bigoplus V_0 \bigoplus V_1$$

the corresponding TKK Lie algebra with involution $\theta$, as constructed in Theorem 4.1.6 where $V_0 = \mathrm{Ind}(V,V)$. The *symmetric part* of $\mathfrak{g}(V)$ is defined to be

$$\mathfrak{s}(V) = \left\{(x, (h^+, h^-), -x) : \;\; x \in V,\, \theta(h^+, h^-) = (h^+, h^-)\right\}.$$

The condition $\theta(h^+, h^-) = (h^+, h^-)$ implies that $h^+ = h^-$, by (4.2). In particular, if $h^+ = \sum_{i=1}^n a_i \,\square\, b_i$, then $h^+ = h^- = -\sum_{i=1}^n b_i \,\square\, a_i$ implies $2h^+ = \sum_{i=1}^n (a_i \,\square\, b_i - b_i \,\square\, a_i)$. Hence we can write $\mathfrak{s}(V)$ as follows

$$\mathfrak{s}(V) = \left\{(x, (h,h), -x) : \, h = \sum_{i=1}^n (a_i \,\square\, b_i - b_i \,\square\, a_i),\; x, a_i, b_i \in V\right\}.$$

A straightforward computation shows that $\mathfrak{s}(V)$ is a Lie subalgebra of $\mathfrak{g}(V)$. Note that the restriction of $\theta$ to $\mathfrak{s}(V)$, also denoted by $\theta$, is an involution on $\mathfrak{s}(V)$.

Analogous to Hilbert ternary algebras, an *inner derivation* of a Jordan triple system $V$ is a linear map $d(x,y) : V \longrightarrow V$ of the form

$$d(x,y) = x \,\square\, y - y \,\square\, x \qquad (x, y \in V).$$

For each $(0, (h,h), 0) \in \mathfrak{s}(V)$, we can write $h = \sum_{i=1}^n d(a_i, b_i)$ in terms of the inner derivations.

We show that the symmetric part $(\mathfrak{s}(V), \theta)$ is a normed Lie algebra if and only if $V$ has continuous inner derivations. The following result is proved in [9] for non-degenerate Jordan triple systems. However, the proof also applies to the case without the assumption of non-degeneracy.

**Proposition 4.2.2.** *Let $V$ be a Jordan triple system and $\mathfrak{g}(V) = V_{-1} \bigoplus V_0 \bigoplus V_1$ its TKK Lie algebra in Theorem 4.1.6, with symmetric part $\mathfrak{s}(V)$. The following conditions are equivalent.*

(i) *$\mathfrak{s}(V)$ is a normed Lie algebra.*

(ii) *$V$ is a normed linear space such that the bilinear map $d : (x, y) \in V \times V \longmapsto d(x, y) \in \mathcal{B}(V)$ is well-defined and continuous.*

*Proof.* (i) $\Rightarrow$ (ii). Let $(\mathfrak{s}(V), \|\cdot\|_{\mathfrak{s}(V)})$ be a normed Lie algebra such that $\|[X, Y]\|_{\mathfrak{s}(V)} \leq C\|X\|_{\mathfrak{s}(V)} \|Y\|_{\mathfrak{s}(V)}$ for some $C > 0$ and $X, Y \in \mathfrak{s}(V)$. We define a norm on $V$ by

$$2\,\|x\| = \|(x, 0, -x)\|_{\mathfrak{s}(V)}$$

for all $x \in V$. We first show that the inner derivation $d(x, y) = x \square y - y \square x : V \longrightarrow V$ is $\|\cdot\|$-continuous. For all $x, y, z \in V$, we have

$$\begin{aligned}
2\,\|d(x, y)(z)\| &= \|(d(x, y)(z), 0, -d(x, y)(z))\|_{\mathfrak{s}(V)} \\
&= \|[(0, (d(x, y), d(x, y)), 0), (z, 0, -z)]\|_{\mathfrak{s}(V)} \\
&= \|[[(y, 0, -y), (x, 0, -x)], (z, 0, -z)]\|_{\mathfrak{s}(V)} \\
&\leq C\,\|[(y, 0, -y), (x, 0, -x)]\|_{\mathfrak{s}(V)}\,\|(z, 0, -z)\|_{\mathfrak{s}(V)} \\
&\leq C^2\,\|(y, 0, -y)\|_{\mathfrak{s}(V)}\,\|(x, 0, -x)\|_{\mathfrak{s}(V)}\,\|(z, 0, -z)]\|_{\mathfrak{s}(V)} \\
&= 8C^2\,\|y\|\|x\|\|z\|.
\end{aligned}$$

This proves that $d(x, y)$ is continuous on $V$ and moreover, $d : V \times V \longrightarrow \mathcal{B}(V)$ is well-defined and continuous.

(ii) $\Rightarrow$ (i). Let $V$ be equipped with a norm $\|\cdot\|_V$ such that the inner derivation map $d : V \times V \longrightarrow \mathcal{B}(V)$ is continuous and let $C > 0$ be such that

$$\|d(x, y)\| \leq C\|x\|_V\|y\|_V \qquad (x, y \in V).$$

Then each $h \in V \square V$ with

$$h = \sum_{i=1}^{n} d(a_i, b_i) : V \longrightarrow V$$

is a continuous linear map with norm $\|h\| \leq \sum_{i=1}^{n} \|d(a_i, b_i)\|$. We define a norm on the Lie algebra $\mathfrak{s}(V)$ by

$$\|(x, (h, h), -x)\|_{\mathfrak{s}(V)} = 2\,\|x\|_V + \sqrt{2}\,\|h\|$$

for all $(x, (h, h), -x) \in \mathfrak{s}(V)$. Let $x \oplus (h, h) \oplus -x$ and $y \oplus (g, g) \oplus -y$ be in $\mathfrak{s}(V)$. Then we have

$$\begin{aligned}
& \|[x \oplus (h,h) \oplus -x, y \oplus (g,g) \oplus -y]\|_{\mathfrak{s}(V)} \\
= \;& \|(h(y) - g(x), ([h,g],[h,g]) + d_{x(-y)} - d_{y(-x)}, h(-y) - g(-x))\|_{\mathfrak{s}(V)} \\
= \;& \|(h(y) - g(x), ([h,g] + y \square x - x \square y, [h,g] + y \square x - x \square y), -(h(y) - g(x)))\|_{\mathfrak{s}(V)} \\
= \;& 2\,\|h(y) - g(x)\|_V + \sqrt{2}\,\|hg - gh + y \square x - x \square y\| \\
\leq \;& 2\,\|h\|\|y\|_V + 2\,\|g\|\|x\|_V + 2\sqrt{2}\,\|h\|\|g\| + C\sqrt{2}\,\|x\|_V\|y\|_V \\
\leq \;& (2 + C)\,\|x \oplus (h,h) \oplus -x\|_{\mathfrak{s}(V)}\|y \oplus (g,g) \oplus -y\|_{\mathfrak{s}(V)}.
\end{aligned}$$

Therefore $\mathfrak{s}(V)$ is a normed Lie algebra. $\square$

**Remark 4.2.3.** In the sequel, if a Jordan triple system $V$ is equipped with a norm $\|\cdot\|$, then we will always equip the Lie algebra $\mathfrak{s}(V)$ with the norm $\|\cdot\|_{\mathfrak{s}(V)}$ defined above.

**Example 4.2.4.** Let $V$ be the Jordan triple system

$$V = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset M_2(\mathbb{R})$$

with the triple product defined by $\{A, B, C\} = \frac{1}{2}ABC + \frac{1}{2}CBA$ and the inner product $\langle A, B\rangle = \mathrm{Trace(A^*B)}$ for $A, B, C \in V$. Then $V$ is degenerate since, for example,

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = 0$$

for any $x, y \in V$. Let

$$\mathfrak{g}(V) = V \bigoplus \mathrm{Ind}(V, V) \bigoplus V$$

be the canonical TKK Lie algebra of $V$ with involution $\theta$ given by

$$\theta\left(\sum_{i=1}^{n} A_i \square B_i\right) = -\sum_{i=1}^{n} B_i \square A_i$$

for all $A_i, B_i \in V$.

Direct computation shows that the box operator $A \square B : V \longrightarrow V$ has matrix representation

$$A \square B = \frac{1}{2}\begin{pmatrix} a_2 b_2 & a_1 b_2 \\ 0 & 2a_2 b_2 \end{pmatrix} \tag{4.4}$$

with respect to the basis $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, where $A = \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix}$. Let

$$\mathfrak{s}(V) = \left\{ (X, (h^+, h^-), -X) : \theta(h^+, h^-) = (h^+, h^-) \in \mathrm{Ind}(V, V),\ X \in V \right\}$$

be the symmetric part of $\mathfrak{g}(V)$. It follows from (4.4) that $A \square B = -B \square A = 0$ for all $A, B \in V$. Hence

$$\mathfrak{s}(V) = \{(X, (0, 0), -X) : X \in V\}.$$

**Example 4.2.5.** Consider $\mathbb{R}$ as a (non-degenerate) Jordan triple system with the inner product $\langle x, y \rangle = xy$ and the triple product $\{x, y, z\} = xyz$ for $x, y, z \in \mathbb{R}$. By Remark 4.1.10, the TKK Lie algebra $(\mathfrak{g}(\mathbb{R}), \theta)$ is given by

$$\mathfrak{g}(\mathbb{R}) = \mathbb{R} \bigoplus \mathbb{R} \square \mathbb{R} \bigoplus \mathbb{R}$$

$$\theta(a \oplus b \oplus c) = c \oplus -b \oplus a.$$

Each box operator $a \,\square\, b : \mathbb{R} \longrightarrow \mathbb{R}$ is represented by the $1 \times 1$ matrix $(ab)$ and we can identify $\mathbb{R} \,\square\, \mathbb{R}$ with $\mathbb{R}$. The TKK Lie product, as in (4.1), is given by

$$[(a, b, c), (x, y, z)] = (bx - ay, az - cx, cy - bz)$$

for $a, b, c, x, y, z \in \mathbb{R}$ (cf. [9]). In fact, $\mathfrak{g}(\mathbb{R})$ is isomorphic to the cross product Lie algebra $\mathbb{R}^3$ via the Lie isomorphism

$$\varphi : (a, b, c) \in \mathfrak{g}(\mathbb{R}) \longmapsto (c, b, a) \in \mathbb{R}^3.$$

The TKK Lie algebra $\mathfrak{g}(\mathbb{R})$ is canonical. The symmetric part of $\mathfrak{g}(\mathbb{R})$ is given by

$$\mathfrak{s}(\mathbb{R}) = \{(a, 0, -a) : \, a \in \mathbb{R}\}.$$

**Example 4.2.6.** Let $V$ be the Jordan triple system

$$V = \left\{ \begin{pmatrix} y & x \\ x & -y \end{pmatrix} : \, x, y \in \mathbb{R} \right\} \subset M_2(\mathbb{R})$$

with the triple product defined by $\{A, B, C\} = \frac{1}{2}ABC + \frac{1}{2}CBA$ for $A, B, C \in V$. Then $V$ is non-degenerate. By Theorem 4.1.6 and Remark 4.1.10, $\mathfrak{g}(V) = V \bigoplus V \,\square\, V \bigoplus V$ is the TKK Lie algebra of $V$ with involution $\theta$ given by

$$\theta\left(\sum_{i=1}^{n} A_i \,\square\, B_i\right) = -\sum_{i=1}^{n} B_i \,\square\, A_i$$

for all $A_i, B_i \in V$. Direct computation shows that the box operator $A \,\square\, B : V \longrightarrow V$ is exactly the left multiplication by the matrix $AB$, that is,

$$(A \,\square\, B)(X) = ABX \qquad (X \in V).$$

Consider $V$ as a real Hilbert space with the inner product

$$\langle A, B \rangle = \mathrm{Trace}(A^* B) \qquad (A, B \in V)$$

and basis $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.

