From Fibonacci to the mathematics of cows and quantum circuitry

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Abstract. The Fibonacci sequence is a famously well-known integer sequence from the thirteenth century which has transcended its original motivation. It possesses many interested and varied applications within architecture, engineering and science. Less well known is the Narayana sequence which itself has interesting and wide-ranging Fibonacci-type connections. In this paper, we shall recall Narayana's original motivation that gives rise to the sequence bearing his name. We also provide an interesting application of this sequence to the construction to quantum gate circuitry used in quantum computation.

1. Introduction

A celebrated feature of mathematics is the way in which it perpetually demonstrates good bridge-building capabilities between seemingly unrelated topics of study. An example of this extraordinary feature relates to the class of sequences called integer sequences which derives its name from the property that its sequence terms are integers. The terms of an integer sequence may be specified explicitly with respect to a formula, or implicitly through a recurrence relation (Sloane 1973). Integer sequences can be analyzed by a variety of techniques, some of which include the application of a data compression algorithm (Bell *et al* 1990), computation of the discrete Fourier transform (Loxton 1989), and evaluation of its generating function (Wilf 2005). Additionally, there are also a large number of transformation methods which can be applied to integer sequences, including the Euler transform, exponential transform and Möbius transform.

The Fibonacci sequence is an integer sequence that itself displays good bridge-building capabilities. Each term in this sequence is produced by adding together the two previous terms and the sequence takes its name from the famous thirteenth century mathematician Fibonacci, whose use he explained in his 1202 Liber abaci (Sigler 2002). The Fibonacci sequence is a sequence that is well-understood. The famous astronomer Johannes Kepler observed that the ratio of consecutive Fibonacci terms approached the famous golden ratio. It is through this connection that the Fibonacci sequence is widely recognized in art and architecture. Interestingly, and equally well-known is that the Fibonacci sequence also helps explain patterns that arise within biological contexts. The applications and important consequences that arise from the Fibonacci sequence demonstrate the this sequence has transcended it original motivation (Liber abaci). We now ask whether there exist other Fibonacci-like sequences possessing important applications which transcend the original motivations of the sequence in question.

Figure 1. Circuit descriptions for the CNOT gate types. (a) The CNOT1 gate; the control system $|m\rangle \in \mathcal{H}_{\mathcal{A}}$ remains unchanged after application whereas the state of the target system $|n\rangle \in \mathcal{H}_{\mathcal{B}}$ is transformed under modular arithmetic to the state $|n \oplus m\rangle$ with $m, n \in \mathbb{Z}_d$. (b) The CNOT2 gate in which the roles of systems $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ are reversed.

2. Narayana's Cows

Narayana was a fourteenth century Indian mathematician who gives name is given to a Fibonacci-inspired problem based on an idealized population of cows. For the sake of completeness, we shall state this problem (Waldschmidt 2009): A cow produces one calf every year. Then, beginning in its fourth year, each calf produces one calf at the beginning of each year. How many cows are there altogether after seventeen years? The Narayana sequence overlaps closely with that of Fibonacci; indeed, its n^{th}-term is defined as

$$a(n) = a(n-1) + a(n-3) \tag{1}$$

with initial conditions $a(0) = a(1) = a(2) = 1$. For what follows, it will be useful to enumerate initial values of the Narayana sequence:

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, \ldots \tag{2}$$

The online encyclopedia of integer sequences records some interesting combinatorial connections offered by the Narayana sequence (Sloane). However, we shall present an application of the Narayana sequence which relates to the construction of optimal gate circuits that can be used for quantum computation. We now present preliminary material which will serve as a basis for our study.

3. Quantum Computation

Crucial for successful quantum computation is the requirement that engineers can build robust multiple-qubit quantum gates. The most elementary of all multiple-qubit quantum gates are given by two-qubit controlled unitary operators, and a classic example of these is the controlled-NOT (CNOT) gate. In terms of a classical perspective, the CNOT gate is the quantum analogue of the classical XOR gate. Barenco *et al* 1995 have shown that any multiple-qubit quantum operation may be restricted to compositions of single-qubit gates and the CNOT gate. It is for this reason that the CNOT gate has acquired the special status as the hallmark of multi-qubit control. Now, researchers in quantum computation have done considerable work in optimizing quantum circuitry networks. An important milestone in this direction related to general two-qubit operations requiring at most three CNOT gates (Vatan and Williams 2004). A crucial aspect of this result is the demand that the qubit SWAP gate requires at least three CNOT gates. Consequently, the SWAP gate has taken a prominent position in many quantum circuitry designs.

