Entire choosability of near-outerplane graphs

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Abstract

It is proved that if G is a plane embedding of a K_4-minor-free graph with maximum degree Δ, then G is entirely 7-choosable if $\Delta \leq 4$ and G is entirely $(\Delta+2)$-choosable if $\Delta \geq 5$; that is, if every vertex, edge and face of G is given a list of $\max\{7, \Delta+2\}$ colours, then every element can be given a colour from its list such that no two adjacent or incident elements are given the same colour. It is proved also that this result holds if G is a plane embedding of a $K_{2,3}$-minor-free graph or a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph. As a special case this proves that the Entire Coluring Conjecture, that a plane graph is entirely $(\Delta + 4)$-colourable, holds if G is a plane embedding of a K_4-minor-free graph, a $K_{2,3}$-minor-free graph or a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph.

Key words: Outerplanar graph, Minor-free graph, Series-parallel graph.
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1 Introduction

Graph colouring problems in which more than one type of element are to be coloured were first introduced by Ringel [12]. (These are sometimes known as *simultaneous colourings*.) Ringel conjectured that the vertices and faces of a plane graph can be coloured with six colours, which was proved by Borodin [2].

For colourings in which edges and faces are to be coloured, Melnikov [11] conjectured that if G is a plane graph with maximum degree Δ, then the number of colours needed for an edge-face colouring of G is at most $\Delta + 3$. This was proved independently by Sanders and Zhao [13] and by Waller [16].

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For *entire colourings*; that is, colourings in which vertices, edges and faces are to be coloured, Kronk and Mitchem [9] proposed the *Entire Colouring Conjecture*, which states that if G is a plane graph, then the number of colours needed for an entire colouring of G is at most $\Delta + 4$. This is still an open problem for graphs with $\Delta = 4$ or 5: see [10] for a proof when $\Delta \leq 3$ and [14] for a proof when $\Delta \geq 6$.

The concept of *list-colouring*, where each element is to be coloured from its own list of colours, was introduced independently by Vizing [15] and by Erdős, Rubin and Taylor [4]. Simultaneous list-colourings are considered in [5].

Formally, let $G = (V, E, F)$ be a plane graph. A *list-assignment* L to the elements of G is the assignment of an unordered list $L(z)$ of colours to each element z of G. If G has a list-assignment L, then an *entire list-colouring* is an assignment of a colour to every vertex v, every edge e and every face f from its own list $L(v)$, $L(e)$ or $L(f)$ of colours. An entire list-colouring of G is *proper* if no two adjacent or incident elements are given the same colour. If $|L(z)| \geq k$ for every element $z \in V \cup E \cup F$, then G is *entirely k-choosable* if G has a proper entire list-colouring from all possible lists. The smallest integer k such that G is entire k-choosable is the *entire list-chromatic number* or *entire choosability* $\mathrm{ch}_{\mathrm{vef}}(G)$ of G. If every list is identical, then $\mathrm{ch}_{\mathrm{vef}}(G) = \chi_{\mathrm{vef}}(G)$, where $\chi_{\mathrm{vef}}(G)$ is the entire chromatic number.

It is well known that a graph is outerplanar if and only if it is both K_4-minor-free and $K_{2,3}$-minor-free. We will call a graph *near-outerplane* if it is a plane embedding of a K_4-minor-free graph or a $K_{2,3}$-minor-free graph. In fact, in the following theorem we will replace the class of $K_{2,3}$-minor-free graphs by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. The graph $\bar{K}_2 + (K_1 \cup K_2)$ can be obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2, or, alternatively, from K_4 by adding a vertex of degree 2 subdividing an edge.

By an abuse of terminology we will call two elements *neighbours* if they are adjacent or incident, since no two such elements can be given the same colour. All other terminology is standard, as defined in the references: for example [1,19].

It was proved by Wang and Zhang [17] that if G is an outerplane graph with maximum degree $\Delta \geq 5$, then $\chi_{\mathrm{vef}}(G) \leq \Delta + 2$. More recently, Wu and Wu [20] proved that if G is a plane embedding of a K_4-minor-free graph with maximum degree Δ, then $\chi_{\mathrm{vef}}(G) \leq \max\{8, \Delta + 2\}$. In this paper we will prove that if G is a near-outerplane graph with maximum degree Δ, then $\mathrm{ch}_{\mathrm{vef}}(G) \leq \max\{7, \Delta + 2\}$. Since $\chi_{\mathrm{vef}}(G) \leq \mathrm{ch}_{\mathrm{vef}}(G)$, this will improve the result of Wu and Wu, and, as a special case, will prove the Entire Colouring Conjecture for all near-outerplane graphs. The coupled choosability of near-outerplane graphs is considered in [6], whilst the edge-face choosability of near-outerplane graphs is considered in [7].

Theorem 1. *Let G be a near-outerplane graph with maximum degree Δ. Then $\mathrm{ch}_{\mathrm{vef}}(G) \leq \max\{7, \Delta+2\}$. In particular,*

(i) if $\Delta = 0$, then $\mathrm{ch}_{\mathrm{vef}}(G) = 2$;
(ii) if $\Delta = 1$, then $\mathrm{ch}_{\mathrm{vef}}(G) = 4$;
(iii) if $\Delta = 2$, then

$$\mathrm{ch}_{\mathrm{vef}}(G) = \begin{cases} 6 & \textit{if } G \textit{ has a component that is a cycle whose} \\ & \textit{length is not divisible by } 3; \\ 5 & \textit{if } G \textit{ has a component that is a cycle and the} \\ & \textit{length of every such cycle is divisible by } 3; \\ 4 & \textit{if } G \textit{ is cycle-free.} \end{cases} \tag{1}$$

It is clear that $\mathrm{ch}_{\mathrm{vef}}(G) \geq \chi_{\mathrm{vef}}(G) \geq \chi_{\mathrm{vef}}(K_{1,\Delta}) = \Delta + 2$, and that the results are sharp when $\Delta = 2$. It remains to show that the results are sharp when $3 \leq \Delta \leq 4$, in which case the upper bound of 7 is attained by any graph with K_4 as a block, and by both embeddings of $K_2 + \bar{K}_3$, which can be obtained from $K_{2,3}$ by adding an edge joining the two vertices of degree 3. It is a fairly straightforward exercise to show that $\mathrm{ch}_{\mathrm{vef}}(K_4) = 7$ and $\mathrm{ch}_{\mathrm{vef}}(K_2 + \bar{K}_3) = 7$, which were both proved in [5]. All of the results in Theorem 1 are sharp for $\chi_{\mathrm{vef}}(G)$ also. Furthermore, these results are sharp for the smaller class of K_4-minor-free graphs if $\Delta \neq 3$, for the smaller classes of both $K_{2,3}$-minor-free graphs and $(\bar{K}_2 + (K_1 \cup K_2))$-minor free graphs, and for the smaller class of outerplane graphs if $\Delta \neq 3$ or 4.

We will make use of the following two theorems. Theorem 2 is a slight extension of a theorem of Dirac [3]. Theorem 3 summarises the results for edge and total choosability of near-outerplanar graphs. In particular we will make use of the well-known result [4,15] that $\mathrm{ch}(C_4) = \mathrm{ch}'(C_4) = 2$, which is included in Theorem 3 since choosability and edge-choosability are equivalent when $\Delta = 2$.

Theorem 2. [18] *A K_4-minor-free graph G with $|V(G)| \geq 4$ has at least two nonadjacent vertices with degree at most 2.*

Theorem 3. [8] *If G is a near-outerplanar graph with maximum degree Δ, then $\mathrm{ch}'(G) = \chi'(G) = \Delta$ and $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$, apart from the following exceptions*:

(i) if $\Delta = 1$ then $\mathrm{ch}''(G) = \chi''(G) = 3 = \Delta + 2$;
(ii) if $\Delta = 2$ and G has a component that is an odd cycle, then $\mathrm{ch}'(G) = \chi'(G) = 3 = \Delta + 1$;

(*iii*) *if* $\Delta = 2$ *and* G *has a component that is a cycle whose length is not divisible by three, then* $\mathrm{ch}''(G) = \chi''(G) = 4 = \Delta + 2$;
(*iv*) *if* $\Delta = 3$ *and* G *has* K_4 *as a component, then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 2 = 5$.