We can embed $V$ into $\mathcal{B}(V)$ by $v = (a_1, a_2) \mapsto \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(V)$ where $v$ is a linear combination of the above basis elements with coefficients $a_1, a_2 \in \mathbb{R}$, and $\mathcal{B}(V)$ identifies with the space $M_2(\mathbb{R})$ of $2 \times 2$ real matrices.

With respect to the above basis the box operator $A \square B : V \longrightarrow V$ can be represented by the matrix

$$\begin{pmatrix} a_2b_2 + a_1b_1 & a_2b_1 - a_1b_2 \\ a_1b_2 - a_2b_1 & a_1b_1 + a_2b_2 \end{pmatrix} \tag{4.5}$$

where $A = \begin{pmatrix} a_2 & a_1 \\ a_1 & -a_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_2 & b_1 \\ b_1 & -b_2 \end{pmatrix}$. We have the embedding

$$\mathfrak{g}(V) = V \bigoplus V \square V \bigoplus V \hookrightarrow \mathcal{B}(V) \bigoplus \mathcal{B}(V) \bigoplus \mathcal{B}(V).$$

Let

$$\mathfrak{s}(V) = \{(X, Z, -X) : \theta Z = Z \in V \square V,\ X \in V\}$$

be the symmetric part of $\mathfrak{g}(V)$. Let $Z = A \square B \in V \square V$. Then $\theta Z = Z$ implies $-B \square A = A \square B$. It follows from (4.5) that $Z = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$ for some $z \in \mathbb{R}$. Hence every element in $\mathfrak{s}(V)$ has the form

$$\begin{pmatrix} \begin{matrix} y & x \\ x & -y \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} 0 & z \\ -z & 0 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} -y & -x \\ -x & y \end{matrix} \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$. Note that $\mathfrak{g}(V)$ is not canonical since

$$\left[\begin{pmatrix} y & x \\ x & -y \end{pmatrix}, \begin{pmatrix} b & a \\ a & -b \end{pmatrix}\right] = \begin{pmatrix} 0 & 2(ay-bx) \\ 2(bx-ay) & 0 \end{pmatrix}$$

implies $[V, V] \neq V \,\square\, V = \mathfrak{g}_0$ in $\mathfrak{g}(V)$.

By [9, Theorem 4.7], the symmetric parts $\mathfrak{s}(V)$ in the above three examples correspond to the Lie algebras of Riemannian symmetric manifolds.

## 4.3 Jordan Hilbert triples

Jordan triples which are Hilbert spaces are called Jordan Hilbert triples. They occur in the study of infinite dimensional manifolds (see [9]). In this section, we study the structures of a subclass of Jordan Hilbert triples, namely, the $JH$-triples introduced in [9].

**Proposition 4.3.1.** *Let $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be a Jordan Hilbert triple which satisfies*

$$\langle \{a, b, x\}, x\rangle = \langle x, \{b, a, x\}\rangle \tag{4.6}$$

*for all $a, b, x \in V$. Then $(V, \langle\cdot,\cdot\rangle)$ is a Hilbert ternary algebra with the ternary product defined by*

$$[a, b, x]_d = d(a, b)x \qquad (a, b, x \in V)$$

*where $d(a, b) = a \,\square\, b - b \,\square\, a$ is an inner derivation.*

*Proof.* We first observe that condition (4.6) is equivalent to

$$\langle d(a, b)x, y\rangle = \langle x, d(b, a)y\rangle \quad (a, b, x, y \in V). \tag{4.7}$$

Indeed, given (4.6), the following identity

$$\langle (a \,\square\, b)(x + y), x + y\rangle = \langle x + y, (b \,\square\, a)(x + y)\rangle$$

implies $\langle (a\square b)x, y\rangle + \langle (a\square b)y, x\rangle = \langle x, (b\square a)y\rangle + \langle y, (b\square a)x\rangle$ and hence

$$\langle (a\square b)x, y\rangle - \langle (b\square a)x, y\rangle = \langle x, (b\square a)y\rangle - \langle x, (a\square b)y\rangle.$$

Conversely, the above identity clearly implies (4.6) and $\langle d(a,b)x, x\rangle = 0$ for all $a, b, x \in V$.

By the above remark, we already have

$$\langle [a,b,x]_d, y\rangle = \langle x, [b,a,y]_d\rangle$$

for all $a, b, x, y \in V$. Using the identity

$$d(a,b)x = d(a,x)b + d(x,b)a,$$

we have

$$\begin{aligned}
\langle x, [b,a,y]_d\rangle &= \langle x, d(b,a)y\rangle \\
&= \langle x, d(y,a)b + d(b,y)a\rangle \\
&= \langle d(a,y)x, b\rangle + \langle d(y,b)x, a\rangle \\
&= \langle d(a,x)y, b\rangle + \langle d(x,y)a, b\rangle + \langle d(x,b)y, a\rangle + \langle d(y,x)b, a\rangle \\
&= 2\langle a, d(y,x)b\rangle + \langle y, d(x,a)b\rangle + \langle y, d(b,x)a\rangle \\
&= 2\langle a, d(y,x)b\rangle + \langle y, d(b,a)x\rangle \\
&= 2\langle a, d(y,x)b\rangle - \langle d(b,a)y, x\rangle \\
&= 2\langle a, d(y,x)b\rangle - \langle x, d(b,a)y\rangle
\end{aligned}$$

which gives $\langle x, [b,a,y]_d\rangle = \langle a, [y,x,b]_d\rangle$. □

The ternary product $[\cdot,\cdot,\cdot]_d$ in Proposition 4.3.1 has been introduced in [8] to study the structure of a $JH$-triple $(V, \{\cdot,\cdot,\cdot\})$, we call $[\cdot,\cdot,\cdot]_d$ the *derived ternary product* of $(V, \{\cdot,\cdot,\cdot\})$, and call $(V, \langle\cdot,\cdot\rangle, [\cdot,\cdot,\cdot]_d)$ the *derived Hilbert ternary algebra* of $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$.

**Corollary 4.3.2.** *Let $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be the Jordan Hilbert triple in Proposition 4.3.1. Then every inner derivation on $V$ is continuous.*

*Proof.* By Proposition 4.3.1, $(V, \langle\cdot,\cdot\rangle, [\cdot,\cdot,\cdot]_d)$, with the derived ternary product $[\cdot,\cdot,\cdot]_d$, is a Hilbert ternary algebra. Then the inner derivation $d(a,b)$ is the left multiplication operator $d(a,b)x = [a,b,x]_d$ on this Hilbert ternary algebra and by Proposition 3.1.10, it is continuous. □

Given the Jordan Hilbert triple $V$ in Proposition 4.3.1 and Corollary 4.3.2, the continuity of the derivations $d(a,b)$ is not sufficient for the symmetric part $\mathfrak{s}(V)$ to be a *normed* Lie algebra. For this, as shown in Proposition 4.2.2, we need continuity of the map $(a,b) \in V \times V \longmapsto d(a,b) \in \mathcal{B}(V)$. This motivates the following definition.

**Definition 4.3.3.** A Jordan Hilbert triple $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ is called a *JH-triple* if its triple product $\{\cdot,\cdot,\cdot\}$ is continuous and the inner product $\langle\cdot,\cdot\rangle$ satisfies condition (4.6) above.

This definition is more general than the one given in [9] where non-degeneracy is assumed in the definition of a $JH$-triple.

Let $V$ be a $JH$-triple and $\mathfrak{g}(V) = V \bigoplus \mathrm{Ind}(V,V) \bigoplus V$ be the corresponding TKK Lie algebra with involution $\theta$. The symmetric part is given by

$$\begin{aligned}
&\mathfrak{s}(V)\\
= &\left\{(x,(h,h),-x) : h = \sum_{i=1}^{n} d(a_i,b_i),\ x, a_i, b_i \in V\right\}\\
= &\left\{(0,(h,h),0) : h = \sum_{i=1}^{n} d(a_i,b_i),\ a_i, b_i \in V\right\} \bigoplus \{(x,0,-x) : x \in V\}\\
= &\ \mathfrak{k} \oplus \mathfrak{p}
\end{aligned}$$

where $\mathfrak{k}$ is the 1-eigenspace and $\mathfrak{p}$ is the $(-1)$-eigenspace of the involution $\theta$. Since we have

$$\|d(a,b)\| \leq C\,\|a\|\,\|b\| \qquad (a,b \in V)$$

for some constant $C > 0$, $(\mathfrak{s}(V), \|\cdot\|_{\mathfrak{s}(V)})$ is a normed Lie algebra by Proposition 4.2.2. By the definition of the norm $\|\cdot\|_{\mathfrak{s}(V)}$ on $\mathfrak{s}(V)$, the inherited norm on $\mathfrak{p}$ is given by

$$\|(x, 0, -x)\| = 2\,\|x\|_V.$$

Since $V$ is a Hilbert space, $(\mathfrak{p}, \|\cdot\|)$ is complete. We have

$$\mathfrak{k} \subset \operatorname{Ind}(V,V) \subset (V \,\square\, V)^2 \subset \mathcal{B}(V)^2.$$

Let $\bar{\mathfrak{k}}$ be the closure of $\mathfrak{k}$ in $\mathcal{B}(V)^2$ where $\mathcal{B}(V)^2$ is equipped with the norm $\|(h,k)\| = \max\{\|h\|, \|k\|\}$. Then the direct sum

$$\bar{\mathfrak{s}}(V) = \bar{\mathfrak{k}} \oplus \mathfrak{p}$$

with the obvious norm (cf. Remark 4.2.3) is called the *completion* of $\mathfrak{s}(V)$. Since the Lie product is continuous on $\mathfrak{s}(V)$, it can be extended to a Lie product on $\bar{\mathfrak{s}}(V)$.

