The

Significance

of

Evidence-based Reasoning

for

Mathematics

Mathematics Education

Philosophy

and the

Natural Sciences

Quantification from an Evidence-based Perspective

First Edition (preprint)

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"*Whereof one cannot speak, thereof one must be silent.*"

<small>. . . *Ludwig Wittgenstein:* [Wi22], *p.90.*.</small>

Author's Preface
(*Print edition*)

This investigation seeks to caution about the consequences, and dangers, of preferring Plato's *knowledge as justified belief* over Piccinini's *knowledge as factually grounded (evidence-based) belief*, in an age of *alternative facts* for which we—at least those of my ilk and generation (born circa 1940)—must shoulder the main responsibility.

Dangers we may have unknowingly *belittled*—even if not *denied* outright—and counter-intuitive consequences we may have not only uncritically *welcomed*, but passionately *nurtured* in our schools and universities, post Cantor, by assuming in classical mathematics, logic, philosophy, and the natural sciences, that the *unspecified* can be treated as *specifiable* without supporting *evidence*.

Fittingly, *ALL* the arguments and formal conclusions in this book follow from just *one*—seemingly trivial— distinction, between algorithmic *verifiability* and algorithmic *computability*, formalised in the paper: *The truth assignments that differentiate human reasoning from mechanistic reasoning: The evidence-based argument for Lucas' Gödelian thesis*, which appeared in the December 2016 issue of *Cognitive Systems Research*.

Bhupinder Singh Anand

Mumbai
28th April 2020

Let not posterity view us as having spent the gift of life polishing the pebbles, and tarnishing the diamonds.

<small>. . . *Anonymous.*</small>

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World Logic Day

UNESCO

**United Nations
Educational, Scientific
and
Cultural Organization**

40th session, Paris, 2019

40C/74

13 November 2019
Original: English

<u>**Item 5.31 of the agenda**</u>

PROCLAMATION OF A WORLD LOGIC DAY

OUTLINE

Source: 207 EX/Decision 42.

Background: At its 207th session, in 207 EX/Decision 42, the Executive Board recommended "that UNESCO's General Conference, at its 40th session, proclaim 14 January "World Logic Day", in association with the International Council for Philosophy and Human Sciences (CIPSH)".

Purpose: The report explains the historical, cultural and intellectual importance of logic for philosophy and the sciences; argues that the proclamation of World Logic Day would contribute to the development of logic, through teaching and research, as well as to public dissemination of the discipline and to the promotion of international cooperation in the field; and offers a rationale for UNESCO celebration of World Logic Day without financial implications, in close synergy with World Philosophy Day, drawing on the commitment of existing academic networks, and working in collaboration with the CIPSH.

Decision required: Paragraph 7.

1. The ability to think is one of the most defining features of humankind. In different cultures, the definition of humanity is associated with concepts such as consciousness, knowledge and reason. According to the classic western tradition, human beings are defined as "rational" or "logical animals". Logic, as the investigation on the principles of reasoning, has been studied by many civilizations throughout history and, since its earliest formulations, logic has played an important role in the development of philosophy and the sciences.

2. Despite its undeniable relevance to the development of knowledge, sciences and technologies, there is little public awareness on the importance of logic. The proclamation of World Logic Day by UNESCO, in association with the International Council for Philosophy and Human Sciences (CIPSH), intends to bring the intellectual history, conceptual significance and practical implications of logic to the attention of interdisciplinary science communities and the broader public.

3. A dynamic and global annual celebration of World Logic Day aims at fostering international cooperation, promoting the development of logic, in both research and teaching, supporting the activities of associations, universities and other institutions involved with logic, and enhancing public understanding of logic and its implications for science, technology and innovation. Furthermore, the celebration of World Logic Day can also contribute to the promotion of a culture of peace, dialogue and mutual understanding, based on the advancement of education and science.

4. On 14 January 2019, the first World Logic Day was celebrated as an initiative of universities, research institutes, foundations and associations active in the field of mathematics, philosophy, computer sciences, engineering, economics and cognitive sciences. The existence of these networks, and their commitment to future annual celebrations of World Logic Day, offer credible assurance that the proclamation of the Day by UNESCO, in association with the relevant member organizations of the CIPSH, will lead to a solid action plan towards impact and added value. UNESCO's commitment to World Logic Day will ensure that it has a broader and more geographically diverse orientation and connects organically to established programme priorities in the humanities.

5. The initiative has no financial implications for UNESCO. Celebrations would be organized, drawing on their own resources, by institutions in Member States and by international partners, including especially CIPSH member organizations, in particular the Division for Logic, Methodology and Philosophy of Science and Technology of the International Union for History and Philosophy of Science and Technology (DLMPST/IUHPST). The work of the Secretariat to coordinate World Logic Day would be absorbed within the human capacities already assigned to development of humanities workstreams, in close synergy with the annual celebration of World Philosophy Day. Coordinated planning of the two days will ensure a stronger presence of philosophy throughout UNESCO's calendar, consolidate relations with academic stakeholders through the association with the CIPSH, and thus contribute to the strengthening of the humanities called for by the General Conference in 39 C/Resolution 29. Organization of specific activities by UNESCO, in particular at Headquarters, would be conditional on extrabudgetary support.

6. The Executive Board considered the above matter at its 207th session and recommended, in 207 EX/Decision 42, that the General Conference proclaim 14 January "World Logic Day", in association with CIPSH.

Proposed draft resolution

7. In light of the above, the General Conference may wish to adopt a resolution along the following lines:

> *The General Conference*,
>
> *Having examined* 40 C/74,
>
> *Noting* the recommendation of the Executive Board as contaned in 207 EX/Decision 42,
>
> *Recalling* UNESCO's mandate in education, sciences and culture,
>
> *Also recalling* that logic, as a discipline that encourages rational and critical thinking, is of paramount importance for the development of human knowledge, science and technology,
>
> *Further recalling* that the dissemination of logic, a discipline that was developed by different civilizations and is based on the virtues of discourse and argumentation, can contribute to the achievement of a culture of peace, dialogue and mutual understanding among nations,
>
> *Noting* that the proclamation of world logic day will have no financial implications for UNESCO,
>
> *Convinced* that the proclamation of world logic day would contribute to the development of logic, through teaching and research, as well as to public dissemination of the discipline and to the promotion of international cooperation in the field,
>
> 1. *Invites* the Member States of UNESCO to celebrate world logic day every year, organizing national events on 14 January with the active participation of governmental and non-governmental, public and private institutions concerned, such as schools, universities, research institutes, philosophical and scientific associations, and others,
>
> 2. *Invites* the Director-General to encourage and publicize initiatives taken in this regard at the national, regional and international levels, working closely with the International Council for Philosophy and Human Sciences (CIPSH),
>
> 3. *Proclaims* 14 January "World Logic Day", in association with the International Council for Philosophy and Human Sciences (CIPSH).

Additional reference:

Jean-Yves Beziau. 2019. *1st World Logic Day: 14 January 2019.* In *Logica Universalis*, Volume 13, pp.120. Springer Nature.

https://link.springer.com/article/10.1007/s11787-019-00221-5

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Bernard of Chartres used to say that we are like dwarfs on the shoulders of giants, so that we can see more than they, and things at a greater distance, not by virtue of any sharpness of sight on our part, or any physical distinction, but because we are carried high and raised up by their giant size.
. . . John of Salisbury: Metalogica, 1159.

Dedicated to UNESCO's 2nd World Logic Day
14th January 2020

With gratitude to—from one who has been privileged a life-time dancing on the shoulders of—some, occasionally indulgent, more often long-suffering, Giants.

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ABSTRACT

In this multi-disciplinary investigation, we addresses the philosophical challenge that arises when an intelligence—whether human or mechanistic—accepts arithmetical propositions as true under an interpretation—either axiomatically or on the basis of subjective self-evidence—*without* any specified methodology for *evidencing* such acceptance. We then show how an *evidence-based* perspective of quantification in terms of:

- algorithmic *verifiability*, and
- algorithmic *computability*

admits *evidence-based* definitions of:

- *well-definedness*, and
- *effective computability*,

which yield two unarguably constructive interpretations of the first-order Peano Arithmetic PA—over the structure $\mathbb{N}$ of the natural numbers—that are complementary, not contradictory:

- The first yields the *weak*, standard, interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA over $\mathbb{N}$, which is *well-defined* with respect to assignments of algorithmically *verifiable* Tarskian truth values to the formulas of PA under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$; and thus constitutes a constructively *weak* proof of consistency for PA.
- The second yields a *strong*, finitary, interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA over $\mathbb{N}$, which is *well-defined* with respect to assignments of algorithmically *computable* Tarskian truth values to the formulas of PA under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$; and thus constitutes a constructively *strong* proof of consistency for PA.

We situate our investigation within a broad analysis of quantification vis à vis:

- Hilbert's ε-calculus
- Gödel's ω-consistency
- The Law of the Excluded Middle
- Hilbert's ω-Rule
- An Algorithmic ω-Rule
- Gentzen's Rule of Infinite Induction
- Rosser's Rule C
- Markov's Principle
- The Church-Turing Thesis
- Aristotle's particularisation (defined as the postulation that from the provability of a formula $[\neg(\forall x)\neg F(x)]$—abbreviated as $[(\exists x)F(x)]$—in a formal theory $\mathcal{L}$, we may conclude the existence of some *unspecified* object a, in the domain of any *well-defined* interpretation $\mathcal{I}$ of $\mathcal{L}$, for which the interpretation $F^*(a)$ of $[F(a)]$ is true)
- Wittgenstein's perspective of constructive mathematics
- An *evidence-based* perspective of quantification.

By showing how these are formally inter-related, we highlight the fragility of both:

- the persisting, *theistic*, classical/Platonic interpretation of quantification grounded in Hilbert's ε-calculus;

and

- the persisting, *atheistic*, constructive/Intuitionistic interpretation of quantification rooted in Brouwer's belief that the Law of the Excluded Middle is non-finitary.

We then consider some consequences for mathematics, mathematics education, philosophy, and the natural sciences, of an *agnostic*, *evidence-based*, finitary interpretation of quantification which challenges classical paradigms in all these disciplines.

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Part I
INTRODUCTION

PREFACE

In this multi-disciplinary investigation of the perception, and role, of quantification *in* formal mathematical languages, *and* under their intended interpretations, we presume familiarity with:

(1) Introductory-level texts on mathematical logic and computability[1];

(2) David Hilbert's, Luitzen Egbertus Jan Brouwer's and Kurt Gödel's deliberations[2] apropos a constructive definition of quantification, over the domain $\mathbb{N}$ of the natural numbers, which might yield a finitary proof of consistency for arithmetic;

and, ***most crucially***, with:

(3) The following (amongst other; see §2.)—paradigm challenging—conclusions of the *evidence-based* definitions of algorithmic *verifiabilty* and algorithmic *computability* introduced in the relatively recent paper [An16], '*The Truth Assignments That Differentiate Human Reasoning From Mechanistic Reasoning: The Evidence-Based Argument for Lucas' Gödelian Thesis*', which appeared in the December 2016 issue of *Cognitive Systems Research*:

   (a) The first-order Peano Arithmetic PA has *two*—hitherto unsuspected and unarguably *constructive*—Tarskian (see §2.A.) interpretations over the domain $\mathbb{N}$ of the natural numbers (compare [Art07]):

      (i) The *standard* interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA, which interprets quantification *weakly* in terms of algorithmic *verifiability* ([An16], Theorem 5.6, p.40); whence PA is *constructively* (*weakly*) consistent ([An16], Theorem 5.7, p.40);

      (ii) A *finitary* interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA, which interprets quantification *strongly* in terms of algorithmic *computability* ([An16], Theorem 6.7, p.41); whence PA is *finitarily* (*strongly*) consistent ([An16], Theorem 6.8, p.41);

   (b) PA is categorical ([An16], Corollary 7.2, p.41).

We shall situate our investigation within a broad analysis of quantification vis à vis:

   • Hilbert's ε-calculus

   • Gödel's ω-consistency

   • The Law of the Excluded Middle LEM

   • Hilbert's ω-Rule

   • An Algorithmic ω-Rule

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[1] We take Elliott Mendelson [Me64], George Boolos et al [BBJ03], and Hartley Rogers [Rg87], as representative—in the areas that they cover—of standard expositions of classical first-order theory and of computability (in particular, of the first-order Peano Arithmetic PA, and of classical Turing-computability).

[2] See, for instance, *van Heijenoort*: [Hei76]; *Davis*: [Da64]; *Benacerraf, Putnam*: [BPu64].

- Gentzen's Rule of Infinite Induction

- Rosser's Rule C

- Markov's Principle

- The Church-Turing Thesis CT

- Aristotle's particularisation (defined as the postulation that from the provability of a formula $[(\forall x)F(x)]$ in a formal theory $\mathcal{L}$, we may conclude the existence of some *unspecified* object a, in the domain of any *well-defined* interpretation $\mathcal{I}$ of $\mathcal{L}$, for which the interpretation $F^*(a)$ of $[F(a)]$ is true)

- Wittgenstein's perspective of constructive mathematics

- An *evidence-based* perspective of quantification.

By showing how these are formally inter-related, we shall highlight the fragility of both:

- the persisting, *theistic* (see §8.A.), classical/Platonic interpretation of quantification grounded in Hilbert's ε-calculus; and the

- persisting, *atheistic* (see §8.B.) constructive/Intuitionistic interpretation of quantification rooted in Brouwer's belief that the Law of the Excluded Middle LEM is non-finitary.

We shall then consider some consequences for:

- Mathematics;

- Mathematics education;

- Philosophy; and the

- Natural sciences;

of an agnostic, *evidence-based*, finitary interpretation of quantification that challenges classical paradigms in all these disciplines.

CHAPTER 1. *INTRODUCTION*

1.   The Complementarity Thesis

The thesis of this investigation is that (as argued in Part III, §9.) current mathematical paradigms of what is entailed by David Hilbert's development of proof theory[3], and by L. E. J. Brouwer's development of constructive mathematics[4], *appear* contradictory (see, for instance, Barendregt [Brd13], Errett Bishop [Bi18], Gila Sher [Shr18], and Penelope Maddy [Ma18], [Ma18a]) *only* because they fail to adequately accommodate that (as reportedly believed by Hilbert prior to 1929; see §6.A.a.):

Thesis 1. (Complementarity Thesis) *Mathematical 'provability' and mathematical 'truth' need to be interdependent and complementary, 'evidence-based', assignments-by-convention towards achieving:*

(1) *The goal of proof theory, post Peano, Dedekind and Hilbert, which is:*

— *to uniquely characterise each informally defined mathematical structure S (e.g., the Peano Postulates and their associated, classical, predicate logic),*

– *by a corresponding, formal, first-order language L, and a set P of finitary axioms/axiom schemas and rules of inference (e.g., the first-order Peano Arithmetic PA and its associated first-order logic FOL),*

- *which assign unique provability values (provable/unprovable) to each well-formed proposition of the language L without contradiction;*

(2) *The goal of constructive mathematics, post Brouwer and Tarski, which must be:*

— *to assign unique, evidence-based, truth values (true/false) to each well-formed proposition of the language L,*

– *under an, unarguably constructive, well-defined interpretation $\mathcal{I}$ over the domain D of the structure S,*

- *such that the provable formulas of L are true under the interpretation.*

In other words (see Part III, §10.C.):

- Whilst the focus of proof theory *may* be viewed as seeking to ensure that any mathematical language intended to represent our conceptual metaphors and their inter-relatedness is unambiguous, and free from contradiction;

- The focus of constructive mathematics *must* be viewed as seeking to ensure that any such representation does, indeed, uniquely identify and adequately represent such metaphors and their inter-relatedness.

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[3]As surveyed, for instance, in *Sieg*: [Si12].

[4]As covered, for instance, in *Kleene*: [Kl52] and *Bauer*: [Ba16].

From the *evidence-based* perspective of this investigation, mathematics is thus merely a set of precise, symbolic, languages (see §10.), intended to serve Philosophy and the Natural Sciences (see §10.C.) by seeking to provide the necessary tools for adequately expressing our sensory observations—and their associated perceptions (and abstractions)—of a 'common' external world (corresponding to what some cognitive scientists, such as Lakoff and Núñez in [LR00], term as *primary* and *secondary* 'conceptual metaphors') in a symbolic language of, ideally, unambiguous communication.

Moreover (see §10.E.), we may need to recognise explicitly in our basic mathematical education (see §20.) that *evidence-based* reasoning:

(a) restricts the ability of highly expressive mathematical languages, such as the first-order Zermelo-Fraenkel Set Theory ZF, to *categorically* communicate abstract concepts (corresponding to Lakoff and Núñez's conceptual metaphors in [LR00]) such as those involving Cantor's first limit ordinal ω[5];

and:

(b) restricts the ability of effectively communicating mathematical languages, such as the first-order Peano Arithmetic PA, to *well-define* infinite concepts such as ω (see §14.A.a.).

In other words (as highlighted by §16.C., Cases 16.C.a. to 16.C.d., and §16.D., Case 16.D.c.), from the *evidence-based* perspective of this investigation and, ideally, that of all disciplines that appeal to currently accepted scientific methods:

— although ZF admits unique, set-theoretical, definitions of—and allows us to unambiguously talk about the *putative* existence of—'*ideal*' real numbers as the *putative* limits of Cauchy sequences of rational numbers, and their *putative* properties, in a Platonically conceived, albeit mathematically defined, universe,

— only PA—by virtue of the Provability Theorem for PA (see §2.E., Theorem 2.17) admits unique, algorithmically *verifiable*, number-theoretic definitions of—and allows us to unambiguously talk about the *categorical* existence of (see §4.H.)—*specifiable* real numbers (see §4.H., Theorem 4.4) and their properties which can be communicated as *knowledge*—in the sense of §3.A.—when describing the actual universe we inhabit.

Mathematics, therefore, needs to be treated as a sub-discipline of linguistics (as suggested in §20.); and any ontological commitments associated with mathematical statements pertain not to the language per se, but to the conceptual metaphors that the language is intended to represent and communicate.

Further, the epistemological perspective (compare Gila Sher [Shr18]) of Thesis 1 is that (see §9. and §10.F.) logic, too, can be viewed as merely a methodological tool that seeks to formalise an intuitive human ability that pertains not to the language which seeks to express it formally, but to the cognitive sciences in which its study is rooted:

Definition 1. (Well-defined logic) *A finite set λ of rules is a well-defined logic of a formal mathematical language L if, and only if, λ assigns unique, evidence-based, values:*

---

[5]See [LR00], Preface, p.*xii-xiii*: "*How can human beings understand the idea of actual infinity?*"

(a) *Of provability/unprovability to the well-formed formulas of L; and*

(b) *Of truth/falsity to the sentences of the Theory $T(U)$ which is defined semantically by the λ-interpretation of L over a given mathematical structure U that may, or may not, be well-defined; such that*

(c) *The provable formulas interpret as true in $T(U)$.*

> **Comment**: We note that although the question of whether or not λ categorically defines a unique Theory $T(U)$ is mathematical, the question of whether, and to what extent, any Theory $T(U)$ succeeds in faithfully representing the structure U mathematically (in the sense of Carnap's *explicatum* and *explicandum* in [Ca62a]), is a philosophical question for the cognitive sciences (cf. [LR00]).

In other words—a view that resonates with Ludwig Wittgenstein's perspective in the *Tractatus* (see Frascolla: [Fra94], § The "Knowledge" of Forms: Vision and Calculation, pp.24-27)—a mathematical or logical truth is merely an assertion of the reliability of a mathematical language to faithfully *express* that which is sought to be expressed formally within the language. It has no bearing on the *ontological status* of that which is sought to be expressed within the language.

The goals of proof theory and constructive mathematics[6] ought to, thus (as argued in Part III, §10.E.), be viewed as necessarily interdependent and complementing, rather than being independent of, or in conflict with, each other as to which is more 'foundational'.

The foundational significance of Thesis 1 for the physical sciences (see Query 13) is highlighted in Part IV, §16.C., Cases 16.C.a. to 16.C.d.; and its paradigm-challenging consequences in §19.

> **Comment**: For instance, one could interpret the first order Peano Arithmetic PA so that the numeral [1] interprets as a grain of sand, and the numeral [n] interprets as a heap of n grains of sand. By the Provability Theorem for PA (see §2.E., Theorem 2.17), the theorems of PA then interpret as:
>
> - all the possible arithmetically definable—in classical mechanics—properties of, and relations between, heaps of sand located *anywhere* in the universe that, by Definition 3, we can *evidence* as algorithmically *computable* truths which are both *determinate* and *predictable*;
>
> as well as:
>
> - all the possible arithmetically definable—in quantum mechanics—properties of, and relations between, *accessible* heaps of sand in the universe that, by Definition 2, we can only *evidence* as algorithmically *verifiable* truths which are *determinate* but not necessarily *predictable*.

Finally, the foundational significance of *evidence-based* reasoning and Thesis 1 for mathematics education, and the philosophy of mathematics education, is addressed in §20.; albeit briefly, since doing justice to the subject of what mathematics is (see, for instance, §10.), and how mathematics and the philosophy of mathematics, respectively, ought to be taught as suggested by Thesis 1, is outside the ambit and competence of this investigation.

We note that the above, *evidence-based*-by-consensual methodology, perspective of this investigation is nothing new. For instance, knowledge as Gualtiero Piccinini's *factually grounded*

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[6]Corresponding to what Gila Sher terms in [Shr18] as 'proof-theoretic' and 'truth-theoretic' approaches to the classical perspective of 'logic'.

belief (see [Pic19]) rather than as Plato's *justified true belief*, seemed natural to ensure the certitude that—given the limitations of languages of common discourse for expressing conceptual metaphors unambiguously and communicating them categorically—early Indian mathematicians (like those elsewhere) sought in both their deliberations and practice. As P. P. Divakaran observes in [Dvk18]:

"As a general rule practising mathematicians in India, like those elsewhere, did not mix their science with its metaphysical and logical foundations, with one fortunate exception. When someone as articulate and as deeply reflective about his vocation as Nilakantha speaks about how we acquire new knowledge and how we are to know it to be true (the three wise books, Chapter 9.2), we have to listen. It is more than likely that his thoughts are not just personal but mirror the epistemic ethos of his time and place and perhaps that of Indian mathematics as a whole from the time of Aryabhata; after a thousand years, the insistence on the supremacy of the intellect (*mati*) continues to echo in these pronouncements. In slightly oversimplified terms, here is the model he proposes (a more analytic account in the canonical philosophical language will be found in Narasimha, cited above, and references therein).

The primary instrumentality of our apprehension of the world is our senses, not only of the astronomical world but even of the mathematical (visual geometry such as in cut-and-shift proofs, 3-dimensional geometric versions of algebraic identities, etc.). The sense data are to be subjected to analysis and tentative inferences drawn by means of our mental faculties, our ability to compute included. These are then exposed to scrutiny by the knowledgeable, debated and revisions made if necessary, and shared with pupils, thus sustaining a living chain of continuity. Nilakantha would not go out of his way to run down revealed wisdom, *śruti*, (his remote predecessors from 9th century Mahodayapuram in Kerala had already done that with their derisive remarks about *paurāṇika śruti*, the supposed revelations of the Puranas) and he has reverence for the words of the great teachers from the past. But he is uncompromising about the need to subject prior knowledge, whether revealed or merely uttered by mortals and lodged in an abstract communal memory (*smṛti*), to the tests of observation and logical inference and rejected if found wanting. In an uncanny paraphrase of Aryabhata's "*svamatināvā*", but more modestly, he says: "Everything here (the proof of the theorem of the diagonal) is rooted in *yukti* (about which, see below) alone, not in the beliefs and practices of yore (*āgama*)". And, as for the primacy of observation, he practised what he preached, taking upon himself the mission of campaigning for his teacher Parameshvara's *Dṛggaṇita* revision, forced by fresh data, of the planetary model of Aryabhata, no less.

The principles on which the practice of mathematics in India was grounded were not, thus, very rigid or 'theoretical' (Nilakantha says that theories are unending and inconclusive) and they could not have been more different from the inflexible frame in which Euclid's austere axiomatic-deductive system was confined. There were no unquestionable first principles to help choose, once and for all, a set of postulates and rules of logic, or to decide what objects needed to be defined. Instead we have a more dynamic and fluid foundation that was built from intelligent, rigorously exercised common sense so to say, and responsive to accumulated experience. The idea of an infallible set of axioms leading, by equally infallible logic, to mathematical truths of unquestionable certitude would not have been given a hearing: if all knowledge is contingent, how can it be otherwise for metaknowledge, the knowledge that some part of that knowledge is (or is not) true? How then did Indian mathematicians know or decide that their mathematical insights were indeed true?

The answer, the only satisfactory one given their philosophical stance, is in the last two of Nilakantha's tests of validation: there is no absolute criterion but only a convergence towards a consensus; the search for the infallible proof is ultimately a futile search, "unending and inconclusive" like all theorising. In practice most mathematicians seem to have been happy to leave the judgement to their peers, as Bhaskara II suggests in his comparison of a putative proposition whose *upapatti* does not get the approval of the assembly of the learned to rice without butter, unpalatable. The miracle is that the mathematics that this philosophical openness produced is true (and interesting) mathematics by contemporary axiomatic-deductive standards. Or, perhaps, there is no miracle; perhaps the universal and immutable truths of mathematics are open to all

gifted and prepared minds—even of those who never heard of Plato and Aristotle."

. . . [Dvk18], Chapter 15.1, pp.402-403

CHAPTER 2. *INTRODUCTION*

2. Revisiting an *evidence-based* paradigm

To place this investigation in its appropriate, *evidence-based*, perspective we first revisit the *evidence-based* paradigm introduced in [An12] and the paper [An16]; a paradigm whose philosophical significance is that it pro-actively addresses the challenge[7] that arises when an intelligence:

— whether human or mechanistic,

— accepts arithmetical propositions as true under an interpretation,

— either axiomatically or on the basis of subjective self-evidence,

— *without* any specified methodology for *evidencing* such acceptance,

— in the sense of, for instance, Chetan Murthy and Martin Löb:

> "It is by now folklore ...that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic ..."
> ...*Murthy: [Mu91], §1 Introduction.*

> "Intuitively we require that for each event-describing sentence, $\phi_{o\iota} n_\iota$ say (i.e. the concrete object denoted by n_ι exhibits the property expressed by $\phi_{o\iota}$), there shall be an algorithm (depending on **I**, i.e. M^*) to decide the truth or falsity of that sentence."
> ...*Löb: [Lob59], p.165.*

The foundational significance of the *evidence-based* definitions of arithmetical truth, introduced in [An12], lies in the fact that the first-order Peano Arithmetic PA—which, by [An16], is *finitarily* consistent (see §2.C.a., Theorem 2.16)—forms the bedrock on which all formal mathematical languages that admit rational and real numbers are founded (see, for instance, Edmund Landau's classically concise exposition [La29] on the foundations of analysis; see also §19.C.c.).

Axioms and rules of inference of the first-order Peano Arithmetic PA

PA$_1$ $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;

PA$_2$ $[(x_1 = x_2) \rightarrow (x_1' = x_2')]$;

PA$_3$ $[0 \neq x_1']$;

PA$_4$ $[(x_1' = x_2') \rightarrow (x_1 = x_2)]$;

PA$_5$ $[(x_1 + 0) = x_1]$;

PA$_6$ $[(x_1 + x_2') = (x_1 + x_2)']$;

PA$_7$ $[(x_1 \star 0) = 0]$;

PA$_8$ $[(x_1 \star x_2') = ((x_1 \star x_2) + x_1)]$;

PA$_9$ For any well-formed formula $[F(x)]$ of PA:
$[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$.

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[7]For a brief review of such challenges, see *Feferman*: [Fe06] and [Fe08]; also [An04] and [Fre18].

Generalisation in PA If $[A]$ is PA-provable, then so is $[(\forall x)A]$.

Modus Ponens in PA If $[A]$ and $[A \rightarrow B]$ are PA-provable, then so is $[B]$.

Hence the reliability of any conceptual metaphors of our observations of physical phenomena which appeal—in their mathematical representations—to mathematical definitions of real numbers (in the sense of [LR00]; see also §10.C.) must be circumscribed by whether, or not, PA can be interpreted categorically (see §2.E., Corollary 2.18) over the domain $\mathbb{N}$ of the natural numbers.

Now conventional wisdom, whilst accepting Alfred Tarski's classical definitions of the satisfiability and truth of the formulas of a formal language under an interpretation as adequate to the intended purpose, *postulates* that under the classical putative standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of the first-order Peano Arithmetic PA over the domain $\mathbb{N}$ of the natural numbers:

(i) The satisfiability/truth of the atomic formulas of PA can be assumed as *uniquely* decidable under $\mathcal{I}_{PA(\mathbb{N},\ S)}$;

(ii) The PA axioms can be assumed to *uniquely* interpret as satisfied/true under $\mathcal{I}_{PA(\mathbb{N},\ S)}$;

(iii) The PA rules of inference—Generalisation and Modus Ponens—can be assumed to *uniquely* preserve such satisfaction/truth under $\mathcal{I}_{PA(\mathbb{N},\ S)}$;

(iv) Aristotle's particularisation can be assumed to hold under $\mathcal{I}_{PA(\mathbb{N},\ S)}$.

> **Comment**. In [An16], Aristotle's particularisation (see also §4., Definition 12) is treated as the implicit, *non-finitary*, assumption that the classical first-order logic FOL is ω-consistent; and so we may always interpret the formal expression '$(\exists x)F(x)$'[8] of a formal language under any *well-defined* interpretation of FOL as 'There exists an object s in the domain of the interpretation such that $F^*(s)$', where the formula $[F(x)]$ of the formal language interprets as $F^*(x)$.
>
> ω-**consistent**. A formal system S is ω-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any *specified* S-term $[a]$.

The seemingly innocent and self-evident assumptions of *uniqueness* in (i) to (iii)—as also the seemingly innocent assumption in (iv) which, despite being obviously *non-finitary*, is unquestioningly (see §4.B.) accepted in classical literature as equally self-evident under any logically unexceptionable interpretation of the classical first-order logic FOL—conceal an ambiguity with far-reaching consequences.

The ambiguity is revealed if we note that Tarski's classic definitions (see §2.A. and §2.A.a.) permit both human and mechanistic intelligences to admit finitary *evidence-based* definitions of the satisfaction and truth of the *atomic* formulas of PA over the domain $\mathbb{N}$ of the natural numbers in *two*, hitherto unsuspected and essentially different, ways:

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[8]We note that, in a formal language, the formula '$[(\exists x)F(x)]$' is merely an abbreviation of the formula '$[\neg(\forall x)\neg F(x)]$. Moreover, that we shall use square brackets to differentiate between a symbolic expression—such as $[F(x)]$—which denotes a formula of a formal language L (treated syntactically as a string of symbols without any associated meaning), and the symbolic expression—such as $F^*(x)$—which denotes its meaning under a *well-defined* (in the sense of §4.F., Definition 19) interpretation; we find such differentiation useful in order to avoid the possibility of conflation between the two, particularly when (as is not uncommon) the same symbolic expressions are used to denote—or are common to—the two.

(1a) In terms of *classical* algorithmic *verifiability* (see §2., Definition 2); and

(1b) In terms of *finitary* algorithmic *computability* (see §2., Definition 3).

Definition 2. (Algorithmic verifiability) *A number-theoretical relation $F(x)$ is algorithmically verifiable if, and only if, for any given natural number n, there is an algorithm $AL_{(F,\ n)}$ which can provide objective evidence for deciding the truth/falsity of each proposition in the finite sequence $\{F(1), F(2), \ldots, F(n)\}$.*

Definition 3. (Algorithmic computability) *A number theoretical relation $F(x)$ is algorithmically computable if, and only if, there is an algorithm AL_F that can provide objective evidence for deciding the truth/falsity of each proposition in the denumerable sequence $\{F(1), F(2), \ldots\}$.*

> **Comment**: In §4.G., Theorem 4.2 (corresponding to [An16], Theorem 2.1, p.37), we shall show that there are *well-defined* number theoretic Boolean functions that are algorithmically *verifiable* but not algorithmically *computable*; and consider some consequences for the classical Church-Turing Thesis in §4.G.a.

2.A.   Reviewing Tarski's inductive assignment of truth-values under an interpretation

The paper [An16] essentially follows standard expositions (such as [Me64]; see §23.) of Tarski's inductive definitions on the 'satisfiability' and 'truth' of the formulas of a formal language under an interpretation where:

Definition 4. *If $[A]$ is an atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of a formal language S, then the denumerable sequence $(a_1, a_2, \ldots)$ in the domain $\mathbb{D}$ of an interpretation $\mathcal{I}_{S(\mathbb{D})}$ of S satisfies $[A]$ if, and only if:*

(i) *$[A(x_1, x_2, \ldots, x_n)]$ interprets under $\mathcal{I}_{S(\mathbb{D})}$ as a unique relation $A^*(x_1, x_2, \ldots, x_n)$ in $\mathbb{D}$ for any witness $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$;*

(ii) *there is a Satisfaction Method that provides objective evidence[9] by which any witness $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$ can objectively $\textbf{\textit{define}}$ for any atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of S, and any given denumerable sequence $(b_1, b_2, \ldots)$ of $\mathbb{D}$, whether the proposition $A^*(b_1, b_2, \ldots, b_n)$ holds or not in $\mathbb{D}$;*

(iii) *$A^*(a_1, a_2, \ldots, a_n)$ holds in $\mathbb{D}$ for any $\mathcal{W}_{\mathbb{D}}$.*

> **Witness:** From an *evidence-based* perspective, the existence of a 'witness' as in (i) above is implicit in the usual expositions of Tarski's definitions.
>
> **Satisfaction Method:** From an *evidence-based* perspective, the existence of a Satisfaction Method as in (ii) above is also implicit in the usual expositions of Tarski's definitions.
>
> **An *evidence-based* perspective:** We highlight the word '$\textbf{\textit{define}}$' in (ii) above to emphasise the *evidence-based* perspective underlying this paper; which is that the concepts of 'satisfaction' and 'truth' under an interpretation are to be explicitly viewed as objective assignments by a convention that is witness-independent. A Platonist perspective would substitute 'decide' for 'define', thus implicitly suggesting that these concepts can 'exist', in the sense of needing to be discovered by some witness-dependent means—eerily akin to a 'revelation'—if the domain $\mathbb{D}$ is $\mathbb{N}$.

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[9]In the sense of [Mu91] and [Lob59] (see §2.).

2.A.a.   Tarski's inductive definitions

Moreover, the truth values of 'satisfaction', 'truth', and 'falsity' are assignable inductively—whether *finitarily* or *non-finitarily*—to the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual[10]:

Definition 5. *A denumerable sequence s of $\mathbb{D}$ satisfies $[\neg A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, s does not satisfy $[A]$;*

Definition 6. *A denumerable sequence s of $\mathbb{D}$ satisfies $[A \rightarrow B]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, either it is not the case that s satisfies $[A]$, or s satisfies $[B]$;*

Definition 7. *A denumerable sequence s of $\mathbb{D}$ satisfies $[(\forall x_i)A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, given any denumerable sequence t of $\mathbb{D}$ which differs from s in at most the i'th component, t satisfies $[A]$;*

Definition 8. *A well-formed formula $[A]$ of $\mathbb{D}$ is true under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, given any denumerable sequence t of $\mathbb{D}$, t satisfies $[A]$;*

Definition 9. *A well-formed formula $[A]$ of $\mathbb{D}$ is false under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, it is not the case that $[A]$ is true under $\mathcal{I}_{S(\mathbb{D})}$.*

The implicit assumption of Aristotle's particularisation in current mathematical paradigms is evidenced in (V)(ii) of Mendelson's assertion—following his formulation of Tarski's definitions (essentially as above)—that:

> "Verification of the following consequences of the definitions above is left to the reader. (Most of the results are also obvious if one wishes to use only the ordinary intuitive understanding of the notions of truth and satisfaction). ...
>
> (V) (i) A sequence s satisfies $\mathcal{A} \wedge \mathcal{B}$ if and only if s satisfies $\mathcal{A}$ and s satisfies $\mathcal{B}$. A sequence s satisfies $\mathcal{A} \vee \mathcal{B}$ if and only if s satisfies $\mathcal{A}$ or s satisfies $\mathcal{B}$. A sequence s satisfies $\mathcal{A} \equiv \mathcal{B}$ if and only if s satisfies both $\mathcal{A}$ and $\mathcal{B}$ or s satisfies neither $\mathcal{A}$ nor $\mathcal{B}$.
>
> (ii) A sequence s satisfies $(Ex_i)\mathcal{A}$ if and only if there is a sequence s' which differs from s in at most the ith place such that s' satisfies $\mathcal{A}$."
>
> *. . . Mendelson: [Me64], pp.51-52.*

2.A.b.   The ambiguity in the classical *standard* interpretation of PA over $\mathbb{N}$

Now, the classical *standard* interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA over the domain $\mathbb{N}$ of the natural numbers (as detailed in [Me64], p.107) is obtained if, in $\mathcal{I}_{S(\mathbb{D})}$:

(a) we define S as PA with the standard first-order predicate calculus FOL as the underlying logic[11];

(b) we define $\mathbb{D}$ as the set $\mathbb{N}$ of natural numbers;

---

[10]See [Me64], p.51; [Mu91].

[11]Where the string $[(\exists \ldots)]$ is defined as—and is to be treated as an abbreviation for—the PA formula $[\neg(\forall \ldots)\neg]$. We do not consider the case where the underlying logic is Hilbert's formalisation of Aristotle's logic of predicates in terms of his ε-operator ([Hi27], pp.465-466).

(c) we assume for any *atomic* formula $[A(x_1, x_2, \ldots, x_n)]$ of PA, and any given sequence $(b_1^*, b_2^*, \ldots, b_n^*)$ of $\mathbb{N}$, that the proposition $A^*(b_1^*, b_2^*, \ldots, b_n^*)$ is decidable in $\mathbb{N}$;

(d) we define the witness $\mathcal{W}_{(\mathbb{N},\ S)}$ *informally* as the 'mathematical intuition' of a human intelligence for whom, classically, (c) has been *implicitly* accepted as '*objectively decidable*' in $\mathbb{N}$.

(e) we postulate that Aristotle's particularisation[12] holds over $\mathbb{N}$.

> **Comment**: Clearly, (e) (which, in [Me64], is implicitly entailed by [Me64], para (V)(ii), p.52) does not form any part of Tarski's inductive definitions of the satisfaction, and truth, of the formulas of PA under the above interpretation. Moreover, its inclusion makes $\mathcal{I}_{PA(\mathbb{N},\ S)}$ extraneously non-finitary[13] (see also §4.B.).

We shall show that the implicit acceptance in (d) conceals an ambiguity that needs to be made explicit since:

Lemma 2.1. *Under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$, an atomic formula $A^*(x_1, x_2, \ldots, x_n)$ is both algorithmically verifiable and algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{(\mathbb{N},\ S)}$.*

Proof. (i) It follows from the argument in §2.B., Theorem 2.3, that $A^*(x_1, x_2, \ldots, x_n)$ is algorithmically *verifiable* in $\mathbb{N}$ by $\mathcal{W}_{(\mathbb{N},\ S)}$.

(ii) It follows from the argument in §2.C., Theorem 2.9, that $A^*(x_1, x_2, \ldots, x_n)$ is algorithmically *computable* in $\mathbb{N}$ by $\mathcal{W}_{(\mathbb{N},\ S)}$. The lemma follows. □

2.B. The *weak*, algorithmically *verifiable*, interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA

The paper [An16] then considers a *weak*, algorithmically *verifiable*, interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA, under which we define:

Definition 10. *An atomic formula $[A]$ of PA is satisfiable under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ if, and only if, $[A]$ is algorithmically verifiable under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

We note that:

Theorem 2.2. *The atomic formulas of PA are algorithmically verifiable as true or false under the algorithmically verifiable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

Proof. It follows by Gödel's definition of the primitive recursive relation xBy[14]—where x is the Gödel number of a proof sequence in PA whose last term is the PA formula with Gödel-number y—that, if $[A]$ is an atomic formula of PA, we can algorithmically verify which one of the PA formulas $[A]$ and $[\neg A]$ is necessarily PA-provable and, ipso facto, true under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. □

---

[12]See §4., Definition 12; which postulates that a PA formula such as $[(\exists x)F(x)]$ can always be taken to interpret under $\mathcal{I}_{PA(\mathbb{N},\ S)}$ as 'There is some natural number n such that $F(n)$ holds in $\mathbb{N}$.

[13]As argued by Brouwer in [Br08].

[14][Go31], p. 22(45).

Comment: We note that the interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ cannot claim to be finitary[15] since it follows from §4.G., Theorem 4.2, that we cannot conclude finitarily from Tarski's Definition 4 (in §2.A.), and Definitions 5 to 9 (in §2.A.a.) whether or not a quantified PA formula $[(\forall x_i)R]$ is algorithmically verifiable as true under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ if $[R]$ is algorithmically verifiable but not algorithmically computable under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$[16].

2.B.a.   The PA axioms are algorithmically *verifiable* as true under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$

The significance of defining satisfaction in terms of algorithmic verifiability under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ is that:

Lemma 2.3. *The PA axioms PA_1 to PA_8 are algorithmically verifiable as true over $\mathbb{N}$ under the interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$.*

Proof. Since $[x + y]$, $[x \star y]$, $[x = y]$, $[x']$ are defined recursively[17], the PA axioms PA_1 to PA_8 (see §2.) interpret as recursive relations that do not involve any quantification. The lemma follows from §2.B., Theorem 2.2, Tarski's Definition 4 (in §2.A.), and Tarski's Definitions 5 to 9 (in §2.A.a.). $\qquad\qquad\square$

Lemma 2.4. *For any given PA formula $[F(x)]$, the Induction axiom schema $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$.*

Proof. We note that, by Tarski's Definition 4 (in §2.A.), and Definitions 5 to 9 (in §2.A.a.):

(a) If $[F(0)]$ interprets as an algorithmically *verifiable* false formula under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ the lemma is proved.

> *Reason*: Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically *verifiable* true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ if, and only if, either $[F(0)]$ interprets as an algorithmically *verifiable* false formula or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically *verifiable* true formula under under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$.

(b) If $[F(0)]$ interprets as an algorithmically *verifiable* true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically *verifiable* false formula, under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically *verifiable* true formulas under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ then, for any natural number n, there is an algorithm which (by Definition 2) will evidence that $[F(n) \rightarrow F(n')]$ is an algorithmically *verifiable* true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$.

(d) Since $[F(0)]$ interprets as an algorithmically *verifiable* true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$, it follows for any natural number n that there is an algorithm which will evidence that each of the formulas in the finite sequence $\{[F(0), \ F(1), \ \ldots, \ F(n)]\}$ is an algorithmically *verifiable* true formula under the interpretation.

---

[15]Since it defines a model of PA if, and only if, PA is ω-consistent and so we may always non-finitarily conclude from $[(\exists x)R(x)]$ the existence of some numeral $[n]$ such that $[R(n)]$.

[16]Although a proof that such a PA formula exists is not obvious, we shall see (§2.F., Corollary 2.21) that Gödel's 'undecidable' arithmetical formula $[R(x)]$ is algorithmically *verifiable* but not algorithmically *computable* under the interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$.

[17]cf. [Go31], p.17.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically *verifiable* true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

Since the above cases are exhaustive, the lemma follows. □

> **Comment**: We note that if $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically verifiable true formulas under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, then we can only conclude that, for any natural number n, there is an algorithm which will give evidence for any $m \leq n$ that the formula $[F(m)]$ is true under $\mathcal{I}_{PA(\mathbb{N},\ S)}$.
>
> We cannot conclude that there is an algorithm which, for any natural number n, will give evidence that the formula $[F(n)]$ is true under $\mathcal{I}_{PA(\mathbb{N},\ S)}$.

Lemma 2.5. *Generalisation preserves algorithmically verifiable truth under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

Proof. The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically *verifiable* true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$[18]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically *verifiable* true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$'

both mean:

$[F(x)]$ is algorithmically verifiable as always true under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. □

It is also straightforward to see that:

Lemma 2.6. *Modus Ponens preserves algorithmically verifiable truth under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.* □

We thus have that:

Theorem 2.7. *The axioms of PA are always algorithmically verifiable as true under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, and the rules of inference of PA preserve the properties of algorithmically verifiable satisfaction/truth under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.* □

By §2.B., Theorem 2.2 we further conclude that PA is *weakly* consistent:

Theorem 2.8. *If the PA formulas are algorithmically verifiable as true or false under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, then PA is consistent.* □

> **Comment**: We note that, unlike Gentzen's argument[19], which appeals to debatably 'constructive' properties of set-theoretically defined transfinite ordinals, such a—strictly arithmetical—*weak* proof of consistency is unarguably 'constructive'; however it is not 'finitary' since we cannot conclude from §2.B., Theorem 2.2 that the quantified formulas of PA are 'finitarily' decidable as true or false under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

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[18]See Definition 8 (in §2.A.a.)
[19]See [Me64], pp.258-271.

2.C.   The *strong*, algorithmically *computable*, interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA

The paper [An16] considers next a *strong*, algorithmically *computable*, interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA, under which we define:

Definition 11. *An atomic formula* $[A]$ *of PA is satisfiable under the interpretation* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ *if, and only if,* $[A]$ *is algorithmically computable under* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

We note that:

Theorem 2.9. *The atomic formulas of PA are algorithmically computable as true or as false under the algorithmically computable interpretation* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

Proof. If $[A(x_1, x_2, \ldots, x_n)]$ is an *atomic* formula of PA then, for any given sequence of numerals $[b_1, b_2, \ldots, b_n]$, the PA formula $[A(b_1, b_2, \ldots, b_n)]$ is an *atomic* formula of the form $[c = d]$, where $[c]$ and $[d]$ are *atomic* PA formulas that denote PA numerals. Since $[c]$ and $[d]$ are recursively defined formulas in the language of PA, it follows from a standard result[20] that, by §2., Definition 3, $[c = d]$ is algorithmically *computable* as either true or false in $\mathbb{N}$ since there is an algorithm that, for any given sequence of numerals $[b_1, b_2, \ldots, b_n]$, will give *evidence* (in the sense of [Mu91] and [Lob59]) whether $[A(b_1, b_2, \ldots, b_n)]$ interprets as true or false in $\mathbb{N}$. The lemma follows.   □

We note that the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ is finitary since:

Lemma 2.10. *The formulas of PA are algorithmically computable finitarily as true or as false under* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

Proof. The Lemma follows by finite induction from by §2., Definition 3, Tarski's Definition 4 (in §2.A.), and Definitions 5 to 9 (in §2.A.a.), and Theorem 2.9.   □

2.C.a.   The PA axioms are algorithmically *computable* as true under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$

The significance of defining satisfaction in terms of algorithmic computability under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ as above is that:

Lemma 2.11. *The PA axioms* PA_1 *to* PA_8 *are algorithmically computable as true under the interpretation* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

Proof. Since $[x + y]$, $[x \star y]$, $[x = y]$, $[x']$ are defined recursively[21], the PA axioms PA_1 to PA_8 (see §2.) interpret as recursive relations that do not involve any quantification. The lemma follows from §2.B., Theorem 2.2 and Tarski's Definition 4 (in §2.A.), and Definitions 5 to 9 (in §2.A.a.).   □

Lemma 2.12. *For any given PA formula* $[F(x)]$, *the Induction axiom schema* $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ *interprets as an algorithmically computable true formula under* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

---

[20]For any natural numbers m, n, if $m \neq n$, then PA proves $[\neg(m = n)]$ ([Me64], p.110, Proposition 3.6). The converse is obviously true.

[21]cf. [Go31], p.17.

Proof. By Tarski's Definition 4 (in §2.A.), and Definitions 5 to 9 (in §2.A.a.):

(a) If $[F(0)]$ interprets as an algorithmically *computable* false formula under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ the lemma is proved.

> *Reason*: Since $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$ interprets as an algorithmically *computable* true formula if, and only if, either $[F(0)]$ interprets as an algorithmically *computable* false formula, or $[((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$ interprets as an algorithmically *computable* true formula, under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$.

(b) If $[F(0)]$ interprets as an algorithmically *computable* true formula, and $[(\forall x)(F(x) \rightarrow F(x'))]$ interprets as an algorithmically *computable* false formula, under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$, the lemma is proved.

(c) If $[F(0)]$ and $[(\forall x)(F(x) \rightarrow F(x'))]$ both interpret as algorithmically *computable* true formulas under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$, then by Definition 3 there is an algorithm which, for any natural number n, will give *evidence* (in the sense of [Mu91] and [Lob59]) that the formula $[F(n) \rightarrow F(n')]$ is an algorithmically *computable* true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$.

(d) Since $[F(0)]$ interprets as an algorithmically *computable* true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$, it follows that there is an algorithm which, for any natural number n, will give *evidence* that $[F(n)]$ is an algorithmically *computable* true formula under the interpretation.

(e) Hence $[(\forall x)F(x)]$ is an algorithmically *computable* true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$.

Since the above cases are exhaustive, the lemma follows.   □

Lemma 2.13. *Generalisation preserves algorithmically computable truth under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$.*

Proof. The two meta-assertions:

'$[F(x)]$ interprets as an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$[22]'

and

'$[(\forall x)F(x)]$ interprets as an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$'

both mean:

$[F(x)]$ is algorithmically computable as always true under $\mathcal{I}_{PA(\mathbb{N}, \ S)}$.   □

It is also straightforward to see that:

Lemma 2.14. *Modus Ponens preserves algorithmically computable truth under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$.* □

We thus have that[23]:

Theorem 2.15. *The axioms of PA are always algorithmically computable as true under the interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$, and the rules of inference of PA preserve the properties of algorithmically computable satisfaction/truth under $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$.* □

We thus have a *finitary* proof that:

Theorem 2.16. *PA is strongly consistent.* □

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[22]See §2.A.a, Definition 8

[23]Without appeal, moreover, to Aristotle's particularisation.

2.D.   Dissolving the Poincaré-Hilbert debate

The significance of *evidence-based* reasoning is also highlighted in the case of the Poincaré-Hilbert debate[24] on whether the PA Axiom Schema of Induction can be labelled 'finitary' or not.

It turns out that the debate dissolves since the Axiom Schema is:

(a) Algorithmically *verifiable* as true under the *weak* standard interpretation of PA by §2.B.a., Lemma 2.4;

(b) Algorithmically *computable* as true under a *strong* finitary interpretation of PA by §2.C., Lemma 2.12.

In other words:

(i) The algorithmically *verifiable*, non-finitary, interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA validates Poincaré's argument that the PA Axiom Schema of Finite Induction could not be justified finitarily with respect to algorithmic *verifiability* under the classical *standard* interpretation of arithmetic[25], as any such argument would necessarily need to appeal to some form of infinite induction[26]; whilst

(ii) The algorithmically *computable* finitary interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA validates Hilbert's belief that a finitary justification of the Axiom Schema was possible under some finitary interpretation of an arithmetic such as PA.

2.E.   Bridging PA Provability and Turing Computability

The paper [An16] then shows that PA can have no non-standard model, since it is 'computably' complete semantically, in the sense that[27]:

Theorem 2.17. (Provability Theorem for PA) *A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$.*

Proof. We have by definition that $[(\forall x)F(x)]$ interprets as true under the interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ if, and only if, $[F(x)]$ is algorithmically *computable* as always true in $\mathbb{N}$.

By §2.C.a., Theorem 2.15, $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ defines a finitary model of PA over $\mathbb{N}$ such that:

(a) If $[(\forall x)F(x)]$ is PA-provable, then $[F(x)]$ interprets as an arithmetic relation $F^*(x)$ which is algorithmically *computable* as always true in $\mathbb{N}$;

(b) If $[\neg(\forall x)F(x)]$ is PA-provable, then it is not the case that $[F(x)]$ interprets as an arithmetic relation $F^*(x)$ which is algorithmically *computable* as always true in $\mathbb{N}$.

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[24]See [Hi27], p.472; also [Br13], p.59; [We27], p.482; [Pa71], p.502-503.

[25]See [Me64], p.107.

[26]Such as, for instance, in Gerhard Gentzen's *non-finitary* proof of consistency for PA, which involves a *non-finitary* Rule of Infinite Induction (see [Me64], p.259(II)(e).) that appeals to the well-ordering property of transfinite ordinals.

[27]We note that Theorem 2.17 (Provability Theorem for PA) offers a solution to Barendregt and Wiedijk's: '*The challenge of computer mathematics*' [BW05].

Now, we cannot have that both $[(\forall x)F(x)]$ and $[\neg(\forall x)F(x)]$ are PA-unprovable for some PA formula $[F(x)]$, as this would yield the contradiction:

(i) There is a finitary model—say $\mathcal{I}'_{PA(\mathbb{N},\ SC)}$—of PA+$[(\forall x)F(x)]$ in which $[F(x)]$ interprets as an arithmetic relation $F^*(x)$ that is algorithmically *computable* as always true in $\mathbb{N}$.

(ii) There is a finitary model—say $\mathcal{I}''_{PA(\mathbb{N},\ SC)}$—of PA+$[\neg(\forall x)F(x)]$ in which it is not the case that $[F(x)]$ interprets as an arithmetic relation $F^*(x)$ that is algorithmically *computable* as always true in $\mathbb{N}$.

The theorem follows.   □

Corollary 2.18. *PA is categorical.*   □

Before considering further (in §2.F.) the immediate, formal, consequences—as detailed in [An16]—of the Provability Theorem for PA, we briefly preview in §2.E.a. its far-reaching consequences that highlight the significance of *evidence-based* reasoning for interpreting mathematical propositions as 'true' in number-theory, computability, philosophy and the natural sciences.

2.E.a.   Preview: The significance of the Provability Theorem for PA for number theory and computability

From the *evidence-based* perspective of this investigation (see §1., Thesis 1), the significance of the Provability Theorem for PA (§2.E., Theorem 2.17) for number theory is that:

(a) although (see §11.G.), there can be *no* PA formula that interprets over $\mathbb{N}$ as the meta-statement 'PA is consistent' *without* inviting the paradoxes (see §16.) of impredicativity;

(b) we *can* interpret the number-theoretic expression $Wid(PA) \equiv (\exists x)[Form(x)\ \&\ \overline{Bew}_{PA}(x)]$ (see [Go31], p.36) over $\mathbb{N}$ as asserting: 'There is a PA-formula that is not PA-provable', which is equivalent to asserting that 'PA is consistent'.

Its corresponding significance for computability theory is that:

(c) whilst (see [Tu36], p.134 and [Me64], p.256) there can be *no* Turing machine U which, given the 'standard description' of any 'arbitrary' Turing machine T and any instantaneous tape description α, can determine whether or not there is a computation of T beginning with α (see §17.F.a., Query 21, Halting-decidability problem for T), *without* inviting the paradoxes (see §16.) of impredicativity;

(d) there *is* a PA formula that will determine whether or not T halts on α (see §17.F.b., Theorem 17.3 and Corollary 17.4).

In other words:

- we can express properties about the natural numbers in the language of recursive functions that cannot be expressed in the language of arithmetic; and

- we can express properties about the natural numbers in the language of arithmetic that cannot be expressed in the language of recursive functions.

2.F.   Gödel's '*undecidable*' formula $[\neg(\forall x)R(x)]$ is provable in PA

We note that the argumentation in §2.E.a. reflects the conclusion in [An16] that:

Lemma 2.19. *If $\mathcal{I}_{PA(\mathbb{N},\ M)}$ defines a model of PA over $\mathbb{N}$, then there is a PA formula $[F]$ which is algorithmically verifiable as always true over $\mathbb{N}$ under $\mathcal{I}_{PA(\mathbb{N},\ M)}$ even though $[F]$ is not PA-provable.*

Proof. Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[28]—such that, if PA is consistent, then $[R(x)]$ is not PA-provable[29], but $[R(n)]$ is instantiationally PA-provable for any given PA numeral $[n]$. Since PA is consistent by §2.C.a., Theorem 2.16, for any given numeral $[n]$, Gödel's primitive recursive relation $xB[[R(n)]]$[30] must hold for some x. The lemma follows.   □

By the argument in Theorem 2.17 it further follows that, contradicting current paradigms:

Corollary 2.20. *The PA formula $[\neg(\forall x)R(x)]$ defined in Lemma 2.19 is PA-provable.*   □

Corollary 2.21. *In any well-defined model of PA, Gödel's arithmetical formula $[R(x)]$ interprets as an algorithmically verifiable, but not algorithmically computable, tautology over $\mathbb{N}$.*

Proof. Gödel has shown that $[R(x)]$[31] always interprets as an algorithmically *verifiable* tautology over $\mathbb{N}$[32]. By Corollary 2.20 $[R(x)]$ is not algorithmically computable as always true in $\mathbb{N}$.   □

Corollary 2.22. *PA is not ω-consistent.*

Proof. Gödel has shown that if PA is consistent, then $[R(n)]$ is PA-provable for any given PA numeral $[n]$[33]. By Corollary 2.20 and the definition of ω-consistency, if PA is consistent then it is *not* ω-consistent.   □

> **Comment**: We prove Corollary 2.22 independently in §9.A.f., Theorem 9.5. We note that this conclusion is contrary to accepted dogma. See, for instance, Davis' remarks in [Da82], p.129(iii) that "...there is no equivocation. Either an adequate arithmetical logic is ω-inconsistent (in which case it is possible to prove false statements within it) or it has an unsolvable decision problem and is subject to the limitations of Gödel's incompleteness theorem".