2 Proof of Theorem 1 if $\Delta \le 3$

It is clear that if $\Delta = 0$, then $\mathrm{ch}_{\mathrm{vef}}(G) = 2$, and if $\Delta = 1$, then $\mathrm{ch}_{\mathrm{vef}}(G) = 4$. If $\Delta = 2$, then let f_0 be the exterior face, let F_1 be set of faces of G that are adjacent to f_0, and, recursively, let F_{k+1} be the set of faces that are adjacent to F_k $(1 \le k \le n-1)$ and that are not in F_j for some $j < k$. We can first colour f_0 and then, in order, each of the sets of faces $F_1, F_2, \ldots, F_n$ since no face is adjacent to more than one coloured face at the time of its colouring. It remains to colour the vertices and edges. So the problem is reduced to total choosability of paths and cycles, and these results are given in Theorem 3. If G is cycle-free, then G has only one face, and so $\mathrm{ch}_{\mathrm{vef}}(G) = \mathrm{ch}''(G) + 1$. If G contains a cycle, then every vertex and every edge of each cycle in G is incident with exactly two faces, and so $\mathrm{ch}_{\mathrm{vef}}(G) = \mathrm{ch}''(G) + 2$. So, if $\Delta = 2$, then (1) holds.

If $\Delta = 3$, then suppose, if possible, that G is a near-outerplane graph with maximum degree 3 such that $\mathrm{ch}_{\mathrm{vef}}(G) > 7$. Assume that every vertex v, every edge e and every face f of G is given a list $L(v)$, $L(e)$ or $L(f)$ of 7 colours such that G has no proper entire colouring from these lists. Since $\mathrm{ch}_{\mathrm{vf}}(G) \le 5$ [6], it follows that the vertices and faces of G can be coloured from their lists. Since every edge is incident with two vertices and at most two faces, every edge has at least 3 usable colours in its list. Since $\mathrm{ch}'(G) = 3$ by Theorem 3, it follows that these edges can be coloured.

We will now prove Theorem 1 for $\Delta \ge 4$. In Section 3 we will prove Theorem 1 for plane embeddings of K_4-minor-free graphs, which is restated in Theorem 6. In Section 4 we will use Theorem 6 to prove Theorem 1 for plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs, which is restated in Theorem 22. This will complete the proof of Theorem 1.

3 K_4-minor-free graphs with $\Delta \ge 4$

Let the *bounding cycle* of a 2-connected block B of a plane graph G be the cycle of B that has the largest area inside it; that is, in a plane embedding of B the bounding cycle forms the boundary of the outer face of B.

Lemma 4. *Every component C of a plane graph with $|V(C)| \geq 3$ is either 2-connected or has an end-block B such that no interior face of B has a block of C embedded in it.*

Proof. It is clear that C is either 2-connected or has an end-block B. If $B \cong K_2$, then B has no interior face, and so we may assume that every end-block B is 2-connected. Select B so that the area inside the bounding cycle of B is as small as possible. Then no interior face of B can have another block of C embedded in it since otherwise B must contain another end-block of C, and this end-block necessarily has a smaller area inside its bounding cycle than B. $\square$

Let C be a component of a plane embedding of a K_4-minor-free graph G such that no interior face of C has another component of G embedded in it. If C is 2-connected, then let $B = C$ and let z_0 be any vertex of maximum degree in C; otherwise, by Lemma 4, let B be an end-block of C with cut-vertex z_0 such that no interior face of B has a block of C embedded in it.

If B contains a vertex with degree at least 3 in G, then let B_1 be the graph whose vertices are the vertices of B that have degree at least 3 in G, where two vertices are adjacent in B_1 if and only if they are connected in G by an edge or by a path whose interior vertices have degree 2.

If $u, x \in V(B)$, then let P_{ux} be the set of paths in B of length 1 or 2 between u and x that contain no interior vertex of degree at least 3; that is, if $uvx \in P_{ux}$ then $d_G(v) = 2$. Also, let p_{ux} be the number of paths in P_{ux}.

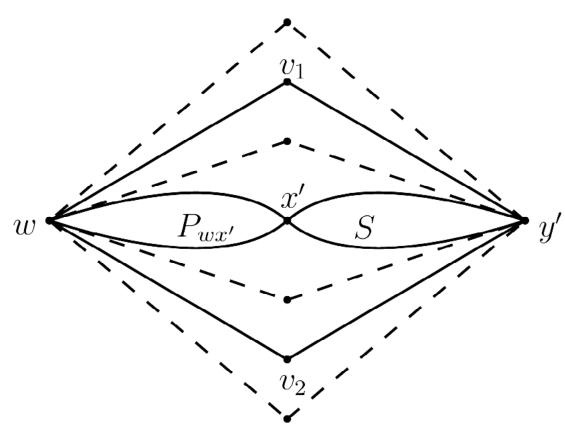


Figure 1

Lemma 5. *Suppose that B does not contain a vertex of degree 1 or two adjacent vertices of degree 2 in G. Then the graph B_1 exists and does not contain a vertex of degree 0. Suppose that B_1 does not contain a vertex of degree 1. Then B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, where $p_{ux} + p_{uy} = d_G(u) \geq 3$, and where $p_{uy} \geq 2$. Moreover, no two paths in*

P_{uy} bound a region that has a path not in P_{uy} embedded in it, and if $p_{ux} \geq 2$, then no two paths in P_{ux} bound a region that has a path not in P_{ux} embedded in it also.

Proof. If B does not contain a vertex of degree 1, then $B \ncong K_2$, and if B does not contain two adjacent vertices of degree 2, then B is not a cycle. So B has at least two vertices with degree at least 3, and so it follows that B_1 exists and does not contain a vertex of degree 0. Since B_1 is a minor of B, it follows that B_1 is K_4-minor-free. Since, by the hypothesis of the lemma, B_1 does not contain a vertex of degree 1, it follows that $B_1 \cong K_3$, or, by Theorem 2, B_1 has at least two nonadjacent vertices with degree exactly 2.

Let w be a vertex of degree 2 in B_1 that is adjacent in B_1 to x' and y'. Then, by the definition of B_1 and since B does not contain two adjacent vertices of degree 2 in G, it follows that $p_{wx'}, p_{wy'} \geq 1$ and $p_{wx'} + p_{wy'} = d_G(w) \geq 3$. Furthermore, since $d_G(w) \geq 3$, we may assume without loss of generality that $p_{wy'} \geq 2$.

By interchanging x' and y' if necessary, we may assume that if no two paths in $P_{wy'}$ bound a region that has a path not in $P_{wy'}$ embedded in it, then no two paths in $P_{wx'}$ bound a region that has a path not in $P_{wx'}$ embedded in it also, and so the proof would be complete. So we may assume that there is a region R bounded by two paths in $P_{wy'}$ that has a path $w \dots y'$ not in $P_{wy'}$ embedded in it. Since $p_{wx'} + p_{wy'} = d_G(w)$ it follows that every such path in R must contain x', and so the bounding cycle of B consists of two paths in $P_{wy'}$. Let S be the subgraph of B obtained by deleting w and all its neighbours of degree 2 in B. An example is shown in Figure 1, where $R = wv_1y'v_2w$, where the dashed edges may or may not be present, and if B is an end-block, then $y' = z_0$.

Since w is adjacent in B_1 to y', and since $B_1 \cong K_3$ or has at least two non-adjacent vertices with degree exactly 2, then there is a vertex $u \neq y'$ in S such that $d_{B_1}(u) = 2$, and where possibly $u = x'$. Let u be adjacent in B_1 to x and y. Then, by what we have proved about w, the result follows since every region bounded by paths in P_{ux} or P_{uy} is inside the bounding cycle of B. This completes the proof of Lemma 5. □

We will now prove Theorem 1 for plane embeddings of K_4-minor-free graphs with $\Delta \geq 4$, which is restated in the following theorem.