3.1. Elementary quantum gates

Let $\mathcal{H}$ represent the d-dimensional complex Hilbert space $\mathbb{C}^d$, and let us fix each orthonormal basis state of the d-dimensional Hilbert space to map to an element of the ring of integers modulo d, $\mathbb{Z}_d$. This yields the basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\} \subset \mathbb{C}^d$ whose elements correspond to the d column vectors of the identity matrix $\mathbb{I}_d$ is called the computational basis. We say a *qudit* is a d-dimensional quantum state $|\psi\rangle \in \mathcal{H}$ which can be expressed as $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$ where

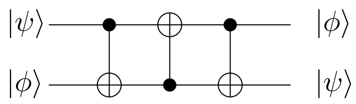

Figure 2. The SWAP gate illustrating the cyclical permutation of two qubits. System $\mathcal{A}$ begins in the state $|\psi\rangle$ and ends in the state $|\phi\rangle$ while system $\mathcal{B}$ begins in the state $|\phi\rangle$ and ends in the state $|\psi\rangle$.

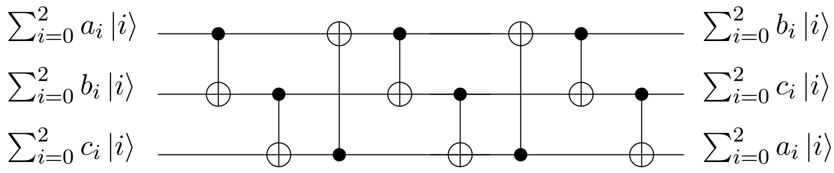

Figure 3. A qutrit SWAP gate that cyclically permutes the states of three qutrits. This gate is composed of eight two-qutrit CNOT gates

$\alpha_i \in \mathbb{C}$ and $\sum_{i=0}^{d-1} |\alpha_i|^2 = 1$. Given d-dimensional Hilbert spaces $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, consider the set of $d^2 \times d^2$ unitary transformations $U \in U(d^2)$ that act on the two-qudit quantum system $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. Let $U_{\text{CNOT1}} \in U(d^2)$ represent the generalized CNOT gate that has control qudit $|\psi\rangle \in \mathcal{H}_\mathcal{A}$ and target qudit $|\phi\rangle \in \mathcal{H}_\mathcal{B}$. The action of U_{CNOT1} on the set of basis states $|m\rangle \otimes |n\rangle$ of $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ is given by

$$U_{\text{CNOT1}} |m\rangle \otimes |n\rangle = |m\rangle \otimes |n \oplus m\rangle \,, \qquad m, n \in \mathbb{Z}_d, \tag{3}$$

with $\oplus$ denoting addition modulo d. Similarly, let $U_{\text{CNOT2}} \in U(d^2)$ denote the generalized CNOT gate having control qudit $|\phi\rangle \in \mathcal{H}_\mathcal{B}$ and target qudit $|\psi\rangle \in \mathcal{H}_\mathcal{A}$. The action of U_{CNOT2} on the set of basis states $|m\rangle \otimes |n\rangle$ of $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ is written

$$U_{\text{CNOT2}} |m\rangle \otimes |n\rangle = |m \oplus n\rangle \otimes |n\rangle \,, \qquad m, n \in \mathbb{Z}_d. \tag{4}$$

Fig. 1 provides the quantum gate circuitry representation for the respective CNOT types while Fig. 2 illustrates the well-known SWAP gate that permutes the states of two qubits, for $d = 2$.

4. The Narayana sequence and a qutrit SWAP gate

Let $d = 3$ and consider the following problem. Given three qutrit quantum systems: system $\mathcal{A}$ in the state $|a\rangle$; system $\mathcal{B}$ in the state $|b\rangle$; and system $\mathcal{C}$ in the state $|c\rangle$. Using only instances of the two-qutrit CNOT gate, determine the gate that implements a SWAP of the input states so that system $\mathcal{A}$ ends in the state $|b\rangle$, system $\mathcal{B}$ ends in the state $|c\rangle$, and system $\mathcal{C}$ ends in the state $|a\rangle$. Our construction method is presented in Fig. 3.