Corollary 2.23. *The classical standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA does not well-define a model of PA.*

> **Comment**: *Well-define* in the sense of §4.F., Definition 19. We note that 'finitists' of all hues—ranging from Brouwer [Br08], to Wittgenstein [Wi78], to Alexander Yessenin-Volpin [He04]—have persistently questioned the assumption that the classical standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ (see [Me64], p.107) can be treated as *well-defining* a model of PA; see also [Brm07].

Proof. If PA is consistent but not ω-consistent, then Aristotle's particularisation does not hold over $\mathbb{N}$. Since the classical *standard* interpretation of PA appeals to Aristotle's particularisation, the lemma follows.   □

---

[28]Gödel refers to the formula $[R(x)]$ only by its Gödel number r ([Go31], p.25(12)).

[29]Gödel's aim in [Go31] was to show that $[(\forall x)R(x)]$ is not P-provable; by Generalisation it follows, however, that $[R(x)]$ is also not P-provable.

[30]Where $[[R(n)]]$ denotes the Gödel-number of the PA formula $[R(n)]$.

[31]Gödel refers to the formula $[R(x)]$ only by its Gödel number r; [Go31], p.25, eqn.12.

[32][Go31], p.26(2): "$(n)\neg(nB_{\kappa}(17\,Gen\,r))$ holds"

[33][Go31], p.26(2).

2.F.a.   An evidence-based perspective of Lucas' Gödelian argument

The paper [An16] concludes from this that Lucas' Gödelian argument (see §17.) can validly claim (as validated in §17.D., Theorem 17.1) that:

Thesis 2. *There can be no mechanist model of human reasoning if the assignment* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ *can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions, and the assignment* $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ *can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.*

Argument: Gödel has shown how to construct an arithmetical formula with a single variable—say $[R(x)]$[34]—such that $[R(x)]$ is not PA-provable, but $[R(n)]$ is instantiationally PA-provable for any given PA numeral $[n]$. Hence, for any given numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$[35] must hold for some natural number m.

If we assume that any mechanical witness can only reason *finitarily* then although, for any given numeral $[n]$, a mechanical witness can give evidence under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ that the PA formula $[R(n)]$ holds in $\mathbb{N}$, no mechanical witness can conclude *finitarily* under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ that, for any given numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$.

However, if we assume that a human witness can also reason *non-finitarily*, then a human witness *can* conclude under the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ that, for any given numeral $[n]$, the PA formula $[R(n)]$ holds in $\mathbb{N}$. $\quad\square$

The above distinction illuminates the argument in [An13] and [An15], where we suggest how such a perspective offers a resolution to the EPR paradox, by recognising that (see also §19.):

(i) the assignment $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ can be viewed as corresponding to the way human intelligence conceptualises, symbolically represents, and logically reasons about, those sensory perceptions that are triggered by physical processes which are representable—not necessarily finitarily—by algorithmically *verifiable* formulas;

(ii) the assignment $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ can be viewed as corresponding to the way human intelligence conceptualises, symbolically represents, and logically reasons about, only those sensory perceptions that are triggered by physical processes which are representable—finitarily—by algorithmically *computable* formulas;

(iii) there are physical processes of a quantum nature that are representable only by determinate, algorithmically *verifiable*, mathematical functions, but not by any predictable, algorithmically *computable*, mathematical function.

---

[34]Gödel refers to this formula only by its Gödel number r ([Go31], p.25(12)).

[35]Where xBy denotes Gödel's primitive recursive relation 'x is the Gödel-number of a proof sequence in PA whose last term is the PA formula with Gödel-number y' ([Go31], p. 22(45)); and $\lceil[R(n)]\rceil$ denotes the Gödel-number of the PA formula $[R(n)]$.

CHAPTER 3. *INTRODUCTION*

3.   Three fragile Hilbertian, Brouwerian, and Gödelian, dogmas

We note that [An16] highlights the fragility of three Hilbertian, Brouwerian, and Gödelian mathematical and philosophical dogmas by showing that, from an—unarguably constructive— *evidence-based* perspective:

(1) Since PA is not ω-consistent (see §2.F., Corollary 2.22; also §9.A.f., Theorem 9.5), we cannot *unrestrictedly* conclude from the provability of $[(\exists x)F(x)]$[36] that $[F(n)]$ is PA-provable for some *unspecified* numeral $[n]$, since such a putative numeral may not always be *specifiable* by the rules that determine the formation of PA-terms.

   Hence *evidence-based* reasoning does not admit the standard—albeit *faith-based* (see §4.B.)—classical interpretation of quantification that Hilbert sought to formalise in his ε-calculus (see §4.);

(2) The classical first-order logic FOL—in which the Law of the Excluded Middle (LEM) is the theorem $[A \lor \neg A]$—is *finitarily* consistent (an immediate consequence of §2.C.a., Theorem 2.15; see also §5.D., Theorem 5.13).

   Thus *evidence-based* reasoning does not admit Brouwer's belief that LEM—which he apparently conflated with Hilbert's 'principle of excluded middle' (see §4.)—is non-constructive. We note that whereas Hilbert's 'principle of excluded middle' entails LEM, the converse does not hold;

(3) PA is categorical (see §2.E., Corollary 2.18); whence there are no formally undecidable arithmetical propositions in PA.

   Thus *evidence-based* reasoning does not admit Gödel's belief that his formal reasoning in [Go31] entails non-standard models of PA (see also §14.).

   The above three examples highlight the significance of *evidence-based* reasoning for asserting any piece of information as *knowledge* (see also §10.F.).

3.A.   What is knowledge?

The point is currently sought to be codified as FGB by philosopher and neuroscientist Gualtiero Piccinini in his thesis [Pic19], where he argues that any piece of information should be treated as *knowledge*:

- *only* if it is '*factually grounded belief*' (FGB),

- and not if it is *only* a '*justified true belief*' (JTB);

---

[36]We note that $[(\exists x)F(x)]$ is merely an abbreviation for the formula $[\neg(\forall x)\neg F(x)]$, i.e., $[(\exists x)F(x) \equiv \neg(\forall x)\neg F(x).]$

and where we treat his phrase '*a truthmaker*' to correspond to a '*methodological evidence-based assignment of a truth-value by a witness*' in the Tarskian sense (see §2.A.):

> "According to the traditional analysis, to know that p is to have a justified true belief (JTB) that p (Plato, *Meno* 98). This traditional analysis seems to be missing something: beliefs can be both true and justified, yet fail to be knowledge. This is the Gettier problem (Gettier 1963). For example, consider someone who looks at a broken clock that displays 4:39 when, coincidentally, it is 4:39 (Russell 1948). Their belief that the time is 4:39 is true and justified; yet it does not amount to knowledge.
>
> Gettier problem notwithstanding, some maintain that knowledge is justified true belief—or something close (e.g., Sellars 1975, p. 99; Hetherington 1999, 2011, 2016;Weatherson 2003; Turri 2012a, Kern 2017; see also Shope 1983, Turri 2012b, Olsson2015, and Ichikawa and Steup 2017). Even virtue epistemology may be summarized as the claim that, in spite of Gettier, knowledge is virtuously produced true belief (Greco 1993, p. 413). This is not far from the traditional analysis.
>
> I will argue that this allegiance is correct to this extent: the traditional analysis has the right ingredients but misses the right *connection* between them. The traditional analysis and cognate views endure despite the Gettier problem because they are tantalizingly close to the correct account.
>
> In order for belief, justification, and truth to constitute knowledge, they must be mutually connected as follows: justification must tie a belief to the facts. Accordingly, I will argue that knowledge is *factually grounded belief*—belief grounded in the facts. This account explains why Gettierized beliefs do not count as knowledge, illuminates the sensitivity of knowledge attributions to epistemic standards, and suggests an improved reply to the skeptic.
>
> A few caveats before we begin. First, I take "fact" as primitive. Second, I take a truthmaker to be whatever fact makes something true. Third, I assume truthmaker maximalism: every truth has a truthmaker. My proposal does not depend on truthmaker maximalism; I assume it because it makes the exposition easier. Fourth, for present purposes, grounding is an epistemically normative relation between a belief and the facts—grounding in this epistemic sense should not to be confused with grounding as ontological dependence."
>
> *. . . Piccinini:* [Pic19] *(preamble).*

Piccinini apparently seeks to distinguish between:

- providing justification for a piece of informtion that is already accepted as an *infallible* truth[37] (a contentious—albeit inherited, and seemingly '*Platonian*'—attitude in the sense of §10.F.(1), which is only too familiar in day to day human conflict);

and

- providing justification for a piece of information that admits labeling it consequently as a *fallible* truth (the ideal attitude sought in any scientific enquiry in the sense of §10.F.(3)):

> "1. *Knowledge is Factually Grounded Belief*
>
> Gettier "successfully refuted the traditional analysis of knowledge as justified true belief" (Sosa et al. 2009, p. 189). Given that Gettier's refutation is widely acknowledged, why have epistemologists continued to tiptoe around the traditional analysis? It's because the traditional analysis has all the right ingredients. It just doesn't connect them in the right way.
>
> The flaw in the traditional analysis derives from what I call Plato's procedure. Plato starts with true belief and tries to improve it; his proposal is to improve true belief by adding justification

---

[37]See, for instance, Chateaubriand: [Cha05], Chapter 24, *Knowledge and Justification*, pp.398-414.

(Plato, *Meno* 98 and *Theaetetus* 201). Taking true belief as the starting point results both in the traditional analysis and an immediate dilemma: either justification entails truth, or it doesn't.

If justification entails truth, then justified beliefs are infallible. As Julien Dutant (2015) argues, some version of infallibilism about justification was widely held until the 1950s. The problem is that, on the face of it, few if any of our beliefs have any kind of justification that entails their truth. Therefore, infallibilism about justification begets skepticism.

To avoid skepticism, many philosophers argued that at least some beliefs about our mental states are justified in the right way. In an effort to preserve knowledge, they attempted to reduce the external world to (bundles of) mental states. Prominent examples of this program are idealism and phenomenalism. When those efforts collapsed, infallibilism about justification collapsed with them. The historical outcome was precisely the fallibilist view that knowledge is JTB even though justification does not entail truth (Malcolm 1952; Chisholm 1956, 1957; Ayer 1956).

This is the view that Gettier refuted: if justification does not entail truth, then JTB falls short of knowledge. Any attempt to patch up the traditional analysis by adding conditions that do not entail truth is bound to encounter the same problem (Zagzebski 1994). To summarize the dialectic, infallibilism about justification is a dead end that ultimately leads either to skepticism or to fallibilism, and fallibilist versions of JTB face the Gettier problem (though see Dodd 2011, Littlejohn 2012, Schroeder 2015, Booth 2017, and Kern 2017 for recent efforts to revive versions of infallibilism).

The Gettier problem is so intractable that some have despaired of analyzing knowledge at all, or at least of analyzing knowledge as a kind of true belief (Williamson 2000). One radical departure from the JTB tradition is due to John Hyman (1999, 2006, 2015). He argues that knowledge is the ability to be guided by the facts in our actions, thoughts, and feelings. Although Hyman's proposal is unviable, it contains a crucial insight that I will incorporate in my proposal.

Having the ability to be guided by the facts is neither necessary nor sufficient for knowledge. It is unnecessary because sometimes we know things without having the ability to be guided by what we know. For instance, someone might know that fire is hot but lack the ability to be guided by that fact in thinking or acting, perhaps because they are in a minimally conscious state. It is insufficient because there are situations in which we are guided by a fact without knowing that fact. For instance, someone's typing might be guided by the relative positions of the keys on the keyboard absent any knowledge *that* the keys are in certain positions. In other words, someone may be guided by the fact that p simply because they know *how* to respond to p; they may not know that p (cf. Hughes 2014). Another limit of Hyman's account is that it calls for an explanation of how we are guided by the facts. In many cases, the explanation involves our ability to represent knowledge in our minds, which pushes us back towards JTB accounts.

Even though knowledge is not the ability to be guided by the facts, there is something right about Hyman's proposal: knowledge is partially constituted by an agent's relation to the facts. I will graft Hyman's insight onto the traditional analysis of knowledge.

The solution is to abandon Plato's starting point. Instead of starting with true belief and trying to improve on true belief, as Plato and his many followers do, let's start with belief alone. A first improvement to belief is justification: a justified belief is better than a belief without justification. To avoid some of the internalist connotations of the term "justification," I use the term "grounding" instead. Thus, a first improvement over belief alone is that a belief be grounded. A second improvement connects the belief to the facts. As I use the term, a belief is factually grounded if and only if that belief's grounding connects it with its truthmaker and other relevant facts. Please note that although for simplicity I will write that having knowledge that p amounts to having a belief grounded in the fact that p, grounding a belief in its truthmaker may require connecting it to other facts besides its truthmaker. When a belief is thus grounded in the facts, it amounts to knowledge. The result is an account of knowledge as belief grounded in the facts—*factually grounded belief*.

More explicitly:

Factually Grounded Belief (FGB)

An agent A knows that p if and only if:

(1) A believes that p

(2) A's belief that p is grounded

(3) A's belief that p is grounded in the fact that p.

Compare this to the traditional analysis:

JTB

An agent A knows that p if and only if:

(4) p is true

(5) A believes that p

(6) A's belief that p is justified.

(1) is the same as (5) and, setting aside the terminological difference, (2) plays the same role as (6). The main difference between the two accounts is that (3) entails (4) but not vice versa.

Requiring that knowledge be grounded in the facts meets Stephen Hetherington's (2016) challenge of explaining why Getterized beliefs do not count as knowledge even though they are justified, true, and yet possibly false—that is, explaining why Gettierized beliefs are not knowledge without presupposing any implicit infallibilism. The explanation is that Gettierized beliefs are grounded to a degree—a degree that varies from Gettier case to Gettier case—but they are not fully grounded in the facts. That's also why Gettierized beliefs could easily have been false: since they are not fully grounded in the facts, minor departures from the alignment of factors that conspires to make them both justified and true would result in their being false."

. . . Piccinini: [Pic19], §1.

Piccinini's further analysis of '*Factually Grounded Belief*' suggests we can cogently argue that:

- Hilbert's (§3., 1),

- Brouwer's (§3., 2), and

- Gödel's (§3., 3),

mathematical and philosophical dogmas are *fragile* because although, prima facie, they appear acceptable as '*Justified True Beliefs*', they cannot claim further to ensure their categorical communication—as they ought to—in order to be treated as '*Factually Grounded Beliefs*':

"2. *Factually Grounded Belief*

Knowledge is factually grounded *belief*. A belief is an agent's cognitive state that represents things as being one way or another and guides the agent, whether consciously or unconsciously. Cognitive states are part of an agent's cognitive economy—they interact with other cognitive states and sensory inputs to form new cognitive states and guide actions. A belief in this sense may guide the agent's action in some circumstances but not others.

. . .

Knowledge is *factually* grounded belief. "Factually" refers first and foremost to the specific fact a true belief is about—a true belief's truthmaker. It also refers to other facts that must be taken into account to fully ground a belief. Knowledge is factive because the beliefs that partially constitute it represent facts. By the same token, false beliefs cannot be knowledge because they do not represent facts."

. . . Piccinini: [Pic19], §2.

Moreover, the *fragility* of Hibert's and Brouwer's dogmatic beliefs (identified in §8.) illustrates Piccinini's argument that:

- although 'a true belief is a belief with the property of being true' in current paradigms such as those admitting *Justified True Belief* as the norm (e.g., in Chateaubriand [Cha05]);

- such a 'belief' may not necessarily be 'true' to the 'facts'—in which it implicitly claims to be 'grounded'—according to Piccinini's suggested paradigm, where *Factually Grounded Belief* is the norm:

> "A true belief is a belief with the property of being true. Being true is being in accordance with the facts—at least on a correspondence theory of truth. So on a JTB account, the facts are relevant to being true, but the facts themselves may not be constitutive of truth, and therefore of being a true belief. In other words, while requiring that a belief be true does relate a belief to the facts, it need not include the facts as a constituent of a true belief. Being grounded in the facts does include the facts as a constituent.
>
> The fact that makes a belief true is that belief's truthmaker. According to the present proposal, a belief amounts to knowledge just in case it's grounded in its truthmaker. Thus, the FGB account is similar to the truthmaker account of knowledge proposed by Adrian Heathcote in a series of recent papers (2006, 2012, 2014a, 2014b, 2016). Heathcote argues that Gettier's counterexamples can be addressed by distinguishing the truthmaker for the belief from the proposition that justifies the belief, showing that these two come apart in Gettier cases, and then requiring that they not come apart in cases of knowledge. In other words, Heathcote argues that in order to constitute knowledge, its not enough that a belief be true and justified—in addition, the truthmaker for the belief and the "state of affairs from which the evidence for the belief is drawn" (2006, 151) must be identical.
>
> In his official formulation, Heathcote requires that the justification agents have for their belief must be evidence of the fact that makes the belief true. He adds this as a fourth condition to JTB: "the evidence that [the agent] has which constitutes the justification is evidence of the very state of affairs that makes [the belief] true" (2006,p. 165). In Heathcote's usage, evidence *of* a fact should not be confused with evidence *for* a fact. Unlike evidence *for*, evidence *of* is factive (2014a, p. 3); so, having evidence of p entails that p. This makes the requirement that the belief be true redundant. If we eliminate this redundancy, Heathcote's truthmaker account overlaps the JTB account in a way that parallels the FGB account:
>
> > **Truthmaker Account** An agent A knows that p if and only if:
> >
> > (5) A believes that p
> > (6) A's belief that p is justified
> > (7) A's belief that p is justified by evidence of p.
>
> By requiring that the evidence justifying a belief be *of* the belief's truthmaker, Heathcote can elegantly address those Gettier cases in which the truthmaker for the belief is different from the proposition that justifies the belief. His truthmaker account does not accommodate Gettier cases in which a belief is justified by evidence of its truthmaker."
>
> . . . *Piccinini:* [Pic19], §2.

3.B.   A removable ambiguity in Brouwer-Heyting-Kolmogorov realizability

We note [An16] also highlights that the roles of classical and constructive mathematics are interdependent and complementary; as evidenced, for instance (see also §2.D., Poincaré-Hilbert

debate), by the Provability Theorem for PA (see §2.E., Theorem 2.17), which bridges formal arithmetical provability and its interpreted, *evidence-based*, arithmetical truth under the *finitary* interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA.

The far-reaching consequences of such complementarity for mathematics, mathematics education, philosophy and the natural sciences, are appreciated once we identify, and remove, the root of a critical ambiguity in interpreting quantification constructively (see §4.C.)—essentially an ambiguity in Brouwer-Heyting-Kolmogorov realizability—which seems to have, for instance, inhibited the recognition of (see §10.E.):

(a) The role of a first-order set theory such as ZFC in identifying those of our mathematical metaphors (as detailed, for instance, in [LR00]) which can be *defined* unambiguously;

vis à vis:

(b) The role of a first-order Peano Arithmetic such as PA in identifying those of our mathematical metaphors which can, further, be *well-defined* and communicated *categorically*.

We identify the root of the ambiguity as lying in the postulation of an *unspecified* value in classical, *faith-based* (see §4.B.), existential quantification—e.g., by appeal to Hilbert's ε-function (see §7.A.), or to Rosser's Rule C (see §5.G.)—without *evidencing* that such an *unspecified* value is *specifiable*—i.e., *well-defined* (see §4.F.; also Theorem 4.4)—by the rules that circumscribe the domain of the quantifier.

3.B.a. Paradigm challenging consequences

It is an ambiguity which, further, by failing to differentiate that:

(a) algorithmic *verifiability* (§2., Definition 2); and

(b) algorithmic *computability* (§2., Definition 3);

are distinct, but *constructive*, concepts (as is the concept algorithmic '*undecidability*'; see [Fe94], p.6), has far-reaching consequences for varied logical, mathematical, philosophical and scientific paradigms (as detailed in Part IV, §12. to §20.).

Paradigms which, we shall argue, have hitherto tolerated unsustainable, *faith-based*, beliefs whose illusory 'self-evidentiary' appeal (for instance, the 'obviousness' of an isomorphism between the structure of the natural numbers and that of the finite ordinals in Goodstein's *curious* argumentation highlighted in Part IV, §15.) could, reasonably, be viewed as owing more to psychological factors than to mathematical ones—as Andrej Bauer ([Ba16]) suggests from a classically constructive perspective (or to theological ones, as Stanislaw Krajewski argues in [Kr16]).

3.C. Overview

To situate our thesis in an appropriate perspective vis à vis current mathematical beliefs and practices, we shall begin our investigation (in Part II) by a broad analysis of quantification vis à vis:

1. Hilbert's ε-function (§4.);

2. Hilbert's principle of excluded middle (§4.);

3. Aristotle's particularisation (§4., Definition 12);

4. An *evidence-based* perspective of quantification (§2., Definitions 2, 3, and §4.C., Definitions 13 to 17);

5. Wittgenstein's perspective of constructive mathematics (§4.I.);

6. An *evidence-based* definition of effective computability (§4.G.a., Definition 21);

7. The Church-Turing Thesis (§4.G.a., Theorem 4.3, and §7.C., Theorem 7.3);

8. Cantor's diagonal argument (§4.H.a.);

9. An Algorithmic ω-Rule (§5.);

10. Hilbert's ω-Rule (§5.C., Lemma 5.5);

11. Gentzen's Rule of Infinite Induction (§5.B., Thesis 3);

12. Gödel's ω-consistency (§5.D., Corollary 5.10);

13. The Law of the Excluded Middle (§5.D., Corollary 5.15);

14. Markov's Principle (§5.E., Corollary 5.16);

15. The Axiom of Choice (§5.F., Lemma 5.18);

16. Rosser's Rule C (§5.G., Corollary 5.21);

17. Hilbert's purported 'sellout' of finitism (§6.).

By showing how these are formally inter-related, we shall highlight (in §8.A.) the fragility of both the persisting, *theistic*, classical/Platonic interpretation of quantification grounded in Hilbert's ε-calculus; and the persisting, *atheistic* (see §8.B.), constructive/Intuitionistic interpretation of quantification rooted in Brouwer's unjustifiable belief that the Law of the Excluded Middle (LEM) is not finitary.

We shall then consider (in §8.C.) an *agnostic*, *evidence-based* interpretation of quantification that:

(a) admits *evidencing* the satisfaction and truth of the *quantified* formulas of the first-order Peano Arithmetic PA, over the domain $\mathbb{N}$ of the natural numbers, in two, hitherto unsuspected and essentially different, ways: namely, in terms of *weak* algorithmic *verifiabilty* (§2., Definition 2) and *strong* algorithmic *computability* (§2., Definition 3);

(b) yields (see §2.C.a., Theorem 2.16) a finitary proof of consistency for PA, as sought by Hilbert in the second of his twenty three Millenium 1900 Problems (see [Hi00]); an immediate consequence of which is that the classical first-order logic FOL—in which LEM is a theorem—too is finitarily consistent (see §5.D., Theorem 5.13).

In Part III, §9. we shall consider some consequences—for mathematics, mathematics education, philosophy, and the natural sciences—of formalising a Wittgensteinian perspective of constructive mathematics; and in §11.B. consider the questions of whether, to what extent, and how, Gödel could be held guilty of implicit obfuscation in the conclusions he draws from his formal reasoning in his seminal paper [Go31] on formally undecidable arithmetical propositions.

In Part IV, §12. to §20., we shall conclude this investigation by analysing the arguments where *evidence-based* reasoning challenges current paradigms in mathematics, mathematics education, philosophy, and the natural sciences, apropos:

- Cantor's Continuum Hypothesis (§12.);

- Gödel's and Rosser's proofs of 'undecidability' (§13.);

- Non-standard models of PA (§14.);

- Goodstein's argument (§15.);

- The logical and semantic paradoxes (§16.);

    - The mythical 'set-theoretical' limits of fractal constructions (§16.C.);
    - The mythical completability of metric spaces (§16.D.b.);

- Lucas' and Penrose's Gödelian Arguments (§17.);

    - A Definitive Turing Test (§17.E.);
    - The Mind-Body problem (§17.J.a.);

- Prime divisibility and integer factorisation (§18.);

    - Estimating primes in an arithmetical progression (§18.C.);
    - Estimating twin primes (§18.D.);

- The EPR paradox (§19.A.);

    - The Bohr-Einstein debate (§19.B.);
    - Dimensionless constants (§19.D.);
    - Conjugate properties (§19.D.f.);
    - Entangled particles (§19.D.g.);
    - Schrödinger's cat 'paradox' (§19.D.h.).

- The philosophy of mathematics education (§20.);

- The significance of *evidence-based* reasoning and the Church-Turing Thesis for quantum computing (§21.);

- The significance of *evidence-based* reasoning for SETI (§22.).

In the concluding Part V we append standard definitions and *evidence-based* perspectives of some concepts to which this investigation critically appeals.

Part II
SOME MATHEMATICAL CONSEQUENCES OF *EVIDENCE-BASED* REASONING

CHAPTER 4. *MATHEMATICAL CONSEQUENCES*

4. Both Hilbert's ε-calculus and Brouwer's Intuitionism are fragile

We begin our investigation by noting that Hilbert formalised quantification in his ε-calculus as follows:

"**Hilbert's formalisation of quantification in his ε-calculus**:

IV. The logical ε-axiom

  13. $A(a) \rightarrow A(\varepsilon(A))$

Here $\varepsilon(A)$ stands for an object of which the proposition $A(a)$ certainly holds if it holds of any object at all; let us call ε the logical ε-function.

    1. By means of ε, "all" and "there exists" can be defined, namely, as follows:

      (i) $(\forall a)A(a) \leftrightarrow A(\varepsilon(\neg A))$

      (ii) $(\exists a)A(a) \leftrightarrow A(\varepsilon(A)) \dots$

    On the basis of this definition the ε-axiom IV(13) yields the logical relations that hold for the universal and the existential quantifier, such as:

        $(\forall a)A(a) \rightarrow A(b)$ ... (Aristotle's dictum),

and:

        $\neg((\forall a)A(a)) \rightarrow (\exists a)(\neg A(a))$ ... (principle of excluded middle)."

    ... Hilbert: [Hi27].

We further note that (see §7.A.), in any formal first order language whose logic subsumes Hilbert's ε-calculus, Hilbert's 'principle of excluded middle' is an intended formalisation of (and interprets as):

Definition 12. (Aristotle's particularisation) *If the formula $[\neg(\forall x)\neg F(x)]$ of a formal first order language L is defined as 'true' under an interpretation, then we may always conclude unrestrictedly that there must be some unspecified object s in the domain D of the interpretation such that, if the formula $[F(x)]$ interprets as the relation $F^*(x)$ in D, then the proposition $F^*(s)$ is 'true' under the interpretation.*

Comment: The significance of the qualification 'unrestrictedly' is that it admits the possibility where an *unspecified* instantiation may sometimes be *unspecifiable*— in the sense of §7.C., Definition 22—within the parameters of a formal system S that subsumes the *classical* first-order logic FOL (thereby implicitly admitting non-standard models of S) which, under its *classical* interpretation, is uncritically *assumed* to admit Aristotle's particularisation (see §4.B.).

Non-standard model: A non-standard model of a formal system S is a model of S that admits objects in the domain D of the interpretation defining the model if, and only if, there is some object in D which is not definable in S.

Notation: We shall use square brackets to differentiate between a symbolic expression—such as $[F(x)]$—which denotes a formula of a formal language L (treated syntactically as a string of symbols without any associated meaning), and the symbolic expression—denoted by $F^*(x)$—that denotes its meaning under a well-defined interpretation; we find such differentiation useful in order to avoid the possibility of conflation between the two, particularly when (as is not uncommon) the same symbolic expressions are used to denote—or are common to—the two.

We note that (compare with §5.G., Corollary 5.20)

Theorem 4.1. *Hilbert's 'principle of excluded middle' is 'stronger' than Aristotle's particularisation.*

Comment: 'Stronger' in the sense that a formal mathematical theory S is 'stronger' than a theory T if every provable formula of T is provable in S. The two are 'equivalent in strength' if T is also 'stronger' than S.

Proof. If the formula $[\neg(\forall x)\neg F(x)]$ is provable in a formal first order language L that admits Hilbert's ε-calculus then:

- in any *well-defined* interpretation of L over a domain D that admits Aristotle's particularisation,

- there is the possibility that, if $[F(x)]$ interprets as $F^*(x)$,

- there *may* be an *unspecified* instantiation $F^*(s)$ in D

  - where s is an element of D

  - that is *unspecifiable* as an L-term (in a 'broader' sense of §7.C., Definition 22).

However, Hilbert's 'principle of excluded middle' entails that the *unspecified* instantiation in Aristotle's particularisation *must* correspond to an *unspecified*—but *specifiable*—term of L[38]. The theorem follows. □

Comment: The significance of Theorem 4.1 is highlighted from a phenomenological perspective by Stathis Livadas in [Lvd16], where he notes—in the context of Paul Cohen's *forcing* argument in [Co63] (see also §10.E.b. to §10.E.e.)—an implicit circularity in ignoring that 'in particularizing a formal individual from a universal sentence of a general form in order to fulfill another predicative sentence or formula we may be subject to the requirement of confirmation by a continuous connection of actual and possible intuitions':

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[38]As is also postulated by Rosser's Rule C; see §5.G., Corollary 5.21.

"In regarding self-constituting temporality as the ultimate common ground of all phenomenologically motivated analysis of logical-mathematical concepts and meanings (which is my position), the issue of the inverse procession, namely that of passing from a general pure concept to its pure possibilities as its particularizations is also conditioned on the phenomenological notion of time. More specifically the logical requirement of individuality in the sense, for instance, of positing an object-individual as the identical substrate of predicates and logical truths is not just a particularization of the universal concept individual in general but may be bound to the conditions of temporal constitution. This means that in particularizing a formal individual from a universal sentence of a general form in order to fulfill another predicative sentence or formula we may be subject to the requirement of confirmation by a continuous connection of actual and possible intuitions. In turn, the possibility of a continuous connection of actual and possible intuitions is conditioned on the existence of a subjectively generated continuous unity and is associated with a sense of inner temporality, one that is not rooted in the 'external' objective temporality. For instance to check that a subset A of a partially ordered space $(X, \leq)$ is dense in X we must take a random element $x \in X$ and prove the existence of another element y, possibly fulfilling some other property, to satisfy the formula $(\forall x \in X)(\exists y \leq x)[y \in A]$ **(1)**. In case such an element y is a free or bounded variable of a second formula its identification as the particular element that fulfills the definition formula **(1)** of density presupposes the confirmation of all actual and possible intuitions relative to its place in the second formula which is conditioned in turn on the continuous (immanent) unity of all possible connections establishing its prior ontological status (see for some technical details "Appendix 2")."

. . . Livadas: [Lvd16], §4 The Question of Universal-Existential Quantification . . . , para (d), pp.20-21.

The implicit circularity is evident when interpreting 'a second-order universal quantification over all subsets of the power-set of the set of natural numbers $\mathcal{P}(N)$' since, as Livadas notes further, 'any universal quantification over such sets, regardless of any temporal or constitutional concerns for this quantification, already establishes their de facto acceptance as completed totalities':

"I complete this section with a review of the role of universal-existential quantification over an indefinite horizon in formulas with ontological claims, in particular, in the proof-theoretic process of generation of Gödel's incompleteness results. In fact, universal quantification over an indefinite horizon plays a major part in the proof of almost all significant infinity results in foundational mathematics, e.g., in certain well-known independence results as it is the *Continuum Hypothesis* (**CH**) and its generalized form **GCH**. In this case one has to go a step further and apply a second-order universal quantification over all subsets of the power-set of the set of natural numbers $\mathcal{P}(N)$, a process considered as as presupposing a concept of completed totality for the uncountably infinite set $\mathcal{P}(N)$ and therefore as losing contact with 'real-world' intuition.[6] Any statement (or relation) expressed by applying universal quantification over sets such as $\mathcal{P}(N)$ or even $\mathcal{P}(\mathcal{P}(N))$ is normally taken as a definite one with legitimate ontological claims which is evidently a circular interplay since any universal quantification over such sets, regardless of any temporal or constitutional concerns for this quantification, already establishes their de facto acceptance as completed totalities.[7]

Consequently any universal-existential quantification over an indefinite horizon, and *a fortiori* a second-order one clearly presupposes a notion of complete totality for the intended scope of its quantifiers which, in view of the previous discussion at the level of constitutional-temporal processes, reduces to the constitution of infinite sets of any order in the form of the continuous unity of completed wholes in presentational immediacy. In turn, this kind of actual infinity far from being a spatio-temporal and causality-generated one, insofar as it is immanent to the self-constituting temporal consciousness, conditions in one way or another not only the already established key foundational results of K. Gödel and P. Cohen but also more recent attempts to achieve enlargements of inner models so as to be consistent with all known large cardinal

axioms."

. . . Livadas: [Lvd16], §4 The Question of Universal-Existential Quantification . . . , pp.21-22.

From the *evidence-based* perspective of this investigation we note that we can interpret Livadas' stipulation that:

> "in particularizing a formal individual from a universal sentence of a general form in order to fulfill another predicative sentence or formula we may be subject to the requirement of confirmation by a continuous connection of actual and possible intuitions"

as requiring that even any intuitive ontological commitment, consequent to the 'particularizing of a formal individual from a universal sentence', must, even if only in principle, necessitate:

— algorithmic *verifiability* of 'actual intuitions' in some sense of §2., Definition 2, and §4.C., Definition 13;

and:

— algorithmic *computability* of 'possible intuitions' in some sense of §2., Definition 3 and §4.C., Definition 16;

in order to claim—'in a next 'second-order' level', as clarified by Livadas[39]—'a continuous connection of actual and possible intuitions'.

We shall further see that, from an *evidence-based* perspective of the Complementarity Thesis 1 (see §1.), both Hilbert's ε-calculus and Brouwer's Intuitionism are fragile since:

(a) On the one hand—as L. E. J. Brouwer had protested (see §7.B.)—Hilbert's intended interpretation of his ε-calculus does not admit (see §5.D., Theorem 5.12), or even seek to admit, a finitary interpretation;

(b) On the other hand, whereas Aristotle's particularisation entails the classical Law of the Excluded Middle LEM (see §7.A., Lemmas 7.1 and 7.2), intuitionistic perspectives—following Brouwer—unjustifiably (see §5.D., Theorem 5.12 and §5.D., Corollary 5.15) believe that the converse must also be true, thereby failing to recognise LEM as finitary.

Classical Law of the Excluded Middle LEM (cf., *[Me64], p.4*): For any well-formed formula [P] of a formal system S that subsumes the first-order logic FOL, [P v ¬P] is a theorem of S.

4.A. Gödel's ω-consistency too is fragile

We note further that, in order to avoid intuitionistic objections to his reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions, Kurt Gödel did not assume that, for instance, the classical 'standard' interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA[40] must be semantically well-definable—essentially since unbounded quantification in arithmetic may not be definable recursively[41], as noted by Gödel when defining his number-theoretical relation '$Bew(x)$', which asserts that x is the Gödel-number of a provable formula of Gödel's formal system P in [Go31]:

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[39] In a private correspondence on 14th April 2020.

[40] As defined in [Me64], p.107; see also §23.(16)/(17).

[41] A consequence of a, subsequent, seminal theorem by Alfred Tarski in [Ta35]:

Tarski's Theorem: The set Tr of Gödel numbers of wfs of S which are true in the standard model is not arithmetical, i.e., there is no wf $A(x)$ of S such that Tr is the set of numbers k for which $A(\overline{k})$ is true in the standard model. . . . *Mendelson: [Me64], p.151, Corollary 3.38.*

"46. $Bew(x) \equiv (Ey)\ y\ B\ x$

x is a PROVABLE FORMULA. [$Bew(x)$ is the only one of the concepts 1-46 which cannot be asserted to be recursive.]"

. . . *Gödel: [Go31], p.22.*

Instead, Gödel introduced the syntactic property of ω-consistency as an explicit assumption in his formal reasoning ([Go31], p.23 and p.28).

ω-**consistency**: A formal system S is ω-consistent if, and only if, there is no S-formula [$F(x)$] for which, first, [$\neg(\forall x)F(x)$] is S-provable and, second, [$F(a)$] is S-provable for any specified S-term [a].

Gödel explained that his reason for introducing ω-consistency as an explicit assumption in his intended, strictly syntactical, reasoning was to avoid appealing to the semantic concept of classical arithmetical truth[42]—a concept based on an intuitionistically objectionable logic which implicitly[43] assumes that Aristotle's particularisation (see §4., Definition 12) holds over the domain $\mathbb{N}$[44] of the natural numbers.

"The method of proof which has just been explained can obviously be applied to every formal system which, first, possesses sufficient means of expression when interpreted according to its meaning to define the concepts (especially the concept "provable formula") occurring in the above argument; and, secondly, in which every provable formula is true. In the precise execution of the above proof, which now follows, we shall have the task (among others) of replacing the second of the assumptions just mentioned by a purely formal and much weaker assumption."

. . . *Gödel: [Go31], p.9.*

We shall see (§5.D., Corollary 5.10) that Gödel's assumption is 'weaker' in the sense that:

- *If* Tarski's inductive definitions (see §2.A.) of the satisfaction and truth of existentially quantified PA formulas under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ *assume* that Aristotle's particularisation is valid over $\mathbb{N}$,

  Comment: '*Assume*', since the following is not *unrestrictedly* entailed by Tarski's inductive definitions:

  "A sequence s satisfies $(Ex_i)A$ if and only if there is a sequence s′ which differs from s in at most the ith place such that s′ satisfies A."

  . . . *Mendelson: [Me64], p.52, V(ii).*

- *Then* PA is consistent if, and only if, it is ω-consistent.

  Comment: It is also 'weaker' in the more formal sense that a formal mathematical theory S is 'weaker' than a theory T if every provable formula of S is provable in T. The two are 'equivalent in strength' if T is also 'weaker' than S.

Now, if there were a finitary proof that PA is ω-consistent, it would follow that Gödel's formula [$(\forall x)R(x)$] (see §2.F., Lemma 2.19) would be formally undecidable in PA by [Go31], Theorem VI, p.24.

However, since [$\neg(\forall x)R(x)$] is PA-provable (§2.F., Corollary 2.20), we conclude that Gödel's assumption of ω-consistency for arithmetic is fragile (§2.F., Corollary 2.22; see also §9.A.f., Theorem 9.5).

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[42]To which Gödel implicitly appeals in his semantic—set-theoretical and debatably (see §11.B.) constructive—proof of the existence of a formally undecidable proposition in Russel and Whitehead's Principia Mathematica (PM); which he defines therein as [$R(q); q$] ([Go31], pp.7-8).

[43]And invalidly, by §5.D., Theorem 5.12.

[44]We take $\mathbb{N}$ to be the classical mathematical structure of the natural numbers that is uniquely defined by Dedekind's second order formalisation of the Peano Postulates.

4.B. *Faith-based* quantification

We note that, unusually for a mathematical assertion, *Aristotle's particularisation* (§4., Definition 12) is not offered in classical mathematics—which subsumes the first-order logic FOL—as an axiom or a thesis, but as a belief that is not unequivocally self-evident, and must be appealed to as an article of unquestioning faith.

A faith which explicitly avoids, yet implicitly follows in essence (see §7.), Hilbert's formalisation of quantification in terms of his ε-operator in [Hi25].

A faith that has subsequently been sanctified by prevailing custom in published classical, and constructivist/intuitionistic, literature and textbooks[45] at such an early stage of any mathematical curriculum, and planted so deeply into students' minds[46], that thereafter most cannot even detect its presence—let alone need for justifying quantification—in a definition or a proof sequence[47]!

A faith, moreover, whose absurdity is illustrated starkly (see [BP14]; [Kr19]) not only in the admittance of a mathematical definition of the most unarguable concept of 'God'—as an 'existence' of which nothing can be greater—but in purported mathematical proofs of the existence of such a Deity!

4.C. *Evidence-based* quantification

The significance of Aristotle's particularisation for Gödel's assumption of ω-consistency—and, as we shall see, for Hilbert's Program (see [Zac07]) in particular and constructive mathematics (see [Kl52], [Ba16]) in general—is highlighted when contrasted (see §2.) with a constructive perspective of quantification that admits *evidencing* the satisfaction and truth of the *quantified* formulas of the first-order Peano Arithmetic PA, over the domain $\mathbb{N}$ of the natural numbers, in two, hitherto unsuspected and essentially different, ways:

(1) in terms of *weak* algorithmic *verifiabilty* (see §2., Definition 2); and

(2) in terms of *strong* algorithmic *computability* (see §2., Definition 3).

The distinction between (1) and (2) illuminates the Brouwerian perspective of quantification that, we argue in §8.B., could be appropriately labeled as 'atheistic' vis à vis the classical, Hilbertian, 'theistic' (see §8.A.) perspective of quantification.

In *evidence-based* reasoning, for instance, if the PA formula $[(\forall x)F(x)]$[48] is intended (see also §7.C.) to be read *weakly* as *'For any specified x, $F^*(x)$ is decidable as true'* under an

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[45] See, for instance: Whitehead/Russell: [WR10], p.20; Hilbert: [Hi25], p.382; Hilbert/Ackermann [HA28], p.48; Skolem: [Sk28], p.515; Gödel: [Go31], p.32; Carnap: [Ca37], p.20; Kleene: [Kl52], p.169; Rosser: [Ro53], p.90; Bernays/Fraenkel: [BF58], p.46; Beth: [Be59], pp.178 & 218; Suppes: [Su60], p.3; Luschei: [Lus62], p.114; Wang: [Wa63], p.314-315; Quine: [Qu63], pp.12-13; Kneebone: [Kn63], p.60; Mendelson: [Me64], pp.4-5, V(ii) (op. cit.); Novikov: [Nv64], p.92; Lightstone: [Li64], p.33; Cohen: [Co66], p.4; Shoenfield: [Sh67], p.13; Davis: [Da82], p.xxv; Rogers: [Rg87], p.xvii; Epstein/Carnielli: [EC89], p.174; Murthy: [Mu91]; Smullyan: [Sm92], p.18, Ex.3; Awodey/Reck: [AR02b], p.94, Appendix, Rule 5(i); Boolos/Burgess/Jeffrey: [BBJ03], p.102; Crossley: [Cr05], p.6; Srivastava, [Sri08], p.3 & p.18; Voevodsky, [Vo10]; Smith, [Smi13], p.39(4); Cho, [Cho18].

[46]Friend and Molinini anecdotally highlight the pernicious influence of such faith-based reasoning on the perspective—and understanding—of even seasoned scholars (see [FM15], pp.201-202). The anecdote about Professor Manohar S. Huzurbazar offers, however, a contrasting example in §25..

[47]See, for instance, Mendelson: [Me64], p.52, V(ii) (op. cit.); [Cho18]; [Dvd19]!

[48]For ease of exposition we consider, without loss of generality, only the case of a PA-formula with a single variable.

interpretation, where the formula $[F(x)]$ interprets as the arithmetical relation $F^*(x)$, then it must be consistently interpreted *weakly* in terms of algorithmic *verifiability* as follows:

Definition 13. (Weak quantification) *A PA formula $[(\forall x)F(x)]$ is algorithmically verifiable as true under an interpretation if, and only if, $F^*(x)$ is algorithmically verifiable as always true.*

> **Definition 14. (Deterministic algorithm)** *A deterministic algorithm is a well-defined mechanical method, such as a Turing machine, that computes a mathematical function which has a unique value for any input in its domain, and the algorithm is a process that produces this particular value as output.*
>
> **Comment**: By 'deterministic algorithm' we mean a '*realizer*' in the sense of the *Brouwer-Heyting-Kolmogorov* rules (see [Kl52], p.503-505).

Moreover, the PA formula $[\neg(\forall x)F(x)]$, if intended to be read *weakly* as *'It is not the case that, for any specified x, $F^*(x)$ is true'* must be consistently interpreted *weakly* in terms of algorithmic *verifiability* as:

Definition 15. (Weak negation) *The PA formula $[\neg(\forall x)F(x)]$ is algorithmically verifiable as true under an interpretation if, and only if, there is no algorithm which will evidence that $[(\forall x)F(x)]$ is algorithmically verifiable as always true under the interpretation.*

> **Comment**: We note that *weak negation* implies that $[(\forall x)F(x)]$ is not provable in PA; it *does not*, however, entail that $F^*(x)$ is not algorithmically *verifiable* as always true.

Similarly, if $[(\forall x)F(x)]$ is intended to be read *strongly* as *'For all x, $F^*(x)$ is decidable as true'*, then it must be consistently interpreted *strongly* in terms of algorithmic *computability* as follows:

Definition 16. (Strong quantification) *A PA formula $[(\forall x)F(x)]$ is algorithmically computable as true under an interpretation if, and only if, $F^*(x)$ is algorithmically computable as always true.*

Whilst the PA formula $[\neg(\forall x)F(x)]$, if intended to be read *strongly* as *'It is not the case that, for all x, $F^*(x)$ is true'* must be consistently interpreted *strongly* in terms of algorithmic *computability* as:

Definition 17. (Strong negation) *The PA formula $[\neg(\forall x)F(x)]$ is algorithmically computable as true under an interpretation if, and only if, there is no algorithm which will evidence that $[(\forall x)F(x)]$ is algorithmically computable as always true under the interpretation.*

> **Comment**: We note that *strong negation*, too, implies that $[(\forall x)F(x)]$ is not provable in PA. By §2.E., Theorem 2.17 (*Provability Theorem for PA*), it *does*, however, entail that $F^*(x)$ is not algorithmically *computable* as always true.

We note that *strong* algorithmic *computability* implies the existence of an algorithm that can *finitarily* decide the truth/falsity of each proposition in a *well-defined* denumerable sequence of number-theoretical propositions, whereas *weak* algorithmic *verifiability* does not imply the existence of an algorithm that can *finitarily* decide the truth/falsity of each proposition in a *well-defined* denumerable sequence of number-theoretical propositions.

> **Comment**: Classically, the concepts 'well-defined' and 'effectively computable' are treated as intuitive, and not expressible formally in a manner subject to verification by a Turing machine. However we show, in §4.F. and §4.G.a., that both concepts can be defined in terms of *weak* algorithmic *verifiability*, even when they are not subject to verification by a Turing machine in terms of *strong* algorithmic *computability*.

The distinction between the concepts of *weak* 'algorithmic verifiability' and *strong* 'algorithmic computability' seeks to eliminate an implicit ambiguity in the classical concept of 'realizability' (see §4.E.; also [Ba16], p.5; [Kl52], p.503-505).

> **Comment**: Although the conventional *set-theoretical* terminology of constructive mathematics—as detailed in [Ba16]—may prefer that §2., Definitions 2, 3, and §4.C., Definitions 13 to 17, be expressed in terms, for instance, of '*verifiable* realizability' and '*computable* realizability' instead of 'algorithmic *verifiability*' and 'algorithmic *computability*', we have preferred the latter terminology as more illuminating from the perspective of this introductory investigation into the philosophical and mathematical significance of post-computational, *evidence-based* and purely *arithmetical*, reasoning (see §8.C.).

We shall see that the distinction was also anticipated by both Brouwer (see §4.K.) and Hilbert (see §5.B.).

Moreover, the significance of the Complementarity Thesis (§1., Thesis 1) is highlighted if we note that the distinction between 'any' and 'all' sought to be made in §4.C., Definitions 13 to 17, pertains to the assignment of truth-values to the formulas of a formal theory under an interpretation of the universal quantifier over a *well-defined* domain of interpretation.

It is to be distinguished from the distinction Bertrand Russell sought to make in [Rus08] (pp.156-163; see also [Fe02], pp.3-4) between 'all' and 'any' with respect to the assignment of provability-values to the formulas of a formal theory that admits 'apparent' and 'real' variables in a proof sequence.

4.D. Quantification in constructive mathematics is ambiguous

The distinction between the concepts of *weak* 'algorithmic verifiability' and *strong* 'algorithmic computability' also illuminates the illusory barriers faced by formal theories of constructive mathematics which seek to interpret formal existential quantification constructively in terms of only algorithmic *computability* or Kleene's *recursive realizability* (see §4.E.a.).

For instance, as observed by E. B. Davies in his defense of pluralism in mathematics, "and in particular Errett Bishop's constructive approach to mathematics":

> "**1. Introduction**
>
> Errett Bishop's book 'Foundations of Constructive Analysis' appeared in 1967 and started a new era in the development of constructive mathematics. His account of the subject was entirely different from, and far more systematic than, Brouwer's programm of intuitionistic mathematics.
>
> . . .
>
> Briefly, I defend what I call pluralism in mathematics—the view that classical mathematics, constructive mathematics, computer assisted mathematics and various forms of finitistic mathematics can coexist. I revive Carnap's dictum that one must decide the framework of discourse *before* questions about existence and truth make sense; see Carnap (1950). In different frameworks the answer to a question may be different, but this in no way implies that one or the other is 'right'. This position is anti-Platonistic."
>
> . . . *Davies*: [Dav05].

From the *evidence-based* perspective of quantification in §4.C., the Complementarity Thesis 1 (in §1.) can be viewed as *the* 'pluralism' and 'constructivism' that Davies seeks, but apparently views as in need of defense since it seemingly limits the classical theorems that can be accepted as constructive.

In other words, Davies seems to be of the view that although every theorem in Bishop's constructive mathematics—in which the existential quantifier $\exists$ is interpreted in terms of algorithmic *computability* only—is a theorem in classical mathematics, not every theorem of classical mathematics can be interpreted constructively under Aristotle's particularisation (see Definition 12):

> "**2. What is constructive mathematics?**
>
> It has often been said that Bishop rejected the law of the excluded middle, but a more useful description of the situation is that he gave the symbol $\exists$ a different meaning from the usual one. In classical mathematics $\exists$ refers to Platonic existence, but Bishop used it to refer to the production of an algorithm for constructing the relevant quantity. In classical mathematics $\exists$ may be defined in terms of $\forall$: the expression $\exists x A$ is logically equivalent to $\neg(\forall a \neg A)$ [sic]. In constructive mathematics, $\exists$ is a new quantifier with stricter conditions for its application. All of the differences between classical and constructive mathematics follow from the new meaning assigned to the new symbol. We wish to emphasize that every theorem in Bishop's constructive mathematics is also a theorem in classical mathematics. Constructive mathematicians have to work harder to prove theorems, because their criteria for existence are stricter; the pay-off is that the statements of the theorems contain more information."
>
> . . . *Davies:* [Dav05].

Moreover, Davies ascribes more to Bishop's *interpretation* of the existential quantifier $\exists$ in terms of algorithmic *computability* (or of Kleene's *recursive realizability*) as the characteristic feature of Bishop's constructivism, rather than to the latter's rejection of the classical Law of the Excluded Middle (LEM) á la Brouwer (see §8.B.a.).

He thus implicitly accepts Bishop's constructive analysis as a formal mathematical system that has a classical interpretation under which $\exists$ entails Aristotle's particularisation—a concept that is *not* 'intelligible' to a machine—and a constructive interpretation under which $\exists$ entails algorithmic *computability*—a concept that *is* 'intelligible' to a machine:

> "The use of a formal mathematical system as a programming language presupposes that the system has a constructive interpretation. Since most formal systems have a classical, or nonconstructive, basis (in particular, they contain the law of the excluded middle), they cannot be used as programming languages.
>
> The role of formalisation in constructive mathematics is completely distinct from its role in classical mathematics. Unwilling—indeed unable, because of his education—to let mathematics generate its own meaning, the classical mathematician looks to formalism, with its emphasis on consistency (either relative, empirical, or absolute), rather than meaning, for philosophical relief. For the constructivist, formalism is not a philosophical out; rather it has a deeper significance, peculiar to the constructivist point of view. Informal constructive mathematics is concerned with the communication of algorithms, with enough precision to be intelligible to the mathematical community at large. Formal constructive mathematics is concerned with the communication of algorithms with enough precision to be intelligible to machines."
>
> . . . *Bishop:* [Bi18], pp.1-2.

From the *evidence-based* perspective of quantification in §4.C., however, Bishop's above notion of 'formal constructive mathematics' implicitly admits an ambiguity that could be viewed as *the* barrier which constrains recognising:

- that PA is a categorical language (see Corollary 2.18)

- which can communicate 'algorithms with enough precision to be intelligible to machines'.

Such a perspective would reflect that:

- a formal system can admit a constructive, *evidence-based*, algorithmically *verifiable* interpretation of PA (see §2.B.a., Theorem 2.7),

    - that admits a 'truth' assignment which is *not* 'intelligible' to a machine (see §17.E., Query 17),
    - where $\exists x A$ (treated as an abbreviation for $\neg(\forall x \neg A)$) entails only a *weak* algorithmic *verifiability* (by Definition 15),
    - which, in turn, entails *strong* algorithmic *computability* (by Definition 17);

- and, moreover, by:

    - the *finitary* proof of consistency for PA (§2.C.a., Theorem 2.16),
    - the Provability Theorem for PA (§2.E., Theorem 2.17),
    - and its immediate corollary (Corollary 2.18),

- the system:

    - *can* admit the classical Law of the Excluded Middle (see §5.D., Corollary, 5.14),
    - without entailing Aristotle's particularisation (see §5.D., Corollary 5.15),
        * which entails ω-consistency if the system is consistent (see §5.D., Corollary 5.10); and
        * which is entailed by Rosser's Rule C (see §5.G., Lemma 5.20).

4.E.   Where realizability in constructive mathematics is ambiguous

We briefly outline the—albeit explicitly unrecognised—ambiguity as highlighted further, for instance, by Bauer's argumentation in [Ba16], where he characterises classical mathematics, vis à vis current paradigms of constructive mathematics, as a compromise on the intuitive notion of 'truth':

> "Classical mathematical training plants excluded middle so deeply into young students' minds that most mathematicians cannot even detect its presence in a proof. In order to gain some sort of understanding of the constructivist position, we should therefore provide a method for suspending belief in excluded middle.
>
> If a geometer tried to disbelieve Euclid's fifth postulate, they would find helpful a model of non-Euclidean geometry—an artificial world of geometry whose altered meanings of the words 'line' and 'point' caused the parallel postulate to fail.
>
> Our situation is comparable, only more fundamental because we need to twist the meaning of 'truth' itself. We cannot afford a full mathematical account of constructive worlds, but we still can distill their essence, as long as we remember that important technicalities have been omitted."
>
> . . . Bauer: [Ba16], p.6.

He then claims that:

> "It is well worth pointing out that constructive mathematics is a generalization of classical mathematics, as was emphasized by Fred Richman, for a proof which avoids excluded middle and choice is still a classical proof. However, trying to learn constructive thinking in the classical world is like trying to learn noncommutative algebra by studying abelian groups."
>
> . . . Bauer: [Ba16], p.6.

Bauer expands on the need of constructive mathematics to 'twist' the meaning of 'truth' as necessitated by the differing modes of truth-assignments required by the gamut of differing constructive worlds which—Bauer ruefully notes—a constructive mathematics that claims to generalise classical mathematics is compelled to accommodate.

He then addresses two such assignments, the first of which appeals to the *computable* properties of realisability.

> "In our first honestly constructive world only that is true which can be computed. Let us imagine, as programmers do, that mathematical objects are represented on a computer as data, and that functions are programs operating on data. Furthermore, a logical statement is only considered valid when there is a program witnessing its truth. We call such programs realizers, and we say that statements are realized by them. The *Brouwer-Heyting-Kolmogorov* rules explain when a program realizes a statement:
>
> (1) falsehood $\bot$ is not realized by anything;
>
> (2) truth $\top$ is realized by a chosen constant, say $\star$;
>
> (3) $P \vee Q$ is realized by a pair (p, q) such that p is a realizer of P and q of Q;
>
> (4) $P \wedge Q$ is realized either by $(0, p)$, where p realizes P, or by $(1, q)$, where q realizes Q;
>
> (5) $P \Rightarrow Q$ is realized by a program which maps realizers of P to realizers of Q;
>
> (6) $\forall x \in A.P(x)$ is realized by a program which maps (a representation of) any $a \in A$ to a realizer of $P(a)$;
>
> (7) $\exists x \in A.P(x)$ is realized by a pair (p, q) such that p represents some $a \in A$ and q realizes $P(a)$;
>
> (8) $a = b$ is realized by a p which represents both a and b.
>
> The rules work for any reasonable notion of 'program'. Turing machines would do, but so would quantum computers and programs actually written by programmers in practice."
>
> . . . Bauer: [Ba16], pp.6-7.

As examples of the use of realizers, Bauer first offers an example of a universally quantified '*for all*' (cf. §7.C.(b)) proposition that, by Definitions 3, is algorithmically *computable* under interpretation:

> "*For every natural number there is a prime larger than it.*
>
> This is a 'for all' statement, so its realizer is a program p which takes as input a natural number n and outputs a realizer for 'there is a prime larger than n', which is a pair (m, q) where m is again a number and q realizes 'm is prime and $m > n$'. If we forget about q, we see that p is essentially a program that computes arbitrarily large primes. Because such a program exists, there are arbitrarily large primes in the computable world."
>
> . . . Bauer: [Ba16], p.7.