The following result characterises $JH$-triples in terms of their TKK Lie algebras. It is an improvement of the result in [9, Lemma 3.7] from which the non-degenerate assumption is removed.

**Theorem 4.3.4.** *Let $V$ be a normed Jordan triple in which the triple product is continuous. Let $\mathfrak{s}(V)$ be the symmetric part of the TKK Lie algebra of $V$ with eigenspace decomposition $\mathfrak{s}(V) = \mathfrak{k} \oplus \mathfrak{p}$. Then the following conditions are equivalent.*

(i) *$V$ is a $JH$-triple.*

(ii) *$\mathfrak{s}(V) = \mathfrak{k} \oplus \mathfrak{p}$ is a normed Lie algebra such that $\mathfrak{p}$ is a Hilbert space in the inherited norm and the completion $\bar{\mathfrak{s}}(V) = \bar{\mathfrak{k}} \oplus \mathfrak{p}$ is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ of $\mathfrak{p}$.*

*Proof.* (i) $\Rightarrow$ (ii). By the above remark, the symmetric part $\mathfrak{s}(V) = \mathfrak{k} \oplus \mathfrak{p}$ is a normed Lie algebra and $\mathfrak{p} = \{(x, 0, -x) \in \mathfrak{s}(V) : x \in V\}$ is complete in the norm

$$\|(x, 0, -x)\|_{\mathfrak{s}(V)} = 2\,\|x\|_V \qquad (x \in V)$$

where $\|\cdot\|_V$ is the Hilbert space norm of $V$. Let $\langle \cdot, \cdot \rangle_V$ be the inner product of $V$. It induces an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{p}} : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathbb{R}$ defined by

$$\langle (x, 0, -x), (y, 0, -y) \rangle_{\mathfrak{p}} = 4\,\langle x, y \rangle_V$$

and $(\mathfrak{p}, \langle \cdot, \cdot \rangle_{\mathfrak{p}})$ is a Hilbert space. For

$$Z = (0, (h, h), 0) \in \mathfrak{k} = \left\{ (0, (h, h), 0) : \ h = \sum_{i=1}^{n} d(a_i, b_i), \ a_i, b_i \in V \right\},$$

we have

$$\begin{aligned} \langle [Z, (u, 0, -u)], (u, 0, -u) \rangle_{\mathfrak{p}} &= \langle (h(u), 0, -h(u)), (u, 0, -u) \rangle_{\mathfrak{p}} \\ &= \langle h(u), u \rangle_V = 0 \end{aligned} \tag{4.8}$$

where for $h = \sum_{i=1}^{n} d(a_i, b_i)$ with $a_i, b_i \in V$, we have

$$\begin{aligned} \langle h(u), u \rangle &= \sum_{i=1}^{n} \langle d(a_i, b_i) u, u \rangle_V \\ &= \sum_{i=1}^{n} (\langle (a_i \square b_i) u, u \rangle_V - \langle (b_i \square a_i) u, u \rangle_V) \\ &= \sum_{i=1}^{n} (\langle (a_i \square b_i) u, u \rangle_V - \langle u, (a_i \square b_i) u \rangle_V) \\ &= 0. \end{aligned}$$

It follows that, for $Z \in \overline{\mathfrak{k}} \subset \mathcal{B}(V) \times \mathcal{B}(V)$, we also have

$$\langle [Z, (u, 0, -u)], (u, 0, -u) \rangle_{\mathfrak{p}} = 0$$

by (4.8) and continuity of the triple product. Therefore $(\overline{\mathfrak{s}}(V), \theta)$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$.

(ii) $\Rightarrow$ (i). Let $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ be the inner product on $\mathfrak{p}$ such that $\langle [Z, X], X \rangle_{\mathfrak{p}} = 0$ for all $Z \in \overline{\mathfrak{k}}$ and $X \in \mathfrak{p}$. Define an inner product on $V$ by

$$\langle x, y \rangle := \langle (x, 0, -x), (y, 0, -y) \rangle_{\mathfrak{p}}.$$

Then $(V, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Let $h = d(a, b)$ with $a, b \in V$. Then $(0, (h, h), 0) \in \overline{\mathfrak{k}}$ and for any $x \in V$, we have $\langle h(x), x \rangle = 0$ since

$$\begin{aligned} 0 &= \langle [(0, (h, h), 0), (x, 0, -x)], (x, 0, -x) \rangle_{\mathfrak{p}} \\ &= \langle (h(x), 0, -h(x)), (x, 0, -x) \rangle_{\mathfrak{p}} \\ &= \langle h(x), x \rangle. \end{aligned}$$

Hence $\langle (a \square b)x, x \rangle - \langle (b \square a)x, x \rangle = 0$, that is,

$$\langle (a \square b)x, x \rangle = \langle x, (b \square a)x \rangle \qquad (\forall a, b, x \in V).$$

Therefore $(V, \langle \cdot, \cdot \rangle)$ is a $JH$-triple. $\square$

For a $JH$-triple $V$, we define

$$Z(V) = \{a \in V : a \square b = b \square a, \forall b \in V\}$$

which is a closed subspace of $V$. Given $a \in Z(V)$, we have

$$\{a, b, x\} = \{a, x, b\}$$

for all $b, x \in V$. In fact, $\{a, b, x\} = \{b, a, x\} = \{x, a, b\} = \{a, x, b\}$.

**Lemma 4.3.5.** *If $(V, \langle \cdot, \cdot \rangle, \{\cdot, \cdot, \cdot\})$ is a $JH$-triple and $[\cdot, \cdot, \cdot]_d$ the derived ternary product on $V$, then $Z(V)$ is exactly the annihilator of the Hilbert ternary algebra $(V, \langle \cdot, \cdot \rangle, [\cdot, \cdot, \cdot]_d)$.*

*Proof.* Let $a \in Z(V)$. Then $a \square b = b \square a$ for any $b \in V$ and we have

$$[a, b, x]_d = d(a, b)x = (a \square b - b \square a)x = 0$$

for all $b, x \in V$. The converse can be proved easily. $\square$

We call a linear operator $D : V \longrightarrow V$ on a $JH$-triple $V$ a *derivation* if it satisfies

$$D\{x, y, z\} = \{Dx, y, z\} + \{x, Dy, z\} + \{x, y, Dz\}$$

for all $x, y, z \in V$.

We give below an interesting application of the derived Hilbert ternary structure in a $JH$-triple.

**Proposition 4.3.6.** *Every derivation on a $JH$-triple $V$ with $Z(V) = \{0\}$ is continuous.*

*Proof.* Let $D$ be a derivation on a $JH$-triple $V$ with inner product $\langle \cdot, \cdot \rangle$ and triple product $\{\cdot, \cdot, \cdot\}$. Let $[\cdot, \cdot, \cdot]_d$ be the derived ternary product on $V$. We have, for all $x, y, z \in V$,

$$\begin{aligned}
D[x, y, z]_d &= D(d(x, y)z) \\
&= D\{x, y, z\} - D\{y, x, z\} \\
&= \{Dx, y, z\} + \{x, Dy, z\} + \{x, y, Dz\} \\
&\quad -\{Dy, x, z\} - \{y, Dx, z\} - \{y, x, Dz\} \\
&= d(Dx, y)z + d(x, Dy)z + d(x, y)Dz \\
&= [Dx, y, z]_d + [x, Dy, z]_d + [x, y, Dz]_d
\end{aligned}$$

which shows that $D$ is a derivation on the Hilbert ternary algebra $(V, [\cdot, \cdot, \cdot]_d)$. By Lemma 4.3.5, the Hilbert ternary algebra $(V, [\cdot, \cdot, \cdot]_d)$ has zero annihilator. Hence $D$ is continuous on $V$ by Theorem 3.3.14. $\square$

**Definition 4.3.7.** We define a *$JH^*$-triple* to be a Hermitian Jordan triple $V$ which is also a complex Hilbert space and the inner product $\langle \cdot, \cdot \rangle$ satisfies

$$\langle (a \square b)x, x \rangle = \langle x, (b \square a)x \rangle \tag{4.9}$$

for all $a, b, x \in V$.

By restricting to the real scalars and real part of the inner product, a $JH^*$-triple can be regarded as a (real) $JH$-triple.

We say that the inner product in a Jordan Hilbert triple $V$ is *associative* if

$$\langle \{a, b, x\}, y\rangle = \langle x, \{b, a, y\}\rangle$$

holds for all $a, b, x, y \in V$.

For a *complex* Jordan Hilbert triple $V$, associativity of the inner product is equivalent to condition (4.9). Indeed, assume (4.9), then for $a, b \in V$, we have

$$\langle (a \square b)(x + y), x + y\rangle = \langle x + y, (b \square a)(x + y)\rangle$$

$$\langle (a \square b)(x + iy), x + iy\rangle = \langle x + iy, (b \square a)(x + iy)\rangle$$

for all $x, y \in V$, which imply

$$\begin{aligned} \langle (a \square b)x, y\rangle + \langle (a \square b)y, x\rangle &= \langle x, (b \square a)y\rangle + \langle y, (b \square a)x\rangle \\ -\langle (a \square b)x, y\rangle + \langle (a \square b)y, x\rangle &= -\langle x, (b \square a)y\rangle + \langle y, (b \square a)x\rangle. \end{aligned}$$

Therefore $\langle (a \square b)x, y\rangle = \langle x, (b \square a)y\rangle$ for all $a, b, x, y \in V$.

Hence a $JH^*$-triple always has an associative inner product. However, we will give examples to show that the inner product in a $JH$-triple need not be associative.

A *real $JH^*$-triple* is defined to be a (real) $JH$-triple with an associative inner product.

We note that a $JH$-triple $(V, \langle \cdot, \cdot\rangle, \{\cdot, \cdot, \cdot\})$ with associative inner product is a real Hilbert ternary algebra (cf. Definition 3.1.1). Conversely, an abelian Hilbert ternary algebra $(V, [\cdot, \cdot, \cdot])$ is a $JH$-triple if the ternary product satisfies

$$[x, y, z] = [z, y, x] \qquad (x, y, z \in V).$$

**Remark 4.3.8.** Given an element $a$ in a $JH$-triple $(V, \{\cdot, \cdot, \cdot\})$, we have $[a, a, x]_d = d(a, a)x = 0$ for all $x \in V$. Hence the derived Hilbert ternary algebra $(V, [\cdot, \cdot, \cdot]_d)$

contains no nonzero tripotent. Therefore the derived Hilbert ternary algebra $(V, [\cdot,\cdot,\cdot]_d)$ cannot be an abelian Hilbert ternary algebra by Corollary 3.4.4.