Theorem 6. *Let G be a plane embedding of a K_4-minor-free graph with maximum degree $\Delta \geq 4$. Then*

(*i*) $\mathrm{ch}_{\mathrm{vef}}(G) \leq \Delta + 2$ *if* $\Delta \geq 5$;
(*ii*) $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$ *if* $\Delta = 4$.

Proof. Fix the value of $\Delta \geq 4$ and suppose, if possible, that G is a plane embedding of a K_4-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that G is a counterexample to either part. Assume that every vertex v, every edge e and every face f of G is given a list $L(v)$, $L(e)$ or $L(f)$ of $\Delta + 2$ or 7 colours as appropriate. Assume also that G has no proper entire colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined before Lemma 5. For each uncoloured element z in G, let $L'(z)$ denote the list of usable colours for z; that is, $L'(z)$ denotes $L(z)$ minus any colours already used on neighbours of z in G.

Claim 7. *G does not contain a vertex of degree 1.*

Proof. Suppose that u is a vertex of degree 1 in G that is adjacent to v. Let $H = G - u$. By hypothesis H has a proper entire colouring from its lists. The edge uv has at most $\Delta + 1$ coloured neighbours, and so uv can be given a colour from its list. Since u now has three coloured neighbours u can be coloured from its list. This contradiction proves Claim 7. □

Claim 8. *B does not contain two adjacent vertices of degree 2 in G.*

Proof. Suppose that $xuvy$ is a path in B (or a cycle if $x = y$) where both u and v have degree 2 in G. If $x \neq y$, let $H = G/uv$. By hypothesis H has a proper entire colouring from its lists. After applying a colouring of H to G, the remaining elements uv, u, v can be coloured in any order since each has at least one usable colour in its list at the time of its colouring. If $x = y$, then $B \cong K_3$. Let f be the interior face of B. Let $H = G - \{u, v\}$ where the face in H in which u and v were embedded is given the same list as the exterior face of B. By hypothesis H has a proper entire colouring from its lists.

Now each of ux, vx, u, v, f, uv has at most Δ, Δ, 2, 2, 2, 1 coloured neighbours in G respectively. So each of the remaining elements

$$ux, vx, u, v, f, uv \tag{2}$$

has a list of at least 2, 2, 5, 5, 5, 6 usable colours respectively. It follows that the remaining elements can be coloured in the order (2). This contradiction proves Claim 8. □

Claim 9. *If B contains the configuration in Figure 2(a), where $xuyvx$ is an interior face, where x is not adjacent to y, and where only x and y are incident with edges in G not shown, then $d_G(x) = d_G(y) = \Delta$ and $\Delta = 5$ or 6.*

Proof. Suppose that B contains the configuration in Figure 2(a), where $xuyvx$ is an interior face, where x is not adjacent to y, and where only x and y are

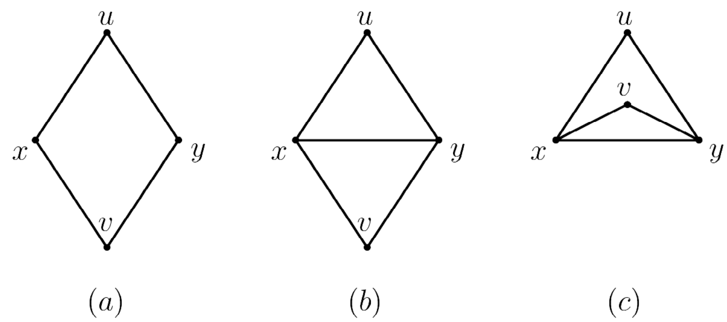


Figure 2

incident with edges in G not shown. Let f be the interior face $xuyvx$. Since, by Claim 8, both x and y have degree at least 3 in G, and if C is not 2-connected then B is an end-block by definition, it follows that f is adjacent to two different faces. Let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xvy in its boundary. Let $H = G - \{u, v\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u and v can be coloured at the end since each has six neighbours and a list of at least seven colours.

(i): Suppose first that $\Delta \geq 7$. Since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3 that these edges can be coloured. We can now colour f since it has only eight coloured neighbours, and then colour u and v. So we may assume that $\Delta = 5$ or 6, and contrary to what we want to prove, that $d_G(x) \leq \Delta - 1$ and that $d_G(y) \leq \Delta$.

Now each of uy, vy, f, ux, vx has at most Δ, Δ, 4, $\Delta - 1$, $\Delta - 1$ coloured neighbours in G respectively. So each of the remaining elements

$$uy, vy, f, ux, vx \tag{3}$$

has a list of at least 2, 2, 3, 3, 3 usable colours respectively. If we try to colour the elements in the order (3) then it is only with vx that we may fail.

If possible, give ux and vy the same colour. The remaining elements can now be coloured in the order (3). So we may assume that $L'(ux) \cap L'(vy) = \emptyset$ so that $|L'(ux) \cup L'(vy)| \geq 5$. Now either $|L'(vx)| \geq 5$, or else ux or vy can be given a colour that is not in $L'(vx)$. In each case the remaining elements can be coloured in the order (3), using a colour that is not in $L'(vx)$ on a neighbour of vx at the first opportunity.

(ii): Colour f, which is obviously possible. Next, since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3 that these edges can be coloured. In every case the colouring can be completed, which is the required contradiction. □

Claim 10. *If B contains the configuration in Figure 2(b) or 2(c), where in each case the faces are as shown and where only x and y are incident with edges in G not shown, then $d_G(x) = d_G(y) = \Delta$ and $\Delta = 5$.*

Proof. Suppose that B contains the configuration in Figure 2(b) or 2(c), where in each case the faces are as shown and where only x and y are incident with edges in G not shown. Let f be the face $xuyx$ or $xuyvx$ as appropriate. Let f' be the face $xvyx$. Let the other face with xuy in its boundary be f_1 and let the other face with xvy or xy in its boundary be f_2 as appropriate. (It is possible that $f_1 = f_2$ but the proof given here is still valid in this case.) Let $H = G - \{u, v\}$. Let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u and v can be coloured at the end since each has six neighbours and a list of at least seven colours.

(i): Suppose first that $\Delta \geq 6$. Since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3 that these edges can be coloured. We can now colour f and then f' since each has at most seven coloured neighbours at the time of its colouring. So we may assume that $\Delta = 5$, and contrary to what we want to prove, that $d_G(x) \leq \Delta - 1$ and that $d_G(y) \leq \Delta$.

If B contains the configuration in Figure 2(b) or 2(c), then each of uy, vy, f, ux, vx, f' has in Figure 2(b) at most 5, 5, 4, 4, 4, 4 coloured neighbours in G respectively, or in Figure 2(c) at most 5, 4, 3, 4, 3, 4 coloured neighbours in G respectively. So each of the remaining elements

$$uy, vy, f, ux, vx, f' \tag{4}$$

has in Figure 2(b) a list of at least 2, 2, 3, 3, 3, 3 usable colours respectively, or in Figure 2(c) a list of at least 2, 3, 4, 3, 4, 3 usable colours respectively. If we try to colour the elements in the order (4) then it is only with f' that we may fail.

If B contains the configuration in Figure 2(b), then, if possible, give vy and f the same colour. The remaining elements can now be coloured in the order (4). So we may assume that $L'(vy) \cap L'(f) = \emptyset$ so that $|L'(vy) \cup L'(f)| \geq 5$. Now either $|L'(f')| \geq 5$, or else vy or f can be given a colour that is not in $L'(f')$. In each case the remaining elements can be coloured in the order (4).

If B contains the configuration in Figure 2(c), then either $|L'(f')| \geq 4$, or else f can be given a colour that is not in $L'(f')$. In each case the remaining elements can be coloured in the order (4).