He then proffers, as a more interesting example, a universally quantified '*for any*' (cf. §7.C.(a)) proposition that, by Definitions 2, is algorithmically *verifiable* but not algorithmically *computable* (see §4.G., Theorem 4.2) under interpretation:

"(1) $\forall x \in R. x = 0 \lor x \neq 0$.

If we define real numbers as the Cauchy completion of rational numbers, then a real number $x \in R$ is represented by a program p which takes as input $k \in N$ and outputs a rational number r_k such that $|x - r_k| \leq 2^{-k}$. Thus a realizer for (1) is a program q which accepts a representation p for any $x \in \mathbb{R}$ and outputs either $(0, s)$ where s realizes $x = 0$, or $(1, t)$ where t realizes $x \neq 0$. Intuitively speaking, such a q should not exist, for however good an approximation r_k of x the program q calculates, it may never be sure whether $x = 0$. To make a water-tight argument, we shall use q to construct the Halting oracle, which does not exist. (The usual proof of nonexistence of the Halting oracle is yet another example of a constructive proof of negation.) Given a Turing machine T and an input n, define the sequence $r_0, r_1, r_2, \ldots$ of rational numbers by

- $r_k = 2^{-j}$ if $T(n)$ halts at step j and $j \leq k$,

- $r_k = 2^{-k}$ otherwise.

This is a Cauchy sequence because $\mid r_k - r_m \mid \leq 2^{-min(k,m)}$ for all $k, m \in N$, and it is computable because the value of r_k may be calculated by a simulation of at most k steps of execution of $T(n)$. The limit $x = \lim_k r_k$ satisfies

- $x = 2^{-j} > 0$, if $T(n)$ halts at step j,
- $x = 0$, if $T(n)$ never halts.

The program p which outputs r_k on input k represents x because $|x - r_k| \leq 2^{-k}$ for all $k \in N$. We may now decide whether $T(n)$ halts by running $q(p)$: if it outputs $(0, s)$, then $T(n)$ does not halt, and if it outputs $(1, t)$, then $T(n)$ halts."

... Bauer: [Ba16], pp.7-8.

Bauer notes that although the above argument needs:

"...the following (valid) instance of excluded middle: for every $k \in \mathbb{N}$, either $r_k = 2^{-k}$ or $r_k = 2^{-j}$ for some $j < k$"

... Bauer: [Ba16], p.8.

the statement (1) is an instance of excluded middle which is not realized.

He concludes with an anti-mechanist thesis that echoes—albeit for debatable reasons—the concluding thesis of [An16] (Lucas' Gödelian Thesis, see §2.F., Thesis 2):

"The strategy to place constructivism inside a box is working! If one takes the limited view that everything must be computed by machines, then excluded middle fails because machines cannot compute everything. Our excluded middle is not affected because we are not machines."

... Bauer: [Ba16], p.8.

Bauer uses the computable world to further explain why the following instance of 'subsets of finite sets are finite' is not realized:

"(2) *All countable subsets of* $0, 1$ *are finite.*

In computable mathematics a finite set is represented by a finite list of its elements, and a countable set by a program which enumerates its elements, possibly with repetitions. The subsets $\{\}$, $\{0\}$, $\{1\}$ and $\{0, 1\}$ are all countable and finite, so (2) looks pretty true. Remember though that in the computable world 'for all' means not 'it holds for every instance' but rather 'there is a program computing witnesses from instances'. A realizer for (2) is a program q which takes as input a program p enumerating the elements of a subset of $\{0, 1\}$ and outputs a finite list of all the elements so enumerated."

... Bauer: [Ba16], p.8.

Bauer argues that:

"To see intuitively where the trouble lies, suppose p starts enumerating zeroes:

$0, 0, 0, 0, 0, 0, \ldots$

The output list should contain 0, but should it contain 1? However long a prefix of the enumeration we investigate, if it is all zeroes, then we cannot be sure whether 1 will appear later. For an actual proof we use the same trick as before: with q in hand we could construct the Halting oracle. Given any Turing machine T and input n, consider the program p which works as follows:

- $p(k) = 1$ if $T(n)$ halts in fewer than k steps,

- $p(k) = 0$ otherwise.

The subset $S \subseteq \{0, 1\}$ enumerated by p is constructed so that

- $1 \in S$ if $T(n)$ halts,

- $1 \notin S$ if $T(n)$ does not halt.

Now scan the finite list computed by $q(p)$: if it contains 1, then $T(n)$ holds, otherwise it does not."
. . . Bauer: [Ba16], p.8.

From the *evidence-based* perspective of §4.C., however, we can '*see intuitively where the trouble lies*' in the above examples if we recognise that:

- the Law of the Excluded Middle (LEM) is *finitarily* valid (an immediate consequence of §2.C.a., Theorem 2.16); and,

- by unnecessarily disallowing appeal to LEM, the *Brouwer-Heyting-Kolmogorov* rules

    - are unable to accommodate
        * algorithmic *verifiability* (i.e., evidencing each *specified* instance by *some* algorithm), and
        * algorithmic *computability* (i.e., evidencing all *intended* instances by a *single* algorithm),
    - as distinct, but constructive, concepts.

The *Brouwer-Heyting-Kolmogorov* rules are, thus, unable to offer a resolution of the query raised in Bauer's second, *more interesting*, example, which is essentially whether:

Halting problem for T (§17.F.a., Query 21): Given a Turing machine T, can one effectively decide, given any instantaneous description alpha, whether or not there is a computation of T beginning with *alpha*?

That the underlying issue is not restricted to Bauer's specific example is seen if we note that, from an *evidence-based* perspective:

- the Halting problem for T (§17.F.a., Query 21)

- is not only answerable in the affirmative (see §17.F.b., Corollary 17.4),

- but is conflated (see §17.F.a.), in current paradigms of both constructive and classical mathematics,

- with Turing's Halting problem (§17.F., Query 20),

- which admits a negative answer (see §17.F.b., Corollary 17.5).

  Turing's Halting problem (§17.F., Query 20) Is it always decidable by a Turing machine whether or not a partial recursive function is total?

4.E.a.   Recursive realizability and the Law of the Excluded Middle

The inability of *Brouwer-Heyting-Kolmogorov* rules to accommodate:

- algorithmic *verifiability* (i.e., evidencing each *specified* instance by *some* algorithm), and

- algorithmic *computability* (i.e., evidencing all *intended* instances by a *single* algorithm),

as distinct, but constructive, concepts reflects, prima facie, the perception that treating 'realizability' as only 'recursive realizability' should suffice in intuitionistic argumentation.

For instance, as detailed by Stephen Cole Kleene:

> "Realizability is intended as an intuitionistic interpretation of a formula; and to say intuitionistically that $\mathbf{A}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y})$ is realizable should imply its being intuitionistically true, i.e. that the proposition $A(x_1, \ldots, x_n, y)$ constituting its intuitionistic meaning holds. The formula $\exists y A(x_1, \ldots, x_n, y)$ asserts the existence, for every $x_1, \ldots, x_n$, of a y depending on $x_1, \ldots, x_n$, such that $A(x_1, \ldots, x_n, y)$; or in other words, the existence of a function $y = \phi(x_1, \ldots, x_n)$ such that, for every $x_1, \ldots, x_n, A(x_1, \ldots, x_n, \phi(x_1, \ldots, x_n))$.
>
> . . .
>
> The formula $\exists y A(x_1, \ldots, x_n, y)$ does not assert the uniqueness of the function $y = \phi(x_1, \ldots, x_n)$ such that $A(x_1, \ldots, x_n, \phi(x_1, \ldots, x_n))$; for this we need $\exists! y A(x_1, \ldots, x_n, y)$ (§41).
>
> Classically, given the existence of some function ϕ such that, for all $x_1, \ldots, x_n, A(x_1, \ldots, x_n, \phi(x_1, \ldots, x_n))$, the least number principle provides formally a method of describing a particular one (*149 § 40, *174b § 41). While we do not have the least number principle intuitionistically, we do know by Corollary 2 that, whenever a particular intuitionistic proof of a formula of the form $\exists y A(x_1, \ldots, x_n, y)$ is given, we can on the basis of that proof describe informally a particular general recursive function $\phi(x_1, \ldots, x_n)$ such that, or all $x_1, \ldots, x_n, A(x_1, \ldots, x_n, \phi(x_1, \ldots, x_n))$."
>
> . . . Kleene: [Kl52], pp.509-510.

However, a consequence for constructive mathematics of limiting the concept of 'realizability' to only 'recursive realizability' (what we have termed as 'algorithmic *computability*') is that—misleadingly in view of §5.D., Theorem 5.13—it does not admit the Law of the Excluded Middle:

> "The formula $\forall x (A(x) \vee \neg A(x))$ is classically provable, and hence under classical interpretations true. But it is unrealizable. So if realizability is accepted as a necessary condition for intuitionistic truth, it is untrue intuitionistically, and therefore unprovable not only in the present intuitionistic formal system, but by any intuitionistic methods whatsoever.
>
> This incidentally implies that our classical formal system reinforced by an intuitionistic proof of simple consistency cannot serve as an instrument of intuitionistic proof, as suggested in §14, except

of formulas belonging to a very restricted class (including those of the form $B(x)$ and $\forall B(x)$ end §
42, but not the present formula $\forall x(A(x) \vee \neg A(x))$).

The negation $\neg \forall x(A(x) \vee \neg A(x))$ of that formula is classically untrue, but (by the corollary)
realizable, and hence intuitionistically true, if we accept realizability (intuitionistically established)
as sufficient for intuitionistic truth.

So the possibility appears of asserting the formula $\neg \forall x(A(x) \vee \neg A(x))$ intuitionistically. Thus we
should obtain an extension of the intuitionistic number theory, which has previously been treated
as a subsystem of the classical, so that the intuitionistic and classical number theories diverge,
with $\neg \forall x(A(x) \vee \neg A(x))$ holding in the intuitionistic and $\forall x(A(x) \vee \neg A(x))$ in the classical."
. . . Kleene: [Kl52], pp.513-514.

Misleading since we cannot, in view of §5.D., Theorem 5.13, admit the 'possibility' entertained
by Kleene that the formula $\neg \forall x(A(x) \vee \neg A(x))$ could be asserted intuitionistically; nor the
further argument that a 'strengthened' intuitionistic system denying $\forall x A(x) \vee \neg A(x)$ can be
'shown by interpretation to be simply consistent':

"Not only is the formula $\neg \forall x(A(x) \vee \neg A(x))$ itself realizable, but by Theorem 62 (a) (taking it
as the Γ), when we add it to the present intuitionistic formal system, only realizable formulas
become provable in the enlarged system. So then every provable formula will be true under the
realizability interpretation. In particular, the strengthened intuitionistic system is thus shown by
interpretation to be simply consistent."
. . . Kleene: [Kl52], p.514.

Moreover, by failing to recognise that algorithmic *verifiability* is not only constructive,
but *the* 'truth' assignment that yields the *weak*, standard interpretation of PA in classical
mathematics, intuitionistic perspectives uncritically accept that, if $\mathbf{A}_p(\mathbf{p})$ corresponds to Gödel's
formula 17 *Gen r* ([Go31], p.25, eqn. 13), then:

". . . we can interpret the formula $\mathbf{A}_p(\mathbf{p})$ from our perspective of the Gödel numbering as expressing
the proposition that $\mathbf{A}_p(\mathbf{p})$ is unprovable, i.e. it is a formula A which asserts its own provability."
. . . Kleene: [Kl52], p.207.

The fallacy in the—not uncommon—conclusion that $\mathbf{A}_p(\mathbf{p})$ 'is a formula A which asserts
its own provability' is highlighted in §11.G..

4.F.   Well-definedness

We note that the distinction between §2., Definition 2, and §2., Definition 3, now allows us to
define:

Definition 18. (Well-defined sequence) *A Boolean number-theoretical sequence $\{F^*(1), F^*(2),$
$\ldots\}$ is well-defined if, and only if, the number-theoretical relation $F^*(x)$ is algorithmically veri-
fiable.*

Moreover, by insisting that, for any specified natural number$i$, there be a deterministic
algorithm that computes $F^*(j)$ as either 0 or 1 for all $j \leq i$ in a Boolean sequence, Definition
18 essentially restricts the introduction of putative, set-theoretically postulated, *unspecified*
real numbers into any *evidence-based* reasoning (see §4.H., Theorem 4.4) unless their binary
representations are presumed *well-defined*.

Comment: The significance of Definition 18 for the natural sciences is seen in the putative resolution that it admits of the EPR paradox in [An15] (see §19.).

In this investigation we shall, accordingly, assume the following definition that, no matter how imprecise, seeks to capture at least the essence of what we intuitively mean by 'well-defined':

Definition 19. (Well-definedness) *A mathematical concept is well-defined if, and only if, it can be defined in terms of algorithmic verifiability.*

4.G.   Algorithmically *verifiable* but not algorithmically *computable*

The following argument (see also [An16], Theorem 2.1, p.37) illustrates that although every algorithmically *computable* relation is algorithmically *verifiable* (hence well-defined), the converse is not true:

Theorem 4.2. *There are well-defined number theoretic functions that are algorithmically verifiable but not algorithmically computable.*

Proof. We note that:

(a) Since any putative, set-theoretically postulated, real number $R = Lt_{n \to \infty} \Sigma_{i=0}^{n} r(i).2^{-i}$ in binary notation is mathematically definable by a correspondingly unique Cauchy sequence[49]:

$$\{\Sigma_{i=0}^{n} r(i).2^{-i} : n = 0, 1, \ldots; r(i) \in \{0, 1\}\}$$

of rational numbers in binary notation, it follows that:

   (i) *If* $r(n)$ denotes the nth digit in the decimal expression of the real number R in binary notation;

   (ii) *Then*, for any *specified* natural number n, Gödel's primitive recursive β-function defines an algorithm $AL_{(R, \, n)}$ that can verify the truth/falsity of each proposition in the finite sequence:

$$\{r(0) = 0, r(1) = 0, \ldots, r(n) = 0\}.$$

Hence, for any real number R, the relation $r(x) = 0$ is algorithmically *verifiable* (hence well-defined) trivially by §2., Definition 2.

(b) Since it follows from Alan Turing's Halting argument ([Tu36], p.132, §8) that there are algorithmically uncomputable real numbers:

   (i) Let $r(n)$ denote the nth digit in the decimal expression of an algorithmically *uncomputable* real number R in binary notation.

   (ii) By (a), the relation $r(x) = 0$ is algorithmically verifiable trivially.

---

[49]**Cauchy sequence**: A sequence $x_1, x_2, x_3, \ldots$ of real numbers is a Cauchy sequence if, and only if, for every real number $\epsilon > 0$, there is a an integer $N > 0$ such that, for all natural numbers $m, n > N$, $|x_m - x_n| \leq \epsilon$.

(iii) However, by definition there is no algorithm AL_R that can decide the truth/falsity of each proposition in the denumerable sequence:

$$\{r(0) = 0, r(1) = 0, \ldots\}.$$

Hence, although the relation $r(x) = 0$ is algorithmically *verifiable* (hence well-defined), it is not algorithmically *computable* by §2., Definition 3.   □

> **Gödel's β-function Lemma**: If f is an arbitrary sequence of natural numbers and k is an arbitrary natural number, then there exists a pair of natural numbers n, d such that $f^{(n,d)}$ and f coincide in their first k terms. ([Go31], Theorem VII, Lemma 1, p.31).

> **Comment**: The significance of the Lemma is that if f is a function defined by primitive recursion on a parameter n, say by $f(0) = c$ and $f(n + 1) = g(n, f(n))$, then to express $f(n) = y$ one would like to say: there exists a sequence $a_0, a_1, \ldots, a_n$ such that $a_0 = c$, $a_n = y$ and for all $i < n$ one has $g(i, a_i) = a_i + 1$. While that is not possible directly, one can say instead: there exist natural numbers a, b, and a primitive recursive function $\beta(b, c, i)$, such that $\beta(a, b, 0) = c$, $\beta(a, b, n) = y$ and for all $i < n$ one has $g(i, \beta(a, b, i)) = \beta(a, b, i + 1)$ (see also [Me64], p.131).

Although a proof that some number-theoretic relation instancing §4.G., Theorem 4.2 can be *well-defined* (in the sense of §4.F., Definition 18) is not obvious, §2.F., Corollary 2.21 shows that Gödel's *meta-mathematically* defined formula $[R(x)]$ is algorithmically *verifiable* as always true under the *weak*, standard, interpretation $I_{PA(N,SV)}$ of PA (see §2.B.), hence *well-defined* (in the sense of Definition §4.F., 18), but not algorithmically *computable* as always true under the *strong*, finitary, interpretation $I_{PA(N,SC)}$ of PA (see §2.C.).

We note that a definition of a set of computationally significant, *well-defined*, number-theoretic functions which are algorithmically *verifiable*, but not algorithmically *computable*, was also given by Gregory Chaitin in [Ct75].

He defined a class of Ω constants such that, if $C(n)$ is the nth digit in the decimal expression of an Ω constant, then the function $C(x)$ is algorithmically *verifiable* but not algorithmically *computable*; where Ω is Chaitin's Halting Probability given by $0 < \Omega = \sum 2^{-|p|} < 1$, the summation is over all self-delimiting programs p that halt, and $|p|$ is the size in bits of the halting program p (see [Ct82]).

> **Comment**: We note that Specker sequences (see also §16.D.) do not instance §4.G., Theorem 4.2, since they are algorithmically *computable*, monotonically increasing, bounded sequences of rational numbers.

Theorem 4.2 now admits a formal definition of *relative* randomness:

Definition 20. (Relative randomness) *A number-theoretic sequence is relatively random if, and only if, it is algorithmically verifiable, but not algorithmically computable.*

A *relatively* random number-theoretic sequence cannot thus be represented—nor defined—as a partial recursive function; since the latter always defines, and is defined by, some deterministic Turing machine (see [Me64], p.237; [Kl52], p.373).

However, complementing the Provability Theorem for PA which models algorithmically *computable* number-theoretic functions arithmetically (see §2.E.a.), any *relatively* random sequence *can* be defined within PA in terms of Gödel's β-functions (as detailed in §12.A. and §12.B.).

4.G.a.   Effective computability and the Church-Turing Thesis

Moreover, §4.F., Definition 18 now admits an *evidence-based* definition of 'effective computability'.

Definition 21. (Effective computability) *A number-theoretical sequence $\{F^*(n)\}$ is effectively computable if, and only if, it is well-defined.*

We conclude that (see also §7.C., Theorem 7.3), contrary to current paradigms[50] (see [OWJ06]; also [Pic11], [Tyl07]):

Theorem 4.3. (CT falsifiable) *The classical Church-Turing Thesis is false in any interpretation of the first-order Peano Arithmetic PA that admits evidence-based reasoning.*

Proof. By §4.G.a., Definition 21 and §2.F., Corollary 2.21, Gödel's meta-mathematically *well-defined* formula $[R(x)]$ is algorithmically *verifiable* as always true under $I_{PA(N,SV)}$, but not algorithmically *computable* as always true under $I_{PA(N,SC)}$. Hence it is not partial recursive. □

4.H.   Every real number is *specifiable* in PA

The distinction between §2., Definition 2, and §2., Definition 3, also allows us to place Cantor's Theorem[51], namely that the real numbers are uncountable, into a coherent constructive perspective.

We note that—as entailed by Cantor's diagonal argument—there is no algorithmically computable function $F(n)$ whose values can be put in a 1-1 correspondence with *all* real numbers that are definable as non-terminating Cauchy sequences.

From an *evidence-based* perspective we *cannot*, however, conclude from this that that there are *unspecifiable* real numbers, since (compare §12.B., Corollary 12.4):

Theorem 4.4. (Specifiability Theorem for Reals) *Every real number is specifiable in PA.*

Proof. Since any real number R is mathematically definable by a correspondingly unique Cauchy sequence:

$$\{\Sigma_{i=0}^{n} r(i).2^{-i} : n = 0, 1, \ldots; r(i) \in \{0, 1\}\}$$

of rational numbers in binary notation, it is *specifiable* in PA because it can be represented in PA by an algorithmically *verifiable* (hence *well-defined*) arithmetical function by means of Gödel's β-function (see [Go31], Lemma 1, p.31; also §12.A., Lemma 12.2). □

> **Comment**: To place §4.H., Theorem 4.4 in an appropriate perspective, we note that, as Bauer remarks:

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[50]Challenged—albeit on different grounds—also by Selmer Bringsjord and Naveen Sundar Govindarajalu in [BG11], [BG11a]: "One of us has previously argued that the Church-Turing Thesis (CTT), contra Elliot Mendelson, is not provable, and is—in light of the mind's ability to effortlessly hypercompute—moreover false".

[51]See [Ru53], §2.16, Theorem, p.23; [Ru53], §2.40, Corollary, p.34; also [BBJ03], p.16.

> "A cursory literature search reveals other bizarre statements considered in constructive mathematics: '$\mathbb{R}$ has measure zero' , 'there is a bounded increasing sequence without an accumulation point', 'ordinals form a set', 'there is an injection of $\mathbb{N}^{\mathbb{N}}$ into $\mathbb{N}$' , and so on."
>
> . . . *Bauer: [Ba16], p.6.*

He defends such constructivist conclusions by arguing that:

> "A constructivist might point out that what counts as bizarre is subjective and remind us that once upon a time the discovery of non-Euclidean geometries was shelved in fear of rejection, that Weierstraß's continuous but nowhere differentiable function was and remains a curiosity, and that the Banach-Tarski theorem about conjuring two balls from one is even today called a 'paradox'."
>
> . . . *Bauer: [Ba16], p.6.*

4.H.a.   *Evidence-based* reasoning does not admit Cantor's theorem

We note that the classical conclusion $\aleph_0 \not\longleftrightarrow 2^{\aleph_0}$ reflects the Platonic assumption that there are 'set-theoretically completed' Cauchy sequences which cannot be expressed in PA.[52]

However, §4.H., Theorem 4.4 shows that the assumption is fragile, since Cauchy sequences which are defined as algorithmically *verifiable*, but not algorithmically *computable*, correspond to *well-defined*, 'essentially incompletable', real numbers whose Cauchy sequences cannot, in a sense, be known 'completely' even to Laplace's *vast intelligence*:

> "We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it—an intelligence sufficiently vast to submit these data to analysis—it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytic expressions the past and future stars of the system of the world. Applying the same method to some other objects of its knowledge, it has succeeded in referring to general laws observed phenomena and in foreseeing those which given circumstances ought to produce. All these efforts in the search for truth tend to lead it back continually to the vast intelligence which we have just mentioned, from which it will always remain infinitely removed."
>
> . . . *Laplace: [Lap02], p.4.*

> **Comment**: We note that Laplace's conception of a 'vast intelligence' to which the 'human mind offers . . . a feeble idea of this intelligence' essentially articulates the Mechanist's argument which Lucas's Gödelian Thesis seeks to refute (see §17.A.a.). An argument echoed by in 2016 by Stephen Hawking:
>
> > "I believe there is no deep difference between what can be achieved by a biological brain and what can be achieved by a computer. It therefore follows that computers can, in theory, emulate human intelligence—and exceed it."
> >
> > . . . *Hawking: Stephen Hawking—Will AI kill or save humankind?*

---

[52]Such a conclusion can also be viewed as another instance (see, for instance, §15.) of ignoring Skolem's cautionary remarks in [Sk22] about unrestrictedly corresponding putative mathematical entities across domains of different axiom systems.

In other words, the numerical values of algorithmically *verifiable*, but not algorithmically *computable*, sequences must be treated as *well-defined*, formally specifiable, first-order, non-terminating processes which are 'eternal work-in-progress' in the sense of §4.H., Theorem 4.4.[53]

Thus, from an *evidence-based* perspective, §4.H., Theorem 4.4 implies that real numbers do not exist in some Platonic, set-theoretic, universe of points that constitute a line, but are arithmetical constructs identifiable as specific number-theoretic definitions that are algorithmically *verifiable* (hence *well-defined*), but not necessarily algorithmically *computable*.

The following theorem shows that, as illustrated further by the gedanken in §16.C.b., Case 2, such arithmetical constructs assume significance as real numbers which can, debatably, be termed as 'existing' mathematically, only when such a definition is made explicit formally in an argumentation (compare with Brouwer's parallel perspective cited in §4.K.).

Theorem 4.5. (Invalid Cauchy Limit Theorem) *The values of n for which Gödel's arithmetic formula $[R(x)]$, with Gödel number r as defined in [Go31], p.25, eqn.12, interprets as a true arithmetic proposition $R^*(n)$ in $\mathbb{N}$ do not define the real number $c = \sum_{n=1}^{\infty}(1/10^n) = 0.c_1 c_2 \ldots c_n \ldots$; which is classically treated as defining the real number $1/9$.*

Proof. We note that:

(i) If $c = \sum_{n=1}^{\infty}(1/10^n) = 0.c_1 c_2 \ldots c_n \ldots$ defines the real number $1/9$ then $c_n = 1$ is a recursive relation that is algorithmically *computable* as always true in $\mathbb{N}$[54] by a Turing machine that computes the decimal representation of $1/9$.

(ii) Hence it is expressible in PA by some PA-formula $[C(x) = 1]$ such that, for any specified $n \in \mathbb{N}$:

- If $c_n = 1$ holds in $\mathbb{N}$, then $[C(n) = 1]$ is PA-provable;
- If $c_n \neq 1$ holds in $\mathbb{N}$, then $[\neg(C(n) = 1)]$ is PA-provable.

(iii) The PA-formula $[C(x) = 1]$ now (see §10.D.b.) interprets as an arithmetical relation $C^*(x) = 1$ such that $C^*(n) = 1$ is an arithmetical relation which is also algorithmically *computable* as always true in $\mathbb{N}$ since $(C^*(n) = 1) \equiv (c_n = 1)$.

(iv) Hence, by the Provability Theorem for PA (§2.E., Theorem 2.17), the arithmetical formula $[C(x) = 1]$ is PA-provable.

(v) Now, Gödel has shown (see [Go31], p.26(2)) that, for any specified PA-numeral $[n]$, the PA-formula $[R(n)]$ is PA-provable.

(vi) Moreover, since $[R(x)]$ is a well-defined PA-formula (see §11.D., Argument B), we can define a PA-formula $[D(x)]$ such that $[(D(x) = 1) \equiv R(x)]$.

(vii) Thus $[D(x) = 1]$ interprets as an arithmetical relation $D^*(x) = 1$ such that $D^*(n) = 1$ holds in $\mathbb{N}$ if, and only if, $[R(n)]$ interprets as true in $\mathbb{N}$.

---

[53]A perspective suggested by the way dimensionless constants are viewed in the physical sciences, as highlighted in §19.D.

[54]See §2.; also [Mu91], §1, Introduction.

(viii) Hence $D^*(n) = C^*(n) = c_n = 1$ for all n in $\mathbb{N}$.

 (ix) If, now, $d = 0.D^*(1)D^*(2)\ldots D^*(n)\ldots$ were to define the real number $c = \sum_{n=1}^{\infty}(1/10^n) = 0.c_1c_2\ldots c_n\ldots = 0.D^*(1)D^*(2)\ldots D^*(n)\ldots$, then $D^*(x) = 1$ would be algorithmically *computable* as always true in $\mathbb{N}$.

  (x) In which case, by the Provability Theorem for PA (§2.E., Theorem 2.17), $[D(x) = 1]$ would be PA-provable;

 (xi) Whence $[R(x)]$—and therefore $[(\forall x)R(x)]$ by Generalisation—would both be provable in PA.

(xii) Since Gödel has shown that $[(\forall x)R(x)]$ is not PA-provable (see [Go31], p.25(1)), neither $[R(x)]$ nor $[D(x) = 1]$ can be PA-provable.

(xiii) Hence d does not define a real number, even though $0.D^*(1)D^*(2)\ldots D^*(n)\ldots$ defines the Cauchy sequence $\sum_{n=1}^{\infty}(1/10^n)$.

The theorem follows.   □

We conclude that whether or not a *well-defined*—hence algorithmically *verifiable* by §4.F., Definition 18—Cauchy sequence can be treated as having a classical, algorithmically *computable*, Cauchy limit that defines a real number depends not only on the terms of the sequence but—as suggested by the Complementarity Thesis (see §1., Thesis 1)—on *how* the terms of the sequence are defined/interpreted.

Since the above distinction cannot be made in any set theory which defines sets extensionally, Theorem 4.5 shows that, contrary to classical arguments, every Cauchy sequence of rational numbers cannot be postulated as defining a real number by the usual set-theoretical arguments *if* the reals are sought to be modeled geometrically as the points on a line.

The significance of Theorem 4.5 for the natural sciences is highlighted in §19.D. (see also §17.G.a.).

> **Comment**: The argument that 'every Cauchy sequence of rational numbers cannot be postulated as defining a real number by the usual set-theoretical arguments' is implicit in physicist Nicolas Gisin's unusual, intuitionistic, interpretation of 'real numbers' as 'the hidden variables of classical physics' in [Gi19]; where he argues (see also [Gi20]) that 'real numbers should not be considered as "physical real" and classical mechanics, like quantum physics, is indeterministic':
>
> > "... At first sight, no doubts, quantum theory imposes limits to what can be known. There are Heisenberg's uncertainty relation and—Copenhagen obliged—Bohr's complementary principle. But is it scientific to believe that scientific theories limit human knowledge? In particular, does quantum theory limit our knowledge or does it faithfully describe an indeterministic world, a world in which objects do not have determined positions, momenta and further properties? In short, should one speak of the uncertainty relation or of the indeterminacy relation?
> >
> > For a realist, like myself, scientific theories describe what there is, not the limits of our knowledge. One cannot simultaneously know with arbitrary precision the position and momentum of particles not because of some fancy limitations to our knowledge, but merely because particles do not have simultaneous precise positions and momenta. Nevertheless, looking for additional variables is highly interesting, because it may allow one to discover new physics. This implies that the hypothetical new variables

should not be hidden, at least not be hidden by essence for ever: they may be hidden today, but the interest is to find and reveal them.[1] At least, this is the rough story. In quantum theory, things are more complex, because of the locality issue, on one side, and for historical reasons on the other side."

. . . *Gisin*: [Gi19], *Introduction.*

We note that such an *evidence-based* perspective reflects Leopold Kronecker's views on what may be treated as a *well-defined* mathematical concept. As detailed by Sieg:

"Kronecker made restrictive demands on the formation of mathematical concepts, e.g., he insisted on their decidability.[6] Being well aware of these demands, Dedekind attached the following footnote to his remark that a system S, as an object of our thinking, is completely determined as soon as "of each thing it is determined, whether or not it is an element of S" (Dedekind 1888, p. 2):

How this determination is brought about, and whether we know of a way of deciding upon it, is a matter of utter indifference for all that follows; the general laws to be developed in no way depend upon it; they hold under all circumstances. I mention this expressly because Kronecker not long ago (in *Kronecker* 1886) has endeavored to impose certain limitations upon the free formation of concepts in mathematics, which I do not believe to be justified . . .

This footnote is directed against one in (Kronecker 1886), where Kronecker argues not only against Dedekind's concepts like module or ideal, but also against a *general concept of irrational number.* Explaining his rejection, Kronecker writes:

Even the *general* concept of an infinite series . . . is in my opinion . . . only admissible on condition that in every special case, on the basis of the arithmetical law for the formation of the terms (or of the coefficients), certain presuppositions are shown to be satisfied, which permit the series to be applied like finite expressions, and which consequently make it really unnecessary to go beyond the concept of a *finite series.* (Kronecker 1886, p. 947)

The effect of such a finiteness condition on the concept of real number is stated in a letter Kummer wrote to Schwartz on 15 March 1872. Kummer remarks that he and Kronecker share the conviction that "the effort to create enough individual points to fill out a continuum, i.e., enough real numbers to fill out a line, is as vain as the ancient efforts to prove Euclid's parallel postulate".[7]

If *individual points* can be created only in accord with Kronecker's finiteness demand, then Kummer's observation is provable. One first notices that the system of real numbers thus created is countable. Next one has to address the question, what is the geometric line that cannot be filled by the individually created points? In 1872, the very year of Kummer's letter, Dedekind had characterized an *arithmetical continuum* as the system of all cuts of rational numbers.[8] As that system is isomorphic to the continuous geometric line and is uncountable, Kummer's observation has been established. The argument I just sketched is, of course, anachronistic, but brings out the strikingly different approaches to the arithmetization of the geometric continuum. In this way, it makes evident the impact of broader foundational views on mathematical practice. At the heart of the difference between these foundational positions is the freedom of introducing abstract concepts—given by *structural definitions.*"

. . . *Sieg*: [Si12], *pp. 10-11.*

In other words, whereas expressing intuitive concepts in the language of a formal mathematical theory on the basis of *structural definitions*[55] have—following Dedekind—been considered justified so long as the introduction of such definitions is consistent with the theory, the

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[55]In the sense of [Si12], §4, Structural Definitions, p.11-14.

interpretations of such concepts—according to Kronecker—are meaningful (in the sense of accommodating §1., Thesis 1(1) and §1., Thesis 1(2); and justifiably so as demonstrated in §16.D.d.) only if the definitions are *well-defined* over the domain of the interpretation in the sense of §4.F., Definition 18.

4.I. From a Wittgensteinian perspective

We shall see that the significance of differentiating between:

(i) the *strong*, algorithmically *computable*, 'truth'—of the provable formulas of a formal mathematical language L—that follows by *finitary mathematical* reasoning from the axioms and rules of inference of L under a well-defined interpretation; and

(ii) the *weak*, algorithmically *verifiable*, 'truth'—of the provable formulas of L—that follows by *non-finitary meta-mathematical* reasoning from the axioms and rules of inference of L under a *well-defined* interpretation;

is implicit in, and illuminates (see §9.), Timm Lampert's interpretation of Ludwig Wittgenstein's objection—in the latter's 'notorious' paragraph in [Wi78] (see §9.A.)—to the, philosophically disquieting, conclusions (see §11.B.) that Gödel drew from his undifferentiated *mathematical* and *meta-mathematical* reasoning in [Go31]:

> "The most crucial aspect of any comparison of two different types of unprovability proofs is the question of what serves as the "criterion of unprovability" (I, §15). According to Wittgenstein, such a criterion should be a purely syntactic criteria independent of any meta-mathematical interpretation of formulas. It is algorithmic proofs relying on nothing but syntactic criteria that serve as a measure for assessing meta-mathematical interpretations, not vice-versa."
>
> *. . . Timm Lampert:* [Lam17].

4.J. Skolem's paradox

Moreover, Wittgenstein's perspective is reflected in, and illuminates (see §11.B., Lemma 11.1), Thoralf Skolem's cautionary remarks against inviting paradox[56] by conflating entailments of formal systems under different interpretations (see §2.D.), or over different domains:

> **Comment**: We note that, in a 1922 address delivered in Helsinki before the Fifth Congress of Scandinavian Mathematicians, Skolem improved upon both the argument and statement of Löwenheim's 1915 theorem ([Lo15], p.235, Theorem 2)—subsequently labelled as the (downwards) Löwenheim-Skolem Theorem ([Sk22], p.293):
>
> > **(Downwards) Löwenheim-Skolem Theorem** ([Lo15], p.245, Theorem 6; [Sk22], p.293): If a first-order proposition is satisfied in any domain at all, then it is already satisfied in a denumerably infinite domain.
>
> Skolem then drew attention to a:
>
> > **Skolem's (apparent) paradox**: ". . . peculiar and apparently paradoxical state of affairs. By virtue of the axioms we can prove the existence of higher cardinalities, of higher number classes, and so forth. How can it be, then, that the entire domain B can already be enumerated by means of the finite positive integers? The explanation is not difficult to find.

---

[56] See, for instance, Goodstein's argument in §15., Theorem 15.1.

> In the axiomatization, "set" does not mean an arbitrarily defined collection; the sets are nothing but objects that are connected with one another through certain relations expressed by the axioms. Hence there is no contradiction at all if a set M of the domain B is non-denumerable in the sense of the axiomatization; for this means merely that *within B* there occurs no one-to-one mapping Φ of M onto Z_o (Zermelo's number sequence). Nevertheless there exists the possibility of numbering all objects in B, and therefore also the elements of M, by means of the positive integers; of course such an enumeration too is a collection of certain pairs, but this collection is not a "set" (that is, it does not occur in the domain B)."
>
> *. . . Skolem:* [Sk22], *p.295.*

4.K. From a Brouwerian perspective

We also note that the distinction in §4.G., Theorem 4.2 between algorithmically *verifiable* number-theoretic functions (and the real numbers defined by them) and algorithmically *computable* number-theoretic functions (and the real numbers defined by them) is, prima facie, similar to the one that, according to Mark van Atten, L. E. J. Brouwer sought to make explicit in his 1907 PhD thesis:

> "The distinction between a construction proper and a construction project was well known to Brouwer. It is essential to his notion of denumerably unfinished sets:
>
>> [H]ere we call a set denumerably unfinished if it has the following properties: we can never construct in a well-defined way more than a denumerable subset of it, but when we have constructed such a subset, we can immediately deduce from it, following some previously defined mathematical process, new elements which are counted to the original set. But from a strictly mathematical point of view this set does not exist as a whole, nor does its power exist; however we can introduce these words here as an expression for a known intention. [10, p.148; trl. 45, p.82]
>
> But in the quotations from 1947 and 1954 above we do not see Brouwer say, analogously, that sequences that are not completely defined do from a strictly mathematical point of view not exist as objects, but that terms for them are introduced as expressions for a known intention (namely, to begin and continue a construction project of a certain kind). This explains the fact noted in the latter half of Gielen, De Swart, and Veldman's reflection.
>
> Still, the distinction at the basis of De Iongh's view between construction processes that are governed by a full definition of the object under construction and those that, as a matter of principle, cannot be thus governed, is a principled one of mathematical relevance, and it is important to realise that, if a proposed axiom turns out not to hold in general, it may still hold for one of these two subclasses.
>
> [. . .]
>
> [10] L. E. J. Brouwer. Over de grondslagen der wiskunde. PhD thesis, Universiteit van Amsterdam, 1907."
>
> *. . . van Atten:* [At18], *pp.67-68.*

CHAPTER 5. *MATHEMATICAL CONSEQUENCES*

5.   The significance of Hilbert's ω-Rule for Gödel's ω-consistency

The distinction between §2., Definition 2, and §2., Definition 3, also allows us to address, and place, Hilbert's ω-Rule of infinite induction ([Hi30], pp.485-494) within a broader *evidence-based* perspective; where we first define an:

Proposition 5.1. (Algorithmic (weak) ω-Rule of infinite induction) *If it is proved that the PA formula $[F(x)]$ interprets as an arithmetical relation $F^*(x)$ that is algorithmically computable as true for any specified natural number n, then the PA formula $[(\forall x)F(x)]$ can be admitted as an initial formula (axiom) in PA.*

The significance of the (*weak*) Algorithmic ω-Rule of infinite induction is that it is defined in terms of algorithmic *computability*; and is an immediate consequence of the Provability Theorem for PA (§2.E., Theorem 2.17) which, essentially, was what Hilbert had sought in his Program (see [Hi00]).

Thus, as part of his program for giving mathematical reasoning a finitary foundation, Hilbert proposed a *stronger* (since it implies the Provability Theorem for PA) ω-Rule of infinite induction as an ad hoc means of extending a Peano Arithmetic to a possible completion[57], which we can rephrase, without loss of generality, in terms of algorithmic *verifiability* as:

Proposition 5.2. (Hilbert's (*strong*) ω-Rule of infinite induction) *If it is proved that the PA formula $[F(x)]$ interprets as an arithmetical relation $F^*(x)$ that is algorithmically verifiable as true for any specified natural number n, then the PA formula $[(\forall x)F(x)]$ can be admitted as an initial formula (axiom) in PA.*

The question of whether or not *weakened* versions of Hilbert's original ω-Rule of infinite induction could be regarded as finitary is addressed in detail by Schirn and Niebergall:

> "Restricted versions of the ω-rule have been suggested both as a means of explicating certain forms of finitary arguments or proofs and as a way of correctly extending a theory already accepted. In this section, we want to deal with the question as to whether weak versions of the ω-rule can be regarded as finitary. For if they can, they may prove useful for the construction of metamathematical theories that clash neither with Hilbert's programme nor with Gödel's Incompleteness Theorems. In pursuing our aim, we align ourselves with Hilbert's programme. By contrast, in his 1931 essay Hilbert himself introduces a restricted ω-rule as a means of extending PA, though he does so in a way which admits different interpretations.
>
> > *Rule ω^**: When it is shown that the formula $A(\mathbf{Z})$ is a correct numerical formula for each particular numeral $\mathbf{Z}$, then the formula $\forall x A(x)$ can be taken as a premise.
>
> Hilbert qualifies this rule expressly as finitary and goes on to remind us that $\forall x A(x)$ has a much wider scope than $A(\tilde{\mathfrak{n}})$, where $\tilde{\mathfrak{n}}$ is an arbitrary given numeral."
>
> *. . . Schirn and Niebergall: [SN01], p.137.*

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[57]In other words, to logically showing that, for any specified arithmetical proposition, either the proposition, or its negation, is formally provable from the axioms and rules of inference of the extended Arithmetic.

Schirn and Niebergall conclude that Hilbert's assumption of Aristotle's particularisation as a valid, and essential, form of reasoning—as evidenced in his definitions of the universal and existential quantifiers in terms of his ε-operator (see §7.A.)—committed him to an essentially non-finitary perspective; reflected also in his ω-rule of infinite induction; both of which we shall show—§5.C., Lemma 5.5 and §5.D., Corollary 5.10—are stronger than Gödel's assumption of ω-consistency in the latter's 1931 paper [Go31] on 'formally undecidable' arithmetical propositions:

> "We venture to surmise that Hilbert qua metalogician relies on existence assumptions of precisely this kind without being haunted by any finitist qualms. And we do think that those assumptions of infinity that are made by accepting one application of rule ω^* are not more far-reaching than those made by accepting transfinite induction upto ε_0.
>
> It should be evident that the ω-rule or even one application of it cannot be accepted from Hilbert's original finitist point of view. Yet both modern metalogic and Hilbert's metamathematics of the 1920s rest on certain assumptions of infinity that clash anyway with his classical finitism (cf. Niebergall and Schirn 1998, section 4). Intuitively speaking, one may tend to believe that the metalogical assumptions of infinity just appealed to, or Hilbert's assumption in his work on proof theory in the 1920's that there are infinitely many stroke-symbols, are slightly weaker than those that we make when we apply an ω-rule. However this may be, we do not rule out that Hilbert wants to commit himself only to the possible existence of infinitely many stroke-figures or, alternatively, to the existence of infinitely many possible stroke-figures. Unless a satisfactory theory of the potential infinite is to hand, it is probably wise to postpone closer scrutiny of the question whether, from the point of view of strength, applications of a given ω-rule and the assumptions of infinity, both made by Hilbert in the 1920s and common in contemporary metalogic, differ essentially from each other."
>
> *. . . Schirn and Niebergall: [SN01], p.141.*

Now, Gödel's 1931 paper can, not unreasonably, be viewed as the outcome of a presumed attempt to formally validate Hilbert's ω-rule of infinite induction finitarily, since:

Lemma 5.3. *If we meta-assume Hilbert's ω-rule of infinite induction for PA, then a consistent PA is ω-consistent.*

Proof. If the PA formula $[F(x)]$ interprets as an arithmetical relation $F^*(x)$ that is algorithmically *verifiable* as true for any specified natural number n, and the PA formula $[(\forall x)F(x)]$ can be admitted as an initial formula (*axiom*) in PA, then $[\neg(\forall x)F(x)]$ cannot be PA-provable if PA is consistent. The lemma follows. $\square$

> **Comment**: We note, however, that we cannot similarly conclude from the the Algorithmic ω-Rule of infinite induction that a consistent PA is ω-consistent.

However, by Gödel's Theorem VI in [Go31], it follows from §5., Lemma 5.3 that one consequence of assuming Hilbert's ω-Rule of infinite induction is that there must, then, be a formally undecidable arithmetical proposition; a further consequence of which would be that any first-order arithmetic such as PA must be essentially incomplete (i.e., not completable by the addition of an ω-Rule of infinite induction as envisaged by Hilbert).

> **Gödel's Theorem VI**: "For every ω-consistent recursive class κ of FORMULAS, there exists a recursive CLASS EXPRESSION r such that neither v Gen r nor Neg(v Gen r) belongs to Flg(κ) (where v is the FREE VARIABLE of r)."
>
> *. . . [Go31], Theorem VI, p.24*

5.A.   Is Hilbert's ω-Rule equivalent to Gentzen's Infinite Induction?

Schirn and Niebergall also address the question of whether Hilbert's ω-rule of infinite induction is weaker than Gentzen's cut-elimination, and consider the argument that:

> "Since we can construe the infinitely many premises of one application and, hence, of finitely many applications of the ω-rule as ordered with order type ω, the proof theorist who intends to employ the ω-rule has to presuppose only (the existence of) ω. By contrast, Gentzen's consistency proof for pure number theory in his 1936 article presupposes (the existence of) ε_0. Moreover, if a proof theorist endorsing the basic tenets of Hilbert's finitism were asked how he brings it about to prove infinitely many premises, he might respond as follows:
>
>> To accept one application of rule ω^* is not more problematic than to make the assumption that one can conclude from the PA-provability of '$\forall x(0 \leq x)$' to the PA-provability of '$0 \leq n$' for every n. Both cases require that *modus ponens* be applied infinitely many times, where the sequence of the prooflines has order-type ω."
>
> *. . . Schirn and Niebergall: [SN01], p.140.*

Schirn and Niebergall remark, and stress, that the issue confronting Hilbert then—as also 'finitists' of all hues since[58]—was that of unambiguously defining a deterministic procedure for interpreting quantification finitarily; both over the numerals and the numbers that they seek to formally represent:

> "It is important to bear in mind that finitist mathematics may be extended by adding well-formed formulae or by adjoining further 'principles'. It is the first that is at issue in Hilbert's proposed finitist interpretation of quantified statements about numerals (Hilbert and Bernays 1934, 32ff.). So, let us begin by taking a closer look at this.
>
> (1) A general statement about numerals '$\forall \tilde{\mathfrak{n}}\, \tilde{\mathfrak{U}}(\tilde{\mathfrak{n}})$' can be interpreted finitistically only as a *hypothetical* statement, i.e. as a statement about every *given* numeral. A general statement about numerals expresses a law that has to be verified for each individual case.[25]
>
> (2) An existential statement about numerals '$\exists \tilde{\mathfrak{n}}\, \tilde{\mathfrak{U}}(\tilde{\mathfrak{n}})$' must be construed, from the finitist point of view, as a 'partial proposition', i.e. 'as an incomplete communication of a more exactly determinate statement, which consists either in the direct specification of a numeral with the property $\tilde{\mathfrak{U}}$ or in the specification of a procedure for gaining such a numeral' (Hilbert and Bernays 1934, 32). The specification of the procedure requires that for the sequence of acts to be carried out a determinate limit be presented.
>
> (3) In like manner we have to interpret finistically statements in which a general statement is combined with an existential statement such as 'For every numeral $\tilde{\mathfrak{r}}$ with the property $\tilde{\mathfrak{U}}(\tilde{\mathfrak{r}})$ there exists a numeral $\tilde{\mathfrak{l}}$ for which $\tilde{\mathfrak{B}}(\tilde{\mathfrak{r}}, \tilde{\mathfrak{l}})$ holds', for example. In the spirit of the finitist attitude, this statement must be regarded as the incomplete communication of a procedure with the help of which we can find for each given numeral $\tilde{\mathfrak{r}}$ with the property $\tilde{\mathfrak{U}}(\tilde{\mathfrak{r}})$ a numeral $\tilde{\mathfrak{l}}$ which stands to $\tilde{\mathfrak{r}}$ in the relation $\tilde{\mathfrak{B}}(\tilde{\mathfrak{r}}, \tilde{\mathfrak{l}})$.
>
> (4) Hilbert points out that negation is unproblematic when applied to what he calls 'elementary propositions', i.e. to statements which can be decided by direct intuitive observation. In the case of universally and existentially quantified statements about numerals, however, it is not immediately clear what ought to be regarded as their negation in a finitist sense. The assertion that a numeral $\tilde{\mathfrak{n}}$ with the property $\tilde{\mathfrak{U}}(\tilde{\mathfrak{n}})$ does not exist has to be conceived of as

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[58]We note that 'finitists'—ranging from Brouwer [Br08], to Wittgenstein [Wi78], to Alexander Yessenin-Volpin [He04]—have persistently (and not unreasonably in view of §2.F., Corollary 2.23) questioned the assumption that the classical 'standard' interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA—which implicitly admits non-constructive entailments such as Aristotle's particularisation—can be treated as constructively *well-defining* a model of PA (in the sense of §9., Definition 26; see also [Brm07], [Pos13]).

the assertion that it is *impossible* that a numeral $\tilde{\mathfrak{n}}$ has the property $\tilde{\mathfrak{U}}(\tilde{\mathfrak{n}})$. Strengthened negation of an existential statement, thus constructed, is not (as in the case of negation of an elementary statement) the contradictory of '$\exists \tilde{\mathfrak{n}} \tilde{\mathfrak{U}}(\tilde{\mathfrak{n}})$'. From the finitist standpoint, we therefore cannot make use of the alternative according to which there either exists a numeral $\tilde{\mathfrak{n}}$ to which $\tilde{\mathfrak{U}}(\tilde{\mathfrak{n}})$ applies or the application of $\tilde{\mathfrak{U}}(\tilde{\mathfrak{n}})$ to a numeral $\tilde{\mathfrak{n}}$ is excluded. Hilbert admits that, from the finitist perspective, the law of the excluded middle is invalid in so far as for quantified sentences we do not succeed in finding a negation of finitist content which satisfies the law.

Fn25   The proposed interpretation of universal quantification is reminiscent of Gentzen's and W. W. Tait's account (See Tait 1981) in that it likewise embodies a version of the ω-rule which rests on the identification of numerals with numbers. Tait's additional idea is that the law in question is to be construed as something given by a finitist function. ...*Schirn and Niebergall*: [SN01], *p.143*.

Comment: We note that Schirn and Niebergall too seem to implicitly subscribe to the—*misleading* by §5.D., Corollary 5.14—admission they ascribe to Hilbert, namely that 'from the finitist perspective, the law of the excluded middle is invalid in so far as for quantified sentences we do not succeed in finding a negation of finitist content which satisfies the law'.

Schirn and Niebergall note that, although Hilbert endeavoured to distinguish between quantified propositions over numerals and quantified propositions over the numbers that they seek to represent (corresponding to what we have termed as *weak* and *strong* interpretations of quantification in §4.C., Definition 13 to §4.C., Definition 17), he could not express the distinction formally:

Now, when we compare (1)-(4) with Hilbert's remarks on what can be formulated finitistically in say, 'Über das Unendliche' (1926), we notice two things. Explication (4) is very much akin to the points made in that paper about the negation of quantified statements. The matter stands differently with (1)-(3). On plausible grounds, one should assume that a finitistically interpreted sentence is capable of being formulated finitistically in the first place. If that is correct, then (1) to (3) ought to be understood in such a way that universally quantified sentences, even sentences whose formalizations are genuine Π_2^0-sentences (cf. (3)), can be formulated in the language of finitist mathematics. Plainly, if around 1934 Hilbert really wished to maintain that quantified sentences of types (1)-(3) have a proper place in the language of finitist metamathematics, he would have departed significantly from his conception of metamthematics in the 1920s. It is quite true that both in 'Über das Unendliche' and in *Grundlagen der Mathematik* (1934) Hilbert spares himself the trouble of developing the language of finitist metamathematics in a systematic way. There is one crucial difference, though. In his celebrated essay, the distinction between real and ideal statements, although chiefly designed to streamline the formalism, provides at least a clue for assessing the scope and the limits of the language of finitist mathematics. By contrast, the reader of Hilbert and Bernays 1934 who is expecting to encounter this helpful distinction again here will be disappointed. In this book, there is not even a trace of it framed in familiar terms.

Admittedly, all this does not exclude that an alternative way of construing the phrase 'finitistically interpretable' can be contrived. Consider sentences of type (1). In 'Über das Endliche' '$\forall x(x + 1 = 1 + x)$' is not a sentence of L_M, and the same applies to an expression like (*) 'For every given $\tilde{\mathfrak{a}}$ '$\tilde{\mathfrak{a}} + 1 = 1 + \tilde{\mathfrak{a}}$' is true'. By contrast, if a numeral $\tilde{\mathfrak{a}}$ is given, the expression '$\tilde{\mathfrak{a}} + 1 = 1 + \tilde{\mathfrak{a}}$' is a sentence of the language of finitist metamathematics. In *Grundlagen der Mathematik* (1934), the question of which language (*) may belong to is passed over in silence. We are only told that a finitist interpretation of (*) requires that it be construed as a hypothetical judgement about every given numeral (cf. (1)) (we aassume that (*) should be considered a general statement about numerals). A similar formulation is employed in 'Über das Endliche' (91 [378]), with the minor difference that here Hilbert talks about interpretation *simpliciter*.[28] And it is almost precisely at this point that he introduces his conception of real and ideal statements, stressing that the latter are, from the finitist point of view, devoid of meaning. This shows: the fact that in 'Über das Endliche' certain sentences of type (1), like (*), are amenable to (a finitist) interpretation is compatible with the fact that the language of finitist metamathematics does not comprise

sentences of this type. The finitist interpretation of (*) proceeds in such a way that for every given numeral $\tilde{\mathfrak{a}}$ (*) is replaced with '$\tilde{\mathfrak{a}} + 1 = 1 + \tilde{\mathfrak{a}}$', and then each of the sentences '$\tilde{\mathfrak{a}} + 1 = 1 + \tilde{\mathfrak{a}}$' is interpreted finitistically. Seen from this angle, we should not take it for granted that in *Grundlagen der Mathematik* (1934) finitist interpretability implies finitist formulability. What we do take for granted is that if this implication holds for sentences of one of these types, then it must also hold for the sentences of the remaining types.

Fn28   It is reasonable to assume that here he likewise has a finitist interpretation in mind. Notice that non-finitary sentences, i.e. ideal sentences, are not interpreted at all."

. . . *Schirn and Niebergall:* [SN01], *p.143.*

Perhaps—as illustrated by §2., Definitions 2 and 3—a transparent and unambiguous description of the deterministic infinite procedures[59] needed to *evidence* the distinction formally becomes available only *after* the realisation that Turing's 1936 paper [Tu36]) admits *evidence-based* reasoning in the sense of [Mu91] and [Lob59]; namely, that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic[60].

5.B.   Hilbert's *weak* proof of consistency for PA

Schirn and Niebergall note further that, in order to argue that every numerical formula derivable from the axioms of a weakened arithmetic H was 'true', Hilbert and Bernays introduced the concept of 'verifiabilty', whose well-definedness, however, appealed to the existence of appropriate 'reduction procedures' in cases where quantification and/or its negation was intended to be interpreted over only all 'numeral' instantiations of the formulas of H:

"In order to find out whether in *Grundlagen der Mathematik* (1934) quantified sentences of types (1)-(4) are indeed regarded to belong to the well-formed sentences of the language of finitist metamathematics, it is useful to take a closer look both at the number-theoretic formalisms presented there and at the corresponding consistency proofs. In §6 (Hilbert and Bernays 1934, 220ff.), Hilbert carries out a consistency proof for a certain weak arithmetical axiom system (cf. 1934, 219) which we call H. The 'proof' is entirely informal, and it is not clear whether Hilbert shows metamathematically 'There is no proof in H for *falsum*' or only for every concretely given proof figure *a* that *a* is no proof for *falsum* in H. The very beginning of the proof speaks in favour of the second option, that is, we conjecture that Hilbert conducts what is in effect an informal version of what in our paper 'Hilbert's finitism and the notion of infinity' (1998) we call an *approximative* consistency proof:[29] 'We now imagine that we are given such a proof figure with the end formula $0 \neq 0$. On this (proof figure) two processes can be effected one after another which we call *dissolution* of the proof figure in "*proof-threads*" and *elimination of the free variables*' (Hilbert and Bernays 1934, 220; cf. 298).

Hilbert and Bernays show, in the first place, that every numerical formula that can be derived from the axioms of H without the use of bound variables is true.[30] In a second step, they demonstrate that every numerical formula provable in H is true even if we drop the restriction concerning the bound variables. They generalize the notion of a true formula in such a way that all formulae of a given proof figure are taken into account, not only the numerical ones (cf. Hilbert and Bernays 1934, 232ff.). This is accomplished by introducing the term 'verifiable'. Confining themselves provisionally to formulae without universal quantifiers, Hilbert and Bernays explain the term as follows: (i) a numerical formula is verifiable, if it is true; (ii) a formula containing one or more free individual variables, but no other variables, is verifiable, if it can be shown that it is true for every replacement of the variables with numerals; and (iii) a formula with bound variables, but without formula variables and without universal quantifiers is verifiable, if the application of a

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[59]Corresponding to Hilbert's 'reduction procedure' quoted in §5.B..