**Lemma 4.3.9.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a JH-triple and let $(V, [\cdot,\cdot,\cdot]_d)$ be the derived Hilbert ternary algebra defined in Proposition 4.3.1. The following are equivalent.*

(i) $[x, y, z]_d = [z, y, x]_d$ *for all* $x, y, z \in V$.

(ii) $(V, [\cdot,\cdot,\cdot]_d)$ *is a real* $JH^*$*-triple.*

(iii) $(V, \{\cdot,\cdot,\cdot\})$ *is flat.*

(iv) $(V, [\cdot,\cdot,\cdot]_d)$ *is a degenerate JH-triple or* $V = \{0\}$.

*Proof.* (i) $\Rightarrow$ (ii). Applying the Jordan triple identity for the triple product $\{\cdot,\cdot,\cdot\}$, we have

$$\begin{aligned}
[a, b, [x, y, z]_d]_d &= \{a, b, \{x, y, z\}\} - \{a, b, \{y, x, z\}\} \\
&\quad - \{b, a, \{x, y, z\}\} + \{b, a, \{y, x, z\}\} \\
&= \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\} \\
&\quad - \{\{a, b, y\}, x, z\} + \{y, \{b, a, x\}, z\} - \{y, x, \{a, b, z\}\} \\
&\quad - \{\{b, a, x\}, y, z\} + \{x, \{a, b, y\}, z\} - \{x, y, \{b, a, z\}\} \\
&\quad + \{\{b, a, y\}, x, z\} - \{y, \{a, b, x\}, z\} + \{y, x, \{b, a, z\}\} \\
&= \{[a, b, x]_d, y, z\} - \{x, [b, a, y]_d, z\} + \{x, y, [a, b, z]_d\} \\
&\quad + \{[b, a, y]_d, x, z\} - \{y, [a, b, x]_d, z\} - \{y, x, [a, b, z]_d\} \\
&= [[a, b, x]_d, y, z]_d - [x, [b, a, y]_d, z]_d + [x, y, [a, b, z]_d]_d
\end{aligned}$$

for all $a, b, x, y, z \in V$. Hence $[\cdot,\cdot,\cdot]_d$ satisfies the Jordan triple identity and $(V, [\cdot,\cdot,\cdot]_d)$ is a Jordan triple. Since $(V, [\cdot,\cdot,\cdot]_d)$ is a Hilbert ternary algebra, the inner product in $V$ is associative with respect to $[\cdot,\cdot,\cdot]_d$. Therefore $(V, [\cdot,\cdot,\cdot]_d)$ is a real $JH^*$-triple.

(ii) $\Rightarrow$ (iii). We have $[x, y, z]_d = [z, y, x]_d$ for all $x, y, z \in V$, that is,

$$\{x, y, z\} - \{y, x, z\} = \{z, y, x\} - \{y, z, x\}$$

which implies $\{y, x, z\} = \{y, z, x\}$. Hence $z \square x = x \square z$ for all $x, z \in V$ and $V$ is flat.

(iii) $\Rightarrow$ (iv). We have $[a, b, x]_d = 0$ for all $a, b, x \in V$ since $(V, \{\cdot, \cdot, \cdot\})$ is flat.

(iv) $\Rightarrow$ (i). By the definition of a Jordan triple, $[x, y, z]_d = [z, y, x]_d$ for all $x, y, z \in V$. $\square$

**Lemma 4.3.10.** *Every one dimensional $JH$-triple has associative inner product and is flat.*

*Proof.* Let $V$ be a one dimensional $JH$-triple. We show

$$\langle (a \square b)x, y\rangle = \langle x, (b \square a)y\rangle$$

for all $a, b, x, y \in V$. Let $V = \mathbb{R}v$ and $v \neq 0$. Then $x = \alpha v$ and $y = \beta v$ for some real numbers $\alpha$, $\beta$. The above identity always holds since $\langle (a \square b)x, y\rangle = \alpha\beta\langle (a \square b)v, v\rangle = \alpha\beta\langle v, (b \square a)v\rangle = \langle x, (b \square a)y\rangle$.

Similar argument proves that $x \square y = y \square x$ for all $x, y \in V$. $\square$

Now we give examples of $JH$-triples of every dimension $\geq 2$, in which the inner product is not associative.

**Example 4.3.11.** Consider the complex numbers $\mathbb{C}$ with the real inner product $\langle x, y\rangle = \mathrm{Re}(\mathrm{x}\overline{\mathrm{y}})$ and triple product $\{x, y, z\} = xyz$. Then $\mathbb{C}$ is a flat $JH$-triple where the inner product is not associative. For instance, if $a = 2$, $b = y = i$ and $x = 3$, we have $\langle (a \square b)x, y\rangle = \langle 6i, i\rangle = 6 \neq -6 = \langle 3, -2\rangle = \langle x, (b \square a)y\rangle$.

Example 4.3.11 shows that flatness of a $JH$-triple does not always imply associativity of its inner product.

**Example 4.3.12.** Let $\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}$ with the usual inner product

$$\langle (a, b, c), (x, y, z) \rangle = ax + by + cz$$

and triple product defined by

$$\{(a, b, c), (x, y, z), (u, v, w)\} = ((ax-by)u-(ay+bx)v, (ax-by)v+(ay+bx)u, czw).$$

Then $\mathbb{R}^3$ is a flat $JH$-triple. In fact, if we consider $\mathbb{R}^3 = \mathbb{C} \bigoplus \mathbb{R}$ as a vector space direct sum, then the triple product defined above is the direct sum of the triple product $\{p, q, r\} = pqr$ in $\mathbb{C}$ and the usual triple product in $\mathbb{R}$. In this case, the above inner product is not associative.

**Example 4.3.13.** Let $\mathbb{H}$ be the real Hilbert space of quaternions with basis $\{\mathbf{1}, i, j, k\}$. One can define the canonical triple product on $H$ by

$$\{x, y, z\} \quad = \quad \frac{1}{2}xyz + \frac{1}{2}zyx.$$

Equip $\mathbb{H}$ with the usual inner product $\langle x, y \rangle = \mathrm{Re}(x\bar{y})$. Then for $x = x_0 + x_1 i + x_2 j + x_3\, k$, we have $\mathrm{Re}(xy\bar{x}) = \sum_{n=0}^{3} x_n \mathrm{Re}(y) x_n$. Using this, it can verified that

$$\langle \{a, b, x\}, x \rangle = \langle x, \{b, a, x\} \rangle$$

for $a, b, x \in \mathbb{H}$. Therefore $\mathbb{H}$ with the above canonical triple product is a $JH$-triple.

It is not flat and its inner product is not associative. Indeed, let $a = i$ and $b = j$. Then $(a \,\square\, b)(i) = j \neq -j = (b \,\square\, a)(i)$ and $\langle (2 \,\square\, k)(1 + 2k), 3 \rangle = -12 \neq 12 = \langle 1 + 2k, (k \,\square\, 2)3 \rangle$.

We show that $\mathbb{H}$ is simple. Let $I \neq \{0\}$ be a closed triple ideal of $\mathbb{H}$. Fix $0 \neq b \in I$. Then for any $x \in \mathbb{H}$, we have

$$\{x, \mathbb{H}, b\} + \{x, b, \mathbb{H}\} \subset I.$$

Hence $\frac{1}{2}xhb + \frac{1}{2}bhx + \frac{1}{2}xbh + \frac{1}{2}hbx \in I$ for any $h \in \mathbb{H}$. Let $h = \overline{b}$. We get $2\|b\|^2 x \in I$ which implies $x \in I$. Therefore $I = \mathbb{H}$ and $\mathbb{H}$ is simple.

**Example 4.3.14.** Let $\mathbb{H}$ be the $JH$-triple defined in Example 4.3.13. Then $\mathbb{H}^n = \{(a_1, \cdots, a_n) : a_i \in \mathbb{H}\}$ with the inner product

$$\langle (a_1, \cdots, a_n), (b_1, \cdots, b_n) \rangle = \mathrm{Re} \sum_{i=1}^{n} a_i \bar{b}_i$$

and the coordinatewise triple product

$$\{(a_1, \cdots, a_n), (b_1, \cdots, b_n), (c_1, \cdots, c_n)\} = (\{a_1, b_1, c_1\}, \cdots, \{a_n, b_n, c_n\})$$

is a $JH$-triple in which the inner product is not associative.

Given $JH$-triples $V_1$, $V_2$, $\cdots$, $V_n$, we define the $JH$-triple structure on the direct sum $\bigoplus_i V_i$ and the tensor product $\bigotimes_i V_i$ coordinatewise. Then $\bigoplus_i V_i$ is a real $JH^*$-triple if, and only if, each $V_i$ is such. The same holds for tensor products.

**Example 4.3.15.** Let $V = \bigoplus_{\alpha \in I} V_\alpha$ be a Hilbert space direct sum, where each $V_\alpha = \mathbb{H}$ for all $\alpha$ in an infinite set $I$. We note that the triple product on $\mathbb{H}$ is continuous. In fact, we have $\|xy\| = \|x\|\,\|y\|$ for all $x, y \in \mathbb{H}$ and $\|\{x, y, z\}\| \leq \|x\|\,\|y\|\,\|z\|$ where $x, y, z \in \mathbb{H}$. Hence the coordinatewise triple product

$$\{(x_\alpha)_{\alpha \in \mathrm{I}}, (y_\alpha)_{\alpha \in \mathrm{I}}, (z_\alpha)_{\alpha \in \mathrm{I}}\} = (\{x_\alpha, y_\alpha, z_\alpha\})_{\alpha \in \mathrm{I}}$$

on $V$ is well-defined since

$$\begin{aligned} \|\{(x_\alpha)_{\alpha \in I}, (y_\alpha)_{\alpha \in I}, (z_\alpha)_{\alpha \in I}\}\|^2 &= \sum_{\alpha \in \mathrm{I}} \|\{x_\alpha, y_\alpha, z_\alpha\}\|^2 \\ &\leq \sum_{\alpha \in \mathrm{I}} \|x_\alpha\|^2 \|y_\alpha\|^2 \|z_\alpha\|^2 \\ &< \infty. \end{aligned}$$

Then $V$ is an infinite dimensional $JH$-triple in which the inner product is not associative.

Let $V$ be a $JH$-triple and $a \in V$. The smallest closed subtriple $V(a)$ of $V$ containing $a$ is the norm closed real linear span of the odd powers $a^1 = a$, $a^3 =$

$\{a,a,a\}$, $a^5 = \{a^3, a, a\}$, $\cdots$. It is a $JH$-triple with the inherited inner and triple products. The $JH$-triple $V(a)$ is flat by power associativity of the triple product.

