## Random graph models for wireless communication networks

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# Random graph models for wireless communication networks 

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17th March 2010

## Abstract

This thesis concerns mathematical models of wireless communication networks, in particular ad-hoc networks and 802.11 WLANs. In ad-hoc mode each of these devices may function as a sender, a relay or a receiver. Each device may only communicate with other devices within its transmission range. We use graph models for the relationship between any two devices: a node stands for a device, and an edge for a communication link, or sometimes an interference relationship. The number of edges incident on a node is the degree of this node. When considering geometric graphs, the coordinates of a node give the geographical position of a node.

One of the important properties of a communication graph is its connectedness whether all nodes can reach all other nodes. We use the term connectivity, the probability of graphs being connected given the number of nodes and the transmission range to measure the connectedness of a wireless network. Connectedness is an important prerequisite for all communication networks which communication between nodes. This is especially true for wireless ad-hoc networks, where communication relies on the contact among nodes and their neighbours.

Another important property of an interference graph is its chromatic number - the minimum number of colours needed so that no adjacent nodes are assigned the same colour. Here adjacent nodes share an edge; adjacent edges share at least one node; and colours are used to identify different frequencies. This gives the minimum number of frequencies a network needs in order to attain zero interference. This problem can be solved as an optimization problem deterministically, but is algorithmically NP-hard. Hence, finding good asymptotic approximations for this value becomes important.

Random geometric graphs describe an ensemble of graphs which share common features. In this thesis, node positions follow a Poisson point process or a binomial point process. We use probability theory to study the connectedness of random graphs and random geometric graphs, which is the fraction of connected graphs among many graph samples. This probability is closely related to the property of minimum node degree being
at least unity. The chromatic number is closely related to the maximum degree as $n \rightarrow \infty$; the chromatic number converges to maximum degree when graph is sparse. We test existing theorems and improve the existing ones when possible. These motivated me to study the degree of random (geometric) graph models.

We study using deterministic methods some degree-related problems for Erdaős-Rényi random graphs $\mathcal{G}(n, p)$ and random geometric graphs $G(n, r)$. I provide both theoretical analysis and accurate simulation results. The results lead to a study of dependence or non-dependence in the joint distribution of the degrees of neighbouring nodes.

We study the probability of no node being isolated in $\mathcal{G}(n, p)$, that is, minimum node degree being at least unity. By making the assumption of non-dependence of node degree, we derive two asymptotics for this probability. The probability of no node being isolated is an approximation to the probability of the graph being connected. By making an analogy to $\mathcal{G}(n, p)$, we study this problem for $G(n, r)$, which is a more realistic model for wireless networks. Experiment shows that this asymptotic result also works well for small graphs.

We wish to find the relationship between these basic features the above two important problems of wireless networks: the probability of a network being connected and the minimum number of channels a network needs in order to minimize interference.

Inspired by the problem of maximum degree in random graphs, we study the problem of the maximum of a set of Poisson random variables and binomial random variables, which leads to two accurate formulae for the mode of the maximum for general random geometric graphs and for sparse random graphs. To our knowledge, these are the best results for sparse random geometric graphs in the literature so far. By approximating the node degrees as independent Poisson or binomial variables, we apply the result to the problem of maximum degree in general and sparse $G(n, r)$, and derived much more accurate results than in the existing literature. Combining the limit theorem from Penrose and our work, we provide good approximations for the mode of the clique number and chromatic number in sparse $G(n, r)$. Again these results are much more accurate than existing ones. This has implications for the interference minimization of WLANs.

Finally, we apply our asymptotic result based on Poisson distribution for the chromatic
number of random geometric graph to the interference minimization problem in IEEE 802.11b/g WLAN. Experiments based on the real planned position of the APs in WLANs show that our asymptotic results estimate the minimum number of channels needed accurately. This also means that sparse random geometric graphs are good models for interference minimization problem of WLANs. We discuss the interference minimization problem in single radio and multi-radio wireless networking scenarios. We study branch-and-bound algorithms for these scenarios by selecting different constraint functions and objective functions.

## Acknowledgements

This project is funded by QMUL, electronic engineering and complexity group BT. I would like to take this opportunity to thank the college, the department and group for the financial support. This thesis would not have been possible without many people being supportive. I am grateful to have this opportunity to thank all those who made this thesis possible.

I would like to show my gratitude to my industrial supervisor Dr Keith Briggs for his continuous help on supervising me all through my MSc to PhD studies, providing resources, offering directions and discussing points. As a result of discussions with me, Dr Keith Briggs wrote the very_nauty graph library ${ }^{1}$ for the basic low-level graph algorithms, which made it possible for me to write all the programs for the examples in the thesis, and these call functions from that library.

I would like to express my gratitude to my university supervisor Dr Raul Mondragon for his encouragement, guidance and support from the initial to the final level of my PhD study.

I would like to show my gratitude to my second university supervisor Professor David Arrowsmith for his support to my PhD study, helping set up the directions and discussing the progresses.

It is an honor for me to express my gratitude to Professor Thomas Prellberg for his contribution on approximation method in the work of Poisson maxima approximation.

It is an honor for me to express my gratitude to Professor Mathew Penrose for his explanation on his theories about random geometric graphs.

I would like to thank Professor Mike Smith and Dr Sverrir Olaffson for giving me the opportunities to doing my MSc placement and later part of my PhD work in BT.

My thanks also go to Kathy Abbot in QMUL Library for her continuous guidance and support.

I would like to express my gratitude to all the colleagues in mobility center, BT for giving me a warm work environment. Special acknowledgement goes to Santosh Kawade

[^0]for discussions and suggestions for help on building the relationship between theories and real world problems.

I am indebted to my many of my student colleagues in Adastral park Shi Jin, Min Xie, Jia Chen and Jiayuan Chen for joyful discussion on algorithms, mathematics and wireless network technologies.

I am grateful to my friends Huifang Kong, Feifei Cao, Qiao Li and Linlin who have given me a warm life experience and happy memories.

I wish to thank my extended families, especially uncle Guang Geng and Aunt Lei Song for sharing their wisdom of life.

Most importantly, I dedicate this thesis to my parents Wenzhen Song and Ling Geng, without their love and support this thesis will never become possible.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the project.

Linlin Song
17th March 2010

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## Symbols used in this thesis

| $G$ | A graph |
| :--- | :--- |
| $V$ | set of vertices (or nodes) |
| $E$ | set of edges (or links) |
| $N$ | $\|V\|$ |
| $M$ | $\|E\|$ |
| $r$ | threshold value for two nodes to be connected in $G(n, r)$ |
| $p$ | probability for two nodes to be connected in $\mathcal{G}(n, p)$ |
| $\mathcal{G}(n, p)$ | Erdős-Rényi random graph |
| $\mathcal{G}(n, m)$ | Erdős-Rényi random graph with fixed number of edges |
| $G(n, r)$ | random geometric graph |
| $\mathcal{G}_{\mathrm{p}}(n, r)$ | random geometric graph on Poisson point process |
| $\mathcal{G}_{\mathrm{b}}(n, r)$ | random geometric graph on binomial point process |
| $G_{g}(n)$ | the giant cluster of $g$ containing $n$ nodes |
| $G_{c}(n)$ | a complete graph of graph $g$ containing $n$ nodes |
| $a_{i, j}$ | adjacency matrix |
| $d_{i}$ | degree of node/vertex $i$ |
| $e_{i, j}$ | edge formed by nodes $i$ and $j$ |
| $s_{i, j}$ | distance between node $i$ and $j$ |
| $f(x)$ | probability distribution of $x$ |
| $f(x, y)$ | joint distribution of $x, y$ |
| $\delta(G)$ | the minimum degree of a graph $g$ |
| $\Delta(G)$ | the maximum degree of a graph $g$ |
| $\chi(G)$ | the chromatic number of a graph $G$ |
| $\chi^{\prime}(G)$ | the edge chromatic number of a graph $G$ |
| $\omega(G)$ | the clique number of a graph $G$ |
| $\omega_{n}$ | the clique number of random geometric graph $G(n, r)$ |
| $\Delta_{n}$ | the maximum degree of random geometric graph $G(n, r)$ |
| $\delta_{n}$ | the minimum degree of random geometric graph $G(n, r)$ |

$\chi_{n} \quad$ the chromatic number of random geometric graph $G(n, r)$
$\rho \quad$ Pearson correlation coefficient
$\hat{\rho} \quad$ Sample Pearson correlation coefficient

## Abbreviations used in this thesis

| UDG | Unit disk graphs |
| :--- | :--- |
| RGG | Random geometric graphs |
| B\&B | Branch-and-bound |
| WLAN | Wireless local area network |
| WMN | Wireless mesh networks |
| AP | Access point |
| 1G | The first generation |
| 2G | The second generation |
| 3G | The third generation |
| FD | Frequency division |
| CD | Code division |
| TD | Time division |
| OFDM | Orthogonal frequency-division multiplexing |
| FDMA | frequency-division multiple access |
| PDA | Personal data assistant |
| DSSS | Direct-sequence spread spectrum |
| WiMax | Worldwide Interoperability for Microwave Access |
| LTE | Long-term evolution |
| GSM | Global System for Mobile communications |
| WiFi | Wireless Fidelity, a trademark of the Wi-Fi Alliance for certified |
|  | products based on the IEEE 802.11 standards. |
| CSMA/CA | Carrier Sense Multiple Access With Collision Avoidance |

## Chapter 1

## Introduction

Wireless communications is a fast-growing industry. It provides analog, digital, voice, data and multimedia services. It has changed how we interact with each other and our way of life. For this reason wireless communication have captured the interest of researchers, media companies, manufacturers and service providers.

Connectivity is a vital prerequisite for any network requires communication between nodes. It is an important characteristic in order to obtain reliable communication. Network connectivity in the literature refers to the probability that all the nodes in a network can communicate with each other at any time. The higher this probability, the more reliable communication the network can provide.

Interference is an important issue of wireless networks; higher interference cause lower throughput and hence lower efficiency of the communication. Interference problems are eliminated given enough channels. But in reality the number of available channels is limited. For an example in wireless networks using frequency division multiple access techniques channels are divided by using different frequencies, because spectrum is scarce and an expensive resource. Under the same conditions, the smaller the number of frequencies needed the better a channel allocation scheme is. The problem of finding the minimum number of channels belongs to the minimum channel allocation problem in the literature.

This chapter briefly reviews some wireless networks, how random graph models are used to represent both the facilities in these networks and the relationship between these facilities. We review the most popular existing wireless networks: cellular, Wireless Local Area Networks (WLAN), ad-hoc and mesh. In following section we start with several most popular wireless networks in operation today: cellular, WLAN, ad-hoc and mesh. These
topologies are used in this thesis as different scenarios to our studies.

### 1.1 Wireless networks overview

A cellular network consists of fixed based stations and mobile devices (such as mobile phone handsets). The mobile devices communicate with each other via the base stations. The spectrum is divided into a set of channels that can be used simultaneously. The techniques used to share the channels include frequency division (FD), time division (TD) or code division (CD) multiplexing. The first generation (1G) of wireless networks were circuit-switched cellular networks. They provided analog voice services only. The second generation of wireless networks (2G) provided digital voice communications. GSM (Global System for Mobile communication) is the most successful 2G technology, it has around 3 billion users in the world. Today's third generation 3G wireless networks integrate voice and data applications. These networks aim to provide better voice quality, high-speed Internet and multimedia services. The future fourth generation 4G wireless networks will use technologies like LTE (Long Term Evolution) and WiMax (Worldwide Inter-operability for Microwave Access) aiming to provide higher data rate than former generations.

Wireless local area networks (WLANs) have high broadband bandwidth and low deployment cost. Mobiles phones, personal data assistants (PDAs) and laptops equipped with a WLAN interface can access the Internet service through the access points (APs). The WLAN is designed to provide high data rate to users in a certain area. The leading standard for WLAN is IEEE 802.11 (WiFi), which operates on the 2 GHz or 5 GHz frequency bands and uses orthogonal frequency division multiplexing (OFDM). Other standards developed around WLANs are: IEEE 802.11b is based on high-rate DSSS (Direct-sequence spread spectrum) on the 2.4 GHz frequency band, and IEEE 802.11 g is based on OFDM operating on the 2.4 GHz band. An example of a WLAN topology is shown in figure $1.1^{1}$.

Wireless mesh networks (WMN) are self-organized and self-configured network that provide Internet backhaul services to clients. The main purpose of using a mesh network

[^1]

Figure 1.1: An example of the topology of the WLAN
is to provide alternative routes, large capacity and high data rates. The three main mesh architecture types are [1]: backbone WMN, client WMN and hybrid WMN. Mesh networks are still developing, so currently there is a large amount of research on their deployment and usage. WMNs can be built using different technologies, of which IEEE 802.11 is the most commonly used. IEEE 802.11s considered the single-radio WMN architecture standard. Researchers in [33, 20] addressed mutli-radio and multi-channels WMNs. WMN based on IEEE 802.16 (WiMax) is studied in [44]. The example of the topology of WMN can be found in figure 1.2.

The cellular, WLAN and mesh networks require the existence of fixed infrastructure for their functionality. The fixed infrastructures are the base stations, access points and mesh points. An ad-hoc network is a wireless local area network where no fixed infrastructures are required for its functionality. The network devices communicate with neighbouring devices [27] and two devices are neighbours if they fall in each other transmission range. The mobile devices can act as sender, relay or receiver of traffic. The network is a self-organising and decentralised system. Compare to other wireless networks it has the advantage of fast deployment and automatic control. These are desirable features in emergency scenarios


Figure 1.2: An example of the topology of wireless mesh networks
such as earthquake, motorway congestion or in environments where human beings cannot access, such as poisonous, low oxygen and high or low temperature places. An example of the topology of a wireless ad-hoc network is in figure 1.3.

### 1.1.1 Connectivity

In ad-hoc networks, connectivity is a prerequisite for providing communication among its members. In a connected ad-hoc or mesh network there must exist at least one route from a node to any other node in the network. However having only one route between two nodes to deliver the traffic is not a desirable property. In this case, the loss of one link or node will create that some part of the network will be disconnected and traffic would be lost. To avoid this loss, the existence of alternative routes is a desirable property. The number of alternative routes is a measurement of the robustness of a network. There are two ways to change the connectivity of a network: remove devices or add devices. A natural question to ask when designing an ad-hoc or mesh network is: how many devices an ad-hoc or mesh should have to be connected?


Figure 1.3: An example of the topology of wireless ad-hoc networks

### 1.1.2 Channel allocation

In wireless systems, frequency is a scarce and expensive resource. In many countries, including UK and US, the spectrum is auctioned to the highest bidder. Therefore the spectrum obtained from the auctions must be used efficiently to gain reasonable profit. Hence one of the basic requirements in wireless system is the reuse of the frequency spectrum. In a wireless receiver device, the amplitude of the radio signal received decays as the distance to the wireless transmitter increases. Hence a communication channel can be reused if two transmitters are far apart from each other so they do not interfere.

There are two potential sources of interference: co-channel and adjacent/inter channel interference. Co-channel interference is when the interference arises from another communication device using the same channel. Adjacent/inter channel interference arises when another communication device(s) are using an adjacent channel in the frequency domain. In this thesis, we study only the co-channel interference problem.

The reuse of channels depends on the reduction of channel interference. Network engineers, designers and operators have to assign the communication channels to the network
devices in such a way that the total amount of interference is minimal. To achieve this goal there still many open questions that need further research; for example, what is the smallest number of channels needed to minimise the interference and maintain connectivity? In particular, this problem needs to be solved when the number of devices increases in time or are contained in a small geographical region. In an operating network: how should these channels be allocated to minimise interference and also minimise network disruptions (for example in mesh networks)? BT is a service provider. It operates many different facilities to provide various services. It is also a WLAN provider; the service is named BT Openzone. The current technique used by WLAN service providers such as BT is to manage interference hot-spots by "manually" allocating the channels at the access points. This method works well if the number of APs is small. For large number of APs this method is not feasible and another approach has to be pursued.

In this thesis we use random graph models as simplified models for wireless networks. These models capture the basic features of a network: the number of nodes and the transmission range. We hope these relationships could provide network planners and researchers useful information on network planning and performance comparing.

### 1.1.3 Graph models of wireless networks

A common approach to study wireless networks is to represent them as a graph. In these graph models, the graph nodes are the APs, mesh routers or ad-hoc devices. If one of the node is covered by the transmission range $r_{c}$ of another node then these two nodes are neighbouring nodes, i.e. there is a link between the nodes. This link can represent the channel or the interference between the nodes. In here we follow the common approach to represent a communication channel or interference as a binary variable. No signal decay or shadowing are considered in this approach. If $r_{i}$ is the threshold value for interference, then, if the distance between two nodes is less than $r_{i}$ the nodes are interfering with each other. Otherwise there is no co-channel interference. A similar approach is used to represent if two nodes can communicate, that is, there is a threshold distance $r_{c}$ for the existence


Figure 1.4: An example of the communication an interference graph. The solid line represents the transmission range of the channels, the dashed line the range of the interference.
of a communication channel between two nodes. Normally, for a wireless network, the interference threshold is larger than the communication threshold $r_{i} \geqslant r_{c}$. A node and its threshold distance can be represented as a disk. An example of the communication and interference graph is shown in figure 1.4. A wireless network can be represented by a random geometric graph. These random graphs are defined by the number of nodes, a distance threshold value (Euclidian distance), the position of the nodes and a probability distribution. In this thesis we represent the connectivity and interference problems as two different graphs.

### 1.2 Thesis organisation

This thesis aims to provide network engineers with guidelines on how to initialise the network settings. The guidelines are based on deterministic theoretical analysis, heuristic analysis, asymptotic results, algorithmic studies and experimental results. Three random
graph models are studied; Erdős-Rényi random graphs, random geometric graphs based on Poisson point process and based on binomial point process. Two properties related to the connectivity of wireless networks are studied: the connectedness and the minimum degree. Three graph properties which are related to interference minimisation are studied: the maximum degree, clique number and chromatic number. Two degree related statistics are studied: degree distribution and degree correlation.

During the prusuit of our analytical results for the above two problems, we assume degree independence. This assumption is widely used when studying this kind of problems. In contrast to existing research, we do not want to simply make this assumption. We could like to study this assumption first. Is it true or not? How significant it is? We do not only obtain the analytical results but also compare the result with and without the assumption in order to know how our results are influenced by this assumption.

We study the interference minimisation problems in various networking scenarios.
Chapter 2 introduces graph preliminaries, the definition of graph connectivity and graph colouring, the random graph models and point processes. It is aimed to familiarise the reader with the definitions and measurements used in the thesis. These definitions defined the graph model mathematically; the studies on these random graph models in later chapters are based on these definitions.

Chapter 3 studies the degree distribution and correlation in three graph models. For random geometric graphs we consider three different situations: infinite area, boundary area, area with edge effect. These three situations correspond to situations that the number of nodes are distributed following a Poisson distribution uniformly on an infinite area; node distributed on a square area and node degree distribution are not affected by the position of the node in the square area (nodes in the centre area are the same as those in the area close to the boundary); node distributed on a square area and node degree distribution are affected by the position of the node in the square. By using a torus surface instead of a square the edge effect problem can be avoided. Both analysis and experiment results are provided. The analysis is studied by probabilistic methods. The independence of the degrees is studied by the measurement of correlation. We provide deterministic
methods, equation and results for the degree correlation between two connected nodes and two overlapping nodes, the transmission range of the nodes are overlapped with each other. This chapter also studies the connectedness of all the three random graph models. The connectedness of random graphs is a measurement of the probability of the graph samples being connected. It is closely related with the probability of the minimum degree larger than 0 (no node isolated). I provide the deterministic probability method for the calculation for the probability of minimum degree larger than 0 for Erdős-Rényi random graph models. I compare by simulation the probability of those graph samples being connected and minimum degree larger than 0 . By assuming the node degree are independent Poisson random variables and binomial random variables, we derive two heuristic methods for the connectedness of the Erdős-Rényi random graph models. We also provide the experiment results comparing the two heuristics with the exact method. By using an analogue to the Erdős-Rényi random graph model, we derive two heuristic methods for the random geometric graphs. We also provide the experiment results comparing the probability of the graph samples being connected and the minimum nodes larger than 0 .

Chapter 4 to chapter 5 are devoted to the interference minimisation problem in wireless networks. Also in this chapter we present the asymptotics for the maxima of $n$ independent identical (iid) Poisson random variables. We derived accurate asymptotic to approximate Poisson maxima. Experiments show that they provide accurate prediction to the maximum of Poisson variables. Chapter 4 studied the new asymptotics based on the assumption of iid Poisson random variable and iid binomial random variable. Generally the theoretical binomial maxima predicts the mode of degree maxima correctly. Experiments show that for sparse graphs there exist a sequence of $n, r$ values where the maximum degree has the focusing phenomenon (the value of the maximum degree concentrate on two adjacent values).

Poisson maxima asymptotics derived in Chapter 4 predict the two adjacent integer modal values of the maximum degree, clique number and chromatic number much more accurately than those in the literature. When graphs are denser, the Poisson maxima also predicts the mode correctly, although there is no focusing phenomenon for maximum degree
in this limiting regime. We studied the asymptotics for the clique number and chromatic number and derive new theorems for the focusing phenomenon of the clique number and chromatic number for $(n, r)$ sequences for sparse graphs. We provide experiment results for sparse, intermediate and dense limit regime.

We study also the interference minimisation problem for wireless networks systematically by using branch-and-bound algorithms in chapter 5. By using different objective functions, this algorithm calculates the chromatic number and edge chromatic number. These are the solutions for the interference minimisation when there is interference graph. The algorithm also minimises the interference when the interference graph is complete; that is, any two nodes using the same channels interfere with each other, unless they are using this channel to communicate. The algorithm also solves the problem when there is a communication graph, in other word, it need to preserve the communication links.

## Chapter 2

## Graph models of wireless networks

Graphs are used to describe many engineering problems. In this thesis we use graphs to study two problems occurring in wireless networks: connectivity and interference. One of our main assumptions is that these problems can be expressed as a Boolean relationship. Two nodes are connected (true) or they are not (false). Two channels interfere with each other (true) or they do not interfere at all (false). The binary Boolean relationship between many objects can be represented with a graph. In this chapter we introduce the graph theoretical definitions used through the thesis to study connectivity and interference. The chapter is self-contained and no previous knowledge of graph theory is assumed. The purpose of this chapter is to quickly familiarise the reader with the terminology, notation and literature. We use terminology and notation that is standard in graph theory $[6,32,17]$. As we introduce new definitions or concepts of graph theory we would explain how they relate to wireless networks.

### 2.1 Graph, vertices, edges and degree

Definition 2.1. A graph $G=(V, E)$ : is a set of vertices (or nodes) $V$ and a set of edges (or links) $E$, each edge being a pair of two vertices from $E$.

In figure 2.1, $V=\{0,1,2,3,4,5\}, E=\{(2,3),(2,5),(3,4),(5,0),(0,1),(1,4),(1,2)\}$. An edge $\left(v_{1}, v_{2}\right)$ is said to be incident on $v_{1}$ and $v_{2}$. Nodes $v_{1}$ and $v_{2}$ are said to be adjacent. Edges are said to be adjacent if they share a vertex. In this thesis the term graph means a simple undirected graph where each edge is a subset of $V$ containing two distinct vertices, and no edge is repeated. We use $n$ to denote the number of vertices and $m$ to denote the


Figure 2.1: Example of a simple undirected graph.
number of edges: $n=|V|, m=|E|$. If $V$ is a finite set, then $G$ is called a finite graph; otherwise the graph is infinite.

Definition 2.2. Degree $d(v)$ : the degree of vertex $v$ is the number of edges incident to it.
The minimum degree of graph $G$ is the minimum value of $d(v)$ over all $v \in V$, denoted as $\delta(G)$. The maximum degree of graph $G$ is the maximum value of $d(v)$ over all $v \in V$, denoted as $\Delta(G)$.

In a wireless network, the set of radio transmitters are represented as nodes. If a transmitter is within another transmitter's communication range, then the transmitters communicate. An edge represents that two nodes communicate. This edge is also known as a communication link. The graph representing a set of transmitters and communicating links is called the communication graph. In a communication graph the degree of a transmitter is the number of other transmitters that communicate with this transmitter.

In general, evaluating interference is a difficult task due to the interaction between the transmitters, the physical characteristics of the surroundings where the network is deployed (buildings, electrostatic noise, etc.) and the different control techniques used in the transmitting devices to reduce the signal-to-noise ration (SNR). In this thesis we use a binary variable to describe the existence of interference and disregard all other properties related to interference. If the distance between two transmitters is less than a given value then they will interfere with each other. If two transmitters interfere with each other, then we say that the nodes interfere. The existence of interference is represented by an edge between these transmitters called the interference link. The degree of a node belonging
to the interference graph is proportional to the amount of interference caused by other nodes. If the distance between transmitters is greater than a given value, then there is no interference.

### 2.2 Graph connectivity and colouring

Connectivity is of particular importance in ad-hoc networks. As nodes (mobile, PDAs or laptops, mesh routers) rely on their neighbours (devices that is within their transmission range) to transmit information. In ad-hoc networks nodes are mobile so even if two nodes can communicate, the communication can break up if any of the nodes involved in the communication path moves out of reach. For this reason is desirable that there is more than one path between two nodes, i.e. the existence of alternative routes.