[60]Moreover, as it turns out, in two, essentially different, ways (see §4.C.).

certain reduction procedure leads to a verifiable formula in the sense of (i) or (ii).[31] In a further step, Hilbert and Bernays show that the end formula of the given proof (in H) is verifiable (cf. Hilbert and Bernays 1934, 244ff.). H is therefore consistent.

As to (ii), it is plain that verifiability is defined through an unbounded quantification over numerals, i.e. for all substitution instances. The phrase 'can be shown' remains unexplained and is possibly meant to impart a 'constructive' or finitist air to unbounded universal quantification over numerals. These belong, in the terminology of Hilbert (1926), to the class of ideal statements and are as such unacceptable for the finitist of the 1920s. We further note that carrying out consistency proofs along the lines of (i)-(iii) requires that the verifiability predicate can be formulated in the language of finitist metamathematics. Hence, this language must contain sentences of type (1)."

Fn29 In Niebergall and Schirn 1998, §6 we define this notion as follows (for axiomatizable theories S and T with representation τ): S proves the approximative consistency of T:$\Leftrightarrow \forall n \ S \vdash \neg Proof_\tau (n, \bot)$. We assume here that the formalized proof predicate is the standard one. In our opinion, the notion of an approximative consistency proof captures the core of the conception of finitary metamathematical consistency proofs which Hilbert developed in his papers on proof theory in the 1920s.

Fn30 Numerical formulae are characterized as quantifier-free sentences; see Hilbert and Bernays 1934, 228. Hilbert emphasizes that this is only a stricter version of the assertion that it is impossible to derive $0 \neq 0$ from the axioms of H without admitting bound variables (Hilbert and Bernays 1934, 230)."

. . . *Schirn and Niebergall:* [SN01]*, pp.144-145.*

Now, if we treat Hilbert and Bernays' intent whilst introducing their concept of 'verifiability' as corresponding to the concept of 'algorithmic *verifiability*' introduced in §2., Definition 2 then—despite Schirn and Niebergall's reservations in [SN01]—it can be argued that Hilbert's reasoning does yield a *weak*, constructive, proof of consistency for PA which is essentially that of §2.B.a., Theorem 2.8; even though it fails to yield the *strong*, finitary, proof of consistency for PA (see §2.C.a., Theorem 2.16) which Hilbert sought in the second of his 23 Millenium 1900 Problems ([Hi00]; see also §23.(7)).

Moreover, from such a perspective Hilbert and Bernays' reasoning would be at least as constructive as Gentzen's, essentially set-theoretical, trans-finitary proof ([Me64], p.258) of consistency for a first-order number theory if we admit Gentzen's Rule of Infinite Induction ([Me64], p.259) in a formal system S_∞ in which all theorems of S are provable ([Me64], p.263, Lemma A-3):

Infinite Induction: $\dfrac{\mathcal{A}(\overline{n})\vee\mathcal{D} \ \text{ for all natural numbers n}}{((x)\mathcal{A}(x))\vee\mathcal{D}}$

Comment: From the *evidence-based* perspective of this investigation, Gentzen's, essentially set-theoretical, trans-finitary proof ([Me64], p.258) of consistency for a first-order Arithmetic can claim to be a proof of consistency for the formal first-order system S of Peano Arithmetic defined by Mendelson (in [Me64], pp.102-103) *only* if the Arithmetic of the set-theoretically-defined finite ordinals is treated as a conservative extension of PA. We note that, by §15.A., Corollary 15.3 and, equivalently, Theorem 15.4, this is not the case.

Further, if we were to interpret Infinite Induction as essentially stating that:

Proposition 5.4. (Gentzen's ω-Rule of infinite induction) *If the S_∞-formula $[\mathcal{A}(\overline{n})]$ interprets as true for any specified natural number n, then we may conclude that $[(\forall x)\mathcal{A}(x)]$ is provable in S_∞.*

then it would immediately follow that:

Thesis 3. *Hilbert's ω-Rule of infinite induction is equivalent to Gentzen's ω-Rule of infinite induction.* □

5.C.   Hilbert's ω-Rule is stronger than ω-consistency

Now we note that, in his 1931 paper [Go31], Gödel constructed an arithmetical formula $[R(x)]$[61] in his formal Peano Arithmetic P and showed that, if P is assumed ω-consistent, then both $[(\forall x)R(x)]$ and $[\neg(\forall x)R(x)]$[62] are unprovable in P ([Go31], p.25(1), p.26(2)), even though $[R(n)]$ is provable in P for any specified numeral $[n]$ (whence $[R(n)]$ would be 'true' under any well-defined interpretation of P).

It immediately follows that:

Lemma 5.5. *Assuming that PA admits Hilbert's ω-Rule of infinite induction is stronger than assuming that PA is ω-consistent.*   □

Proof. If PA admits Hilbert's ω-rule of infinite induction, then Gödel's arithmetical formula $[(\forall x)R(x)]$ would be PA-provable. The lemma follows.   □

> **Comment**: We note that the 'constructiveness' of mathematical rigour in *evidence-based* arithmetical reasoning reflected in §5.C., Lemma 5.5 and §5.B., Thesis 3 is in striking contrast to the—debatable to the point of inconsistency, as argued by Vladimir Voevodsky in [Vo10]—'constructiveness' of classically accepted mathematical rigour in set-theoretically-based arithmetical reasoning such as is reflected in [Cho18], §*Implications of Gentzen's Proof*: "*Gentzen's proof certainly meets ordinary standards of mathematical rigor ...*".

Lemma 5.5 can be viewed as justifying Gödel's claim that his argument in [Go31]—from which he concludes the existence of an undecidable arithmetical proposition—is based on the weaker[63] premise that a consistent PA can be ω-consistent.

The question arises whether an even weaker Algorithmic ω-Rule—as defined above[64]—can yield a finitary completion for PA as sought by Hilbert, albeit for an ω-*in*consistent PA.

It is a question that can now be answered in the affirmative, since PA is not only 'algorithmically' complete in the sense of the *weak* Algorithmic ω-Rule (see §2.E., Theorem 2.17), but categorical (see also §5.G.) with respect to algorithmic *computability* (see §2.E., Corollary 2.18)!

5.D.   Aristotle's particularisation entails ω-consistency

In this investigation we argue that these issues are related, and placing them in an appropriate perspective requires any constructive perspective of mathematics to question (see §8.) not only the persisting, *theistic*, belief in classical mathematics that Aristotle's particularisation remains valid even when applied over an infinite domain such as $\mathbb{N}$, but also the basis of Brouwer's unjustifiable, *atheistic*, belief that the Law of the Excluded Middle is non-constructive, following his challenge of the classical belief in [Br08] (see §7.).

> **Comment**: Unjustifiable, since §2.C.a., Theorem 2.16, finitarily establishes that PA is consistent. Hence the underlying first-order logic FOL—in which the Law of the Excluded Middle is a theorem—too is finitarily consistent. Consequently, whereas the sole target of Brouwer's objection to Hilbert's formalisation of quantification—Aristotle's particularisation—implies the Law of the Excluded Middle, the converse is not true.

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[61]Which, in his terminology, Gödel defined, and referred to only indirectly, in [Go31] by its Gödel-number r (see eqn.(12), p.25).

[62]Which Gödel defined, and referred to, in [Go31] only by their Gödel-numbers 17 *Gen r* and *Neg*(17 *Gen r*), respectively (see eqn.(13), p.25).

[63]i.e., weaker than assuming Hilbert's ω-rule, which entails §5., Lemma 5.3.

[64]Which, prima facie, does not imply that a consistent PA is necessarily ω-consistent.

For instance, we note that:

Lemma 5.6. *If PA is consistent but not ω-consistent, then there is some PA formula $[F(x)]$ such that, under any interpretation—say $\mathcal{I}_{PA(N)}$—of PA over $\mathbb{N}$:*

(i) *the PA formula $[\neg(\forall x)F(x)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$;*

(ii) *for any specified numeral $[n]$, the PA formula $[F(n)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$.*

Proof. If PA is consistent then, by definition, a provable PA-formula is true in any *well-defined* interpretation of PA under which the PA-axioms interpret as true, and the PA rules of inference preserve such truth. The lemma follows immediately from the definition of ω-consistency, and from Tarski's standard definitions of the satisfaction, and truth, of the formulas of a formal system such as PA under an algorithmically *verifiable* interpretation (see §2.B.).   □

Further:

Lemma 5.7. *If PA is consistent and the interpretation $\mathcal{I}_{PA(N)}$ admits Aristotle's particularisation over $\mathbb{N}$[65], then:*

(i) *if the PA formula $[\neg(\forall x)F(x)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$,*

(ii) *then there is some unspecified natural number m such that the interpreted arithmetical proposition $F^*(m)$ is algorithmically verifiable as false in $\mathbb{N}$.*

Proof. The lemma too follows immediately from the definition of Aristotle's particularisation and Tarski's standard definitions of the satisfaction, and truth, of the formulas of a formal system such as PA under an algorithmically *verifiable* interpretation (see §2.B.).   □

It follows immediately from §5.D., Lemma 5.7 that:

Corollary 5.8. *If PA is consistent and Aristotle's particularisation holds over $\mathbb{N}$, then there can be no PA formula $[F(x)]$ such that, under any interpretation $\mathcal{I}_{PA(N)}$ of PA over $\mathbb{N}$:*

(i) *the PA formula $[\neg(\forall x)F(x)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$;*

(ii) *for any specified numeral $[n]$, the PA formula $[F(n)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$.*   □

In other words[66]:

Corollary 5.9. *If PA is consistent and Aristotle's particularisation holds over $\mathbb{N}$, then PA is ω-consistent.*   □

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[65]Such as, for instance, any interpretation that defines the existential quantifier as in [Me64], pp.51-52 V(ii).

[66]We note that §5.D., Corollary 5.9 negates Martin Davis' speculation in [Da82], p.129, that such a proof of ω-consistency may be "...open to the objection of *circularity*".

It follows that:

Corollary 5.10. *If Aristotle's particularisation holds over $\mathbb{N}$, then PA is consistent if, and only if, it is ω-consistent.*

Proof. We note first that, by §5.D., Corollary 5.9, if PA is consistent and Aristotle's particularisation holds over $\mathbb{N}$, then PA is ω-consistent.

We note next that if PA is ω-consistent then, since $[n = n]$ is PA-provable for any specified PA numeral $[n]$, we cannot have that $[\neg(\forall x)(x = x)]$ is PA-provable. Since an inconsistent PA proves $[\neg(\forall x)(x = x)]$, an ω-consistent PA cannot be inconsistent. □

It also follows that:

Corollary 5.11. *If PA is consistent but not ω-consistent, then Aristotle's particularisation does not hold in any interpretation of PA over $\mathbb{N}$.* □

Finally since, by §2.F., Corollary 2.22, PA is finitarily consistent but not ω-consistent, it further follows that (compare [An13a], §2, Theorem 1, p.6):

Theorem 5.12. *Aristotle's particularisation does not hold in any finitary interpretation of PA under which the PA-axioms interpret as true, and the PA rules of inference preserve such truth.*
 □

Moreover:

Theorem 5.13. *The first-order logic FOL is finitarily consistent.*

Proof. The theorem follows from the finitary proof that the first-order Peano Arithmetic is consistent (§2.C.a., Theorem 2.16)—whence FOL too is *finitarily* consistent. □

Further since, by definition, $[P \rightarrow P \equiv \neg P \vee P]$:

Corollary 5.14. *The Law of the Excluded Middle $[P \ \vee \neg P]$ is a theorem of the first-order logic FOL.* □

we conclude that:

Corollary 5.15. *The Law of the Excluded Middle does not entail Aristotle's particularisation.*
 □

5.E.   Markov's principle does not hold in PA

We note that an immediate consequence of §5.D., Theorem 5.12 is that Markov's principle does not—contrary to what has been argued by some advocates of intuitionistic logic—hold in PA:

> "Mathematicians of the Russian school accept the following principle: if $[n]$ is a recursive binary sequence (i.e., for each i, $n_i = 0$ or $n_i = 1$), and if we know that not for all i does $n_i = 0$, then we may say that there is an i such that $n_i = 1$. Formally, in terms of a binary number-theoretic function, f:
>
> $$\neg \forall x(f(x) = 0) \rightarrow \exists n(f(n) = 1).$$

> Advocates of intuitionistic logic often find this unpalatable. Existential statements should be harder to prove. But in fact this is the principle that allows one to prove in constructive recursive analysis that every real valued function is continuous at each point in which it is defined. This was first proved by Tseĭtin. Markov himself had proved weaker versions, which are classically but not constructively equivalent."
>
> *. . . Posy: [Pos13], p.112.*

Corollary 5.16. *Markov's principle:* $\neg(\forall x)(f(x) = 0) \rightarrow (\exists n)(f(n) = 1)$, *where* $f(n)$ *is a Boolean number-theoretic function such that* $f(n) = 0$ *or* $f(n) = 1$, *does not hold in PA.*

Proof. Gödel has shown in [Go31] how to construct an arithmetical formula with a single variable—say $[R(x)]$[67]—such that $[(\forall x)R(x)]$ is not PA-provable[68], but $[R(n)]$ is instantiationally PA-provable for any specified PA numeral $[n]$[69].

Hence, for any specified numeral $[n]$, treating Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$ as a Boolean number-theoretical function that takes the value 0 if 'true', and the value 1 if 'false', $xB\lceil[R(n)]\rceil$ must hold for some x (where $\lceil[R(n)]\rceil$ denotes the Gödel-number of the formula $[R(n)]$).

The corollary follows since, by §2.F., Corollary 2.20, the PA formula $[\neg(\forall x)R(x)]$ is also PA-provable. □

5.F. Aristotle's particularisation is 'stronger' than the Axiom of Choice

To appreciate the extent of non-constructivity implicit in the concept, we note that the postulation of an 'unspecified' object in Aristotle's particularisation is 'stronger' than the usual set-theoretical Axiom of Choice.

This follows from Rudolf Carnap's analysis in a 1962 paper on the use of Hilbert's ε-operator in scientific theories ([Ca62], pp.157-158; see also Wang's remarks [Wa63], pp.320-321):

> "What now is the connection between the ε-operator and the axiom of choice? Is the acceptance of the former tantamount to that of the latter? In more formal terms, is the axiom of choice derivable from the other axioms of set theory if the underlying logic contains the ε-operator with its axioms? In some sense, this is the case, but the assertion needs some qualifications. ... The decisive point for this question of derivability is the specific form of the axiom schema of subsets (*Aussonderungsaxiom*). In the customary language L it may be formulated as follows, where "Su" stands for "u is set":
>
> (4) '$Su \supset (\exists y) [Sy \cdot (v)(v \in y \equiv v \in u \cdot \phi)]$' where ϕ is any sentential formula of language L containing 'v' as the only free variable.
>
> If L_ε is taken as the axiomatic language, there is the choice of two versions of the axiom schema, differing in the kinds of formulas admitted as ϕ. The first version is the same as (4): only the formulas of L_ε without 'ε' are admitted; in other words, formulas of L (as a sub-language of L_ε). The second version, which we shall call (4_ε), is formed from (4) by replacing 'L' with 'L_ε'. (4_ε) is

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[67]Gödel refers to the formula $[R(x)]$ only by its Gödel number r ([Go31], p.25, eqn.12). Although Gödel's aim in [Go31] was to show that $[(\forall x)R(x)]$ is not P-provable, it follows that $[R(x)]$ is also, then, not P-provable.

[68]Which corresponds to Gödel's proof in [Go31] that (p.26(2)): $(n)\overline{nB_\kappa(17Gen\ r)}$ holds.

[69]Which corresponds to Gödel's proof in [Go31] that (p.26(2)): $(n)Bew_\kappa\left[Sb\left(r\ {17 \atop Z(n)}\right)\right]$ holds.

stronger than (4). But to accept this version seems natural, once the ε-operator has been accepted as a primitive logical constant.

Consider now the principle of choice:

(5) If x is a set such that:

    (a) any element of x is non-empty,

    (b) any two distinct elements of x are disjoint,

  then there is a set y (called a selection set of x) such that

    (c) $y \subset \bigcup x$,

    (d) for any element z of x, $y \cap z$ has exactly one element.

It can now be seen easily that, if the axiom schema of subsets is taken in the stronger form (4_ε), then (5) is derivable. The derivation is as follows. Let x be any set satisfying the conditions (a) and (b) in (5). According to the axiom of the union set, $\bigcup x$ is a set. Therefore, by (4_ε), there is a set y containing exactly those elements v of $\bigcup x$ for which

$$(\exists z) \; [z \in x \; \cdot \; v = \varepsilon_u (u \in z)],$$

(This last formula is taken as ϕ in (4_ε).) Thus y is a subset of $\bigcup x$ containing just the representative of the elements of x. Hence y satisfies the conditions (c) and (d) in (5). Thus (5) is derived."

$\ldots$ *Carnap: ([Ca62], pp.157-158)*

Now, it follows from Carnap's analysis that, if we define a formal language ZF_ε by replacing:

$$[(\forall x) F(x)] \text{ with } [F(\varepsilon_x (\neg F(x)))]$$

$$[(\exists x) F(x)] \text{ with } [F(\varepsilon_x (F(x)))]$$

in the Zermelo-Fraenkel set theory ZF, then it can be seen that:

Lemma 5.17. *The Axiom of Choice is true in any well-defined interpretation of the Zermelo-Fraenkel set theory ZF_ε that admits Aristotle's particularisation.* □

Lemma 5.18. *The postulation of an 'unspecified' object in Aristotlean particularisation is a stronger postulation than the Axiom of Choice.* □

5.G.   Rosser's Rule C is *stronger* than Gödel's ω-consistency

Clearly the proof that PA is categorical with respect to algorithmic *computability* (§2.E., Corollary 2.18) conflicts immediately with the conventional wisdom that J. Barkley Rosser's proof of undecidability ([Ro36]) successfully avoids the assumption of ω-consistency.

    Comment: It also conflicts with conventional set-theoretical wisdom: (a) that ZF provides a relative proof of consistency for PA (see, for instance, [Cho18]); and (b) that ZF entails the existence of non-standard models of PA that admit elements other than the natural numbers (see §14.).

However, we note that:

Lemma 5.19. *If:*

(i) *from the P-provability of* $[(\exists x)F(x)]$ *we can always conclude the existence within a proof sequence of an unspecified P-term* $[a]$ *such that* $[F(a)]$ *is provable;*

then:

(ii) *we cannot have that a P-formula* $[(\exists x)F(x)]$ *is P-provable and also that* $[\neg F(a)]$ *is P-provable for any specified, constructively well-defined, term* $[a]$ *of P;*

Proof. The lemma follows immediately from its statement.   □

We note that (ii) is Gödel's definition of ω-consistency, which he explicitly assumed when deriving his 'formally undecidable' arithmetical formula (which involves a universal quantifier) in [Go31].

We also note that (i) is essentially Hilbert's definition of existential quantification in his ε-calculus (see §3.C.), which Rosser enunciated as Rule C ([Ro53], pp.127-130), and tacitly assumed (see §13.) as a valid deduction rule of FOL—albeit restricted as an eliminable 'catalyst' to *strictly* within a proof sequence, in the sense that whatever is assumed under Rule C does not appear in the final formula of the sequence—when deriving his 'formally undecidable' arithmetical formula (which involves an existential quantifier) in [Ro36], where he explicitly assumed only that P is simply consistent:

Rosser's Rule C (*Excerpted from Mendelson [Me64], p.73-74, §7, Rule C.*)

"It is very common in mathematics to reason in the following way. Assume that we have proved a wf of the form $(Ex)\mathcal{A}(x)$. Then, we say, let b be an object such that $\mathcal{A}(b)$. We continue the proof, finally arriving at a formula which does not involve the arbitrarily chosen element b. ...

In general, any wf which can be proved using arbitrary acts of choice, can also be proved without such acts of choice. We shall call the rule which permits us to go from $(Ex)\mathcal{A}(x)$ to $\mathcal{A}(b)$, Rule C ("C" for "choice"). More precisely, the definition of a Rule C deduction in a first-order theory K is as follows:

$\Gamma \vdash_c \mathcal{A}$ if and only if there is a sequence of wfs $\mathcal{B}_1, \ldots, \mathcal{B}_n = \mathcal{A}$

such that the following four statements hold.

(I) For each i, either

    (i) $\mathcal{B}_i$ is an axiom of K, or

    (ii) $\mathcal{B}_i$ is in Γ, or

    (iii) $\mathcal{B}_i$ follows by MP or Gen from preceding wfs in the sequence, or

    (iv) There is a preceding wf $(Ex)\mathcal{C}(x)$ and $\mathcal{B}_i$ is $\mathcal{C}(d)$, where d is a new individual constant. (Rule C)

(II) As axioms in (I)(i), we can also use all logical axioms involving the new individual constants already introduced by applications of (I)(iv), Rule C.

(III) No application of Gen is made using a variable which is free in some $(Ex)\mathcal{C}(x)$ to which Rule C has been previously applied.

(IV) $\mathcal{A}$ contains none of the new individual constants introduced in any application of Rule C.

Fn†   *The first formulation of a version of Rule C similar to that given here seems to be due to Rosser ([Ro53], pp.127-130).*"

Comment: We note that, by admitting introduction of an *unspecified* new individual constant d into the formal reasoning, Rule C(I)(iv) implicitly assumes—without proof (see §5.H.a.), and without formally admitting an axiom of choice into K which is equivalent to Hilbert's ε-based choice axiom (see §7.A.)—that such a d can, indeed, be recursively constructed—at least in principle—as a K-term by the first-order construction of terms permitted within K, since any putative K-formula which could define a K-term such as d can denote only algorithmically *computable* constants if K is first-order.

However, Rosser's belief that simple consistency suffices for establishing his 'formally undecidable' arithmetical formula (which involves an existential quantifier) in P is illusory since (compare with §4., Theorem 4.1):

Lemma 5.20. *Rosser's Rule C entails Aristotle's particularisation.*

Proof. If P is simply consistent, the introduction of an *unspecified* P-term into the formal reasoning under Rule C entails, by §4., Definition 12, Aristotle's particularisation in any interpretation of P. □

Corollary 5.21. *Rosser's Rule C is stronger than Gödel's ω-consistency.*

Proof. If P is simply consistent, the introduction of an *unspecified* P-term into the formal reasoning under Rule C entails Aristotle's particularisation in any interpretation of P, which in turn entails that P is ω-consistent (see §5.D., Corollary 5.9). The corollary follows by §5.G., Lemma 5.19. □

Although the *implicit* assumption of ω-consistency—entailed by Rosser's Rule C—is not immediately obvious in Rosser's original proof (see §13.)—nor in Kleene's proof of 'Rosser's form of Gödel's theorem' ([Kl52], Theorem 29, pp.208-209)—it is seen to be implicit in §5.H.a. (i) - (ix), which is an essential step in Mendelson's argument for Proposition 3.32 (Gödel-Rosser Theorem) in [Me64].

We note that, in a relatively recent paper [SS17], Saeed Salehi and Payam Seraji claim that Rosser's Incompleteness Theorem '*does not generally hold for definable non-recursively enumerable theories*', whilst Gödel's First Incompleteness Theorem could hold for some such theory; thus implicitly reflecting (compare Corollary 5.21) that the former is 'stronger' than the latter:

> "Gödel's First Incompleteness Theorem is generalized to definable theories, which are not necessarily recursively enumerable, by using a syntactic-semantic notion (that is the consistency of a theory with the set of all true Π_n sentences or equivalently the Σ_n soundness of the theory) that corresponds to Gödel's notion of ω-consistency in an appropriate way. It is also shown that Rosser's Incompleteness Theorem does not generally hold for definable non-recursively enumerable theories; whence Gödel-Rosser's Incompleteness Theorem is optimal in a sense."
>
> *. . . Saeed and Seraji: [SS17], Abstract.*

5.H.   Mendelson's proof highlights where Rosser's argument presumes ω-consistency

We analyse Mendelson's meticulously detailed expression ([Me64], p.145, Proposition 3.32) of Rosser's argument—since it is more transparent than Kleene's ([Kl52], Theorem 29, pp.208-209)—and highlight where it tacitly presumes that P is ω-consistent.

Now, Gödel defines a formal Peano Arithmetic P, and a primitive recursive relation, $q(x, y)$, that holds if, and only if, x is the Gödel-number of a well-formed P-formula, say $[H(w)]$—which has a single free variable, $[w]$—and y is the Gödel-number of a P-proof of $[H(x)]$.

So, for any natural numbers h, j:

(a) $q(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of $[H(h)]$.

Rosser's argument defines an additional primitive recursive relation, $s(x, y)$, which holds if, and only if, x is the Gödel-number of $[H(w)]$, and y is the Gödel-number of a P-proof of $[\neg H(x)]$.

Hence, for any natural numbers h, j:

(b) $s(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of $[\neg H(h)]$.

Further, it follows from Gödel's Theorems V ([Go31], p.22) and VII ([Go31], p.29) that the primitive recursive relations $q(x, y)$ and $s(x, y)$ are instantiationally equivalent to some arithmetical relations, $Q(x, y)$ and $S(x, y)$, such that, for any natural numbers h, j:

(c) If $q(h, j)$ holds, then $[Q(h, j)]$ is P-provable;

(d) If $\neg q(h, j)$ holds, then $[\neg Q(h, j)]$ is P-provable;

(e) If $s(h, j)$ holds, then $[S(h, j)]$ is P-provable;

(f) If $\neg s(h, j)$ holds, then $[\neg S(h, j)]$ is P-provable;

Now, whilst Gödel defines $[H(w)]$ as:

$$[(\forall y)\neg Q(w, y)],$$

Rosser's argument defines $[H(w)]$ as:

$$[(\forall y)(Q(w, y) \rightarrow (\exists z)(z \leq y \wedge S(w, z)))],$$

Further, whereas Gödel considers the P-provability of the Gödelian proposition,:

$$[(\forall y)\neg Q(h, y)],$$

Rosser's argument considers the P-provability of the proposition:

$$[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))].$$

We note that, by definition:

(i) $q(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of:

$$[(\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z)))];$$

(ii) $s(h, j)$ holds if, and only if, j is the Gödel-number of a P-proof of:

$$[\neg((\forall y)(Q(h, y) \rightarrow (\exists z)(z \leq y \wedge S(h, z))))].$$

5.H.a.   Where Mendelson's proof tacitly assumes ω-consistency

(a) We assume, first, that r is the Gödel-number of some proof sequence in P for the Rosser proposition $[(\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z)))]$.

Hence $q(h,r)$ is true, and $[Q(h,r)]$ is P-provable.

However, we then have that $[Q(h,r) \rightarrow (\exists z)(z \leq r \wedge S(h,z))]$ is P-provable.

Further, by Modus Ponens, we have that $[(\exists z)(z \leq r \wedge S(h,z)))]$ is P-provable.

Now, if P is simply consistent, then $[\neg((\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z))))]$ is not P-provable.

Hence, $s(h,n)$ does not hold for any natural number n, and so $\neg s(h,n)$ holds for every natural number n.

It follows that $[\neg S(h,n)]$ is P-provable for every P-numeral $[n]$.

Hence, $[\neg((\exists z)(z \leq r \wedge S(h,z)))]$ is also P-provable—a contradiction.

Hence, $[(\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z)))]$ is not P-provable if P is simply consistent.

(b) We assume next that r is the Gödel-number of some proof-sequence in P for the proposition $[\neg((\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z))))]$.

Hence $s(h,r)$ holds, and $[S(h,r)]$ is P-provable.

However, if P is simply consistent, $[(\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z)))]$ is not P-provable.

Hence, $\neg q(h,n)$ holds for every natural number n, and $[\neg Q(h,n)]$ is P-provable for all P-numerals $[n]$.

(i) The foregoing implies $[y \leq r \rightarrow \neg Q(h,y)]$ is P-provable, and we consider the following deduction ([Me64], p.146):

   (1) $[r \leq k]$ ... *Hypothesis*
   (2) $[S(h,r)]$ ... *By 3(b)*
   (3) $[r \leq k \wedge S(h,r)]$ ... *From (1), (2)*
   (4) $[(\exists z)(z \leq k \wedge S(h,z))]$ ... *From (3)*

(ii) From (1)-(4), by the Deduction Theorem, we have that $[r \leq k \rightarrow (\exists z)(z \leq k \wedge S(h,z))]$ is provable in P for any P-numeral $[k]$;

(iii) Now, $[k \leq r \vee r \leq k]$ is P-provable for any P-numeral $[k]$;

(iv) Also, $[(k \leq r \rightarrow \neg Q(h,k)) \wedge (r \leq k \rightarrow (\exists z)(z \leq k \wedge S(h,z)))]$ is P-provable for any P-numeral $[k]$.

(v) Hence $[(\neg(k \leq r) \vee \neg Q(h,k)) \wedge (\neg(r \leq k) \vee (\exists z)(z \leq k \wedge S(h,z)))]$ is P-provable for any P-numeral $[k]$.

(vi) Hence $[\neg Q(h,k) \vee (\exists z)(z \leq k \wedge S(h,z))]$ is P-provable for any P-numeral $[k]$.

(vii) Hence $[(Q(h,k) \rightarrow (\exists z)(z \leq k \wedge S(h,z)))]$ is P-provable for any P-numeral $[k]$.

(viii) Now, (vii) contradicts our assumption that $[\neg((\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z))))]$ is P-provable.

(ix) Hence $[\neg((\forall y)(Q(h,y) \rightarrow (\exists z)(z \leq y \wedge S(h,z))))]$ is not P-provable if P is simply consistent.

However, the claimed contradiction in (viii) *only* follows if we assume that P is ω-consistent, and *not* if we assume only that P is simply consistent.

In other words, Mendelson's step (viii) implicitly appeals to Rosser's Rule C, and assumes that the formula $[\neg(\forall y)(Q(h,y)]$ entails the formula $[\neg(Q(h,k)]$ for some *unspecified* term $[k]$ of P—which entails that Aristotle's particularisation holds in any model of P (see §5.G.)—without justifying that such a $[k]$ can, indeed, be *specified* in P without inviting contradiction.

We note that the appeal to Rosser's Rule C in Mendelson's step (viii) is obscured in Kleene's proof of 'Rosser's form of Gödel's theorem' ([Kl52], Theorem 29, pp.208-209), where he appeals to an earlier lemma *169 in order to conclude from $\vdash \forall b[b < \mathbf{k} \supset \neg\mathbf{A}(\mathbf{q},b)]$ that $\vdash \forall b[\neg\mathbf{A}(\mathbf{q},b) \vee \exists c(c \leq b \ \& \ \mathbf{B}(\mathbf{q},c))]$.

That Kleene implicitly appeals to Rosser's Rule C in his proof is seen in his introducory remarks to his Theorem 29, where he explicitly appeals to Aristotle's particularisation (see §4., Definition 12) in his interpretation of the Rosser formula:

> "We have given the original Gödel form of the theorem first, as the proof is intuitively simpler and follows the heuristic outline. Rosser 1936 has shown, however, that by using a slightly more complicated example of an undecidable formula, the hypothesis of ω-consistency can be dispensed with, and the incompleteness proved from the (simple) consistency alone. Consider the formula $\forall b[\neg\mathbf{A}(a,b) \vee \exists c(c \leq b \ \& \ \mathbf{B}(a,c))]$. This has a Gödel number, call it q. Now consider the formula $\mathbf{A}_q(\mathbf{q})$, i.e.
>
> $\mathbf{A}_q(\mathbf{q})$: $\forall b[\neg\mathbf{A}(\mathbf{q},b) \vee \exists c(c \leq b \ \& \ \mathbf{B}(\mathbf{q},c))]$
>
> We can interpret the formula $\mathbf{A}_q(\mathbf{q})$ from our perspective of the Gödel numbering as asserting that to any proof of $\mathbf{A}_q(\mathbf{q})$ there exists a proof of $\neg\mathbf{A}_q(\mathbf{q})$ with an equal or smaller Gödel number, which under the hypothesis of simple consistency implies that $\mathbf{A}_q(\mathbf{q})$ is unprovable."
>
> *. . . Kleene: [Kl52], p.208.*

CHAPTER 6. *MATHEMATICAL CONSEQUENCES*

6.   Hilbert's purported 'sellout' of finitism

We digress here slightly to assess the '*weak*' proof of consistency for PA in §2.B.a., Theorem 2.8, and the '*strong*' proof of consistency for PA in §2.C.a., Theorem 2.16, from the perspective of what Schirn and Niebergall—in their analysis of Hilbert's finitism ([SN01])—term as 'The sellout of finitism' by Hilbert and Bernays, where they note that:

> "In §5.2 of Hilbert and Bernays (1939), entitled 'The formalized metamathematics of the number-theoretic formalism' (cf. 302ff.), the authors introduce a notational variant of PA which they call Z_μ. Its purported drawback for metamathematical purposes rests on the fact 'that in the formalization of finitist reasoning in the system (Z_μ) the characteristic of the finitist argumentation is, for the most part, lost' (1939, 361). Nonetheless, Z_μ is regarded as setting a provisional upper limit for a finitistically acceptable metatheory (Hilbert and Bernays 1939, 353ff., 361ff.).
>
> At the beginning of the section 'Eliminability of the "tertium non datur" for the investigation of the consistency of the system (Z_μ)', Hilbert and Bernays observe that the 'proof-theoretic methods hitherto applied (by them) , even though they partially go beyond the domain of recursive number theory, apparently do not transcend the domain of those concept formations and modes of inference that can still be presented within the formalism Z_μ' (Hilbert and Bernays 1939, 361).[50] On the face of it, this passage suggests that Hilbert and Bernays are here operating with a twofold notion of extending proof-theory or metamathematics: the extension involves both the language of metamathematics and the metamathematical theory itself. Unfortunately, they do not distinguish clearly between these two methods of extending metamathematics; their respective remarks give rise to ambiguity.
>
> Hilbert and Bernays sketch, in the first place, an extension L_{PRA}^+ of L_{PRA} which is supposed to contain only 'finitary' statements. Taking L_{PRA} as the starting point, L_{PRA}^+ is arrived at in two stages: first, symbols for certain computable number-theoretic functions are adjoined to L_{PRA} (call the set of formulae thereby defined L_{PRA}'). Second, L_{PRA}' is converted into L_{PRA}^+ by way of adding to L_{PRA}' only those statements that can be 'interpreted in a strict sense' by a statement of L_{PRA}' (cf. Hilbert and Bernays 1939, 362). Hilbert and Bernays do not explain the phrase 'interpreted in a strict sense', but their ensuing exposition suggests that it is at least formulae of the type '$\forall x \exists y\ \psi(x,y)$' with quantifier-free formula ψ that aare capable of being 'interpreted in a strict sense' in L_{PRA}'. The interpretation can be given by choosing for such a '$\forall x \exists y\ \psi(x,y)$' the quantifier-free formula '$\psi(x, f(x))$' in L_{PRA}', where f is a function-sign for a recursive function which has already been introduced in L_{PRA}'. That these two formulae are equivalent to one another in some sense of 'equivalent' is suggested by the phrase 'strict interpretation', but the authors do not argue for this 'equivalence'.[51]

> Fn50   The authors also argue that the proof-theoretical methods have been extended from PRA to PA without infringing the 'methodic fundamental idea of finitist proof theory' (1939, 362).

> Fn51   Obviously, the conception of the finitistically admissible presented in this example is akin to the position Hilbert and Bernays advocate in 1934, but deviates from Hilbert's finitism in the 1920s. The truly original, austere notion of a finitary statement embodies less than what can be expressed in L_{PRA}^+."

> . . . *Schirn and Niebergall:* [SN01], *p.154.*

6.A.   *Evidence-based* reasoning meets Gödel's criteria for constructivity

What is noteworthy—from the *evidence-based* perspective of [An16] (see §2.)—about the above account is that the search for finitary means of reasoning in the first volume of *Grundlagen*

der Mathematik (1934)—which even then conflicted with Hilbert's enthusiastic espousal of Cantor's set theory, thereby leading to what came to be known as 'Hilbert's Program'—was apparently abandoned around the period of the second volume of *Grundlagen der Mathematik* (1939); influenced in part, perhaps, by developments following Gödel's 1931 incompleteness theorems which seemed to suggest—as Gödel reportedly remarked in his 1938 Zilsel lecture— that "intuitionistic methods went beyond finitist ones" (as Gödel had analysed formally in [Go33]).

In a detailed account of these developments, and their impact on Hilbert's Program, Wilfried Sieg refers to a lecture Gödel delivered in Vienna on 29 January 1938:

> "... to a seminar organized by Edgar Zilsel. The lecture presents an overview of possibilities for continuing Hilbert's program in a revised form. It is an altogether remarkable document: biographically, it provides, together with (1933b) and (1941), significant information on the development of Gödel's foundational views; substantively, it presents a hierarchy of constructive theories that are suitable for giving (relative) consistency proofs of parts of classical mathematics (see §§2-4 of the present note); and, mathematically, it analyzes Gentzen's (1936) proof of the consistency of classical arithmetic in a most striking way (see §7). A surprising general conclusion from the three documents just mentioned is that Gödel in those years was intellectually much closer to the ideas and goals pursued in the Hilbert school than has been generally assumed (or than can be inferred from his own published accounts). ...
>
> The Zilsel lecture gives, as we remarked, an overview of possibilities for a revised Hilbert program. The central element of that program was to prove the consistency of formalized mathematical theories by finitist means. Gödel's 1931 incompleteness theorems have been taken to imply that for theories as strong as first-order arithmetic this is impossible, and indeed, so far as Gödel ventures to interpret Hilbert's finitism, that is Gödel's view in the present text as well as earlier in (1933b) (though not in (1931d)) and later in (1941), (1958) and (1972). The crucial questions then are what extensions of finitist methods will yield consistency proofs, and what epistemological value such proofs will have.
>
> Two developments after (Gödel 1931d) are especially relevant to these questions. The first was the consistency proof for classical first-order arithmetic relative to intuitionistic arithmetic obtained by Gödel (1933d). The proof made clear that intuitionistic methods went beyond finitist ones (cf. footnote 10 below). Some of the issues involved had been discussed in Gödel's lecture (1933b), but also in print, for example in (Bernays 1935b) and (Gentzen 1936). Most important is Bernays's emphasis on the "abstract element" in intuitionistic considerations. The second development was Gentzen's consistency proof for first-order arithmetic using as the additional principle—justified from an intuitionistic standpoint—transfinite induction up to ε_0. Already in (1933b, p. 31) Gödel had speculated about a revised version of Hilbert's program using constructive means that extend the limited finitist ones without being as wide and problematic as the intuitionistic ones:
>
>> But there remains the hope that in future one may find other and more satisfactory methods of construction beyond the limits of the system A [[capturing finitist methods]], which may enable us to found classical arithmetic and analysis upon them. This question promises to be a fruitful field for further investigations.
>
> The Cambridge lecture does not suggest any intermediate methods of construction; by contrast, Gödel presents in the Zilsel lecture two "more satisfactory methods" that provide bases to which not only classical arithmetic but also parts of analysis might be reducible: quantifier-free theories for higher-type functionals and transfinite induction along constructive ordinals. Before looking at these possibilities, we sketch the pertinent features of the Cambridge talk, because they give a very clear view not only of the philosophical and mathematical issues Gödel addresses, but also of the continuity of his development."
>
> *... Sieg: [Si12], Chapter II.4, pp.193-195.*

Comment: Of interest in this context is Vladimir Voevodsky's remark (see also [Cho18]) on what Gentzen's proof of consistency for first-order arithmetic entails from a univalent foundational perspective:

"There is another argument which is often cited as a proof of consistency of first-order arithmetic which has been invented by Gerhard Gentzen (1909-1945).

While Gentzen's reduction argument leads to many very interesting developments it can not be used as a proof of consistency. In relation to the consistency issue the only thing which it shows is that any inconsistency will define a non-terminating decreasing sequence of 'ordinals less that ε_0'."

... *Voevodsky [Vo10].*

The above account also raises the following point of interest from the *evidence-based* perspective of [An16]:

For any integer $n \geq 0$, and integers $x_i \geq 0$, we denote the ordinal $W < \omega^\omega$ by $(x_0, x_1, x_2, x_3, x_4, \ldots, x_n)$, where:

$$W = \omega^n.x_n + \ldots + \omega^4.x_4 + \omega^3.x_3 + \omega^2.x_2 + \omega.x_1 + x_0$$

Define:

$$S_k = \{(x_0, x_1, x_2, x_3, x_4, \ldots, x_n)\} \ni (x_0 + x_1 + x_2 + x_3 + x_4 + \ldots + x_n) = k$$

Then S_k is a finite set of n-tuples for any $k \geq 0$. Hence $\{S_k\}$ is denumerable.

Now we note that $\omega^i \in S_1$ for all $n \geq i \geq 1$, and it is reasonable to assume that some finite initial segment of any denumerable ordering of the ordinals below ω^ω, which does not appeal (non-constructively) to an axiom of choice, must include an ordinal $\omega^i.x_j$ for some $x_j > 0$ corresponding to each $n \geq i \geq 1$.

Query 1. *Can the above argument in §6.A. be extended to ordinals below ϵ_0 by defining higher order ordinals similarly in terms of the ordered n-tuples $(W, W_1, W_2, \ldots, W_n)$, where $W_i = \omega_i^n.x_{i,n} + \ldots + \omega_i^4.x_{i,4} + \omega_i^3.x_{i,3} + \omega_i^2.x_{i,2} + \omega_i.x_{i,1}$, and so on recursively?*

Since transfinite induction can reasonably be considered constructive only if the induction is definable in terms of an *evidence-based* procedure over an *algorithmically verifiable* (even if not *algorithmically computable*) ordering of the ordinals which does not appeal to an axiom of choice, it is not obvious in what sense Gentzen's proof—unlike the *weak* proof of consistency in §2.B.a., Theorem 2.8—can be considered constructive.

Sieg notes that the issue of constructivity was addressed by Gödel earlier in his 1933 'Cambridge' lecture as follows:

Understanding by mathematics "the totality of the methods of proof actually used by mathematicians", Gödel sees the problem of providing a foundation for these methods as falling into two distinct parts (p. 1):

At first these methods of proof have to be reduced to a minimum number of axioms and primitive rules of inference, which have to be stated as precisely as possible, and then secondly a justification in some sense or other has to be sought for these axioms, i.e., a theoretical foundation of the fact that they lead to results agreeing with each other and with empirical facts.

The first part of the problem is solved satisfactorily through type theory and axiomatic set theory, but with respect to the second part Gödel considers the situation to be extremely unsatisfactory. "Our formalism", he contends, "works perfectly well and is perfectly unobjectionable as long as we consider it as a mere game with symbols, but as soon as we come to attach a meaning to our symbols serious difficulties arise" (p. 15). Two aspects of classical mathematical theories (the non-constructive notion of existence and impredicative definitions) are seen as problematic because of a necessary Platonist presupposition "which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent" (p. 19). This analysis conforms with that given in the Hilbert school, for example in (Hilbert and Bernays 1934), (Bernays 1935b) and

(Gentzen 1936). Gödel expresses the belief, again as the members of the Hilbert school did, that the inconsistency of the axioms is most unlikely and that it might be possible "to prove their freedom from contradiction by unobjectionable methods".

. . . Sieg: [Si12], Chapter II.4, pp.195-196.

We note that the *strong* (intuitionistically unobjectionable) finitary proof of consistency for PA in §2.C.a., Theorem 2.16, justifies the optimism Gödel shared in 1933 with Hilbert and Bernays over a positive outcome for Hilbert's Program.

Moreover, §2.C.a., Theorem 2.16, underscores another implicit thesis of this investigation that:

The deterministic infinite procedures (corresponding to Hilbert's 'reduction procedure' quoted in §5.B.) needed to *formalise* the distinction between 'constructive' and 'finitary' reasoning (as illustrated for quantification in §7.A., and generally by §2., Definitions 2 and 3), involve a paradigm shift in recognising that:

- Turing's 1936 paper [Tu36]) admits *evidence-based* reasoning for assigning the values of 'satisfaction' and 'truth' to the formulas of a first-order language such as PA,

- in the sense that one can view the *values* of a simple functional language as specifying *evidence* for propositions in a constructive logic ([Mu91], §1 Introduction; [Lob59], p.165),

- which yields two constructively well-defined, hitherto unsuspected, *complementary* interpretations of PA (as defined in §2.B. and §2.C.)

- under Tarski's inductive definitions of the satisfiability and truth of the PA-formulas under an interpretation.

We note further that, according to Sieg, Gödel's focus in 1933 was already on identifying the minimum requirements that *any* method claiming to prove consistency of a system must satisfy in order to be considered constructive:

"Clearly, the methods whose justification is being sought cannot be used in consistency proofs, and one is led to the consideration of parts of mathematics that are free of such methods. Intuitionistic mathematics is a candidate, but Gödel emphasizes (p. 22) that

the domain of this intuitionistic mathematics is by no means so uniquely determined as it may seem at first sight. For it is certainly true that there are different notions of constructivity and, accordingly, different layers of intuitionistic or constructive mathematics. As we ascend in the series of these layers, we are drawing nearer to ordinary non-constructive mathematics, and at the same time the methods of proof and construction which we admit are becoming less satisfactory and less convincing.

The strictest constructivity requirements are expressed by Gödel (pp. 2325) in a system A that is based "exclusively on the method of complete induction in its definitions as well as in its proofs". That implies that the system A satisfies three general characteristics: (A1) Universal quantification is restricted to "infinite totalities for which we can give a finite procedure for generating all their elements"; (A2) Existential statements (and negations of universal ones) are used only as abbreviations, indicating that a particular (counter-)example has been found without—for brevity's sake—explicitly indicating it; (A3) Only decidable notions and calculable functions can be introduced. As the method of complete induction possesses for Gödel a particularly high degree of evidence, "it would be the most desirable thing if the freedom from contradiction of ordinary non-constructive mathematics could be proved by methods allowable in this system A" (p. 25)."

. . . Sieg: [Si12], Chapter II.4, p.196.

If we apply Gödel's stipulations (A1), (A2) and (A3) to the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA defined in §2.B., and the *strong* finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA defined in §2.C., we note that they can be viewed as broadly meeting Gödel's criteria of constructivity:

(A1) *Universal quantification is restricted to infinite totalities for which we can give a finite procedure for verifying all assertions about their elements.*

The *weak* interpretation of universal quantification under the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA (see §7.D.), as well as the *strong* interpretation of universal quantification under the *strong* finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA (see §7.E.), are both defined constructively in terms of finitely determinate algorithms over the respective domains of quantification;

(A2) *Existential statements (and negations of universal ones) are used only as abbreviations, indicating that a particular (counter-)example has been found without—for brevity's sake— explicitly indicating it.*

Existential quantification in each case is used only as an abbreviation for the negation of universal quantification such that:

(a) The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$, and is *defined* as *verifiably* true in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ relative to its truth assignment T_V if, and only if, it is not the case that, for any *specified* natural number n, we may conclude on the basis of *evidence-based* reasoning that the proposition $\neg F^*(n)$ holds in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$; where the proposition $F^*(n)$ is *postulated* as holding in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ for some *unspecified* natural number n if, and only if, it is not the case that, for any *specified* natural number n, we may conclude on the basis of *evidence-based* reasoning that the proposition $\neg F^*(n)$ holds in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$;

  (i) However, we note that we cannot assume that the satisfaction and truth of quantified formulas of PA are always finitarily decidable—in the sense of being algorithmically *computable*—under the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA over $\mathbb{N}$ (as defined in §23.), since we cannot prove finitarily from *only* Tarski's definitions and the assignment T_V of algorithmically *verifiable* truth values to the atomic formulas of PA under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ whether, or not, a specified quantified PA formula $[(\forall x_i)R]$ is algorithmically *verifiable* as true under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$;

  (ii) Moreover, it is not unreasonable to conclude—in the light of Gödel's stipulation (A2) in the previous quote—that the failure to successfully carry out Hilbert's Program may be attributed to an unawareness of the *evidence-based* distinction between algorithmically *computable* truth and algorithmically *verifiable* truth (see §4.C.(1) and §4.C.(2)).

(b) The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$, and is *defined* as true in $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ relative to its truth assignment T_C if, and only if, we may conclude on the basis of *evidence-based* reasoning that it is not the case, for any *specified* natural number n, that the proposition $\neg F^*(n)$ holds in $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.

We note that $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ is a *strong* finitary interpretation of PA since—when interpreted suitably—all theorems of first-order PA interpret as *finitarily* true in $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ relative to T_C (see §2.C.a., Theorem 2.15).

(A3) *Only decidable notions and calculable functions can be introduced.*

Only decidable notions are used to establish that the PA axiom schema of induction interprets as *verifiably* true under the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA (§2.B.a., Lemma 2.4); and as *computably* true under the *strong* finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA (2.C.a., Lemma 2.12).

To an extent, the above explains in hindsight why, according to Sieg, Gödel's focus shifted from seeking the consistency sought originally by Hilbert's Program to assessing the relative consistency of various systems and proofs:

> "Gödel infers that Hilbert's original program is unattainable from two claims: first, all attempts for finitist consistency proofs actually undertaken in the Hilbert school operate within system A; second, all possible finitist arguments can be carried out in analysis and even classical arithmetic. The latter claim implies jointly with the second incompleteness theorem that finitist consistency proofs cannot be given for arithmetic, let alone analysis. Gödel puts this conclusion here quite strongly: ".... unfortunately the hope of succeeding along these lines [[using only the methods of system A]] has vanished entirely in view of some recently discovered facts" (p. 25). But he points to interesting partial results and states the most far-reaching one, due to (Herbrand 1931) in a beautiful and informative way (p. 26):
>
>> If we take a theory which is constructive in the sense that each existence assertion made in the axioms is covered by a construction, and if we add to this theory the non-constructive notion of existence and all the logical rules concerning it, e.g., the law of excluded middle, we shall never get into any contradiction.
>
> Gödel conjectures that Herbrand's method might be generalized to treat Russell's "ramified type theory", i.e., we assume, the theory obtained from system A by adding ramified type theory instead of classical first-order logic.[9]
>
> There are, however, more extended constructive methods than those formalized in system A; this follows from the observation that system A is too weak to prove the consistency of classical arithmetic together with the fact that the consistency of classical arithmetic can be established relative to intuitionistic arithmetic.[10] The relative consistency proof is made possible by the intuitionistic notion of absurdity, for which "exactly the same propositions hold as do for negation in ordinary mathematics—at least, this is true within the domain of arithmetic" (p. 29). This foundation for classical arithmetic is, however, "of doubtful value": the principles for absurdity and similar notions (as formulated by Heyting) employ operations over all possible proofs, and the totality of all intuitionistic proofs cannot be generated by a finite procedure; thus, these principles violate the constructivity requirement (A1).
>
> Despite his critical attitude towards Hilbert and Brouwer, Gödel dismisses neither in (1933b) when trying to make sense out of Hilberts program in a more general setting, namely, as a challenge to find consistency proofs for systems of "transfinite mathematics" relative to "constructive" theories. And he expresses his belief that epistemologically significant reductions may be obtained.

Fn9 In Konzept, p. 0.1, Godel mentions Herbrands results again and also the conjecture con- cerning ramified type theory. The obstacle for an extension of Herbrands proof is the principle of induction for transfinite statements, i.e., formulae containing quantifiers. Interestingly, as discovered in (Parsons 1970), and independently by Mints (1971) and Takeuti (1975, p. 175), the induction axiom schema for purely existential statements leads to a conservative extension of A, or rather its arithmetic version, primitive recursive arithmetic. How Herbrands central considerations can be extended (by techniques developed in the tradition of Gentzen) to obtain this result is shown in (Sieg 1991).

Fn10 In his introductory note to (1933d), Troelstra (1986, p. 284) mentions relevant work also of Kolmogorov, Gentzen and Bernays. Indeed, as reported in (Gentzen 1936, p. 532), Gentzen and Bernays discovered essentially the same relative consistency proof independently of Godel. According to Bernays (1967, p. 502), the above considerations made the Hilbert school distinguish intuitionistic from finitist methods. Hilbert and Bernays (1934, p. 43) make the distinction without referring to the result discussed here."
... Sieg: [Si12], Chapter II.4, pp.196-197.

We also note that—according to Carl J. Posy's implicitly empathetic account of Hilbert's Program—prior to publication of the second volume of the *Grundlagen der Mathematik* in 1929, Hilbert was yet 'confident in our ability to produce provably adequate formal systems':

Hilbert's Program: Constructivism of the Right

"It might seem strange to call Hilbert a constructivist. After all, he himself introduced nonconstructive methods into algebra, he was unfriendly towards the Kroneckerian restrictions, and—in opposition to Brouwer—he was a staunch supporter of classical logic. Indeed, Hilbert did not practice or condone "constructive mathematics" in the sense that I have been using the term. Nevertheless, he was a constructivist: he saw infinity as a problem for mathematics (or, more precisely, as the source of mathematics' problems), and as a solution he aimed to found mathematics on a base of intuition, just as do all the constructivists we have considered.

Hilbert in fact was driven by an opposing pair of pulls, and his program for the foundation of mathematics was the result of those pulls.

On the one hand, Hilbert held that there is no infinity in physical reality, and none in mathematical reality either. Only intuitable objects truly exist, and only an intuitively grounded process (he spoke of "finitary thought") can keep us within the realm of the intuitable. This is his constructivism. Mathematical paradox arises, he said, when we exceed those bounds. And indeed, he held that infinite mathematical objects do go beyond the bounds of mathematical intuition. For him finite arithmetic gave the basic objects, and he held that arithmetic reasoning together was the paradigm of finitary thought. Together this comprised the "real" part of mathematics. All the rest—set theory, analysis, and the like—he called the "ideal" part, which had no independent "real content".

On the other hand, Hilbert also believed that this ideal mathematics was sacrosant. No part of it was to be jettisoned or even truncated. This is why I dub it "constructivism of the right". "No one will expel us," he famously declared, "from the paradise into which Cantor has led us (Hilbert 1926).

Hilbert's program, which was first announced in 1904 and was further developed in the 1920s, was designed to reconcile these dual pulls.[35] outline of the program for a branch of mathematics whose consistency is in question is generally familiar: axiomatize that branch of mathematics; formalize the axiomatization in an appropriate formal language; show that the resulting formal system is adequate to the given branch of mathematics (i.e., sound and complete); and then prove the formal system to be consistent.

The important assumptions here are that formal systems are finitely graspable things and that the study of formal systems is a securely finitary study. Thus, he is proposing to use the finitary, trustworthy part of mathematics to establish the consistency of the ideal part.

Today, of course, we know that the program as thus formulated cannot succeed. Gödel's theorems tell us that. But in the late 1920s, Hilbert still had ample encouraging evidence. Russell and Whitehead's *Principia Mathematica* stood as a monument to formalization. He and his students successfully had axiomatized and formalized several branches of mathematics. Moreover, he firmly believed that within each branch of mathematics we can prove or refute any relevant statement. He believed that is, optimistically, in the solvability of all mathematical problems. And so he was confident in our ability to produce provably adequate formal systems. And—assuming in advance the success of his program—he was comfortable in developing the abstract, unanchored realms of ideal mathematics.

Fn35   It was announced in Hilbert's lecture "Über die Grundlagen der Logik und der Arithmetik" (published as Hilbert 1905). He developed the Program more fully in the 1920s. Hilbert and Bernays' book *Grundlagen der Mathematik* (1934) contains the most mature statement of the program."

. . . Posy: [Pos13], pp.119-120.

Comment: We note that Posy's conclusion, 'we know the program as thus formulated cannot succeed. Gödel's theorems tell us that', can no longer be treated as definitive in view of the finitary,

evidence-based, proofs of PA-consistency in §2.C.a., Theorem 2.16, and of PA-categoricity in §2.E., Corollary 2.18.

6.A.a.   Shift in Hilbert's focus

In other words, around 1929 Hilbert's focus, and that of mainstream classical meta-mathematics thereafter, apparently shifted from seeking finitary means of reasoning (compare with §1., Thesis 1) to where it has resided ever since: determining the relative proof-theoretic strengths of formal systems, irrespective of whether or not they have any *evidence-based* interpretation that would assure the soundness—and hence the consistency—of the concerned systems.

> **Comment**: We can, not unreasonably, view such a shift as an attempt to justify that a formal system (which we can informally view as corresponding to Carnap's *explicatum* in [Ca62a]; or to Gamez's 'P-description' and 'C-description' in [Gam18], Fig.5.2, p.79) does indeed represent that which (viewed as corresponding informally to Carnap's *explicandum* in [Ca62a]; or to Gamez's 'C-theory' in [Gam18], F, p.79; or to what some cognitive scientists, such as Lakoff and Núñez in [LR00], term as 'conceptual metaphors') it seeks to express formally.

Schirn and Niebergall deplore at length this weakening of Hilbert's finitary resolve[70], which they implicitly seem to also ascribe to efforts by Hilbert and Bernays to contain the perceived negative implications of Gödel's 1931 paper [Go31] on finitism, whilst at the same time unquestioningly accepting the validity of Gödel's conclusions therein; even though such acceptance entailed accepting non-standard integers, such as Cantor's transfinite ordinals 'ω' and 'ε_0' as legitimate objects in 'constructive' reasoning.