(ii): Colour f and f' which is obviously possible. Next, since each edge of the 4-cycle $xuyvx$ has at least two usable colours in its list, it follows from Theorem 3 that these edges can be coloured. In every case the colouring can be completed, which is the required contradiction. □

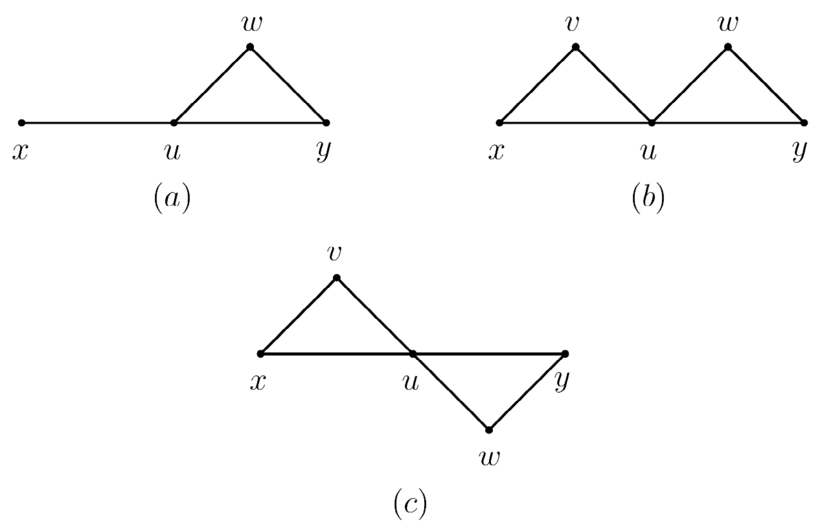


Figure 3

Claim 11. *B does not contain the configuration in Figure $3(a)$, where $uwyu$ is a face in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure $3(a)$, where $uwyu$ is a face in G, and where only x and y are incident with edges in G not shown. Let f be the face $uwyu$, let f_1 be the face with $xuwy$ in its boundary and let f_2 be the face with xuy in its boundary. Since B is a block it follows that f_1 and f_2 are distinct. Let $H = G - w$ and let the faces in H that have xuy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists.

Now each of wy, f, uw, w has at most $\Delta + 1$, 5, 4, 3 coloured neighbours in G respectively, and so each has a list of at least 1, 2, 3, 4 usable colours respectively; so these elements can be coloured in this order. This contradiction proves Claim 11. □

Claim 12. *B does not contain the configuration in Figure $3(b)$ or Figure $3(c)$, where in each case $xvux$ and $uwyu$ are faces in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure $3(b)$ or Figure $3(c)$, where in each case $xvux$ and $uwyu$ are faces in G, and where only x and y are incident with edges in G not shown. Let f be the face $xvux$ and let

f' be the face $uwyu$. If G contains the configuration in Figure 3(b), let f_1 be the face with $xvuwy$ in its boundary and let f_2 be the face with xuy in its boundary. If G contains the configuration in Figure 3(c), let f_1 be the face with $xvuy$ in its boundary and let f_2 be the face with $xuwy$ in its boundary. Let $H = G - \{v, w\}$. Since, by Claim 11, both x and y have degree at least 4 in G, and since B is a block, it follows that f_1 and f_2 are distinct. Let the faces in H that have xuy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that v and w can be coloured at the end since each has six neighbours and a list of at least seven colours.

First uncolour ux, u and uy. Now each of wy, uy, ux, vx, u, f, uv, uw, f' has at most Δ, Δ, Δ, Δ, 4, 3, 1, 1, 3 coloured neighbours in G respectively. So each of the remaining elements

$$wy, uy, ux, vx, u, f, uv, uw, f' \tag{5}$$

has a list of at least 2, 2, 2, 2, 3, 4, 6, 6, 4 usable colours respectively. If we try to colour the elements in the order (5) then it is only with f' that we may fail.

If possible, give ux and wy the same colour. The remaining elements can now be coloured in the order (5) with the exception that uw is coloured last. So we may assume that $L'(ux) \cap L'(wy) = \emptyset$. If possible, give u and wy the same colour. Since the colour on u is not in $L'(ux)$ the remaining elements can now be coloured in the order (5). So we may assume that $L'(u) \cap L'(wy) = \emptyset$ so that $|L'(u) \cup L'(wy)| \geq 5$. Now either $|L'(f')| \geq 5$, or else u or wy can be given a colour that is not in $L'(f')$. If $|L'(f')| \geq 5$, or if wy is given a colour that is not in $L'(f')$, then the remaining elements can be coloured in the order (5). So we may assume that u is given a colour α that is not in $L'(f')$. If $\alpha \notin L'(uy)$, then the remaining elements can be coloured in the order (5) with the exception that both ux and uy are coloured before wy *in that order*. If $\alpha \in L'(uy)$, then give uy the colour α and uncolour u. The remaining elements can now be coloured in the order (5). This contradiction proves Claim 12. $\square$

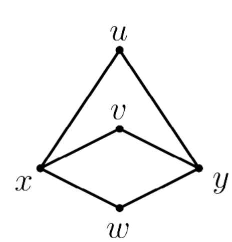


Figure 4

Claim 13. *If B contains the configuration in Figure 4, where $xuyvx$ and $xvywx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown, then $d_G(x) = d_G(y) = \Delta$ and $\Delta = 5$.*

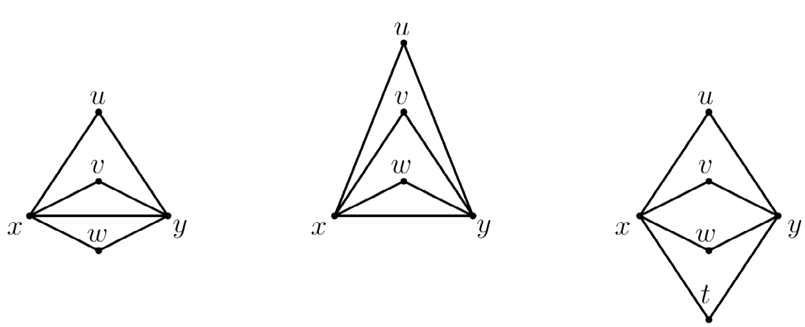
Proof. Suppose that B contains the configuration in Figure 4, where $xuyvx$ and $xvywx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$ and let f' be the face $xvywx$. Let the other face with xuy in its boundary be f_1 and let the other face with xwy in its boundary be f_2. Since, by Claim 9, $d_G(x) = d_G(y) = \Delta$ and $\Delta = 6$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w can be coloured at the end since each has six neighbours and a list of eight colours.

Now each of wy, wx, ux, uy, vy, vx, f, f' has at most 5, 5, 5, 5, 4, 4, 3, 3 coloured neighbours in G respectively. So each of the remaining elements

$$wy, wx, ux, uy, vy, vx, f, f' \tag{6}$$

has a list of at least 3, 3, 3, 3, 4, 4, 5, 5 usable colours respectively. If we try to colour the elements in the order (6) then it is only with f' that we may fail.

If possible, colour both vx and vy so that vx is given a colour that is not in $L'(f')$. Next, since each edge of the 4-cycle $xuywx$ has at least two usable colours in its list, it follows from Theorem 3 that these edges can be coloured. We can now colour f and then f' since each has at least one usable colour in its list at the time of its colouring. So we may assume that $L'(vx) \subseteq L'(f')$. If possible, give vx and wy the same colour. The remaining elements can now be coloured in the order (6). So we may assume that $L'(vx) \cap L'(wy) = \emptyset$ so that $|L'(vx) \cup L'(wy)| \geq 7$. Now either $|L'(f')| \geq 7$, or else wy can be given a colour that is not in $L'(f')$ since $L'(vx) \subseteq L'(f')$. In each case the remaining elements can be coloured in the order (6). This contradiction proves Claim 13. $\square$



(a) (b) (c)

Figure 5

Claim 14. *B does not contain the configuration in Figure* $5(a)$, *where $xuyvx$, $xvyx$ and $xywx$ are faces in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure $5(a)$, where $xuyvx$, $xvyx$ and $xywx$ are faces in G, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$, let f' be the face $xvyx$ and let f'' be the face $xywx$. Also, let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xwy in its boundary. Since, by Claim 10, $d_G(x) = d_G(y) = \Delta = 5$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w\}$ and let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w can be coloured at the end since each has six neighbours and a list of seven colours. First uncolour xy.