### 2.2.1 Connectivity

In this thesis a connected network means that there is at least one available route between any pair of nodes. The existence of more than one path (route) between any pair of nodes makes the network robust to link or node failures. The following definitions are from Bollobás [7].

Definition 2.3. Path: A path in a graph is a sequence of vertices, each of which is adjacent to its successor in the sequence. Two paths are edge-disjoint if there is no shared edge between them.

Definition 2.4. Connected ${ }^{1}$ : A graph is connected if there is a path between all pairs of vertices.

Definition 2.5. Subgraph $S(G)$ : The set $V^{\prime}$ of vertices of $S(G)$ is a subset of the vertices $V$ of graph $G$, the set $E^{\prime}$ of edges of $S(G)$ is a subset of the edges $E$ of graph $G$, see figure $2.2(\mathrm{a})$ and $2.2(\mathrm{~b})$

[^2]
(a)

(b)

(c)

Figure 2.2: (a): a graph drawn on the plane; (b): a subgraph of the graph; (c): the induced subgraph of the chosen subgraph.

An induced subgraph combines all edges in $G$ that join the vertices in $S(G)$, see figure 2.2(c).

Definition 2.6. Cluster: The connected induced subgraph of a graph is a cluster of this graph.

Definition 2.7. Giant cluster $G_{g}(n)$ : The largest connected component of a graph.
Definition 2.8. $k$-vertex-connectivity: A graph is said to be $k$-vertex connected if and only if there is no set of $k-1$ vertices whose removal will disconnect the graph. Equivalently a graph is said to be $k$-vertex connected if for each pair of nodes there existence at least $k$ mutually independent paths connecting them.

Definition 2.9. $k$-edge-connectivity: A graph is $k$-edge connected if and only if there is no set of $k-1$ edges whose removal would disconnect the graph, which is equivalent to saying that for each pair of nodes there existence at least $k$ edge-disjoint paths connecting them. Figure 2.3 shows a 3 -edge connected and a 2 -vertex connected graphs.

In a communication network which is represented by a graph, the existences of a path between two nodes $A$ and $B$ means there is a route available for a message to be sent between transmitter $A$ and receiver $B$. If the graph edges are undirected that means that network channels are bidirectional. In here we are going to assume that the channels are always bidirectional, hence we always use an undirected graph. If the communication

(a)

(b)

Figure 2.3: (a): a 3-edge-connected graph; (b): a 2-vertex-connected graph.
graph is $k$-vertex-connected then there exist least $k$ alternative routes for each pair of nodes. None of these alternative routes have common nodes (apart from the source and destination nodes). A communication graph is $k$-edge-connected if there exist at least $k$ alternative routes for every pair of nodes. None of these routes have common communication links. The robustness of a wireless network to node or link removal is related to the $k$-connectivity of the communication graph.

### 2.2.2 Colouring

A possible way to eliminate the interference in a network is to assign to each of the communication links a different frequency (or time slot). However this is not feasible as the spectrum is a scarce resource and the number of channels is limited. Even if a wireless network has enough channels to eliminate interference, if the network size increases, the interference will increase up to a point where there are not enough channels to avoid interference. A method to improve the usage of network and avoid interference is scheduling. The scheduler assigns different frequencies/time/code to the communication channels. The number of different channels available to the scheduler depends on the wireless technology used. For example the standard IEEE 802.11.a allows 12 non-overlapping channels. Clearly if the number of neighbour transmitters increases to be greater than twelve then it is possible that the scheduler cannot resolve all the interference or collisions problems.

The IEEE 802.11 b standard provides 3 non-overlapping channels and 12 overlapping
channels. IEEE 802.11a standard provide 3 non-overlapping channels and 13 - 19 nonoverlapping channels. This will allow APs in WLAN network based on IEEE 802.11 to transmit simultaneously in multi-channels. IEEE 802.11 standard proposed a CSMA/CA mechanism in order to reduce collision. But this mechanism require neighbouring nodes to remain silent when one node is sending. This will cause severe delay to the network traffic. Scheduling is the method of assigning different frequency/time/code to potentially communication channels. In our model the scheduling can be realized by the method of graph colouring. The minimum number of frequencies/time slots/codes is the chromatic number of interference graphs.

In the past decades, graph colouring has been used to study the scheduling problem [19, 30]. In this representation a channel is represented with a colour. So for example, in IEEE802.11a there are 12 different colours. The scheduling problem becomes a colouring problem; that is, how to assign different colours such that we do not use the same colour for adjacent channels. The smallest number of colours needed to colour a graph is called the chromatic number. In general the chromatic number is difficult to evaluate (NP-hard problem).

(a)

(b)

Figure 2.4: Graph colouring: (a): a vertex colouring of a graph where $\chi=3$.; (b): An edge colouring of a graph where $\chi^{\prime}=3$.

Definition 2.10. (proper) ${ }^{2}$ vertex colouring: an allocation of a finite set of colours to the vertices of the graph, so that no two adjacent vertices are assigned the same colour.

[^3]Definition 2.11. (proper) edge colouring: an allocation of a finite set of colours to the edges of the graph such that two adjacent edges are not assigned the same colour.

Definition 2.12. vertex chromatic number $\chi$ : is the smallest number of colours needed to vertex-colour a graph.

Definition 2.13. edge chromatic number $\chi^{\prime}(G)$ : is the smallest number of colours needed to edge-colour a graph. Figures 2.4 are an examples of vertex and edge colouring.

(a)

(b)

Figure 2.5: (a) A graph and (b) its Line Graph.

The edge colouring problem can be transformed into vertex colouring via the line graph $L(G)$ of graph $G$, such that $\chi^{\prime}(G)=\chi(L(G))$. Figure 2.5(b) shows an example of graph and its line graph.

Definition 2.14. Line graph $L(G)$ : of an undirected graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of $G$ and any two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in $G$.

The problem of how to allocate the network channels to eliminate interference can be mapped into a vertex/edge colouring problem in the corresponding interference graph. For example the colours can be thought of as either disjoint time slots or frequency bands. For example in figure 2.4(a) the vertex colouring can be thought as an example of channel allocation to minimise the co-channel interference caused by neighbouring access points
(APs). A vertex represents the AP and the links represent the interference if they are all transmitting in the same frequency. After colouring the interference graph, each vertex is assigned a different colour. In this example the minimum number of colours is three.

The edge colouring in figure 2.4(b) can represent the interference between multiradio, multichannel wireless mesh networks (WMN). A mesh router can talk simultaneously to other routes but using different channels. In the figure each link colour represent a different frequency. Three different colours are needed to solve the edge colouring problem; that is, three different frequencies to avoid interference.

### 2.2.3 Complete graphs and cliques

In the last three sections, we have introduced graphs, communication graphs and interference graphs. We have related the robustness of a wireless network with the $k$-connectivity of the communication graph, and the channel allocation with the colouring of the interference/communication graph. In this section we introduce the concepts of complete graph and clique which are used in the calculation of the chromatic number (edge and vertex).

Definition 2.15. Complete graph $G_{c}(n)$ : is a graph in which all pairs of vertices are adjacent. Figure 2.6(a) contains a four-node complete graph and a three-node complete graph.

Definition 2.16. Clique: a clique is a complete induced subgraph [34] (see figure 2.6(b) for an example).

Definition 2.17. Clique number $\omega(G)$ : is the number of vertices in a largest clique of the graph.

Clique number is quite often used as a lower bound for chromatic number; for all graphs, it is true that $\omega \leqslant \chi \leqslant \Delta+1$ [34]. For an example, figure 2.6 shows for the graph in figure 2.2 (a) $\omega(G)=4$.

(a)

(b)

Figure 2.6: Complete graph and clique. (a): two complete subgraphs contained in the graph shown in figure $2.2(\mathrm{a})$; (b): the largest complete graph is the clique of graph shown in figure 2.2(a).

### 2.3 Random graph models

In a real wireless network two nodes are more likely to communicate with each other if they are geographically close. When studying wireless networks is common to make assumptions about the distribution of distances between the nodes, usually this distribution is defined using a random process. In this section we introduce and define the terminology of two random graph models. Random graphs are often used to model complex networks [7]. However as the concept of distance is vital in the representation of a wireless network they tend to be described via a random geometric graph model. The connectivity of the graphs obtained by these models depends on the distance between the nodes. A random geometric graph is used to describe a simplified wireless network where shadowing or decay of the signal is not considered. Penrose [32] used random geometric graphs to study the channel allocation problem in wireless network in the case that the number of nodes go to infinity. Similar work has been done by McDiarmid and Müller [25, 24].

Definition 2.18. Random graph: a graph in which vertices and/or edges are generated by a random process [7].

Let $n$ be the number of vertices and $M=n(n-1) / 2$ be the total number of possible

(a)

(b)

(c)

(d)

Figure 2.7: Example of random graphs $\mathcal{G}(n, p)$ instances: unit disk graphs. (a): $\mathrm{n}=10$, $\mathrm{p}=0.3$; (b): $\mathrm{n}=10, \mathrm{p}=0.3(\mathrm{c}): \mathrm{n}=10, \mathrm{p}=0.6(\mathrm{~d}): \mathrm{n}=10, \mathrm{p}=0.6$
edges then, the probability that the graph has $m$ edges is given by the binomial distribution:

$$
\begin{equation*}
\mathbf{P}(m=x)=\binom{M}{x} p^{x}(1-q)^{M-x}, q=1-q, x=1,2, \ldots, M \tag{2.1}
\end{equation*}
$$

Figure 2.7 shows four different instances of a random graph. Each edge exist with probability $p$, in figure $2.7(\mathrm{a})$ and 7 (b) $p=0.3$. If $e_{i, j}$ is the edge connecting nodes $i, j$ then $p=0.3$ means that in 100 independent trials, there are 30 graphs that $e_{i, j}$ exists. Since the existence of each of the possible edges is independent of the trail, the total number of edges is not necessary the same for each graph instance. Figure 2.7(c) and 2.7(d) show the case $p=0.6$. It is clear that when probability increases the existence of an edge increases.

Definition 2.19. Geometric graph: A geometric graph is a graph which is embedded in the plane. The vertices are points in general position and edges are segments connecting two chosen points [14].

Definition 2.20. Disk graph: a graph in which vertices are assigned a position in Euclidean space. Each vertex is associated with a disk of a specific radius cantered on it, an edge occur when a vertex is inside the disk of another vertex [43].

Definition 2.21. Unit disk graph (UDG): a unit disk graph is a graph where all disks have the same radius [9].

Unit disk graphs are a specific type of geometric disk graph. Unit disk graphs have been
used in the study of wireless networks [42,12]. Hale demonstrated that many frequencyrelated problems are NP-hard[42]; Clifford and Leith studied the convergence of distributed algorithms for wireless channel allocation problems [12].

Definition 2.22. Random Graphs, Erdős-Rényi ${ }^{3} \mathcal{G}(n, p)$ : a graph with $n$ vertices and each edge occurs with probability $p$ independently of all other edges. The probability of $m$ edges given by the binomial distribution $\left.m \sim \operatorname{Bi}\binom{n}{2}, p\right)[7]$.

Definition 2.23. Random geometric graphs $G(n, r)$ : a geometric graph with $n$ vertices and with threshold distance $r$. The vertices are distributed randomly. An edge occurs iff the distance between two vertices are less than $r$ (see figure 2.8).


Figure 2.8: Example of random geometric graphs $G(n, r)$. (a): $n=70, r=0.1$; (b): $n=70, r=0.1$ (c): $n=70, r=0.2$ (d): $n=10, r=0.2$

Random geometric graphs are an ensemble of unit disk graphs, where node positions are randomly distributed. In the application to wireless networks if the distance between two nodes is smaller than $r$, these two nodes communicate/interfere with each other; this is modelling the pathloss of radio signals. Random geometric graphs belong to intersection graphs of various geometric objects, which have been used to model real-life problems [34].

We assume that the value of the transmission/interference range $r$ can be provided by service provider, wireless device manufacturers and standard decision makers. This model is a simplified model since in the real world the value of $r$ is hard to measure, and also wireless devices receive signals when the SINR is not lower than a certain level. There are other models making use of SINR in the literature [35, 30].

[^4]In the next section we will describe various random point processes for generating the vertex sets. The underlying point process will be a key factor for determining the degree distribution of random geometric graph.

### 2.3.1 Point process and random graph models

Point processes are one of the random processes in $\mathbb{R}^{d}$. Point processes in $\mathbb{R}^{2}$ define the number of nodes and how the nodes are distributed on a surface. Point processes model the geographical location of nodes positions of wireless networks. In these networks node positions are assumed to be random and may involves random changes; that is, wireless ad-hoc networks.

A general definition of the point process $\mathcal{P}$ in $\mathbb{R}^{d}$ is given in [32]. In this thesis, we use two point processes: homogeneous Poisson point process and binomial point process. Here homogeneous means that the node density is constant at all points of space.

We next introduce the definition of the two point processes. As we introduce the definitions, the properties and the simulation methods are also introduced. The simulation method refers to the method of assigning the $x, y$ coordinates to a number of points, so that the distribution of $x$ and $y$ satisfies the statistics of this point process. For a thorough introduction of geometric statistics and simulation methods, see [23, chap.11].

Definition 2.24. Poisson/binomial deviate: in numerical analysis is a value generated by a Poisson/binomial generator.

Definition 2.25. Homogeneous Poisson point process: The number of points $N(A)$ falling into any bounded region in $\mathbb{R}^{2} A$ is a Poisson variable with mean $\lambda|A|$. Thus $N(A) \sim$ $\operatorname{Poi}(\lambda|A|)$. For any two non-intersecting Borel sets ${ }^{4}, A_{1}$ and $A_{2}$, the random variables $N\left(A_{1}\right)$ and $N\left(A_{2}\right)$ are independent. Method of generation: Get a Poisson deviate $n$ with mean $\lambda|A|$, where $|A|$ is the area of the region $A$ in which the points will be placed. Place $n$ points uniformly and randomly in $A$.

[^5]Definition 2.26. Homogeneous binomial point process: The number of points $N$ falling in the plane is a binomial random variable, where $N$ is the number of successes out of $|A|$ independent Bernoulli trials and $p$ is the probability of success in a single Bernoulli trial. For any two non-overlapping regions $A$ and $B$, the random variables $N_{A}$ and $N_{B}$ are not independent, since $N(A)=n$ implies $N(\bar{A})=N-n$. Method of generation: place $N$ points uniformly and randomly in $A$.

Two different point processes are of concern here: Poisson point process and binomial point process. The Poisson point process is used as an approximation of binomial point process in the literature both pure theoretically and with the application to wireless networks [32, 4, 24, 21]. The Poisson point process is popular with researchers because of its nice property of dependence between the number of points falling into two non-overlapping area $N(A)$. This is because working with independent random variables is easier than dependent random variables. In random geometric graph models for wireless networks $N(A)$ represent the number of nodes falls into a area $A$, since in this model every node $i$ is surrounded with a circle $A_{i}$ of equal radius.

The total number of neighbours node $i$ has is its degree. So node degrees are $n$ independent Poisson random variable in $G(n, r)$ model. Node degrees are $n$ dependent binomial random variable in $\mathcal{G}_{\mathrm{b}}(n, r)$ model. More detailed study of stochastic geometric can be found in [11]. The study of the binomial distribution of node degrees can be also found in the literature in $[11,37]$.

### 2.4 Summary

In past decades, graphs have been used to model wireless networks. In this chapter we reviewed the concept of a graph, the parameters and measures to analysis a graph and the how these graphs concepts are used to describe a wireless networks; for example, the routing, robustness and channel allocation in a wireless networks are mapped into the connectivity and colouring of graphs. As in a wireless network the position of the
nodes and the distance between one another are of fundamentally important we reviewed random graph models and random geometric graph models. These models capture the unpredictability of the nodes position and also take into consideration the transmission range of the nodes. The generation of the nodes position are obtained from a point process (Poisson or binomial).

The graph property that plays a major role when studying connectivity and interference is the degree of the nodes. How robust or well connected is a network ( $k$-connectivity) is strongly dependent on the nodes degree. The scheduling mechanism to avoid interference are mapped to a colouring graph problem, which in turn is related to the chromatic number which value is bounded by the degree of the nodes. The distribution of nodes degrees depend on how these nodes are placed in the space; that is, which point process was used to generate the random geometric graph. The distribution of nodes degree will be a Poisson or binomial if the point process to generate the graph is Poisson or binomial, respectively.

## Chapter 3

## The connectivity of wireless ad-hoc networks

In the previous chapter we introduced the concept of random graph and random geometric graph as models of wireless networks. In this chapter we study the connectivity of wireless networks using these models. Here connectivity refers to the probability that graphs drawn from an ensemble, which have the same parameters, are connected. We are interested $n$ the probability that a graph chosen at random from an ensemble of graphs is connected. All the graphs belonging to the ensemble have the same parameters. The parameters for random graphs $\mathcal{G}(n, p)$ are the number of nodes $n$ and the probability $p$ that they share a link; for $G(n, r)$ the parameters are the number of nodes $n$ and the transmission radius $r$. In a wireless ad-hoc networks the probability that a graph is connected can be used as a reference point by a network planner before deploying a network: given number of nodes and their transmission range what is the chance that the network will be connected?

In this chapter we consider two problems related to the overall connectivity of a network: the degree distribution and the minimum node degree.

The connectivity problem for wireless ad-hoc networks has been studied by Bettstetter [4]. In this paper a random geometric graph model based on binomial point process $\mathcal{G}_{\mathrm{b}}(n, r)$ was used to model an ad-hoc wireless networks.

Random geometric graph model based on Poisson point process $\mathcal{G}_{\mathrm{p}}(n, r)$ is used as an approximation to $\mathcal{G}_{\mathrm{b}}(n, r)$ when $r$ is fixed and $n$ is large. Similarly, in one dimension, a Poisson distribution is used as an approximation for binomial distribution when $p$ is small $n$ is large. This is why $\mathcal{G}_{\mathrm{p}}(n, r)$ has been used for modeling wireless ad-hoc/sensor networks, where nodes are randomly positioned, the number of nodes are large and the physical networks are sparse. The assumption of the node degrees being independent random
variables is made to derive an asymptotic result of the $k$-connectivity of wireless ad-hoc networks. The $k$-connectivity is a measurement of network robustness. The probability of graphs being $k$-connected (connected) are closely related with the minimum node degree being larger than $k$ for both $\mathcal{G}(n, p)$ and $G(n, r)$ [7,31]. Knowing the degree distribution, the simplest way of calculating this, is to assume statistically independence of node degrees. This is in fact not correct by the properties of $\mathcal{G}_{\mathrm{b}}(n, r)$ and $\mathcal{G}_{\mathrm{p}}(n, r)$. It is known for large random graphs [29] the degree correlation of two connected nodes is zero. The studies of random graphs [7] deal with asymptotic behavior of large graphs, so it does not help us on the exact method. We study the degree dependence and connectivity deterministically using probability theories. By this method, what we called exact methods/results, we obtained not only the result for limit cases and also result apply to small graphs.

We first study the assumption of degree dependence for $\mathcal{G}(n, p)$ and $\mathcal{G}_{\mathrm{p}}(n, r)$, then we study the relation between connectivity and minimum degree. During the study of degree dependence, we derive the exact formula for the joint distribution of any two chosen node by considering the connected nodes and unconnected separately for random graph $\mathcal{G}(n, p)$. We derive the exact formula for the correlation coefficient between the degree of two connected nodes for random graph $\mathcal{G}(n, p)$. An exact formula for the probability minimum node degree larger than 0 is derived. These work are to our knowledge new in the literature. We derive the exact formula for the correlation of two exact formula of the correlation between two overlapping nodes ${ }^{1}$ for $\mathcal{G}_{\mathrm{p}}(n, r)$. The result shows that our results shows that node degree are dependent for all above three random graph models. For large graphs, the correlation between two connected nodes is negligible. Both of the $G(n, r)$ models have positive correlation coefficient, which means the high degree nodes tends to connect with high degree nodes as the transmission range increases. We then used the assumption and derived asymptotics for $\mathcal{G}(n, p)$ by first using the binomial degree distribution and then the Poisson distribution as an approximation to binomial. It is also shown that though the assumption of the node degree being statistically independent is wrong, the probability derived based on this assumption are not significantly influenced.

[^6]Experiments are used to verify our analytical results. Special attention should be made to the simulation method when dealing with different analytical results. We show how the edge effect influence the degree distribution both analytically and experimentally. The simulation involves the procedure of assign coordinates to nodes, deciding the existence of edges, and node degree counting. We hope that out analytical method and results will provide researchers more insight into these models.

### 3.1 Degree dependence of random graph $\mathcal{G}(n, p)$

We start this section by recalling some results from random graph theory. In a graph with $n$ nodes the maximum number of possible edges is $M=n(n-1) / 2$. If $p$ is the probability that an edge exists between any two nodes, then the probability that a graph has $m$ edges is $P(m)=\binom{M}{m} p^{m}(1-p)^{M-m}$, where we have used that the existence of an edge is independent of existence the other edges. Given $n$ and $p$, the expectation value for the total number of edges is $\mathbb{E}(|E|)=n(n-1) p / 2$.

Let $\eta_{i}$ denote the event that there is an edge between node $i$ and any other node, and let us assume that the existence of this event is approximated by the binomial distribution $P\left(\eta_{i}\right) \approx \operatorname{Binomial}(1, p)$. The degree $d_{i}$ of node $n_{i}$ is the total number edges incident to this node. Assuming that the degree of a node is related to the probability that the event $\eta_{i}$ occurs then the probability that node $i$ has degree $k$ is given by the binomial distribution

$$
\begin{equation*}
\mathbf{P}\left(d_{i}=k\right)=\binom{n-1}{k} p^{k}(1-p)^{n-1-k} . \tag{3.1}
\end{equation*}
$$

In this case the expectation that a node will have degree $d_{i}$ is $\mathbb{E}\left(d_{i}\right)=(n-1) p$.
Figure 3.1(a) shows the degree distribution of one graph from the ensemble of $\mathcal{G}(n, p)$. Figure 3.1(b) shows the average behaviour of the degree distribution obtained from 500 graphs of the ensemble of $\mathcal{G}(n, p)$. In both figures the parameters are $n=100$ and $p=0.2$. Comparing both figures it seems that the assumption that the degree of a node is independent from the degree of the other nodes is correct. In figure 3.1(a) it seems the fluctuations


Figure 3.1: $\mathcal{G}(n, p)$ degree distribution. (a): all degree distribution in one graph with $n=100, p=0.2$; (b): one node degree distribution in 500 graphs with $n=100, p=0.2$. E is the expectation of the degree distribution; mean is the average value of all the node degrees.
in the degree distribution compared with the theoretical prediction (equation 3.1) are specific to the sampled graph chosen from the ensemble, and that if we consider the average of many samples (figure 3.1(b)) then equation (3.1) is correct due to the degree independence between nodes. However this is not true, as the averaging process is masking the degree dependence. To verify if the degree of a node is independent of the degree of other nodes we need to evaluate the correlation between node degrees.

We have evaluated the correlation between node degrees and obtained the following theorem:

Theorem 3.1. The degree of any two chosen nodes in $\mathcal{G}(n, p)$ are dependent random variables.

Proof 3.1. First lets recall some definitions: Two discrete random variables $X$ and $Y$ are
independent if for all $x, y$,

$$
\begin{equation*}
\mathbf{P}(X=x, Y=y)=\mathbf{P}(X=x) \mathbf{P}(Y=y) \tag{3.2}
\end{equation*}
$$

If $X$ and $Y$ are conditionally independent given $Z$, then

$$
\begin{equation*}
\mathbf{P}(X=x \cap Y=y \mid Z=z)=\mathbf{P}(X=x \mid Z=z) \mathbf{P}(Y=y \mid Z=z) \tag{3.3}
\end{equation*}
$$

If $n$ random variables $\left(X_{1}, \ldots, X_{n}\right)$ are independent, then any two of them are also independent random variables. If two random variables are correlated, then they are dependent.