> "We observe that in Hilbert and Bernays 1939 the authors pass easily from the determination of what is finitistically formulable to a characterization of what is finitistically provable. We are told that for the formalization of certain general results of proof theory it is *desirable* to obtain as mathematical theorems conditionals containing a universally quantified sentence as antecedent (Hilbert and Bernays 1939, 358, 362). Such sentences are for example (formalizations of) assertions concerning the unprovability or verifiability of formulae or the computability of functions. To illustrate the idea, Hilbert and Bernays sketch a formalization of the informal consistency proof for H in *Grundlagen der Mathematik* (1934), to which we have already referred in §2. The formalization is carried out in PA, and it is shown by means of a complexity analysis that a fragment of PA, though extending PRA, would actually suffice for the consistency proof.. Proof-theoretic means extending PRA, including a form of complete induction which cannot be formalized by the induction schema of recursive number theory (Hilbert and Bernays 1939, 358), are said to be useful or desirable for conducting certain formal consistency proofs.
>
> However this may be, the crucial question for Hilbert and Bernays is whether the so-called finitary methods may go beyond the scope of the modes of inference formalizable in Z_μ. The question is said to lack a precise formulation, on the grounds that 'finitary' has not been introduced as a sharply defined termed, but only as a label for a 'methodic guideline'. It serves merely to recognize certain forms of concept formation and of inference definitely as finitary and certain others definitely as non-finitary. It is not appropriate, though, for drawing an exact dividing line between modes of inference which meet the requirements of the finitist method and modes of inference which do not.[52]

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[70]Not entirely unreasonably, since an unintended consequence of such a shift for the natural sciences—which use formal mathematical structures to express, and communicate, their observations of a commonly accepted reality—might, for instance, be viewed as the arguably unreasonable (according to Sabine Hossenfelder in [Hos18a]; see also §16.D.e.) focus of particle physicists on only the proof-theoretic properties of their theories, irrespective of whether or not such theories have any *evidence-based* interpretation that would assure their soundness in mirroring the external reality they seek to express.

It is in this connection that Hilbert and Bernays mention a typical borderline-case; it concerns the question whether conditionals with a universally quantified sentence as antecedent can be formulated finitistically. They claim to have removed this indeterminacy by distinguishing between sentences and inference rules (Hilbert and Bernays 1939, 358f., 361). Hilbert and Bernays admit, though, that in some cases this distinction may strike us as *forced*, and all this is said to require that the bounds of the finitist framework hitherto established be somewhat loosened, that is, that we go beyond what can be formulated in L_{PRA}^{+} and proved in recursive number theory.

Two comments on these and similar remarks and ideas in Hilbert and Bernays (1939) are in order here. First, what the authors may make clear with them is at best that, compared with Hilbert's finitism of the 1920s, the *language* of finitist metamathematics must be extended; for instance, unbounded quantifications should now be finitistically formulable. Yet Hilbert and Bernays do not even address the issue why in that case all theorems of PA should be sound from a finitist point of view. Moreover, remarks to the extent that it is useful or desirable that the language of metamathematics has a certain expressive power and that the metamathematical theory itself includes a certain repertoire of proof-theoretic means convey nothing about the assumed finitary character of both the metamathematical language and the metamathematical theory under consideration.

Second, Hilbert's and Bernay's remarks presented above suggest that the old *foundational* view dominating the pre-Gödelian period of Hilbertian proof theory has been replaced with a view like this: we are accustomed to certain informal metamathematical considerations, and experience teaches us that they can be formalized in PA. Hence, we are entitled to use them in metamathematical reasoning. Whether Hilbert and Bernays do not care any longer much about questions of finitist justifiability, or whether they leave their readers with a principle of the following kind: *what is not definitely infinitistic may be regarded as finitist*, remains unclear. Deplorably, this is not the only place where Hilbert and Bernayshedge instead of putting their cards on the table. Surely Hilbert, as the founder of the finitist point of view, should feel called upon to give a clear-cut explication of 'finitist' allowing a fair assessment of his programme. So, it could seem that the appeal to the alleged indefinability of 'finitist' is meant to serve as a safeguard against possible objections. This may come out a little clearer in Hilbert's and Bernays's treatment of transfinite induction to which we now turn.

Possibly guided by some principle of the kind just mentioned and the desire to be able to formalize metatheoretical considerations to as high a degree as possible, Hilbert and Bernays arrive at PA (or Z_{μ}, respectively) as a provisional boundary within which a finitist metatheory may be developed (1939, 354, 361). The crucial question for Hilbert and Bernays is now whether the so-called finitary methods may go beyond the scope of the modes of inference formalizable in Z_{μ}. (Remember that, owing to the vagueness of the word 'finitary', they do not consider this question to be formulated in precise terms.) For, as they point out (1939, 353f.), a (formal) metamathematical consistency proof for PA cannot be carried out in PA itself. Nevertheless, Hilbert and Bernays do not rest content with the idea that there can be no finitary proof for PA. Accordingly, they insist that 'in any case, it is possible [. . .] to surpass the modes of inference formalizable in (Z_{μ}) without using the typically non-finitary inferences. And in this way we succeed in giving a very simple consistency proof for the system (Z)' (1939, 362). Hilbert and Bernays refer in this connection to an arithmetical version of transfinite induction.[53] The line of thought which leads them eventually to considering transfinite induction, in particular up to ε_0, as a possibly 'legitimate' method of proof theory deserves close attention."

[. . .]

"At the very end of the last chapter of *Grundlagen der Mathematik* (1939), Hilbert and Bernays make a concluding (but convoluted) remark on Gentzen's (1936) consistency proof, which suggests that it was no longer their serious concern to argue for the finitist nature of the proof-theoretic means applied in consistency proofs for mathematical theories they consider important. We are told that it is a consequence of Gödel's Theorem that

the more comprehensive the formalism to be considered is, the higher are the order types,

> i.e. forms of the generalized induction principle, that must be used. [...] The methodic requirements for the contentual proof of that higher induction principle supply the standard for [determining] which kind of methodic assumptions must be taken as a basis for the contentual attitude, if the consistency proof for the formalism in question is to be successful, (Hilbert and Bernays 1939, 387)

> Fn52 We think that in Hilbert's classical papers the expression 'finitary' is much less vague than in *Grundlagen der Mathematik* (1939). In spite of its vagueness both during the pre-Gödelian and post-Gödelian period of Hilbertian proof theory, it is reasonable to say that it had undergone a thorough shift of meaning by 1939.

> Fn53 Therefore the remark just quoted seems to suggest that PA+TI[ε_0] could be treated as a finitistically admissible theory."

> *...Schirn and Niebergall: [SN01], pp.154-157.*

However, since:

(i) Schirn and Niebergall observe that, regarding the consistency of PA, 'Hilbert and Bernays do not rest content with the idea that there can be no finitary proof for PA'; and

(ii) Hilbert's and Bernays' 'informal' proof of the consistency of arithmetic in the *Grundlagen der Mathematik*—as analysed in [SN01] (see §5.B.)—can be viewed as essentially outlining a proof of §2.B.a., Theorem 2.8;

a more appropriate perspective may be that Hilbert's weakened finitism in 1939 reflected, as we noted earlier, the circumstance that the deterministic infinite procedures (corresponding to Hilbert's 'reduction procedure' quoted in §5.B.) needed to *formalise* the distinction between 'constructive' and 'finitary' reasoning (as illustrated for quantification in §7.A.; and generally by §2., Definitions 2 and 3) were already intuited by Hilbert, even though they become explicit only *after* the realisation that Turing's 1936 paper [Tu36]) admits *evidence-based* reasoning—in the sense that one can view the *values* of a simple functional language as specifying *evidence* for quantified propositions in a constructive logic in two, essentially different, ways (see §4.C.).

CHAPTER 7. *MATHEMATICAL CONSEQUENCES*

7.   Hilbert's and Brouwer's interpretations of quantification

It is not entirely unreasonable (as argued in [An15a], [An15b]) to view Hilbert's evolving finitism as a search for a definitive response to Brouwer's objection to his classical interpretation of quantification; which Hilbert formalised in [Hi27] by defining a formal ε-calculus L_ε where he sought to capture the essence:

— of Aristotle's *unspecified* x in Definition 12,

— as an *unspecified* term $[\varepsilon_x(F(x))]$.

Hilbert then defined:

1. $[(\forall x)F(x) \leftrightarrow F(\varepsilon_x(\neg F(x)))]$

2. $[(\exists x)F(x) \leftrightarrow F(\varepsilon_x(F(x)))]$

and showed that Aristotle's logic is a well-defined interpretation of L_ε:

— *if* $[\varepsilon_x(F(x))]$ can be interpreted as some, *unspecified*, x satisfying $F(x)$.

7.A.   Hilbert's interpretation of quantification

Formally, Hilbert interpreted quantification in terms of his ε-function as follows:

"IV. The logical ε-axiom

    13. $A(a) \rightarrow A(\varepsilon(A))$

Here $\varepsilon(A)$ stands for an object of which the proposition $A(a)$ certainly holds if it holds of any object at all; let us call ε the logical ε-function.

    1. By means of ε, "all" and "there exists" can be defined, namely, as follows:

        (i) $(\forall a)A(a) \leftrightarrow A(\varepsilon(\neg A))$

        (ii) $(\exists a)A(a) \leftrightarrow A(\varepsilon(A)) \ldots$

    On the basis of this definition the ε-axiom IV(13) yields the logical relations that hold for the universal and the existential quantifier, such as:

        $(\forall a)A(a) \rightarrow A(b) \ldots$(Aristotle's dictum),

    and:

        $\neg((\forall a)A(a)) \rightarrow (\exists a)(\neg A(a)) \ldots$(principle of excluded middle)."

    . . . Hilbert: [Hi27].

Thus, Hilbert's interpretation (i) of universal quantification—under any objective (i.e., *evidence-based*) method T_H of assigning truth values to the sentences of a formal logic L—is that the sentence $(\forall x)F(x)$ can be defined as holding (presumably under a well-defined interpretation $\boldsymbol{H}$ of L with respect to T_H) if, and only if, $F(a)$ holds whenever $\neg F(a)$ holds for some *unspecified* a (under $\boldsymbol{H}$); which would imply that $\neg F(a)$ does not hold for any specified a (since $\boldsymbol{H}$ is well-defined), and so $F(a)$ holds for any specified a (under $\boldsymbol{H}$).

Further, Hilbert's interpretation (ii) of existential quantification, with respect to T_H, postulates that $(\exists x)F(x)$ holds (under $\boldsymbol{H}$) if, and only if, $F(a)$ holds for some *unspecified* a (under $\boldsymbol{H}$).

> **Comment**: The consequent—and continuing—influence of Hilbert's interpretation of quantification on mathematics and philosophy is illustrated by Hilary Putnam's 1971 remark that:
>
> > "Quantification over mathematical entities is indispensable for science—but this commits us to—the [independent] existence of the mathematical entities [that satisfy our theories]. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes."
> >
> > . . . Putnam: [Pu71], p.347.

We can express this formally as (compare with §5.D., Theorem 5.12):

Lemma 7.1. *Aristotle's particularisation holds under every well-defined interpretation of Hilbert's ε-calculus L_ε.* □

We also have further that (compare with §5.D., Corollary 5.14):

Lemma 7.2. *The Law of the Excluded Middle (LEM) holds under every well-defined interpretation of Hilbert's ε-calculus L_ε.*

Proof The lemma follows since the principle of excluded middle is a theorem of L_ε and entails LEM. □

7.B. Brouwer's objection

Brouwer's objection to such an *unspecified* and 'postulated' interpretation of quantification was that, for an interpretation to be considered *constructively well-defined* relative to T_H when the domain of the quantifiers under an interpretation is infinite, the decidability of the quantification under the interpretation must be constructively verifiable in some intuitively, and mathematically acceptable, sense of the term 'constructive' ([Br08]).

In other words (as highlighted by the semantic and logical paradoxes analysed in §16.), any assumption of an *unspecified* object under an interpretation of a formal language L is valid if, and only if, such an element is shown to be *specifiable* as a term in L by the rules for the formation of L-terms.

Two questions arise:

(a) Is Brouwer's objection relevant today?

(b) If so, can we interpret quantification finitarily?

7.C.   Is the PA-formula $[(\forall x)F(x)]$ to be interpreted *weakly* or *strongly*?

The perspective we choose for addressing these issues is that of the structure $\mathbb{N}$, defined by:

1. $\mathbb{N}$ (*the set of natural numbers*);

2. $=$ (*equality*);

3. S (*the successor function*);

4. $+$ (*the addition function*);

5. $*$ (*the product function*);

6. 0 (*the null element*)

which serves for a definition (see §23.(16)) of the, classical, standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of the first-order Peano Arithmetic PA.

However, if we are to avoid intuitionistic objections to the admitting of *unspecified* natural numbers in the definition of quantification under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, we are faced with the ambiguity where *if*:

— $[(\forall x)F(x)]$ and $[(\exists x)F(x)]$ denote PA-formulas; and

— The relation $F^*(x)$ denotes the interpretation in the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of the PA-formula $[F(x)]$ under an inductive assignment of Tarskian truth values T_{SV}; where

— The underlying first-order logic FOL of PA admits *evidence-based* interpretation (in the sense of §4.C.);

then the question arises[71]:

(a) Is the PA-formula $[(\forall x)F(x)]$ to be interpreted *weakly* as:

- '*For any specified n, $F^*(n)$*',
    - which holds if, and only if,
    - for *any specified n* in $\mathbb{N}$,
    - there is *algorithmic evidence* that $F^*(n)$ holds in $\mathbb{N}$,

    and the PA-formula $[\neg(\forall x)F(x)]$ interpreted *weakly* as:

    - *There is no algorithm which will evidence that the PA-formula $[(\forall x)F(x)]$ can be interpreted weakly[72];*

---

[71]We reiterate that the distinction between 'any' and 'all' made below pertains to the assignment of truth-values to the formulas of a formal theory under an interpretation of the universal quantifier over a *well-defined* domain of interpretation. It is to be distinguished from the distinction Russell makes in [Rus08] (pp.156-163) between 'all' and 'any' with respect to the assignment of provability-values to the formulas of a formal theory that admits 'apparent' and 'real' variables in a proof sequence. See also [Fe02], pp.3-4.

[72]Which implies *only* that $[(\forall x)F(x)]$ is not provable in PA; it *does not* entail that $F^*(x)$ is not algorithmically *verifiable*.

or:

(b) is the formula $[(\forall x)F(x)]$ to be interpreted *strongly* as:

- '*For all n, $F^*(n)$*',

  — which holds if, and only if,

  — there is *algorithmic evidence* that,

  — for *any specified n* in $\mathbb{N}$,

  — $F^*(n)$ holds in $\mathbb{N}$?

and the PA-formula $[\neg(\forall x)F(x)]$ interpreted *strongly* as:

- *There is no algorithm which will evidence that the PA-formula $[(\forall x)F(x)]$ can be interpreted strongly*[73];

where:

Definition 22. (Integer specifiability) *A natural number n in $\mathbb{N}$ is specifiable if, and only if, it can be explicitly denoted as a PA-numeral by a PA-formula that interprets as an algorithmically computable constant.*

7.C.a. The Church-Turing Thesis entails Aristotle's particularisation

The significance of §7.C., Definition 22 for *evidence-based* reasoning (which admits *only* §4.C.) is that (compare with the conclusions in §5. and §5.D.):

Theorem 7.3. *The Church-Turing Thesis entails Aristotle's particularisation.*   □

Proof. If we accept the Church-Turing Thesis (§23.(2)), then admitting a natural number as *unspecified* in $\mathbb{N}$ (as in §4., Definition 12), implies that, by §7.C., Definition 22, it is *specifiable* in PA and, ipso facto, *specified* under any well-defined interpretation of PA.   □

The significance of Theorem 7.3 is, further, that it emphasises the extent to which *faith-based* reasoning—which appeals *unrestrictedly* to Aristotle's particularisation (see §4., Definition 12; also §4.B.)—is implicit even in argumentation that—contradicting §4.G.a., Theorem 4.3 (CT falsifiable)—seeks to 'prove' the Church-Turing Thesis, or equivalent theses, from finitary, or constructive, premises.

For instance, in her article [Du14], Marie Duží attempts 'to define the notion of *algorithm/effective procedure*'; in this case by applying 'a *procedural theory of concepts*' which was 'formulated by Materna using Transparent Intensional Logic (TIL) as a background theory', to support her hypothesis that, consequently, 'the *Church-Turing thesis* becomes *provable*' in such a context:

> "We considered four ways of construing the notion of computability:
>
> 1) *EP*—analytical concept of effective procedure, algorithm

---

[73]Which, too, implies that $[(\forall x)F(x)]$ is not provable in PA. By §2.E., Theorem 2.17 (*Provability Theorem for PA*), this *does*, however, entail that $F^*(x)$ is not algorithmically *computable*.

2) *TM*—Turing machine, *GR*—general recursivity, λD—lambda definability

3) *MN*—machine-computable in the narrow sense (for instance with laws of physics imposing limitations on the machine)

   MW—machine-computable in the wide sense (for instance involving infinitely small times ...)

4) *O*-machines with an oracle.

The Church-Turing thesis claims the equivalence of (1) and (2). Thus the Church-Turing thesis proposes three kinds of a refinement of the concept of effective procedure/algorithm.

At this point we can formulate a *hypothesis*: if the concept of an effective procedure (algorithm) is sufficiently refined and delimited, for instance, as proposed above by our *refined definition*, then the *Church-Turing thesis* becomes *provable*."

...*Duží: [Du14], §6. Summary and concluding remarks.*

In an earlier paper [DG08] (subsequently qualified, and clarified, by Gurevich in [Gu19]), Nachum Dershowitz and Yuri Gurevich too argue that Church's Thesis provably follows from—in their case—four, seemingly 'undeniably' computable, postulates:

"The first issue that needs to be addressed when axiomatizing effective computation is: What kind of object is a "computation"? Once we agree that it is some sort of state transition system (Postulate I in what follows), we need to formalize the appropriate notions of "state" and of "transition". To model states, we take the most generic of mathematical objects, namely, logical structures (Postulate II). To ensure that each transition step is effective, we require only that it not entail an unbounded amount of exploration of the current state (Postulate III). Finally, we need to make sure that a computation does not start out with any magical abilities (Postulate IV). We will demonstrate that under these very natural and general hypotheses regarding algorithmic activity, which certainly suffice for the computation of all recursive functions, the recursiveness of the computed function is in fact guaranteed.

More precisely, but still informally, the postulates say the following about algorithms:

    I. *An algorithm determines a sequence of "computational" states for each valid input.*

   II. *The states of a computational sequence are structures. And everything is invariant under isomorphism.*

 III. *The transitions from state to state in computational sequences are governable by some fixed, finite description.*

 IV. *Only undeniably computable operations are available in initial states.*

Postulates IIII are called the "Sequential Postulates" [42]. They axiomatize (deterministic, sequential) algorithms in general, not only those for computable functions; they apply equally to algorithms dealing with complex numbers, say, as to those for integers only. Postulate IV, which will be fleshed out later, ensures that an algorithm is not endowed from the outset with uncomputable oracles, such as infinite precision operations on real numbers, or a solvability decider for Diophantine equations. We will show in this paper that Church's Thesis provably follows from these four postulates."

...*Dershowitz/Gurevich: [DG08], §1.3. Sketch of axioms, p.306.*

The implicit non-constructivity imputed to the above argumentations in [Du14] and [DG08]—as being entailed by Theorem 7.3—was sought, in the case of [DG08], to be identified by Doukas Kapantaïs in [Kap16]; where, in his refutation of the Church-Turing Thesis (compare §4.G.a., Theorem 4.3), he argues against Dershowitz and Gurevich's 'Postulates' (in [DG08]) by proposing 'an effective computation that cannot be translated *salva* isomorphism into a computation of a machine with this abstract structure':

"I now present three different interpretations of the Church-Turing thesis and single out the one I believe that my paper refutes.

In Church's (1936) initial formulation, the Thesis consists in the claim that the class of effectively calculable numeric functions is identical to the class of recursive numeric functions, which is also identical to the class of λ-definable numeric functions. Turing (1936) proved that the latter two are identical to the class of Turing Machine computable numeric functions, and, like Church, he assumed that they all coincide with the effectively calculable numeric functions (although, unlike Church, he did not confine the domains of functions to numeric ones). Among the notions appearing in all these identity statements the only one that is not formal is the notion of "effectively calculable numeric function".

Several alternative ways of interpreting the Thesis have been suggested. I will now classify them under three major categories/interpretations.

INT1. The Church-Turing thesis is in reality a definition. It has been proved that there is an idealized calculator with huge computational powers (i.e. the Turing Machine) and a family of equipotent machines[8] and formalisms. We should all agree to name the functions these machines and formalisms can compute "effectively calculable functions".

INT2. The Church-Turing thesis is in fact a conjecture. After the discovery of this idealized calculator with these huge computational powers, and the discovery of a family of other equipotent machines and formalisms, the conviction grew that these machines/formalisms really exhaust the computational powers in general. Now, if they really do so, anything that can be effectively calculated can be computed by a Turing Machine.

INT3. The Church-Turing thesis is the conjecture according to which not only anything that can be calculated can be computed by a Turing Machine, but also any formalism or machine that is equipotent to a Turing Machine is equivalent to it up to isomorphism. It comes down to the conjecture that all these maximal computational systems and machines share the same abstract structure, and so they do not only have the same computational power but, which is more, they compute in the same abstract way.

Were one to put INT1 to INT3 into slogans, INT1 would be: "Turing Machine computable functions are called 'effectively calculable' ", INT2: "No numeric function that can be calculated cannot be computed by a Turing Machine", and INT3: "All maximal models of computation share the same abstract structure with Turing Machines.

In what follows, I will be referring to the Church-Turing thesis as the thesis behind INT3, an interpretation well attested in the literature ...

INT3 goes beyond INT2 in the following two respects. First, it tries to provide an explanation of the otherwise mysterious co-extensiveness of this family of formalisms and machines. For to say that they are all equipotent, because they exhaust the limits of computation, is not an explanation of why they exhaust the limits of computation; it's just some further evidence that they do so. On the other hand, bringing forward what they themselves share in common (i.e. a common structure) is much more ambitious. For, after having brought forward what these formalisms/machines share in common, one could further suppose that any formalism/machine that exhausts the limits of computation shares this element in common too. Notice here that, if this last supposition turns out to be correct, one would also have a formal proof of the Thesis. For consider it this way. *Prima facie*, the Church-Turing thesis cannot be formally proved, since it claims that a non-formal item (i.e. "effectively calculable") is identical with a formal one (i.e. "Turing Machine computable"). Now, there can be no formal proof of any identity statement relating a formal and a non-formal item. The only identity statements that can be formally proved are statements relating items within a formal language of a theory. So, suppose that some explanation has been provided as for why these formalisms/machines are equipotent. If this explanation consists in the finding of yet another formal item, i.e. their common structure, and you further assume that any formalism/machine that exhausts the limits of computation must be characterized by this item

too, then, what you are actually doing is proposing a formal interpretation for the "effectively calculable". I.e. you do not only prove that all these formalisms/machines share a formal element in common, you further propose that this element is shared by all formalisms and machines that exhaust the limits of computation. So, you can now formally define "effectively calculable" through this.

An enterprise of this sort has been undertaken in Dershowitz & Gurevich (2008).[11] What these authors did is the following. On the one hand, they have proposed a specific axiomatization as the formal counterpart of "effectively calculable".[12] On the other, they have proved that all the formalisms/machines of the second part of the equivalence are interpretations of this axiomatization. So, in case this axiomatization really captures the informal notion of "effectively calculable", then, their proof must also be a formal proof of the Thesis. Additionally, they have informally argued that this axiomatization must indeed be capturing the informal notion of "effectively"calculable", for, in order not to be capturing it, one would need to be able to imagine a computational method that is both effective and, at the same time, falsifies at least one among the four Postulates of their system.[13] So, now, the burden of (dis)proof is on the opponent, who must either argue against these Postulates directly, or come forward with an effective computation that cannot be translated *salva* isomorphism into a computation of a machine with this abstract structure.

In what follows, I will do the latter by indicating a certain way of mechanically calculating the original Ackermann function that we, humans, can perform and that cannot be mimicked by any Turing Machine. If this is exact, the isomorphism between us, as calculators, and these formalisms/machines fails. More precisely, I will claim that there are some updates in this particular way of computing the original Ackermann function that have no isomorphic counterparts in any computation of the same function as performed by these machines. This implies not only that the set of Postulates of Dershowitz and Gurevich need to be loosened in order to be able to capture "effectively calculable", but also that INT3 is false, since the same Postulates provably capture the abstract structure behind Turing Machines and equipotent machines and formalisms.[14]"

. . . Kapantaïs: [Kap16], §3 What the Church-Turing thesis says. Quotation accessed Apr 18 2020 from ResearchGate.

Intriguingly—from the *evidence-based* perspective of this investigation—Kapantaïs argues in [Kap16] from an unusual anthropomorphic image[74] of a *function* 'as a person to whom you give some items and who, then, returns some items back':

What is a "function"? In essence, what a function is comes down to this: Something, an item, the so-called "argument" of the function, is substituted by another (not necessarily different) thing, item. What performs this substitution is the function itself. The "not necessarily different" clause above is suggestive and not contradictory with respect to the verb "is substituted" it depends upon. For the important thing, as far as functions are concerned, is only this: they take some arguments as their input and they return some values as their output. So, the function is an operation that operates on items (trivially, on what else could an operation operate upon?) and produces items. The set of items, from which the function takes the items it operates upon, is called the "domain of arguments", and the set of items, from which the function takes the items that result from the operation, is called "the domain of values". It is not compulsory that the operation is fruitful for all arguments. Some functions produce no result, when given some argument values. These functions are called "partial". The rest are called "total".

You can easily—and safely, from the formal point of view—make an anthropomorphic model of what a function is. You can imagine it as a person to whom you give some items and who, then, returns some items back.

So far we've only said that functions operate on items and return items. A natural question is what kind of items these items might be. The straightforward answer is: "any kind". Provided

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[74]Functional and, seemingly, intentionally non set-theoretical to avoid any implicit appeal to non-constructive concepts as, we have imputed above, is the case in Dershowitz and Gurevich's argumentation in [DG08].

they satisfy suitable identity criteria, people, atoms, streets, sets, numbers, functions, intensions, feelings—you name it!—might serve as the arguments or values of functions.

The way we have presented things thus far suggests that we consider functions to be items in their own right, and so, since no item can be an argument/value of a function, unless it satisfies appropriate identity criteria, we now ask: What kind of criteria are these with respect to functions? The criteria we will employ are these: two functions will be identical, if and only if they return the same values for same arguments. This is called an "extensional" criterion, since it ignores the way the function arrives at its values. The only thing that matters is that, on being given these specific arguments, it provides these specific values.

Seen thus, functions exist "out there", along with trees, human beings, sets, numbers and any other item that belongs to our world. Upon the same extensional criterion, functions can ultimately be reduced to sets of ordered pairs.

Functions from natural numbers to natural numbers are called "numeric functions"."

. . . Kapantaïs: [Kap16], §1 What a function is. Quotation accessed Apr 18 2020 from ResearchGate.

What is intriguing about Kapantaïs' unusual anthropomorphic image of a function 'as a person to whom you give some items and who, then, returns some items back', is that if we, instead, treated the latter as the image of an algorithmically *verifiable* function ([An16], Definition 1, p.37; see also §2., Definition 2), then the definition of 'effective computability' (as defined by §4.F., Definition 18, and §4.G.a., Definition 21) corresponds faithfully to Kapantaïs' meaning of the term, 'the idea being that a function that cannot be effectively calculated/computed is not calculable/computable at all'.

Moreover, Kapantaïs' 'conclusion that the human brain/mind is not equivalent up to isomorphism to Turing Machines' is essentially the validation of Lucas' Gödelian argument as detailed in the concluding thesis of [An16] (Thesis 1, p.42; see also §2.F.a.; §17.D., Theorem 17.1, and §17.E., Query 17):

"What the above proof establishes is that the Machine cannot mimic the way the human calculator finds An in the following essential respect. The human calculator launches a program, implementing an algorithm such that it builds recursively the function An by n consecutive updates A0 that involve no 'do until' commands. Had the operation of the Machine and of the human calculator been isomorphic, there should be a translation function such that it takes each element of the stages the human calculator goes through, while computing An, to the stages the Machine goes through,while computing f(n), and this translation would have left the abstract algorithm they both implement, by their distinct programs, intact. Suppose that there is such a translation function τ. Obviously, $\tau(0)=[0]$, $\tau(1)=[1]$, ... Say that "u" denotes the update function upon states during a calculation. What is missing for the isomorphism to be preserved is $\tau(u(Ax))$. For example, u(Ax) is Ax+1, but u([Ax]) is not [Ax+1]. There is simply no way for the Machine to update [Ax] in a way that preserves isomorphism.[30] That is to say that, there can be no translation *of theway* the human calculator updates states along her way to An to a similar way that the Machine updates states along its way to [An]. The particular details as for how exactly the Machine arrives at [An] are of no importance. The "dumbest" but still effective way would be to check all numbers from 0 onwards to see whether they are [An] or not.If f(x)=y is decidable, the Machine will eventually stop at [An]. One can imagine several interesting shortcuts, but, still, there would be no program available to the Machine in order to calculate an upper bound for (or to calculate the exact number of) the updates it needs for reaching [An], and, so, the way it updates its states must be structurally different.[31]

Notice that this is stronger than saying that the Turing Machine as a calculator and the human being as a calculator differ. After all, a one dimensional Turing Machine also differs from a two dimensional Turing Machine, and both differ from a Post Machine. However, these machines are interpretations of the same abstract structure, and, so, their calculations can be translated

into one another. Strings are strings, graphs are graphs, sets are sets, but this variety reflects no structural difference.

On the other hand, the difference between the idealized human calculator and the Turing Machine is structural, and, so:

  (i) Not all maximal models of mechanical computation are equivalent up to isomorphism to Turing Machines (This is a refutation of the Church-Turing thesis in the form of INT3).

And:

  (ii) The human brain/mind is not equivalent up to isomorphism to Turing Machines. (Notice that the counterexample to the Turing Machine was an algorithm implemented by the idealized human calculator.)"

. . . *Kapantaïs: [Kap16], §5 Conclusions. Quotation accessed Apr 18 2020 from ResearchGate.*

7.D.   The standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA interprets $[(\forall x)F(x)]$ *weakly*

Keeping the distinction between §7.C.(a) and §7.C.(b) in mind, it would seem that classically, under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA:

(1a) The formula $[(\forall x)F(x)]$ is *defined* as true in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ relative to an assignment T_{SV} of truth values under interpretation over $\mathbb{N}$ if, and only if, for any *specified* natural number n, we may conclude on the basis of *evidence-based* reasoning that the proposition $F^*(n)$ holds in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$;

(1b) The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$, and is *defined* as true in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ relative to T_{SV} if, and only if, we may conclude on the basis of *evidence-based* reasoning that it is not the case there is an algorithm which will evidence the formula $[(\forall x)F(x)]$ as true in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ relative to T_M;

(1c) The proposition $F^*(n)$ is *postulated* as holding in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ for some *unspecified* natural number n if, and only if, it is not the case that, for any *specified* natural number n, we may conclude on the basis of *evidence-based* reasoning that the proposition $\neg F^*(n)$ holds in $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

If we *assume* that Aristotle's particularisation holds under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA (as defined in §23.(16)), then (1a), (1b) and (1c) together interpret $[(\forall x)F(x)]$ and $[(\exists x)F(x)]$ under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ *weakly*, as seems implicitly intended by Hilbert's ε-function; whence they attract Brouwer's objection.

This would, then, answer question §7.B.(a).

7.E.   A finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA which interprets $[(\forall x)F(x)]$ *strongly*

Now, our thesis is that the implicit target of Brouwer's objection[75] is the unqualified semantic *postulation* of Aristotle's particularisation entailed by §7.D.(1c), which appeals to Platonically non-constructive, rather than intuitively constructive, plausibility.

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[75]And perhaps of parallel objections perceived generically as "*Limitations of first-order logic*"; see [AR02b], p.78, §2.1.

We note that this conclusion about Brouwer's essential objection apparently differs from conventional intuitionistic wisdom (i.e., perspectives based essentially on Brouwer's explicitly stated objection to the Law of the Excluded Middle (LEM) as expressed in [Br23], p.335-336):

— which would presumably deny appeal to §7.D.(1c) in an interpretation of FOL by denying that the FOL theorem $[P \ v \ \neg P]$ (*Law of the Excluded Middle*) is *finitary*;

— even though denying appeal to §7.D.(1c) in an interpretation of FOL does *not* entail denying the FOL theorem $[P \ v \ \neg P]$ (a consequence of §5.D., Corollary 5.15).

We can thus re-phrase question §7.B.(b) more specifically:

• Can we define an interpretation of PA over $\mathbb{N}$ without appealing to §7.D.(1c)?

We note that we *can*, indeed, define another—hitherto unsuspected—*evidence-based* interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA under an inductive assignment of Tarskian truth values T_{SC} over the structure $\mathbb{N}$, where:

(2a) The formula $[(\forall x)F(x)]$ is *defined* as true in $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ relative to T_{SC} if, and only if, we may conclude on the basis of *evidence-based* reasoning that there is an algorithm which, for any *specified* natural number n, will evidence that the proposition $F^*(n)$ holds in $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$;

(2b) The formula $[(\exists x)F(x)]$ is an abbreviation of $[\neg(\forall x)\neg F(x)]$, and is *defined* as true in $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ relative to T_{SC} if, and only if, we may conclude on the basis of *evidence-based* reasoning that it is not the case there is an algorithm which will evidence the formula $[(\forall x)F(x)]$ as true in $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ relative to T_{SC}.

We note that $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ is a *strong* finitary interpretation of PA since—when interpreted suitably—all theorems of first-order PA interpret as *finitarily* true in $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ relative to T_C (an immediate consequence of §2.C.a., Theorem 2.16).

This answers question §7.B.(b).

7.F.   Are *both* interpretations $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ and $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA over $\mathbb{N}$ *well-defined*?

The question arises:

Query 2. *Are both the interpretations $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ and $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA over the structure $\mathbb{N}$ well-defined, in the sense that the PA axioms interpret as true, and the rules of inference preserve truth, relative to each of the assignments of truth values T_V and T_C respectively?*

We note that [An16] answers the question affirmatively by showing that the two interpretations $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ and $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA over the structure $\mathbb{N}$ can be viewed as complementary, since (see §2.A.) Tarski's classic definitions permit an intelligence—whether human or mechanistic—to admit finitary, *evidence-based*, inductive definitions of the satisfaction and truth of the *atomic* formulas of the first-order Peano Arithmetic PA, over the domain $\mathbb{N}$ of the natural numbers, in two, hitherto unsuspected and essentially different, ways:

(1) in terms of *weak* algorithmic *verifiabilty*; and

(2) in terms of *strong* algorithmic *computability.*

However, we note that, from the PA-provability of $[\neg(\forall x)F(x)]$, we may *only* conclude under the finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, on the basis of *evidence-based* reasoning, that it is not the case $[F(n)]$ interprets as always true in $\mathbb{N}$.

We may *not* conclude further, in the absence of *evidence-based* reasoning, that $[F(n)]$ interprets as false in $\mathbb{N}$ for some numeral $[n]$.

More precisely, we may not conclude from the PA-provability of $[\neg(\forall x)F(x)]$, in the absence of *evidence-based* reasoning, that the proposition $F^*(n)$ does not hold in $\mathbb{N}$ for some *unspecified* natural number n since, by §2.F., Corollary 2.22, PA is *not* ω-consistent.

CHAPTER 8. *MATHEMATICAL CONSEQUENCES*

8. Three perspectives of logic

We conclude from the foregoing considerations that the common perceptions of a mutual inconsistency between classical and constructive mathematical philosophies—vis à vis 'omniscient' mathematical truth, and 'omniscient' mathematical ontologies, decried by Krajewski in [Kr16]—are illusory; they merely reflect the circumstance that (see §1., Thesis 1), to date, all such philosophies do not explicitly—and unambiguously (as proposed in §9.)—define the relations between a language and the logic that is necessary to assign unequivocal, *evidence-based*, truth-values of both 'provability' *and* 'truth' to the propositions of the language.

8.A. Hilbert's theism

For instance, classical perspectives which admit Hilbert's formal definitions of quantification can be labelled 'theistic', since they implicitly assume—without providing *evidence-based* criteria for interpreting quantification constructively—both that:

 (a) the first-order logic FOL is consistent;

and that:

 (b) Aristotle's particularisation (see §4., Definition 12)—which postulates that '[¬∀¬x]' can unrestrictedly be interpreted as 'there exists an *unspecified* instantiation of x'—holds under any interpretation of FOL.

The significance of the label 'theistic'[76] is that conventional wisdom 'omnisciently' believes that Aristotle's particularisation remains valid—sometimes without qualification—even over infinite domains; a belief that—as highlighted in §4.A.—is not unequivocally self-evident, but must be appealed to as an article of unquestioning faith (see §4.B.).

8.B. Brouwer's atheism

In sharp contrast, constructive approaches based on Brouwer's philosophy of Intuitionism can be labelled 'atheistic'[77] because they—also without providing adequate *evidence-based* criteria for interpreting quantification constructively—deny both that[78]:

 (a) FOL is consistent (since they *omnisciently* deny that the Law of the Excluded Middle LEM—which is a theorem of FOL—holds under any *well-defined* interpretation of FOL);

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[76]Although intended to highlight an entirely different distinction, that the choice of the label 'theistic' may not be totally inappropriate is suggested by Tarski's reported point of view to the effect (Franks: [Fr09], p.3): "...that Hilbert's alleged hope that meta-mathematics would usher in a 'feeling of absolute security' was a 'kind of theology' that 'lay far beyond the reach of any normal human science' ...".

[77]As can other 'constructive' approaches such as those analysed by Posy in [Pos13] (p.106, §5.1).

[78]But see also Maietti: [Mt09] and Maietti/Sambin: [MS05].

> **Comment**: The *omniscience* of the belief is reflected in [Kl52]:

> "The formula $\forall x(A(x) \lor \neg A(x))$ is classically provable, and hence under classical interpretation true. But it is unrealizable. So if realizability is accepted as a necessary condition for intuitionistic truth, it is untrue intuitionistically, and therefore unprovable not only in the present intuitionistic formal system, but by any intuitionistic methods whatsoever."
>
> . . . *Kleene: [Kl52], p.513.*

and that:

(b) Aristotle's particularisation holds under any interpretation of FOL that has an infinite domain.

Although Brouwer's explicitly stated objection appeared to be to the Law of the Excluded Middle as expressed and interpreted at the time (Brouwer: [Br23], p.335-336; Kleene: [Kl52], p.47; Hilbert: [Hi27], p.475), some of Kleene's remarks ([Kl52], p.49), some of Hilbert's remarks (e.g., in [Hi27], p.474) and, more particularly, Kolmogorov's remarks (in [Ko25], fn. p.419; p.432) suggest that the intent of Brouwer's fundamental objection can also be viewed today as being limited only to the (yet prevailing) classical belief—as an article of Hilbertian faith—that the validity of Aristotle's particularisation can be extended without qualification to infinite domains.

The significance of the label 'atheistic' is that whereas intuitionistic approaches to mathematics deny the faith-based belief in the unqualified validity of Aristotle's particularisation over infinite domains, their denial of the Law of the Excluded Middle is itself an 'omniscient' belief that is also not unequivocally self-evident, and must be appealed to as an article of unquestioning faith[79].

8.B.a.   Denial of an unrestricted applicability of the Law of the Excluded Middle is a *belief*

The perspective is implicit in Bauer's unusually candid acknowledgment in [Ba16] that constructive mathematics holds denial or acceptance of the Law of the Excluded Middle (LEM) as an optional belief that is open to persuasion:

> "Unless we already believe in $\neg\neg P \Leftrightarrow P$, we cannot get one from the other by exchanging P and $\neg P$."
>
> . . . *Bauer: [Ba16], p.2.*

> "Classical mathematical training plants excluded middle so deeply into young students' minds that most mathematicians cannot even detect its presence in a proof. In order to gain some sort of understanding of the constructivist position, we should therefore provide a method for suspending belief in excluded middle."
>
> . . . *Bauer: [Ba16], p.6.*

We note, however, that Bauer's admission masks an atheistic disbelief that is embedded equally deeply in constructive mathematics.

Reason: It is constructive mathematics that mistakenly equates denial of the ε-axioms in Hilbert's ε-calculus ([Hi27]) with denial of LEM in *well-defined* interpretations (in the sense of §1., Definition 1, and §9., Definitions 26 and 27) of formal theories; where the logical axioms

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[79]Thus lending justification to Krajewski's comment in [Kr16]: "Brouwer created mathematical intuitionism and was a mystic".

and rules of inference are those of the standard first-order logic FOL which—as defined in introductory logical texts (e.g., [Me64])—forms an essential part of classical mathematical training.

The root of this conflation lies in the fact that Brouwer's original objection (in [Br08]) was to the definition of existential quantification in terms such as those of Hilbert's ε-operator in the latter's ε-calculus, in which LEM is a theorem.

Denying LEM is thus *sufficient* for Brouwer's purpose of denying validity to any interpretation of Hilbert's definition of existential quantification over any putative structure in which the calculus is satisfied.

However it is not *necessary* since—as entailed by §5.D., Corollary 5.15—the converse does not hold.

In other words, denying validity to any interpretation of Hilbert's definition of existential quantification over a structure in which the calculus FOL is satisfied does not entail that LEM is not satisfied over the structure.

Moreover, as observed by Gödel in [Go33], such a denial of tertium non datur *misleadingly* compelled Arend Heyting to admit an intuitionistic notion of "absurdity" into his formalisation of intuitionistic arithmetic, which entailed that "all of the classical axioms become provable propositions for intuitionism as well":

> "If one lets correspond to the basic notions of Heyting's propositional calculus the classical notions given by the same symbols and to "absurdity" ($\neg$), ordinary negation ($\sim$), then the intuitionistic propositional calculus $\boldsymbol{A}$ appears as a proper subsystem of the usual propositional calculus $\boldsymbol{H}$. But, using a different correspondence (translation) of the concepts, the reverse occurs: *the classical propositional calculus is a sub-system of the intuitionistic one.* For, one has: Every formula constructed in terms of conjunction ($\wedge$) and negation ($\neg$) alone which is valid in $\boldsymbol{A}$ is also provable in $\boldsymbol{H}$. For each such formula must be of the form: $\neg A_1 \wedge \neg A_2 \wedge \ldots \wedge \neg A_n$, and if it is valid in $\boldsymbol{A}$, so must be each individual $\neg A_i$; but then by Gilvenko $\neg A_i$ is also provable in $\boldsymbol{H}$ and hence also the conjunction of the $\neg A_i$. From this, it follows that: if one translates the classical notions $\sim p$, $p \rightarrow q$, $p \vee q$, $p.q$ by the following intuitionistic notions: $\neg p$, $\neg(p \wedge \neg q)$, $\neg(\neg p \wedge \neg q)$, $p \wedge q$ then each classically valid formula is also valid in $\boldsymbol{H}$.
>
> The aim of the present investigation is to prove that something analogous holds *for all of arithmetic and number theory*, as given e.g. by the axioms of Herbrand. Here also one can give an interpretation of the classical notions in terms of intuitionistic notions, *so that all of the classical axioms become provable propositions for intuitionism as well.*
>
> [...]
>
> Theorem I, whose proof has now been completed, shows that *intuitionistic arithmetic and number theory are only apparently narrower than the classical versions, and in fact contain them (using a somewhat deviant interpretation).* The reason for this lies in the fact that the intuitionistic prohibition against negating universal propositions to form purely existential propositions is made ineffective by permitting the predicate of absurdity to be applied to universal propositions, which leads formally to exactly the same propositions as are asserted in classical mathematics. Intuitionism would seem to result in genuine restrictions only for analysis and set theory, and these restrictions are the result, not of the denial of tertium non datur, but rather of the prohibition of impredicative concepts. The above considerations, of course, yield a consistency proof for classical arithmetic and number theory. However, this proof is certainly not "finitary" in the sense given by Herbrand, following Hilbert."
>
> *...Gödel: [Go33], pp.75 & 80.*

Thus, from an *evidence-based* perspective, Gödel's demonstration of an equivalence between classical arithmetic and Heyting's Arithmetic emphasises the thesis of this investigation that

denial of LEM (tertium non datur) was unnecessary for ensuring finitism; especially since such denial denied formal, finitary, argumentation to Intuitionism for much of that which it apparently sought to protect.

8.B.b.   Brouwerian interpretations of $\wedge, \vee, \rightarrow, \exists, \forall$

The significance of the label 'atheistic' is also seen in the following, presumably standard, intuitionistic interpretations of $\wedge, \vee, \rightarrow, \exists, \forall$, as detailed by Bishop in [Bi18]:

> "Each formula of Σ represents a constructively meaningful assertion, in that it denotes a constructively meaningful assertion for given values of the free variables, if we interpret $\wedge, \vee, \rightarrow, \exists, \forall$ in the constructive (Brouwerian) sense. Here is a brief summary of Brouwer's interpretations. (The interpretations hold for all fixed values of the free variables.)
>
> (a) $A \wedge B$ asserts A and also asserts B.
>
> (b) $A \vee B$ either asserts A or asserts B, and we have a finite method for deciding which of the two it does assert.
>
> (c) $A \rightarrow B$ asserts that if A is true, then so is B. (To prove $A \rightarrow B$ we must give some method that converts each proof of A into a proof of B.)
>
> (d) $\forall x A(x)$ asserts that $A(f)$ holds for each (constructively) defined functional f of the same type as the variable x, where $A(f)$ is obtained from $A(x)$ by substituting f for all free occurrences of x.
>
> (e) $\exists x A(x)$ asserts that we know an algorithm for constructing a functional f for which $A(f)$ holds."
>
> *...Bishop: [Bi18], pp.6-7.*

We note that although Bishop asserts the above interpretations as constructive, they are ambiguous as to the intended meaning of the words 'all' and 'each', since the interpretations do not distinguish between:

(i) *algorithmic computability*, i.e., whether there is an algorithm which, for 'all' specifiable values of the free variables, evidences that the formula Σ denotes a constructively meaningful assertion; and

(ii) *algorithmic verifiability*, i.e., whether, for 'any/each' specified value of the free variables, there is an algorithm which evidences that the formula Σ denotes a constructively meaningful assertion.

Accordingly, they cannot accommodate a coherent computational interpretation of Gödel's first-order arithmetical formula $[R(x)]$, which:

(1) is such that the PA-formula $[R(n)]$ is PA-provable for any substitution of the numeral $[n]$ for the variable $[x]$ in the PA-formula $[R(x)]$, even though the formula $[(\forall x)R(x)]$ is not PA-provable;

and which:

(2) interprets as an arithmetical relation, say $R^*(x)$, such that, for any specified natural number n, there is *always* some algorithm that will evidence the proposition $R^*(n)$ as true, but there is *no* algorithm that, for any specified natural number n, will evidence $R^*(n)$ as a true arithmetical proposition (an immediate consequence of §2.F., Corollary 2.21).

Curiously, although (1) is essentially the first half of Gödel's 'undecidability' argument in [Go31][80], the significance of interpretation (2) apparently escaped Gödel's attention; even though what we have termed as an ambiguity—reflecting a failure to constructively define, and distinguish between, the concepts 'for each/any' and 'for all'—in the intuitionistic interpretation of quantification can, reasonably, be seen as something that Gödel too viewed with disquietude as a 'vagueness' in Heyting's formalisation of intuitionistic logic—a vagueness which he, however, seemed at the time to view as an *unsurmountable* barrier[81] towards the furnishing of a constructive intuitionistic proof of consistency for classical arithmetic. As remarked by Mark van Atten in [At17]:

> "Gödel's 1933 lecture is concerned with the question of a constructive consistency proof for classical arithmetic. In considering what should count as constructive mathematics, Gödel there argues against accepting impredicative definitions, and insists on inductive definitions. Gödel discusses the prospects for a consistency proof for classical arithmetic using intuitionistic logic, then best known from Heyting's formalisation 'Die formalen Regeln der intuitionistischen Logik' (Heyting, 19301,b,c), as well as Heyting's Königsberg lecture of 1931, 'Die intuitionistiche Grundlegung der Mathematik', published as Heyting 1931.
>
> [. . .]
>
> The principles in Heyting's formalisation that have Gödel's special interest are those for 'absurdity', that is, intuitionistic negation. But Gödel goes on to argue that this notion is not constructive in his sense, and hence of no use for a constructive consistency proof of classical arithmetic. The problem he sees is that their intuitionistic explanation involve a reference to the totality of all constructive proofs. The example he gives is

$$p \supset \neg\neg p$$

> which, he says, means 'If p has been proved, then the assumption $\neg p$ leads to a contradiction. Gödel says that these axioms are not about constructions on a substrate of numbers but rather on a substrate of proofs, and therefore the example may be explicated as 'Given *any* proof for a proposition p, you can construct a reductio ad absurdum for the proposition $\neg p$'. He then comments that

>> Heyting's axioms concerning absurdity and similar notions [. . .] violate the principle, which I stated before, that the word 'any' can be applied only to those totalities for which we have a finite procedure for generating all their elements [. . .] The totality of all possible proofs certainly does not possess this character, and nevertheless the word 'any' is applied to this totality in Heyting's axioms [. . .] Totalities whose elements cannot be generated by a well-defined procedure are in some sense vague and indefinite as to their borders. And this objection applies particularly to the totality of intuitionistic proofs because of the vagueness of the notion of constructivity. Therefore this foundation of classical arithmetic by means of the notion of absurdity is of doubtful value. (Gödel, 1933b, p.53)

> A draft of this passage in Gödel's archive does not quite end with rejection of Heyting's logic. Instead, it reflects:

>> Therefore you may be doubtful [sic] as to the correctness of the notion of absurdity and as to the value of a proof for freedom from contradiction by means of this notion. But nevertheless it may be granted that this foundation is at least more satisfactory than the ordinary platonistic interpretation [. . .]

---

[80]p.25: "1. 17 *Gen r* is not κ-provable".

[81]Albeit surmountable today, once the source of the ambiguity is identified and removed, since PA is finitarily consistent (see §2.C., Theorem 2.16).

Either way, the doubt about, or objection to, the notion of absurdity immediately generalises to implication as such.

It is remarkable, given the construction of Gödel's talk, in which the discussion of the intuitionistic logical connectives is preceded by an argument against the use of impredicative definitions for foundational purposes, that the objection Gödel puts forward is not that Heyting's principles for absurdity are impredicative, but that they are vague. Impredicativity of course entails constructive undefinability and in that sense vagueness, and it is possible that Gödel had seen the problem of impredicativity but thought that, in the context of a consistency proof that is looked for because of its epistemic interest, vagueness is the more important thing to note, even if impredicativity is the cause of it."

. . . van Atten: [At17], pp.6-7.

8.B.c.   Conception of Truth in Intuitionism

The following review of the concept of truth in Intuitionism by Panu Raatikainen, in [Raa04], highlights that the consequence, of what we have termed as the 'atheistic' denial of the Law of the Excluded Middle in Intuitionistic calculi, has been to treat the concept of 'truth' in Intuitionism as a constructive alternative—rather than as the complement suggested by the Complentarity Thesis (§1., Thesis 1)—to the concept of 'proof' in formal argumentation, entailed by what we have termed as the 'theistic' assumption of the validity of Aristotle's particularisation in any well-defined interpretation of Hilbert's ε-calculus:

"I will end by recalling what Heyting once said is the aim of intuitionism: 'We look for a basis of mathematics which is directly given and which we can immediately understand without philosophical subtleties (*Heyting 1974*, p. 79). It is arguable that after almost a hundred years of intensive attempts, intuitionism has not yet succeeded in this. Above, we have examined the three basic choices there are for the intuitionistic theory of truth, the strict actualism, the liberalized actualism and possibilism, and found all them wanting.

In this Appendix, I shall deal solely with the later intuitionism which has a more positive view of logic than the orthodox intuitionism of Brouwer, according to which mathematics is absolutely independent of logic. Indeed, much of contemporary intuitionism, or constructivism, views mathematics simply as deriving theorems with the help of intuitionistic logic from intuitionistically acceptable axioms (and whatever principles used in proofs not covered by intuitionistic logic). Thus e.g. Bridges says that in practice what the contemporary constructive mathematicians are doing amounts to 'doing mathematics with intuitionistic logic' (*Bridges 1997*).

Another popular trend in the present-day intuitionism is to emphasize that one should recognize a proof when one sees one.[28] This idea derives from Kreisel, and has been pressed repeatedly especially by Dummett. More formally, it is expressed by the requirement that the proof relation must be decidable. It is indeed arguable that such a requirement is necessary for the intuitionistic epistemology. It also harmonizes well with Heyting's view that '[a] mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever' (*Heyting1956b*, p. 6).

The whole picture I want to consider here is beautifully expressed by Sundholm: 'Proofs begin with immediate truths (axioms), which themselves are not justified further by proof, and continue with steps of immediate inference, each of which cannot (be) further justified by proof' (*Sundholm 1983*, p. 162). I shall next argue that the two above ideas are incompatible. (Interestingly, also Beeson (*1985*) denies the decidability of proof relation. He ends up with this conclusion somewhat differently than the way I do.)

For simplicity, let us focus on the provability in the language of arithmetic L(HA). Now given a finite sequence of formulas, it is certainly possible to check effectively whether every step in it is an application of intuitionistically acceptable rule of inference. But how about the premises? Only if one can in addition see that all the premises of a derivation are intuitionistically true one can

say that one has a proof of the conclusion at hand. This is at least in principle possible if axiom hood is a decidable property. However, in the intuitionistic setting, it cannot be! For if it was, the intuitionistic provability could be captured by a formalized system. And then, by Gödel's theorem, there would be truths that are unprovable, contrary to the basic principle of intuitionism, which equates truth with provability.

The situation is actually even much worse—I doubt that it is generally realized how bad it really is. Not only must the set of admissible axioms be undecidable. It cannot be semi-decidable, i.e. recursively enumerable (Σ_1^0), it cannot be Trial-and-Error decidable (Δ_2^0); it cannot be anywhere in the arithmetical hierarchy (not Σ_n^0 for any n). (Here I assume that the notions of arithmetical hierarchy, or at least the idea of being definable in the language of arithmetic, make sense; in practice quite many contemporary intuitionists seem to accept them.) For assume that the property of being an admissible axiom were definable by an arithmetical formula (however complex). This implies that also provability is definable in the language of arithmetic. Then one can apply Gödel's technique and construct a statement of the language which is unprovable but true.

Thus the totality of intuitionistically provable sentences (already, restricted to L(HA) i.e. the arithmetical sentences) necessarily is non-arithmetical, i.e. at least hyperarithmetical (Δ_1^1). But this means that they are just as abstract and inaccessible as truth in classical arithmetic. The same holds already for the alleged axioms, that is, 'the immediate truths'. But certainly non-arithmeticality makes the sphere of 'the immediate truths' implausibly complex and inaccessible. If one cannot tell whether the premises used in a derivation are acceptable, that is, true, or not, one cannot tell whether one has a genuine proof before one's eyes or not, contrary to the standard assumption of contemporary intuitionism."

. . . *Raatikainen:* [Raa04]

8.C.   Finitary agnosticism

Finally, the *evidence-based* reasoning in [An16] shows that we can avoid both Hilbertian and Brouwerian 'omniscience', in our foundational assumptions, by adopting what may be labelled as a finitarily 'agnostic'—essentially Wittgensteinian (see also §9.A.; §10.)—perspective; and noting that although, if Aristotle's particularisation holds in an interpretation of a FOL then LEM must also hold in the interpretation, the converse is not true (see §5.D., Corollary 5.15).

"What really matters to Wittgenstein is not the denial that an arithmetical term has a meaning, but rather that this meaning is an ideal object, named or described by the term. In his view, even the distinction between a numeral and a number is quite legitimate, in perfect analogy with the distinction, explicitly made by him, between a propositional sign (*Satzzeichen*) and a proposition (*Satz*). That between numerals and numbers is not an ontological distinction between two kinds of entities (material entities versus ideal entities), but a distinction between two ways of considering the one linguistic reality: the way in which one considers a sign as a mere physical entity and the way in which one takes into account its role of notational device to represent a certain formal property.[59] Nonetheless, a point has to be stressed, which to a large extent shortens Wittgenstein's distance from formalism and justifies, in my opinion, the description of his earlier conception of mathematics as quasi-formalistic (a label that, as we shall see, is fitting for every stage of the whole development of his philosophy of mathematics). The recognition of the mutual reducibility of two models of linguistic construction—when calculation is required—is the outcome of a rule-governed process of transformation of a certain grouping of the elements of a string of "Ω'", *exhibiting* the first model, into a different one, *exhibiting* the second model. Thus, although arithmetical signs within calculations are not considered as mere physical structures, doing mathematics is appropriately described as a sign manipulation activity.