Now each of vy, vx, f' has 2 coloured neighbours in G, each of wy, wx, f'', ux, uy, f has 3 coloured neighbours in G, and xy has 4 coloured neighbours in G. So each of the remaining elements z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 5$ if $z \in \{vy, vx, f'\}$, $|L'(z)| \geq 4$ if $z \in \{wy, wx, f'', ux, uy, f\}$, and $|L'(xy)| \geq 3$. Now either $|L'(f)| \geq 5$, or else vy can be given a colour that is not in $L'(f)$. In each case colour vy. At this point, each of the remaining elements

$$xy, wy, wx, f'', ux, vx, uy, f, f' \tag{7}$$

has a list L'' of at least 2, 3, 4, 4, 4, 4, 3, 4, 4 usable colours respectively.

If possible, give f'' and vx the same colour. The remaining elements can now be coloured in the order (7) with the exception that if we fail at uy, then since $|L(uy)| = 7$ and at the time of its colouring uy has seven coloured neighbours in G, we can uncolour vy and give uy the colour that was on vy. We can now recolour vy since it has six coloured neighbours in G and a list of seven colours. Finally, we can give colours to f and then f'. So we may assume that $L''(f'') \cap L''(vx) = \emptyset$ so that $|L''(f'') \cup L''(vx)| \geq 8$. Now either $|L''(f')| \geq 8$, or else f'' or vx can be given a colour that is not in $L''(f')$. In each case the remaining elements can be coloured in the order (7), although, as above, it may be necessary to give uy the colour that is on vy and to recolour vy. This contradiction completes the proof of Claim 14. $\square$

Claim 15. *B does not contain the configuration in Figure* $5(b)$, *where $xuyvx$, $xvywx$ and $xwyx$ are faces in G, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 5(b), where $xuyvx$, $xvywx$ and $xwyx$ are faces in G, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$, let f' be the face $xvywx$ and let f'' be the face $xwyx$. Also, let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xy in its boundary. Since, by Claim 10, $d_G(x) = d_G(y) = \Delta = 5$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w\}$ and let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w can be coloured at the end since each has six neighbours and a list of seven colours. First uncolour xy.

Now each of wy, wx, vy, vx, f' has 2 coloured neighbours in G, each of uy, ux, f, f'' has 3 coloured neighbours in G, and xy has 5 coloured neighbours in G. So each of the remaining elements z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 5$ if $z \in \{wy, wx, vy, vx, f'\}$, $|L'(z)| \geq 4$ if $z \in \{uy, ux, f, f''\}$, and $|L'(xy)| \geq 2$. Now either $|L'(f'')| \geq 5$, or else wy can be given a colour that is not in $L'(f'')$. In each case colour wy, and then colour xy. At this point, each of the remaining elements

$$uy, ux, f, vy, vx, wx, f', f'' \tag{8}$$

has a list L'' of at least 2, 3, 4, 3, 4, 3, 4, 3 usable colours respectively.

If possible, give f and wx the same colour. The remaining elements can now be coloured in the order (8). So we may assume that $L''(f) \cap L''(wx) = \emptyset$ so that $|L''(f) \cup L''(wx)| \geq 7$. Now either $|L''(f')| \geq 7$, or else f or wx can be given a colour that is not in $L''(f')$. In each case the remaining elements can be coloured in the order (8) with the exception that if wx is given a colour that is not in $L''(f')$ and we fail at vx, then since $|L(vx)| = 7$ and at the time of its colouring vx has seven coloured neighbours in G, we can uncolour wx and give vx the colour that was on wx. We can now recolour wx since it has six coloured neighbours in G and a list of seven colours. Finally, we can give colours to f' and then f''. This contradiction proves Claim 15. □

Claim 16. *B does not contain the configuration in Figure 5(c), where $xuyvx$, $xvywx$ and $xwytx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain the configuration in Figure 5(c), where $xuyvx$, $xvywx$ and $xwytx$ are faces in G, where x is not adjacent to y, and where only x and y are incident with edges in G not shown. Let f be the face $xuyvx$, let f' be the face $xvywx$ and let f'' be the face $xwytx$. Also, let f_1 be the other face with xuy in its boundary and let f_2 be the other face with xty in its boundary. Since, by Claim 9, $d_G(x) = d_G(y) = \Delta = 5$, and by the definition of B, it follows that f_1 and f_2 are distinct. Let $H = G - \{u, v, w, t\} + xy$ and embed xy where xuy was embedded in G. Let xy in H have the same list as ux in G. Also, let the faces in H that have xy in their boundary have the same lists as f_1 and f_2 in G. By hypothesis H has a proper entire colouring from these lists. Note that u, v, w and t can be coloured at the end since each has six neighbours and a list of seven colours.

Now each of wy, wx, vx, vy, f' has 2 coloured neighbours in G, and each of ty, tx, ux, uy, f, f'' has 3 coloured neighbours in G. So each of the remaining elements z has a list $L'(z)$ of usable colours, where $|L'(z)| \geq 5$ if $z \in \{wy, wx, vx, vy, f'\}$, and $|L'(z)| \geq 4$ if $z \in \{ty, tx, ux, uy, f, f''\}$. Now either $|L'(f)| \geq 5$, or else vy can be given a colour that is not in $L'(f)$. Similarly, either $|L'(f'')| \geq 5$, or else wx can be given a colour that is not in $L'(f'')$. In each case colour both vy and wx. At this point, each of the remaining elements

$$ty, tx, wy, ux, vx, uy, f', f, f'' \tag{9}$$

has a list L'' of at least 3, 3, 3, 3, 3, 3, 3, 4, 4 usable colours respectively.

If possible, give uy and vx the same colour. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z, where $|L'''(wy)| \geq 2$, $|L'''(tx)| \geq 2$, and $|L'''(f'')| \geq 4$. If $|L'''(wy)| = 2$ and $|L'''(tx)| = 2$, then it follows that the colour on wx was in both $L'(wy)$ and $L'(tx)$. So it is possible to give both wy and tx the colour on wx and to recolour wx. The remaining elements can now be coloured in the order (9). So we may assume that at least one of $L'''(wy)$ and $L'''(tx)$ has at least three colours. If possible, give wy and tx the same colour. The remaining elements can now be coloured in the order (9). So we may assume that $L'''(wy) \cap L'''(tx) = \emptyset$ so that $|L'''(wy) \cup L'''(tx)| \geq 5$. Now either $|L'''(f'')| \geq 5$, or else wy or tx can be given a colour that is not in $L'''(f'')$. In each case the remaining elements can be coloured in the order (9). So we may assume that this is not possible so that $L''(uy) \cap L''(vx) = \emptyset$, and, by symmetry, that $L''(wy) \cap L''(tx) = \emptyset$.

Since $|L''(uy) \cup L''(vx)| \geq 6$, either $|L''(f)| \geq 6$, or else uy or vx can be given a colour that is not in $L''(f)$. If $|L''(f)| \geq 6$, or uy can be given a colour that is not in $L''(f)$, then colour uy. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z. Now $|L'''(wy) \cup L'''(tx)| \geq 5$, so either $|L'''(f'')| \geq 5$, or else wy or tx can be given a colour that is not in $L'''(f'')$. In each case the remaining elements can be coloured in the order (9). So we may assume that vx can be given a colour that is not in $L''(f)$. Again, at this point, $|L'''(wy) \cup L'''(tx)| \geq 5$, so either $|L'''(f'')| \geq 5$, or else wy or tx can be given a colour that is not in $L'''(f'')$. In each case colour both wy and tx. The remaining elements can now be coloured in the order (9) with the exception that if we fail at uy, then since $|L(uy)| = 7$ and at the time of its colouring uy has seven coloured neighbours in G, we can uncolour vy and give uy the colour that was on vy. We can now recolour vy since it has six coloured neighbours in G and a list of seven colours. Finally, we can give colours to f', f, f'' *in that order*. This contradiction proves Claim 16. $\square$