Let $f(x)$ denote the marginal distribution of the degree $d_{i}$ :

$$
\begin{equation*}
f(x)=\mathbf{P}\left(x=d_{i}\right)=\binom{n-1}{x} p^{x} q^{n-1-x}, n \geqslant 2 \tag{3.4}
\end{equation*}
$$

where $q=1-p$. Let $d_{i}^{\prime}$ stand for the remaining degree; remaining means the degrees which are not shared by the connected nodes $i$ and $j$. Let $d_{i, j}$ be the number of neighbour(s) added to nodes $i$ and $j$ due to the presence of edge $e_{i, j}$. Then:

$$
\begin{align*}
i \text { and } j \text { are connected } & \Rightarrow d_{i, j}=1  \tag{3.5}\\
i \text { and } j \text { are not connected } & \Rightarrow d_{i, j}=0 \tag{3.6}
\end{align*}
$$

The marginal distribution of the remaining degrees $d_{i}^{\prime}$ is:

$$
\begin{equation*}
f_{p}(x)=\mathbf{P}\left(d_{i}^{\prime}=x\right)=\binom{n-2}{x} p^{x-1} q^{n-2-x} \tag{3.7}
\end{equation*}
$$

The remaining degree of any two nodes are independent, hence the degree of any two nodes are conditional independent. The joint mass function of $\mathcal{G}(n, p)$ can be calculated as follows: Case 1.-two nodes are connected

$$
f_{p}(x, y)=\mathbf{P}\left(d_{i}=x, d_{j}=y \mid i \text { and } j \text { are connected }\right)
$$

$$
\begin{align*}
& =f_{p}(x) f_{p}(y) \\
& =\mathbf{P}\left(d_{i}^{\prime}=x-d_{i, j} \mid d_{i, j}=1\right) \mathbf{P}\left(d_{j}^{\prime}=y-d_{i, j} \mid d_{i, j}=1\right) \\
& =\mathbf{P}\left(d_{i}^{\prime}=x-1, d_{j}^{\prime}=y-1\right) \tag{3.8}
\end{align*}
$$

Case 2.- two nodes are not connected

$$
\begin{align*}
f_{q}(x, y) & =\mathbf{P}\left(d_{i}=x, d_{j}=y \mid i \text { and } j \text { are not connected }\right) \\
& =f_{q}(x) f_{q}(y) \\
& =\mathbf{P}\left(d_{i}^{\prime}=x-d_{i, j} \mid d_{i, j}=0\right) \mathbf{P}\left(d_{j}^{\prime}=y-d_{i, j} \mid d_{i, j}=0\right) \\
& =\mathbf{P}\left(d_{i}^{\prime}=x, d_{j}^{\prime}=y\right) \tag{3.9}
\end{align*}
$$

The joint distribution of two nodes:

$$
\begin{align*}
f(x, y)= & \mathbf{P}\left(d_{i}=x, d_{j}=y\right) \\
= & \mathbf{P}\left(d_{i}=x, d_{j}=y \mid i \text { and } j \text { are connected }\right) \mathbf{P}(i \text { and } j \text { are connected }) \\
& +\mathbf{P}\left(d_{i}=x, d_{j}=y \mid i \text { and } j \text { are not connected }\right) \mathbf{P}(i \text { and } j \text { are not connected }) \\
= & p \mathbf{P}\left(d_{i}^{\prime}=x-1\right) \mathbf{P}\left(d_{j}^{\prime}=y-1\right)+q \mathbf{P}\left(d_{i}=x\right) \mathbf{P}\left(d_{j}=y\right) \\
= & p\binom{n-2}{x-1} p^{x-1} q^{n-2-(x-1)}\binom{n-2}{y-1} p^{y-1} q^{n-2-(y-1)} \\
& +q\binom{n-2}{x} p^{x} q^{n-2-x}\binom{n-2}{y} p^{y} q^{n-2-y} \\
= & \binom{n-2}{x-1}\binom{n-2}{y-1} p^{x+y-1} q^{2 n-2-x-y} \\
& +\binom{n-2}{x}\binom{n-2}{y} p^{x+y} q^{2 n-3-x-y} \\
\neq & f(x) f(y) \tag{3.10}
\end{align*}
$$

Therefore in $\mathcal{G}(n, p)$, the degrees of any two nodes are not independent.
To evaluate the correlation between node degrees first we recall that the correlation
coefficient $\rho$ is defined as:

$$
\begin{equation*}
\rho(X, Y)=\frac{\operatorname{cov}(X, Y)}{(\operatorname{var}(X) \operatorname{var}(Y))^{1 / 2}}=\frac{\mathbb{E}(X Y)-\mathbb{E}((X) \mathbb{E}((Y)}{\left((\operatorname{var}(X) \operatorname{var}(Y))^{1 / 2}\right.} . \tag{3.11}
\end{equation*}
$$

The nodes' degrees satisfty the following theorem:
Theorem 3.2. The correlation between the degree of any two nodes in $\mathcal{G}(n, p)$ is $\frac{1}{n-1}$.
Proof 3.2. Let $d_{i}$ denote the degree of node $i, d_{i, j}$ denote the number of possible edges between node $i$ and node $j$ and $d_{i}^{\prime}$ denote the number of possible edges between node $i$ and the other $n-2$ nodes (except node $j$ ). According to the definition of random graphs, edges between any pair of nodes exist independently with probability $p$, hence $d_{i, j}, d_{i}^{\prime}$ and $d_{j}^{\prime}$ are independent random variables and they follow binomial distributions:

$$
\begin{align*}
\mathbf{P}\left(d_{i, j}=k\right) & =\binom{1}{k} p^{k}(1-p)^{k}, k=0,1  \tag{3.12}\\
\mathbf{P}\left(d_{i}^{\prime}=k\right) & =\binom{n-2}{k} p^{k}(1-p)^{k}, k=0 \ldots n-2  \tag{3.13}\\
\mathbf{P}\left(d_{i}=k\right) & =\binom{n-1}{k} p^{k}(1-p)^{k}, k=0 \ldots n-1 . \tag{3.14}
\end{align*}
$$

Hence the variances for the possible number of nodes between nodes $i$ and $j$ and the the nodes' degree are

$$
\begin{equation*}
\operatorname{var}\left(d_{i, j}\right)=p q, \operatorname{var}\left(d_{i}\right)=(n-1) p q, \text { textwhere } q=1-p . \tag{3.15}
\end{equation*}
$$

Let $d_{i}$ denote the degree of node $i$, then we have the following relationships

$$
\begin{align*}
d_{i} & =d_{i, j}+d_{i}^{\prime}  \tag{3.16}\\
d_{j} & =d_{i, j}+d_{j}^{\prime}  \tag{3.17}\\
d_{i} d_{j} & =d_{i}^{\prime} d_{j}^{\prime}+d_{i, j} d_{j}^{\prime}+d_{i, j} d_{i}^{\prime}+d_{i, j}^{2}  \tag{3.18}\\
\mathbb{E}\left(\left(d_{i} d_{j}\right)\right. & =\overline{d_{i}^{\prime} d_{j}^{\prime}}+\overline{d_{i, j} d_{j}^{\prime}}+\overline{d_{i, j} d_{i}^{\prime}}+\overline{d_{i, j}^{2}}  \tag{3.19}\\
\mathbb{E}\left(( d _ { i } ) \mathbb { E } \left(\left(d_{j}\right)\right.\right. & =\left(\overline{d_{i}^{\prime}}+\overline{d_{i, j}}\right)\left(\overline{d_{j}^{\prime}}+\overline{d_{i, j}}\right) \tag{3.20}
\end{align*}
$$

$$
=\overline{d_{i}^{\prime} d_{i, j}}+\overline{d_{i, j} d_{j}^{\prime}}+\overline{d_{i}^{\prime} d_{j}^{\prime}}+{\overline{d_{i, j}}}^{2}
$$

Hence the covariance

$$
\begin{align*}
\operatorname{cov}\left(d_{i}, d_{j}\right) & =\mathbb{E}\left(\left(d_{i} d_{j}\right)-\mathbb{E}\left(( d _ { i } ) \mathbb { E } \left(\left(d_{j}\right)\right.\right.\right. \\
& =\overline{d_{i} d_{i, j}}+\overline{d_{i, j} d_{j}^{\prime}}+\overline{d_{i}^{\prime} d_{j}^{\prime}}+\overline{d_{i, j}^{2}}-\left(\overline{d_{i}^{\prime} d_{i, j}}+\overline{d_{i}^{\prime} d_{j}^{\prime}}+\overline{d_{j}^{\prime} d_{i, j}}+\overline{d_{i, j}^{2}}\right) \\
& =\overline{d_{i, j}^{2}}-\overline{d_{i, j}^{2}} \\
& =\mathbb{E}\left(\left(d_{i, j}^{2}\right)-\mathbb{E}\left(\left(d_{i, j}\right)^{2}\right.\right. \\
& =\operatorname{var}\left(d_{i, j}\right) . \tag{3.21}
\end{align*}
$$

Finally the correlation is

$$
\begin{align*}
\rho & =\frac{\operatorname{cov}\left(d_{i}, d_{j}\right)}{\sqrt{\left(\operatorname{var}\left(d_{i}\right) \operatorname{var}\left(d_{j}\right)\right)}}=\frac{\operatorname{var}\left(d_{i, j}\right)}{\sqrt{\left(\operatorname{var}\left(d_{i}\right) \operatorname{var}\left(d_{j}\right)\right)}}=\frac{p q}{(n-1) p q}=\frac{1}{n-1}  \tag{3.22}\\
& \approx 1 / n, \text { when } n \gg 1 \tag{3.23}
\end{align*}
$$

Therefore in $\mathcal{G}(n, p)$, the degree of any two chosen nodes are not independent. Although the degree independence is not true, when $n$ is large the correlation is negligible.

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \rho=0 \tag{3.24}
\end{equation*}
$$

### 3.2 Degree correlation of random geometric graphs based on Poisson point processes $\mathcal{G}_{\mathbf{p}}(n, r)$

A Poisson point process can be considered as an approximation to a binomial point process; the binomial point process is a special Poisson point process with condition that $n$ is fixed. Because the property of the number of nodes falling into a non-overlapping area is independent, $\mathcal{G}_{\mathrm{p}}(n, r)$ is often used as an approximation for a random geometric graph based on a binomial process $\mathcal{G}_{\mathrm{b}}(n, r)[32,4]$.

The infinite random geometric graph based on stationary Poisson point process is known as continuum percolation [26]. or as a Boolean model [11]. According to the definition of $\mathcal{G}_{\mathrm{p}}(n, r)$, the number of nodes that fall into a bounded region are described by a Poisson distribution. This means that the total number of nodes in a bounded region and the degree of any chosen node in this region are also described by a Poisson distribution.

In real problems defined on an infinite area the random geometric graph $\mathcal{G}_{\mathrm{p}}(n, r)$ is the appropriate model to study very large networks such as wireless sensor networks.

In this section we evaluate the degree-degree correlation of random geometric graphs $\mathcal{G}_{\mathrm{p}}(n, r)$. Several cases are going to be considered:

1. two nodes $i, j$ on an infinite area with overlapping disks $\left(d_{i, j} \leqslant 2 r\right)$,
2. two connected nodes $i, j\left(d_{i, j} \leqslant r\right)$ and

Theorem 3.3. The degree-degree correlation of nodes $i$ and $j$ with degree $d_{i}$ and $d_{j}$ is independent of $\lambda$ and the radius $r, \lambda$ is the density of nodes, and only depends on the distance $s$ between the nodes $i$ and $j$.

Proof 3.3. Let $A_{i}$ and $A_{j}$ denote the disk area of node $i$ and $j ; A_{i, j}$ denote the overlapping area of node $i$ and node $j ; A_{i}^{\prime}, A_{j}^{\prime}$ denote the non-overlapping areas, $A_{i}^{\prime}+A_{i, j}=A_{i, j}+A_{j}^{\prime}=$ $\pi r^{2}$. Let $d_{i}^{\prime}, d_{i}, j, d_{j}^{\prime}$ denotes the number of nodes falling into areas $A_{i}^{\prime}, A_{i, j}, A_{j}^{\prime}$. Since $A_{i}, A_{i, j}, A_{j}^{\prime}$ are non-overlapping areas, $d_{i}, d_{i, j}, d_{j}^{\prime}$ are independent random variables. Let $d_{i}$ denote the sum of the number of nodes falling into $A_{i}^{\prime}$ and $A_{i, j}$, which is the degree of node i. According to the definition of Poisson point process, random variables $d_{i}, d_{j}, d_{i}^{\prime}, d_{j}^{\prime}, d_{i, j}$ follow a Poisson distribution.

$$
\begin{align*}
\mathbf{P}\left[d_{i}=x\right] & =\frac{e^{-\lambda A_{i}}\left(\lambda A_{i}\right)^{x}}{x!} \quad x=0,1,2, \ldots  \tag{3.25}\\
\mathbf{P}\left[d_{i}^{\prime}=x\right] & =\frac{e^{-\lambda A_{i}^{\prime}}\left(\lambda A_{i}^{\prime}\right)^{x}}{x!} \quad x=0,1,2, \ldots  \tag{3.26}\\
\mathbf{P}\left[d_{j}=x\right] & =\frac{e^{-\lambda A_{j}}\left(\lambda A_{j}\right)^{x}}{x!} \quad x=0,1,2, \ldots \tag{3.27}
\end{align*}
$$

$$
\begin{align*}
\mathbf{P}\left[d_{j}^{\prime}=x\right] & =\frac{e^{-\lambda A_{j}^{\prime}}\left(\lambda A_{j}^{\prime}\right)^{x}}{x!} \quad x=0,1,2, \ldots  \tag{3.28}\\
\mathbf{P}\left[d_{i, j}=x\right] & =\frac{e^{-\lambda A_{i, j}}\left(\lambda A_{i, j}\right)^{x}}{x!} \quad x=0,1,2, \ldots \tag{3.29}
\end{align*}
$$

The degree of the nodes $i$ and $j$ satisfy:

$$
\begin{align*}
d_{i} & =d_{i, j}+d_{i}^{\prime}  \tag{3.30}\\
d_{j} & =d_{i, j}+d_{j}^{\prime}  \tag{3.31}\\
d_{i} d_{j} & =d_{i}^{\prime} d_{j}^{\prime}+d_{i, j} d_{j}^{\prime}+d_{i, j} d_{i}^{\prime}+d_{i, j}^{2}  \tag{3.32}\\
\mathbb{E}\left(d_{i} d_{j}\right) & =\overline{d_{i}^{\prime} d_{j}^{\prime}}+\overline{d_{i, j} d_{j}^{\prime}}+\overline{d_{i, j} d_{i}^{\prime}}+\overline{d_{i, j}^{2}}  \tag{3.33}\\
\mathbb{E}\left(d_{i}\right) \mathbb{E}\left(d_{j}\right) & =\left(\overline{d_{i}^{\prime}}+\overline{d_{i, j}}\right)\left(\overline{d_{j}^{\prime}}+\overline{d_{i, j}}\right)  \tag{3.34}\\
& =\overline{d_{i}^{\prime} d_{i, j}}+\overline{d_{i, j} d_{j}^{\prime}}+\overline{d_{i}^{\prime} d_{j}^{\prime}}+{\overline{d_{i, j}}}^{2} . \tag{3.35}
\end{align*}
$$

Hence the covariance is

$$
\begin{align*}
\operatorname{cov}\left(d_{i}, d_{j}\right)= & \mathbb{E}\left(\left(d_{i} d_{j}\right)-\mathbb{E}\left(\left(d_{i}\right) \mathbb{E}\left(d_{j}\right)\right.\right. \\
= & \overline{d_{i} d_{i, j}}+\overline{d_{i, j} d_{j}^{\prime}}+\overline{d_{i}^{\prime} d_{j}^{\prime}}+\overline{d_{i, j}^{2}} \\
& -\left(\overline{d_{i}^{\prime} d_{i, j}}+\overline{d_{i}^{\prime} d_{j}^{\prime}}+\overline{d_{j}^{\prime} d_{i, j}}+\overline{d_{i, j}^{2}}\right) \\
= & \overline{d_{i, j}^{2}}-\overline{d_{i, j}^{2}} \\
= & \mathbb{E}\left(\left(d_{i, j}^{2}\right)-\mathbb{E}\left(\left(d_{i, j}\right)^{2}\right.\right. \\
= & \operatorname{var}\left(d_{i, j}\right) . \tag{3.36}
\end{align*}
$$

Finally the correlation

$$
\begin{equation*}
\rho=\frac{\operatorname{cov}\left(d_{i}, d_{j}\right)}{\sqrt{\left(\operatorname{var}\left(d_{i}\right) \operatorname{var}\left(d_{j}\right)\right)}}=\frac{\operatorname{var}\left(d_{i, j}\right)}{\sqrt{\left(\operatorname{var}\left(d_{i}\right) \operatorname{var}\left(d_{j}\right)\right)}}=\frac{\lambda A_{i, j}}{\lambda \pi r^{2}}=\frac{A_{i, j}}{\pi r^{2}} \tag{3.37}
\end{equation*}
$$

The overlapping area $A_{i, j}$ can be calculated directly. Let $\alpha=\angle D A C$; see figure 3.2. If the nodes are uniformly distributed on a plane, then the probability that one node falls


Figure 3.2: The geometry of two overlapping circles in $\mathcal{G}_{\mathrm{p}}(n, r)$. (a): Example of $\mathcal{G}_{\mathrm{p}}(n, r)$; (b): the geometry of the two overlapping circles.
into the area of another node is proportional to the radius $r$. Let $s$ be the distance between nodes $i$ and $j$ which is independent of the angle $\alpha$. The probability function describing $s$ is equivalent to the probability function describing the distance between two uniformly distributed nodes in one dimension within range $r$, then

$$
\begin{equation*}
\alpha=2 \arccos \left(\frac{\|A B\|}{2\|A D\|}\right)=2 \arccos (s / 2 r) . \tag{3.38}
\end{equation*}
$$

Hence equation 3.3 can be rewritten as the function of the radius $r$ and angle $\alpha$ if

$$
\begin{equation*}
\rho=\frac{\alpha r^{2}-\sin (\alpha) r^{2}}{\pi r^{2}}=\frac{\alpha-\sin \alpha}{\pi}, \text { where } \alpha=2 \arccos (s / 2 r) \tag{3.39}
\end{equation*}
$$

Let $g(s, r)$ denote the function of the distance $s$ and radius $r$ and

$$
\begin{equation*}
\rho=g(s, r)=\frac{2 \arccos (s / 2 r) r^{2}-s \sqrt{4 r^{2}-s^{2}} / 2}{\pi} \tag{3.40}
\end{equation*}
$$

Here $s$ is a random variable, $r$ is a constant, hence

$$
\begin{equation*}
\rho=\int g(s, r) f(s) d s \tag{3.41}
\end{equation*}
$$

We now derive the distribution function of $s$.


Figure 3.3: Two connected nodes in $\mathcal{G}_{\mathrm{p}}(n, r)$.

Lemma 3.2.1. The probability distribution of the distance $s$ is $f(s)=2 s / r^{2}$ for $0 \leqslant s \leqslant r$.

## Proof 3.2.1.

$$
\begin{equation*}
\mathbf{P}\left(s<\left\|x_{0}-x_{1}\right\|<s+d s \mid\left\|x_{0}-x_{1}\right\|<s\right) \propto c s \quad 0 \leqslant s \leqslant r \tag{3.42}
\end{equation*}
$$

see figure 3.3 for the relationship between $s$ and $\left\|x_{0}-x_{1}\right\|$ and $c$ is the normalisation constant. Solving $\int_{0}^{r} c s d s=1$, where $\left[1 / 2 c s^{2}\right]_{0}^{r}=1$, we have: $c=2 / r^{2}$; hence $f(s)=$ $2 s / r^{2}$.

Theorem 3.4. The degree-degree correlation $\rho$ between two connected nodes on a infinite plane is $\rho=(4 \pi-3 \sqrt{3}) /(4 \pi)$

## Poisson point process on infinite plane



Figure 3.4: Degree correlation for random graphs based on a Poisson point process $\mathcal{G}_{\mathrm{p}}(n, r)$ $s<r$. Red is the analytical result; green is the simulation result, in the graph the analytic and simulation result overlap.

Proof 3.4.

$$
\begin{align*}
\rho= & \int_{0}^{r} g(s, r) f(s) d s \\
= & \int_{0}^{r} \frac{\left.2 \arccos (s / 2 r)-s \sqrt{4 r^{2}-s^{2}} / 2\right)}{\pi} \frac{2 s}{r^{2}} d s \\
= & {\left[\frac{2 s^{2} \arccos (s / 2 r)}{\pi r^{2}}-\frac{2 s \sqrt{1-s^{2} / 4 r^{2}}}{\pi r}-\frac{4 \arcsin (s / 2 r)}{\pi}\right.} \\
& \left.+\frac{\left(s\left(4 r^{2}-s^{2}\right)\right)^{3} / 2}{4 \pi r^{4}}-\frac{s \sqrt{4 r^{2}-s^{2}}}{\pi r^{2}}-\frac{2 \arctan \left(\frac{s}{\sqrt{4 r^{2}-s^{2}}}\right)}{\pi}\right]_{0}^{r} \\
= & \frac{4 \pi-3 \sqrt{3}}{4 \pi}  \tag{3.43}\\
\simeq & 0.5865033288 \tag{3.44}
\end{align*}
$$

The previous results were obtained under the condition that $s<r$, this condition can be relaxed. Following the same approach we have that

Lemma 3.2.2. the probability function of $s$ is

$$
\begin{equation*}
f(s)=s / 2 r^{2} \quad 0 \leqslant s \leqslant 2 r \tag{3.45}
\end{equation*}
$$

## Proof 3.2.2.

$$
\begin{equation*}
\mathbf{P}\left(s<\left\|x_{0}-x_{1}\right\|<s+d s \mid\left\|x_{0}-x_{1}\right\|<s\right) \propto c s \quad 0 \leqslant s \leqslant r . \tag{3.46}
\end{equation*}
$$

(See figure 3.3 for the relationship between $s$ and $\left\|x_{0}-x_{1}\right\|$.)
Solve:

$$
\begin{equation*}
\int_{0}^{2 r} c s \mathrm{~d} s=1\left[1 / 2 c s^{2}\right]_{0}^{2 r}=1 \tag{3.47}
\end{equation*}
$$

then the normalisation constant $c$ is $c=1 / 2 r^{2}$ hence

$$
\begin{equation*}
f(s)=\frac{s}{2 r^{2}} \tag{3.48}
\end{equation*}
$$

Theorem 3.5. In $\mathcal{G}_{\mathrm{p}}(n, r)$ the degree-degree correlation $\rho$ between two overlapping nodes on a infinite plane is $\rho=1 / 4$

Since $\rho$ is defined on $0 \leqslant s \leqslant 2 r$, we integrate over $[0,2 r]$ instead of $(-\infty,+\infty)$.

## Proof 3.5.

$$
\begin{align*}
\rho= & \int_{0}^{2 r} g(s, r) f(s) d s \\
= & \int_{0}^{2 r} \frac{2 \arccos (s / 2 r) r^{2}-\frac{1}{2} s \sqrt{4 r^{2}-s^{2}} s}{2 r^{2} \pi} d s \\
= & {\left[\frac{2 s^{2} \arccos (s / 2 r)}{\pi r^{2}}-\frac{2 s \sqrt{1-\left(s^{2} / 4 r^{2}\right)}}{\pi r}-\frac{4 \arcsin (s / 2 r)}{\pi}\right.} \\
& \left.+\frac{s^{3}\left(4 r^{2}-s^{2}\right)^{3}}{8 \pi r^{4}}-\frac{s \sqrt{4 r^{2}-s^{2}}}{\pi r^{2}}-2 \pi \arctan \left(s / \sqrt{4 r^{2}-s^{2}}\right)\right]_{0}^{2 r} \\
= & 1 / 4 \tag{3.49}
\end{align*}
$$



Figure 3.5: The degree correlation for random graphs based on a Poisson point process $\mathcal{G}_{\mathrm{p}}(n, r)$ on an infinite plane $s<2 r$. Red is the analytical result; green is the simulation result, the two results overlap.

### 3.3 Bound and edge effect in the degree correlation

Wireless networks, such as ad-hoc/mesh networks are better described using a finite area. To model this situation the nodes are contained in a bounded area. Even if the nodes are contained in a finite region, there may be nodes outside the region that have to be considered when evaluating the degree correlation. It is possible that that the nodes are contained in a bounded region and there are no nodes outside this region, in this case there is an edge in the distribution of nodes, creating an edge effect in the degree-degree correlation. Notice that the edge effect problem is different from the bounded region problem, an example of this difference is in figure 3.6(b).

The effects of having a bounded region have to be considered when, for an example, we would like to study the interference problem of the BT Openzone services in the Soho area in London. Beyond the Soho area there are other wireless devices installed. Outside the Soho area, APs (access points) may still interfere with the ones in the boundary of Soho. These APs close to the boundary of Soho contribute to the interference problem so they

### 3.3. BOUND AND EDGE EFFECT IN THE DEGREE CORRELATION 54



Figure 3.6: (a): node degree in boundary region; (b): node degree when edge effect occur. The degree is related to the number of nodes inside the small disks.
are consider as of equal importance as the node in the central part of Soho.
The edge effect is important when studying a situation like the BT Openzone in the city of Edinburgh. In this case there are no more APs installed in Edinburgh suburban area. The APs located close to the boundary of the city should not be treated as those in a more central area, since they receive and create less interference than those in the central area.

Figure 3.6(a) is an example of the bounded region and figure 3.6 of the edge effect. The smaller circles represent the interference range of the APs. The blue dots represent the APS in BT Openzone Soho and its surrounding area and the APs of the city of Edinburgh.

### 3.3.1 Bounded region

In this case the correlation between two nodes will depend on the transmission radius $r$. For this case we have obtained the following theorem:

Theorem 3.6. The degree-degree correlation between two connected nodes in $\mathcal{G}_{\mathrm{p}}(n, r)$ on a bounded square region increases as $r$ increases.