**Lemma 4.3.16.** *Let $V$ be a $JH$-triple and $a \in V$. Then the inner product of $V(a)$ is associative if and only if $a \,\square\, a : V(a) \longrightarrow V(a)$ is a symmetric operator.*

*Proof.* It suffices to prove the sufficiency. If the box operator $a \,\square\, a$ is symmetric on $V(a)$, then by power associativity of the triple product and induction, $a^n \,\square\, a$ is also symmetric for odd $n$. Power associativity also implies that

$$a^n \,\square\, a^m = a^{n+m+1} \,\square\, a$$

for odd $n$ and $m$. Hence $a^n \,\square\, a^m : V(a) \longrightarrow V(a)$ is symmetric.

Since $V(a)$ is the real closed linear span of odd powers of $a$, it follows from continuity of the triple product that

$$\langle (x \,\square\, y)z, w\rangle = \langle z, (y \,\square\, x)w\rangle$$

for all $x, y, z, w \in V(a)$. $\square$

**Example 4.3.17.** Let $M_2(\mathbb{R})$ be the Jordan Hilbert triple of $2 \times 2$ real matrices, with the usual triple product $\{A, B, C\} = \frac{1}{2}ABC + \frac{1}{2}CBA$ and inner product $\langle A, B\rangle = \text{Trace}(\text{AB}^*)$. Let $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$. Then the closed subtriple $V(a)$ generated by $a$ is a $JH$-triple. However, the inner product in $V(a)$ is not associative. We have $\langle (a \,\square\, a)a, a^3\rangle = 11 \neq 7 = \langle a, (a \,\square\, a)a^3\rangle$.

**Proposition 4.3.18.** *Let $V$ be a $JH$-triple. Then the following statements are equivalent.*

(i) *$V$ is simple and abelian with associative inner product.*

(ii) *$V$ is the one dimensional space $\mathbb{R}$ equipped with the usual triple product $\{x, y, z\} = \pm\, xyz$ and inner product*

$$\langle x, y \rangle_k = k\, xy \qquad (x, y \in \mathbb{R})$$

*for some $k > 0$.*

*Proof.* (i) $\Rightarrow$ (ii). Let $(V, \{\cdot, \cdot, \cdot\})$ have associative inner product. Then it is a Hilbert ternary algebra. Further, if it is simple and abelian, then there is a ternary isomorphism $\tau : V \longrightarrow \mathcal{C}_2^k(H, K)_\pm$ with

$$\tau\{x, y, z\} = (\tau x)(\tau y)^*(\tau z),$$

where $H$ and $K$ are $\mathbb{F}$-Hilbert spaces with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and $k > 0$ by Theorem 3.4.15. This implies $[a, b, c] = ab^*c = cb^*a$ in $\mathcal{C}_2^k(H, K)_\pm$ and it is only possible if $\dim H = \dim K = 1$. Indeed, suppose $\dim H \geq 2$. Let $e_1$ and $e_2$ be two nonzero orthonormal vectors in $H$. Pick $f \in K$ with $\|f\| = 1$. Then we have

$$[f \otimes e_2, f \otimes e_1, f \otimes e_1] = 0 \quad \text{and} \quad [f \otimes e_1, f \otimes e_1, f \otimes e_2] = f \otimes e_2.$$

Since the triple product in $V$ is symmetric in outer variables, we have $f \otimes e_2 = 0$. This gives $f\langle h, e_2 \rangle = 0$ and hence $\langle h, e_2 \rangle = 0$ for all $h \in H$. Therefore $e_2 = 0$ which is a contradiction. Likewise $\dim K = 1$. Since the identity $ab^*c = cb^*a$ is not true in $\mathcal{C}_2^k(\mathbb{C}) = \mathcal{C}_2^k(\mathbb{R}^2)$ and $\mathcal{C}_2^k(\mathbb{H}) = \mathcal{C}_2^k(\mathbb{R}^4)$, we have $H = K = \mathbb{R}$.

(ii) $\Rightarrow$ (i). The one dimensional space $(\mathbb{R}, \langle \cdot, \cdot \rangle_k, \{\cdot, \cdot, \cdot\})$ is a simple abelian $JH$-triple and its inner product is associative by Lemma 4.3.10. □

We have seen in Proposition 4.3.1 that if $(V, \{\cdot, \cdot, \cdot\})$ is a $JH$-triple, then $V$ is a Hilbert ternary algebra in the derived ternary product $[a, b, x]_d := d(a, b)x$. Conversely, every abelian Hilbert ternary algebra gives rise to a $JH$-triple.

The following result has been proved in [8], we include the proof for completeness.

**Proposition 4.3.19.** *Let $(V, [\cdot,\cdot,\cdot])$ be an abelian Hilbert ternary algebra. Then $V$ is a real $JH^*$-triple under the symmetrized ternary product*

$$\{x,y,z\}_s := \frac{1}{2}[x,y,z] + \frac{1}{2}[z,y,x]$$

*where $x,y,z \in V$. Moreover, if $V$ is simple with respect to $[\cdot,\cdot,\cdot]$, then $V$ is simple with respect to $\{\cdot,\cdot,\cdot\}_s$.*

*Proof.* Continuity of the triple product $\{\cdot,\cdot,\cdot\}_s$ follows from that of the ternary product $[\cdot,\cdot,\cdot]$, by Proposition 3.1.10. For all $a,b,x,y,z \in V$, we have

$$\begin{aligned} & \{a,b,\{x,y,z\}_s\}_s \\ = \; & \frac{1}{4}\left([a,b,[x,y,z]] + [[x,y,z],b,a] + [a,b,[z,y,x]] + [[z,y,x],b,a]\right) \end{aligned}$$

and since the ternary product $[\cdot,\cdot,\cdot]$ is abelian, we deduce that

$$\begin{aligned} & \{\{a,b,x\}_s,y,z\}_s - \{x,\{b,a,y\}_s,z\}_s + \{x,y,\{a,b,z\}_s\}_s \\ &= \frac{1}{4}\big([[a,b,x],y,z] + [z,y,[a,b,x]] + [[x,b,a],y,z] + [z,y,[x,b,a]] \\ &\quad -[x,[b,a,y],z] - [z,[b,a,y],x] - [x,[y,a,b],z] - [z,[y,a,b],x] \\ &\quad +[x,y,[a,b,z]] + [[a,b,z],y,x] + [x,y,[z,b,a]] + [[z,b,a],y,x]\big) \\ &= \frac{1}{4}\big([a,b,[x,y,z]] + [z,[b,a,y],x] + [x,[y,a,b],z] + [[z,y,x],b,a] \\ &\quad -[x,[b,a,y],z] - [z,[b,a,y],x] - [x,[y,a,b],z] - [z,[y,a,b],x] \\ &\quad +[x,[b,a,y],z] + [a,b,[z,y,x]] + [[x,y,z],b,a] + [z,[y,a,b],x]\big) \\ &= \{a,b,\{x,y,z\}_s\}_s \end{aligned}$$

by canceling terms of opposite sign. Hence $\{\cdot,\cdot,\cdot\}_s$ satisfies the Jordan triple identity. Evidently, the triple product $\{\cdot,\cdot,\cdot\}_s$ is symmetric in the first and third variables. If $a,b,x,y \in V$, then

$$\begin{aligned} \langle \{a,b,x\}_s, y\rangle &= \frac{1}{2}\langle [a,b,x],y\rangle + \frac{1}{2}\langle [x,b,a],y\rangle \\ &= \frac{1}{2}\langle x,[b,a,y]\rangle + \frac{1}{2}\langle x,[y,a,b]\rangle \\ &= \langle x, \{b,a,y\}_s\rangle. \end{aligned}$$

Hence $(V, \{\cdot,\cdot,\cdot\}_s)$ is a $JH$-triple with associative inner product.

Now let $V$ be a simple Hilbert ternary algebra with respect to $[\cdot,\cdot,\cdot]$ and let $I$ be a nonzero closed triple ideal of $(V, \{\cdot,\cdot,\cdot\}_s)$. Then we have

$$\{I, V, V\}_s + \{V, I, V\}_s \subset I$$

or

$$\frac{1}{2}[I, V, V] + \frac{1}{2}[V, V, I] + [V, I, V] \subset I.$$

Hence $I$ is a nonzero closed ternary ideal of the Hilbert ternary algebra $(V, [\cdot,\cdot,\cdot])$. It follows that $I = V$ and $JH$-triple $(V, \{\cdot,\cdot,\cdot\}_s)$ is simple. $\square$

Given a subspace $W$ of a Hilbert space $V$, we will always denote by $W^\perp$ the orthogonal completement of $W$ in $V$.

The following result has been proved in [8], we include the proof for completeness.

**Proposition 4.3.20.** *Let $V$ be a $JH$-triple and let*

$$V_s = \{a \in V : (a \,\square\, x)^* = x \,\square\, a, \forall x \in V\}.$$

*Then $V_s$ is a closed subtriple of $V$ and for every nonzero $a \in V_s^\perp$, we have $(a \,\square\, y)^* \neq y \,\square\, a$ for some $y \in V_s^\perp$.*

*Proof.* Let $a, b \in V_s$. We show that $\{a, b, a\} \in V_s$. Using the Jordan triple identity, we have

$$
\begin{aligned}
&\langle(\{a,b,a\}\,\square\, x)y, z\rangle \\
&= \langle\{\{a,b,a\},x,y\},z\rangle \\
&= \langle\{a,b,\{a,x,y\}\} + \{a,\{b,a,x\},y\} - \{a,x,\{a,b,y\}\},z\rangle \\
&= \langle(a\,\square\, b)\{a,x,y\},z\rangle + \langle(a\,\square\,\{b,a,x\})y,z\rangle - \langle(a\,\square\, x)\{a,b,y\},z\rangle \\
&= \langle(a\,\square\, x)y,(b\,\square\, a)z\rangle + \langle y,(\{b,a,x\}\,\square\, a)z\rangle - \langle(a\,\square\, b)y,(x\,\square\, a)z\rangle \\
&= \langle y,(x\,\square\, a)\{b,a,z\}\rangle + \langle y,\{\{b,a,x\},a,z\}\rangle - \langle y,(b\,\square\, a)\{x,a,z\}\rangle \\
&= \langle y,\{x,a,\{b,a,z\}\} + \{\{b,a,x\},a,z\} - \{b,a,\{x,a,z\}\}\rangle \\
&= \langle y,\{x,\{a,b,a\},z\}\rangle \\
&= \langle y,(x\,\square\,\{a,b,a\})z\rangle
\end{aligned}
$$

for all $a, b, x, y, z \in V$. This proves that $\{a, b, a\} \in V_s$. Hence $V_s$ is a subtriple of $V$. It is closed by the continuity of the triple product in $V$.