Construction Method: Let k and l be positive integers. For non-negative integers j, consider the function

$$f(j) = \binom{j}{k} \bmod l. \tag{5}$$

Integer functions of this type are periodic and we will make special use of this fact in the construction. In particular, we shall focus on the set of modular binomial coefficients given by

$$a_j = \sum_{i=0}^{j/3} \binom{j-2i}{i} \bmod 3 \tag{6}$$

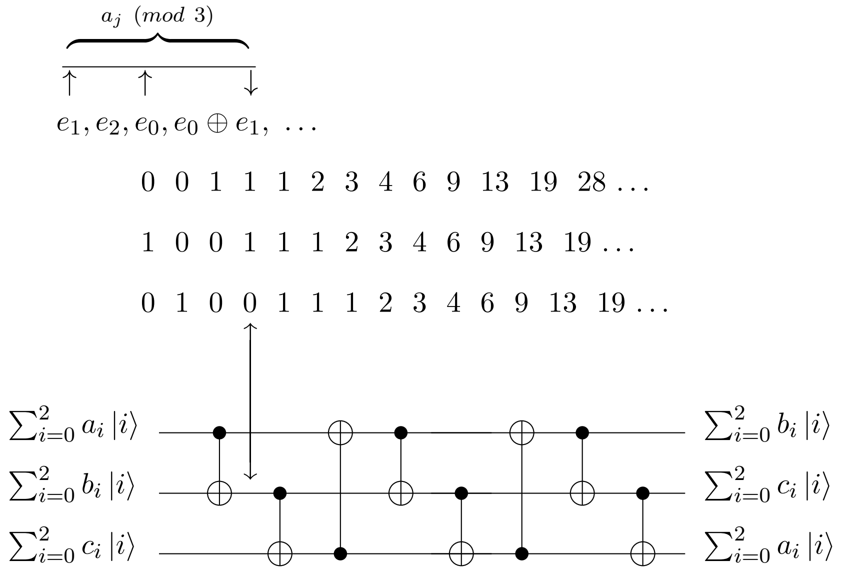

Figure 4. A quantum network composed entirely in terms of CNOT gates illustrating a cyclic SWAP of three 3-dimensional states. The columns of the array above describe the state of the system following the application of respective CNOT gates. In particular, system $\mathcal{A}_1$ is in the state $|e_0 + e_1\rangle_1$ following the application of the first CNOT gate. As the columns of array are periodic (modulo 3), a repeated application of generalized CNOTs on successive pairs of quantum systems induces a network design that eventually permutes the states of 3 qutrits

where we show how these coefficients, the terms of the Narayana sequence, can be used to construction of a regular generalized SWAP gate for qudits.

Fig. 4 describes our circuit design for a generalized SWAP gate for three 3-dimensional quantum states. The design is outlined in terms of a regular sequence of CNOT gates (modulo 3) that act on successive pairs of quantum states. The columns of the array in Fig. 4 describe the states of the target quantum systems after the corresponding CNOT gates have been applied. The up-down arrow between the array and network indicates the correspondence between system $\mathcal{A}_1$ being in the state $|e_0 + e_1\rangle_1$, where $e_0 + e_1$ is calculated modulo 3, and the application of the first CNOT gate. The sum $e_0 + e_1$ is represented as the dot product of the row vector (e_0, e_1, e_2) and the corresponding column vector $(1, 1, 0)^{\mathrm{T}}$. A similar case holds for subsequent CNOT applications. As a result of taking the dot product between regularly repeating column vectors representing three translations of the Narayana sequence modulo 3 and the row vector (e_0, e_1, e_2), we can trace the Narayana sequence to an ordering of eight qutrit CNOT gates which induce the required cyclic SWAP of original qutrit input states.

References

Barenco A, Bennett C H, Cleve R, DiVincenzo D P, Margolus N, Shor P, Sleator T, Smolin J A and Weinfurter H 1995 *Phys. Rev. A* **52** 5 3457

Bell T C, Cleary J G and Witten I H 1990 *Text Compression* (Englewood Cliffs, NJ)

Liber abaci `http://www-groups.dcs.st-andrews.ac.uk/~history/Quotations/Fibonacci.html`

Loxton J H 1989 *Spectral Studies of Automata. Irregularities of Partitions* (ed G Halasz and V T Sos) (New York: Springer-Verlag) pp 115-128

Sloane N J A 1973 *The Handbook of integer sequences* (Boston, MA: Academic Press)

Sloane N J A `https://oeis.org/A000930`

Sigler L E (2002) *Fibonacci's Liber Abaci* (Springer-Verlag)

Vatan F and Williams C 2004 *Phys. Rev. A* **69** 3 032315

Waldschmidt M `https://www.imj-prg.fr/~michel.waldschmidt/AgendaArchives.html`

Wilf H S 2005 *generatingfunctionology* (A K Peters/CRC Press; 3rd ed)

Wilmott C and Wild P 2014 *Quantum Inf. Process.* **13** 6