The last remark leads us to the comparison with intuitionism. We have said that the reference to vision, intuition or immediate recognition plays a decisive role in Wittgenstein's conception of our "knowledge" of the formal domain. And there is also a certain similarity between Wittgenstein's theses on the purely instrumental role assigned to mathematical notation and even to the formulation

of theorems (equations), on the one hand, and certain typical ideas of Brouwer concerning logical and mathematical language, on the other. However, a decisive element of disagreement between the two conceptions can be easily identified. When Wittgenstein, dealing with mathematics and logic, speaks of vision, he does not intend to supply a psychological foundation to mathematical activity. Rather, he resorts to the notion of intuition or to the metaphor of vision to describe the relationship between speakers and what is shown by language (the domain of necessity) and to contrast it with the meaningful expression of a thought (the picture of a contingent state of affairs). It goes without saying that all this has nothing to do with the psychologization of mathematics of intuitionists. [60]"

... Frascolla: [Fra94], § Foundations of Mathematics (I), p.40.

The significance of the label 'agnostic' is that an *evidence-based* perspective:

(a) Neither shares an ascetic Brouwerian faith which unnecessarily denies appeal to LEM— and, ipso facto, to the consistency of FOL—since [An16] shows that such consistency follows immediately from a finitary proof of consistency of the first order Peano Arithmetic PA (see §2.C.a., Theorem 2.16);

(b) Nor shares a libertarian Hilbertian faith that unrestrictedly admits Aristotle's particularisation over infinite domains (see §5.D., Corollary 5.12).

Moreover, recognising such distinction has significant consequences for the the natural sciences in general, and cognitive sciences in particular, since they yet subscribe unquestioningly to the, not uncommon, perception[82] that Gödel's Incompleteness Theorems set absolute limits on the ability of the brain to express and communicate mental concepts verifiably.

On the contrary, both the classical and intuitionistic interpretations of quantification yield interpretations of the first-order Peano Arithmetic PA—over the structure $\mathbb{N}$ of the natural numbers—that are complementary, not contradictory.

The former yields the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA over $\mathbb{N}$, which is *well-defined* with respect to *weak* non-finitary assignments of algorithmically verifiable Tarskian truth values T_V to the formulas of PA under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ and which, from a perspective such as that of John Lucas' Gödelian Thesis ([Lu61]; [Lu96]), can be viewed as circumscribing the ambit of *non-finitary* human reasoning about 'true' arithmetical propositions (see §2.F.).

The latter yields a *strong* finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA over $\mathbb{N}$, which is *constructively well-defined* (in the sense of §9., Definitions 24 to 26) with respect to *strong* finitary assignments of algorithmically computable Tarskian truth values T_C to the formulas of PA under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ and which, from the perspective of Lucas' Gödelian Thesis, can be viewed as circumscribing the ambit of *finitary* mechanistic reasoning about 'true' arithmetical propositions; where we tentatively define:

Definition 23. (Well-defined interpretation) *An interpretation $\mathcal{I}$ of a formal language L, over a domain D of a structure $\mathcal{S}$, is constructively well-defined relative to an assignment of truth values $T_\mathcal{I}$ to the formulas of L if, and only if, the provable formulas of L interpret as true over D under $\mathcal{I}$ relative to the assignment of truth values $T_\mathcal{I}$.*

Of interest is Frank Waaldijk's perspective, which particularly emphasises the need for such a unified, constructive, foundation for the mathematical representation of elements of reality such as those considered in [LR00]:

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[82]Addressed informally from a naive perspective in [An04].

"Our investigations lead us to consider the possibilities for 'reuniting the antipodes'. The antipodes being classical mathematics (CLASS) and intuitionism (INT). . . . It therefore seems worthwhile to explore the 'formal' common ground of classical and intuitionistic mathematics. If systematically developed, many intuitionistic results would be seen to hold classically as well, and thus offer a way to develop a strong constructive theory which is still consistent with the rest of classical mathematics. Such a constructive theory can form a conceptual framework for applied mathematics and information technology. These sciences now use an ad-hoc approach to reality since the classical framework is inadequate. . . . [*and can*] easily use the richness of ideas already present in classical mathematics, if classical mathematics were to be systematically developed along the common grounds before the unconstructive elements are brought in."

. . . Waaldijk: [Wl03], §1.6, p.5).

Part III
THE PHILOSOPHICAL FOUNDATIONS OF *EVIDENCE-BASED* REASONING
CHAPTER 9. *PHILOSOPHICAL FOUNDATIONS*

9.   What is logic?

In Part III of this investigation we consider, and speculate upon, some philosophical consequences of finitary *agnosticism* that, by removing the ambiguity in the rules for Brouwer-Heyting-Kolmogorov realizability as indicated in §4.C., now allows us to formalise a Wittgensteinian perspective of constructive mathematics, and of its goal, by distinguishing between:

- a *first-order theory*—such as the Set Theory ZFC or the Peano Arithmetic PA—that admits a *proof-theoretic logic* which, on the basis of *evidence-based* reasoning, assigns the values 'provable/unprovable' to the well-formed formulas of the theory;

- a *first-order theory*—such as the Peano Arithmetic PA, that further admits a *model-theoretic logic* which, on the basis of *evidence-based* reasoning, assigns the values 'true/false' to the well-formed formulas of the theory;

where we address §1., Thesis 1 formally by qualifying §1., Definition 1:

Definition 24. (Proof-theoretic logic) *The proof-theoretic logic of a first-order theory S is a set of rules consisting of:*

- *a selected set of well-formed formulas of S labelled as 'axioms/axiom schemas' that are assigned the value 'provable'; and*

- *a finitary set of rules of inference in S;*

that assign evidence-based values of 'provable' or 'unprovable' to the well-formed formulas of S by means of the axioms and rules of inference of S.

Definition 25. (Model-theoretic logic) *The model-theoretic logic of a first-order theory S with a proof-theoretic logic is a set of rules that assign evidence-based truth values of 'satisfaction', 'truth', and 'falsity' to the well-formed formulas of S under an interpretation $\mathcal{I}$ such that the axioms of S interpret as 'true' under $\mathcal{I}$, and the rules of inference of S preserve such 'truth' under $\mathcal{I}$.*

We contrast §1., Definition 1, with the epistemically grounded perspective of conventional wisdom (such as, for instance, [Mur06]; see also [Brd13], *§2. Foundational Formalisms, Logic*; [BW05]) when it fails to distinguish between the multi-dimensional nature of the logic of a formal mathematical language (as defined above), and the one-dimensional nature of the veridicality of its assertions (articulated either informally as in, for example, Lakoff and Núñez [LR00], or implicitly as, for instance, in Gila Sher [Shr13]):

"Logic, the investigation suggests, is grounded in the formal aspect of reality, and the outline proposes an account of this aspect, the way it both constrains and enables logic (gives rise to logical truths and consequences), logic's role in our overall system of knowledge, the relation between logic and mathematics, the normativity of logic, the characteristic traits of logic, and error and revision in logic.

. . .

It is an interesting fact that, with a small number of exceptions, a systematic philosophical foundation for logic, a foundation for logic rather than for mathematics or language, has rarely been attempted (fn1: One recent exception is Maddy [2007, Part III], which differs from the present attempt in being thoroughly naturalistic. Another psychologically oriented attempt is Hanna [2006]. Due to limitations of space and in accordance with my constructive goal, I will limit comparisons and polemics to a minimum).

. . .

By a philosophical foundation for logic I mean in this paper a substantive philosophical theory that critically examines and explains the basic features of logic, the tasks logic performs in our theoretical and practical life, the veridicality of logic - including the source of the truth and falsehood of both logical and meta-logical claims, . . . the grounds on which logical theories should be accepted (rejected, or revised), the ways logical theories are constrained and enabled by the mind and the world, the relations between logic and related theories (e.g., mathematics), the source of the normativity of logic, and so on. The list is in principle open-ended since new interests and concerns may be raised by different persons and communities at present and in the future. In addition, the investigation itself is likely to raise new questions (whether logic is similar to other disciplines in requiring a grounding in reality, what the distinctive characteristics of logical operators are, etc.).

. . .

The motivation for engaging in a foundational project of this kind is both general and particular, both intellectual and practical, both theoretical and applicational. Partly, the project is motivated by an interest in providing a foundation for knowledge in general - i.e., a foundation both for human knowledge as a whole and for each branch of knowledge individually (logic being one such branch). Partly, the motivation is specific to logic, and is due to logic's unique features: its extreme "basicness", generality, modal force, normativity, ability to prevent an especially destructive type of error (logical contradiction, inconsistency), ability to expand all types of knowledge (through logical inference), etc. In both cases the interest is both intellectual and practical. Finally, our interest is both theoretical and applicational: we are interested in a systematic theoretical account of the nature, credentials, and scope of logical reasoning, as well as in its applications to specific fields and areas.

. . .

If the bulk of our criticisms is correct, the traditional foundationalist strategy for constructing a foundation for logic (and for our system of knowledge in general) should be rejected. It is true that for a long time the foundationalist strategy has been our only foundational strategy, and as a result many of its features have become entangled in our conception of a foundation, but this entanglement can and ought to be unraveled. . . . My goal is an epistemic strategy that is both free of the unnecessary encumbrances of the foundationalist strategy and strongly committed to the grounding project. Following Shapiro [1991], I will call such a strategy a *foundation without foundationalism.*"

. . . *Sher: [Shr13], pp.145-146, 151.*

For a formal mathematical language $\mathcal{L}$ to, then, precisely express and objectively (i.e., on the basis of *evidence-based* reasoning) communicate categorically characteristics of some structure $\mathcal{U}$ that may, or may not, be constructively *well-defined* (in the sense of §4.F., Definition 19), it must be able to categorically represent some Theory $T(\mathcal{U})$ whose characteristic is that:

Definition 26. (Well-defined model) *The Theory $T(\mathcal{U})$ defined semantically by the λ-interpretation of a formal mathematical language $\mathcal{L}$ over the structure $\mathcal{U}$ is a constructively well-defined model of $\mathcal{L}$ if, and only if, λ is a constructively well-defined Logic of $\mathcal{L}$.* □

This, now, yields the perspective that:

Definition 27. (Constructive mathematics) *Constructive mathematics is the study of formal mathematical languages that have a constructively well-defined logic.*   □

The significance of §9., Definitions 24 to 27 is illustrated by the following account by Carl J. Posy of the purported ways in which:

> "...adopting intuitionistic logic limits the ways in which a constructivist can carry out a mathematical proof. A standard example is the classical proof that there are irrational r and s such that r^s is a rational number: either $\sqrt{2}^{\sqrt{2}}$ is rational or it is irrational. If it is rational, then take $r = s = \sqrt{2}$. If it is irrational, then take $r = \sqrt{2}^{\sqrt{2}}$ and $s = \sqrt{2}$. In this case $r^s = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$. The constructivist cannot make that initial assumption that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational."
>
> *...Posy:* [Pos13], *p.109.*

Though—as the author notes—this theorem is in fact constructively recoverable, the question—left unaddressed here by both classical and constructive theories—is not whether a particular formula is rational or irrational, but whether the logic that assigns truth assignments to the formulas of the concerned language is sufficiently *well-defined* so as to *evidence* the decidability of whether a formula is either rational or irrational.

9.A. Wittgenstein's 'notorious' paragraph about 'the Gödel Theorem'

We note that such an *evidence-based* perspective reflects in essence the views Ludwig Wittgenstein emphasised in his 'notorious paragraph', #8 in [Wi78][83], where he essentially argues—albeit obscurely—that mathematical truth must necessarily be an 'assignment-by-convention' (in the sense, for instance, of §1., Thesis 1) which, in the absence of an associated *evidence-based* methodology for finitarily *verifying* any assignment of truth values to the propositions of a language (such as those defined for PA in 9.A.d.; see also §2.B., §2.C., and §4.C.), might legitimately entail contradictions which appeal to putative truth-values, and from which we can conclude *no more* than the *absence* of an associated *evidence-based* methodology for *verifying* an assignment; and certainly not the *presence* of an inherent *to-be-avoided-at-all-costs* inconsistency:

> "1. It is easy to think of a language in which there is not a form for questions, or commands, but question and command are expressed in the form of statements, e.g. in forms corresponding to our "I should like to know if ..." and "My wish is that ...".
>
> No one would say of a question (e.g. whether it is raining outside that it was true or false. Of course it is English to say so of such a sentence as "I want to know whether ..." But suppose this form were always used instead of the question?
>
> 2. The great majority of sentences we speak, write and read, are statement sentences.
>
> And—you say—these sentences are true or false. OR, as I might also say, the game of truth-functions is played with them. For assertion is not something that gets added to the proposition, but an essential feature of the game we play with it. Comparable, say, to the that characteristic of chess by which there is winning and losing in it, the winner being the one who takes the other's king. Of course, there could be a game in a certain sense very near akin to chess, consisting in

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[83]In footnote 9 of [FP00], Floyd and Putnam note that: "The 'notorious' paragraph RFM I Appendix III 8 was penned on 23 September 1937, when Wittgenstein was in Norway (see the Wittgenstein papers, CD Rom, Oxford University Press and the University of Bergen, 1998, Item 118 (Band XIV), pp. 106ff)".

making the chess moves, but without there being any winning and losing in it; or with different conditions for winning.

3. Imagine it were said: A command consists of a proposal ('assumption') and the commanding of the thing proposed.

4. Might we not do arithmetic without having the idea of uttering arithmetical propositions, and without ever having been struck by the similarity between a multiplication and a proposition?

Should we not shake our heads, though, when someone shewed us a multiplication done wrong, as we do when someone tells us it is raining, if it is not raining?—Yes; and there is a point of connexion. But we also make gestures to stop our dog, e.g. when he behaves as we do not wish.

We are used to saying "2 times 2 is 4" and the verb "is" makes this into a proposition, and apparently establishes a close kinship with everything we call a 'proposition'. Whereas it is only a matter of a very superficial relationship.

5. Are there true propositions in Russell's system, which cannot be proved in his system?—What is called a true proposition in Russell's system, then?

6. For what does a proposition's '*being true*' mean? '*p is true = p*. (That is the answer.)

So we want to ask something like: under what circumstances do we assert a proposition? Or: How is the assertion of the proposition used in the language game? And the 'assertion of the proposition' is here contrasted with the utterance of the sentence, e.g. as practice in elocution,— or as *part* of another proposition, and so on.

If, then, we ask in this sense: "Under what circumstances is a proposition asserted in Russell's game?" the answer is: at the end of one of his proofs, or as a 'fundamental law' (Pp.) There is no other way in this system of employing asserted propositions in Russell's symbolism.

7. "But may there not be true propositions which are written in this symbolism, but are not provable in Russell's system?"—'True propositions', hence propositions which are true in another system, i.e. can rightly be asserted in *another* game. Certainly; why should there not be such propositions; or rather: why should not propositionsof physics, e.g.be written in Russell's symbolism? The question is quite analogous to: Can there be true propositions in the language of Euclid, which are not provable in his system, but are true?—Why, there are even propositions which are provable in Euclid's system, but are *false* in another system. May not triangles be—in another system—similar (*very* similar) which do not have equal angles?—"But that's just a joke! For in that case they are not 'similar' to one another in the same sense!"—Of course not; and a proposition which cannot be proved in Russell's system is "true" or "false" in a different sense from a proposition of *Principia Mathematica*.

8. I imagine someone asking my advice; he says: "I have constructed a proposition (I will use '*P*' to designate it) in Russell's symbolism, and by means of certain definitions and transformations it can be so interpreted that it says: '*P* is not provable in Russells system'. Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable."

Just as we ask, " 'Provable' in what system?", so we must also ask: " 'True' in what system?" 'True in Russell's system' means, as was said: proved in Russell's system; and 'false in Russell's system' means: the opposite has been proved in Russell's system.—Now what does your "suppose it is false" mean? *In the Russell sense* it means, 'suppose the opposite is proved in Russell's system'; *if that is your assumption* you will now presumably give up the interpretation that it is unprovable. And by 'this interpretation' I understand the translation into this English sentence.—If you assume that the proposition is provable in Russell's system, that means it is true *in the Russell sense*, and the interpretation "*P* is not provable" again has to be given up. If you assume that the proposition is true in the Russell sense, *the same* thing follows. Further: if the proposition is supposed to be

false in some other than the Russell sense, then it does not contradict this for it to be proved in Russell's system (What is called "losing" in chess may constitute winning in another game.)

9. For what does it mean to say that P and "P is unprovable" are the same proposition? It means that these *two* English sentences have a single expression in such-and-such a notation.

10. "But surely P cannot be provable, for, supposing it were proved, then the proposition that it is not provable would be proved." But if this were now proved, or if I believed—perhaps through an error—that I had proved it, why should I not let the proof stand and say I must withdraw my interpretation "*unprovable*"?

11. Let us suppose I prove the unprovability (in Russell's system) of P; then by this proof I have proved P. Now if this proof were one in Russell's system—I should in that case have proved at once that it belonged and did not belong to Russell's system.—That is what comes of making up such sentences.—But there is a contradiction here!—Well, then there is a contradiction here. Does it do any harm here?

12. Is there harm in the contradiction that arises when someone says: "I am lying.—So I am not lying.—So I am lying.—etc."? I mean: does it make our language less usable if in this case, according to the ordinary rules, a proposition yields its contradictory, and vice versa?— the proposition itself is unusable, and these inferences equally; but why should they not be made?—It is a profitless performance!—It is a language-game with some similarity to the game of thumb-catching.

13. Such a contradiction is of interest only because it has tormented people, and because this shews both how tormenting problems can grow out of language, and what kind of things can torment us.

14. A proof of unprovability is as it were a geometrical proof; a proof concerning the geometry of proofs. Quite analogous e.g. to a proof that such-and-such a construction is impossible with ruler and compass. Now such a proof contains an element of prediction, a physical element. For in consequence of such a proof we say to a man: "Dont exert yourself to find a construction (of the trisection of an angle, say)—it can be proved that it can't be done". That is to say: it is essential that the proof of unprovability should be capable of being applied in this way. It must—we might say—be a forcible reason for giving up the search for a proof (i.e. for a construction of such-and-such a kind).

A contradiction is unusable as such a prediction.

15. Whether something is rightly called the proposition "X is unprovable" depends on how we prove this proposition. The proof alone shews what counts as the criterion of unprovability. The proof is part of the system of operations, of the game, in which the proposition is used, and shews us its 'sense'. Thus the question is whether the 'proof of the unprovability of P' is here a forcible reason for the assumption that a proof of P will not be found.

16. The proposition "P is unprovable" has a different sense afterwards—from before it was proved. If it is proved, then it is the terminal pattern in the proof of unprovability.—If it is unproved, then what is to count as a criterion of its truth is not yet clear, and—we can say—its sense is still veiled.

17. Now how am I to take P as having been proved? By a proof of unprovability? Or in some other way? Suppose it is by a proof of unprovability. Now, in order to see *what* has been proved, look at the proof. Perhaps it has here been proved that such-and-such forms of proof do not lead to P.—Or, suppose P has been proved in a direct way—as I should like to put it—and so in that case there follows the proposition "P is unprovable", and it must now come out how this interpretation of the symbols of P collides with the fact of the proof, and why it has to be given up here.

Suppose however that not-P is proved.—Proved how? Say by P's being proved directly—for from that follows that it is provable, and hence not-P. What am I to say now, "P" or "not-P"? Why

not both? If someone asks me "Which is the case, P, or not-P?" then I reply: P stands at the end of a Russellian proof, so you write P in the Russellian system; on the other hand, however, it is then provable and this is expressed by not-P, but this proposition does not stand at the end of a Russellian proof, and so does not belong to the Russellian system.

—When the interpretation "P is unprovable" was given to P, this proof of P was not known, and so one cannot say that P says: *this* proof did not exist.—Once the proof has been constructed, this has created *a new situation*: and now we have to decide whether we will call *this* a proof (a *further* proof), or whether we will still call this the statement of unprovability.

Suppose not-P is directly proved; it is therefore proved that P can be directly proved! So this is once more a question of interpretation—unless we now also have a direct proof of P. If it were like that, well, that is how it would be.

(The superstitious dread and veneration by mathematicians in face of contradiction.)

18. "But suppose, now, that the proposition were *false*—and hence provable?"—Why do you call it 'false'? Because you can see a proof?—Or for other reasons? For in that case it doesn't matter. For one can quite well call the Law of Contradiction false, on the grounds that we very often make good sense by answering a question "Yes and no". And the same for the proposition sim '$\sim\sim p = p$' because we employ double negation as a *strengthening* of the negation and not merely as its cancellation.

19. You say: "..., so P is true and unprovable". That presumably means: "Therefore P". That is all right with me—but for what purpose do you write down this 'assertion'? (It is as if someone had extracted from certain principles about natural forms and architectural style the idea that on Mount Everest, where no one can live, there belonged a châlet in the Baroque style. And how could you make the truth of the assertion plausible to me, since you can make no use of it except to do these bits of legerdemain?

20. Here one needs to remember that the propositions of logic are so constructed as to have *no* application as *information* in practice. So it could very well be said that they were not *propositions* at all; and one's writing them down at all stands in need of justification. Now if we append to these 'propositions' a further sentence-like structure of another kind, then we are all the more in the dark about what kind of application this system of sign-combinations is supposed to have; for the mere *ring of a sentence* is not enough to give these connexions of signs any meaning."

... Wittgenstein: [Wi78], Appendix III.

In their paper 'A note on Wittgenstein's 'notorious paragraph' about the Gödel Theorem', Juliet Floyd and Hilary Putnam draw attention to Wittgenstein's remarks, and argue that this paragraph contains a "philosophical claim of great interest" which:

"...is simply this: if one assumes (and, a fortiori if one actually finds out) that $\neg P$ is provable in Russell's system one should ...give up the "translation" of P by the English sentence "P is not provable"."

... Floyd and Putnam: [FP00].

In §11. we shall show that Wittgenstein's reservations on Gödel's interpretation of his own formal reasoning are not only of historical importance, but presciently reflect the justifiable uneasiness that academicians and philosophers such as Floyd and Putnam—and, more recently, Timm Lampert in [Lam17]—have continued to sense, express, and debate (as reviewed, for instance, in [Mat13]), over standard (text-book) interpretations of Gödel's formal reasoning even eighty five years after the publication of the latter's seminal 1931 paper ([Go31]) on formally undecidable arithmetical propositions:

"Contrary to Wittgenstein's early critics, Shanker [1988], Floyd & Putnam[2000] and Floyd [2001] argue that Wittgenstein does not question Gödel's undecidability proof itself. Instead, they say, Wittgenstein's remarks are concerned with the semantic and philosophical consequences of Gödel's proof; those remarks represent, according to Floyd and Putnam, a "remarkable insight" regarding Gödel's proof. I share the view that Wittgenstein believed that it is not the task of philosophy to question mathematical proofs. However, I argue that from Wittgenstein's perspective, Gödel's proof is not a mathematical proof. Instead, it is a proof that relies on "prose" in the sense of meta-mathematical interpretations, and thus, it is a valid object of philosophical critique.Thus, I deny that Wittgenstein views Gödel's undecidability proof as being just as conclusive as mathematical impossibility proofs. Wittgenstein's simplied, rather general way of referring to an ordinary language interpretation of G without specifying exactly where questionable meta-mathematical interpretations are relevant to Gödel's proof might have led to the judgment that Wittgenstein's critique is not relevant to Gödel's proof.

Contrary to Floyd and Putnam, Rodych [1999] and Steiner [2001] assume that Wittgenstein argues against Gödel's undecidability proof. According to their interpretation, Wittgenstein's objection against Gödel's proof is that from proving G or $\neg G$, it does not follow that PM is inconsistent or ω-inconsistent. Instead, one could abandon the meta-mathematical interpretation of G. However, according to both authors, this critique is inadequate because Gödel's proof does not rely on a meta-mathematical interpretation of G. By specifying where Wittgenstein's critique is mistaken, they wish to decouple Wittgenstein's philosophical insights from his mistaken analysis of Gödel's mathematical proof. I agree with Rodrych and Steiner that Wittgenstein's critique does not offer a sufficient analysis of the specific manner in which a meta-mathematical interpretation is involved in Gödel's reasoning. However, in contrast to these authors, I will explain why both Gödel's semantic proof and his so-called syntactic proof do rely on a meta-mathematical interpretation.

Priest [2004], Berto [2009a] and Berto [2009b] view Wittgenstein as a pioneer of paraconsistent logic. They are especially interested in Wittgenstein's analysis of Gödel's proof as a proof by contradiction. Like Rodych and Steiner, they maintain that Wittgenstein's remarks are not, in fact, pertinent to Gödel's undecidability proof because Wittgenstein refers not to a syntactic contradiction within PM but rather to a contradiction between the provability of G and its meta-mathematical interpretation. However, according to them, Wittgenstein's critique is not mistaken. Rather, it is concerned with the interpretation and consequences of Gödel's undecidability proof. Presuming Wittgenstein's rejection of any distinction between (i) metalanguage and object language and (ii) provability and truth, they show that engaging with Gödel's proof depends on philosophical presumptions. I do not question this. However, I will argue that Wittgenstein's critiqued can be interpreted in a way that is indeed relevant to Gödel's undecidability proof.

The intention of this paper is not to enter into an exegetical debate on whether Wittgenstein understands Gödel's proof and whether he indeed objects to it. For the sake of argument, I assume that to be given. Furthermore, similarly to, e.g., Rodych and Steiner, I take "Wittgenstein's objection" to Gödel's proof to be as follows: "Instead of inferring the incorrectness or (ω-)inconsistency of PM (or PA) from a proof of G (or $\neg G$), one might just as validly abandon the meta-mathematical interpretation of G. Therefore, Gödel's proof is not compelling because it rests on a doubtful meta-mathematical interpretation." I recognize that this is highly controversial, to say the least. However, the literature seems to agree that such an objection, be it Wittgenstein's or not, has no relation to Gödel's undecidability proof and thus is not reasonable. The intention of this paper is to show that this is not true. This objection can, indeed, be related to Gödel's method of defining provability within the language of PM, and it questions this essential element of Gödel's meta-mathematical proof method by measuring its reliability on the basis of an algorithmic conception of proof."

. . . Lampert: [Lam17].

    We shall argue further that Wittgenstein's reservations in [Wi78], as also the uneasiness expressed by, amongst others, Floyd and Putnam in [FP00] and Lampert in [Lam17], can—and arguably must, as we advocate in this investigation—be seen as indicating specific points of ambiguity that need to be addressed on both technical *and* philosophical grounds, rather than

be dismissed on mere technicalities, since both Wittgenstein and Gödel can be held guilty of conflating 'ω-consistency' with 'correctness'.

In his master's thesis [Mat13], Ásgeir Berg Matthíasson seeks to give a fairly balanced account of where Wittgenstein could justifiably have been held to account:

> "As we saw in the introduction and the first chapter of this thesis, the early debate on Wittgenstein's remarks on Gödel was mostly negative and characterized by the claim that Wittgenstein misunderstood Gödel's theorem in essentially two ways: that (a) he did not see how consistency was a necessary premise of Gödel's proof, or simply did not understand what consequences inconsistency would have on a formal system, and (b) the claim that he mistakenly thought Gödel's proof used some kind of natural language interpretation of the sentence P in order to show that it was true, but unprovable, thus by abandoning this interpretation, Gödel's result could be avoided. It is almost certain that the widespread acceptance of these claims severely damaged Wittgenstein's reputation as a philosopher of mathematics.
>
> The later debate, however, offered a much more nuanced and broader view, more sensitive to both Wittgenstein's other philosophical writings and the overall spirit in which he wrote. Despite this, they are not all without their shortcomings. Many of them almost seem made especially for the purpose of vindicating Wittgenstein and absolving the great philosopher from all charges of technical incompetence and misunderstanding. Others have been much more elaborate in their exegesis of Wittgenstein's remarks than the text could possibly support and are thus very unlikely to persuade anyone not already convinced of the truth of their conclusions.
>
> In this thesis a fairly comprehensive critical overview of this debate has been given. Unfortunately, the conclusion of this discussion however is undeniably a certain *aporia*—it is still difficult to give a precise account of what Wittgenstein actually did say in his infamous remarks on Gödel, and given their cryptic and unpolished nature, this will most likely always be the case. However, a few things can be gleaned from the preceding discussion.
>
> First of all, we can see from the discussion on Berto and the dialetheists that Wittgenstein's remarks on consistency are far from being as outrageous as they were originally seen, and that the charge of technical incompetence on Wittgenstein's part was overstated by the early commentators. Of course the philosophical position of dialetheism is far from being the accepted position in mainstream analytic philosophy, but at least it [is] taken seriously nowadays and more logicians are interested in paraconsistent logic than ever before. This should vindicate Wittgenstein's remarks somewhat in the mind of modern readers, even though they are perhaps not ready to accept his claims from a philosophical standpoint.
>
> On the other hand, the interpretations which have tried to interpret Wittgenstein's remarks in such a way as to deny claim (b), namely the interpretations of Shankar, Floyd and Floyd and Putnam, must—when all is considered—be taken to have failed in this task. This reading is simply the most natural one, and they only manage to avoid this conclusion by building very elaborate interpretations with many implausible assumptions and textual problems. In this respect the unequivocal conclusion must be that the early commentators (and Victor Rodych, of course) were simply right: When Wittgenstein wrote the remarks contained in (RFM I, App. III) he did not have a good understanding of Gödel's proof and this led him to say wrong things about it.
>
> This mistake, on Wittgenstein's part, is however not necessarily so severe, as to justify the damage it has done to his reputation as a philosopher of mathematics. It is undoubtedly true that Gödel's proof causes, as it is often seen, serious problems for the positions of logicism and formalism in the philosophy of mathematics, if it is not simply a refutation of those views. The matter is however far from being clear whether or not Wittgenstein's philosophy of mathematics entails such a position, as is often assumed, and many reasons to suppose that this is not the case—at least it is not what one would expect from the author of the *Philosophical Investigations*."
>
> . . . Matthíasson: [Mat13], §*Concluding remarks.*

From the *evidence-based* perspective of this investigation however, that the onus of guilt must fall heavier on Gödel (see §11.B.) follows not only from his misleading remark that

the semantic concept of 'truth' might be replaceable by the 'purely formal and much weaker assumption' of ω-consistency:

> "The method of proof which has just been explained can obviously be applied to every formal system which, first, possesses sufficient means of expression when interpreted according to its meaning to define the concepts (especially the concept "provable formula") occurring in the above argument; and, secondly, in which every provable formula is true. In the precise execution of the above proof, which now follows, we shall have the task (among others) of replacing the second of the assumptions just mentioned by a purely formal and much weaker assumption."
>
> . . . Gödel: [Go31], p.9.

but also from his implicit—and equally misleading—footnote 48a on page 28 of [Go31], which suggests that assuming any formal system of arithmetic—such as, for instance, the first-order Peano Arithmetic PA—to be ω-consistent is intuitionistically unobjectionable, and may be treated *as a matter of fact*:

> "In the proof of Theorem VI no properties of the system P were used other than the following:
>
> 1. The class of axioms and the rules of inference (i.e. the relation "immediate consequence") are recursively definable (when the primitive symbols are replaced in some manner by natural numbers).
>
> 2. Every recursive relation is definable within the system P (in the sense of Theorem V).
>
> Hence, in every formal system which satisfies assumptions 1, 2 and is ω-consistent, there exist undecidable propositions of the form $(x)F(x)$, where F is a recursively defined property of natural numbers, and likewise in every extension of such a system by a recursively definable ω-consistent class of axioms. To the systems which satisfy assumptions 1, 2 belong, as one can easily confirm, the Zermelo-Fraenkel and the v. Neumann axiom systems for set theory, and, in addition, the axiom system for number theory which consists of Peano's axioms, recursive definitions (according to schema (2)) and the logical rules. Assumption 1 is fulfilled in general by every system whose rules of inference are the usual ones and whose axioms (as in F) result from substitution in finitely many schemata.[48a]
>
> [Footnote 48a] The true reason for the incompleteness which attaches to all formal systems of mathematics lies, as will be shown in Part II of this paper, in the fact that the formation of higher and higher types can be continued into the transfinite."
>
> . . . Gödel: [Go31], p.28.

That both of Gödel's assertions are misleading follows since PA is *both* strongly (finitarily) consistent by §2.C.a., Theorem 2.16—hence 'correct'—*and ω-in*consistent by §2.F., Corollary 2.22 (and, as we shall show, independently, by §9.A.f., Theorem 9.5).

We note, moreover, that the latter proof appears to reflect Lampert's interpretation of Wittgenstein's argument in [Wi78]:

> "In I, §17, Wittgenstein suggests to look at proofs of unprovability "in order to see *what* has been proved". To this end, he distinguishes two types of proofs of unprovability. He mentions the first type only briefly: "Perhaps it has here been proved that such-and-such forms of proof do not lead to P." (P is Wittgenstein's abbreviation for Gödel's formula G). In this section, I argue that Wittgenstein refers in this quote to an algorithmic proof proving that G is not provable within PM. Such a proof of unprovability would, to Wittgenstein, be a compelling reason to give up search for a proof of G within PM. Wittgenstein challenges Gödel's proof because it is not an unprovability proof of this type. This is also why Wittgenstein does not consider algorithmic proofs of unprovability in greater detail in his discussion of Gödel's proof. Such proofs represent the background against which he contrasts Gödel's proof to a type of proof that is beyond question.

Unfortunately, Wittgenstein does not follow his own suggestion to more carefully evaluate unprovability proofs with respect to Gödel's proof. Instead, he distinguishes different types of proofs of unprovability in his own words and in a rather general way; cf. I, §8-19. His critique focuses on a proof of unprovability that relies on the representation of provability within the language of the axiom system in question. Thus, following his initial acknowledgement of algorithmic unprovability proofs in I, §17, Wittgenstein repeats, at rather great length, his critique of a meta-mathematical unprovability proof. It is this type of unprovability proof that he judges unable to provide a compelling reason to give up the search for a proof of G. The most crucial aspect of any comparison of two different types of unprovability proofs is the question of what serves as the "criterion of unprovability" (I, §15). According to Wittgenstein, such a criterion should be a purely syntactic criteria independent of any meta-mathematical interpretation of formulas. It is algorithmic proofs relying on nothing but syntactic criteria that serve as a measure for assessing meta-mathematical interpretations, not vice-versa.

[...]

Gödel's proof is not an algorithmic unprovability proof. Instead, Gödel's proof is based on the representation of provability within the language of PM. Based on this assumption, Gödel concludes that PM would be inconsistent (or ω-inconsistent) if G (or $\neg G$) were provable. Thus, given PM's (ω)-consistency, G is undecidable. This reasoning is based on the purely hypothetical assumption of the provability of G; it does not consider any specific proof strategies for proving formulas of a certain form within PM.

Given an algorithmic unprovability proof for G, the meta-mathematical statement that G is provable would be reduced to absurdity. This would be a compelling reason to abandon any search for a proof. Such a proof by contradiction would contain a "physical element" (I, §14) because a *meta*-mathematical statement concerning the provability of G is reduced to absurdity on the basis of an algorithmic, and thus purely mathematical, proof. Wittgenstein does not reject such a proof by contradiction in §14."

... Lampert: [Lam17].

We note further that, according to Lampert, Wittgenstein's remarks in [Wi78] can be interpreted as claiming that any 'intended interpretation' of quantification in 'an *instance* of a formula or of its abbreviation, such as G or $\neg \exists y B(y, \lceil G \rceil)$' in Gödel's reasoning, would introduce an element of 'prose' which—in the context of the *evidence-based* perspective of this investigation—may reasonably be taken to be an assumption such as that of Aristotle's particularisation (§4., Definition 12), which is stronger than both Gödel's ω-consistency (see §5.D.) and Rosser's Rule C (see §5.G.):

"The proofs by contradiction of the type to which Wittgenstein objects are proofs that involve *interpretation* of logical formulas: the inconsistency concerns the relation between the provability of a formula (proven or merely assumed) and its interpretation. Here, "interpretation" is not to be understood in terms of purely formal semantics underlying proofs of correctness or completeness. Formal semantics assign extensions to formal expressions without considering specific instances of formal expressions that are meant to refer to extensions. Instead, in proofs of contradiction Wittgenstein is concerned with an "interpretation of a formula" refers to an *instance* of a formula or of its abbreviation, such as G or $\neg \exists y B(y, \lceil G \rceil)$, stated as a sentence in ordinary language or a standardized fragment of an ordinary language. Interpretations of this kind are so-called "intended interpretations" or "standard interpretations", which are intended to identify extensions such as truth values, truth functions, sets or numbers by means of ordinary expressions. As soon as interpretations of this kind become involved, one departs from the realm of mathematical calculus and "prose" comes into play, in Wittgenstein's view. Therefore, Wittgenstein's "non-revisionist" attitude does not apply to proofs by contradiction that rest on intended interpretations. A rigorous mathematical proof should not be affected by the problem that some intended interpretation may not refer to that to which it is intended to refer, which is a genuinely philosophical problem."

... Lampert: [Lam17].

9.A.a.   Wittgenstein's objection to Gödel's reasoning

The substance of Wittgenstein's objection on philosophical considerations to Gödel's reasoning—essentially to the latter's argument that a Peano Arithmetic such as PA can be assumed ω-consistent without inviting contradiction—emerges if we note that, apart from:

  (a) the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA (see §7.D.) where decidability is defined *weakly* in terms of algorithmic *verifiabilty* by §2., Definition 2; and

  (b) the *strong* finitary interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA (see §7.E.) where decidability is defined *strongly* in terms of algorithmic *computability* by §2., Definition 3;

there is a *weak* 'Wittgensteinian' interpretation $\mathcal{I}_{PA(PA, \ W)}$ of PA where 'satisfaction/truth' is defined, also *weakly*, in terms of algorithmic *verifiabilty* over the structure of the PA-numerals in PA.

The interpretation $\mathcal{I}_{PA(PA, \ W)}$ of PA reflects in essence the views Wittgenstein emphasised in his 'notorious paragraph' ([Wi78], Appendix III 8; see also §9.A.), where he seems to suggest that the 'truth' of a proposition of a mathematical system must be definable in terms of its 'provability' within the system.

9.A.b.   Interpreting Tarski's Theorem constructively

The significance of the interpretation $\mathcal{I}_{PA(PA, \ W)}$ is that standard expositions of Tarski's Theorem ([Ta35]) appear to *implicitly* suggest[84] that—contrary to §9.A.d., Definition 28—a verifiable *evidence-based* truth of the formulas of a first-order Arithmetic such as PA, under a well-defined interpretation, cannot be defined algorithmically in the Arithmetic.

> ***Tarski's Theorem***: *"The set Tr of Gödel numbers of wfs of S which are true in the standard model is not arithmetical, i.e., there is no wf $A(x)$ of S such that Tr is the set of numbers k for which $A(\bar{k})$ is true in the standard model."*
>
> *. . . Mendelson: [Me64], p.151, Corollary 3.38.*

However, we now show why it follows from Gödel's reasoning in [Go31] that such an *implicit* inference cannot be justified by appeal to Tarski's Theorem.

9.A.c.   Tarski's definitions under the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA

We note first that Tarski's definitions (see §2.A.; also §23.(14)) of the *satisfiability* and *truth* of the formulas of the Peano Arithmetic PA under an interpretation over $\mathbb{N}$ are significant only if, for any *specified* PA-formula $[A(x)]$ and any *specified* n in $\mathbb{N}$, there is a deterministic algorithm which will *evidence* whether or not the interpretation $A^*(n)$ of $[A(n)]$ holds under the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA.

Classically, such determination is implicitly assumed to be algorithmically *computable* by appeal to the Church-Turing Thesis. However, in this investigation we argue that, by the principle of Ockham's Razor:

---

[84]We note that both John Lucas ([Lu61], [Lu96]) and Roger Penrose ([Pe90], [Pe94]) accept this seeming implication unquestioningly (see [An07a], [An07b], [An07c]), and use it explicitly as an arguable cornerstone of their respective defence of their respective Gödelian Theses (see also §17.).

(i) there is no justification for such a presumption of *strong* algorithmic *computability* when we can define 'effective computability' (see §4.G.a.) in terms of *weak* algorithmic *verifiability* (see §2., Definition 2);

(ii) the minimum requirement of Tarski's definitions under the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA (as defined in §2.B.) is only *weak* algorithmic *verifiability* (see also §7.D.).

Thus, a formula $[A(x)]$ of PA is defined as *satisfied* under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ if, and only if, for any *specified* value n that lies within the range of the variable x in the domain $\mathbb{N}$ of $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, there is a deterministic algorithm which will *evidence* that the interpretation $A^{*}(n)$ of $[A(n)]$ is *true* under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

The formula $[(\forall x)A(x)]$ of PA is then defined as *true* under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ if, and only if, $[A(x)]$ is *satisfied* under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. Other definitions follow as usual (see §2.A.a.).

9.A.d.   A *weak* 'Wittgensteinian' interpretation $\mathcal{I}_{PA(\mathbb{N},\ W)}$ of PA

We note next that, just as we can interpret PA without relativisation in ZF (in the sense indicated by Feferman in [Fe92]), we can also interpret PA in PA where—also under Tarski's standard definitions (see §2.A.)—we now define the *satisfiability* and *truth* of the formulas of PA under a *weak* 'Wittgensteinian' interpretation $\mathcal{I}_{PA(\mathbb{N},\ W)}$ of PA over the structure of the PA numerals by appeal to the *provability* of a formula in PA.

Definition 28. (Wittgensteinian interpretation) *An atomic formula $[A(x)]$ of PA is satisfied under $\mathcal{I}_{PA(PA,\ W)}$ if, and only if, for any substitution of a specified PA-numeral $[n]$ for the variable $[x]$, there is a deterministic algorithm[85] which will evidence that the formula $[A(n)]$ is provable in PA.*

We note that:

Theorem 9.1. *The atomic formulas of PA are algorithmically verifiable as true or false under the algorithmically verifiable, weak 'Wittgensteinian', interpretation $\mathcal{I}_{PA(\mathbb{N},\ W)}$.*

Proof. It follows from Gödel's definition of the primitive recursive relation xBy[86]—where x is the Gödel number of a proof sequence in PA whose last term is the PA formula with Gödel-number y—that, if $[A]$ is an atomic formula of PA, we can algorithmically verify which one of the PA formulas $[A]$ and $[\neg A]$ is necessarily PA-provable and, ipso facto, true under $\mathcal{I}_{PA(\mathbb{N},\ W)}$. □

We further note that—as in the case of the *weak* standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA (see §2.B.)—the requirement of Tarski's definitions under the *weak* 'Wittgensteinian' interpretation $\mathcal{I}_{PA(PA,\ W)}$ of PA is also only *weak* algorithmic *verifiability*. Further:

Definition 29. *The formula $[(\forall x)A(x)]$ of PA is true under $\mathcal{I}_{PA(PA,\ W)}$ if, and only if, $[A(x)]$ is satisfied under $\mathcal{I}_{PA(\mathbb{N},\ W)}$.*

Other definitions too follow as usual.

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[85]e.g., Gödel's primitive recursive relation xBy in [Go31], Definition 45, p.22.

[86][Go31], p. 22(45).

Comment: We note that the truth of $[(\forall x)A(x)]$ under $\mathcal{I}_{PA(PA,\ W)}$ does not entail that $[(\forall x)A(x)]$ is provable in PA, since it only entails that $[(\forall x)A(x)]$ is algorithmically *verifiable* as always *true* under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$. For the truth of $[(\forall x)A(x)]$ under $\mathcal{I}_{PA(PA,\ W)}$ to entail that $[(\forall x)A(x)]$ is provable in PA (by appeal to the Provability Theorem for PA—see §2.E., Theorem 2.17), it would have to entail further that $[(\forall x)A(x)]$ is algorithmically *computable* as always *true* under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

9.A.e.   *Weak* truth under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ is equivalent to *weak* truth under $\mathcal{I}_{PA(PA,\ W)}$

It follows that:

Theorem 9.2. *The interpretations* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ *and* $\mathcal{I}_{PA(PA,\ W)}$ *of PA are isomorphic.*

Proof. By definition, the domain of the PA numerals under $\mathcal{I}_{PA(PA,\ W)}$ is isomorphic to the domain $\mathbb{N}$ of the natural numbers under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.

Further, both $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ and $\mathcal{I}_{PA(PA,\ W)}$ are interpretations of PA such that:

(i)   each predicate letter A_j^n of PA under $\mathcal{I}_{PA(PA,\ W)}$ interprets as an n-place relation under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ in $\mathbb{N}$;

(ii) each function letter f_j^n of PA under $\mathcal{I}_{PA(PA,\ W)}$ interprets as an n-place operation under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ in $\mathbb{N}$ (i.e., a function from $\mathbb{N}$ into $\mathbb{N}$);

(iii) each individual constant a_i of PA under $\mathcal{I}_{PA(PA,\ W)}$ interprets as some fixed element under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ in $\mathbb{N}$;

(iv) the provable formulas of PA are locally 'true' respectively by definition under each of the interpretations $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ and $\mathcal{I}_{PA(PA,\ W)}$.

The theorem follows.   □

It immediately follows that:

Corollary 9.3. *A formula of PA is true (by §9.A.d., Definition 28) under the weak 'Wittgensteinian' interpretation* $\mathcal{I}_{PA(PA,\ W)}$ *if, and only if, it is true (by §2.B., Definition 10) under the weak standard interpretation* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.   □

Moreover, it also follows that, by the classical definition of a 'model' (see §23.(10)):

Corollary 9.4. *The weak standard interpretation* $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, *and the weak 'Wittgensteinian' interpretation* $\mathcal{I}_{PA(PA,\ W)}$, *are both weak models of PA.*

Proof. By §2.B.a., Theorem 2.7, the axioms of PA interpret as true, and the PA rules of inference preserve such truth, under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, which thus defines the *weak* 'standard' model of PA. By §9.A.e., Corollary 9.3, the axioms of PA interpret as true, and the PA rules of inference preserve such truth, under $\mathcal{I}_{PA(PA,\ W)}$, which too is thus a *weak* model of PA.   □

9.A.f. PA is not ω-consistent

We recall that, in order to avoid intuitionistic objections to his reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions, Gödel introduced the syntactic property of ω-consistency as an explicit assumption in his formal reasoning ([Go31], p.23 and p.28).

> ω**-consistency**: A formal system S is ω-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any specified S-term $[a]$.

The significance of such an assumption of ω-consistency for constructive mathematics is highlighted in §4.A.. We now show that (independently of §2.F., Corollary 2.22) it follows from §9.A.e., Corollary 9.4 that:

Theorem 9.5. *PA is not ω-consistent.*

Proof. Assume PA is ω-consistent.

(i) If $[(\forall x)A(x)]$ is a provable formula of PA, then $[A(0)]$, $[A(1)], [A(2)], \ldots$, are all PA-provable and so $[(\forall x)A(x)]$ is true under $\mathcal{I}_{PA(PA,\ W)}$ (see §9.A.d.).

(ii) Hence $[\neg(\forall x)A(x)]$ cannot be PA-provable if PA is ω-consistent.

(iii) By Gödel's reasoning in [Go31], if PA is ω-consistent, then there is a PA-formula $[R(x)]$ such that both $[(\forall x)R(x)]$ and $[\neg(\forall x)R(x)]$ are not provable in PA, even though $[(\forall x)R(x)]$ is true under $\mathcal{I}_{PA(\mathbb{N},\ W)}$.

(iv) Hence $[\neg(\forall x)R(x)]$ can be added to PA as an axiom without inviting inconsistency.

(v) However, if $[\neg(\forall x)R(x)]$ were to be added as a PA axiom, it would follow that $[(\forall x)R(x)]$ is not true under $\mathcal{I}_{PA(PA,\ W)}$—a contradiction.

The theorem follows. $\square$

CHAPTER 10. *PHILOSOPHICAL FOUNDATIONS*

10.   What *is* mathematics?

The question of what mathematics is, and what it is not, remains a serious concern of mathematics educators, since the faith-based foundations (see §4.B.) of classical mathematical paradigms do not offer any coherent (see, for instance, §16.) philosophical perspective of the nature of mathematics to both teachers and students of mathematics (see 20.). As remarked by mathematics educationist Laxman Luitel:

> "Talking about the nature of mathematics is not a new agenda. It had been discussed even before the fourth century. Plato and his student Aristotle are the first who provided the space to discuss nature of mathematics. From Plato's point of view, objects of mathematics had an existence of their own, behind the mind, in the external world (Dossey, 1992). As a mathematics student from the school level, now I am realizing that my schooling was shaped by Plato's point of view. In my schooling, I thought that mathematics was beyond of our thinking, abstract and discovered. In this context, a student of Plato, Aristotle, had different views. His views of mathematics were not based on a theory of an external, independent, unobservable body of knowledge but were based on experienced reality where knowledge is obtained from the experimentation, observation, and abstraction (Dossey, 1992). From this, I want to say that the observation of any object differs from person to person. This is based on their experience or related to how they perceive. Thus, there is no objective truth related to mathematics or there is no any objective answer to what mathematics is. What is mathematical knowledge? How is the mathematical knowledge constructed?"
>
> . . . Luitel: *[Lui19], pp.4-5.*

Without attempting to address the issue in its broader dimensions, we take Wittgenstein's remarks in [Wi78] (see §9.A.) as implicitly suggesting that (see also §8.C.):

(i) Mathematics is to be considered as a set of precise, symbolic, languages.

(ii) Any language of such a set, say the first order Peano Arithmetic PA (or Russell and Whitehead's PM in Principia Mathematica, or the Set Theory ZF) is, ideally, intended to *adequately express* and *effectively communicate*—in a finite and unambiguous manner—relations between elements that are external to the language PA (or to PM, or to ZF).

(iii) Moreover, each such language is two-valued if we assume that, again ideally, there is some *evidence-based* methodology that defines/determines whether a specific relation either holds (is true) or does not hold (is false) externally under any well-defined interpretation of the language.

(iv) Further:

   (a) A selected, finite, number of primitive formal assertions about a finite set of selected primitive relations of, say, a language $\mathcal{L}$ are defined as axiomatically $\mathcal{L}$-provable;

   (b) All assertions about relations that can be effectively defined in terms of the primitive relations are termed as $\mathcal{L}$-provable if, and only if, there is a finite sequence of assertions of $\mathcal{L}$, each of which is either a primitive assertion or which can effectively be determined in a finite number of steps as an immediate consequence of any two assertions preceding it in the sequence by a finite set of finitary rules of consequence;

   (c) All $\mathcal{L}$-provable relations interpret as true under any well-defined interpretation of $\mathcal{L}$.

As expressed by Pasquale Frascolla when analysing Ludwig Wittgenstein's perspective on the 'truth' values assignable to the formal expressions of a mathematical language that are intended to represent 'real facts':

"To speak of a contingent state of affairs pictured by "$\Omega^{(2x2)}$'x"= Ω^4'x" is like admitting the conceivability of a situation in which this equation is incorrect; but this would imply that the whole logical space in which real facts are placed would no longer be the same. The possibility that two different facts are pictured, respectively, by a proposition in which an expression generated by a double application of the second iteration of a given operation (when applied to an initial symbol) occurs, and by the proposition obtained from the former by replacing that same expression with the expression generated by three applications of the same operation to its own result—starting from the result of its application to the same initial symbol—would be conceded. But, in virtue of the general properties of the concept of operation, this is an impossible world, exactly in the same way in which, according to the view of propositions held in the *Tractatus*, no possible world can contain two different facts corresponding to a proposition and to its double negation. In conclusion, the attempt to attribute a contingent status to numerical identities is bound to fail because we cannot conceive formal properties of the world alternative to those which constitute our logical space and, at the same time, remain comfortably immersed in the latter. To attribute to our world formal properties different from those which we actually acknowledge would mean simply to abandon our logical space for a new one; and, according to Wittgenstein, this cannot be done.

The fundamental assumption of picture theory entail that one cannot speak meaningfully, in language, of the forms of language; this restriction of the sayable is tantamount to ruling out the possibility that forms be conceived as a peculiar sort of objects (as simple constituents of contingent ideal states of affairs), and thus to ruling out the possibility that they provide "the substance" of a second world. This circumstance explains the radical, extreme nature of the consequences deriving from the anti-Platonic conception of forms maintained by Wittgenstein in the *Tractatus*. Given the three following premises:

1 logic and mathematics deal (in a manner we are shortly going to examine in detail) with formal properties and relations of linguistic expressions;

2 the role of Wittgensteinian objects of an ideal world cannot be assigned to forms;

3 a thought, a meaningful proposition, is the logical picture of a contingent configuration of objects and is true if this configuration exists, false otherwise;

only one conclusion can be drawn, if the coherence of the whole has to be saved: the results achieved in logic and mathematics cannot be formulated in meaningful propositions expressing a thought (and, *a fortiori*, the predicates "true" and "false" cannot be appropriately applied). This is exactly the drastic and somewhat disconcerting inference that Wittgenstein makes, as regards mathematics, in 6.2 and 6.21: "The propositions of mathematics are equations (*Gleichungen*), and therefore pseudo-propostions (*Scheinsätze*). A proposition of mathematics does not express a thought."

Now the reason becomes clear why it is quite correct to speak of the existence of a logicist point of view in the philosophy of mathematics of the *Tractatus*. Both in logic and mathematics, suitable notations are constructed in order to perspicuous those formal properties of linguistic expressions which, for the reasons mentioned above, cannot be described meaningfully. In logic, the notation of propositional variables, of symbols of propositional functions, of sentential connectives, of quantifiers, etc., is needed, according to Wittgenstein, to construct formulae which clearly exhibit forms of propositions. These formulae can be used to check whether a given proposition has a certain metalogical property, or whether certain metalogical relations hold between two or more given propositions. The method of checking for a metalogical property or relation is either a mechanical procedure of decision, as occurs with the truth-table method and with any other equivalent method; or a semi-mechanical procedure of generation, as occurs with the derivation of tautological formulae in an axiomatized logical calculus. In exactly the same way, arithmetical notation (numerals and complex arithmetical terms) is introduced as part of a symbolism devoted

to exhibiting perspicuously the forms of the results of the successive application of all sorts of iterations and compositions of logical operations. Arithmetical calculation has a perfectly analogous role to that of logical calculation: to ascertain that the relation of identity of meaning—which, according to Wittgenstein, cannot be meaningfully be spoken of—holds between any two given expressions having certain specified forms. Thus the first part of 6.2 states: "Mathematics is a logical method". However, for a thorough understanding of the content of the group of propositions 6.23-6.241, a further development of the comparison between logic and mathematics is required. To this purpose it is expedient to verify whether, and to what extent, Wittgenstein's view on logical *sinnlos* proposition applies also to the equations into which numerical identities are translated. As it concerns logic, the pivotal thesis is what Wittgenstein himself calls the "fact" which "contains in itself the whole philosophy of logic" (*T* 6.113). The "fact" in question concerns the process of recognition of the truth of a tautology (and of the falsity of a contradiction) and, more generally, the process of recognition of the formal properties of a proposition and of the formal relations between propositions. Suppose that a certain proposition is given. After its form has been perspicuously exhibited by means of logical notation, we are able to decide, using the method of truth-tables or some other equivalent procedure, whether the proposition under consideration is true for all the truth-possibilities of its component propositions, or whether it is false for all the truth-possibilities, or whether it is true for some and false for some others. At this point, if only values **T** or only values **F** have been obtained, and if this procedure has been carried out in order to settle the truth-value of the given proposition, then our work can be considered concluded. In these two limiting cases (tautology and contradiction) the truth-values can be settled by applying suitable procedures of sign manipulation without "going out of language". In contrast, if the tested proposition is true for some of the truth-possibilities of its component propositions (true in some possible worlds) and false for others of them (false in some other possible worlds), then the only method to decide its truth-value is "to go out and see" which of the possible has actually come true, or, in other words, what is the effective configuration of the world. Although Wittgenstein speaks only of the possibility of recognizing that logical propositions are true "from the symbol alone" (*am Symbol allein*), it is obvious that he considers the peculiar mark of all formal properties and relations precisely the possibility of being recognized in this way.[42] This is a well-known aspect of Wittgenstein's philosophy of logic and corresponds to the traditional opposition between *a priori* and *a posteriori* knowledge."

. . . Frascolla: [Fra94], § *The "Knowledge" of Forms: Vision and Calculation, pp.24-27.*

10.A. An empathetic perspective of Wittgenstein's views on what mathematics is, and what it is not

In his doctoral thesis [Daw15], Ryan Dawson seeks to give an empathetic perspective of how Wittgenstein's views on what mathematics is, and what it is not, may be viewed as enlightening, rather than as specifying:

"Wittgenstein's later philosophy of mathematics has been widely interpreted to involve Wittgenstein's making dogmatic requirements of what can and cannot be mathematics, as well as involving Wittgenstein dismissing whole areas (e.g. set theory) as not legitimate mathematics. Given that Wittgenstein promised to 'leave mathematics as it is', Wittgenstein is left looking either hypocritical or confused.

This thesis will argue that Wittgenstein can be read as true to his promise to 'leave mathematics as it is' and that Wittgenstein can be seen to present coherent, careful and non-dogmatic treatments of philosophical problems in relation to mathematics. If Wittgenstein's conception of philosophy is understood in sufficient detail, then it is possible to lift the appearance of confusion and contradiction in his work on mathematics. Whilst apparently dogmatic and sweeping claims figure in Wittgenstein's writing, they figure only as pictures to be compared against language-use and not as definitive accounts (which would claim exclusive right to correctness).