Now if $a \in V_s^{\perp}$ and $a \neq 0$, by definition of $V_s$, there exists some $y \in V$ such that $(a\,\square\, y)^* \neq y\,\square\, a$. Let $y = x + z$ for $x \in V_s$ and $z \in V_s^{\perp}$. Then we have

$$(a\,\square\, x)^* + (a\,\square\, z)^* = (a\,\square\, y)^* \neq y\,\square\, a = x\,\square\, a + z\,\square\, a$$

where $(a\,\square\, x)^* = x\,\square\, a$. Hence $(a\,\square\, z)^* \neq z\,\square\, a$. $\square$

It is evident that the inner product in $V_s$ defined above is associative. We call $V_s$ the *associative part* of $V$. For the $JH$-triple $V$ in Examples 4.3.11, 4.3.13 and 4.3.17, we have $V_s = \{0\}$. But in Example 4.3.12, the associative part of $\mathbb{R}^3$ is given by $\mathbb{R}^3_s = \{(0, 0, x) : x \in \mathbb{R}\}$.

**Lemma 4.3.21.** *Let $V$ be a $JH$-triple. Let $V_s$ be the associative part of $V$ defined in Proposition 4.3.20. Then*

$$\{V_s^{\perp}, V_s, V_s\} = \{0\} = \{V_s, V_s^{\perp}, V_s\}.$$

*Proof.* Let $a \in V_s^\perp$. Since $V_s$ is a subtriple of $V$ by Proposition 4.3.20, we have

$$\langle a, \{x, y, z\}\rangle = 0$$

where $x, y, z \in V_s$. Also, by definition of $V_s$, we get $\langle \{a, x, y\}, z\rangle = \langle a, \{x, y, z\}\rangle$ which implies that $\{V_s^\perp, V_s, V_s\} \subset V_s^\perp$. On the other hand, for $a \in V_s^\perp$ and $b, c \in V_s$, we have

$$\begin{aligned}
& \langle \{\{a, b, c\}, x, y\}, z\rangle \\
= \;& \langle \{a, b, \{c, x, y\}\} + \{c, \{b, a, x\}, y\} - \{c, x, \{a, b, y\}\}, z\rangle \\
= \;& \langle \{c, x, y\}, \{b, a, z\}\rangle + \langle y, \{\{b, a, x\}, c, z\}\rangle - \langle \{a, b, y\}, \{x, c, z\}\rangle \\
= \;& \langle y, \{x, c, \{b, a, z\}\} + \{\{b, a, x\}, c, z\} - \{b, a, \{x, c, z\}\}\rangle \\
= \;& \langle y, \{x, \{a, b, c\}, z\}\rangle
\end{aligned}$$

for all $x, y, z \in V$, which implies $\{V_s^\perp, V_s, V_s\} \subset V_s$. Hence

$$\{V_s^\perp, V_s, V_s\} \subset V_s \cap V_s^\perp = \{0\}.$$

For $a \in V_s^\perp$, we know that $0 = \langle a, \{x, y, z\}\rangle = \langle \{y, x, a\}, z\rangle = \langle y, \{x, a, z\}\rangle$ for arbitrary elements $y, x, z \in V_s$. Thus $\{V_s, V_s^\perp, V_s\} \subset V_s^\perp$. Likewise we can show that $b \in V_s^\perp$ and $a, c \in V_s$ imply

$$\langle (\{a, b, c\} \square x)y, z\rangle = \langle y, (x \square \{a, b, c\})z\rangle$$

for all $x, y, z \in V$. This gives $\{V_s, V_s^\perp, V_s\} \subset V_s$. Therefore

$$\{V_s, V_s^\perp, V_s\} \subset V_s \cap V_s^\perp = \{0\}.$$

□

**Corollary 4.3.22.** *Let $V$ be a JH-triple and let $V_s$ be the associative part of $V$ defined in Proposition 4.3.20. Then we have*

$$[V_s^\perp, V_s, V_s]_d = [V_s, V_s^\perp, V_s]_d = [V_s, V_s, V_s^\perp]_d = \{0\}.$$

*Proof.* By Lemma 4.3.21, we have

$$[V_s, V_s^\perp, V_s]_d = \{V_s, V_s^\perp, V_s\} - \{V_s^\perp, V_s, V_s\} = \{0\}.$$

The other identities can be shown in the same way. $\square$

**Proposition 4.3.23.** *Let $V$ be a $JH$-triple. Then the associative part $V_s$ is a ternary subalgebra of the derived Hilbert ternary algebra $(V, [\cdot,\cdot,\cdot]_d)$.*

*Proof.* By the above corollary, we have

$$\langle [V_s, V_s, V_s]_d, V_s^\perp\rangle = \langle V_s, [V_s^\perp, V_s, V_s]_d\rangle = \{0\}$$

which implies $[V_s, V_s, V_s]_d \subset V_s$. $\square$

The above result can be strengthened as follows.

**Proposition 4.3.24.** *Let $V$ be a $JH$-triple and let $V_s$ be the associative part of $V$. Then we have $[V, V, V_s]_d \subset V_s$.*

*Proof.* To show $d(a,b)x \in V_s$ for any arbitrary element $x \in V_s$, we prove that, for all $y \in V$,

$$\langle (d(a,b)x \,\square\, y)z, w\rangle = \langle z, (y \,\square\, d(a,b)x)w\rangle \qquad (z, w \in V).$$

By the Jordan triple identity and (4.7), we have

$$\begin{aligned}
& \langle z, (y \,\square\, d(a,b)x)w\rangle \\
= \; & \langle z, \{y, \{a,b,x\}, w\} - \{y, \{b,a,x\}, w\}\rangle \\
= \; & \langle z, -\{b,a,\{y,x,w\}\} + \{\{b,a,y\},x,w\} + \{y,x,\{b,a,w\}\} \\
& \quad +\{a,b,\{y,x,w\}\} - \{\{a,b,y\},x,w\} - \{y,x,\{a,b,w\}\}\rangle \\
= \; & \langle z, d(a,b)\{y,x,w\} + \{d(b,a)y, x, w\} + \{y, x, d(b,a)w\}\rangle \\
= \; & \langle d(b,a)z, \{y,x,w\}\rangle + \langle \{x, d(b,a)y, z\}, w\rangle + \langle \{x,y,z\}, d(b,a)w\rangle \\
= \; & \langle \{x, y, d(b,a)z\} + \{x, d(b,a)y, z\} + d(a,b)\{x,y,z\}, w\rangle
\end{aligned}$$

where

$$
\begin{aligned}
& \{x, y, d(b,a)z\} + \{x, d(b,a)y, z\} + d(a,b)\{x, y, z\} \\
= \quad & \{x, y, \{b, a, z\}\} - \{x, y, \{a, b, z\}\} + \{x, \{b, a, y\}, z\} \\
& -\{x, \{a, b, y\}, z\} + \{a, b, \{x, y, z\}\} - \{b, a, \{x, y, z\}\} \\
= \quad & -\{\{b, a, x\}, y, z\} + \{\{a, b, x\}, y, z\} \\
= \quad & \{d(a,b)x, y, z\}.
\end{aligned}
$$

This completes the proof. □

**Proposition 4.3.25.** *Let $V$ be a $JH$-triple. Then the orthogonal complement $V_s^{\perp}$ is a ternary ideal of the derived Hilbert ternary algebra $(V, [\cdot,\cdot,\cdot]_d)$.*

*Proof.* Let $a, b \in V$. Then we have

$$\langle [a, b, V_s^{\perp}]_d, V_s \rangle = \langle V_s^{\perp}, [b, a, V_s]_d \rangle \subset \langle V_s^{\perp}, V_s \rangle = \{0\}$$

by Proposition 4.3.24. This shows that $[V, V, V_s^{\perp}]_d \subset V_s^{\perp}$ and $V_s^{\perp}$ is a left ideal of $(V, [\cdot,\cdot,\cdot]_d)$. Again using Proposition 4.3.24 and also Corollary 4.3.22, we get

$$
\begin{aligned}
\langle [V_s^{\perp}, V, V]_d, V_s \rangle &= \langle V, [V, V_s^{\perp}, V_s]_d \rangle \\
&= \langle V_s, [V, V_s^{\perp}, V_s]_d \rangle + \langle V_s^{\perp}, [V, V_s^{\perp}, V_s]_d \rangle \\
&= \langle V_s, [V, V_s^{\perp}, V_s]_d \rangle \\
&= \langle V_s, [V_s, V_s^{\perp}, V_s]_d \rangle + \langle V_s, [V_s^{\perp}, V_s^{\perp}, V_s]_d \rangle \\
&= \langle V_s, [V_s^{\perp}, V_s^{\perp}, V_s]_d \rangle \\
&= \langle [V_s, V_s, V_s^{\perp}]_d, V_s^{\perp} \rangle \\
&= \{0\}
\end{aligned}
$$

and $[V_s^{\perp}, V, V]_d \subset V_s^{\perp}$. Hence $V_s^{\perp}$ is also a right ideal of $(V, [\cdot,\cdot,\cdot]_d)$. Therefore $V_s^{\perp}$ is an ideal of $V$ with respect to ternary product $[\cdot,\cdot,\cdot]_d$ by Lemma 3.2.5. □

**Remark 4.3.26.** Note that $V_s = (V_s^\perp)^\perp$ is a closed ternary ideal of $(V, [\cdot,\cdot,\cdot]_d)$ by Lemmas 3.2.4 and 3.2.5.

**Lemma 4.3.27.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a $JH$-triple with the associative part $V_s$. The following statements are equivalent.*

(i) $V = V_s$.

(ii) $(V_s^\perp, [\cdot,\cdot,\cdot]_d)$ *is an abelian Hilbert ternary algebra.*

*Proof.* (ii) $\Rightarrow$ (i). If $(V_s^\perp, [\cdot,\cdot,\cdot]_d)$ is an abelian Hilbert ternary algebra and $V_s^\perp \neq \{0\}$, then it has a nonzero tripotent $a$ by Corollary 3.4.4, which gives the contradiction

$$a = [a, a, a]_d = d(a, a)a = 0.$$

Hence $V_s^\perp = \{0\}$ and $V = V_s$. □

In particular, if the inner product of a $JH$-triple $V$ is nonassociative, then $(V_s^\perp, [\cdot,\cdot,\cdot]_d)$ is never abelian.

**Lemma 4.3.28.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a $JH$-triple and $V_s$ the associative part of $V$. Then*

$$\{V_s, V_s^\perp, V_s^\perp\} \subset V_s^\perp.$$

*Proof.* By Lemma 4.3.21, we have

$$\langle \{V_s, V_s^\perp, V_s^\perp\}, V_s\rangle = \langle V_s^\perp, \{V_s^\perp, V_s, V_s\}\rangle = \{0\}$$

which yields $\{V_s, V_s^\perp, V_s^\perp\} \subset V_s^\perp$. □

Likewise, one can see that $\{V_s^\perp, V_s, V_s^\perp\} \subset V_s^\perp$ and also $\{V_s^\perp, V_s^\perp, V_s\} \subset V_s^\perp$.