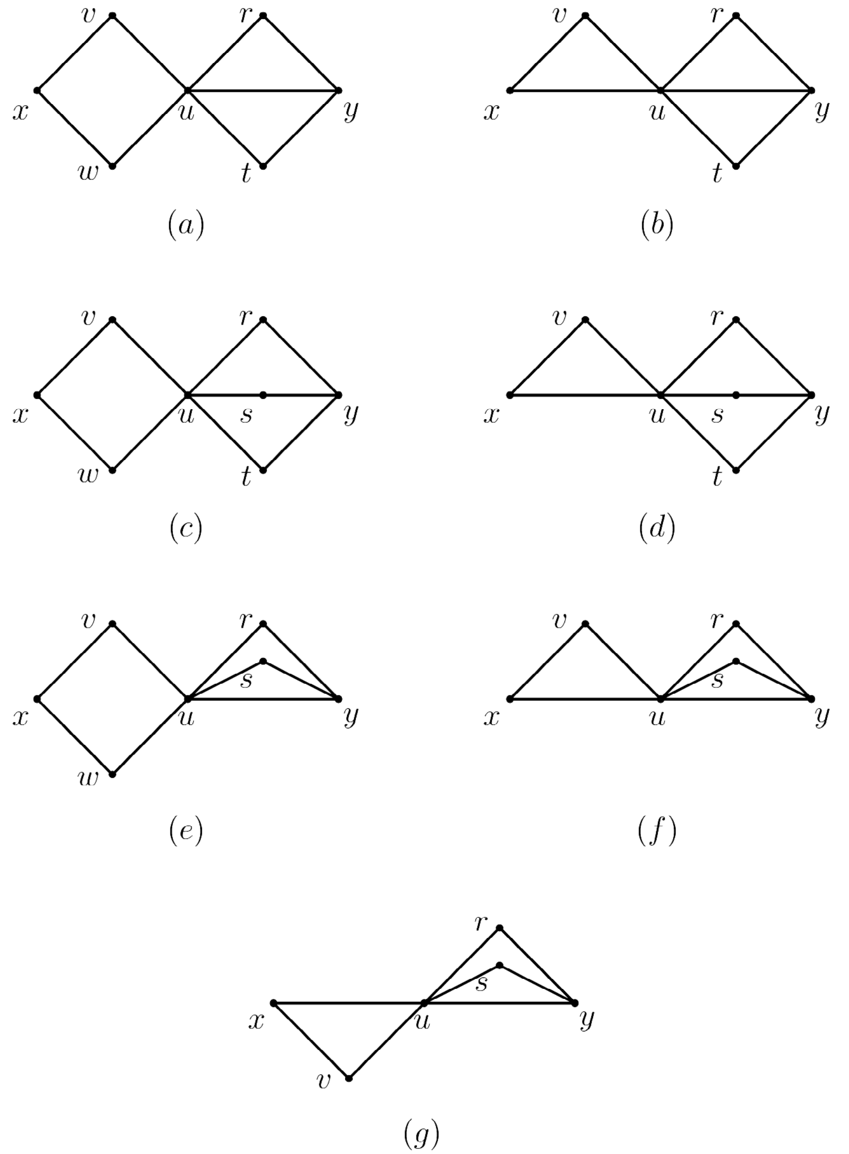


Figure 6

Claim 17. *B does not contain one of the configurations in Figures* 6(a)–6(d), *where the faces are as shown and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain one of the configurations in Figures 6(a)–6(d), where the faces are as shown and where only x and y are incident with edges in G not shown. Let f be the face $uryu$ or $urysu$ as appropriate. Let f' be the face $utyu$ or $utysu$ as appropriate and let f'' be the face $xvuwx$ or $xvux$ as appropriate. Also, let f_1 be the face with xvu in its boundary that is different from f'' and let f_2 be the face with uty in its boundary that is different from f'. Since B is a block it follows that both x and y are incident with edges not shown and that f_1 and f_2 are distinct. Let $H = G - r$ and let the faces in H that have xvu and uty in their boundary have the same lists as f_1 and f_2 in G respectively. By hypothesis H has a proper entire colouring from these lists. First uncolour all elements of the configuration being considered except for x, y, f_1 and f_2. Note that where present, each of v, w, r, s, t can be coloured at the end since each has six neighbours and a list of seven colours.

	vx	wx	ux	uv	uw	f''	u	ru	su	uy	tu	ry	sy	ty	f	f'
(a)	5	5		1	1	3	3	1		3	1	4		4	2	2
(b)	5		5	1		3	4	1		3	1	4		4	2	2
(c)	5	5		1	1	3	2	1	0		1	4	3	4	2	2
(d)	5		5	1		3	3	1	0		1	4	3	4	2	2
(a)	2	2		6	6	4	4	6		4	6	3		3	5	5
(b)	2		2	6		4	3	6		4	6	3		3	5	5
(c)	2	2		6	6	4	5	6	7		6	3	4	3	5	5
(d)	2		2	6		4	4	6	7		6	3	4	3	5	5

Table 1

For each of the configurations in Figures 6(a)–6(d) the maximum number of coloured neighbours of the remaining elements is given in the first half of Table 1, and the minimum number of usable colours in the list of each remaining element is given in the second half of Table 1.

Now either $|L'(f')| \geq 6$, or else tu can be given a colour that is not in $L'(f')$. In each case colour tu.

If B contains the configuration in Figure 6(a) or 6(c), then we can colour in order uw, wx, vx, f'', u, uv since each has at least one usable colour in its list at the time of its colouring.

If B contains the configuration in Figure 6(b) or 6(d), then either $|L'(f'')| \geq 5$, or else uv can be given a colour that is not in $L'(f'')$. In each case colour in order ux, vx, u, uv, f'' so that, where possible, at least one of these is given a colour that is not in $L'(f'')$.

At this point, if B contains the configuration in Figure 6(a) or 6(b), then each of the remaining elements

$$ru, uy, ry, ty, f, f' \tag{10}$$

has a list L'' of at least 2, 0, 3, 2, 4, 4 usable colours respectively.

Since $d_G(y) = \Delta = 5$ by Claim 10, it follows that uy has seven coloured neighbours. If $|L''(uy)| = 0$, then since $|L(uy)| = 7$, it follows that the colour on tu is in $L(uy)$ and is not used on any other neighbours of uy. So we can give uy the colour on tu and uncolour tu. At this point, since each edge of the 4-cycle $urytu$ has at least two usable colours in its list, it follows from Theorem 3 that these edges can be coloured. We can now colour f and then f' since each has at least one usable colour in its list at the time of its colouring.

So we may assume that $|L''(uy)| \geq 1$, and so we can colour uy. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z. If $|L'''(ty)| \geq 2$, then the remaining elements can be coloured in the order (10). So we may assume that $|L'''(ty)| = 1$. Since ty has six coloured neighbours and $|L(ty)| = 7$, it follows that the colour on tu is in $L(ty)$ and is not used on any other neighbour of ty. So if the colour on tu is in $L'''(ry)$, then give this colour to ry; otherwise give this colour to ty and recolour tu. In each csse the remaining elements can be coloured in the order (10).

So we may assume that B contains the configuration in Figure 6(c) or 6(d). Now each of the remaining elements

$$ry, ru, su, sy, ty, f, f' \tag{11}$$

has a list L'' of at least 3, 2, 3, 4, 2, 4, 4 usable colours respectively.

If possible, give f and ty the same colour. The remaining elements can now be coloured in the order (11) with the exception that ru is coloured first. So we may assume that $L''(f) \cap L''(ty) = \emptyset$ so that $|L''(f) \cup L''(ty)| \geq 6$.

Now either $|L''(f')| \geq 6$, or else f or ty can be given a colour that is not in $L''(f')$. If $|L''(f')| \geq 6$, or ty can be given a colour that is not in $L'(f')$, then colour ty. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z. If possible, give ru and sy the same colour. The remaining elements can now be coloured in the order (11). So we may assume that $L'''(ru) \cap L'''(sy) = \emptyset$ so that $|L'''(ru) \cup L'''(sy)| \geq 5$. Now either $|L'''(f)| \geq 5$, or else ru or sy can be given a colour that is not in $L'''(f)$. In each case the remaining elements can be coloured in the order (11).