Wittgenstein emphasises the importance of the applications of mathematics and he feels that our inclination to overlook the connections of mathematics with its applications is a key source of a

number of philosophical problems in relation to mathematics. Wittgenstein does not emphasise applications to the exclusion of all else or insist that nothing is mathematics unless it has direct applications. Wittgenstein does question the alleged importance of certain non-applied mathematical systems such as set theory and the logicist systems of Frege and Russell. But his criticism is confined to the aspirations towards philosophical insight that has been attributed to those systems. This is consonant with Wittgenstein's promises in (*PI*, §124) to 'leave mathematics as it is' and to see 'leading problems of mathematical logic' as 'mathematical problems like any other.' It is the aim of this thesis to see precisely what Wittgenstein means by these promises and how he goes about keeping them."

. . . Dawson: [Daw15], Abstract.

Thus, Dawson not only implicitly endorses Frascolla's view that, for Wittgenstein, a mathematical language is intended to represent 'real facts' but, as this extended extract from [Daw15] shows, reflects the *evidence-based* perspective posited in §10.(i) to §10.(iv) above (see also §10.C. and §10.C.a.), insofar 'that mathematics is made up of a large number of systems, each of which has the meaning of its symbols set by the rules of the system':

"Wittgenstein's notion of 'prose' seems to be intended in part to contrast exact mathematical language with ambiguous ordinary (non-mathematical) language. Writing concerning certain work of Skolem's, he comments:

An explanation in word-language of the proof (of what it proves) only translates the proof into another form of expression: because of this we can drop the explanation altogether. And if we do so, the mathematical relationships become much clearer, no longer obscured by the equivocal expressions of word-language. (*PG*, p.422)

One might be tempted to therefore take Wittgenstein's notion of 'prose' to be based upon a prior theory or conception of what mathematics is, this theory being such as to bring out the superior precision of mathematical language. This is how Shanker (1987, p.209) takes the notion and it is worth considering Shanker's view in some detail since Shanker's view risks making it impossible to take Wittgenstein seriously in his promise not to disagree with mathematicians about mathematics (*PG*, p.369). If Wittgenstein were to be read, as Shanker reads him, as having a theory of what mathematics is, then it would have to be answered how that theory could be understood in terms of Wittgenstein's conception of philosophy and how it could avoid being dogmatic. We shall explore how Shanker goes about reading Wittgenstein as having a theory of mathematics and this will provide a useful way to approach the question of what role the notion of 'prose' has for Wittgenstein. The discussion will naturally lead us on to the question of how the role of 'prose' can be understood as non-dogmatic.

As Shanker describes him, Wittgenstein holds that mathematics is made up of a large number of systems, each of which has the meaning of its symbols set by the rules of the system. The most important rules of a system are the axioms and any further rules are derived from there. Whilst this view needs to be explained at more length[40], the part which is most important for the purpose of how Shanker distinguishes 'prose' from 'proof' is that the meanings of mathematical terms are taken to be fixed by the axioms of the system in question. In order to put across this picture, Shanker lays a particular stress upon remarks like:

Mathematics consists entirely of calculations. In mathematics everything is algorithm and nothing is meaning: even when it doesn't look like that because we seem to be using words to talk about mathematical things. Even these words are used to construct an algorithm. (*PG*, p.208)

In remarks like this Wittgenstein can be read as articulating a conception of mathematical propositions as rules that fix the ways that terms are to be used within a mathematical system. Under this view mathematical systems can be seen to be 'autonomous', in that each system is not reliant upon anything other than the propositions of the system itself for its validity (*PR*, §111;

Shanker 1987, p.305-306).”
. . . Dawson: [Daw15], §2.4. Reading and Misreading Wittgenstein on 'prose', pp.32-33.

We note that such a reading of Wittgenstein's view of:

— mathematics as 'made up of a large number of systems, each of which has the meaning of its symbols set by the rules of the system';

— 'mathematical propositions as rules that fix the ways that terms are to be used within a mathematical system';

— mathematical systems as autonomous, in the sense that 'each system is not reliant upon anything other than the propositions of the system itself for its validity';

can be seen as implicit in the Complementarity Thesis §1., Thesis 1, and §1., Definition 1 (see also §9.); since the *evidence-based* perspective of this investigation—all of whose arguments and formal conclusions seek to follow from, or be entailed by, distinguishing between algorithmic *verifiability* (§2., Definition 2) and algorithmic *computability* (§2., Definition 3)—can be viewed as 'rule-based' from a Wittgensteinian perspective.

On the other hand, the significance of making a distinction between languages of adequate expression and languages of categorical communication—as in §10.E.—is reflected in Dawson's argument that:

> "The pictures of mathematical propositions as rules and of mathematical systems as autonomous are important parts of Wittgenstein's thinking. Before directly considering how these ideas relate to the role of the notion of 'prose', it is worth considering the relationship to some other key themes, especially Wittgenstein's thoughts on the significance of metamathematics and the possibility of scepticism in relation to mathematics. This will help us to better understand what motivates Shanker's view and allow us to do justice to these motivations without following Shanker in ascribing a thesis concerning the nature of mathematics to Wittgenstein.
>
> If mathematics is not seen as a set of autonomous systems and is instead seen as a single global system (perhaps unified by a single set of axioms) then the edifice of mathematics might seem to be open to the possibility of global doubts—problems that could bring down the entire edifice. This kind of picture might seem tempting when talking about 'mathematics' in very general terms, perhaps looking upon 'the body of mathematics' as akin to 'the body of history.' If it were to turn out that some crucial detail of history had been gotten wrong, say that Julius Caesar was not a real person, then this would force us to revise vast amounts of history. But if mathematical systems are autonomous then no analogous relationship holds with problems in mathematical systems (especially contradictions) since then at worst only the system in question could be affected.
>
> This kind of system-specific thinking can be found in Hilbert's writing as well and Friederich (2011, p.5, p.8) suggests that Hilbert may have been an influence on Wittgenstein's development of the idea. But the picture of autonomy that Wittgenstein articulates goes further, since Wittgenstein also stresses that mathematical systems are only related to one another by relationships of analogy or by transformation of one system into another. Hilbert, by contrast, wanted to develop mathematics which would be 'about' mathematical systems. His idea was that mathematical techniques could be used to show whether certain important mathematical systems were consistent. These metamathematical techniques were intended to be part of a foundational programme of putting mathematics on a solid footing by showing mathematical systems to be consistent."
> . . . Dawson: [Daw15], §2.4. Reading and Misreading Wittgenstein on 'prose', p.33.

Further, the significance of Dawson's argument that:

— 'pictures of mathematical propositions as rules and of mathematical systems as autonomous are important parts of Wittgenstein's thinking'

since, otherwise:

— 'the edifice of mathematics might seem to be open to the possibility of global doubts— problems that could bring down the entire edifice',

too is implicit in the *evidence-based* perspective of this investigation, which argues that it is non-falsifiable (in the sense of §4.B.), philosophical, assumptions such as:

— Aristotle's particularisation (§4., Definition 12) is valid in any *well-defined* interpretation of the first-order Peano Arithmetic PA; and

— an axiom of infinity interprets as a *well-defined* mathematical object in any *well-defined* interpretation of set theories such as ZF and ACA_0;

that have admitted false conclusions such as:

(i) Gödel's formula $[(\forall x)R(x)]$, and its negation $[\neg(\forall x)R(x)]$, are both not provable in PA. (which contradicts [An16], Corollary 8.2, p.42; see also §2.F., Corollary 2.20);

(ii) The structure of the finite ordinals under any putative interpretation of ZF is isomorphic to the structure $\mathbb{N}$ of the natural numbers (which contradicts §11.B., Lemma 11.1);

(iii) The subsystem ACA_0 of second-order arithmetic is a conservative extension of PA (which contradicts §15.A., Theorem 15.4).

The obscured significance of Dawson's argument emerges if we view such, *faith-based*, assumptions as metamathematical propositions which would meet his criteria of 'descriptive propositions and not rules':

> "If metamathematical expressions were mathematics 'about' mathematical systems, at least in the referential sense, then metamathematical propositions would be descriptive propositions and not rules. The metamathematical expressions would not be parts of an autonomous system and would instead be dependent upon other systems for their meaning (namely the systems which they are 'about'). This would run contrary to the picture expressed in (*PG*, p.208) and hence one might wonder whether Wittgenstein would therefore have to reject metamathematics or whether he might instead philosophically interpret metamathematics in a different way from Hilbert.
>
> Whilst Wittgenstein acknowledges the validity of metamathematics, he can be seen as disputing its alleged significance. Rather than interpreting metamathematics to be mathematics 'about' mathematics, he presses a picture in which metamathematical techniques just appear as more techniques. The picture is that when one employs a metamathematical technique, one is introducing a new technique and thereby adding something that enables one to do things in the system that one could not do before. In this sense one is creating a new system in which the old system might be seen to figure as a part—one now has a larger system in which a simulation of the old system can be seen. So the metamathematical technique allows one to prove results in the expanded system but it is a matter of prose to say that they are results 'about' the original system, at least if 'about' is meant referentially. One may well say this but saying it would not be to give an interpretation of the mathematical result rather than to simply state the result. One could put the point by saying that metamathematics for Wittgenstein would be mathematics 'about' mathematics in a very different way from how a description is a statement about its subject-matter.[41] So it seems that Wittgenstein could have maintained a view of mathematical systems as autonomous (Shanker 1987, p.305-306) and still acknowledged the validity of metamathematics."

> . . . Dawson: [Daw15], §2.4. Reading and Misreading Wittgenstein on 'prose', p.34.

For instance, §9.A.f., Theorem 9.5—PA is not ω-consistent—appeals to a rule-based, *weak* 'Wittgensteinian' interpretation $\mathcal{I}_{PA(\mathbb{N},\ W)}$ of PA (see §9.A.d.), which illustrates the argument that 'when one employs a metamathematical technique, one is introducing a new technique and thereby adding something that enables one to do things in the system that one could not do before'.

We contrast this with the metamathematical assertions that Gödel made with respect to his own formal reasoning in [Go31] which, from the perspective Dawson attributes to Wittgenstein, can be seen as 'descriptive propositions and not rules'.

For instance, as argued in §11. to §11.G.i., these would include:

(i) "Theorem VIII: There exist undecidable arithmetical propositions in each of the formal systems[53] mentioned in Theorem VI."

Gödel: [Go31], p.31.

(ii) "Let ω be the SENTENCE by which $Wid(\kappa)$ is expressed in P."

Gödel: [Go31], p.37.

(iii) "That the truth of $\omega\ Imp\ (17\ Gen\ r)$ can be deduced from (23) rests simply on the fact that the undecidable proposition $17\ Gen\ r$, as was remarked at the very beginning, asserts its own unprovability."

Gödel: [Go31], p.37, fn.67.

The technical nuances implicit in the above examples could explain why, according to Dawson:

> "Wittgenstein's thoughts related to metamathematics and its use can be hard to follow, especially for contemporary mathematicians, because of the way in which techniques related to metamathematics have become widespread. Wittgenstein's thinking might even appear confused to a contemporary mathematician but it is worth noting that Wittgenstein's thinking does not have to be seen as confused. The development that produces the most confusion in relation to understanding Wittgenstein is that it has become common to distinguish between the syntax and the semantics of a mathematical system. In loose terms the idea is that a system is syntactically a set of expressions the system is associated with the possible structures (described set-theoretically) which would satisfy the expressions—these are the models of the system. A sentence of the system is then described as true if it is satisfied in all consistent models of the system. This mathematical approach post-dates Wittgenstein's work but it is plausible that if Wittgenstein had encountered the approach then he would have acknowledged the validity of these techniques and denied that they offer a definitive analysis of 'truth.'[42] He would perhaps say that the move to considering models for the system effectively moves us to an expanded system, allowing us to show things in the expanded system that relate by means of analogy to the original system. This sort of question will become particularly important in chapter 10, where we will see that some of the objections to Wittgenstein's remarks on Gödel's incompleteness theorems have originated from a failure to see how Wittgenstein understands such metamathematical techniques.[43]"

. . . Dawson: [Daw15], §2.4. Reading and Misreading Wittgenstein on 'prose', p.34.

Moreover, from the *evidence-based* perspective of this investigation, Dawson's view that what 'produces the most confusion in relation to understanding Wittgenstein is that it has become common to distinguish between the syntax and the semantics of a mathematical system', is reflected in the argument, in §1., that such distinction fails to recognise that mathematical 'provability' and mathematical 'truth' need to be interdependent and complementary, rule-based, assignments-by-convention.

As expressed by Dawson:

"If one looks at the picture of mathematical propositions as rules, of mathematical systems as autonomous and the criticism of interpretations of metamathematics as mathematics 'about' mathematics then a picture might seem to emerge of Wittgenstein as advocating a conception of mathematical systems as self-defining. According to such a picture, the axioms of a system might be said to be rules which fix certain aspects of the way that the terms should be used and then the propositions are further rules which are derived from the initial rules. Wittgenstein might then be seen, as he is portrayed by Shanker (1987, p.305-306) as advocating an account of mathematics as the totality of all such systems. Any expressions that are not part of these systems would be non-mathematical, even if they appeared to have some connection to the systems. Such expressions which appeared to be connected to the systems but were not themselves expressions of the systems could be distinguished by being referred to as 'prose.'"

. . . Dawson: [Daw15], §2.4. Reading and Misreading Wittgenstein on 'prose', p.35.

From such a perspective, 'if mathematical propositions were simply rules that set up a system then the contrary of a mathematical proposition would not be a part of a system at all and would thus be meaningless', it would follow that, as argued in §16., the well-known semantic and putatively logical paradoxes cannot be considered as posing serious linguistic or philosophical concerns from an *evidence-based* perspective of constructive mathematics:

"Whilst the notion of mathematical systems as autonomous systems of rules has various advantages (such as, as we shall see in chapter 7, undermining the notion that contradictions might pose a threat to all of mathematics) and it undoubtedly figures in Wittgenstein's thought, the key question is whether this conception is the basis of Wittgenstein's notion of 'prose' in the way that Shanker suggests (1987, p.209). Revealing of the difficulty with this claim is when Shanker says that Wittgenstein's idea of mathematical statements as rules is not an "alternative picture" (p.64) but "a precise philosophical clarification of mathematical syntax" (p.65). The trouble with Shanker's claim is that it if the idea of mathematical propositions as rules is part of a clarification then it surely is just one possible picture and cannot be said to be simply part of "mathematical syntax." Otherwise comments like the following would ring hollow:

> . . . the whole point is that I must not have an opinion . . . I have no right to want you to say that mathematical propositions are rules of grammar. I only have the right to say to you, "Investigate whether mathematical propositions are not rules of expression . . ." (*LFM*, p.55)

If Wittgenstein has to assume that mathematical systems are bodies of rules in order to say what is prose and what is not, then any of his claims concerning what is prose and what is not are just matters of opinion that he would have 'no right' to present as clarifications. The danger here is the danger of supposing that mathematical statements are rules rather than proposing the picture as an object of comparison with which to model mathematical statements as rules. Taking mathematical propositions to be rules would be problematic both because of Wittgenstein's promises not to be dogmatic, and also because Wittgenstein at points seems to point out the limitations of this picture. As Floyd (2000, p.251) notes (in criticism of Shanker), if mathematical propositions were simply rules that set up a system then the contrary of a mathematical proposition would not be a part of a system at all and would thus be meaningless. She cites the following remark by Wittgenstein:

> My explanation mustn't wipe out the existence of mathematical problems.

> That is to say, it isn't as if it were only certain that a mathematical proposition made sense when it (or its opposite) had been proved. (This would mean that its opposite would never have a sense (Weyl).) On the other hand, it could be that certain apparent problems lose their character as problems—the question as to Yes or No. (*PR*, p. 170)

This kind of self-cautioning remark makes much more sense if Wittgenstein is saying that mathematical propositions are 'like' rules or that it can help us see past certain problems if we see

mathematical propositions as akin to rules. This strongly suggests that Shanker's reading is attributing a thesis to Wittgenstein which Wittgenstein is keen to avoid. But if mathematical propositions are only akin to rules (rather than actually being rules) then we are left with the problem of how Wittgenstein does go about using the term 'prose.' Contrary to Shanker, I want to suggest that Wittgenstein's use of this notion can be understood without any need to invoke a prior conception of mathematics. "

. . . Dawson: [Daw15], §2.4. Reading and Misreading Wittgenstein on 'prose', pp.35-36.

As Dawson remarks in [Daw15] with reference to Wittgenstein's remarks (*PR*, p. 170) cited by Juliet Floyd, this 'kind of self-cautioning remark makes much more sense if Wittgenstein is saying that mathematical propositions are 'like' rules or that it can help us see past certain problems if we see mathematical propositions as akin to rules'.

That Wittgenstein might, indeed, have intended what Dawson attributes to him, is suggested by Timm Lampert's 2019 paper [Lam19], where he notes that:

"In two letters to Russell from 1913, Wittgenstein conjectured that first-order logic is decidable. His conjecture was based on his conviction that a decision procedure amounts to an equivalence transformation that converts initial formulas into ideal symbols of a proper notation that provides criteria for deciding the logical properties of the initial formulas. According to Wittgenstein, logical properties are formal properties that are decidable on the basis of pure manipulations of symbols. This understanding of logical properties (such as provability or logical truth/falsehood) is independent of and prior to any interpretation or application of logic."

. . . Lampert: [Lam19], Abstract.

If so, Wittgenstein can be seen as essentially, informally and presciently, articulating in 1913 the Complementarity Thesis (§1., Thesis 1) in anticipation of the *evidence-based* 2016 proof of consistency for PA detailed in [An16] (Theorem 6.8, p.41; see also §2.C.a., Theorem 2.16).

10.B.   The semiotics of an *evidence-based* perspective of logic and mathematics

We note that the semiotics of the *evidence-based* perspective of §9. to §10.A. is reflected in Brian Rotman's broader analysis:

"Insofar, the, as the subject matter of mathematics is the whole numbers, we can say that its objects—the things which it countenances as existing and which it is said to be 'about'—are unactualized possibles, the potential sign production of a counting subject who operates in the presence of a notational system of signifiers. Such a thesis, though, is by no means restricted to the integers. Once it is accepted that the integers can be characterized in this way, essentially the same sort of analysis is available for numbers in general. The real numbers, for example, exist and are created as signs in the presence of the familiar extension of Hindu numerals—the infinite decimals—which act as their signifiers. Of course, there are complications involved in the idea of signifiers being infinitely long, but from a semiotic point of view the problem they present is no different from that presented by *arbitrarily* long finite signifiers. And moreover, what is true of numbers is in fact true of the entire totality of mathematical objects: they are all signs—thought/scribbles—which arise as the potential activity of a mathematical subject.

Thus mathematics, characterized here as a discourse whose assertions are predictions about the future activities of its participants, is 'about'—insofar as this locution makes sense—itself. The entire discourse refers to, is 'true' about, nothing other than its own signs. And since mathematics is entirely a human artefact, the truths it establishes—if such is what they are—are attributes of the mathematical subject: the tripartite agency of Agent/Mathematician/Person who reads and writes mathematical signs and suffers its persuasions.

But in the end, 'truth' seems to be no more than the unhelpful relic of the platonist obsession with a changeless eternal heaven. The question of whether a mathematical assertion, a prediction, can be said to *be* 'true' (or accurate or correct) collapses into a problem about the tense of the verb. A prediction—about some determinate world for which true and false make sense—might in the future be seen to be true, but only *after* what it foretold has come to pass; for only then, and not before, can what was pre-dicted be dicted. Short of fulfillment, as is the condition of all but trivial mathematical cases, predictions can only be believed to be true. Mathematicians believe because they are persuaded to believe; so that what is salient about mathematical assertions is not their supposed truth about some world that precedes them, but the inconceivability of persuasively creating a world in which they are denied. Thus, instead of a picture of logic as a form of truth-preserving inference, a semiotics of mathematics would see it as an inconceivability-preserving mode of persuasion—with no mention of "truth' anywhere."

> . . . *Rotman:* [*Rot88*], *pp.33-34.*

The concepts sought to be formalised in §9., Definitions 24 to 26, and illuminated in §10.(i) to §10.(iv), are also reflected in an informal survey of John von Neumann's 'Methodology of Science', where Giambattista Formica makes an intriguing perspective comparing Hilbert's and von Neumann's foundational 'standpoints' on the 'nature of mathematics', and remarks:

Comparison between Hilbert's and von Neumann's foundational standpoints

	Hilbert *Methodology of the Finitist Program*	**von Neumann** *Oppurtunistic Methodology of Science*
Foundations Of Mathematics	*a)* - Axiomatic presentation of informal mathematical theories *b)* - Justification through finitist consistency proof	*a)* - Axiomatic presentation of informal mathematical theories *b)* - Justification through the criterion of success in science
Foundations of sciences	*a)* - Axiomatic presentation of informal scientific theories *b)* - Justification through relative consistency proof in mathematics	*a)* - Axiomatic presentation of informal scientific theories *b)* - Justification through the criterion of success in applications
Outcomes	*Internalization* of the epistomological problem: finitist mathematics is a part of mathematics	*Externalization* of the epistomological problem: mathematics in science and science in applications

> . . . *Formica:* [*For10*]

"...I would like to stress the fact that von Neumann, just like Hilbert, believes that there is this *quite peculiar duplicity in the nature of mathematics*, namely that its origin is to be found both in human experience and in pure thought. In other words, besides its abstract nature, mathematics has also an empirical nature. Otherwise, he would have talked about the success that

mathematics - and through mathematics, science - achieves in applications in terms of a miracle or an unreasonable effectiveness. In *The Mathematician* (1947) he writes:

> I think that it is a relatively good approximation to truth [. . .] that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost entirely aesthetical motivations, than to anything else and, in particular, to an empirical science. [. . .]. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art*. [. . .]. [A]t a great distance from its empirical source, or after much "abstract" inbreeding, a mathematcal subject in danger of degeneration. [. . .]. In any event, whether this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less directly empirical ideas. (von Neumann 1947, p. 9)"
>
> . . . *Formica:* [For10]

Formica concludes that:

> "What I have tried to do is to describe John von Neumann's oppurtunistic methodology of science. His closeness to Hilbert's spirit is evident. Von Neumann always shared with Hilbert a conservative attitude towards science, combined with a strong faith in the axiomatic method. Yet he never became dogmatic. After Gödel's remarkable results he began a profound revaluation of the concept of mathematical rigour in order to find a solution to the episttomological problem left open in Hilbert's program, i.e. the justification of mathematical and scientific theories formulated in an axiomatic fashion. He found the solution in the criterion of success - not alien to Hilbert's school - to which he ascribed a key foundational task. With regards to Hilbert's program, he made exactly the opposite move: he thought that through the externalization of the epistomological problem, mathematics finds its justification in science, while science finds its own in the realm of applications. In spite of his radical opportunism, von Neumann also believed that mathematical models satisfy certain standards of objectivity. However, in his view these two apparently contrasting ideas - opportunism and objectivity - do not contradict each other, but can be reconciled on the basis of certain ontological assumptions on the nature of mathematics which he, once again, shared wth Hilbert. Mathematics has a double origin, which is to be found pure thought and in human experience. Once this has been accepted, success in the realm of applications is not a miracle, but is probably due to the original empirical source, sometimes remote and obscure, latent in all mathematical thought."
>
> . . . *Formica:* [For10]

10.C. Mathematics must *serve* Philosophy and the Natural Sciences

Formica's comparison between Hilbert's and von Neumann's foundational standpoints[87] reflects another tacit thesis of this investigation, which is the making of an—admittedly arbitrary—distinction between (compare [Ma08]; see also [Fe99]):

- The **natural scientist's hat**, whose wearer's responsibility is recording—as precisely and as objectively as possible—our sensory observations (corresponding to computer scientist David Gamez's 'Measurement' in [Gam18], Fig.5.2, p.79) and their associated perceptions of a 'common' external world (corresponding to Gamez's 'C-report' in [Gam18], Fig.5.2, p.79; and to what some cognitive scientists, such as Lakoff and Núñez in [LR00], term as 'primary', and 'secondary', *conceptual metaphors*);

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[87]Which merit further comparison with [Brd13] that lies, however, beyond the immediate scope of this investigation.

> **Comment**: By 'primary conceptual metaphors', we refer here to metaphors that a human mind conceives, corresponding to physical states of an individual brain which are rooted in the mind's awareness (compare §17.J.a., Hypothesis 2) of phenomena that are accepted as existing *in* a commonly accepted, external, physical reality.
>
> By 'secondary conceptual metaphors', we refer here to metaphors that a human mind conceives, corresponding to physical states of an individual brain which are rooted in the mind's awareness (compare §17.J.a., Hypothesis 3) of phenomena that are accepted as existing *only in the individual mind's perceptions of the symbolic representations* of a commonly accepted, external, physical reality.

- The ***philosopher's hat***, whose wearer's responsibility is abstracting a coherent—albeit informal and not necessarily objective—holistic perspective of the external world from our sensory observations and their associated perceptions (corresponding to Carnap's *explicandum* in [Ca62a]; and to Gamez's 'C-theory' in [Gam18], F, p.79); and

- The ***mathematician's hat***, whose wearer's responsibility is providing the tools for adequately expressing such recordings and abstractions in a symbolic language of unambiguous communication (corresponding to Carnap's *explicatum* in [Ca62a]; and to Gamez's 'P-description' and 'C-description' in [Gam18], Fig.5.2, p.79).

  > **Comment**: We could view this distinction as seeking to address the questions of:
  >
  > - ***What*** we do in scientific disciplines;
  > - ***Why*** we do what we do in scientific disciplines; and
  > - ***How*** we express and communicate whatever it is that we do in scientific disciplines.
  >
  > Philosophically, we could go even further and view the above activities holistically: as providing merely the means by which an intelligence, such as that of homo sapiens, instinctively strives to realise its own creative—largely latent—potential within the evolutionary arrow of a, perpetually-changing, environment that not only gives birth to, but nurtures and encourages, a species to continually adapt to survive unforeseen and unforeseeable life-threatening challenges (including those that may be created by the species shooting itself in the foot).
  >
  > 'Largely latent' since it is scientific folklore that an 'average' human brain, during its lifetime, is estimated to have perceptively—as measured by posthumous brain scans—used only about 4-10% of its capacity to address the—not unreasonably, corresponding 4-10% of—challenges that, over the estimated millions/billions of years of its evolution, the brain has met and adapted to in order to survive.

That this distinction may not reflect conventional wisdom is highlighted in the following argument that:

(a) if mathematics is to *serve* as a lingua franca for the physical sciences,

(b) then it can only represent physical phenomena *unambiguously* by insistence upon *evidence-based* reasoning

(c) which, in some cases (see §16.C. to §16.D.), may prohibit us from building a mathematical theory of a physical process

   - based on the assumption that the limiting behaviour of *every* physical process which can be described by a Cauchy sequence

- *must* be taken to correspond to the behaviour of the classically defined Cauchy limit of the sequence.

Reason: If the discrete states of some physical phenomena are describable only in terms of arithmetical functions that are algorithmically *verifiable*, but not algorithmically *computable*—which, as suggested in [An15], §4 (see §19.D.), may be the case for quantum phenomena, or for the values of some fundamental physical constants—then although, as seen in §4.G., Theorem 4.2, such functions may define Cauchy sequences that, by definition, have algorithmically *uncomputable* mathematical limits, such abstract postulations may not correspond to any limiting state of the phenomena which might, for instance, involve a phase change (see §16.C. to §16.D.)!

The above attempts to crystalise Hermann Weyl's perspective that:

"...I believe the human mind can ascend toward mathematical concepts only by processing reality as it is given to us. So the applicability of our science is only a symptom of its rootedness, not a genuine measure of its value. It would be equally fatal for mathematics—this noble tree that spreads its wide crown freely in the ether, but draws its strength from the earth of real intuitions and perceptions (*Anschauungen und Vorstellungen*)—if it were cropped with the shears of a narrow-minded utilitarianism or were torn out of the soil from which it grew."

...Weyl: [We10], p.10.

From such an *evidence-based* perspective, eliminating ambiguity in critical cases—such as communication between mechanical artefacts, or a putative communication between terrestrial and/or extra-terrestrial intelligences (whether mechanical or organic)—may be viewed as the very raison d'être of mathematical activity.

An activity which aspires:

(1) First, to the construction of mathematical languages that can symbolically express those of our abstract concepts (corresponding to what Lakoff and Núñez term as 'primary', and 'secondary', *conceptual metaphors*, and to Carnap's *explicandum*) which can be subjectively addressed unambiguously.

Languages such as, for instance, the first-order Set Theory ZF, which can be well-defined formally but which have no constructively well-defined model that would admit *evidence-based* assignments of 'truth' values to set-theoretical propositions by a mechanical intelligence.

Author's comment: By 'subjectively address unambiguously' I intend in this context that there is essentially a subjective acceptance of identity by me between:

- an abstract concept in my mind (corresponding to Lakoff and Núñez's 'conceptual metaphor' in [LR00], p.5) that I intended to express symbolically in a language; and
- the abstract concept created in my mind each time I subsequently attempt to understand the import of that symbolic expression (a process which can be viewed in engineering terms as analogous to my attempting to formalise the specifications, i.e., *explicatum*, of a proposed structure from a prototype; and which, by the 'Sapir-Whorf Hypothesis', then determines that my perception of the prototype is, to an extent, essentially rooted in the symbolic expression that I am attempting to interpret).

Sapir-Whorf Hypothesis

"The hypothesis of linguistic relativity, part of relativism, also known as the SapirWhorf hypothesis . . . is a principle claiming that the structure of a language affects its speakers' world view or cognition, and thus people's perceptions are relative to their spoken language.

The principle is often defined in one of two versions: the strong hypothesis, which was held by some of the early linguists before World War II[1], and the weak hypothesis, mostly held by some of the modern linguists.[1]

— The strong version says that language determines thought and that linguistic categories limit and determine cognitive categories.

— The weak version says that linguistic categories and usage only influence thought and decisions."

. . . *Wikipedia: Linguistic Relativity, downloaded 24/10/2019.*

Comment: From the *evidence-based* perspective of this investigation we would, however, qualify the (Sapir-Whorf) hypothesis of linguistic relativity as the assertion that language:

∗ first, *limits* the expression of a thought (as distinct from the thought itself) to what can be expressed within the language by the vocabulary and grammar of the language; and,

∗ second, *influences* how such an expression is understood under a subsequent interpretation of the expression,

so that—even to the originator of the thought—that which is communicated in a subsequent interpretation of the expression need not necessarily reflect faithfully that which was sought to be expressed in the first place.

(2) Second, to study the ability of a mathematical language to objectively communicate the formal expression (corresponding to Carnap's *explicatum*) of *some* such concepts categorically.

A language such as, for instance, the first order Peano Arithmetic PA, which can not only be well-defined formally, but which has a finitary model that admits *evidence-based* assignments of 'truth' values to arithmetical propositions by a mechanical intelligence, and which is categorical (albeit, with respect to algorithmic computability (see §2.E., Corollary 2.18).

Author's comment: By 'objectively communicate categorically' I intend in this context that there is essentially:

(a) first, an objective (i.e., on the basis of *evidence-based* reasoning in the sense of §4.C.) acceptance of identity by another mind between:

- the abstract concept created in the other mind when first attempting to understand the import of what I have expressed symbolically in a language; and

- the abstract concept created in the other mind each time it subsequently attempts to understand the import of that symbolic expression (a process which can also be viewed in engineering terms as analogous to confirming that the formal specifications, i.e., *explicatum*, of a proposed structure do succeed in uniquely identifying the prototype, i.e., *explicandum*);

and:

(b) second, an objective acceptance of functional identity between abstract concepts that can be 'objectively communicated categorically' based on the *evidence* provided by a commonly accepted doctrine such as, for instance, the view that a simple functional language can be used for specifying *evidence* for propositions in a constructive logic (in the sense of §4.C.).

10.C.a.   Conclusion: What is mathematics?

We conclude, from the *evidence-based* perspective of this investigation, that any 'meaningful' dialogue as to 'What is mathematics' ought to admit—i.e., be consistent with—the interpretation where:

1. Mathematics is a language which, by implicit definition of a language, cannot entail an ontology. It can only denote/refer to 'objects' outside the language whose properties and inter-relationships the language seeks/is intended to first, unambiguously express and, second categorically communicate, symbolically.

2. We could also view such 'objects' as the fleeting—but, in principle, measurable/recordable and, hopefully, 'identifiable'—physical brain patterns of individual brains that correspond to the *primary* and *secondary* concepts/conceptual metaphors created in the associated mind which, then, seeks to symbolically 'capture' these fleeting sensory perceptions.

> **Comment**: We note that, in his paper [Lvd16], Stathis Livadas essentially seeks to address, from a more fundamental phenomenological perspective, the nature—and conceivable significance (for what is commonly referred to as 'mathematics/mathematical activity')—of precisely such, primarily set-theoretically originating, 'fleeting brain patterns' which could, conceivably, be postulated as corresponding to putative 'mathematical-logical objects':
>
> > "A major question that can be raised concerning the essential nature of mathematical-logical objects is their relation to objective reality, to the capacities of the mind and ultimately to temporality conceived in the dual sense of both a 'real world' factor underlying spatio-temporal phenomena and as an intrinsic property of a self-constituting temporal consciousness. On the one hand, they can be regarded in terms of the 'external' objective temporality as immutable objects occupying each time an absolute temporal position with regard to the reality of [the] objective world and on the other hand as temporally constituted re-identifications of appearing profiles with regard to the inner temporality of consciousness. In the following, I will consider the term 'mathematical-logical objects' as identical in meaning to 'mathematical objects' insofar as these are taken in the sense of objects of a formal axiomatical theory, their sense bearing a certain affinity with the one attributed to them, for instance, in Gödel's *Is mathematics syntax of language?* (Feferman et al. 1995, pp.334-363), and, more closely, in Tieszen's *After Gödel* (Tieszen 2011). In this respect, I will talk about mathematical-logical objects primarily as objects of formal theories, consequently not in the sense of objects of sensuous observation within objective reality, e.g., as geometrical figures in plane or space or space-filling graphs in a digital screen, etc. On account of this general position one is confronted with a host of philosophical attitudes that range between pure platonism in which mathematical objects are perfect, immutable objects of an ideal world transcending human experience and naive empiricist approaches in which mathematical-logical objects are merely elaborated representations in our mind of what physical experience brings to us through our sense organs."
> >
> > . . . *Livadas: [Lvd16], §Introduction, p.4.*
>
> Livadas clarifies that:
>
> > "In general, constituted platonism to a significant extent meets my own view of mathematical-logical objects, for instance, in categorizing mathematical and logical objects as mind-dependent$_1$ and mind-independent$_2$ with the meaning given to these terms (Sect. 5, §8). I would like to add, though, that Tieszen's denying of Hilbert's approach to objects or constructions of mathematical theories as reducible to immediate sensory perceptions of sign tokens and the proposition of categorial intuition and certain intensional forms of consciousness to accede, for instance, to 'second order' mathematical concepts such as the transfinite sets enters us more generally into some tricky questions with to the fundamentals of Husserlian approach. For example, it is not very clear how it is possible to talk, e.g., about formal individuals as objects of a 'lowest-level' intentionality which are not necessarily causally related to the subject, namely about those individuals generated non-arbitrarily in imagination and yet not possessing an absolute 'real-world' temporal position. What I want to say is that one may propose the a priori

directedness of intentionality to account for the possibility of grasping and reflecting on mathematical objects-individuals (and generally on collections of such objects and their categorical properties) and yet we have no means to describe this a priori directedness but in terms of the contents of its enactments considered as already objectified. However, in being objectified the intentional contents in question are already constituted as real objectivities and 'spatio-temporal like'."

. . . Livadas: [Lvd16], §5 Conclusion, p.29.

Livadas notes further that 'an interesting question to raise is the extent to which a further quest on the character of inner temporality of consciousness might further clarify the inner horizon especially of transfinite objects':

"In view of all that was argued in this paper, mathematical and in a wider sense mathematical-logical objects established as such within formal theories are largely shaped by the constitutional-intentional capacities of each one's consciousness in intersubjective coincidence in a way that their formation is constrained by each one's specific presence in the world (as the soil of primitive experience) and also by the mathematical intuitions associated with this kind of presence. These intuitions corresponding to such features of reason as abstraction, idealization, invariability in transformation, the sense of symmetry, permutation, uniformity, etc., are not intentional capacities of consciousness, at least in a pure reality-independent sense, as they are not conceivable without reference to a reality transcendent to a self-constituting subjectivity, even non-conceivable without reference to a reality impregnated with a sort of historicity with respect to the existence of all conscious beings as its co-constituting and reduction performing factors. Given that mathematical-logical objects, as long as we go up the level of abstraction and complexity, acquire a widening inner horizon of content and properties, an interesting question to raise is the extent to which a further quest on the character of inner temporality of consciousness might further clarify the inner horizon especially of transfinite objects. For what is an undeniable fact reducible to the evidence of cogito is the non-eliminable 'superfluity' of an objective whole in actual reflection, be it in extreme cases a transfinite set or a huge cardinal on the level of constituted, with regard to the generating predicative activities of the mind in discrete steps within objective time. This kind of 'deficiency' in subjective constitution that ultimately seems as temporal in nature can be partly accountable for a characterization of mathematical-logical within the context of formal theories as intersubjectively identical, transtemporal ones and yet provided with an 'inner' and 'outer' horizon open to potentially new insights and clarifications corresponding to possible further refinements in the future to come of the intentional-predicative capacities of the mind. In this view the non-eliminability of the mathematical transfinite and the associated meanings of non-finitistic objects might seem as a question pointing to the need for a further clarification, if feasible at all on the constitutional level, of the notion of a self-constituting temporality and the way it underlies the meaning-giving acts of the mathematical mind."

. . . Livadas: [Lvd16], §5 Conclusion, p.33.

3. The *primary* concepts/conceptual metaphors are those that can be expressed unambiguously, and communicated categorically, to other minds and, ipso facto, treated by convention as referring to 'objects'—and their inter-relationships—that we accept by consensus as 'existing' in a common external reality that living organisms experience through their individual sensory organs; and which we attempt to quantify, and record, by means of standardised methods/apparatus of measurement (using a language intended for that precise purpose).

4. The *secondary* concepts/conceptual metaphors are those that can be well-defined unambiguously within a language, by virtue of the fact that a rich enough language can

well-define expressions ('square circle', 'present king of France') within the language that do not denote/refer—under any possible well-defined interpretation of the language—to anything in our commonly accepted reality. However, some such expressions may in turn create concepts/conceptual metaphors of 'objects' conceivable in an individual mind ('present king of France', 'unicorn') that are associated with fleeting brain patterns in the individual mind which, even if assumed capable of being 'identified' by that mind with a specific symbolic expression, cannot claim to be capable of any well-defined interpretation under which the symbolic expression denotes/refers to anything (mathematical/physical) other than the expression itself.

5. We distinguish here between the concept/conceptual metaphor created in a mind when attempting to interpret the word 'unicorn' from the concept/conceptual metaphor created in a mind when viewing, say, a sculpture depicting an artist's conception of a 'unicorn'; in the latter the concept/conceptual metaphor relates to a physical object that is the 'artist's unicorn'. The latter is, of course capable of being communicated categorically (for instance, photographically).

6. Another way of expressing this is to say that the representation of *primary* concepts/conceptual metaphors in a language always has an *evidence-based* Tarskian interpretation; whereas any representation of a *secondary* concept/conceptual metaphor has no evidence-based Tarskian interpretation.

7. In other words, *primary* concepts/conceptual metaphors are grounded in our consciousness (putatively corresponding to a fleeting brain pattern) of an 'objective' reality; *secondary* concepts/conceptual metaphors in our consciousness (putatively corresponding to a fleeting brain pattern) of only a 'subjective' reality.

8. The above distinction assumes significance, of course, only in view of the *fintary* proof of consistency for the first-order Peano Arithmetic PA (see §2.C., Theorem 2.16)—which entails that PA is *categorical* (see §2.E., Corollary 2.18)— since it enables a distinction to be made between what is only assignable as *provable* within an axiomatic mathematical theory, and what is additionally assignable as *true* under any *well-defined* (i.e., *evidence-based* in the sense of §4.F., Definition 19) interpretation of the theory.

10.D.   When can a formal assertion claim to 'mean' what it represents?

The above perspective raises another important philosophical issue which is implicit in the key thesis of Floyd and Putnam's paper [FP00], and reflected in Wittgenstein's remark (see §9.A.):

> "If you assume that the proposition is provable in Russell's system, that means it is true in the Russell sense, and the interpretation '*P* is not provable' ... has to be given up."

We may state this issue explicitly as:

Query 3. *When can a formal assertion claim, under a well-defined interpretation, to 'mean' what it represents?*

Now if, as argued earlier, we accept that PA formalises our intuitive arithmetic of the natural numbers, and that there is a *well-defined* interpretation $\mathcal{I}$ of PA, it follows that every well-formed formula of PA interprets under $\mathcal{I}$ as a *well-defined* arithmetical expression in $\mathbb{N}$, and every *well-defined* arithmetical expression in $\mathbb{N}$ can be represented as a PA-formula.

The question then arises:

Query 4. *When is a specified number-theoretic function or relation representable in PA?*

10.D.a.   Formal expressibility and representability

Now, the classical PA-expressibility and representability of number-theoretic functions and relations is addressed by the following three conventional definitions[88]:

(a) A number-theoretic relation $R(x_1, \ldots, x_n)$ is said to be expressible in PA if, and only if, there is a well-formed formula $[A(x_1, \ldots, x_n)]$ of PA with n free variables such that, for any natural numbers $k_1, \ldots, k_n$:

   (i) if $R(k_1, \ldots, k_n)$ is true[89], then PA proves: $[A(k_1, \ldots, k_n)]$;

   (ii) if $R(k_1, \ldots, k_n)$ is false, then PA proves: $[\neg A(k_1, \ldots, k_n)]$.

(b) A number-theoretic function $f(x_1, \ldots, x_n)$ is said to be representable in PA if, and only if, there is a well-formed formula $[A(x_1, \ldots, x_n, y)]$ of PA, with the free variables $[x_1, \ldots, x_n, y]$, such that, for any natural numbers $k_1, \ldots, k_n, l$:

   (i) if $f(k_1, \ldots, k_n) = l$, then PA proves: $[A(k_1, \ldots, k_n, l)]$,

   (ii) PA proves: $[(\exists! l) A(k_1, \ldots, k_n, l)]$.

(c) A number-theoretic function $f(x_1, \ldots, x_n)$ is said to be strongly representable in PA if, and only if, there is a well-formed formula $[A(x_1, \ldots, x_n, y)]$ of PA, with the free variables $[x_1, \ldots, x_n, y]$, such that, for any natural numbers $k_1, \ldots, k_n, l$:

   (i) if $f(k_1, \ldots, k_n) = l$, then PA proves: $[A(k_1, \ldots, k_n, l)]$,

   (ii) PA proves: $[(\exists! l) A(x_1, \ldots, x_n, y)]$,

10.D.b.   When may we assert that $A(x_1, \ldots, x_n)$ 'means' $R(x_1, \ldots, x_n)$?

We can, thus, re-phrase Query 4 as:

Query 5. *If a number-theoretic relation $R(x_1, \ldots, x_n)$ is expressible by a PA-formula $[A(x_1, \ldots, x_n)]$, when may we assert that, under a well-defined interpretation $\mathcal{I}$ of PA, $A(x_1, \ldots, x_n)$ 'means' $R(x_1, \ldots, x_n)$?*

---

[88]cf. [Me64], p117-118.
[89]We can assume without any loss of generality that such 'truth' is implicitly *evidence-based*.

Now we note that, if $R(x_1, \ldots, x_n)$ is arithmetical[90], then one of its PA-representation is $[R(x_1, \ldots, x_n)]$, whose interpretation is $R(x_1, \ldots, x_n)$ under $\mathcal{I}$. Hence every arithmetical relation $R(x_1, \ldots, x_n)$ is the interpretation of some PA-formula that expresses it in PA, and we can adapt this to give a formal definition of the term 'means':

Definition 30. (Meaning of a formula under interpretation) *If a number-theoretic relation $R(x_1, \ldots, x_n)$ is expressible by a PA-formula $[A(x_1, \ldots, x_n)]$ then, under any well-defined interpretation of PA, $A(x_1, \ldots, x_n)$ means $R(x_1, \ldots, x_n)$ if, and only if, $R(x_1, \ldots, x_n)$ is the well-defined interpretation, under $\mathcal{I}$ in $\mathbb{N}$, of some PA-formula that expresses $R(x_1, \ldots, x_n)$ in PA.*

Query 4 can now be expressed precisely as:

Query 6. *When is a number-theoretic relation a well-defined interpretation of some PA-formula that expresses it in PA?*

Now, by definition, the number-theoretic relation $R(x_1, \ldots, x_n)$, and the arithmetic relation $A(x_1, \ldots, x_n)$, can be effectively shown as equivalent for any specified set of natural number values for the free variables contained in them.

However, for $A(x_1, \ldots, x_n)$ to *mean* $R(x_1, \ldots, x_n)$, we must have, in addition, that $R(x_1, \ldots, x_n)$ can be effectively transformed into an arithmetical expression, such that it can be a *well-defined* interpretation of some PA-formula that expresses it in PA.

The significance of this is seen in §11.G., where we show how Gödel's sketch of his proof of Theorem XI in [Go31] violates this restriction.

10.E.   Recognising the strengths and limitations of ZF and PA

Another significan consequence of the preceding section is that we may need to recognise explicitly in our basic mathematical education (see §20.) that *evidence-based* reasoning:

(a) restricts the ability of *highly expressive* mathematical languages, such as the first-order Zermelo-Fraenkel Set Theory ZF, to *categorically* communicate abstract concepts (corresponding to Lakoff and Núñez's conceptual metaphors in [LR00]) such as those involving Cantor's first limit ordinal ω[91];

> **Comment**: From the *evidence-based* perspective of this investigation, the restricted ability of languages such as ZF to categorically communicate even a putative semantic interpretation of set-theoretical proofs, reflects Piccinini's distinction (see §3.A.) between *Justified True Belief* and *Factually Grounded Belief*, and his thesis in [Pic19] that we can only term—and, implicitly, categorically communicate—the latter as '*knowledge*'.
>
> In other words, the provable propositions of any set theory which admits an axiom of infinity can be seen to be axiomatically *Justified True Beliefs* that cannot, however, be treated as *knowledge*, and categorically communicated as *Factually Grounded Beliefs*, since such a theory does not admit a *well-defined* model (see Definition 26).

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[90]A number-theoretic relation is arithmetical if, and only if, it can be defined by means of only the concepts $+, \cdot$ (addition and multiplication of natural numbers) and the logical constants $\vee, \neg, \forall x, =$, where $\forall x$ and $=$ are to refer to natural numbers. The definiens of such a concept must therefore be constructed only by means of the indicated symbols, variables for natural numbers $x, y, \ldots$ and the symbols 0 and 1 (function variables and set variables must not occur). (cf. [Go31], p.29.)

[91]See [LR00], Preface, p.*xii-xiii*: "*How can human beings understand the idea of actual infinity?*"

and:

(b) restricts the ability of *effectively communicating* mathematical languages, such as the first-order Peano Arithmetic PA, to *well-define* infinite concepts such as ω (see §14.A.a.).

> **Comment**: From the *evidence-based* perspective of this investigation, the restricted ability of languages such as PA to *well-define* infinite concepts—such as Cantor's first transfinite ordinal ω—reflects the fact that PA is categorical (see Corollary 2.18), and cannot therefore admit properties of terms that, by Piccinini's distinction (see §3.A.), interpret as *Justified True Beliefs* but not *Factually Grounded Beliefs*.
>
> In other words, the provable propositions of any arithmetic such as PA are axiomatically *Justified True Beliefs* that *can*, further, be treated under any *well-defined* interpretation of PA as *knowledge* which can be categorically communicated as *Factually Grounded Beliefs*.

> **Comment**: The need for making the above distinction—between languages of adequate expression and languages of categorical communication vis à vis our conceptual metaphors—explicit in scientific discourse, is highlighted by linguist David Adger's attempt to address—in an interview on the on-line blog Nautilus—the challenges faced in, first, faithfully representing in a language of ordinary discourse our sensory perceptions of the external world; and, second, in communicating such representations categorically:

>> "**You write that language, like fiction, creates meaning where none existed before. Have you ever thought that when we talk to each other, we're just throwing fictional worlds at one another?**
>>
>> I totally think thats what we're doing! This goes back to what the limits of humans are. Our own internal worlds are what we represent and think about the external world. They're probably all wrong to start with, and then we try and link those fictions with other people's fictions. I think most of our interaction is an attempt to align the fictions that we build to be able to survive in the world. And this goes back to culture wars. People have different fictions of the world and sometimes they are pretty brutally out of alignment. Like now. And that's quite terrifying, right?
>>
>> **So how do people understand each other?**
>>
>> Our languages allow us to both create these new ways of thinking and to maintain the ways of thinking we've already built. So there's another paradox there, which is that language maintains as well as creates. But then our individual worlds can be distinct. And trying to pull them into cohesion with other people is a huge amount of work."
>> ...*Adger: [Adg19]*.

We argue, further, that from an *evidence-based* perspective, the notorious semantic and logical paradoxes arise out of a blurring of this distinction, and an attempt to ask of a language more than it is designed to deliver (see §16.).

> **Comment**: To situate this in the wider perspective of a common challenge faced by societies in holistically resolving—without conflict—paradoxes arising out of humankind's subjective spiritual, and objective material, yearnings, one could also argue analogously that we may need to recognise explicitly in our basic education the healthy humility induced in a society that seeks to appreciate—and emphasise the importance of—an *evidence-based* reasoning which:
>
> - restricts the claim of religious education to *categorically* communicate abstract, essentially-subjective, concepts (corresponding to Lakoff and Núñez's conceptual metaphors), such as, for instance, those involving spiritual concepts concerning the qualities and nature of a putative Deity (e.g., see [BP14]);

and:

- restricts the claim of scientific education to *well-define* such concepts (see §4.B.).

Such paradoxes dissolve once we accept that the ontology which we seek to represent in any interpretation of a symbolic language is determined not by the 'logic' of the language—which, contrary to conventional wisdom, we may take as intended solely to assign unique *evidence-based* 'truth' values to the declarative sentences of the language (in the sense of §9., Definitions 24 to 26)—but by the rules that determine the 'terms' which can be admitted into the language (see, for instance, §7.C., Definition 22) without inviting contradiction, in the broader sense of how, or even whether, the brain—viewed as the language defining and logic processing part of any intelligence—can address contradictions.

> **Comment**: In other words, as emphasised succinctly by Wittgenstein in the concluding one-line paragraph #7 of his seminal 1922 Tractatus Logico-Philosophicus:
>
> > "Whereof one cannot speak, thereof one must be silent."
> > . . . *Wittgenstein:* [Wi22], *p.90.*.

Our concerns in these areas have been those commonly shared by scholars of all disciplines—including challenged graduate-level students—with a more than passing interest in the reliability, for their intended individual purposes, of the mathematical languages which any scientific enquiry—by implicit definition—finds essential for attempting unambiguous expression of abstract thought and, subsequently, its unequivocal communication to an other.

We thus argue the thesis that the obstacles to such expression and communication are rooted in the disconcerting perceptions of mutual inconsistency between various 'classical' and 'constructive' philosophies of mathematics vis à vis the disquieting, and seemingly 'omniscient', status accorded classically to both mathematical truth and mathematical ontologies (highlighted by Krajewski in [Kr16] and Lakoff and Núñez in [LR00]); and that such perceptions are, at heart, illusions.

They merely reflect the circumstance that, to date, all such philosophies—whether due to explicitly or implicitly held beliefs—do not unambiguously *define* the relations between a language and the 'logic' (in the sense of §1., Definition 1, and §9., Definitions 26, 27) that is *necessary* to assign unequivocal truth-values of 'satisfaction' and 'truth' to the propositions of the language under a *well-defined* interpretation.

Moreover, an epistemically grounded perspective of conventional wisdom—as articulated, for instance, in [LR00] or [Shr13]—might inadvertently be ignoring the possibility of a distinction between the multi-dimensional nature of the logic of a formal mathematical language (in the sense of §1., Definition 1), and the one-dimensional nature of the veridicality of its assertions.

Similarly, current mathematical paradigms of what is entailed by Hilbert's development of proof theory, and by Brouwer's development of constructive mathematics, may appear contradictory only because they inadvertently fail (see, for example, [RS17]; also [Mycl]) to adequately recognise that 'provability' and 'truth' need to be complementary, '*evidence-based*', assignments-by-convention towards achieving (cf., §1., Complementarity Thesis 1):

(1) The goal of proof theory, post Peano, Dedekind and Hilbert, which has so far been:

— to uniquely characterise each informally defined mathematical structure S (e.g., the Peano Postulates and their associated classical predicate logic),

– by a corresponding formal first-order language L, and a set P of finitary axioms/axiom schemas and rules of inference (e.g., the first-order Peano Arithmetic PA and its associated first-order logic FOL),

- which assign unique provability values (provable/unprovable) to each well-formed proposition of the language L without contradiction;

(2) The goal of constructive mathematics, post Brouwer and Tarski, which has so far been:

— to assign unique, *intuitionistic*, truth values (true/false) to each well-formed proposition of the language L,

– under an *intuitionistically* constructive interpretation I over the domain D of the structure S (when viewed as a 'conceptual metaphor' in the terminology of [LR00]),

- such that the *intuitionistically* true formulas of L are true under the interpretation.

In other words, whilst the focus of proof theory can be viewed as seeking to ensure that any mathematical language intended to represent our conceptual metaphors is unambiguous, and free from contradiction, the focus of constructive mathematics must be viewed as seeking to ensure that any such representation does, indeed, uniquely identify such metaphors.

> **Comment**: The, seemingly unqualified, acceptance of *essentially* Platonic elements such as inaccessible cardinals in set theory can be viewed not only as a failure, but as a refusal to admit, the necessity of the above distinction. For instance Colin J. Rittberg notes that:
>
> > "Non-platonist conceptions of the metaphysics of set theory are dismissed by Woodin as basically untenable and are not discussed in this paper. See Woodin (2009a) for Woodin's argument against non-platonism."
> >
> > *. . . Rittberg: [Rit15], p.126.*

The goals of the two activities ought to, thus, be viewed as necessarily complementing each other, rather than being independent of, or in conflict with, each other as to which is more 'foundational'—as is implicitly argued, for instance, in the following remarks of constructivist Errett Bishop in [Bi66][92]:

> "The constructive point of view is that all mathematics should have numerical meaning. In other words, every mathematical theorem should admit an ultimate interpretation to the effect that certain finite computations within the set of positive integers will give certain results. In contrast, classical (that is, contemporary) mathematics is idealistic: there is no requirement that theorems and their proofs have a numerical meaning, or any predicative content whatever. For instance, the theorem that the real numbers can be well-ordered is evocative (or idealistic), rather than descriptive (or constructive). So is the theorem that a bounded monotone sequence of real numbers converges.
>
> Brouwer has shown that the idealizations involved in classical mathematics, can, in most instances, be traced to the use of a certain logical principle—the principle of the excluded middle. It is perhaps more natural to trace them to a closely related principle—the principle of omniscience—which

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[92]We note that Bishop erroneously treats the Law of the Excluded Middle LEM—ergo the classical first-order logic FOL in which LEM is a theorem—as 'nonconstructive'. Erroneously, since the finitary consistency of PA (see §2.C.a., Theorem 2.16) entails the finitary consistency of FOL.

states that an arbitrary set A either has an element with a given property P or it does not. In case A is an infinite set this principle is not constructively valid, because the examination of each element of A to see whether one of them has property P is not something that can necessarily be done by a finite, routine process.

The constructivist replaces such transcendent logical principles as the principle of omniscience by common sense. The common sense, or operational, meanings of the standard mathematical quantifiers and connectives have been established by Brouwer. Brouwer undertook to develop mathematics along constructive lines. His development, which was not systematic, was impeded by a revolutionary, semi-mystical theory of the continuum. This theory, which in retrospect seems so unnecessary, was repellant to most mathematicians.

In addition to Brouwer, others have espoused more or less constructive points of view, usually less. There are the formalizers of constructivity, whose formal systems have little relevance to the constructivization of existing mathematics; the recursive-function theorists, who base constructivitan ad hoc assumption which is more of an impediment than a tool; Hilbert, who believed the price of a constructive mathematics was too great; and various other groups, none of which is content to let constructive mathematics follow its natural course of development. This paper describes an attempt to redevelop abstract analysis along straightforward constructive lines, rather than forcing it to support a burden of philosophical preconceptions."

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. . . Bishop: [Bi66], pp.308-309.

</div>

and of Gila Sher in [Shr18](as also by Penelope Maddy's perspective in [Ma18], [Ma18a]):

"Philosophers are divided on whether the proof-theoretic or truth-theoretic approach to logic is more fruitful. The proof-theoretic approach has its roots in Gentzen (1934-35) and Prawitz (1965). The truth-theoretic or semantic approach has its roots in Tarski (1936). More recently, the proof-theoretic approach has begun to encroach on semantics itself, with Dummett (1991), Brandom (2000), and others advocating proof-theoretic or inferentialist semantics, which they contrast with truth-theoretic, and in particular truth-conditional, semantics. Thematically, proof-theoretic semantics is associated with verificationism, the meaning-as-use approach to language, assertibilism, anti-realism, anti-representationalism, pragmatist approach to truth, and/or epistemic approach to logic. Truth-theoretic semantics is often associated with a truth-conditional theory of meaning, representational approach to mind and language, realism, correspondence truth, and/or metaphysics. Although the debate on the preferable approach to semantics goes beyond logic, it is often focused on logic, logical constants, logical inference, etc.