Proof 3.6. The exact distance-probability function between two randomly chosen nodes

### 3.3. BOUND AND EDGE EFFECT IN THE DEGREE CORRELATION 55



Figure 3.7: degree-degree correlation of two overlapping nodes of random graphs based on a Poisson point process $\mathcal{G}_{\mathrm{p}}(n, r)$ on a bounded square region. Red is the analytical result; green is the simulation result.
in a square has been studied by [16] and is given by

$$
f(s)= \begin{cases}2 s\left(s^{2}-4 s+\pi\right) & 0 \leqslant s \leqslant 1  \tag{3.50}\\ 2 s\left(4 \sqrt{s^{2}-1}-\left(s^{2}+2-\pi\right)-4 \arctan \sqrt{s^{2}-1}\right) & 1 \leqslant s \leqslant \sqrt{2}\end{cases}
$$

Since $0<s<2 r$, we separate the integral into two parts: $0<r<1 / 2$ and $r>1 / 2$. Following the method used to derive equation 3.43 then:

For $0<r<1 / 2$ :

$$
\begin{align*}
\rho & =\int_{0}^{2 r} g(s, r) f(s) d s  \tag{3.51}\\
& =\int_{0}^{2 r} \frac{\left.2 \arccos (s / 2 r) r^{2}-s \sqrt{4 r^{2}-s^{2}} / 2\right)}{\pi} 2 s\left(s^{2}-4 s+\pi\right) d s \\
& =1 / 45 \frac{r^{2}\left(-512 r+45 r^{2} \pi+45 \pi^{2}\right)}{\pi}
\end{align*}
$$

For $r>1 / 2$ :

### 3.3. BOUND AND EDGE EFFECT IN THE DEGREE CORRELATION

$$
\begin{align*}
\rho= & \int_{0}^{2 r} g(s, r) f(s) d s \\
= & \int_{0}^{1} \frac{\left.2 \arccos (s / 2 r) r^{2}-s \sqrt{4 r^{2}-s^{2}} / 2\right)}{\pi} 2 s\left(s^{2}-4 s+\pi\right) d s \\
& +\int_{0}^{2 r} \frac{\left.2 \arccos (s / 2 r) r^{2}-s \sqrt{4 r^{2}-s^{2}} / 2\right)}{\pi} d s \\
& +2 s\left(4 \sqrt{s^{2}-1}-\left(s^{2}+2-\pi\right)-4 \arctan \sqrt{s^{2}-1}\right) \\
= & \frac{1}{180 \pi r^{2}}\left(\left(360 r^{6}+780 r^{2}-360 \pi r^{2}+360 \pi r^{4}\right) \arcsin (0.5 / r)\right. \\
& \left.-2048 r^{5}+934 r^{4} s+113 s r^{2}-45 \pi s+180 r^{2} \pi^{2}-390 r^{2} \pi-90 \pi r^{2} s+114 s\right)  \tag{3.52}\\
& +\frac{1}{r^{2} \pi} \int_{1}^{\min (2 r, \sqrt{2})} s\left(-4 \sqrt{s^{2}-1}+s^{2}+2-\pi+4 \arctan \left(\sqrt{s^{2}-1}\right)\right)\left(-4 r^{2} \arccos \left(\frac{s}{2 r}\right)\right. \\
& \left.+s \sqrt{4 r^{2}-s^{2}}\right) d s
\end{align*}
$$

Figure 3.7 shows in green $\rho(s)$ obtained using equation 3.52.

### 3.3.2 Edge effect for $G(n, r)$

Using figure 3.8 as a reference. The large circle with radius $R$ denotes the bounded region. The smaller circles with radius $r$ represent the nodes and their interference range. For the nodes near the boundary, the edge effects means that their degree is proportional to the overlapping area of the small circles and the bounded region. Recall that the degree of a node is proportional to its coverage area.

Figure 3.9 shows the geometry of the bounded region and a node which we use to evaluate the area $A(r, R, d)$. This area is the overlap area between the node an bounded region and is this coverage area related to the degree of the node. Let $d$ denote the distance between the centre of the large circle and the small circle. As in section 3.2, the distance


Figure 3.8: Geometry of edge effect: shaded area presents the area where there is no edge effect, ring area is the area where edge effect occurs


Figure 3.9: Geometry of edge effect: overlapping circles: the geometry of the two overlapping circles with different radius.

### 3.3. BOUND AND EDGE EFFECT IN THE DEGREE CORRELATION



Figure 3.10: The effect of edge effect on the degree distribution for a randomly chosen point in $\mathcal{G}_{\mathrm{p}}(n, r)$.red: exact (using numerical integration); blue: simulation; green: theoretical Poisson distribution.
between these two circles only depends on the distance $d$, then the overlapping area is

$$
\begin{align*}
A(r, R, d)= & r^{2} \arccos \left(\frac{d^{2}+r^{2}-R^{2}}{2 D r}\right)+R^{2} \arccos \left(\frac{d^{2}+R^{2}-r^{2}}{2 d R}\right)  \tag{3.53}\\
& -\frac{1}{2} \sqrt{\left((R+r)^{2}-d^{2}\right)\left(d^{2}-(R-r)^{2}\right)}
\end{align*}
$$

Let $f(x)$ denote the probability distribution function of $d$. As was proved in section 3.2 $f(x)$ is linear in $x$, hence $f(x)=\alpha x+\beta$. Following the same approach as in Lemma 3.2.1 we have that

$$
\begin{equation*}
f(x)=\frac{2(x(r-2)+(1-r) r)}{r\left(-2+2 r-r^{2}\right)} \tag{3.54}
\end{equation*}
$$

Now we calculate the degree distribution of a randomly chosen node or a node uniformly placed on a bounded region defined by the condition $\pi R^{2}=1$ :

$$
\begin{equation*}
\mathbf{P}(\text { node } i \text { in ring })=\frac{A(x)}{\pi R^{2}}=1-\frac{\pi(R-r)^{2}}{\pi R^{2}} \tag{3.55}
\end{equation*}
$$

$$
\begin{align*}
\mathbf{P}\left(d_{i}=k\right)= & \mathbf{P}(\text { node } i \text { in circle }(R-r)) \mathbf{P}\left(d_{i}=k\right)  \tag{3.56}\\
& +\mathbf{P}\left(\text { node } i \text { in ring }(r) \int_{1-r}^{r} \mathbf{P}\left(d_{i}=k\right) f(x) d x\right. \\
= & (1-r)^{2} \frac{e^{-\lambda \pi r^{2}}\left(\lambda \pi r^{2}\right)^{k}}{k!} \\
& +\left(1-(1-r)^{2}\right) \int_{1-r}^{r} \frac{e^{-\lambda A(x)}(\lambda A(x))^{k}}{k!} \frac{2(x(r-2)+(1-r) r)}{r\left(-2+2 r-r^{2}\right)} d x
\end{align*}
$$

where we used

$$
\begin{equation*}
\mathbf{P}(\text { node } i \text { in circle })=\frac{\pi(R-r)^{2}}{\pi R^{2}} \tag{3.57}
\end{equation*}
$$

Following directly from equation 3.55 the degree distribution for $\mathcal{G}_{\mathrm{b}}(n, r)$ is

$$
\begin{align*}
\mathbf{P}\left(d_{i}=k\right)= & \mathbf{P}(\text { node } i \text { in circle }(R-r)) \mathbf{P}\left(d_{i}=k\right) \\
& +\mathbf{P}\left(\text { node } i \text { in ring }(r) \int_{1-r}^{r} \mathbf{P}\left(d_{i}=k\right) f(x) d x\right. \\
= & (1-r)^{2}\binom{n-1}{k}\left(\pi r^{2}\right)^{k}\left(1-\left(\pi r^{2}\right)\right)^{n-1-k}  \tag{3.58}\\
& +\left(1-(1-r)^{2}\right) \int_{1-r}^{r}\binom{n-1}{k} A(x)^{k}(1-A(x))^{n-1-k} \frac{2(x(r-2)+(1-r) r)}{r\left(-2+2 r-r^{2}\right)} d x .
\end{align*}
$$

Figure 3.10 shows the degree distribution obtained from equation 3.58 compared with the solution obtained by numerical simulations.

### 3.4 Minimum node degree and connectivity

In this section of the chapter we study the connectivity of wireless ad-hoc networks based on random geometric graphs. Over the last ten years random geometric graphs have been widely studied [31, 15]. Due to the broadcast nature of radio transmissions, a random geometric graph model is a simplified model of a wireless telecommunication network [38, 41]. In all networks, connectivity is a prerequisite in order to provide reliable applications to users [13, 18, 5]. In a connected network, there must exist at least one route from one
node to any other node. When there is no route from one node to any other node, the network is called disconnected. For a random graph $\mathcal{G}(n, p)$ there is a theorem that relates the minimum degree $\delta(\mathcal{G}(n, p))$ present in the graph with its connectivity [7]. This theorem has been extended to include the case of random geometric graphs [31]:

Theorem 3.7. [7, pp.168] Starting with $N$ disconnected nodes, if edges are added one by one to the empty graph where the 2 nodes of an edge are chosen uniformly at random from the $\binom{N}{2}$ possibilities, then almost surely the resulting graph becomes $k$-connected when it achieves a minimum degree $k$.

Proof 3.7. The proof of this theorem is in the book by Bollobás [7].


Figure 3.11: Probability that $\mathcal{G}(n, p)$ is connected. From right to the left in each figure the curves represent $\mathrm{n}=2,5,10,20,50,100,200$. (a): is the experimental result for the graph being 1-connected from equation 3.59; (b): is the experimental result for no node is isolated obtained from equation 3.64 ;

Recall that two paths are said to be mutually independent if and only if they have no common nodes apart from the end nodes,

The vertex connectivity of a graph is the minimum number of vertices whose removal disconnects the graph. A graph is $k$-vertex connected if there is no set of $k-1$ vertices
whose removal would disconnect the graph. Similarly, a graph is $k$-edge connected if and only if there is no set of $k-1$ edges whose removal would disconnect the graph, which is equivalent to saying that for each pair of nodes there exists at least $k$ edge-disjoint paths connecting them. Recall that two paths are said to be edge disjoint if there is no shared edge between them. If $d_{\text {min }}$ is the smallest degree that appears on the graph, then $k$-edge $\leq k$-vertex $\leq d_{\text {min }}$ (Menger's Theorem). Hence if a graph is $k$-vertex connected, if must be $k$-edge connected.

A recursive formula for calculating the probability $P_{n}$ that the random graph $\mathcal{G}(n, p)$ is 1 -connected is [17];

$$
\begin{align*}
\mathbf{P}_{0} & =1  \tag{3.59}\\
& \vdots \\
\mathbf{P}_{n} & =1-\sum_{k=1}^{n-1}\binom{n-1}{k-1} \mathbf{P}_{k} q^{k(n-k)} . \tag{3.60}
\end{align*}
$$

In this section, we study the deterministic probability distribution function of the minimum node degree equal to 1 for $\mathcal{G}(n, p)(\delta(G))$, then by making the assumption of degree independence and Poisson approximation we derive two asymptotics for the pdf of $\delta(G)$. We compare our approximations with the our deterministic result for no node isolated and the deterministic result provided in [17] for the probability that $\mathcal{G}(n, p)$ is 1-connected. The asymptotic analysis for $G(n, r)$ is made by analogy to $\mathcal{G}(n, p)$, and compared to experimental results. We derive several asymptotics based on the assumption of degree independence. We would like to see how those assumption compare with the deterministic results.

In the next section we study the probability of random graph being 1-connected or connected ${ }^{2}$ experiment results.

[^7]

Figure 3.12: Probability that $\mathcal{G}(n, p)$ is connected. From right to the left in each figure the curves represent $\mathrm{n}=2,5,10,20,50,100,200$. (a): is the approximation of no node is isolated based on iid binomial distribution (equation 3.72); (b): is the approximation of no node is isolated based on iid Poisson degree distribution (equation 3.74).

### 3.4.1 The exact method for no node being isolated in $\mathcal{G}(n, p)$

In this section we evaluate the probability that no node is isolated in a random graph $\mathcal{G}(n, p)$. We compare this result with the approximation obtained by assuming that the node degrees are independent and that for large networks their connectivity can be evaluated by using a Poisson approximation.

Theorem 3.8. In a complete graph the number of edges $m^{\prime}$ needed to be removed to generate $i$ isolated nodes (i.e. $G_{c}(n) \equiv G(n, 1)$ ) is:

$$
\begin{equation*}
m_{i}^{\prime}=\left(i^{2}+2 n i-i\right) / 2 \text { for } i=1 \ldots n \tag{3.61}
\end{equation*}
$$

Proof 3.8. Disconnecting $i$ nodes generates $i$ isolated nodes and a new complete sub-graph $G(n-i, 1)$. Let $m(n)$ denote the number of edges $m(n)=\binom{n}{2}$ for the graph $G(n, 1)$, Let
$m^{\prime}(n)$ be the difference between the total number of edges in $G(n, 1)$ and $G(n-i, 1)$ then

$$
\begin{align*}
m^{\prime} & =m(n)-m(n-i) \\
& =\binom{n}{2}-\binom{n-i}{2} \\
& =(n)(n-1) / 2-(n-i)(n-i-1) / 2 \\
& =i(2 n-i-1) / 2 \quad \text { for } i=1 \ldots n \tag{3.62}
\end{align*}
$$

We introduce the probability $f^{\prime}(n, p)$ that $i$ nodes are isolated which can be expressed as

## Lemma 3.4.1.

$$
\begin{align*}
f^{\prime}(n, p) & =\mathbf{P}(i \text { nodes are isolated in } G(n, p)) \\
& =\mathbf{P}(\text { disconnect } m \text { edges from } G(n, 1)) \\
& =\mathbf{P}(G(n-i, p) \text { remains connected }) \tag{3.63}
\end{align*}
$$

Now to evaluate if the graph $\mathcal{G}(n, p)$ is connected we need to evaluate if minimum degree $\delta(\mathcal{G}(n, p))$ is greater than one $(\delta(\mathcal{G}(n, p))>1)$, which is equivalent to evaluate the probability $f(n, p)$ that no node in $\mathcal{G}(n, p)$ is isolated:

Theorem 3.9. For a random graph $\mathcal{G}(n, p)$ with $n$ nodes, the probability $f(n, p)$ that no node is isolated is given by

$$
\begin{align*}
f(n, p) & =\mathbf{P}(\text { No node isolated in } G(n, p)) \\
& =\mathbf{P}\left(d_{1} \geqslant 0, d_{2} \geqslant 0, \ldots, d_{n} \geqslant 0\right) \\
& =1-\left[q^{m_{n}^{\prime}}+\sum_{i=1}^{n-1}\binom{n}{i} q^{m_{i}^{\prime}} \mathbf{P}\left(d_{i+1} \geqslant 0, \ldots, d_{n} \geqslant 0\right)\right] \\
& =1-\left[q^{m_{n}^{\prime}}+\sum_{i=1}^{n-1}\binom{n}{i} q^{m_{i}^{\prime}} \mathbf{P}(\text { No node isolated in } G(n-i, p))\right] \\
& =1-\left[q^{m_{n}^{\prime}}+\sum_{i=1}^{n-1}\binom{n}{i} q^{m_{i}^{\prime}} f(n-i, p)\right] \tag{3.64}
\end{align*}
$$

Proof 3.9. From Theorem 3.8 we have that in a complete graph, disconnecting $m^{\prime}$ edges will generated $i$ isolated nodes and a fully connected sub-graph. Now we only need $i$ isolated nodes and the remaining $n-i$ nodes to be no isolated. Recall that each edge from the $\binom{n(n-1) / 2}{1}$ possible links exists with probability $p$. From Lemma 3.4.1 and as the events of $i$ nodes isolated $(i \in 1 \ldots n)$ are mutually exclusive, then:
$f(n, p)=1-\mathbf{P}($ only 1 node is isolated in $\mathcal{G}(n, p)) \mathbf{P}($ No node isolated in $\mathcal{G}(n-1, p))-$

- $\mathbf{P}($ only 2 nodes are isolated in $\mathcal{G}(n, p)) \mathbf{P}($ No node isolated in $\mathcal{G}(n-2, p))-$ $\vdots$
- $\mathbf{P}($ only $n-1$ nodes are isolated in $\mathcal{G}(n, p)) \mathbf{P}($ No node isolated in $\mathcal{G}(1, p))-$
- $\mathbf{P}($ only $n$ nodes are isolated in $\mathcal{G}(n, p)) \mathbf{P}($ No node isolated in $\mathcal{G}(0, p))$
$=1-\sum_{i=1}^{n-2} \mathbf{P}(m$ edges are disconnected $) f(n-1, p)$
$-\mathbf{P}$ (all edges are disconnected)
$=1-\left[q^{m_{n}^{\prime}}+\sum_{i=1}^{n-2}\binom{n}{i} q^{m_{i}^{\prime}} f(n-1, p)\right]$
Notice that in the above equation we dealt with $i=n, n-1$ separately. The reason is because we define $f(1, p)=0$. We give no definition for $f(0, p)$, since it does not have any meaning to calculate the probability that no nodes are isolated when there are no nodes, and the probability of no node isolated when there is only one node is 0 . By defining $f(1, p)=1, f(0, p)=0, f(n, p)$ can be rewritten as:

$$
\begin{equation*}
f(n, p)=1-\sum_{i=1}^{n}\binom{n}{i} q^{m_{i}^{\prime}} f(n-i, p) \tag{3.66}
\end{equation*}
$$

The values of $f(n, p)$ for the first four values $(n=1,2,3,4)$ can be computed directly by looking at all the possible graph topologies (A gallery of these topologies can be found in the book by Ronald [8]).

$$
\begin{equation*}
f(1, p)=0 \tag{3.67}
\end{equation*}
$$

$$
\begin{align*}
& f(2, p)=p  \tag{3.68}\\
& f(3, p)=3 p^{2} q+p^{3}  \tag{3.69}\\
& f(4, p)=1-q^{6}-6 p q^{5}-4 p^{3} q^{3}-12 p^{2} q^{4} \tag{3.70}
\end{align*}
$$

Previously in this chapter, in section 3.1, we studied the degree distribution, joint distribution and degree correlation for $\mathcal{G}(n, p)$ and concluded that there is weak dependence between the nodes' degrees as the network increases in size, $n \rightarrow \infty$. In fact for $n=100$ this degree dependence is very close to zero. Hence for large $n$, we can assume that the degrees $d_{i}$ are independent random variables. In section 3.1 we introduced the marginal distribution of node's degree as

$$
\begin{equation*}
\mathbf{P}\left(d_{i}=k\right)=\binom{n-1}{k} p^{k}(1-p)^{n-1-k}, \quad i=1 \ldots, n \tag{3.71}
\end{equation*}
$$

where the node's degree was approximated by the binomial $d_{i} \sim \operatorname{Bi}(n-1, p)$. From this marginal distribution we can evaluate the asymptotic behaviour $n \rightarrow \infty$ of $f(n, p)$ as

$$
\begin{align*}
f(n, p) & =\mathbf{P}\left(d_{1}>0\right) \mathbf{P}\left(d_{2}>0\right), \ldots, \mathbf{P}\left(d_{n}>0\right) \\
& =\left(\mathbf{P}\left(d_{i} \geqslant 1\right)\right)^{n} \\
& =\left[\sum_{k=1}^{n-1}\binom{n-1}{k} p^{k}(1-p)^{n-1-k}\right]^{n} \\
& =\left(1-\mathbf{P}\left(d_{i}=0\right)\right)^{n} \\
& =\left(1-(1-p)^{n-1}\right)^{n} \tag{3.72}
\end{align*}
$$

It is also possible to approximate $f(n, p)$ for large $n$, by using a Poisson distribution as the asymptotic behaviour of the binomial distribution, which is

$$
\begin{equation*}
\mathbf{P}\left(d_{i}=k\right)=\frac{((n-1) p)^{k} e^{-(n-1) p}}{k!} \tag{3.73}
\end{equation*}
$$

where the node's degree $d_{i} \sim \operatorname{Poi}(\lambda)$ and $\lambda=(n-1) p$. In this case the asymptotic
behaviour of $f(n, p)$ is

$$
\begin{align*}
f(n, p) & =\mathbf{P}\left(d_{1}>0\right) \mathbf{P}\left(d_{2}>0\right), \ldots, \mathbf{P}\left(d_{n}>0\right) \\
& =\left(\mathbf{P}\left(d_{i} \geqslant 1\right)\right)^{n} \\
& =\left(\sum_{k=1}^{n-1} \frac{((n-1) p)^{k} e^{-(n-1) p}}{k!}\right)^{n} \\
& =\left(1-\mathbf{P}\left(d_{i}=0\right)\right)^{n} \\
& =\left(1-\left(1-e^{-(n-1) p}\right)^{n-1}\right)^{n} \tag{3.74}
\end{align*}
$$

Figure 3.11 and 3.12 compares the minimum degree and the 1 -connected property for a $\mathcal{G}(n, p)$ graph. Figure 3.11 (a) shows the probability that the graph is connected obtained from equation 3.59. Figure $3.11(\mathrm{~b})$ is the probability that no node is isolated obtained from equation 3.64. Figures 3.12 (a) and 3.12 (b) are the probability that no node is isolated using the binomial and Poisson approximation respectively. Comparing the figures 3.11(b), $3.12(\mathrm{a})$ and $3.12(\mathrm{~b})$ it is clear that the binomial and Poisson approximation give a good description of the probability that no node is isolated for large values of $n$.

### 3.4.2 Minimum degree and the 1 -connected property for random geometric graphs.

Before embarking evaluating the minimum degree and the 1 -connected property for random geometric graphs we have to consider the topology of the space where the geometric graph is defined. The main problem is that if the graph is defined in a flat surface, then there will be an edge effect that has to be taken into consideration. A simple way to avoid this edge effect is to lay the graph in a toroidal surface, in this case there is no edge effect $[32,4]$. Figure 3.13 shows the degree distribution for a random geometric graph defined in a square and in a torus. Figure 3.13(a) shows the degree distribution obtained from numerical experiments when using a square surface (in green) compared with the equivalent binomial distribution (in red). Clearly the binomial distribution does not describe


Figure 3.13: Example of how the edge effect can be avoided by laying a random geometric graph $\mathcal{G}_{\mathrm{b}}(n, r)$, generated using a binomial point process, on a torus. (a): the degree distribution of one node of $\mathcal{G}_{\mathrm{b}}(n, r), n=500, r=0.2$, on Cartesian distance system; (b): the degree distribution of one node of $\mathcal{G}_{\mathrm{b}}(n, r), n=500, r=0.2$, on toroidal distance system.
the degree distribution obtained numerically, this is due to the edge effect. Figure 3.13(b) shows the degree distribution obtained from numerical experiment (in green) when using a torus compared with the equivalent binomial distribution (in red), in this case the binomial distribution and the distribution obtained numerically are undistinguishable. In here we define the random geometrical graph on a torus to avoid the edge effect.

According to the definition of a binomial point process, the marginal degree distribution is,

$$
\begin{equation*}
\mathbf{P}\left(d_{i}=k\right)=\binom{n-1}{k} p^{k} q^{n-1-k}, \tag{3.75}
\end{equation*}
$$

where $p=\pi r^{2}$ and $r$ is the radius of the disks centred on the nodes. If we assume that the node's degree are independent and the degree is approximated by a binomial $d_{i} \sim \operatorname{Bi}(n-1, r)$, then the probability that no node is isolated is

$$
f(n, r)=\mathbf{P}\left(d_{1}>0\right) \mathbf{P}\left(d_{2}>0\right), \ldots, \mathbf{P}\left(d_{n}>0\right)
$$

$$
\begin{align*}
& =\left(\mathbf{P}\left(d_{i} \geqslant 1\right)\right)^{n} \\
& =\left[\sum_{k=1}^{n-1}\binom{n-1}{k}\left(\pi r^{2}\right)^{k}\left(1-\pi r^{2}\right)^{n-1-k}\right]^{n} \\
& =\left(1-\mathbf{P}\left(d_{i}=0\right)\right)^{n} \\
& =\left(1-\left(1-\pi r^{2}\right)^{n-1}\right)^{n} \tag{3.76}
\end{align*}
$$

Similarly as for the case of $\mathcal{G}(n, p)$, the binomial distribution can be approximated using a Poisson distribution where $d_{i} \sim \operatorname{Poi}(\lambda)$, and $\lambda=(n-1) \pi r^{2}$, an approximation for $f(n, r)$ for large $n$ is

$$
\begin{align*}
f(n, r) & =\mathbf{P}\left(d_{1}>0\right) \mathbf{P}\left(d_{2}>0\right), \ldots, \mathbf{P}\left(d_{n}>0\right), d_{i} \sim \operatorname{Poi}(\lambda), \lambda=(n-1) \pi r^{2} \\
& =\left(\mathbf{P}\left(d_{i} \geqslant 1\right)\right)^{n} \\
& =\left(\sum_{k=1}^{n-1} \frac{((n-1) p)^{k} e^{-(n-1) \pi r^{2}}}{k!}\right)^{n} \\
& =\left(1-\mathbf{P}\left(d_{i}=0\right)\right)^{n} \\
& =\left(1-\left(1-e^{-(n-1) \pi r^{2}}\right)^{n-1}\right)^{n} . \tag{3.77}
\end{align*}
$$

Figure 3.14 and 3.15 shows then numerical and theoretical results for the relationship between minimum node degree and the graph being 1 -connected. For the numerical experiments we considered $10^{6}$ graphs generated on a torus.