So we may assume that $L''(ty) \subseteq L''(f')$. If $|L''(ty) \cap L''(ry)| \geq 1$, then we can give f' and ry the same colour. The remaining elements can now be coloured in the order (11) with the exception that ty is coloured first. So we may assume that $L''(ty) \cap L''(ry) = \emptyset$. We can now give f a colour that is not in $L''(f')$ so that the remaining elements can be coloured in the order (11) with the exception that ru is coloured first. In every case the colouring can be completed, which is the required contradiction. $\square$

Claim 18. *B does not contain one of the configurations in Figures* 6(e)–6(g), *where the faces are as shown and where only x and y are incident with edges in G not shown.*

Proof. Suppose that B does contain one of the configurations in Figures 6(e)–6(g), where the faces are as shown and where only x and y are incident with edges in G not shown. Let f be the face $urysu$, let f' be the face $usyu$. Let f'' be the face $xvuwx$ or $xvux$ as appropriate. Also, let f_1 be the face with ury in its boundary that is different from f and let f_2 be the face with uy in its boundary that is different from f'. Since B is a block it follows that both x and y are incident with edges not shown and that f_1 and f_2 are distinct. Let $H = G - r$ and let the faces in H that have usy and uy in their boundary have the same lists as f_1 and f_2 in G respectively. By hypothesis H has a proper entire colouring from these lists. First uncolour all elements of the given configurations except for x, y, f_1 and f_2. Note that where present, each of v, w, r, s, can be coloured at the end since each has six neighbours and a list of seven colours.

	vx	wx	ux	uv	uw	f''	u	ru	su	uy	ry	sy	f	f'
(e)	5	5		1	1	3	3	1	0	4	4	3	2	2
(f) and (g)	5		5	1		3	4	1	0	4	4	3	2	2
(e)	2	2		6	6	4	4	6	7	3	3	4	5	5
(f) and (g)	2		2	6		4	3	6	7	3	3	4	5	5

Table 2

For each of the configurations in Figures 6(e)–6(g) the maximum number of coloured neighbours of the remaining elements is given in the first half of Table 2, and the minimum number of usable colours in the list of each remaining element is given in the second half of Table 2.

If B contains the configuration in Figure 6(e), then either $|L'(f')| \geq 7$, or else su can be given a colour that is not in $L'(f')$. In each case colour su, u, uy. At this point each of the elements

$$vx, wx, f'', uv, uw \tag{12}$$

has a list L'' of at least 2, 2, 3, 3, 3 usable colours respectively. If we try to colour these elements in the order (12) then it is only with uw that we may fail.

If possible, give uv and wx the same colour. The remaining elements can now be coloured in the order (12). So we may assume that $L''(uv) \cap L''(wx) = \emptyset$ so that $|L''(uv) \cup L''(wx)| \geq 5$. Now either $|L''(uw)| \geq 5$, or else uv or wx can be given a colour that is not in $L''(uw)$. In each case the remaining elements can be coloured in the order (12), using a colour that is not in $L''(uw)$ on a neighbour of uw at the first opportunity.

If B contains the configuration in Figure 6(f) or 6(g), then first we will colour the elements

$$ux, vx, u, uv, uy, f'', su. \tag{13}$$

Now either $|L'(f')| \geq 7$, or else su can be given a colour that is not in $L'(f')$. If $|L'(f')| \geq 7$, then colour uy; otherwise, at the first opportunity, colour exactly one of uy, u, su using a colour that is not in $L'(f')$. At this point, let $L''(z)$ be the list of usable colours for each remaining element z. Now either $|L''(f'')| \geq 5$, or else uv can be given a colour α that is not in $L''(f'')$. In all cases the remaining elements in (13) can be coloured in order, using a colour that is not in $L''(f'')$ at the first opportunity, and with the exception that if it were su that was given a colour that is not in $L'(f')$, and hence not in $L'(uy)$ or $L'(u)$, then uy is coloured immediately after vx with a colour that is different from α.

At this point, if the configuration is in Figure 6(e), 6(f) or 6(g), then each of the remaining elements

$$ru, ry, sy, f, f' \tag{14}$$

has a list L''' of at least 1, 2, 2, 3, 3 usable colours respectively. If we try to colour the elements in the order (14) then it is only with f that we may fail.

Let β be the colour given to su. Suppose that $\beta \notin L(sy)$ or that β is used on another neighbour of sy so that $|L'''(sy)| \geq 3$. The remaining elements can now be coloured in the order (14) with the exception that sy is coloured immediately after f. So we may assume that $\beta \in L(sy)$ and that β is not used on any other neighbour of sy. Suppose that $\beta \notin L(ru)$ or that β is used on another neighbour of ru so that $|L'''(ru)| \geq 2$. If possible, give ru and sy the same colour. The remaining elements can now be coloured in the order (14). So we may assume that $L'''(ru) \cap L'''(sy) = \emptyset$ so that $|L'''(ru) \cup L'''(sy)| \geq 4$. Now either $|L'''(f)| \geq 4$, or else ru or sy can be given a colour that is not in $L'(f)$. In each case the remaining elements can be coloured in the order (14) with the exception that ry is coloured first. So we may assume that $\beta \in L(ru)$ and that β is not used on any other neighbour of ru. So we can give ru and sy the colour β and recolour su. The remaining elements can now be coloured in the order (14). In every case the colouring can be completed, which is the required contradiction. $\square$

Claim 7 implies that $B \not\cong K_2$ and Claim 8 implies that B is not a cycle; so B has at least two vertices with degree at least three and $d_G(z_0) \geq 3$. Let B_1 be the graph as defined before Lemma 5.

Claim 19. *B_1 is not K_4-minor-free.*

Proof. Since B has at least two vertices with degree at least 3, it follows that B_1 exists and has no vertex of degree 0. Suppose that x is a vertex of degree 1 in B_1. Then x is adjacent in B_1 to z_0. By the definition of B_1 and by Claim 8, it follows that $p_{xz_0} \geq 3$, and that every path between x and z_0 is in P_{xz_0}. So, by the definition of B, it follows that x must occur in B as vertex x in Figure 2(b), 2(c) or 4, where the faces are as shown and where only x and y may be incident with edges in G not shown. Since, by Claims 10 and 13, both x and z_0 must have degree $\Delta = 5$ in G, it follows that $p_{xz_0} = 5$. So B must contain one of the configurations in Figure 5, where the faces are as shown and where only x and y are incident with edges in G not shown. However, Claims 14–16 show that this is impossible. So B_1 has no vertex of degree 1.

In view of Claims 7 and 8, it follows from Lemma 5 that B_1 contains a vertex u of degree 2 that is adjacent in B_1 to x and y say, where $p_{ux} + p_{uy} = d_G(u) \geq 3$, where $p_{uy} \geq 2$, and where no two paths in P_{uy} bound a region that has a path not in P_{uy} embedded in it, and no two paths in P_{ux} bound a region that has a path not in P_{ux} embedded in it also.

By Claims 14–16, it follows that $p_{uy} \leq 3$. First suppose that $p_{uy} = 3$. Then, by Claims 10 and 13, it follows that $d_G(u) = \Delta = 5$ and that u must occur in B as vertex u in one of the configurations in Figure 6, where the faces are as shown and where only x and y are incident with edges in G not shown. However, Claims 17 and 18 show that this is impossible. So we may assume that $p_{uy} = 2$ and $p_{ux} \leq 2$, and so $d_G(u) \leq 4$. By Claim 9, it follows that u must occur in B as vertex u in Figure 3(a), 3(b), or 3(c), where the faces are as shown and where only x and y are incident with edges in G not shown. (Note that w, and v if present, have degree 2 in G and are therefore different from z_0.) However, Claims 11 and 12 show that this is impossible. This contradiction completes the proof of Claim 19. $\Box$

Since B_1 is a minor of G, Claim 19 implies that G is not K_4-minor-free. This contradiction completes the proof of Theorem 6. $\Box$

4 $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs with $\Delta \geq 4$

We will make use of Theorem 6. For each uncoloured element z in G, let $L'(z)$ denote the list of usable colours for z; that is, $L'(z)$ denotes $L(z)$ minus any colours already used on neighbours of z in G.