**Lemma 4.3.29.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a $JH$-triple and $V_s$ the associative part of $V$. Then*

$$[V_s, V_s^\perp, V_s^\perp]_d = [V_s^\perp, V_s, V_s^\perp]_d = [V_s^\perp, V_s^\perp, V_s]_d = \{0\}$$

*and for $x \in V_s$ and $a, b \in V_s^{\perp}$, we have*

$$\{x, a, b\} = \{a, x, b\} = \{a, b, x\}.$$

*Proof.* By Proposition 4.3.25 and Remark 4.3.26, both $V_s$ and $V_s^{\perp}$ are closed ternary ideals of the derived Hiberty ternary algebra $(V, [\cdot,\cdot,\cdot]_d)$. Then

$$[V_s, V_s^{\perp}, V_s^{\perp}]_d \subset V_s \cap V_s^{\perp} = \{0\}.$$

Consequently, we have

$$\{x, a, b\} - \{a, x, b\} = [x, a, b]_d = 0$$

for $x \in V_s$ and $a, b \in V_s^{\perp}$. Hence $\{x, a, b\} = \{a, x, b\}$.

Similar argument yields that $[V_s^{\perp}, V_s, V_s^{\perp}]_d = [V_s^{\perp}, V_s^{\perp}, V_s]_d = \{0\}$ as well as $\{a, x, b\} = \{a, b, x\}$. $\square$

Applying the above lemmas to the orthogonal decomposition $V = V_s \bigoplus V_s^{\perp}$, a straightforward computation gives the following identities:

$$\begin{aligned}
&\{V, V, V\} = \{V_s, V_s, V_s\} + \{V_s, V_s^{\perp}, V_s^{\perp}\} + \{V_s^{\perp}, V_s^{\perp}, V_s^{\perp}\} \\
&[V, V, V]_d = [V_s, V_s, V_s]_d + [V_s^{\perp}, V_s^{\perp}, V_s^{\perp}]_d \\
&[V_s, V_s, V_s]_d = [V_s, V, V]_d = [V, V_s, V]_d = [V, V, V_s]_d.
\end{aligned}$$

Two nonempty subsets $A, B$ of a Jordan triple $V$ is said to be *triple orthogonal* if $A \square B := \{a \square b : a \in A, b \in B\} = \{0\}$. We give criteria below for $V_s$ and $V_s^{\perp}$ to be triple ideals of $V$.

**Proposition 4.3.30.** *Let $V$ be a JH-triple and $V_s$ the associative part of $V$. Then we have*

(i) *$V_s$ is a triple ideal of $(V, \{\cdot,\cdot,\cdot\})$ if and only if $V_s \square V_s^{\perp} = \{0\}$, that is, $\{V_s, V_s^{\perp}, V\} = \{0\}$.*

(ii) $V_s^\perp$ *is a triple ideal of* $(V, \{\cdot,\cdot,\cdot\})$ *if and only if* $V_s^\perp$ *is a subtriple of* $V$.

*Proof.* (i). By Lemma 4.3.21 and Lemma 4.3.29, we have

$$\begin{aligned}
\{V_s, V, V\} &= \{V_s, V_s, V_s\} + \{V_s, V_s^\perp, V_s^\perp\} \\
\{V, V_s, V\} &= \{V_s, V_s, V_s\} + \{V_s^\perp, V_s, V_s^\perp\} \\
&= \{V_s, V_s, V_s\} + \{V_s, V_s^\perp, V_s^\perp\}.
\end{aligned}$$

Since $V_s$ is a subtriple of $V$, the above identities imply that $V_s$ is an ideal of $V$ if and only if $\{V_s, V_s^\perp, V_s^\perp\} = \{0\}$ by Lemma 4.3.28. The last equality is equivalent to $\{V_s, V_s^\perp, V\} = \{0\}$ since $\{V_s, V_s^\perp, V_s\} = \{0\}$ by Lemma 4.3.21.

(ii). We only need to show sufficiency. We have, from $V = V_s \bigoplus V_s^\perp$,

$$\begin{aligned}
\{V_s^\perp, V, V\} &= 2\{V_s^\perp, V_s^\perp, V_s\} + \{V_s^\perp, V_s^\perp, V_s^\perp\} \\
\{V, V_s^\perp, V\} &= 2\{V_s^\perp, V_s^\perp, V_s\} + \{V_s^\perp, V_s^\perp, V_s^\perp\}
\end{aligned}$$

by Lemma 4.3.21 and 4.3.29. Therefore Lemma 4.3.28 implies that $V_s^\perp$ is an ideal of $V$ whenever $V_s^\perp$ is a subtriple of $V$. □

We note that, for the $JH$-triple $\mathbb{R}^3$ in Example 4.3.12, the associative part $\mathbb{R}^3_s$ is a proper triple ideal of $\mathbb{R}^3$ with respect to the triple product $\{\cdot,\cdot,\cdot\}$.

## 4.4 Nonabelian Hilbert ternary algebras

We have seen that $JH$-triples and Hilbert ternary algebras are closely related. Moreover, $JH$-triples provide examples of nonabelian Hilbert ternary algebras, namely, the derived Hilbert ternary algebra $(V, [\cdot,\cdot,\cdot]_d)$ of a $JH$-triple $(V, \{\cdot,\cdot,\cdot\})$. The *abelian* Hilbert ternary algebras have been classified in Chapter 3. To complete the picture of Hilbert ternary algebras, we give systematically examples of nonabelian Hilbert ternary algebras in this section. They can be divided into two classes, the class of alternative algebras and the nonalternative ones.

A Hilbert ternary algebra $V$ is called *alternative* if it satisfies the following identities

$$\begin{aligned}
&[a,b,[x,y,z]] = [[a,b,x],y,z] + [x,[b,a,y],z] - [x,y,[a,b,z]] \\
&[a,b,[x,y,x]] = [[a,b,x],y,x] \qquad\qquad (4.10)\\
&[a,b,[a,b,x]] = [[a,b,a],b,x]
\end{aligned}$$

for all $a,b,x,y,z \in V$.

Alternative Hilbert ternary algebras need not satisfy the Jordan triple identity, and hence need not be Jordan triples. However, they contain tripotents (cf. [7, Proposition 2]).

In [7], a so-called *$H^*$-triple system* is defined to be a real or complex Hilbert space $V$ with an involution $^* : V \longrightarrow V$ and a trilinear map $[\cdot,\cdot,\cdot] : V^3 \longrightarrow V$ satisfying

$$[x,y,z]^* = [x^*,y^*,z^*]$$

and

$$\langle [x,y,z],w\rangle = \langle z,[y^*,x^*,w]\rangle = \langle y,[z^*,w,x^*]\rangle = \langle x,[w,z^*,y^*]\rangle$$

where $x,y,z,w \in V$.

If $(V,[\cdot,\cdot,\cdot])$ is a simple alternative Hilbert ternary algebra, then with the involution being the identity map $\mathrm{id} : V \longrightarrow V$ or $-\mathrm{id}$, $(V,[\cdot,\cdot,\cdot],^*)$ is a simple alternative $H^*$-triple system according to the above definition. Conversely, we have the following result.

**Lemma 4.4.1.** *If an $H^*$-triple system $(V,[\cdot,\cdot,\cdot],^*)$ is a Hilbert ternary algebra with zero annihilator, then $^*$ is either the identity map* $\mathrm{id}$ *or* $-\mathrm{id}$.

*Proof.* Let $V$ be an $H^*$-triple system with involution $^*$. If $V$ is a Hilbert ternary algebra as well, then we have

$$[x,y,z]^* = [x^*,y,z] = [x,y,z^*] \qquad (x,y,z \in V)$$

which implies that $^*$ is a centralizer of Hilbert ternary algebra $V$. Also, we have

$$[x, y, z]^* = [x, y^*, z] \qquad (x, y, z \in V).$$

The latter implies that $^*$ is self-adjoint on $V$, by Lemma 3.3.8, since the annihilator of $V$ is zero. By Lemma 3.3.5, every self-adjoint centralizer of a Hilbert ternary algebra is a scalar multiple of the identity map. Hence the involution $^* = \pm\mathrm{id}$. $\square$

Using the classification of alternative $H^*$-triple systems given in [7, Theorem 8], alternative nonabelian Hilbert ternary algebras can be classified, the *simple* alternative nonabelian Hilbert ternary algebras of dim $\geq 4$, up to a positive multiple of the inner product, are isomorphic to the following algebra $(V, [\cdot, \cdot, \cdot])$ or its dual (cf. Definition 3.1.13):

$V$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, equipped with an anticonjugation $j$ on $V$ and the ternary product

$$[x, y, z] = \langle y, z \rangle x - \langle x, jz \rangle jy$$

for all $x, y, z \in V$.

We now turn to *nonalternative* Hilbert ternary algebras. We first consider real $JH^*$-triples. The following lemma shows that an alternative Hilbert ternary algebra satisfies the Jordan triple identity if and only if it is abelian. Therefore nonabelian real $JH^*$-triples are nonalternative nonabelian Hilbert ternary algebras.

**Lemma 4.4.2.** *An alternative Hilbert ternary algebra* $(V, [\cdot, \cdot, \cdot])$ *is abelian if and only if the ternary product* $[\cdot, \cdot, \cdot]$ *satisfies the Jordan triple identity.*

*Proof.* Evidently, every abelian Hilbert ternary algebra satisfies the Jordan triple identity. Conversely, let $V$ satisfy the Jordan triple identity. Comparing the Jordan triple identity

$$[a, b, [x, y, z]] = [[a, b, x], y, z] - [x, [b, a, y], z] + [x, y, [a, b, z]]$$

and the first alternative condition in (4.10)

$$[a, b, [x, y, z]] = [[a, b, x], y, z] + [x, [b, a, y], z] - [x, y, [a, b, z]],$$

for all $a, b, x, y, z \in V$, we have

$$[a, b, [x, y, z]] = [[a, b, x], y, z]$$

which is the abelian condition for a Hilbert ternary algebra. $\square$

In Proposition 4.3.19, we showed that, given a simple abelian Hilbert ternary algebra $(V, [\cdot, \cdot, \cdot])$, the Hilbert space $V$ equipped with the triple product

$$\{a, b, c\}_s = \frac{1}{2}[a, b, c] + \frac{1}{2}[c, b, a]$$

is a real $JH^*$-triple which is usually nonabelian. In fact, we know from Proposition 4.3.18 that the only simple abelian real $JH^*$-triples are $\mathcal{C}_2^k(\mathbb{R})_\pm$.