The probability that no node is isolated is shown in figure 3.14(a), in 3.14(b) is the probability that the graph is 1 -connected. The analytical result from Formula 3.76 is shown in figure 3.15(a). The analytical result from Formula 3.77 is shown in figure 3.15(b). These results show that both binomial degree distribution with degree assumption and the Poisson approximation works well for predicting the probability that $\mathcal{G}_{\mathrm{b}}(n, r)$ being connected for large graphs.

### 3.5 The experiments: methods and results

In the previous sections we compared our analytical results with the sample statistics obtained from simulations of the different random graphs. In this section we explain how these simulations where performed. We like to remark that the numerical simulations are not straightforward. The two statistics we would like to study are the degree-degree correlation and degree distribution with edge effect. For the evaluation of the degree-degree correlation the numerical experiments simulate the degree of two nodes in an infinite plane, finite plane (bounded square region) and bounded region with edge effect. It is important that the experimental methods are carefully designed so that the correct statistics can be collected.


Figure 3.14: The probability of $G(n, r)$ being connected: exact results obtained from experiments. From right to left the curves in each figures represent $n=5,10,20,50,100,200,500$. (a): is the experimental result for $G(n, r)$ no node isolated. (b): is the experimental results for $G(n, r)$ being 1-connected.

We have included the pseudo-codes ${ }^{3}$ used in our experiments. We assume that the network parameters $n$ and $r$ are given by the network planners. For verification purposes we

[^8]

Figure 3.15: The probability of $G(n, r)$ being connected: results from theoretical estimation. From right to left the curves in each figures represent $n=5,10,20,50,100,200,500$. (a) is obtained using the approximation based on binomial degree distribution (equation 3.76 for $\left.\mathcal{G}_{\mathrm{b}}(n, r)\right)$. (b) is obtained by the theoretical approximation based on Poisson degree distribution (equation 3.77 for $\mathcal{G}_{p}(n, r)$ ).
assigned arbitrary value to these two parameters. The discrepancy between the theoretical results and the experimental results reflect how accurate the theoretical results are. The sample size used is $10^{6}$ events which guarantees convergence and good accuracy.

The correlation coefficient was obtained using the sample correlation coefficient defined as

$$
\begin{equation*}
\hat{\rho}=\frac{z^{-1} \sum_{i} d_{i} d_{j}-\left[z^{-1} \sum_{i} 1 / 2\left(d_{i}+d_{j}\right)\right]^{2}}{z^{-1} \sum_{i} 1 / 2\left(d_{i}^{2}+d_{j}^{2}\right)-\left[z^{-1} \sum_{i} 1 / 2\left(d_{i}+d_{j}\right)\right]^{2}} \tag{3.78}
\end{equation*}
$$

where $z$ denotes the number of edges that have been sampled.

### 3.5.1 Degree-degree correlation for $\mathcal{G}_{\mathbf{p}}(n, r)$ on an infinite plane

In reality it is not possible to generate an infinite plane, thus we generate the coordinates of two nodes $i$ and $j$ accordingly to the distance distribution shown in Lemma 3.2.1. Then we used the radial generation method to generate other nodes' coordinates: choose at random
a set of coordinates in the area defined by a circular area cantered at the middle point of the line joining the nodes $i$ and $j$, the radius of this circular area is $2 r$. The advantage of using this method is that points are generated only when needed, this is especially useful when dealing with infinite area. Then we count the number of nodes that fall into the disk of $i$ and $j$ separately.

Let $\left(x_{i}, y_{i}\right)$ denote the coordinates of node $i ; d_{i}$ denote the degree of node $i$, the pseudocodes of the radial generation methods is shown in Algorithm 1.

```
Algorithm 1 Radial generation for Poisson point process generation.
Globals: \(S_{i}\) is a constant, \(S_{0}=0 ; n:=\) \#nodes
    for \(i=1 \ldots n\) do
        \(S_{i}:=S_{i-1}+\log (\) Uniform \((0,1))\)
        \(\theta_{i}=2 \pi(\operatorname{Uniform}(0,1))\)
        \(x_{i}=\sqrt{-S_{i} /(\pi \lambda)} \cos \left(\theta_{i}\right)\)
        \(y_{i}=\sqrt{-S_{i} /(\pi \lambda)} \sin \left(\theta_{i}\right)\)
    end for
```

In the above code Uniform $(0,1)$ refers to the generation of uniform random number in the interval $(0,1)$.

### 3.5.2 Degree-degree correlation for $\mathcal{G}_{\mathbf{p}}(n, r)$ on a bounded region

The method to evaluate the degree-degree correlation of any two connected nodes in a bounded region is similar to the method described in algorithm 1 . We first generate the coordinates of the two nodes we are going to be considering and then evaluate the correlation function.

Let 0 and 1 denote the labels of two randomly chosen nodes in a square area. We use a Cartesian coordinate system to describe the position of these two nodes. Both the $x$ and $y$ coordinates of node 0 and node 1 are drawn from an uniform random generator. Let $D_{i}$ denotes the distance between node $i$ and the origin. Then neighbouring nodes are generated using the radial generation described in Algorithm 1. After the coordinates of a new node (referred as $i$ ) are generated then we evaluated if the distance between nodes $i, 0$ and $i, 1$ is within the range $r$. If the nodes are within the range $r$ then add one to the
degree of node 0 or node 1 , respectively. We continue generating nodes until the distances between the newly generated nodes and the origin are larger than $D_{0}+r$ and $D_{1}+r$. So we have enough nodes to study the degrees of node 0 and 1 , since new nodes generated will be to far apart and will not contribute to the calculations of the node degrees, neither for 0 nor 1. The pseudo-code for this procedure is shown in Algorithm 2. In the code, hypot refers to the hypotenuse obtained from the segment $\left(x_{0}-x, y_{0}, y\right)$

```
Algorithm 2 Square Bounded region: Polar coordinate system
Globals: \(S_{0}=0 ; z:=\#\) edges;
    for \(i=1 \ldots m\) do
        \(x_{0}=a \times \operatorname{Uniform}(0,1)\)
        \(x_{1}=a \times \operatorname{Uniform}(0,1)\)
        \(x_{1}=a \times \operatorname{Uniform}(0,1)\)
        \(y_{1}=a \times \operatorname{Uniform}(0,1)\)
        \(\rho_{0}=\operatorname{hypot}\left(x_{0}, y_{0}\right)<r\)
        \(\rho_{1}=\operatorname{hypot}\left(x_{1}, y_{1}\right)<r\)
        while 1 do
            \(s+=\log (\operatorname{Uniform}(0,1))\)
            \(\theta=2 \pi(\operatorname{Uniform}(0,1))\)
            \(\rho=\sqrt{-s /(\pi \lambda)}\)
            \(x=\rho \cos \left(\theta_{i}\right)\)
            \(y=\rho \sin \left(\theta_{i}\right)\)
            if \(\rho>\rho_{0}+r\) and \(\rho>\rho_{1}+r\) then
                break
            end if
            if \(\operatorname{hypot}\left(x_{0}-x, y_{0}-y\right)<r\) then
                \(d_{0}++\)
            end if
            if \(\operatorname{hypot}\left(x_{1}-x, y_{1}-y\right)<r\) then
                \(d_{1}++\)
            end if
        end while
    end for
```


### 3.5.3 Edge effect influences node degree distribution $\mathcal{G}_{\mathbf{p}}(n, r)$ on a circular area

The edge effect problem is shown in figure 3.6(b). The evaluation of the edge effect is simple. We first generate the total number of nodes $n$ using a Poisson random generator. Then we assign to each of the nodes a set of random coordinates $\left(x_{i}, y_{i}\right)$. Then from node label as 1 to node $n$ we count the number of nodes falling in the node's disk area.

```
Algorithm 3 Edge effect simulation: Polar coordinate system, circular area.
Globals: \(a:=1\) length of the square; \(n:=\) \#nodes;
    for \(i=1 \ldots n\) do
        \(\theta=2 \pi(\operatorname{Uniform}(0,1))\)
        \(x_{i}:=\cos (\theta)\)
        for \(j=1 \ldots i\) do
            \(y_{i}:=\sin (\theta)\)
            \(d_{x}=\left\|x_{i}-x_{j}\right\|\)
            if \(d x<r\) then
            \(d y=\left\|y_{i}-y_{j}\right\|\)
            if \(d y<r\) and \(d x^{2}+d y^{2}<r^{2}\) then
                    \(d_{i}++\)
                    \(d_{i}++\)
            end if
            end if
        end for
    end for
```


### 3.5.4 Minimum degree and 1-connected property.

The evaluation of the minimum degree $\delta(G)$ and the 1-connected property are based in the following approach

## Generating random graphs

The pseudocode to generate an instance of the random graph $\mathcal{G}(n, p)$ is shown in Algorithm 4 (more graph algorithms can be found in the book [36]), where $\operatorname{rand}()$ generates a random uniform number in the interval $(0,1)$.

```
for \(z=1 \ldots 10^{6}\) do
    Generate a graph;
    Find the minimum node degree;
    Check whether this graph is connected;
    end for
```

    Calculate the frequency of graphs having minimum node degree larger than zero
    Calculate the frequency of graphs being 1-connected.
    ```
Algorithm \(4 \mathcal{G}(n, p)\) graph instance generation
Globals: \(S_{0}=0 ; n:=\#\) nodes
    for \(i=1 \ldots n\) do
        for \(i=1 \ldots n\) do
            if \(\operatorname{rand}()<p \times R A N D_{M A X}\) then
            \(d_{i}++\)
            \(d_{i}++\)
            end if
        end for
    end for
```


## Generating random geometric graphs

To generate a random geometric graph on a torus, first we distribute at random nodes in a square. To evaluate the distance between these nodes, the square area is projected into eight copies of itself, see figure 3.16. The distance between nodes $A$ and $B$ is chosen to be the smallest distance between all nine $A, B$ pairs. The pseudocode describing this procedure is in Algorithm 5, Algorithm 6 shows the modifications of algorithm 5 to evaluate the edge effect.

```
Algorithm 5 Experiment avoiding edge effect: Cartesian coordinate system, toroidal
distance.
Globals: \(S_{0}=0 ; n:=\#\) nodes
    for \(i=1 \ldots n\) do
        \(x_{i}:=a \operatorname{Uniform}(0,1)\)
        for \(i=1 \ldots n\) do
        \(y_{i}:=a \operatorname{Uniform}(0,1)\)
        \(d_{x}=\left\|x_{i}-x_{j}\right\|\)
        if \(d_{x}>\frac{a}{2}\) then
            \(d x=a-d x\)
        end if
        if \(d x<r\) then
            \(d y=\left\|y_{i}-y_{j}\right\|\)
        end if
        if \(d y>\frac{a}{2}\) then
            \(d y=a-d y\)
        end if
        if \(d y<r\) and \(d x^{2}+d y^{2}<r^{2}\) then
            \(d_{i}++\)
            \(d_{i}++\)
        end if
        end for
    end for
```