Let C be a component of a plane embedding of a $(\bar{K}_2+(K_1\cup K_2))$-minor-free graph G such that no interior face of C has another component of G embedded in it. If C is 2-connected, then let $B=C$ and let z_0 be any vertex of maximum degree in C; otherwise, by Lemma 4, let B be an end-block of C with cut-vertex z_0 such that no interior face of B has a block of C embedded in it.

Lemma 20. *Let G be a $(\bar{K}_2+(K_1\cup K_2))$-minor-free graph. Then each block of G is either K_4-minor-free or else isomorphic to K_4.*

Proof. Suppose that B is a block of G that has a K_4 minor. Since $\Delta(K_4)=3$, it follows that B has a subgraph B' that is homeomorphic to K_4. If an edge of K_4 is subdivided, or if a path is added joining two vertices of K_4, then a $\bar{K}_2+(K_1\cup K_2)$ minor is formed. So $B'\cong K_4$ and $B=K_4$. $\square$

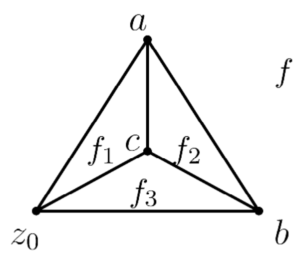


Figure 7

Lemma 21. *Let G be a plane embedding of K_4, as shown in Figure 7. If both f and z_0 are precoloured, and each of the elements az_0, bz_0, cz_0, f_1, f_3, f_2, a, b, c, ab, ac, bc has a list of at least 3, 3, 4, 5, 5, 6, 5, 5, 6, 6, 7, 7 usable colours respectively, then any given colouring of f and z_0 can be extended to the remaining elements of G.*

Proof. First colour in order az_0, bz_0, cz_0, f_1, f_3, which is obviously possible. Now each of the remaining elements

$$a, b, c, f_2, ab, ac, bc \tag{15}$$

has a list of at least 3, 3, 3, 4, 4, 4, 4 usable colours respectively.

If possible, give a and bc the same colour. At this point, each of the remaining elements

$$b, c, f_2, ab, ac \tag{16}$$

has a list L'' of at least 2, 2, 3, 3, 3 usable colours respectively. If possible, give b and ac the same colour. The remaining elements can now be coloured in the order (16). So we may assume that $L''(b)\cap L''(ac)=\emptyset$ so that $|L''(b)\cup L''(ac)|\geq 5$. Now either $|L''(ab)|\geq 5$, or else b or ac can be given a colour that is not in $L''(ab)$. In each case the remaining elements can be coloured in the order (16), using a colour that is not in $L''(ab)$ on either b, f_2 or ac at the first opportunity, where if ac is required to have a colour that is not in $L''(ab)$, then b and c are coloured so that this colour is not given to c. So we may assume that this is not possible so that $L'(a) \cap L'(bc) = \emptyset$, and, by symmetry, that $L'(b) \cap L'(ac) = \emptyset$ and $L'(c) \cap L'(ab) = \emptyset$.

If possible, give f_2 a colour so that each of the remaining elements has a list of at least three usable colours. Since $\mathrm{ch}''(K_3) = 3$, by Theorem 3, it follows that the remaining elements can be coloured from their lists. So we may assume that after colouring f_2, at least one of a, b, c has only two usable colours in its list. Suppose that each of a, b, c has only two usable colours in its list. Then since $|L'(f_2)| \geq 4$ we can change the colour on f_2 so that at least one of a, b, c has three usable colours in its list.

Suppose first that f_2 is given a colour that is in only one of $L'(a)$, $L'(b)$, $L'(c)$. By symmetry we may assume that this colour is in $L'(a)$, and hence not in $L'(bc)$. At this point, let $L''(z)$ be the list of usable colours for each remaining element z, where $|L''(z)| \geq 3$ if $z \in \{b, c, ab, ac\}$, $|L''(a)| = 2$, and $|L''(bc)| \geq 4$. So both b and ac can be given a colour that is not in $L''(a)$. Note that the remaining elements are equivalent to a 4-cycle. At this point, let $L'''(z)$ be the list of usable colours for each remaining element z, where $|L'''(a)| = 2$, $|L'''(bc)| \geq 2$, and $|L'''(c) \cup L'''(ab)| \geq 4$ since $L'(c) \cap L'(ab) = \emptyset$. If each of c and ab has at least two usable colours in its list, then it follows from Theorem 3 that the remaining elements can be coloured. So we may assume that one of c and ab has only one usable colour in its list, and so the other has at least three usable colours in its list. So, starting with whichever has only one usable colour in its list, the remaining elements can be coloured in the order c, a, bc, ab or ab, a, bc, c.

So we may assume that f_2 is given a colour that is in exactly two of $L'(a)$, $L'(b)$, $L'(c)$. By symmetry we may assume that this colour is in $L'(a)$ and $L'(b)$, and hence not in $L'(bc)$ or $L'(ac)$. At this point, let $L''(z)$ be the list of usable colours for each remaining element z, where $|L''(z)| \geq 3$ if $z \in \{c, ab\}$, $|L''(z)| \geq 4$ if $z \in \{ac, bc\}$, and $|L''(a)| = |L''(b)| = 2$. If possible, give b a colour that is in $L''(a)$ and hence not in $L''(bc)$. The remaining elements can now be coloured in the order (15). So we may assume that $L''(a) \cap L''(b) = \emptyset$. If possible, give c a colour that is in $L''(a)$, and hence not in $L''(bc)$ or $L''(b)$. The remaining elements can now be coloured in the order (15). So we may assume that $L''(a) \cap L''(c) = \emptyset$, and, by symmetry, that $L''(b) \cap L''(c) = \emptyset$. So the remaining elements can be coloured in the order (15) with the exception that c is coloured last. In every case the colouring can be completed. This completes the proof of Lemma 21. □

We will now prove Theorem 1 for plane embeddings of $(\bar{K}_2 + (K_1 \cup K_2))$-

minor-free graphs with $\Delta \geq 4$, which is restated in the following theorem.

Theorem 22. *Let G be a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree $\Delta \geq 4$. Then*

(i) $\mathrm{ch}_{\mathrm{vef}}(G) \leq \Delta + 2$ *if* $\Delta \geq 5$;
(ii) $\mathrm{ch}_{\mathrm{vef}}(G) \leq 7$ *if* $\Delta = 4$.

Proof. Fix the value of $\Delta \geq 4$ and suppose, if possible, that G is a plane embedding of a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with the smallest number of vertices and maximum degree at most Δ such that G is a counterexample to either part. Assume that every vertex v, every edge e and every face f of G is given a list $L(v)$, $L(e)$ or $L(f)$ of $\Delta + 2$ or 7 colours as appropriate. Assume also that G has no proper entire colouring from these lists. Clearly G has neither a trivial component nor a K_2 component; so every component C of G has at least three vertices. Let C and B be as defined before Lemma 20.

Claim 23. *$B \ncong K_4$.*

Proof. Suppose that $B \cong K_4$ and let the elements of B be labelled as in Figure 7. Then, by hypothesis, $G - (B - z_0)$ has a proper entire colouring from its lists in which both f and z_0 are coloured. Since $d_G(z_0) \leq \Delta$, there are at most $\Delta - 3$ coloured edges of $G - (B - z_0)$ incident with z_0. So each of the remaining elements az_0, bz_0, cz_0, f_1, f_3, f_2, a, b, c, ab, ac, bc, ab has a list of at least 3, 3, 4, 5, 5, 6, 5, 5, 6, 6, 7, 7 usable colours respectively, and so it follows from Lemma 21 that G can be coloured from its lists. This completes the proof of Claim 23. □

By Lemma 20 and Claim 23, it follows that B is K_4-minor-free. Claim 7 implies that $B \ncong K_2$ and Claim 8 implies that B is not a cycle; so B has at least two vertices with degree at least 3 and $d_G(z_0) \geq 3$. Let B_1 be as defined before Lemma 5. By Claim 19 B_1 is not K_4-minor-free. However, since B_1 is a minor of B this implies that B is not K_4-minor-free. This contradiction completes the proof of Theorem 22. □

Since we have now proved Theorems 6 and 22 this completes the proof of Theorem 1.

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