From [30], we have the following list of simple nonabelian real $JH^*$-triples, they are of the form $(V, \{\cdot, \cdot, \cdot\}, k \langle \cdot, \cdot \rangle)$ where $k > 0$ and $(V, \langle \cdot, \cdot \rangle)$ is one of the following real $JH^*$-triples or their dual (cf. Definition 3.1.13).

1. $V = \mathcal{C}_2(H, K)$ where $H$ and $K$ are Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, but cannot be both $\mathbb{R}$. Equip $V$ with the inner product

$$\langle a, b \rangle = \operatorname{Re} \operatorname{Trace}(ab^*)$$

and the triple product

$$\{a, b, c\} = \frac{1}{2}ab^*c + \frac{1}{2}cb^*a$$

where $a^*$ denotes the adjoint of $a$ in $\mathcal{C}_2(H, K)$.

2. $V = \{a \in \mathcal{C}_2(H) : a^t = -a\}$ where $\mathcal{C}_2(H) = \mathcal{C}_2(H, H)$, $V$ inherits the Jordan triple structures of $\mathcal{C}_2(H)$ and $H$ is a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Each $a \in \mathcal{C}_2(H)$ has a matrix representation $a = (a_{\alpha\beta})$ with $a_{\alpha\beta} \in \mathbb{R}$ or $\mathbb{C}$. Denote by $a^t = (a_{\beta\alpha})$ the transpose of $a$.

3. $V = \{a \in \mathcal{C}_2(H) : a^t = (a^\delta_{\alpha\beta})\}$ where $\delta$ is an involution of $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and we have the following different choices of $\delta$, namely, the usual involution for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the identity for $\mathbb{F} = \mathbb{C}$ or the involution of the second kind for $\mathbb{F} = \mathbb{H}$, as defined in Section 2.1.

4. $V$ is one of the real $JH^*$-triples $V_1, V_2$ and $V_3$ defined below. Let $M_3(\mathbb{K})$ be the real vector space of $3 \times 3$ matrices over $\mathbb{K}$ where $\mathbb{K}$ is the octonions $\mathbb{O}$, real split octonions $\mathbb{O}_s$ or complex split octonions $\mathcal{O}_s$. Let $\delta$ be an involution on $\mathbb{K}$ which commutes with the usual involution $^-$ for $\mathbb{K} = \mathbb{O}$; but commutes with the involution $'$ for $\mathbb{K} = \mathbb{O}_s$ or $\mathcal{O}_s$, defined in Section 2.1. For $a = (a_{ij}) \in M_3(\mathbb{K})$, we let $a^* = (a^\delta_{ji})$ and define the inner and triple products on $M_3(\mathbb{K})$ by

$$\langle a, b\rangle_\delta = \operatorname{Re}\operatorname{Trace}(ab^*) \tag{4.11}$$

and

$$\{a, b, c\}_\delta = (a \circ b^*) \circ c + a \circ (b^* \circ c) - b^* \circ (a \circ c) \tag{4.12}$$

where $a \circ b = \frac{1}{2}(ab + ba)$ and $a, b, c \in M_3(\mathbb{K})$. We define the following real $JH^*$-triples with the above inner and triple products.

$V_1 = \{(a_{ij}) \in M_3(\mathbb{O}) : (a_{ij}) = (\overline{a}_{ji})\}$ for which $\delta =^-$ in (4.11) and (4.12);

$V_2 = \{(a_{ij}) \in M_3(\mathbb{O}_s) : (a_{ij}) = (a'_{ji})\}$ for which (4.11) and (4.12) are defined by the involution $\delta =^\times$ defined in Section 2.1;

$V_3 = \{(a_{ij}) \in M_3(\mathcal{O}_s) : (a_{ij}) = (a'_{ji})\}$ for which (4.11) and (4.12) are defined by the involution $\delta =^\times$ defined in Section 2.1.

5. $V = M_{1\times 2}(\mathbb{K})$ which consists of $1 \times 2$ matrices over $\mathbb{K}$ where $\mathbb{K}$ is the octonions $\mathbb{O}$, real split octonions $\mathbb{O}_s$ or complex split octonions $\mathcal{O}_s$. Define $(a_1, a_2)^* = \begin{pmatrix} a_1^\delta \\ a_2^\delta \end{pmatrix}$ where $\delta$ is the usual involution for $\mathbb{K} = \mathbb{O}$, and $\delta =^\times$

for $\mathbb{K} = \mathbb{O}_s$ or $\mathcal{O}_s$. Equip $V$ with the inner product

$$\langle a, b\rangle = \operatorname{Re}\operatorname{Trace}(ab^*)$$

and triple product

$$\{a, b, c\} = \frac{1}{2}a(b^*c) + \frac{1}{2}c(b^*a)$$

for $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$ in $M_{1\times 2}(\mathbb{K})$.

6. $V$ is the real restriction of a complex spin factor, the latter is a complex Hilbert space $V_{\mathbb{C}}$ with complex inner product $\langle \cdot, \cdot\rangle$ and a conjugation $J : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ (see Definition 2.2.1) satisfying $\langle Jx, Jy\rangle = \langle y, x\rangle$ for $x, y \in V_{\mathbb{C}}$, and is equipped with the triple product

$$\{x, y, z\} = \langle x, y\rangle z + \langle z, y\rangle x - \langle x, Jz\rangle Jy.$$

7. $V$ is a real Hilbert space with a linear isometry $J$ on $V$ such that $J^2 = \mathrm{id}$ and is equipped with the triple product

$$\{x, y, z\} = \langle x, Jy\rangle z + \langle z, Jy\rangle x - \langle x, z\rangle Jy$$

where $\langle \cdot, \cdot\rangle$ is the inner product of $V$.

Apart from real $JH^*$-triples, we have other examples of nonabelian Hilbert ternary algebras. We give examples of nonabelian Hilbert ternary algebras which contain tripotents. Let $(V, \langle \cdot, \cdot\rangle)$ be a real Hilbert space and let $j : V \longrightarrow V$ be an anticonjugation (cf. Definition 2.2.1) such that $\langle jx, jy\rangle = \langle y, x\rangle$. Then $V$ with the ternary product defined by

$$[x, y, z] = \frac{1}{2}\langle x, y\rangle z + \frac{1}{2}\langle z, y\rangle x - \frac{1}{2}\langle x, jz\rangle jy \qquad (x, y, z \in V)$$

is a Hilbert ternary algebra. Indeed, we have $\langle x, jz\rangle\langle jy, w\rangle = \langle jw, y\rangle\langle z, jx\rangle$ which implies that $\langle [x, y, z], w\rangle = \langle z, [y, x, w]\rangle$ for all $x, y, z, w \in V$. Similarly $\langle [x, y, z], w\rangle = \langle x, [w, z, y]\rangle$. If $\dim V \geq 3$, then $V$ is nonabelian. For if

$[a, b, [x, y, z]] = [[a, b, x], y, z]$, we get

$$\langle x, y\rangle(\langle z, b\rangle a - \langle a, jz\rangle jb) - \langle x, jz\rangle(\langle a, y\rangle jb + \langle jy, b\rangle a) =$$
$$\langle x, b\rangle(\langle a, y\rangle z - \langle a, jz\rangle jy) + \langle a, jx\rangle(\langle z, b\rangle jy - \langle jb, y\rangle z).$$

Let $a = b = x = y$ and $\langle z, ja\rangle = 0$. We have

$$z = \frac{\langle z, a\rangle}{\|a\|^2} a$$

which is not true if $a$ and $z$ are nonzero and orthogonal.

Every nonzero element of $V$ with norm 1 is a tripotent. Note that $\langle z, jx\rangle = -\langle x, jz\rangle$ implies that $[x, y, z] \neq [z, y, x]$ for some $x, y, z \in V$, that is, $(V, [\cdot, \cdot, \cdot])$ is not a Jordan triple, in particular, not a $JH$-triple. We also know that $V$ is not alternative. For instance, the identity $[a, b, [x, y, x]] = [[a, b, x], y, x]$ gives us

$$\langle x, y\rangle\langle x, b\rangle a + \langle x, y\rangle\langle x, ja\rangle jb = \langle x, b\rangle\langle a, y\rangle x + \langle x, ja\rangle\langle jb, y\rangle x$$

which is not always true if $\dim V \geq 3$. For example, let $a \neq 0$ and $x = b = y \neq 0$ such that $\langle x, a\rangle = \langle x, ja\rangle = 0$. Then we have

$$\|x\|^2 a = 0$$

which is impossible.

Finally, apart from the derived Hilbert ternary algebra of a $JH$-triple and the Hilbert ternary algebra $(V, [\cdot, \cdot, \cdot]_1)$ in Section 3.1, which are not ternary isomorphic to each other since the latter contains nonzero box operator $a \,\square\, a$, we have the following examples of nonabelian Hilbert ternary algebras without nonzero tripotents.

Let $(V, [\cdot, \cdot, \cdot])$ be any Hilbert ternary algebra. The same Hilbert space $V$ with each of the following ternary products

$$\begin{aligned}
[a,b,c]_2 &= \frac{1}{2}[a,b,c] - \frac{1}{2}[b,a,c] \\
[a,b,c]_3 &= \frac{1}{2}[a,b,c] - \frac{1}{2}[a,c,b] \qquad (a,b,c \in V) \\
[a,b,c]_4 &= \frac{1}{2}[a,c,b] - \frac{1}{2}[c,a,b]
\end{aligned}$$

is a nonabelian Hilbert ternary algebra without nonzero tripotent. Each ternary product in the above is not symmetric in its first and third variables. Hence $V$ cannot be a $JH$-triple.

If $(V, [\cdot,\cdot,\cdot])$ is abelian, then $[\cdot,\cdot,\cdot]_2$ does not satisfy the Jordan triple identity. Hence $(V, [\cdot,\cdot,\cdot]_2)$ is non-isomorphic to the derived Hilbert ternary algebra of a $JH$-triple. Also, $(V, [\cdot,\cdot,\cdot]_2)$ is non-isomorphic to $(V, [\cdot,\cdot,\cdot]_3)$ since the box operator $a \,\square\, a$ need not vanish on $(V, [\cdot,\cdot,\cdot]_3)$. The quadratic operator $Q_a$ vanishes on $(V, [\cdot,\cdot,\cdot]_4)$ but not always on $(V, [\cdot,\cdot,\cdot]_2)$. If $(V, [\cdot,\cdot,\cdot])$ is abelian, then $(V, [\cdot,\cdot,\cdot]_1)$ satisfies the Jordan triple identity while $(V, [\cdot,\cdot,\cdot]_4)$ does not. Hence $(V, [\cdot,\cdot,\cdot]_1)$ and $(V, [\cdot,\cdot,\cdot]_4)$ are non-isomorphic.

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