```
Algorithm 6 Experiment with edge effect: Cartesian coordinate system, square distance.
Globals: \(S_{0}=0 ; n:=\) \#nodes
    for \(i=1 \ldots n\) do
        \(x_{i}:=a \operatorname{Uniform}(0,1)\)
        for \(i=1 \ldots n\) do
            \(y_{i}:=a \operatorname{Uniform}(0,1)\)
            \(d_{x}=\left\|x_{i}-x_{j}\right\|\)
            \(d y=\left\|y_{i}-y_{j}\right\|\)
            if \(d y<r\) and \(d x^{2}+d y^{2}<r^{2}\) then
            \(d_{i}++\)
            \(d_{i}++\)
            end if
        end for
    end for
```



Figure 3.16: The Euclidean distance and toroidal distance: (a): a stage of binomial point process on Euclidean distance on a square; (b): a stage binomial point process on toroidal distance. A and B stand for two randomly chosen nodes, to avoid edge effect $\|A-B\|$ is chosen so that it is always the smallest value among all the nine $A, B$ pairs.

### 3.6 Conclusions

We have studied the degree dependence and connectivity of random graph models and random geometric graph models. We proved that the degree distribution are dependent for above models. During the proof we derive exact formulas for the joint distribution and correlation coefficient. The correlation coefficient between any two chosen nodes in random graphs is in fact $\frac{1}{n-1}$, which means this correlation is negligible for large graphs. We also derived the exact result for the correlation coefficient for random geometric graph $\mathcal{G}_{\mathrm{p}}(n, r)$ for two overlapping nodes in infinite graphs. The results show that there is positive constant correlation between two overlapping nodes. For $\mathcal{G}_{\mathrm{p}}(n, r)$ in a bounded region, the correlation is monotonically increasing to the limit of 1 . We explained the difference between edge effect and bounded region and showed how edge effect influence the node degree distribution for $\mathcal{G}_{\mathrm{p}}(n, r)$ both analytically and experimentally.

We derived an exact formula for the probability that $\mathcal{G}(n, p)$ has no isolated nodes
deterministically. We compared this deterministic result to existing asymptotic results by assuming that node degrees are statistically independent with each other. We derived an approximation formula for the connectivity of $\mathcal{G}(n, p)$. We compared the Poisson approximation and binomial distribution with the deterministic results. The experiments show that the assumption of degree independent does not have significant impact on this particular problem. The above results and methodologies provided useful information for network planners who wish to initiate some basic parameters such as the number of ad-hoc devices and the power levels to pursue good overall network connectivity.

## Chapter 4

## The minimum number of channels to minimize co-channel interference in WLAN

In this chapter we estimate the minimum number of non-overlapping channels a network may have, such that no neighbouring access points (APs) operate on the same channel. For example, in IEEE 802.11 WLAN, if a service provider plans to install a number of APs, it is important to know how many APs can be installed in the target area given the limited number of non-overlapping channels. The IEEE 802.11b/g standards provide 3 non-overlapping and 13 overlapping channels. The IEEE 802.11a standard provides 3 non-overlapping and $13-19$ overlapping channels. The existance of different channels allows APs to transmit simultaneously. More APs could install more wireless devices into the same target area to increase their capacity but this could cause severe interference to all the wireless users. When one AP is transmitting under interference, the IEEE 802.11-CSAM/CM (Carrier Sense Multiple Access with Collision Avoidance) mechanism will requires that the neighbours APs wait for a long time to establish a connection. A better scheduling method is to assign the channels to the APs in such a way that there is no co-channel interference, for example to assign channels that not overlap to neighbouring APs. As the typical transmission range of an APs in an urban regime is between 50 m to 100 m , then two APs can use the same channel if they are a distance apart greater than 100 m . If the number of APs increases and the transmission range stays the same, the network will reach a limit when the number of available non-overlapping channels will run out. It is possible to allow more APs into the target area but this will require to decrease the transmission range of some or all the APs or, to increase the number of non overlapping
channels. So the natural questions to ask are: what is the density of APs where there will be always interference and how many channels we would need to overcome this interference.

In the literature the scheduling problem for wireless network has been mapped into the colouring problem of random geometric graphs. Different colours represent non-overlapping channels. The chromatic number of a graph $\chi$ is the minimum number of colours needed so that no adjacent nodes/edges are assigned the same colour. Therefore the minimum number of channels needed to avoid interference is equivalent to the chromatic number of the equivalent graph. However graph colouring (or equivalently minimizing interference) is a NP-hard (non-polynomial) problem. In spite of this it is possible to obtain an estimate on how many colours are needed for colouring a graph. These estimates are based on the close relationship between the chromatic number, the maximum degree $\Delta_{n}$ present in the graph and the clique number $\omega_{n}$. In here we would use this relationship to estimate the chromatic number and hence the number of non-overlapping channels to avoid interference.

The interference in the network will depend on how densely the nodes are distributed in space. Clearly we expect more interference if the density of nodes is high. Thus, the colouring problem has to be related to this density of nodes. In this chapter, following Penrose [32], we define the subconnective regimes as a function of the node density when the number of nodes tends to infinity. The summary for all connective regimes can be found in table 6.1 in Appendix 1. Examples of graphs in connective and superconnective regime are shown in figure 6.1 in Appendix 1. In the case that the density is low (sparse graph) the value of the chromatic number $\chi_{n}$ has a desirable property known as focusing, which means that as $n \rightarrow \infty$, with high probability, the value of the chromatic number falls between two adjacent integer values, in other words, the value of $\chi_{n}$ is virtually predictable. This kind of results are very useful because they give a bound of the chromatic number without explicitly evaluating it (which is NP-hard).

In this chapter we explain how to obtain accurate asymptotic bounds for the chromatic number of sparse random geometric graphs $\mathcal{G}_{\mathrm{b}}(n, r)$. To do so we also obtained accurate asymptotic approximations for $\Delta_{n}$ and $\omega_{n}$ for sparse graphs. We investigate the accuracy of our asymptotic results via extensive simulations. The numerical evaluation of
the chromatic number of a random geometric graph is obtained by taking a sample graph from the ensemble of geometric graphs; calculating the chromatic number of this graph, repeating this procedure for a large number of times and evaluating the average behaviour of the chromatic number. Here we assume that the node degrees are independent and can be described with a binomial random distribution, or approximated with a Poisson distribution if the number of nodes $n$ is large. The asymptotic bounds obtained here are considerably more accurate than the existing results [32]. Our numerical results show that the asymptotic approximations obtained here are accurate even for small graphs $(n=10)$.

### 4.1 Related works

In practise the graphs representing the interference in WLANs are sparse [40]. This is because in WLAN nodes have limited resources. The degree of one node in the interference graphs stand for the number of other nodes that are interfering with this node. Higher node degree means more interference. In general the sparseness of graphs indicates the graph is with high probability not connected. Here the sparseness is the same definition with that of [32] and [24].

In random geometric graphs these three properties $\left(\omega_{n}, \Delta_{n}\right.$ and $\left.\chi_{n}\right)$ are often treated together. We use $\omega_{n}, \Delta_{n}$ and $\chi_{n}$ here to differentiate the properties of an ensemble of graphs with that of one graph. Asymptotic results $(n \rightarrow \infty)$ have been derived by Penrose [32] for $\Delta_{n}, \omega_{n}$ and $\chi_{n}$ separately for different connectivity regimes. There are three connectivity regimes defined by Penrose: subconnective regime, connective regime and superconnctive regime. These connectivity regimes are defined by how fast the mean degree $\overline{d_{i}}$ of a node $i$ grows as the network grows. The growth of the network is characterised by the logarithm of the number of nodes.

Recall that for random geometric graph $\mathcal{G}_{\mathrm{b}}(n, r)$ the degree $d_{i}$ of a randomly chosen node $i$ is described by

$$
\begin{equation*}
\mathbf{P}\left(d_{i}=k\right)=\binom{n-1}{k} p^{k}(1-p)^{n-1-k} \tag{4.1}
\end{equation*}
$$



Figure 4.1: Example of a subconnective graph drawn on a $1 \times 1$ square.
where $p=\pi r^{2}$ is related to the node's area and the average degree is $\overline{d_{i}}=\mathbb{E}\left(d_{i}\right)=n \pi r^{2}$.

### 4.1.1 Subconnective regime

In the subconnective regime the average degree $\overline{d_{i}}$ grows more slowly than $\log (n)$. Figure 4.1 shows an example of a subconnected graph for $n=50, r=0.09$. The mean degree $\overline{d_{i}}$ satisfies $n r^{2}=\exp (-\sqrt{\log (n)})$. Graphs in the subconnective regime are sparse.

From Penrose results we have that in a subconnective regime: $n r^{2}=o(\log n)$, which is equivalent to saying $\lim _{n \rightarrow \infty}\left(\frac{n r^{2}}{\log n}\right)=0$. There are two theorems for $\Delta_{n}$ and $\omega_{n}$ for this regime:

Theorem 4.1 (Penrose). Suppose that $r$ satisfies both

$$
\begin{equation*}
n r^{2}=o(\log n) \tag{4.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\log \left(\frac{1}{n r^{2}}\right)=o(\log n) \tag{4.3}
\end{equation*}
$$

4.2. NEW ASYMPTOTIC RESULT FOR MAXIMUM DEGREE FOR GENERAL $\mathcal{G}_{\mathbf{B}}(N, R)$
then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \Delta_{n}, \omega_{n}, \chi_{n}=\frac{\log n}{\log \left(\log n /\left(n r^{2}\right)\right)} \tag{4.4}
\end{equation*}
$$

Theorem 4.2 (Penrose). If $n^{(k+1 / k) / k} r^{2} \rightarrow \lambda^{1 / k}$ as $n \rightarrow \infty, k=1,2,3, \ldots, \lambda \geqslant 0$ then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbf{P}\left(\Delta_{n}, \omega_{n}=k-1, k\right)=1 \tag{4.5}
\end{equation*}
$$

### 4.2 New asymptotic result for maximum degree for general $\mathcal{G}_{\mathbf{b}}(n, r)$

The focusing phenomenon in graphs in the subconnective regime given by Penrose for $\chi_{n}$ in section 4.1.1 states that $\chi_{n}, \omega_{n}$ and $\Delta_{n}$ have the same limit as $n \rightarrow \infty$. Since for any graph $\omega \leqslant \chi \leqslant \Delta+1$, the focusing phenomenon should occur for $\chi_{n}$ and should be the same for $\Delta$. From our experimental results we observed that bounds obtained by Penrose for $\Delta_{n}, \omega_{n}$ and $\chi_{n}$ are not accurate (an example of our numerical experiments are figures 4.9, 4.10 and 4.11 which will be explained later in the chapter). We would like to find more accurate bounds for $\chi_{n}$. In this section we will introduce our new asymptotic results for $\chi_{n}$ and $\Delta_{n}$.

According to the definition of a random geometric graph generated using a binomial point process, the node degrees form $n$ dependent binomial random variables. For large networks, we can assume that the node degrees are independent and use the Poisson distribution as an approximation of the binomial distribution. In this case we have that if $p=\pi r^{2}$ and $\lambda=n \pi r^{2}$ as $n \rightarrow \infty$ then degree distribution $d_{i}$ is described by

$$
\begin{equation*}
\mathbf{P}\left(d_{i}=k\right)=\binom{n-1}{k} p^{k}(1-p)^{n-1-k} \tag{4.6}
\end{equation*}
$$

which can be approximated as

$$
\begin{equation*}
\mathbf{P}\left(d_{i}=k\right) \approx e^{-\lambda} \lambda^{k} / k! \tag{4.7}
\end{equation*}
$$

### 4.2. NEW ASYMPTOTIC RESULT FOR MAXIMUM DEGREE FOR GENERAL $\mathcal{G}_{\mathbf{B}}(N, R)$

To evaluate the maximum degree $\Delta$ notice that the degree distribution for $n$-node graphs is equivalent to $n$ random Poisson variables. If $\left\{X_{1}, X_{2}, \ldots, X_{n}\right\}$ denotes $n$ independent Poisson random variables with common mean $\lambda$ then $M_{n}=\max \left(X_{i}\right)=\Delta_{n}$. In this respect we have obtained the following theorem:

Theorem 4.3. Let $\left\{X_{1}, X_{2}, \ldots, X_{n}\right\}$ be i.i.d. Poisson random variables with common mean $\lambda$, so that $\operatorname{Pr}\left[X_{i}=k\right]=e^{-\lambda} \lambda^{k} / k!$. Let $M_{n}=\max \left(X_{i}\right)$ and $\lambda$ fixed, then, with high probability, $M_{n}$ is focused on two adjacent integer values as $n \rightarrow \infty$, one of these values is:

$$
\begin{equation*}
M_{n} \sim x_{0} \equiv \frac{\log n}{W\left(\frac{\log n}{\exp (1) \lambda}\right)} . \tag{4.8}
\end{equation*}
$$

or more accurately

$$
\begin{equation*}
M_{n} \sim x_{1}=x_{0}+\frac{\log \lambda-\lambda-\log (2 \pi) / 2-3 \log \left(x_{0}\right) / 2}{\log \left(x_{0}\right)-\log \lambda} . \tag{4.9}
\end{equation*}
$$

Proof 1.5 We have

$$
\begin{equation*}
\operatorname{Pr}\left[X_{i}<k\right]=Q(k, \lambda) \equiv \Gamma(k, \lambda) / \Gamma(k) \tag{4.10}
\end{equation*}
$$

where $Q$ and $\Gamma(\cdot, \cdot)$ are incomplete Gamma functions; that is,

$$
\begin{equation*}
\Gamma(a, x)=\int_{x}^{\infty} t^{a-1} e^{-t} \mathrm{~d} t \tag{4.11}
\end{equation*}
$$

From the independence of Poisson variables we have that

$$
\begin{equation*}
\operatorname{Pr}\left[M_{n} \leqslant k\right]=\operatorname{Pr}\left[X_{1} \leqslant k\right]^{n}=Q(k+1, \lambda)^{n}=\Gamma(k+1, \lambda)^{n} / \Gamma(k+1)^{n} . \tag{4.12}
\end{equation*}
$$

and that

$$
\begin{align*}
\operatorname{Pr}\left[M_{n}=k\right] & =\operatorname{Pr}\left[M_{n} \leqslant k\right]-\operatorname{Pr}\left[M_{n} \leqslant k-1\right] \\
& =Q(k+1, \lambda)^{n}-Q(k, \lambda)^{n} \tag{4.13}
\end{align*}
$$



Figure 4.2: The distribution of the maximum values of Poisson distribution where $\lambda=5$. From left to right $n=10^{0}, 10^{2}, 10^{4}, \ldots, 10^{24}$. Note that there is an error in Fig 1 in [3], where the curves labelled $k=6$ and $k=8$ are incorrect.

Examples of the distributions given by equation (4.13) are shown in figure 4.2. These numerical results demonstrate the focusing effect. The maxima $M_{n}$ are concentrated on at most two adjacent integers for large $n$. We call these adjacent integers the modal values. The focusing phenomenon allows us to estimate the modal values precisely.

Previously Anderson [2] proved the existence of integers $I_{n}$ and $I_{n+1}$ such that $\operatorname{Pr}\left[M_{n} \in\right.$ $\left.\left(I_{n}, I_{n}+1\right)\right] \rightarrow 1$ as $n \rightarrow \infty$ for fixed $\lambda>0$; and that $I_{n} \sim \beta_{n}$, where $\beta_{n}$ is defined as the unique solution of

$$
\begin{equation*}
Q\left(\beta_{n}, \lambda\right)=1 / n . \tag{4.14}
\end{equation*}
$$

Following Anderson's work, Kimber [22] showed that $I_{n} \sim \log n / \log \log n$ and $P_{n} \sim$ $\left(k / I_{n}\right)^{1+B_{n}}$ with $B_{n}$ dense in $[-1 / 2,1 / 2]$, and concluded that to first order, the rate of growth of $I_{n}$ is independent of the Poisson parameter $\lambda$. Kimber concluded that $P_{n}$, defined as $P_{n}=\operatorname{Pr}\left[M_{n} \in\left(I_{n}, I_{n}+1\right)\right]$, oscillates and this oscillation persists for arbitrarily large $n$. Figure 4.3 illustrates exactly how this probability oscillates as $n \rightarrow \infty$. In figure 4.4, the cyan curve shows that $\frac{\log n}{\log \log n}$ estimates $I_{n}$ very poorly.


Figure 4.3: The maximal probability (with respect to $I_{n}$ ) that $M_{n} \in\left\{I_{n}, I_{n}+1\right\}: \lambda=5$. From left to right $10^{0} \leqslant n \leqslant 10^{40}$. The curves show the probability that $M_{n}$ takes either of its two most frequently occurring values.


Figure 4.4: Exact values and asymptotics of $I_{n}: \lambda=5$. From left to right $n=10^{0}, \ldots, 10^{40}$. The staircase red line (almost hidden by the cyan line) represents the exact mode $I_{n}$. The other lines represent asymptotic approximations: green for the result of [22] (which is independent of $\lambda$ ), dark blue and cyan for our new results $x_{0}$ and $x_{1}$ respectively. The cyan curve always sits between the steps of $I_{n}$, meaning that $x_{1}$ has error less than unity.

### 4.2. NEW ASYMPTOTIC RESULT FOR MAXIMUM DEGREE FOR

 GENERAL $\mathcal{G}_{\mathbf{B}}(N, R)$We aim here to refine Kimber's results and obtain a more accurate asymptotic equation. We consider a continuous distribution $g$ for $\operatorname{Pr}\left[M_{n}=k\right]$ by inserting equation 4.12 into equation 4.13

$$
\begin{align*}
g_{\lambda}(x) & =\operatorname{Pr}\left[M_{n}=k\right]  \tag{4.15}\\
& \equiv 1-\Gamma(x+1, \lambda) / \Gamma(x+1) . \tag{4.16}
\end{align*}
$$

Following Anderson's approach [2] (equation 4.14), we solve $g(x)=1 / n$. For fixed $\lambda \in \mathbb{R}^{+}$, equation 4.16 is a strictly decreasing function on $(0, \infty)$. If $\epsilon=1 / n$ is a small, positive and real, then $g_{\lambda}(x)$ has a unique root $x(\epsilon)$ which increases as $\epsilon \rightarrow 0^{+}$. What we need to find is an asymptotic expansion of this root $x(\epsilon)$ as $\epsilon \rightarrow 0$.

We have that the logarithm of

$$
\begin{equation*}
g_{\lambda}(x)=\exp (-\lambda) \lambda^{x} \sum_{i=1}^{\infty} \frac{\lambda^{i}}{\Gamma(x+i+1)} \tag{4.17}
\end{equation*}
$$

can be expanded as

$$
\begin{align*}
\log \left(g_{\lambda}(x)\right)=-x \log (x)+(1+ & \log \lambda) x-\frac{3}{2} \log (x) \\
& +\left(\log \lambda-\lambda-\frac{\log (2 \pi)}{2}\right)+\frac{\lambda-13 / 12}{x}+\mathcal{O}\left(x^{-2}\right) . \tag{4.18}
\end{align*}
$$

This last equation was obtained using Maple, the code producing this result is shown in figure 4.5. The first approximation to the solution of $\log \left(g_{\lambda}(x)\right)=-\log n$ is given by keeping only the dominant $x \log (x)$ term in equation (4.18):

$$
\begin{equation*}
M_{n} \sim x_{0} \equiv \frac{\log n}{W\left(\frac{\log n}{\exp (1) \lambda}\right)} \tag{4.19}
\end{equation*}
$$

In this last equation $W(\cdot)$ is the principal branch of Lambert's $W$ function [10]. Figure 4.4 compares $M_{n}$ obtained numerically against the asymptotic approximations. In the figure the dark blue curve is $x_{0}$. The stair-wise curve shows the mode for the Poisson max-

### 4.2. NEW ASYMPTOTIC RESULT FOR MAXIMUM DEGREE FOR GENERAL $\mathcal{G}_{\mathbf{B}}(N, R)$

Figure 4.5: Maple worksheet for the expansion of $\log (g)$
ima. From the figure it is clear that equation 4.19 is quite accurate. Ideally the difference between the approximation and the real value of $M_{n}$ should be less than unity, so that the mode of the distribution is correctly identified. A refinement of the asymptotic approximation $x_{1}$ is obtained by using Newton's method; that is, $x_{1}=x_{0}-\left(h\left(x_{0}\right)+\log n\right) / h^{\prime}\left(x_{0}\right)$, where $h$ is another approximation to $\log \left(g_{\lambda}\right)$. For example, by keeping only the terms proportional to $x$ and $\log x$ from equation (4.18) in $h$, we obtain

$$
\begin{align*}
M_{n} & \sim x_{1}=x_{0}+\frac{\log \lambda-\lambda-\log (2 \pi) / 2-3 \log \left(x_{0}\right) / 2}{\log \left(x_{0}\right)-\log \lambda}  \tag{4.20}\\
I_{n} & =\left\lfloor M_{n}\right\rfloor, I_{n+1}=\left\lceil M_{n}\right\rceil \tag{4.21}
\end{align*}
$$

In figure 4.4 the mode of the Poisson maxima is represented by the staircase curve. Each step in the stair represents an increase of 1 from the previous value. Our approximation equation 4.20 seems to have an error less than unity and it is much more accurate than previous work. If further accuracy is required it may be obtained by additional Newton steps. This ends the proof of the theorem.
4.2. NEW ASYMPTOTIC RESULT FOR MAXIMUM DEGREE FOR GENERAL $\mathcal{G}_{\mathbf{B}}(N, R)$

### 4.2.1 The maximum degree in random geometric graphs

In the previous section we developed an accurate approximation to estimate the maximal value in a sequence of independent Poisson random variables. In this section, under the same assumptions as the previous section, we develop a new asymptotic approximation for the maximum degree in random geometric graphs. For binomial distributions, Nadarajah and Mitov [28], based on the Gumbel distribution have proved the following theorem for extreme values of the distribution:

Theorem 4.4. Let $X_{1}, X_{2}, \ldots, X_{n}$ be i.i.d. binomial random variables with parameter $N=N(n) \rightarrow \infty$ and parameter $p$ fixed and $q=1-p$. If $N(n)$ grows with $n$ according to $(\log n)^{3}=o(N(n))$ then

$$
\begin{align*}
M_{n} & \rightarrow p N(n)+b(n) \sqrt{N(n) p q}  \tag{4.22}\\
b(n) & =\sqrt{2 \log n}-\frac{\log \log n+\log 4 \pi}{2 \sqrt{2 \log n}} \tag{4.23}
\end{align*}
$$

Proof 1.6 For the above $n$ i.i.d. binomial distribution, Nadarajah and Mitov [28] have prove the following for extreme values of the distribution base on the Gumbel distribution. If $N(n)$ grows with $n$ according to $(\log n)^{3}=o(N(n))$ then there are sequences:

$$
\begin{equation*}
a(n)=\frac{1}{\sqrt{2 \log n}}, b(n)=\sqrt{2 \log n}-\frac{\log \log n+\log 4 \pi}{2 \sqrt{2 \log n}} \tag{4.24}
\end{equation*}
$$

such that:

$$
\begin{equation*}
\mathbf{P}\left(M_{n} \leqslant \sqrt{p q N(n)} a(n) x+p N(n)+\sqrt{p q N(n)} b(n) \rightarrow \exp (\exp (-x))\right) \tag{4.25}
\end{equation*}
$$

as $n \rightarrow \infty$.
Now we prove that the maxima of such distribution occurs at $x=0$. The equation of

$$
\begin{equation*}
\mathbf{P}\left(\frac{M_{n}-p N-b(n) \sqrt{p q N(n)}}{\sqrt{p q N(n)} a(n)} \leqslant x\right) \rightarrow \exp (-\exp (-x)) \tag{4.26}
\end{equation*}
$$



Figure 4.6: (a): the Gumbel or type I extreme value distribution as the continuous approximation of the cdf (cumulative distributed function) for binomial maxima. (b): the pdf (probability distribution function) shows that the mode of the binomial maxima occurs at $x=0$.

Let $Y=\frac{M_{n}-p N-b(n) \sqrt{p q N(n)}}{\sqrt{p q N(n) a(n)}}$ in the previous equation; we then have

$$
\begin{equation*}
\mathbf{P}(Y \leqslant x) \rightarrow \exp (-\exp (-x)) \tag{4.27}
\end{equation*}
$$

Therefore the cdf of $Y$ is obtained by differentiating equation (4.2.1)

$$
\begin{equation*}
\operatorname{cdf}(Y)=\exp (-x) \exp (-\exp (-x)) \tag{4.28}
\end{equation*}
$$

The mode of binomial maximum occurs when $x=0$. Figure (4.6) shows the cumulative distribution function and pdf of the Gumbel distribution. From figure (4.6)(b) it is clear that the maximum occurs at $x=0$. To obtain the bounds of the maximum degree we evaluate equation 4.25 at $x=0$, that gives

$$
\begin{align*}
& M_{n}-p N(n)-b(n) \sqrt{N(n) p q} \rightarrow 0  \tag{4.29}\\
& M_{n} \rightarrow p N(n)+b(n) \sqrt{N(n) p q} \tag{4.30}
\end{align*}
$$

Since the modal value of binomial distribution must be integral, it should be between the two integer part $I_{n}$ and $I_{n+1}$ of $M_{n}$, where

$$
\begin{equation*}
I_{n}=\left\lfloor M_{n}\right\rfloor, I_{n+1}=\left\lceil M_{n}\right\rceil . \tag{4.31}
\end{equation*}
$$

### 4.3 Experimental results for maximum degree and chromatic number

In this section we verify the validity of our approximations to the mode of the maximum degree. We developed two different approaches, one based using a Poisson distribution as a simplification of the Binomial process (equations 4.20 and 4.20) and the other approximation using the extreme values of a Gumbel distributions (equations 4.30 and 4.31). Firstly we verified if these approximation work well for any random geometrical graphs
$G(n, r) n=20$

(a)
$G(n, r) n=200$

(b)

Figure 4.7: Experimental and numerical results for the mode of the maximum degree of $\mathcal{G}_{\mathrm{b}}(n, r)$. The triangles show the numerical result and the ?squares are the experimental results (a): the number of nodes $n=20$; (b): the number of nodes $n=200$.
and secondly, for sparse graphs we verified if the approximations describe well the focusing phenomenon, that is the bounding of the chromatic number using the maximum degree.

### 4.3.1 Maximum degree for a general random geometric graphs

We now apply a binomial/Gumbel-based method to estimate the maximum degree of a general $\mathcal{G}_{\mathrm{b}}(n, r)$ graph. Here we continue to use the same set-up as the experiments in Chapter 3. This time instead of collecting the information of the minimum degree we collect the information of the maximum degree. The experiment procedure follows Algorithm 5. We have two sets of experiments, where the number of nodes equals to 20 and 200 respectively. In both experiments the disk radius $r$ obey a linear growth, that is, $r$ increases by $1 \%$ each step, starting from 0.01 for 500 steps. The experiment results are shown in figure 4.7. Figure 4.7 (a) and 4.7 (b) corresponds to the number of node equals to 20 and 200 respectively. The red triangles stand for the maximum degree obtained
$G(n, r) n=20$

(a)
$G(n, r) n=200$

(b)

Figure 4.8: Experimental and numerical results for the mode of the maximum degree of $\mathcal{G}_{\mathrm{b}}(n, r)$. The triangles show the numerical result and the squares are the experimental results (a): the number of nodes $n=20$; (b): the number of nodes $n=200$.
using the binomial/Gumbel approximation. The black squares stand for the experimental results of the mode of the maximum degree. Each black square in these results stand for the most frequent value (the model value) of the maximum degree for 2000 graph samples. The discrepancy between the red-triangles line and the black-squares line show that binomial/Gumbel based method approximate the mode of the maximum degree better when $n$ is larger. This is because the degree dependence diminishes as $n \rightarrow \infty$.

Next we use Poisson distribution as an approximation for the binomial distribution and evaluate the maximum degree for $\mathcal{G}_{\mathrm{b}}(n, r)$. We use the same experimental setup as for the previous result. The experimental (black-squares) and the maximum degree obtained from the Poisson approximation (red-triangles) are shown in figure 4.8. These results show that Poisson distribution is not a good approximation for binomial distribution for dense graphs.

### 4.3.2 Focusing phenomenon for sparse networks

Sparse graphs is more interesting in practise since the interference graph representing the interference in WLANs is sparse. Also the limit of $\Delta_{n}$ and $\chi_{n}$ have the focusing feature in sparse graphs. $\chi$ is the minimum number of channels needed for the interference problem in WLANs. We know that the chromatic number is NP-hard problem. Hence we wish to find a better approximation for $\Delta_{n}$. Sparse graphs refers those graphs with few edges. $\pi n r^{2}$ is average degree or the average number of edges $(\overline{|E|})$ in a graph. How $\overline{|E|}$ defers in general $\mathcal{G}_{\mathrm{b}}(n, r)$ and sparse $\mathcal{G}_{\mathrm{b}}(n, r)$ can be found in figure 6.2(a) and 6.2(b) in Appendix-1.

There are two theorems in [32]: the limiting theorem and the focusing theorem. The limiting theorem states that $\Delta_{n}, \omega_{n}$ and $\Delta_{n}$ converge to the same limit in subconnective regime. The focusing phenomenon states that $\Delta_{n}, \omega_{n}$ in sparse $\mathcal{G}_{\mathrm{b}}(n, r)$ has sharp distribution and the modal value falls into two adjacent integers. If we can find a good approximation to $\Delta_{n}$, then we finds a good approximation for $\chi_{n}$ as well. We now compare Poisson based method, binomial/Gumbel based method and with Theroem 4.2 on $\Delta_{n}$ and $\omega_{n}$. Then we test it on $\chi_{n}$. We continue to use the ensemble of graphs as stated in this theorem.

$$
\begin{equation*}
n r^{2}=(1 / n)^{1 / k}, k=3,5,7,9 \tag{4.32}
\end{equation*}
$$

The experiments are done for $n=2,5,20,50,100$. We generate 2000 geometric graphs for each $n, k$ combinations and calculate the chromatic number. The algorithm for generate the histograms of the chromatic numbers is shown in Algorithm 7.

Experimental results ${ }^{1}$ and the numerical results can be found in figure 4.9 for $\Delta_{n}$ and figure 4.11 for $\chi_{n}$. In these results, the green and red horizontal lines represents two integer values $k, k+1$ in the focusing Theorem 4.2. The blue and magenta lines represent the numerical results of Poisson based method. The cyan lines represent the numerical results of binomial/Gumbel based method. More figures for larger $n$ can be found in figure 6.3 to 6.5 in Appendix-1.

[^9]```
Algorithm 7 Generate histograms of chromatic numbers
Globals: i:= #trials, n:= #nodes
    for }k=3,5,7,9 d
        OpenAFile
        for n=2,5,20,50,100 do
            Calculate r
            GenerateAHistogram
            for i=1,\ldots,,2000 do
                GenerateAGraph
                Calculate chi
                Add chi to histogram
            end for
            Write histogram to file
        end for
    end for
```

The presentation of all the sub-graphs in figure 4.9, 4.10 and 4.11 are all the same. They shows that both Poisson based method and B approximate the two values which $\Delta_{n}$ is focused on accurately. Take figure 4.9(d) for an example. Each blue vertical dashed lines has a histogram growing horizontally towards the left. For an example the dashed line pointing to 60 in the $x$ axis is the the distribution of $\Delta_{n}$ for $n r^{2}=(1 / n)^{1 / k}, k=9, n=60$. The histogram is scaled to the length between every two vertical dashed blue lines. In the histogram the center of each bars is sitting on an integer on the $y$ axis, which is horizontally pointed by the bar itself. For an example in figure 4.9 (d), when $n=60$ the possible values for $\Delta_{n}$ is $3,4,5$, with high probability (about $85 \%$ ) that $\Delta_{n}$ is either $4,5.4,5$ is the two integers $I_{n}$ and $I_{n+1}$ we would like to predict. Our Poisson maxima results represented by the magenta and blue lines which is calculated from equation 4.21 predict 4,5 accurately. The cyan lines represent the numerical results from binomial/Gumbel based method. Both approximations (Gumbel/Binomial and Poisson) are equally good at approximating the maximum degree for sparse $\mathcal{G}_{\mathrm{b}}(n, r)$ graphs.


Figure 4.9: The focusing phenomenon of $\Delta$ in random geometric graph $\mathcal{G}_{\mathrm{b}}(n, r)$ in subconnective limit regime, $n r^{2}=(1 / n)^{1 / k}$. Green and red lines: the two integer values described in the focusing theorem in Theorem (4.2); magenta and blue: the two integer values from the Poisson maxima approximation in equation (4.9); the red histograms are the distribution of $\Delta$ from $10^{5}$ trials on number of node equals to $n, n$ is the integer which the dotted blue vertical lines point to. $n=2,5,10,20,50,100$, (a): $k=3$; (b): $k=5$; (c): $k=7$; (d): $k=9$.
$G(n, r), n r^{2}=(1 / n)^{(l /(3-1))}, k=3$

(a)
$G(n, r), n r^{2}=(1 / n)^{(1 /(7-1))}, k=7$

(c)

$$
G(n, r), n r^{2}=(1 / n)^{(1 /(5-1))}, k=5
$$


(b)

(d)

Figure 4.10: Green and red lines: the two integer values described in the focusing theorem in Theorem (4.2); magenta and blue: the two integer values from our Poisson maxima approximation in equation (4.9); the red histograms are the distribution of $\Delta$ from $10^{5}$ trials on number of node equals to $n, n$ corresponds to the integer to which the dotted blue vertical lines point; $n=2,5,10,20,50,100$. (a): $k=3$; (b): $k=5$; (c): $k=7$; (d): $k=9$.
$G(n, r), n r^{2}=(1 / n)^{(l /(3-1))}, k=3$

(a)

(c)

$$
G(n, r), n r^{2}=(1 / n)^{(1 /(5-1))}, k=5
$$


(b)
$G(n, r), n r^{2}=(1 / n)^{(1 /(9-1))}, k=9$

(d)

Figure 4.11: Green and red lines: the two integer values described in the focusing theorem in Theorem (4.2); magenta and blue: the two integer values from our Poisson maxima approximation in equation (4.9); the red histograms are the distribution of $\Delta$ from $10^{5}$ trials on number of node equals to $n, n$ corresponds to the integer to which the dotted blue vertical lines point. $n=2,5,10,20,50,100,(\mathrm{a}): k=3$; (b): $k=5$; (c): $k=7$; (d): $k=9$.

### 4.4 Conclusions

In this chapter we used random geometric graph models for the problem of co-channel interference in WLANs. Under this model the minimum number of channels needed for WLANs to have no co-channel interference equals to the chromatic number of the interference graph. To compute the chromatic number is an NP-hard problem; hence a good approximation is very important, especially for large graphs. Sparse random geometric graphs are more interesting since in reality the interference graph of WLANs are sparse because the resources are limited.

Sparse $\mathcal{G}_{\mathrm{b}}(n, r)$ have the nice property that $\Delta_{n}$ and $\chi_{n}$ converge to the same limit asymptotically, hence we study the approximation of $\Delta_{n}$ (Problem DSRB). We derived an approximation for the maximum value for general $n$ i.i.d. binomial distributions and Poisson distributions. With the assumption of degree independence we applied both of our methods to estimate $\Delta_{n}$. We compared our approximation to $\Delta_{n}$ with the Penrose approximation. Our experiments show that our approximations (based on Poisson and Gumbel) are better than previous approximations. Our experiments also show that for Poisson distribution is not a good approximation to binomial distribution for the problem of evaluating the maximum degree.

## Chapter 5

## Interference management for wireless networks

In the previous chapters we used random geometric graphs as a simplified model of wireless networks. We studied the chance that a wireless ad-hoc network is connected and the minimum number of channels needed for a WLAN to have no co-channel interference as the density of nodes increases. We used probability and statistical methods to prove theorems and asymptotic approximation for minimum degree, maximum degree, clique number and chromatic number.

In this chapter, first we show how our analytical results for the chromatic number $\chi_{n}$ can be used to estimate the minimum number of channels needed for an IEEE 802.11 WLAN. Then, we consider a BT Openzone network as an example of how to use our theoretical results to minimize co-channel interference at the planning stage of the network. Finally, we study some branch-and bound-algorithms for co-channel interference minimisation in different network scenarios.

### 5.1 Interference management based on theoretical estimation

In this section we show how one of the theoretical results derived in Chapter 4 can be applied in a practical situation.

BT Openzone has installed 25 access point (APs) in the city of Edinburgh and 157 APs in Soho at central London. The APs use IEEE $802.11 \mathrm{~g} / \mathrm{b}$ standard. Currently these APs

### 5.1. INTERFERENCE MANAGEMENT BASED ON THEORETICAL ESTIMATION

are operating and providing wireless Internet services to customers.
We are interested in two problems. First we would like to know if our theoretical result on the chromatic number for random geometric graphs can describe properties of real networks like BT Openzone; that is, the number of non-overlapping channels in the network, so that there is no co-channel interference. One property of a random geometric graph model is that its node positions are chosen at random. In the BT Openzone network the existing APs positions are not distributed randomly, but they are allocated according to various commercial preferences and requirements. Also some of our theoretical results were based on the assumption that the network was physically constrained to the surface of a torus. The reason of this assumption was to avoid edge effects. Here we would take into consideration the edge effect. This means that the APs located near the edge will suffer less interference compared to the central APs.

Second, if more APs are installed to increase the network capacity; how many more APs can be installed without running out of non-overlapping channels?

Regarding this question we are going to use the interference approximations developed in Chapter 4 to adjust the transmission range of the APs such that, as the network grows, the number of non-overlapping channels is not larger than 3. This approach can be applied to other 802.11 standards and other similar problems.

### 5.1.1 Chromatic number and Interference for a BT Openzone network

Figure 5.1(a) shows the geographical location of the APs of BT Openzone in central London and its interference graph. The area where the networks is deployed is a square of $2 \times 2$ $\mathrm{km}^{2}$, the transmission radius is $r=50 \mathrm{~m}$. In this area there are 157 wireless APs. To evaluate the chromatic number using a random geometric graph we scale the networks to a $1 \times 1$ square with $n=157$ and transmission radius $r=0.025$. In the graph the nodes represent the APs, the radius of the nodes represent the transmission range of the APS and the links represent pairwise interference. Two APs interfere if they are within each other's

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transmission range. Typically the transmission range of a AP is between 50 m to 100 m in an urban area [39] which in our representation means a radius between $(0.025,0.05)$. To evaluate the chromatic number we make the following assumptions:

- The APs are all equipped with single radio and omnidirectional antennas.
- Their sensing range, transmission power, the background noise and the interference from other users are assumed to be at the same level. We do not consider shadowing or signal decay.
- All APs are assumed to transmit using the same channels initially.
- If the distance between two APs is less than the transmission range, these two APs interfere with each other, otherwise there is no interference.

In order to minimise the interference caused by using the same channel, we assign different channels to the interfering APs. A valid colouring of the resulting interference graph corresponds to the successful channel assignment which eliminates the co-channel interference. In general, the interference graph of a WLAN is sparse. In this case, equation 4.20 allows us to calculate the integer bounds $I_{n}$ and $I_{n+1}$ for the maximum degree $\Delta_{n}$. Since for any graph there is $\omega \leqslant \chi \leqslant \Delta+1$. The focusing phenomenon in graphs in the subconnective regime given by Penrose in section 4.1.1 states that, $\omega_{n}$ and $\Delta_{n}$ have the same limit as $n \rightarrow \infty$. We replace $\omega_{n}$ by $\Delta_{n}$

$$
\begin{equation*}
\Delta_{n} \leqslant \chi_{n} \leqslant \Delta_{n}+1 \tag{5.1}
\end{equation*}
$$

since

$$
\begin{equation*}
\Delta_{n} \sim\left[I_{n}, I_{n+1}+1\right] \tag{5.2}
\end{equation*}
$$

So we use this accurate asymptotic result for the approximation of $\chi_{n}$ and gain an more tight bound for $\chi_{n}$ :

$$
\begin{equation*}
\chi_{n} \sim\left[I_{n}, I_{n+1}+1\right] \tag{5.3}
\end{equation*}
$$



Figure 5.1: BT open zone in central London and its interference graph: (a): The physical location of the access points of the BT open zone in central London; (b): The geometric graph of access points in BT open zone central London.
or

$$
\begin{equation*}
\chi_{n} \in\left(I_{n}, I_{n+1}, I_{n+1}+1\right) \tag{5.4}
\end{equation*}
$$

For an example, when $n=157$ and $r=0.025$, from equation 4.20 we have that $I_{n}=1$, $I_{n+1}=2, I_{n+1}+1=3$. From equation 5.4 we estimated that the range of chromatic number is $\chi_{n} \in(1,2,3)$. To evaluate whether the values obtained using equation 4.20 are useful we calculate the chromatic number algorithmically. Figure 5.1(b) shows the results evaluating the chromatic number algorithmically. In the figure the node colours represent different non-overlapping channels. In this case $\chi=3$ which is correctly estimated by the range obtained above.

Figure 5.2 shows the physical location of the 23 APs of the BT Openzone network in Edinburgh. The APs are distributed in an area of $1.3 \times 1.3 \mathrm{~km}^{2}$. The APs transmission range is 50 m . Figure 5.3 (a) shows the inteference graph if the APs transmission range is 50 m and figure $5.3(\mathrm{~b})$ for transmission range of 100 m . This network is represented with a random geometrical graph where the nodes are distributed in a $1 \times 1$ square area, the num-


Figure 5.2: The map of BT Openzone Edinburgh
ber of nodes is 23 and the transmission range is $r \in(0.34,0.68)$. Equation 4.20 gives that the chromatic number is $0,1,2$ for all possible transmission ranges. This prediction agrees with the chromatic number obtained algorithmically $\chi=2$, showing that the chromatic bounds developed in previous chapters can be used in practical situations.

### 5.1.2 Transmission constraints for the BT Openzone network

Suppose that the APs have been installed and are transmitting at their maximum power without any interference. If new APs are added to the network it is possible that there will be more and more channel interference. If the number of non-overlapping channels is fixed to 3 , then the only way to avoid interference is to reduce the transmission range. However, by how much the transmission range should be reduced is not a simple problem. In this subsection we are going to use our theoretical results to predict what is the maximum number of APs that can be added to the network with the restrictions that the number non-interfering channels is 3 and that the transmission range can be modified but only within the range $(50 \mathrm{~m}, 100 \mathrm{~m})$.

(a)

(b)

Figure 5.3: (a): 23 APs transmit using the minimum transmission range 50m; (b): 23 APs transmit using the maximum transmission range 100 m .

As an example we consider the interference graph that models the BT Openzone in Edinburgh. The growth of the network is done by adding nodes at random positions, three nodes at the time. Then we evaluate the interference graph and its chromatic number as a function of the number of nodes, their positions and their transmission range. The chromatic number obtained from this growing network is compared with the bounds obtained using equation 4.20. The total number of added nodes, i.e. new APs, is $29 \times 3=87$.

Figure 5.4 shows the change in the chromatic number as the networks grows for fixed transmission range. The red filled circles are the chromatic number obtained algorithmically. The open symbols (triangles, circle, rhomboids) are the bounds obtained using equation 4.20, that is triangles for $I_{n}$, circles for $I_{n+1}$ and rhomboids for $I_{n+1}+1$. If the red filled circles overlap with the open circles, it means that the prediction of the chromatic number using the approximation gives the correct value.

In figure 5.4(b), when the number of APs equal to 100 the filled circle do not always fall into any of the empty symbols, meaning that we cannot estimate successfully the value of the chromatic number. We have calculated the frequency of getting the correct answer: for APs with transmission range $r=50 \mathrm{~m}$ we obtained $P_{\text {correct }}=100 \%$, for transmission


Figure 5.4: Interference estimation for the APs in Edinburgh's BT Openzone as the number of APs increases. The minimum number of channels needed such that there is no interference between the APs: theoretical result compared with the simulation result using a unit disk graph model. The red filled circles are the chromatic number obtained algorithmically. The open symbols (triangles, circle, rhomboids) are the bounds obtained theoretically: triangles for $I_{n}$, circles for $I_{n+1}$ and rhomboids for $I_{n+1}+1$. (a): APs transmit using the minimum transmission range 50 m ; (b): APs transmit using the minimum transmission range 100 m .


Figure 5.5: Interference graph Edinburgh's BT Openzone in the case that it has grow from 23 to 80 APs. (b): 80 APs transmit using the minimum transmission range 50 m ; (a): 80 APs transmit using the maximum transmission range 100 m .
range $r=100 m$ we obtained $P_{\text {correct }}=25 / 29=86 \%$.
In figure $5.4(\mathrm{~b})$ when the number of nodes exceeds 97 our prediction of the chromatic number starts to fail. This is because the interference graph is getting more and more dense and our approximations are for sparse graphs. From our evaluations of the chromatic number we noticed that even when all nodes are transmitting at the minimum transmission range, the networks will run out of non-overlapping channels when the number of node reaches 83 .

Next we would like to do interference management. Equation 4.20 gives the relationship between the number of nodes, the transmission range and the minimum number of channels needed. This allows us to change one parameter by controlling the other two. Our aim is to have only 3 non-overlapping channels as the network increases. As we know that we can estimate the chromatic number correctly when we have 23 nodes transmitting to the maximum transmission range $(100 \mathrm{~m})$, this will be our starting configuration. Then we increase the number of APs and if the upper bound $I_{n+1}+1$ is greater than 3, then we reduce the transmission range by $\epsilon_{x}$, until $I_{n+1}+1$ is less or equal to 3 and if the new

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Figure 5.6: Minimum number of channels needed such the BT Edinburg's Openzone APs do not interfere between each other: theoretical result compared with the simulation result using unit disk graph model. (a): APs transmit using the adjusted radius to keep number of channels needed within 3; (b): The number of channels as the number of APs increases. The red filled circles are the chromatic number obtained algorithmically. The open symbols (triangles, circle, rhomboids) are the bounds obtained theoretically: triangles for $I_{n}$, circles for $I_{n+1}$ and rhomboids for $I_{n+1}+1$.
transmission range is not less than 50 m . When the transmission range reaches its minimum value $(50 \mathrm{~m})$ and if the theoretical upper bound of the chromatic number is larger than 3 , then the network has reached its maximum size in terms of the number of APs. The $\epsilon_{x}$ value we used for figure 5.6 is 4.1 m .

When the APs operate to their maximum transmission range the network will run out of channels as the number of nodes reaches 40 . After adjusting the transmission radius the number of APs can be increased to 100 APs. When the number of APs increase to 100 the transmission range is reduced to $54 m$.

Figure 5.6(a) shows how the transmission range is adjusted as the number of APs increases. Figure 5.6(b) shows the change in the chromatic number as the number of number of APs increases.

For an example when the number of APs is 80 and the transmission range is 58 m , the

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theoretical value is 4 . After reducing the transmission range by 4 m , the theoretical value of the minimum number channels needed is reduced to 3 . The interference graph of 80 nodes transmit with an transmission range of 53 m is shown in figure 5.8(a). Figure 5.4 shows that when the number of nodes is less than 97 , the transmission range is between 50 m to 100 m , our equation 4.21 estimates the number of non-overlapping channels needed in order to overcome the co-channel interference correctly.

The bounds for the chromatic number that we developed in previous chapters work very well if the graph is in the subconnective region. This allows us to use a different approach to adjust the transmission range of the APs. In this case we would control the growth of the network under the condition that the networks is always in the subconnective region. In other words, instead of reducing the transmission range by $\epsilon_{x}$, one could choose an average degree growth function for $n r^{2}$. In the example of figure 5.7, the average degree growth function $n r^{2}=(1 / n)^{1 / k}$ is used to increase the network size and equation 4.20 to adjust the transmission range. This approach will guarantee that the network always stays in the subconnective regime and hence guarantees the accuracy of the chromatic number. This approach is very useful when considering large graphs, since for large graphs the evaluation of the chromatic number may not be feasible.

The result of this approach for the Edinburgh Openzone networks is shown in figure 5.7. The interference graph after the radius adjustment for 80 nodes is shown in figure 5.8(b). figure 5.7 shows that the maximum number of APs in Edinburgh is 98 with the transmission range of 52 m . This result is similar to the one obtained using our previous method, where we have 98 nodes with a transmission range of 54 m .

This last method of guarantee that the network is always in the subconnecivity region introduces a problem with the network coverage. To guarantee maximum coverage we can use. One solution to this problem could be to carefully control the growth of the average degree in order to keep the interference graph sparse and the transmission range as large as possible. this thesis.

In term of the coverage, the fixed-value approach is better than the second approach. Since the larger the transmission range the better coverage the network have, see figure 5.5.

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Figure 5.7: The minimum number of channels needed for the BT open zone APs has no interference between each other: theoretical result compared with the simulation result using unit disk graph model. (a): APs transmit using the adjusted radius to keep number of channels needed within 3; (b): The radius adjusted as the number of APs increases. The red filled circles are the chromatic number obtained algorithmically. The open symbols (triangles, circle, rhomboids) are the bounds obtained theoretically: triangles for $I_{n}$, circles for $I_{n+1}$ and rhomboids for $I_{n+1}+1$.

One solution could be carefully switch among a several average degree growth functions in order to keep the interference graph sparse but transmission range as large as possible. This is out of the scope of this thesis.

### 5.2 Interference minimisation and branch-and-bound algorithms

In the previous section, we have discussed the approach of the interference management using our theory. This approach is especially useful when the interference graph of the WLANs are large and sparse. In this section we discuss the algorithmic approach for more general co-channel interference minimisation problem in wireless networks. The aim is to provide an algorithmic framework for various similar problems. The framework we


Figure 5.8: Interference graph of 80 APs BT open zone in Edinburgh: for management purpose. (a): 80 APs transmit using the adjusted transmission range $54 m$; (b): 80 APs transmit using the adjusted transmission range 52 m .
use is the prototype algorithm branch-and-bound $(B \& B)$. The interference minimisation problems are actually combinatorial optimisation problems. The optimisation solutions can be obtained by comparing the complete enumerations of all the available channels. In contrat to the last section, this approach is more suitable for small interference graphs, which may not be sparse.

### 5.2.1 Branch-and-bound: depth-first search

Branch-and-bound is a technique which finds the optimum solution among the complete enumerations. It is the best-known algorithm for many NP-hard combinatorial optimisation problems. The advantage of this method is that it does not require checking every entry of the complete enumeration. The complete enumeration of $n$ objects contains an exponentially large number of items. The actual complexity of the algorithm depends on the particular problem which this algorithm applied to. The number of entries in the search space could be reduced by the procedure of branching, bounding and preprocessing.

In our application to the co-channel interference minimisation for WLAN. The objects

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refer to the APs. The number of objects $n$ corresponds to the number of APs. In figure 5.9(a) each node in the tree represent one stage of the channel assignment. This is an example of assigning channels to 3 APs. The vector $X=\left[x_{1}, x_{2}, x_{3}\right]$ is used to identify the different channels/colours, where 0 means no channel is assigned. The complete enumerations can be obtained by an (DFS) depth-first search, (BFS) breath-first search and other algorithms [39]. This procedure is branching, which means to visit branches from the current node.

The DFS algorithm start with empty assignment $X=[0,0,0]$ as the root node in layer 0 . Then assign 1 to the first APs in layer 1. No channels are assigned to the other APs at this level. Then we assign 1 to the second AP in layer 2. Finally we assign channel 1 to AP 3. The DFS aims to reach from one node to the leaf as soon as possible. Once it reaches the leaf the algorithm backtracks one layer up and tres to reach the next leaf as soon as possible. When finishing travel through all the leaves in layer 2, the algorithm backtracks to another layer up until reaches the last leaf on the bottom right of the tree. In this example APs are named by the index of the vector $X$. The value of each of the element in $X$ is only a symbol to identify different channels. It is the pattern of the vector $X$ that influences the final result. that is, $[1,0,0],[2,0,0],[3,0,0]$ all means a stage that a channel to AP 0, no channels have been assigned to the other two APs (nodes) yet. When the total enumeration procedure is finished there are $3^{3}=27$ entries all together. In figure 5.9 (a) the complete enumeration is represented by the green nodes. The searching space can be reduced by cutting off the duplicated branches originated from node $1,0,0$ and $2,0,0$ in layer 1 . This is the simplest preprocessing of the tree. After preprocessing the searching space is reduced by $2 / 3$ of the original size. The new tree structure is shown in figure 5.9(b). More detailed discussion of branch-and-bound algorithm with other search method can be found in [39].

Our aim is to find a valid colouring of the interference graph with the smallest number of different colours. This is achieved by comparing the current bound $f=|X|$ with the

(a)

(b)

Figure 5.9: Enumeration: deep first search. (a): the tree structure of branch-and-bound complete enumeration of three objects; (b): The tree structure of branch-and-bound enumeration of three objects.
local bound $\alpha$ and the global bound $\omega$. The objective function is

$$
\begin{equation*}
f(x)=\min |X| . \tag{5.5}
\end{equation*}
$$

The local bound is updated as traversing along the tree. It is the currently smallest value of $|X| . \omega$ is only updated at the leaf level. The algorithm will only continue when both $\alpha$ and $\omega$ are decreasing, which mean the local minimum should always be smaller than the global minimum. When $\alpha>\omega$ or $f>\alpha$ the currently node can be discarded together with the branches originating from it, since none of them will lead to the global minimum. The requirement of valid colouring is an constraint which may cause more cutting of the searching space. If an channel assignment assigns the same colours to two adjacent APs, the current node is discarded together with the branches below it, since none of them is a valid colouring.

The pseudocode for the general recursive branch-and-bound algorithm is provided in Algorithm 8. This algorithm uses depth-first search (DFS) and allows constraints. Let $c()$ denote constraint function, and $f()$ denote the objective function. $x[]$ denotes the

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colour assigned to each APs. The updated local bound of $x[]$ is stored in $\alpha[]$. The final channel allocation is stored in vector Solution[ ]. Let depth denote the index of $x[]$ starting from 1 , so depth $=1 \ldots n, n$ is the number of objects. depth is the layer number of the tree, figure $5.9(\mathrm{~b})$. The root is in layer $0 . \omega$ is the global bound of the objective function, initialised to $\infty$.

```
Algorithm 8 recursive DFS branch-and-bound
Globalise: \(\omega:=\infty\); Solution \(:=[0,(\) for \(i \ldots n)]\); depth \(:=0 ; k:=\max\) colour \(; x:=\)
[0, (for \(i \ldots n)] ; n:=\#\) objects.
Function optimise( )
    if \(\alpha()>\omega\) then
        return
    end if
    if depth is \(n\) then
        if \(f() \leqslant \omega\) then
                \(\omega:=f()\)
                Solution \(:=X\)
        end if
        return
    end if
    depth \(+=1\)
    for \(i=1 \ldots k\) do
        \(x[\) depth -1\(]:=i+1\)
        if \(c()\) and \(\alpha() \leqslant \omega\) then
            optimise( )
        end if
    end for
    \(x[--\) depth \(]:=0\)
```


### 5.2.2 Branch-and-bound for channel allocation problems

Now we introduce several optimisation problems related to the channel allocation problem. As we introduce the problems, we show how branch-and-bound can be adapted to solve these problems. The network scenarios include FDMA cellular network, IEEE 802.11 WLAN, wireless ad-hoc networks and WMNs. We investigate how branch-and-bound algorithms can be applied to different network scenarios. We group these four networking

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scenarios into two groups. Group 1: the FDMA cellular networks and the single radio IEEE 802.11 WLAN. They share the common characteristics that the base stations/APs provide coverage to mobile devices. Usually the base stations/APs do not communicate with each other by wireless. Group 2: the IEEE 802.11 multi-radio wireless ad-hoc networks and multi-radio WMNs. In these networks laptops/mesh routers communicate with each other wirelessly.

We first look at the co-channel interference minimisation problem for Group 1 wireless networks. There are two type of problems we discuss here. Type $a$ is the minimum number of channels that a network needed to eliminate the co-channel interference. We use the unit disk graphs as the models for the interference relationship between nodes. Here base stations, APs, mobile devices and mesh routers are the nodes. The edges are defined by the threshold value $r_{i}$. If the distance between two nodes is below the threshold value, there is a link between these two nodes.

Let $s_{i, j}$ stand for the distance between node $i$ and $j$. The existence of a edge is given a step function $f$ :

$$
f= \begin{cases}0 & \text { if } s_{i, j}<r  \tag{5.6}\\ 1 & \text { if } s_{i, j}>r\end{cases}
$$

The task of channel allocation to avoid co-channel interference in FDMA cellular network is to assign different channels to all the base stations so that no two adjacent cells are using the same channel. In WLAN the task is to assign different channels to access points so that no two neighbouring APs are assigned the same channels. The neighbouring APs here means the distance between two nodes are within the range of $r_{i}$. The problem of finding the minimum number of channels needed can be mapped into the chromatic number problem. The solution could be obtained by the branch-and-bound algorithm in Algorithm 8. The constraint in Algorithm 8 is no adjacent nodes can be assigned the same colour. The objective function is to minimise the colour matrix $|X|$. Let $T_{c}$ stand for the total number of colours used in the non-complete interference graph, $x_{i}$ stand for the colour assigned to node $i .|X|$ is the number of different values in vector $X . g$ is a given function

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of $\left(x_{i}-x_{j}\right)$, which describes how the interference is influenced by the channels assigned to node $i$ and $j . g()$ is step function, if $x_{i}=x_{j}$ then there is interference, if $x_{i} \neq x_{j}$ there is no interference.

$$
\begin{gather*}
T_{c}=\left|x_{1}, x_{2}, \ldots, x_{i}\right|, i=1,2, \ldots  \tag{5.7}\\
g= \begin{cases}0 & \text { if } x_{i}=x_{j} \\
1 & \text { if } x_{i} \neq x_{j}\end{cases} \tag{5.8}
\end{gather*}
$$

The objective function $f(x)$ for the problem $a$ is

$$
\begin{equation*}
f(x)=\min \left(T_{c}\right) . \tag{5.9}
\end{equation*}
$$

The Type $b$ problem is to minimise the total interference received by each nodes. In this problem the threshold distance $r_{i}$ is not provided by the network planners. There is no interference graphs involved. Two nodes transmit using the same channels always interfere. This problem can be solved by the Algorithm 8 without the constraint.

Let $T_{i}$ stand for the total amount of interference generated in a wireless network. $\alpha$ is a constant factor. $\alpha$ describes how the interference decreases as the distance increases.

$$
\begin{equation*}
T_{i}=\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{g\left(\left|x_{i}-x_{j}\right|\right)}{r^{\alpha}} . \tag{5.10}
\end{equation*}
$$

The objective function in problem $b$ is

$$
\begin{equation*}
f(x)=\min \left(T_{i}\right) \tag{5.11}
\end{equation*}
$$

Now let us look at the Group 2 wireless networks. In these networks, nodes communicate with each other by wireless. The multi-radio allow nodes to communicate with neighbouring nodes simultaneously. There are also three types of problem we would like to discuss here. The Type c problem: assuming the threshold value $r_{c}$ is provided by network work planner, we would like to know the minimum number of channels these network needed to preserve the communication links. We use unit disk graph model as our com-
munication graphs. The task here is to assign channels to all the communication links so that no adjacent links are assigned the same channels and minimise the total number of channels needed. This problem can be mapped into the edge chromatic number problem and then solved by Algorithm 8. The object is the communication links, the constraint is that no adjacent links may be assigned the same channel. The objective function is to minimise the total number of colours. The Type d problem: assuming that the threshold value $r_{i}$ for interference is proved, what is the minimum number of channels needed to avoid the co-channel interference? In this situation only the interference graph needs to be considered, since $r_{i} \geqslant r_{c}$. This means the interference graph is always denser than the communication graph. The interference graphs always needs more non-overlapping channels than in the communication graph, so that there are no adjacent links assigned the same channes. This problem can be mapped into the edge chromatic number of the interference graph and solved by the same approach as problem $c$.

The Type e problem: assuming that the threshold value $r_{c}$ for communication is provided, what is the minimum total amount of interference nodes received in Group 2 networks. In this scenario the communication graph need to be considered for the constraint, to preserve the communication links. Then the problem can be solved in the similar approach as for problem $b$. An complete code for this problem implemented in C can be found in Appendix 2.

Above are five different interference minimisation problems. They can all be solved systematically by branch-and-bound algorithms.

### 5.3 Conclusions

In this chapter we have demonstrated how our new asymptotic based on Poisson distribution for chromatic number of random geometric graphs works in real-life scenarios. Our asymptotic results provide the information on the relationship between the number of APs, the transmission range and the minimum number of non-overlapping channels needed. Our asymptotic result for chromatic number estimate the minimum number of channels needed
for a network to have no interference accurately. This method is useful especially for large sparse interference graphs. We demonstrate how to use our theory for the co-channel interference minimisation for IEEE 802.11 WLAN. This method can be applied to other IEEE 802.11 standards. We also discuss a more general approach for the interference problem based on a branch-and-bound approach. We provide an recursive depth-first search branch-and-bound algorithm as a prototype to different interference minimisation problem. We discuss the constraint and objective function for each minimisation problems.

## Chapter 6

## Conclusions and future works

In this thesis we studied how graph models can be applied when studying wireless networks. We studied the chance that a wireless ad-hoc network is connected and the co-channel interference problem. We used random geometric graph models to describe the communication and the interference relationship in wireless networks. We derived analytical results for the above two problems. During the analysis of the random geometric graphs, we derived analytical results for the degree correlation, the degree distribution of any randomly chosen node with edge effect and a new asymptotic result for the Poisson maxima. The new asymptotics of the Poisson maxima estimated the two adjacent integer values of the chromatic number accurately for the sparse random geometric graphs. Experiments show that our asymptotics estimate the minimum number of channels needed for a network to have no co-channel interference accurately for real planned positions of the APs in the BT Openzone. The main conclusions are listed in the next section.

### 6.1 Conclusions

The node degrees in Erdős-Rényi random graphs $\mathcal{G}(n, p)$ forms $n$ dependent binomial random variables. The degree correlation of any two chosen nodes is $\frac{1}{n}$. This means for large graphs, the correlation between node degrees is negligible. We studied the chance of ErdősRényi random graph being connected. We derived the exact equation for the chance of no node being isolated for $\mathcal{G}(n, p)$. We assume the degree independence and derived an asymptotic equation. Then we approximated the binomial distribution by Poisson distribution and obtained another asymptotic result. We compare the above three equations
with the exact equation for $\mathcal{G}(n, p)$ being connected numerically. The experiments show that all the above results approximate the chance of $\mathcal{G}_{\mathrm{p}}(n, r)$ being connected accurately for large graphs.

The node degree in random geometric graphs based on a binomial point process $\mathcal{G}_{\mathrm{b}}(n, r)$ forms $n$ dependent binomial random variables. The node degree of random geometric graphs based on a Poisson point process $\mathcal{G}_{\mathrm{p}}(n, r)$ forms $n$ independent Poisson random variables. For infinite $\mathcal{G}_{\mathrm{p}}(n, r)$, the degree correlation of two neighbouring/overlapping nodes are constants, namely $\frac{4 \pi-3 \sqrt{3}}{4 \pi}$ and $\frac{1}{4}$. For $\mathcal{G}_{\mathrm{p}}(n, r)$ on a bounded area, the correlation between two overlapping nodes increases as the transmission range increases; for $\mathcal{G}_{\mathrm{p}}(n, r)$ an exact equation was derived when the graph is contained in a square area. We study the degree distribution with edge effects. We derive an exact equation for any two chosen nodes of $\mathcal{G}_{\mathrm{p}}(n, r)$ and $\mathcal{G}_{\mathrm{b}}(n, r)$ on a circular area. The analytical results are justified by experiments.

We used the random geometric graphs as the model of wireless ad-hoc networks. We study the chance of wireless ad-hoc network being connected by making use of the relationship between no node being isolated and the graph being connected. We assumed the independence of node degrees and derived two asymptotics to evaluate the chance that $\mathcal{G}_{\mathrm{b}}(n, r)$ is connected. One is based on i.i.d. binomial degree distribution; the other is based on i.i.d. Poisson degree distribution. The analytical results were verified by experimental results. The experiments showed that both of the asymptotics predicted the chance of $\mathcal{G}_{\mathrm{b}}(n, r)$ being connected accurately for large sparse graphs. The toroidal distance system was used in these experiments to avoid edge effects. These experiments imply that the degree correlation is negligible for large graphs. Poisson distribution gives a good approximation for this problem on large sparse graphs.

We studied the co-channel interference minimisation for IEEE 802.11b/g WLAN. We mapped this problem into the chromatic number problem of random geometric graphs. We studied the asymptotic approximation for chromatic number of $\mathcal{G}_{\mathrm{b}}(n, r)$. To evaluate the chromatic number for sparse graph we used the focusing phenomenon: the maximum degree and clique number converge to the same limit as the chromatic number. By assuming
the independence of node degrees we derived an equation for binomial maxima. We then approximated the binomial distribution using a Poisson distribution. We then derived an accurate asymptotic approximation for the Poisson maxima. Poisson maxima and binomial maxima are, to best of our knowledge, the most accurate method to evaluate the maximum degree in sparse $\mathcal{G}_{\mathrm{b}}(n, r)$. This statement is based on our comparison of our asymptotic results with the ones existing in the literature. Our new asymptotic for the maximum degree is much more accurate than the existing ones. Experiments also show that Poisson distribution is not a good approximation to binomial distribution for general $\mathcal{G}_{\mathrm{b}}(n, r)$ on the maximum degree calculation.

We applied our asymptotic of the chromatic number to the real APs positions in a BT Openzone network. Experiments showed that our asymptotics estimate the minimum number of channels accurately. We demonstrated the interference management using our asymptotic for future BT Openzone in the city of Edinburgh. This method is very useful for large sparse wireless networks. We studied the algorithmic approach to the interference minimisation problems in IEEE 802.11 wireless networks. We gave several networking scenarios and showed how to adapt the branch-and-bound algorithms to different interference minimisation problems.

### 6.2 Future work

Random geometric graphs are a simplified model of wireless networks, although not necessarily trivial. It is on its own an interesting subject. There are many other ways of generating random geometric graphs. We may take line intersection graphs as an example. These should not be confused with line graphs. We call it a street graph $\left(G_{s} t(n)\right)$. The construction of $\left(G_{s} t(n)\right)$ is as follows: start with an empty graph, spread $n$ dots randomly into the square area, draw a line through each dot with a randomly chosen angle $\alpha, \alpha \in(0,2 \pi)$. Draw a node where two lines intersect. Of the line erase the parts which are outside the most distanced two nodes on this line. The remaining is the $G_{s} t(n)$. This model could represent a wireless mesh network, where the mesh points are located at the junctions
for convenience of installation. The mesh points on the same street are assumed to have wireless connection, whereas the mesh points on different streets do not. The advantage of model is that there is only one parameter involves. It is more close to reality, since in the real world mesh points are more likely to be installed on the street lamps.

Other random graph models such as fan graphs are also interesting graph models. Instead of modelling omnidirectional antennas, fan graphs model directional antennas. Multi-graphs are another interesting graph model which could be used to model the multipath problem in wireless networks. Weighted graph could be used to model the path loss in different directions. Inhomogeneous graphs are also interesting, because nodes have different transmission range. They can be used to model the emerging future networks which contain network facilities coming from different manufacture or service providers with different standards.

## Appendix-1

| Asymptotic results for $\Delta_{n}, \omega_{n}$ and $\chi_{n}$ of $G(n, r)$ |  |  |  |
| :---: | :---: | :---: | :---: |
| Parameters | subconnective | connectivity | superconnective |
| $n r_{n}^{d} \rightarrow$ | $n^{o(1)}$ [1] | c | $\infty$ |
| $n r_{n}^{d} / \log (n) \rightarrow$ | 0 | $\alpha / \theta$ | $\alpha / \theta, \alpha \rightarrow \infty$ |
| $\Delta$ | $F_{j}(\Delta)[4]$ | $L(x, 1)$ [3] | $L^{\prime}(x, 1)$ [2] |
| $\omega$ | $F_{j}(\omega)$ | $L\left(x, 2^{d}\right)$ | $L^{\prime}\left(x, 2^{d}\right)$ |
| $\chi$ | $F_{j}(\chi)$ | open question | $L^{\prime}\left(x, \phi(B) 2^{d}\right)$ |
| Table 6.1: The summary of the theorems from Penrose[32] |  |  |  |

Notes ${ }^{1}$. for the table 6.1:
$[1] n r_{n}^{d} \rightarrow n^{o(1)} \quad$ means $\quad \log \left(1 /\left(n r_{n}^{d}\right)\right) / \log (n) \rightarrow 0$
[2] $L^{\prime}(x, a)$ means $\lim _{n \rightarrow \infty} \frac{x}{n \theta r_{n}^{d}} \stackrel{\text { a.s. }}{=} a^{-1} f_{\max }, \quad r \rightarrow 0$
[3] $L(x, a)$ means $\lim _{n \rightarrow \infty} \frac{x}{n \theta r_{n}^{d}} \stackrel{\text { a.s. }}{=} a^{-1} f_{\max } H_{+}^{-1}\left(\frac{a}{f_{\max } \alpha}\right)$
[4] $F_{j}(x)$ means $\left\{\begin{array}{l}\lim _{n \rightarrow \infty} \mathrm{P}[x \in(j, j+1)]=1 \\ \lim _{n \rightarrow \infty}\left(\frac{x \log \left(\log \left(\frac{\log (n)}{\left(r^{2}\right)}\right)\right.}{\log (n)}\right)=1\end{array}\right.$
[5] $\theta=\pi, f_{\max }=1, \phi(B)=\frac{\pi}{\sqrt{2}}$,

[^10]

Figure 6.1: The example of graph instances for $\mathcal{G}_{\mathrm{b}}(n, r)$ in connective and superconnective regime drawn on a $1 \times 1$ square area. (a): example of graph in the superconnective regime, $n=50$ and $r=0.34$. (b): example of a connective graph drawn on a $1 \times 1$ square area, $n=50$ and $r=0.27$.


Figure 6.2: The average number of edges changes as $n, r$ changes. (a): $r$ grows linearly. Green curve: $n=200$; red curve: $n=20$. (b): $n r^{2}=(1 / n)^{1 / k}, k=3,5,7,9$.


Figure 6.3: The focusing phenomenon of $\Delta$ in random geometric graph $\mathcal{G}_{\mathrm{b}}(n, r)$ in subconnective limit regime, $n r^{2}=(1 / n)^{1 / k}$. Green and red lines: the two integer values described in the focusing theorem in Theorem (4.2); magenta and blue: the two integer values from the Poisson maxima approximation in Equation (4.9); the red histograms are the distribution of $\Delta$ from $10^{5}$ trials on number of node equals to $n, n$ is the integer which the dotted blue vertical lines point to. $n=2,5,10,20,50,100$, (a): $k=3$; (b): $k=5$; (c): $k=7$; (d): $k=9$.


Figure 6.4: The focusing phenomenon of $\omega$ in random geometric graph $\mathcal{G}_{\mathrm{b}}(n, r)$ in subconnective limit regime, $n r^{2}=(1 / n)^{1 / k}$. Green and red lines: the two integer values described in the focusing theorem in Theorem (4.2); magenta and blue: the two integer values from our Poisson maxima approximation in Equation (4.9); the red histograms are the distribution of $\Delta$ from $10^{5}$ trials on number of node equals to $n, n$ corresponds to the integer to which the dotted blue vertical lines point; $n=2,5,10,20,50,100$. (a): $k=3$; (b): $k=5$; (c): $k=7 ;(\mathrm{d}): k=9$.


Figure 6.5: The focusing phenomenon of $\chi$ in random geometric graph $\mathcal{G}_{\mathrm{b}}(n, r)$ in sub connective limit regime, $n r^{2}=(1 / n)^{1 / k}$. Green and red lines: the two integer values described in the focusing theorem in Theorem (4.2); magenta and blue: the two integer values from our Poisson maxima approximation in Equation (4.9); the red histograms are the distribution of $\Delta$ from $10^{5}$ trials on number of node equals to $n, n$ corresponds to the integer to which the dotted blue vertical lines point. $n=2,5,10,20,50,100,(\mathrm{a}): k=3$; (b): $k=5$; (c): $k=7$; (d): $k=9$.

## Appendix-2

## The code of recursive dfs B\&B for wireless mesh networks

The following is the complete code for the interference minimization problem in mesh networks. This code generate a random graph. The command for running this program and the code of the program is as follows:
// gcc -Wall -O graph-interference-5.c -lm -o graph-interference-5
//./graph-interfernce-with-constraint
//./draw_edge_colouring.py graph graph
// acroread graph_edge_coloured.pdf

```
#include <limits.h>
#include <float.h>
#include <math.h>
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
```

typedef int ${ }^{\text {ivector; }}$
typedef int ${ }^{* *}$ imatrix;
typedef double *dvector;
typedef double ${ }^{* *}$ dmatrix;
ivector new_ivector(int n) \{
return calloc(n,sizeof(int));
\}
imatrix new_imatrix(int n) \{
int i;
imatrix $\mathrm{a}=$ malloc(n*sizeof(int*));
for ( $\mathrm{i}=0 ; \mathrm{i}<\mathrm{n} ; \mathrm{i}++$ ) a[i]=calloc(n,sizeof(int));
return a ;
\}
dvector new_dvector(int n) \{
return calloc(n,sizeof(double));
\}
dmatrix new_dmatrix(int n) \{
int i;

```
    dmatrix a=malloc(n*sizeof(double*));
    for (i=0; i<n; i ++) a[i]=calloc(n,sizeof(double));
    return a;
}
void show_ivector(ivector v, int n, char *lab) { 40
    int i;
    if (lab) printf("%s",lab);
    for (i=0; i<n; i++) printf("%d ",v[i]);
}
void show_dmatrix(dmatrix a, int n, char *lab) {
    int i,j;
    if (lab) printf("%s\n",lab);
    for (i=0; i<n; i++) {
        for (j=0; j<n; j++) printf("%f ", a[i][j]);
        printf("\n");
    }
}
void show_dvector(dvector v, int n, char *lab) {
    int i;
    if (lab) printf("%s",lab);
    for (i=0; i<n; i++) printf("%f ",v[i]);
}
void show_imatrix(imatrix a, int n, char *lab) {
    int i,j;
    if (lab) printf("%s\n",lab);
    for (i=0; i<n; i++) {
        for (j=0; j<n; j++) printf("%d ", a[i][j]);
        printf("\n");
    }
}
double optimize(int n, double omega, imatrix a,
                                    ivector e0, ivector e1, int max_degree) {
    // n: number of edges
    int depth,i;
    ivector x=new_ivector(n);
    ivector z=new_ivector(n);
    int dbg=0;
```

```
double f() {
    int i,j,k,l;
    double Ind(int i,int j) {
        int xik,xjl;
        double I=0.0;
        for (k=0; k<d[i];k++) {
            xik=x[a[i][k]];
            if (xik) for (l=0; l<d[j]; l++) {
                xjl=x[a[j][1]];
                if (xjl && xik==xjl && a[i][k]!=a[j][l]) {
                        I}+=\textrm{g}[\textrm{i}][\textrm{j}]
                if (dbg) printf("%d--%d, I=%g\n",i,j,I);
                }
            }
        }
        return I;
    }
    double Isum() {
        double sum=0.0;
        int max;
        if (e0[depth-1]>==e1[depth-1]) max=e0[depth-1];
        else max=e1[depth-1];
        for (i=0; i<max; i++) {
            for (j=i+1; j<max; j++) {
                sum+=Iind(i,j);
                if (dbg) printf("depth=%d, %d--%d, sum=%g\n",depth,i,j,sum);
            }
        }
        return sum;
    }
    return 2*Isum();
}
int c() {
    int i, dm1=depth }-1,\mathrm{ color }=x[dm1]
    int *ae0=a[e0[dm1]];
    int *ae1=a[e1[dm1]];
    for (i=0; i<d[e0[dm1]]; i++)
        if (ae0[i]!=dm1 && x[ae0[i]]==color) return 0;
    for (i=0; i<d[e1[dm1]]; i++)
        if (ae1[i]!=dm1 && x[ae1[i]]==color) return 0;
    return 1;
```

}
void opt() {
double fx;
int j;
if (depth==n) {
fx=f();
if (dbg) {
printf("depth=%d, f()=%g\n", depth,fx);
}
if (fx<omega) {
omega=fx;
printf("omega=%g\n",omega);
memcpy(z,x,n*sizeof(int));
}
return;
}
depth++;
for ( }\textrm{j}=1;\textrm{j}<=5; j++)
if (dbg) {
printf("depth=%d, j=%d\n", depth,j);
}
x[depth-1]=j;
if (c() \&\& f()<omega) { // f is alpha
if (dbg) {
printf("f()=%g, omega=%g\n", f(),omega);
printf("c()=%\mp@code{<br>n",c());}
}
opt();
}
}
x[--depth]=0;
return;
}
x[0]=1;
depth=1;
if (dbg) {
printf("total interference=%g\n",f());
}
opt();
show_ivector(z,n,"channels are allocated as: \n"); printf("\n");
/************************************************

```
```

    *Print your channel allocation into a file below
    *****************************************************/
    FILE* f1=fopen("graph.dat","a");
    for (i=0; i < n; i++) {
        fprintf(f1,"%d %d %d %d\n", i, e0[i], e1[i], z[i]);
    }
    fclose(f1);
    return omega;
    }
int grg_O_L(int n, double r, imatrix a, ivector deg, dmatrix $g$, ivector $e 0$, ivector e1, int max_degree) \{
// a: adjacent list. The row of a contains the index of edges
// incident on this row-number'th node
// d: the degree of node
int i,j,m=0;
double d,x[n],y[n];
for (i=0; i<n; i++) {
x[i]=rand()/(double)RAND_MAX;
y[i]=rand()/(double)RAND_MAX;
}
for (i=0; i<n; i++) {
for (j=i+1; j<n; j++) {
d=hypot((x[i]-x[j]),(y[i]-y[j]));
g[i][j]=g[j][i]=1.0/d/d;
if (d<=r) {
printf("%d--%d\n",i,j);
a[i][deg[i]++]=m;
a[j][deg[j]++]=m;
e0[m]=i;
e1[m]=j;
m++;
}
if (deg[i]>max_degree) max_degree=deg[i];
}
}
printf("max_degree=%d\n",max_degree);
/****************************************************
*Print your graph into a file below *
****************************************************/

```
FILE* f=fopen("graph.dat", "w") ;
fprintf(f, \(1 \%\) d \% \(\mathrm{d} \backslash \mathrm{n} ", n, m\) );
for ( \(i=0 ; i<n ; i++\) ) \{
```

        fprintf(f,"%d %g %g %g %d\n", i, x[i], y[i], r, 1);
    }
    //fprintf(f,"\n");
    fclose(f);
    return m;
    }
int main(int argc, char* argv[]) {
int i,j,max_degree=0;
double result;
/*******************************************************
*Initialize your graph parameter below *
**************************************************/
int nnodes=5;
const double r=0.65;
/******************************************************
*Present your graph below
*
**************************************************/
ivector e0=new_ivector(nnodes*(nnodes-1)/2);
ivector e1=new_ivector(nnodes*(nnodes-1)/2);
ivector d=new_ivector(nnodes);
imatrix a=new_imatrix(nnodes);
dmatrix g=new_dmatrix(nnodes);
int nedges=grg_0_L(nnodes,r,a,d,g,e0,e1,max_degree);
/****************************************************
* Choose the vertex or edge coloring *
* and initialize search space *
*****************************************************/
int nobjs=nedges;
/****************************************************
*Print your graph below
*
*****************************************************/
printf("My graph:\n");
printf("node edge\n");
for (i=0; i<nnodes; i++) {
printf("%d: ",i);
for (j=0; j<d[i]; j++) printf(" %d ",a[i][j]);

```
```

    printf("\n");
    }
    printf("My gmatrix:\n");
    for (i=0; i<nnodes; i++) {
    printf("%d: ",i);
    for (j=0; j<nnodes; j++) printf(" %f ",g[i][j]);
    printf("\n");
    }
for (i=0; i<nedges; i++) {
printf("edge %d: %d--%d\n", i, e0[i], e1[i]);
}
show_ivector(d,nnodes,"degree: "); printf("\n");
/****************************************************
*Set your upper bound below
*
double omega=DBL_MAX;
/***************************************************
*print your result below
***************************************************/
result=optimize(nobjs,omega,a,d,g,nnodes,e0,e1,max_degree);
printf("Interference=%g\n", result);
FILE* f=fopen("graph.dat","a");
fprintf(f,"%g\n",result);
fclose(f);
return 0;
}

```

\section*{Bibliography}
[1] Ian F. Akyildiz, Xudong Wang, and Weilin Wang. Wireless mesh networks: a survey. Comput. Netw. ISDN Syst., 47(4):445-487, 2005.
[2] C. W. Anderson. Extreme Value Theory for a Class of Discrete Distributions with Applications to Some Stochastic Processes. Journal of Applied Probability, Vol. 7:99113, Apr 1970.
[3] Clive W. Anderson, Stuart G. Coles, and Jürg Hüsler. Maxima of Poisson-like variables and related triangular arrays. The Annals of Applied Probability, 7:953-971, Nov 1997.
[4] Christian Bettstetter. On the minimum node degree and connectivity of a wireless multihop network. In MobiHoc '02: Proceedings of the 3rd ACM international symposium on Mobile ad hoc networking \(\mathcal{E}\) computing, pages 80-91, New York, NY, USA, 2002. ACM.
[5] Christian Bettstetter. On the connectivity of ad hoc networks. The Computer Journal, 47(4):432-447, 2004.
[6] Béla Bollobás. Modern Graph Theory. Springer, 1998.
[7] Béla Bollobás. Random Graphs. Cambridge, 2001.
[8] by Ronald C. Read and Robin J. Wilson. An Atlas of Graphs. Clarendon Press, 2002.
[9] Brent N. Clark, Charles J. Colbourn, and David S. Johnson. Unit disk graph. Discrete Applied Mathematics, 86(1-3):165-177, December 1990.
[10] Robert M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert \(w\) Function. Advances in Computational Mathematics, 5:329-359, 1996.
[11] Wilfrid S. Kendall Dietrich Stoyan and Joseph Mecke. Stochastic geometry and its applications. Wiley, 1987.
[12] P. Clifford D.J. Leith. Convergence of Distributed Learning Algorithms for Optimal Wireless Channel Allocation. Proceedings of the 45th IEEE Conference on Decision E Control, December 2006.
[13] O. Dousse and P. Thiran. Connectivity vs capacity in dense ad hoc networks. INFOCOM 2004. Twenty-Third AnnualJoint Conference of the IEEE Computer and Communications Societies, 1:476-486, March 2004.
[14] Jakub erný, Zdenk Dvořák, Vít Jelínek, and Jan Kára. Noncrossing hamiltonian paths in geometric graphs. Discrete Appl. Math., 155(9):1096-1105, 2007.
[15] Lorna Booth; Jehoshua Bruck; Massimo Franceshetti and Ronald Meester. Covering algorithms, continuum percolation and the geometry of wireless networks. The Annals of Applied Probability, 13(2):722-741, 2003.
[16] B. Ghosh. Random distances within a rectange and between two rectangles. Calcutta Math. Soc., 43:17-24, 1951.
[17] E. N. Gilbert. Random Graphs. The Annals of Mathematical Statistics, 30(4):11411144, 1959.
[18] P. Gupta and P. R. Kumar. Critical Power for Asymptotic Connectivity in Wireless Networks. In Decision and Control, 1998. Proceedings of the 37th IEEE Conference on, volume 1, pages 1106-1110, 1998.
[19] Qiao Li Gyouhwan Kim and Robit Negi. A graph-based algorithm for scheduling with sum-interference in wireless networks. Global Telecommunications Conference, \(200 \%\). GLOBECOM '07. IEEE, pages 5059-5063, 2007.
[20] Samik Ghosh Habiba Skalli and Marco Conti Sajal K. Das, Luciano Lenzini. Channel Assignment Strategies for Multiradio Wireless Mesh Networks: Issues and Solutions. IEEE Communications Magazine, 2007.
[21] Martin Haenggi. On distances in uniformly random networks. IEEE Transactions on Information Theory, 51:3584-3586, 2005.
[22] A. C. Kimber. A Note on Poisson Maxima. Probability Theory and Related Fields, 63(4):551-552, December 1983.
[23] Christian Lantuejoul. Geostatistical Simulation: Models and Algorithms. Springer, 2002.
[24] Colin McDiarmid. Random channel assignment in the plane. Random Struct. Algorithms, 22(2):187-212, 2003.
[25] Colin J. H. McDiarmid and Tobias Müller. Colouring random geometric graphs. In Stefan Felsner, editor, 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), volume AE of DMTCS Proceedings, pages 1-4. Discrete Mathematics and Theoretical Computer Science, 2005.
[26] Ronald Meester. Continuum Percolation. Cambridge Tracts in Mathematics, 1996.
[27] C. Silva Ram Murthy and B.S. Manoj. Ad Hoc wireless networks : architectures and protocols. Prentice Hall communications engineering and emerging technologies series. Prentice Hall, 2004.
[28] Saralees Nadarajah and Kosto Mitov. Asymptotics of maxima of discrete random variables. Extremes, 5(3):287-294, Sep 2002.
[29] M. E. J. Newman. Assortative Mixing in Networks. Phys. Rev. Lett., 89(20):208701, Oct 2002.
[30] Choong Seon Hong Nguyen H. Tran. Joint scheduling and channel allocation in wireless mesh networks. Consumer Communications and Networking Conference, 2008. CCNC 2008. 5th IEEE, pages 760-764, 2008.
[31] Mathew Penrose. On \(k\)-connectivity for a geometric random graph. Random Struct. Algorithms, 15(2):145-164, 1999.
[32] Mathew Penrose. Random Geometric Graphs. Oxford studies in probablity, 2003.
[33] A. Raniwala and T. Chiueh. Architecture and algorithms for an ieee 802.11-based multi-channel wireless mesh networks. In Proc. IEEE Infocom, 2005.
[34] F.S. Roberts. Graph theory and its applications to problems of society. SIAM, 1978.
[35] Arjunan Rajeswaran Robit Negi. Physical layer effect on MAC performance in adhoc wireless networks. Proceeding (408) Communications, Internet, and Information Technology, 2003.
[36] Robert Sedgewick. Algorithms in C, Part 5. Addison Wesley Professional, 2008.
[37] Sunil Srinivasa and Martin Haenggi. Distance distributions in finite uniformly random networks: theory and applications. arXiv, 2008.
[38] Olivier Dousse; Patrick Thiran and Martin Hasler. Connectivity in ad-hoc and hybrid networks. INFOCOM 2002. Twenty-First Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE, 2:1079 - 1088, June 2002.
[39] M. R. Tijmes. Simutaneous optimisation of channel and power allocation for wireless cities. BSc report.
[40] Timo D. Hamalainen Timo Vanhatupa, Marko Hannikainen. Performance model for ieee 802.11s wireless network depolyment design. Journal of Parallel and Distributed Computing, 2007.
[41] Ozan K. Tonguz and Gianluigi Ferrari. Is the number of neighbors in ad hoc wireless networks a good indicator of connectivity? In Proc. of the IEEE International Zurich Seminar on Communications: Access-Transmission-Networking, pages 40-43, February 2004.
[42] W.F.Hale. Frequency Assignment: Theory and Applications. Proc. IEEE, 68:14971514, 1980.
[43] Y. Wang X. Y. Li. Simple heuristics and PTASs for intersection graphs in wireless ad hoc networks. Proceedings of the 6th international workshop on Discrete algorithms and methods for mobile computing and communications, 2002.
[44] Shaoqiu XIAO Yan ZHANG, Mingtuo ZHOU and Masayuki FUJISE. An effective wimax qos scheme in mesh networking for maritime its. 6th International Conference on ITS Telecommunications Proceedings, 2006.```


[^0]:    ${ }^{1}$ http://keithbriggs.info/very_nauty.html

[^1]:    ${ }^{1}$ All graphs in this Chapter are made using ConceptDraw and MS Word

[^2]:    ${ }^{1}$ Notice the difference between a connected graph and a complete graph is that in a complete graph there is an edge/arc between every pair of nodes, while in a connected graph there is a path/route between any pair of two nodes

[^3]:    ${ }^{2}$ Converesly, improper colouring means adjacent nodes are allowed to have the same colour

[^4]:    ${ }^{3}$ Erdős and Rényi are the names of the two people who started the theory of random graphs.

[^5]:    ${ }^{4}$ are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections

[^6]:    ${ }^{1}$ Nodes that have overlapping disks

[^7]:    ${ }^{2}$ Graph being 1-edge connected and 1-vertex connected are equivalent to the size of the largest cluster equal to $n$.

[^8]:    ${ }^{3}$ The pseudo-codes can be programmed into any language. We used C programs for our experiments.

[^9]:    ${ }^{1}$ The program that calculates the chromatic numbers can be downloaded from the website http: //keithbriggs.info/verynauty.html

[^10]:    ${ }^{1}$ The details of the above theorems can be found in Random geometric graphs [32]