. . .

The high explanatory power of the truth-theoretic approach does not rule out an equal or even higher explanatory power of the proof-theoretic approach. But to the extent that the truth-theoretic approach is shown to be highly explanatory, it sets a standard for other approaches to logic, including the proof-theoretic approach."

<div style="text-align:right">

. . . Sher: [Shr18].

</div>

10.E.a.   Reductionism in mathematics

In a broad-ranging analysis of the issue, Soloman Feferman questions the rationale for even seeking 'to reduce all of mathematics to one supposedly "universal" system or foundational scheme', vis à vis reductive proof theory that merely 'pursues local reductions of one formal system to another which', Feferman remarks, 'is more justified in some sense':

"In this direction, two specific rationales have been proposed as aims for reductive proof theory, the constructive consistency-proof rationale and the foundational reduction rationale. However, recent advances in proof theory force one to consider the viability of these rationales. Despite the genuine problems of foundational significance raised by that work, the paper concludes with a defense of reductive proof theory at a minimum as one of the principal means to lay out *what rests on what* in mathematics. In an extensive appendix to the paper, various reduction relations

between systems are explained and compared, and arguments against proof-theoretic reduction as a "good" reducibility relation are taken up and rebutted."

... *Feferman: [Fe00], Abstract.*

To illustrate his, seemingly anti-foundational—i.e., anti one-solution-for-all—thesis, Feferman distinguishes between the aims of reductionism in the natural sciences and in mathematics:

"The purposes of reduction in the natural sciences and in mathematics are quite different. In the natural sciences, one main purpose is to *explain* certain phenomena in terms of more basic phenomena, such as the nature of the chemical bond in terms of quantum mechanics, and of macroscopic genetics in terms of molecular biology. In mathematics, the main purpose is *foundational*. This is not to be understood univocally; as I have argued in (Feferman 1984), there are a number of foundational ways that are pursued in practice. One such way is organizational; in that enterprise, reduction in the number of basic concepts and principles is valued, as is ease and naturalness of development. The purpose of other foundational ways is to deal with problematic concepts or principles by special kinds of reduction, such as (historically) the reduction of the complex numbers to the real numbers, or the reduction of the use of infinitesimals to the systematic use of limits. More recent examples from logic are the reduction of set theory with the axiom of choice to that without, or the reduction of classical arithmetic to intuitionistic arithmetic (insofar as the law of excluded middle is problematic for the constructivist). Foundational concerns are of course also important in the natural sciences, such as that of providing a philosophically satisfactory and physically adequate foundation for quantum mechanics; but one does not necessarily think of this as a reductive project. And, explanation is ubiquitous in mathematics, such as in the use of Galois theory to explain the unsolvability of the quintic, or of combinatorial topology to explain the Descartes-Euler formula for polyhedra; again, these are not usually thought of reductively.

Another contrast to be made is that between piecemeal or *local* projects of reduction in both the natural sciences and mathematics, and *global reductionist* programs in both. In the natural sciences, the philosophy of reductionism calls for a level-by-level theoretical of the hierarchy of sciences to a basic material monism. This is envisioned, for example, by Openheim and Putnam (1958) as proceeding from social groups on down through multi-cellular organisms, living cells, molecules and atoms, all the way to elementary particles. (Nowadays, that is to be capped by the physicists' holy grail of the GUT, the "Grand Unified Theory", which is then to be the TOE, the "Theory of Everything".) I am personally very skeptical of this kind of reductionist program in science, for reasons that I have not tried to articulate, at least not in writing (and won't try to here).[1]

Global reductionist programs in the foundations of mathematics share the monistic view with the scientific reductionism, but there is no real analogy with the hierarchical account. The most prominent examples to consider are the logicist program, set-theoretical foundations, functional foundations[2], categorical foundations and—in a sense—Hilbert's program (in its original conception). There are well-known problems with each of these that I shall not repeat here, though advocates of one or another of these programs persist in pushing them. My own view is again skeptical, and leads me to pursue local projects of reduction instead. But I would hope that even those who don't share my general skepticism as to global reductionist programs see the interest of such local projects as an illumination of what rests on what in mathematics. To the extent that various parts of mathematics are represented by formal systems, that comes down to considering relations of reduction between such systems."

... *Feferman: [Fe00], §1, Reduction and reductionism in the natural sciences and in mathematics.*

From the *evidence-based* perspective of the Complementarity Thesis (see §1., Thesis 1) of this investigation, what Feferman refers to as 'natural science' could be viewed essentially as interpretations—of formal mathematical theories of natural phenomena—where the theorems interpret as *evidence-based* truths that we, by convention, refer to as 'scientific knowledge' that corresponds to 'factually grounded beliefs' in the sense of Piccinini [Pic19] (see §3.A.).

Reduction in the interpreted theories of natural sciences would, then, consist of identifying, within a putative global reductionist program in the formal theories of mathematics, those formal propositions that would interpret as *evidence-based* truths in the above sense.

Feferman raises serious doubts concerning the possibility of identifying such a global reductionism; essentially since any such formal mathematical theory that admits interpretations which have an 'explanatory' element that, in human cognition, transcends our direct perceptions of natural phenomena, would necessarily need to accommodate the transfinite elements of set theory that can have no *evidence-based* interpretations in our perceptions of natural phenomena (and would not therefore constitute scientific 'knowledge' in Piccinini's sense):

> "Hilbert's program was motivated by the view that the "actual infinite" in mathematics is problematic, leading in some cases to contradictions. The program aimed to justify various parts of mathematics that make implicit or explicit use of principles based on the actual infinite, by representing them in formal systems which would be shown to be consistent by purely finitistic arguments. Patently, the use of set theory and even of impredicative principles (such as that of the least upper bound) in analysis would require such justification on this view. But, according to Hilbert, already arithmetic (as represented in the system PA) makes use of the actual infinite in its application of classical logic to statements involving quantification over the natural numbers. That is already seen in the assumption of the law of excluded middle for statements of the form $\forall x R(x)$ with R quantifier-free; such cannot in general be decided in a finite number of steps—one must "run through" the totality of natural numbers to determine their truth or falsity. The Hilbert school did not delimit finitist proofs by defining finitist methods in terms of a formal system. In practice, at least in the early stages, it did not go beyond PRA, and that has been argued by (Tait 1981) to be the upper limit of finitism, a thesis which is largely accepted these days. In any case, whatever formal system S_0 would be determined to represent finitism, Gödel's second incompleteness theorem showed that one would not be able establish the consistency of systems stronger than S_0 by the means available in S_0. In other words, to continue the consistency program, the idea of a privileged basis for that kind of justification for all of mathematics would have to be abandoned, and would have to be replaced by a shifting basis on some other sort of constructive principles."
>
> . . . *Feferman: [Fe00], §4, The constructive consistency-proof rationale for reductive proof theory.*

Now we note, first, that Feferman's implicit reservations concerning a finitary justification:

- for the 'application of classical logic to statements involving quantification over the natural numbers'; and

- for the classical Law of the Excluded Middle

are belied (see §2.D., Poincaré-Hilbert debate, and §5.D., Theorem 5.13/Corollary 5.14, respectively) by the *evidence-based* perspective of this investigation, which admits a *finitary* proof of *strong* consistency for PA (see §2.C.a., Theorem 2.16) based on distinguishing in arithmetic between:

- *classical* algorithmic *verifiability* (see §2., Definition 2); and

- *finitary* algorithmic *computability* (see §2., Definition 3).

However, we also note further that, from such a perspective, Feferman's reservations about the possibility of identifying a global reductionism for mathematics are justified since:

- whilst a case could conceivably be made for the applicability of Gödel's incompleteness theorems to set theory (see §11.B., Query 9 and Lemma 11.1),

- the Provability Theorem for PA (see §2.E., Theorem 2.17) entails that PA is categorical (see §2.E., Corollary 2.18),

- whence there are no formally undecidable arithmetical propositions (see also §2.F., Lemma 2.19 and Corollary 2.20);

and, moreover, since we can show that:

- *transfinite* reasoning entails that a Goodstein sequence over the finite ZF ordinals must terminate finitely *within* ZF,

  - *even if* the corresponding Goodstein sequence over the PA numerals did not terminate finitely *within* PA (see §15., Theorem 15.1 and Corollary 15.2).

In other words, Feferman's reservations can be seen to intuit the *evidence-based* conclusions of this investigation that:

- PA admits only *finitary* reasoning, and thus has a *finitary* model in which the *evidence-based* truths of the natural sciences can be interpreted categorically;

- ZF admits *transfinite* reasoning, and thus has no *finitary* model in which the *evidence-based* truths of the natural sciences can be interpreted categorically.

That, in the absence of reasoning which admits *evidence-based* quantification (see §4.C.), the distinction between finite and transfinite reasoning remains blurred can be seen in Feferman's analysis of Gentzen's reasoning:

> "In Gerhard Gentzen's groundbreaking 1936 article "Die Widerspruchsfreiheit der reine Zahlentheorie" (referred to here through its English translation in Gentzen 1969, pp. 132-201), it was shown how, in the case of arithmetic, the consistency program might be extended while hewing to finitist principles as closely as possible. Gentzen's paper contains several sections discussing the aims and significance of his consistency proof—besides its extensive technical work whose general character was indicated in sec. 3 above. Under the heading, "How are consistency proofs possible?", Gentzen says: "*There can be no 'absolute consistency proof'*. A consistency proof can merely *reduce* the correctness of certain forms of inference to the correctness of other forms of inference. . . . in a consistency proof we can use only forms of inference that count as considerably *more secure* than the forms of inference of the theory whose consistency is to be proved." (op. cit., p. 138) Gentzen then goes on to say that because of Gödel's incompleteness theorem, it is not possible to establish the consistency of arithmetic using a part or all of the methods used in that system, but: "[it] remains quite conceivable that the consistency of elementary number theory can in fact be verified by means of techniques which, in part, no longer belong to elementary number theory, but which can nevertheless be considered to be *more reliable* than the doubtful components of elementary number theory itself." (op. cit., p. 139) He later argues (op. cit., pp. 193ff) that his entire proof is finitistic except possibly for the application of transfinite induction up to ε_0, and that inspection of the argument for that principle shows it to be "*indisputable*", in contrast to the "transfinite" principles of the formal system of number theory."
>
> . . . Feferman: [Fe00], §4, The constructive consistency-proof rationale for reductive proof theory.

However if, from the *evidence-based* perspective of this investigation, we identify '*constructive*' with '*well-defined*' (see §4.F., Definition 19), then we can distinctly distinguish between:

1. Gentzen's *transfinite* proof of the consistency of the arithmetic of the finite ordinals *relative* to reasoning, *within* ZF, that appeals to set-theoretical properties of transfinite ordinals which, by definition, are *non-constructive* since they can have no *well-defined* interpretations under which they can, even in some extended sense of 'number', be identified as algorithmically *verifiable* (see §2., Definition 2);

2. the *weak* proof of consistency of PA (see §2.B.a., Theorem 2.8) *relative* to the *evidence-based*, algorithmically *verifiable* and unarguably *constructive*, *number-theoretic* interpretation of PA (see §2.B., Theorem 2.7);

3. the *strong* proof of consistency of PA (see §2.C.a., Theorem 2.16) *relative* to the *evidence-based*, algorithmically *computable* and unarguably *finitary*, *number-theoretic* interpretation of PA (see §2.C., Theorem 2.15).

Although (3) establishes the *finitary* consistency of the first-order Peano Arithmetic PA as implicitly sought by Hilbert in [Hi00], whilst (2) establishes the *constructive* nature of the classical, standard interpretation of PA, (1) leaves open the question of the *finitary* consistency of a second-order Peano Arithmetic, such as ACA_0 (see §15.A.), that would yield a *constructive* model for the properties of the real numbers in any putative model of ACA_0.

Prima facie, the utility of any such model would still be questionable since, as shown in §16.C., Cases 16.C.a. to 16.C.d. and §16.D., Case 16.D.c., the limit of a Cauchy sequence of real numbers in any set-theoretical representation of the states of a physical phenomena does not entail that the limiting state of the phenomena must correspond to the Cauchy limit of the sequence.

Moreover, from the *evidence-based* perspective of this investigation, every real is specifiable in PA (see §4.H., Theorem 4.4), since it can be uniquely corresponded to a Gödel-β-function.

This suggests that it may be the failure to adequately recognise that 'provability' and 'truth' need to be complementary, '*evidence-based*', assignments-by-convention towards achieving the correspondingly complementary roles, and goals (see §10.E.(1) and §10.E.(2)), of a first-order set theory such as ZF vis à vis a first-order Peano arithmetic such as PA that, according to Feferman, obscures the role of 'consistency' in 'finitism' and vice versa:

> "In the further pursuit of the consistency program for analysis and various of its subsystems, Schütte and Takeuti provided similar rationales for their work. Schütte's explanation of the rationale for his proof-theoretic work is that "Gödel's investigations (1931) have shown that the strictest finitist methods are basically inadequate for carrying out the consistency proof required by Hilbert's programme. So proof theory needs not only the very strict finitist methods of a combinatorial nature but also higher level proof procedures. Thus we arrive at methods, first used by Gentzen (1936), using induction which in fact goes beyond the usual complete (mathematical) induction but still has a constructive character ... We use inductive methods for the consistency proofs but do not admit *Tertium non datur* as a proof procedure." (Schütte 1977, p. 3) The consistency-proof rationale was explained as follows by Takeuti (1987), p. 101: "We believe that our standpoint is a natural extension of Hilbert's finitist standpoint, similar to that introduced by Gentzen, and so we call it the Hilbert-Gentzen finitist standpoint. Now a Gentzen-style consistency proof is carried out as follows: (1) Construct a suitable standard ordering, in the strictly finitist standpoint. (2) Convince oneself, in the Hilbert-Gentzen standpoint, that it is indeed a well-ordering. (3) Otherwise use only strict finitist means in the consistency proof." Takeuti then goes on (loc. cit.) to explain what is supposed to be admitted under (2): these are "concrete" constructive methods, in contrast to those admitted to intuitionism, which calls on abstract notions of proof and construction.

In summary, the Gentzen-Schütte-Takeuti modified form of the consistency program, that I shall call the *extended Gentzen program*, comes down to carrying out the following three things:

1. Describe *finitistically* the ordering relation of a notation system for ordinals up to an ordinal α_s.

2. Give a *finitist* proof that the principle of transfinite induction up to α_s, TI(α_s), implies the consistency, Con$_S$, of S.

3. Give a *constructive proof* of (the instances used in (2) of) TI(α_s).

The first obvious criticism to be made of the extended Gentzen program is that the notions of finitist and constructive proof required for it are vague. In particular, there are many varieties of constructivism, which on the one hand do not always square with each other, and on the other hand are not simply characterized by saying that one uses only inductive methods without the law of excluded middle. However, I think we can be charitable to an extent here, for two reasons. First of all, one can inspect specific executions of the program and, for the most part, see that the arguments employed in (1) and (2) are prima-facie finitist, while those employed in (3) are prima-facie constructive. Secondly, we now have a great deal of knowledge of formal systems which explicate finitism and constructivism in their various forms, with respect to which we can locate more precisely what a given execution of the program succeeds in doing. Still, one can anticipate that questions will have to be raised in borderline cases, as we shall see in sec. 6.

A second obvious criticism is that there is no reason given for the asymmetry of methods in (1), (2) as against those in (3). If one is to admit constructive proofs in (3), why not allow constructive definitions and proofs in (1) and (2), not just those that are finitist? And, if one takes that step, why not consider quite different constructive foundations, such as the simple reduction of classical to intuitionistic arithmetic (i.e., PA to HA) by the Gödel translation? (Interestingly, Gentzen describes the translation (1969, pp. 169-170), but doesn't make an argument why that is insufficient for his aims.)"

. . . Feferman: [Fe00], §4, The constructive consistency-proof rationale for reductive proof theory.

Feferman notes further—apropos his criticism of the extended Gentzen program, that it perceives 'the criterion of consistency itself as its be-all and end-all'—that (echoing essentially the raison d'être of the Complementarity Thesis 1 in §1. if we replace 'consistency' in Feferman's analysis by 'ω-consistency'[93]) it was 'Brouwer who first objected that consistency is insufficient to guarantee "correctness" in some intuitive interpretation'; a criticism, according to Feferman, that was most endorsed later by Kreisel who—also echoing the raison d'être of the Complementarity Thesis 1 in §1. if we, additionally, take Kreisel's 'functional interpretation' to mean '*evidence-based* interpretation (see §2.)—sought to provide 'general formal criteria such as functional interpretations to replace the incomparable condition of consistency':

"But the main point of criticism of the extended Gentzen program, as of the Hilbert program which it modifies, is the criterion of consistency itself as its be-all and end-all. This had its origin in Hilbert's early identification[6] of the "existence" of mathematical concepts with the consistency and completeness of axiom systems for them. Later, in the mature formulation of his program, Hilbert only emphasized the consistency criterion in service of an instrumentalist justification of formal systems, though he still presumed that completeness would also be established in the cases of interest. The idea was to eliminate the "ideal" statements of a system in favour of the "real" statements, which we can identify with Π_1^0 formulas (treated as open statements). Indeed, for systems S containing a modicum of arithmetic, if S is consistent, then every Π_1^0 statement provable in S is valid. It was Brouwer who first objected that consistency is insufficient to guarantee "correctness" in some intuitive interpretation. Then Gödel's incompleteness theorems bore out

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[93]Since, by definition, a consistent system has a *well-defined* (see §4.F., Definition 19)—which we treat as corresponding to Kreisel's 'functional'—interpretation within an *evidence-based* paradigm.

this criticism with the construction of a consistent system extending PA which is not valid in the natural numbers (namely PA + $\neg Con_{PA}$).

> The most vocal critic of the consistency criterion (in numerous essays) has been Georg Kreisel, saying for example in a late survey of his own work that "I was repelled by Hilbert's exaggerated claim for consistency as a sufficient condition for mathematical validity or some kind of existence" (Kreisel 1987, p. 395). Kreisel aimed instead to use proof theory to make "explicit the additional knowledge provided by those proofs." (loc. cit.) More explicitly, he sought to "unwind" mathematical proofs on the one hand and to provide "general formal criteria such as functional interpretations to replace the incomparable condition of consistency; 'incomparable' because the aim of functional interpretations is meaningful without restriction on metamathematical methods." (loc. cit.)[7]"
>
> *...Feferman: [Fe00], §4, The constructive consistency-proof rationale for reductive proof theory.*

We note that Feferman's observation that:

> "...Gödel's incompleteness theorems bore out this criticism with the construction of a consistent system extending PA which is not valid in the natural numbers (namely PA + $\neg Con_{PA}$)"

is falsified by the Provability Theorem for PA (see §2.E., Theorem 2.17), which entails that PA is categorical (see §2.E., Corollary 2.18).

More significantly, it is a telling instance of the failure to adequately recognise the distinctly different (see §11.B., Lemma 11.1), albeit complementary, roles and goals (see §10.E.(1) and §10.E.(2)) of a first-order set theory such as ZF, vis à vis a first-order Peano arithmetic such as PA since, paraphrasing Feferman, what we *can* conclude instead in the above quote is that (see §11.B.(a) and §11.B.(b)):

- 'Gödel's incompleteness theorems bore out this criticism with the construction of a consistent system extending ZF which is not valid over the ordinals (namely ZF + $\neg Con_{ZF}$)'.

It is a similar failure to adequately recognise the distinctly different, albeit complementary, roles and goals of a second-order arithmetic such as PA_2 (or ACA_0), vis à vis a first-order Peano arithmetic such as PA, that misleads Feferman to treat, for instance, the consistency of 'PA + $\neg$TP, where TP is the Twin Prime conjecture' as open to 'genuine question'[94] as that of 'PA_2 + $\neg$TP':

> "Of course, consistency itself is meaningful without restriction on metamathematical methods, and one can point to systems of possible mathematical interest for which there may be genuine question as to their consistency, e.g. Quine's system NF, or PA + $\neg$TP, where TP is the Twin Prime conjecture, or—more ambitiously—PA_2 + $\neg$TP, where PA_2 is full 2nd order arithmetic, or—still more ambitiously—ZF + $\neg$TP. (One may substitute for TP here any currently open problem in number theory, such as Goldbach's Conjecture (GC) or the Riemann Hypothesis (RH), that is strongly suspected of being true but difficult to prove.[8])"
>
> *...Feferman: [Fe00], §4, The constructive consistency-proof rationale for reductive proof theory.*

A failure that, moreover, further misleads Feferman into overlooking Skolem's caution (see §4.J.)—against drawing finitary conclusions from transfinitary reasoning—and implicitly asserting (contrary to §11.B., Lemma 11.1; to §15., Corollary 15.2; and to §15.A., Corollary 15.3/Theorem 15.4) that PA can be 'expanded through the notion of arbitrary subset of the natural numbers' (see also [Fe97], p.18:):

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[94]'Misleads', since PA is *finitarily* consistent by §2.C.a., Theorem 2.16; and, by §18.D., Theorem 18.44, TP is provable.

"But what about the consistency of PA and PA_2 and ZF? The most advanced current work in proof theory that may contribute to the extended Gentzen program hardly reaches beyond the subsystem Π_2^1-CA of $(\Pi_\infty^1-$CA$)$. I, for one, have absolutely no doubt that PA and even PA_2 are consistent, and no genuine doubt that ZF is consistent, and there seems to be hardly anyone who seriously entertains such doubts. Some may defend a belief in the consistency of these systems by simply pointing too the fact that no obvious inconsistencies are forthcoming in them, or that these systems have been used heavily for a long time without leading to an inconsistency. To an extent, those kinds of arguments apply to NF, which has been studied and worked on by a number of people. My own reason for believing in the consistency of these systems is quite different. Namely, in the case of PA, we have an absolutely clear intuitive model in the natural numbers, which in the case of PA_2 is expanded through the notion of arbitrary subset of the natural numbers. Finally, ZF has an intuitive model in the transfinite iteration of the power set operation taken cumulatively. This has nothing to do with a belief in a platonic reality whose members include the natural numbers and arbitrary sets of natural numbers, and so on. On the contrary, I disbelieve in such entities. But I have as good a conception of what arbitrary subsets of natural numbers are *supposed* to be like as I do of the basic notions of Euclidean geometry, where I am invited to conceive of points, lines and planes as being utterly fine, utterly straight, and utterly flat, resp. What is *not* evident on the latter conception without special work is the consistency of the system of Euclidean geometry with the parallel axiom replaced by its negation. Similarly, while the notion of arbitrary set and the cumulative hierarchy argues for believing straight off not only in the consistency of ZF but also of ZFC (= ZF + AC), much additional work had to be done to establish the consistency of ZFC + ¬CH (namely Paul Cohen's method of forcing). To return, for example, to NF, that has no intuitive model to support our direct belief in its consistency, and the problem of establishing such a result, it is to be established at all, will no doubt require special metamathematical work, for which restriction in advance to constructive methods would be irrelevant. But if, say, we find out that $ZFC \vdash Con_{NF}$ and we accept the consistency of ZF then we must accept the consistency of NF, since Con_{NF} is a Π_1^0 statement."

. . . Feferman: [Fe00], §4, *The constructive consistency-proof rationale for reductive proof theory.*

We note that, from the *evidence-based* perspective of this investigation, Feferman's belief in:

- the consistency of PA is justified by the *finitary* proof of consistency for PA (see §2.C.a., Theorem 2.16);

- the consistency of systems such as ZF (or PA_2, ACA_0) follows from Gödel's set-theoretical proof of arithmetic undecidability in PM ([Go31], pp.7-9). This is, in effect (see §11.B.), a set-theoretical proof of arithmetic undecidability over the finite ordinals that is formally entailed by the axioms of a set-theory such as ZF (or PA_2, ACA_0), since all the number-theoretic propositions that Gödel assumes as true in PM correspond to propositions over finite ordinals that are formally provable in ZF (or PA_2, ACA_0); whence we can treat the system as either inconsistent, or, ω-consistent and, therefore, consistent.

10.E.b.   Cohen on Skolem and pessimism about proof in mathematics

In his penetrating analysis [Co05] of the limiting consequences of divorcing proof-theoretic considerations from model-theoretic considerations of formal mathematical languages, Paul J. Cohen raises several issues with a pessimism that the Complementarity Thesis 1 in §1. seeks to highlight, and show as unwarranted, in this *evidence-based* investigation into the role, goal, and raison d'ètre, of mathematical languages in philosophy and the natural sciences:

"Attitudes towards formalization and proof have gone through large swings during the last 150 years. We sketch the development from Frege's first formalization, to the debates over intuitionism and other schools, through Hilbert's program and the decisive blow of the Gödel Incompleteness Theorem. A critical role is played by the Skolem-Lowenheim Theorem, which showed that no

first-order axiom system can characterize a unique infinite model. Skolem himself regarded this
as a body blow to the belief that mathematics can be reliably founded only on formal axiomatic
systems. In a remarkably prescient paper, he even sketches the possibility of interesting new
models for set theory itself, something later realized by the method of forcing. This is in contrast
to Hilbert's belief that mathematics could resolve all its questions. We discuss the role of new
axioms for set theory, questions in set theory itself, and their relevance for number theory. We
then look in detail at what the methods of the predicate calculus, i.e. mathematical reasoning,
really entail. The conclusion is that there is no reasonable basis for Hilbert's assumption. The
vast majority of questions even in elementary theory, of reasonable complexity, are beyond the
the reach of any such reasoning. Of course this cannot be proved and we present only plausibility
arguments. The great success of mathematics comes from considering 'natural problems', those
which are related to previous work and offer a good chance of being solved. The great glories of
human reasoning, beginning with the Greek discovery of geometry, are in no way diminished by
this pessimistic view."

. . . Cohen: [Co05], Abstract.

From the *evidence-based* perspective of this investigation, Cohen presciently focuses upon
what he considers 'undoubtedly the most basic result about general axiomatic systems', when
he:

"...alludes to both the work of Thoralf Skolem, and, perhaps even more, to the conclusions he
came to at a rather early stage of the development of mathematical logic. The work is, of course,
the famous Lowenheim-Skolem Theorem, for which Skolem gave a simplified proof, and which
is undoubtedly the most basic result about general axiomatic systems. It can be given various
formulations, but the form which Skolem himself attributes to Lowenheim is that 'every first order
expression is either contradictory or satisfiable in a denumerably infinite domain' (Skolem 1970).
As Skolem showed, there is a natural extension to the case of countably many such expressions.
'Contradictory' here is defined by reference to the rules of the predicate calculus, i.e. normal
mathematical reasoning. The startling conclusion that Skolem drew is the famous Skolem Paradox,
that any of the usual axiom systems for set theory will have countable models, unless they are
contradictory."

. . . Cohen: [Co05], Introduction.

Cohen's pessimism about the role of proof in mathematics—in resolving 'all but the tiniest
fraction of possible mathematical questions'—is seemingly rooted in his interpretation of
Skolem's 'startling' conclusion (see also §4.J.):

"Since I will not assume that my audience are all trained logicians, I point out that though the set
of reals from the countable model is countable seen from outside, there is no function 'living in
the model' which puts it one-to-one correspondence with the set of integers of the model. This
fact and other considerations led Skolem to this viewpoint:

"I believed that it was so clear that axiomatization in terms of sets was not a satisfactory
ultimate foundation of mathematics, that mathematics would, for the most part, not
be very much concerned by it."

The view that I shall present differs somewhat from this, and is in a sense more radical, namely
that it is unreasonable to expect that any reasoning of the type we call rigorous mathematics can
hope to resolve all but the tiniest fraction of possible mathematical questions."

. . . Cohen: [Co05], Introduction.

However, Cohen's pessimism in this instance is misplaced since, from the *evidence-based*
perspective of this investigation, the reals *can* be suitably defined in terms of algorithmically
verifiable Gödel β-functions *within* the first-order Peano Arithmetic PA (see §4.H., Theorem 4.4)

and, ipso facto, placed in 1-1 correspondence with the natural numbers (see §12.B., Theorem 12.3 and Corollary 12.4).

Cohen further refers to 'the Incompleteness Theorem of Gödel' as 'the first, and perhaps the only, proved statement supporting the basic pessimism of Skolem's viewpoint':

> "Now, no discussion of proof can fail to refer to the Incompleteness Theorem of Gödel. The result states that no reasonable system of mathematics can prove its own consistency, where the latter is stated as a theorem about proofs in its own formal system, and hence can be construed as a result in combinatorics or number theory. The Incompleteness Theorem is a theorem of mathematics, and not a philosophical statement. Thus, in this sense, it is unassailable, but, in another sense, since it refers to such a specific question, it is not really relevant to to the question which I am addressing in this talk, namely the extent to which problems in mathematics can reasonably be expected to be settled by mathematical reasoning. It is, of course, the first, and perhaps the only, proved statement supporting the basic pessimism of Skolem's viewpoint."
> . . . Cohen: [Co05], Introduction.

Cohen's pessimism in this instance too is misplaced since, from the *evidence-based* perspective of this investigation, it is based on the common mis-perception (see §11.G.d.) that, reflecting Gödel's presumption in [Go31], "P is consistent" is always expressible in the formal Peano Arithmetic P defined by him in [Go31]. This, however, does not follow from Gödel's formal reasoning in [Go31] (see §11.G.f.).

Cohen observes that the earlier (beginning with the Greek mathematicians and philosophers), optimistically inspiring, stages of the development of formal proofs and the axiomatic method were motivated primarily (compare with the Complementarity Thesis 1 in §1.) by the need to precisely formulate intuitive concepts of mathematical truth and logical deduction; culminating in the 'Gödel Completeness Theorem, which many people regard as implicit in Skolem's work':

> "With the publication of Frege's epic work 'Begriffschrift' in 1879, the notion of a formal system was given a definite form. Imported related work was done by Boole, and Pierce, and later Peano presented a similar approach, but with Frege's work, for the first time in the history of human thought, the notion of logical deduction was given a completely precise formulation. Frege's work not only included a description of the language (which we might nowadays call the 'machine language'), but also a description of the rules for manipulating this language, which is nowadays known as the predicate calculus. . . . this was a major landmark. For the first time one could speak precisely about proofs and axiomatic systems. The work was largely duplicated by others, e.g. Russell and Whitehead, who gave their own formulations and notations, and even Hilbert made several attempts to reformulate the basic notions of a formal system. The variety of such attempts relates to the problem of clearly distinguishing between the axioms which are assumed as the starting point of the theory and the methods of deduction which are to be used. The Gödel Completeness Theorem, which many people regard as implicit in Skolem;s work, explicitly shows that there is no ambiguity in the rules of deduction. This is in marked contrast to the Incompleteness Theorem, which shows that no reasonable axiom system can be complete."
> . . . Cohen: [Co05], Introduction.

In other words, what are today referred to as 'proof-theoretic' considerations, and 'model-theoretic' considerations, were apparently treated as two sides of the same coin until the unsettling nature—and, largely, the commonly accepted, albeit misleading (see §11.), interpretations—of the entailments of Gödel's Incompleteness Theorems for axiomatic systems.

These focused attention, almost entirely, on Hilbert's (Program influenced) attempts to justify the axioms of formal systems without any reference to their intended interpretations (except, perhaps, as dispensable intuitive models to be used for conceptual guidance only).

Thus, Cohen remarks that:

"Alongside these developments, there raged a lively debate, continuing almost to the onset of World War 2, about the ultimate validity of mathematics. This debate saw the emergence of formalism, logicism and intuitionism as competitors for the correct foundation of mathematics. I will briefly discuss these competing philosophies, noting at the outset that each seems to focus on proofs rather than models. In this respct Skolem's ideas were in sharp contrast to those of most of his contemporaries. I believe that today the situation is rather the reverse, due in part to my own work, showing how many models of set theory can be constructed using the notion of forcing (Cohen 1966). Indeed, Skolem even foresaw, in his 1922 paper, the construction of new models of set theory, for there he states:

> "It would in any case be of much greater interest if one could prove that a new subset of Z could be adjoined without giving rise to contradictions; but this would probably be very difficult."

As I said, his interest in models was perhaps ahead of his time, so let me discuss now some of the common viewpoints on foundations."

. . . Cohen: [Co05], Introduction.

However, it is debatable whether Skolem's, seemingly finitary, perspective would have admitted a 'model' in the same—intuitively non-constructive and Hilbertian—sense as Cohen; since the model 'forced' by the latter in [Co66] admits Aristotle's particularisation (see §4., Definition 12 and [Co66], p.4).

Moreover, by §5.D., Theorem 5.12, not only would Aristotle's particularisation not hold in any *well-defined* interpretation of a formal system that claims to also model the PA numerals, but 'forcibly' admitting Aristotle's particularisation in any putative model of the system would, in turn, 'force' the syntactic property of ω-consistency upon the system (see §5.D.); thereby making the interpretation proof-theoretic, rather than model-theoretic (in the sense of [Shr18]) as claimed by Cohen.

10.E.c.   Cohen and The Axiom of Choice

The significance of this is seen in the accepted interpretation of Cohen's argument in his 1963-64 papers[95]; the argument is accepted as definitively establishing that the Axiom of Choice is essentially independent of a set theory such as ZF.

Now, Cohen's argument—in common with the arguments of many important theorems in standard texts on the foundations of mathematics and logic—appeals to the *unspecified* object in Aristotle's particularisation under any putative interpretion of the existential axioms of ZF (or statements about ZF ordinals).

This is seen in his proof (see [Co66], p.19) and application of the—seemingly paradoxical (see Skolem's remarks [Sk22], p295; also [Co66], p.19)—(downwards) Löwenheim-Skolem Theorem (see [Lo15], p.245, Theorem 6; [Sk22], p.293):

> **(Downwards) Löwenheim-Skolem Theorem**: If a first-order proposition is satisfied in any domain at all, then it is already satisfied in a denumerably infinite domain.

Cohen appeals to this theorem for legitimising putative models of a language—such as the standard model 'M' of ZF (see [Co66], p.19 and p.82), and its forced derivative 'N' (see [Co66], p.121)—in his argument (see [Co66], p.83 and p.112-118).

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[95][Co63] and [Co64].

Now, the significance of Hilbert's formalisation of Aristotle's particularisation by means of the ε-function (see §4.) is seen in Cohen's following remarks, where he explicitly appeals in the above argument to a semantic—rather than formal—definition of the *unspecified* object in Aristotle's particularisation:

> "When we try to construct a model for a collection of sentences, each time we encounter a statement of the form $(\exists x)B(x)$ we must invent a symbol $\overline{x}$ and adjoin the statement $B(\overline{x})$. ... when faced with $(\exists x)B(x)$, we should choose to have it false, unless we have already invented a symbol $\overline{x}$ for which we have strong reason to insist that $B(\overline{x})$ be true."
>
> [Co66], *p.112; see also p.4.*

Cohen, then, argues that:

Lemma 10.1. *The Axiom of Choice is false in N.*

10.E.d. No putative mode of ZF can appeal to Aristotle's particularisation

We note, however, that (see §4., Definition 12) such an *unspecified* instantiation may sometimes be *unspecifiable*— in the sense of §7.C., Definition 22—within the parameters of the formal system ZF, thereby implicitly admitting non-standard elements—in any putative interpretation of ZF—that cannot be recursively defined as 'terms' within a first-order theory such as ZF.

In other words, since the ZF-formula $[(\exists x)B(x)]$ is only an abbreviation for the ZF-formula $[\neg(\forall x)\neg B(x)]$, we can only conclude from the ZF-provability of $[(\exists x)B(x)]$ that $[(\forall x)\neg B(x)]$ is not ZF-provable if ZF is consistent; and so $[\neg B(a)]$ is not provable for *any* well-defined term $[a]$ of ZF.

We cannot conclude that—if the formula $[B(x)]$ interprets as the relation $B^*(x)$ under a putative interpretation, say M, of ZF—the ZF-provability of $[(\exists x)B(x)]$ entails there is some element $\overline{x}$ in the model M for which $B^*(\overline{x})$ holds in M; and which is such that we can introduce a term $[a]$ into ZF by adding $[B(a)]$ as an additional axiom to ZF without inviting inconsistency, since such a term may not be recursively definable by the rules for formation of first-order terms within ZF.

Moreover, since Hilbert's ε-function (see §4.) formalises precisely Cohen's concept of '$\overline{x}$'—more properly, '$\overline{x}_B$'—as $[\varepsilon_x B(x)]$, it follows that:

Theorem 10.2. *Any model of ZF, in which the quantifiers are interpreted according to Aristotle's particularisation, is a model of ZF_ε if the expression $[\varepsilon_x B(x)]$ is interpreted to yield Cohen's symbol '$\overline{x}_B$' whenever $[B(\varepsilon_x(B(x)))]$ interprets as true in the model.*

Hence Cohen's argument is also applicable to ZF_ε. However since, by [Ca62] (see §5.F., Lemma 5.17), the Axiom of Choice is true in any putative interpretation of ZF_ε which appeals to Aristotle's particularisation, Cohen's argument (see [Co63] and [Co64]; [Co66])—when applied to ZF_ε—actually shows that:

Corollary 10.3. *ZF_ε has no model that admits Aristotle's particularisation.*

Corollary 10.4. *ZF has no model that admits Aristotle's particularisation.*

We cannot, therefore, conclude that the Axiom of Choice is essentially independent of the axioms of ZF, since none of the 'models forced' by Cohen (in his argument for such independence) can be treated as putative interpretation of ZF without inviting inconsistency.

10.E.e.   Cohen and the Gödelian Argument

At the conclusion of his lectures on "Set Theory and the Continuum Hypothesis", delivered at Harvard University in the spring term of 1965, Cohen remarked:

> "We close with the observation that the problem of CH is not one which can be avoided by not going up in type to sets of real numbers. A similar undecidable problem can be stated using only the real numbers. Namely, consider the statement that every real number is constructible by a countable ordinal. Instead of speaking of countable ordinals we can speak of suitable subsets of ω. The construction $\alpha \rightarrow F_\alpha$ for $\alpha \leq \alpha_0$, where α_0 is countable, can be completely described if one merely gives all pairs (α, β) such that $F_\alpha \in F_\beta$. This in turn can be coded as a real number if one enumerates the ordinals. In this way one only speaks about real numbers and yet has an undecidable statement in ZF. One cannot push this farther and express any of the set-theoretic questions that we have treated as statements about integers alone. Indeed one can postulate as a rather vague article of faith that any statement in arithmetic is decidable in "normal" set theory, i.e., by some recognizable axiom of infinity. This is of course the case with the undecidable statements of Gödel's theorem which are immediately decidable in higher systems."
>
> . . . *Cohen:* [Co66], *p.151.*

Cohen appears to assert here that if ZF is consistent, then we can 'see' that the Continuum Hypothesis is subjectively true for the integers under some model of ZF, but—along with the Generalised Continuum Hypothesis—we cannot objectively 'assert' it to be true for the integers since it is not provable in ZF, and hence not true in all models of ZF.

However, by this argument, Gödel's undecidable arithmetical propositions, too, can be 'seen' to be subjectively true for the integers in the standard model of PA, but cannot be 'asserted' to be true for the integers since the statements are not provable in an ω-consistent PA, and hence they are not true in all models of an ω-consistent PA!

The latter is, essentially, John Lucas' well-known Gödelian argument (see [Lu61]; also §17.A.), forcefully argued by Roger Penrose in his popular expositions, 'Shadows of the Mind' ([Pe94]) and 'The Emperor's New Mind' ([Pe90]).

The argument seems plausible, but is unsound (see [An07a]; [An07b]; [An07c]), since PA is ω-*in*consistent (see §2.F., Corollary 2.22). It is based on a misinterpretation—of what Gödel actually proved formally in his 1931 paper—for which, moreover, neither Lucas nor Penrose ought to be taken to account (see [An07b]; [An07c]; also §11.C.).

The distinction sought to be drawn by Cohen is curious, since we have shown that his argument—which assumes that putative interpretations of ZF can appeal to Aristotle's particularisation—actually establishes that putative interpretations of ZF cannot appeal to Aristotle's particularisation (compare with §5.D., Theorem 5.12, that any *well-defined* interpretation of PA, too, cannot appeal to Aristotle's particularisation).

Loosely speaking, the cause of the putative undecidability of the Continuum Hypothsis—and of the Axiom of Choice—in ZF as shown by Cohen, and that of Gödel's (also Rosser's; see §13.) putative 'undecidable proposition' in Peano Arithmetic, is common; it is interpretation of the existential quantifier under an interpretation as Aristotlean particularisation.

In Cohen's case, such interpretation is made explicitly and unrestrictedly in the underlying predicate logic (see [Co66], p.4) of ZF, and in its interpretation in Aristotle's logic of predicates (see [Co66] p.112).

In Gödel's case it is made explicitly—but formally to avoid attracting intuitionistic objections—through his specification of what he believed to be a 'much weaker assumption' of ω-consistency for his formal system P of Peano Arithmetic (see [Go31], p.9 and pp.23-24).

The significance of Gödel's 'much weaker assumption' is seen in §5.D., Corollary 5.9; namely that, if the Peano Arithmetic PA admits a *well-defined* interpretation that appeals to Aristotle's particularisation, then the Arithmetic is ω-consistent.

10.F.   Three categories of information

From a psychological perspective (see, for instance, [Ba16]), we would thus argue that, both qualitatively and quantitatively, any piece of information (i.e., the perceived content of a *well-defined* declarative sentence) that we treat as a 'fact'[96] is necessarily associated with a suitably *well-defined* truth assignment which must fall into one or more of the following three categories:

> **Comment** The implicit thesis here is that the universe is what it is; any piece of 'information' relating to it should, by definition, be our perception of some part of it. If so, such perceptions might be based on *primary* conceptual metaphors arising from an external reality, or on *secondary* conceptual metaphors arising from symbolic expressions in a symbolic language of our original, *primary*, conceptual metaphors. In either case, by definition, such a perception would reflect a physical state of an individual intelligence in the universe at some instant of time. Again by definition, such a state would be a state of energy[97], irrespective of whether or not the perception itself is claimed to 'exist' only fleetingly in an unverifiable, Platonic, realm of abstract ideas of the individual intelligence. However, see also [Lon14] for a contrary perspective.

(1) Information that we[98] zealotly *believe* to be 'true' in an, absolute, Platonic sense, and have in common with others holding similar *beliefs* zealotly;

(2) Information that we prophetically *hold* to be 'true'—short of Platonic *belief*—since it *can* be treated as *self-evident*, and have in common with others who also *hold* it as similarly *self-evident*[99];

(3) Information that we scientifically *agree* to *define* as 'true' on the basis of an *evidence-based convention* (such as, for instance, suggested in §1., Thesis 1), and have in common with others who accept the same *convention* for assigning *truth* values to such assertions.

Clearly the three categories of information have associated truth assignments with increasing degrees of objective (*evidence-based*) accountability that must, in turn, influence the perspective—and understanding (in the cognitive sense of [LR00])—of whoever is exposed to a particular category at a particular moment of time.

In mathematics, for instance, Platonists who hold even axioms which are not immediately self-evident as 'true' in some absolute sense—such as Gödel ([Go51]) and Saharon Shelah ([She91])—might be categorised as accepting all three of (1), (2) and (3) as definitive; those who hold axioms as reasonable hypotheses only if self-evident—such as Hilbert ([Hi27])[100]—as

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[96]In the sense of Piccinini [Pic19]; see §3.A.. For the purposes of this investigation, we ignore the nuances involved in such a concept as detailed, for instance, in [SP10].

[97]Contrary to Norbert Weiner's reported perspective in [Lon14], p.3: "Information is information, not matter or energy . . .".

[98]Admittedly, though, it would be more realistic to read 'we' as 'influential opinion leaders', thereby implicitly subscribing to the point sought to be made by Melvyn B. Nathanson [Na08] in 'Desperately Seeking Mathematical Truth'.

[99]Compare with Chateaubriand: [Cha05], Chapter 25, *The Tyranny of Belief*, pp.351-359.

[100]See also §25. for a remarkably candid confession of why one might, on occasion, *choose* to believe an axiom to be self-evident even in the absence of a corresponding, self-evident, belief!

holding only (2) and (3) as definitive; and those who hold axioms as necessarily *evidence-based* propositions—such as Brouwer ([Br13])—as accepting only (3) as definitive.

10.F.a. The value of contradiction

In the first case, it is obvious that contradictions between two intelligences which arise solely on the basis of conflicting beliefs—such as, for instance, the classical debate between 'creationists' and 'evolutionists'[101] or, currently, that between proponents of the theory of 'alternative facts' and those of 'scientific facts'[102]—cannot yield any productive insight on the nature of the contradiction.

Although not obvious, it is the second case (2)—of contradictions between two intelligences that arise on the basis of conflicting 'reasonability'—which yields the most productive insight on the nature of the contradiction; essentially *perceived* contradictions such as, for instance:

(a) The *perceived* conflict detailed in §7. between Hilbert's and Brouwer's interpretation of quantification; or

(b) The *perceived* conflict detailed in §2.D. between Hilbert and Poincaré on the finitary interpretability of the axiom schema of induction of the first-order Peano Arithmetic PA; or

(c) The *perceived* conflict (see §19.) between Bohr and Einstein on whether the mathematical representation of some fundamental laws of nature can only be expressed in terms of functions that are essentially unpredictable, or whether all the laws of nature can be expressed in terms of functions that are essentially deterministic;

Reason: Such conflicts compel us to address the element of implicit subjectivity in the individual conceptual metaphors (see [LR00]) underlying the contradictory perspectives that, then, motivates us to seek (3) for an appropriate resolution of the corresponding contradiction, as in the case of:

(i) The argument in §8. that Hilbert's and Brouwer's interpretations of quantification are complementary and not contradictory; and

(ii) The dissolving of the Hilbert-Poincaré debate by virtue of §2.B.a., Lemma 2.4 and §2.C.a., Lemma 2.12;

(iii) The dissolving of the Bohr-Einstein debate by the argument in [An15] (see §19.B.b.) that any mathematical representation of a law of nature is necessarily expressed in terms of functions that are algorithmically *verifiable*—hence *deterministic*—but that such functions need not be algorithmically *computable*—and therefore *predictable*.

The third case (3) is thus the holy grail of communication[103]—one that admits unambiguous and effective communication without contradiction; and which is the focus of this investigation.

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[101]Typical of a phenomena whose topical dimensions are insightfully—and sensitively—addressed by Harvey Whitehouse for a lay audience—from the perspective of Cognition and Evolutionary Anthropolgy—in an interview in [Gal18].

[102]As addressed, for instance, by physicists Steven Vigdor and Tim Londergan in [VL17].

[103]Critically so in any search for an extra-terrestrial intelligence.

10.F.b.   The Case for Professors of Stupidity

The pernicious influence of differing dogmas, which arise solely on the basis of conflicting faith-based beliefs, on the perspective—and understanding—of even seasoned scholars is anecdotally highlighted by Michèle Friend and Daniele Molinini:

> "While the text we refer to for a traditional explanation is meant as a textbook for students, not for professional scientists in the field; their explanations end with physical observations and 'physical' constants (such as 'observation', 'particle','travel', and 'light'), not mathematical constants. And there is not much further explanation of these in the standard professional literature.
>
> Our claim here is that when there is no further explanation we are left with the following reactions:
>
> (i)   we give up further search since we cannot 'understand' (any better);
>
> (ii)   we develop an 'intuition' which corresponds to the constants; or
>
> (iii)   we seek further explanation in the form of concepts we already understand.
>
> Many students and less formally educated people fall in to (i). Most professional physicists fall into (ii). The Andréka-Németi group fall into (iii). In the absence of *already* having the required intuitions, when we are confronted with these possibilities we experience what we shall call a '*malaise*'. It is this malaise which motivates the Andréka-Németi project.
>
> Let us illustrate the *malaise* with a story told by Németi. In a class on relativity theory attended by Németi, the professor explained the twin paradox to the students. The students were puzzled, wondered at this 'paradox', and generally experienced a sense of *malaise*. This is all we mean by '*malaise*' here. Németi then asked the professor for a better explanation. Instead of an explanation, he was told the following: continue with your courses on relativity theory. Write a Ph.D. thesis in relativity theory. Become a professor teaching relativity theory. Then if you are very fortunate, after a few years, you will understand the twin paradox.
>
> We do not think that the story is unrepresentative of relativity theory (as it is usually presented and taught). We saw an example of a standard explanation for particles not travelling faster than light earlier. We interpret the story in the following way. The professor himself was unable to give a better explanation. But he had 'gotten used to it'. He had followed (ii) in the above methodology which is standard in the practice of physics. Or, he observed in his colleagues that they had 'gotten used to it' and was waiting for the day when he would 'get used to it'. 'Getting used to' something is a matter of time and exposure, and either is a type of explanation or *substitutes* for explanation!
>
> Take the first disjunct. If intuition, or a sense of familiarity is a type of explanation, then with the intuition, the *explanation* has come to an end, maybe a temporary end. The twin paradox is after all called a 'paradox'. It is supposed to be puzzling. It invites why questions. That is, it invites further explanation; so at best it is an incomplete explanation. It is interesting to observe that labelling it a 'paradox' seems to license an authority not to have a further explanation! This little story is about a *lack* of explanation in a perfectly robust scientific theory. So at least the Andréka-Németi explanation is more complete than the standard one, since it does not rely on physical intuitions, but rather, on logic and mathematics. Maybe these are intuitions too, but they are *arguably* more fundamental or more basic.
>
> After all, where could we look for a better explanation than the one given in the story? Not to the laws of relativity theory, since they promptly lead us to the paradox, and leave some physical constants without further explanation, except implicitly through the other laws. Instead, we have to question the physical laws themselves, and ask for explanations of those. How can we do this? The answer turns on what we think is 'more primitive' or 'more basic' than a scientific law. The answer we (and many scientists) give is: mathematics and logic are more primitive. But ultimately, of course, this can be disputed. In this case we have a draw."
>
> <small>. . . Friend and Molinini: [FM15], pp.201-202.</small>

The relevance, and increasing gravity, of this point—equally for the *evidence-based* challenge to the unverifiable assumption of Aristotle's particularisation in conventional wisdom (see §4.A. and footnote 45), as for the current challenges faced globally by democratic institutions which are constitutionally rooted in consensus of the lowest common denominator—is dramatically illuminated[104] by science editor Brian Gallagher in a commentary where, referring to Bertrand Russell's *The Triumph of Stupidity*, and to one of Russell's "most famous" quips that the "fundamental cause of the trouble is that in the modern world the stupid are cocksure while the intelligent are full of doubt", he writes:

> "Russell's quip prefigured the scientific discovery of a cognitive bias—the Dunning-Kruger effect—that has been so resonant that it has penetrated popular culture, inspiring, for example, an opera song (from Harvard's annual Ig Nobel Award Ceremony): "Some people's own incompetence somehow gives them a stupid sense that anything they do is first rate. They think it's great." No surprise, then, that psychologist Joyce Ehrlinger prefaced a 2008 paper she wrote with Dunning and Justin Kruger, among others, with Russell's comment—the one he later made in his 1951 book, *New Hopes for a Changing World:* "One of the painful things about our time is that those who feel certainty are stupid, and those with any imagination and understanding are filled with doubt and indecision." "By now," Erlinger noted in that paper, "this phenomenon has been demonstrated even for everyday tasks, about which individuals have likely received substantial feedback regarding their level of knowledge and skill." Humans have shown a tendency, in other words, to be a bit thick about even the most mundane things, like how well they drive.
>
> Russell, who died in 1970 at 97 years of age, probably would not be surprised to hear news of this new study, published in *Nature Human Behaviour*: "Extreme opponents of genetically modified foods know the least but think they know the most." The researchers, led by Philip Fernbach, cognitive scientist and co-author of *The Knowledge Illusion: Why We Never Think Alone*, analyzed survey responses from a nationally representative sample of U.S. adults. They obtained similar results, they write, "in a parallel study with representative samples from the United States, France and Germany, and in a study testing attitudes about a medical application of genetic engineering technology (gene therapy)."
>
> Fernbach called their result "perverse." It was nevertheless consistent with prior work exploring the Dunninger-Kruger effect and the psychology of extremism, he said. "Extreme views often stem from people feeling they understand complex topics better than they do." Now as ever, societies need to know how to combat this.
>
> But what exactly is stupidity? David Krakauer, the President of the Santa Fe Institute, told interviewer Steve Paulson, for *Nautilus*, stupidity is not simply the opposite of intelligence. "Stupidity is using a rule where adding more data doesn't improve your chances of getting [a problem] right," Krakauer said. "In fact, it makes it more likely you'll get it wrong." Intelligence, on the other hand, is using a rule that allows you to solve complex problems with simple, elegant solutions. "Stupidity is a very interesting class of phenomena in human history, and it has to do with rule systems that have made it harder for us to arrive at the truth," he said. "It's an interesting fact that, whilst there are numerous individuals who study intelligence—there are whole departments that are interested in it—if you were to ask yourself what's the greatest problem facing the world today, I would say it would be stupidity. So we should have professors of stupidity—it would just be embarrassing to be called the stupid professor"."
>
> *. . . Gallagher: Nautilus, 30 Jan 2019, http://nautil.us//blog/the-case-for-professors-of-stupidity.*

10.G.   An interpretation must be effectively decidable

We take Rotman's semiotic perspective [Rot88] (see §10.) as echoing the essence of Wittgenstein's remarks in [Wi78] (see §9.A.), if we view the latter as indicating that an effective interpretation

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[104]If we read 'stupidity' as 'stupid behaviour', and 'intelligence' as 'intelligent behaviour'!

$\mathcal{I}_{L(D)}$ of a language L into the domain D of another language L' with a *well-defined* logic is essentially the specification of an effective method by which any assertion of L is translated unambiguously into a unique assertion in L'.

Clearly, if an assertion is provable in L, then it should be effectively decidable as true under any *well-defined* interpretation of L in the domain D of L'—since a finite deduction sequence of L would, prima facie, translate as a finite logical consequence in D under the interpretation.

10.G.a.   Is the converse necessarily true?

The question arises:

Query 7. *If an assertion of L is decidable as true/false under an interpretation $\mathcal{I}_{L(D)}$ in the domain D of L', then does such decidability also ensure an effective method of deciding its corresponding provability/unprovability in L?*

Obviously, such a question can only be addressed unambiguously if there is an effective method for determining whether an assertion of L is decidable as true/false in D under the interpretation $\mathcal{I}_{L(D)}$. If there is no such effective method, then we are faced with the following thesis that is implicit in, and central to, Wittgenstein's 'notorious' paragraph in [Wi78] (see §9.A.):

Thesis 4. *If there is no effective method for the unambiguous decidability of the assertions of a mathematical language L under any interpretation $\mathcal{I}_{L(D)}$ of L in the domain D of a language L', then L can only be considered a mathematical language of subjective expression, but not a mathematical language of effective, and unambiguous, communication under interpretation in L'.*

What this means is that, in the absence of an effective method of decidability of the truth/falsity of the formulas of a mathematical language such as PA in the domain $\mathbb{N}$ of the natural numbers under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA, it is meaningless to ask whether, in general, a specific assertion of PA is decidable as true or not in $\mathbb{N}$ under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ (the question of whether the assertion is decidable in PA as provable or not is, then, an issue of secondary consequence).

10.G.b.   Tarskian truth under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA

The philosophical dimensions of this thesis emerge if we consider the classical, *weak*, standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA over the structure of the natural numbers where (cf. [Me64]):

(a) The set of non-negative integers is the domain $\mathbb{N}$;

(b) The integer 0 is the interpretation of the symbol '0' of PA;

(c) The successor operation (addition of 1) is the interpretation of the '′' function (i.e. of f_1^1 in [Me64]);

(d) Ordinary addition and multiplication are the interpretations of '+' and '$\star$';

(e) The interpretation of the predicate letter '=' is the equality relation.

Now, post-Gödel, classical theory seems to hold that:

(f) $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ is a *well-defined* interpretation of PA in $\mathbb{N}$;

(g) PA formulas are decidable under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ in $\mathbb{N}$ by Tarski's definitions of satisfiability and truth (cf. [Me64], p49-53);

(h) The truth and satisfiability of a PA formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ is not always effectively verifiable in $\mathbb{N}$[105].

However, the question, implicit in Wittgenstein's argument regarding the possibility of a semantic ambiguity in Gödel's reasoning in [Go31], then arises[106]:

Query 8. *How can we assert that a PA formula (whether PA-provable or not) is true under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA, so long as such truth remains effectively unverifiable under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$?*

Since the issue is not resolved unambiguously by Gödel in his 1931 paper (nor, prima facie, by subsequent standard interpretations of his formal reasoning and conclusions), Wittgenstein's 'notorious' remark in [Wi78] (see §9.A.) can be taken to argue that, although we may validly draw various conclusions from Gödel's formal reasoning and conclusions, the Platonic existence of a true or false assertion under the classical, *weak*, standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA cannot (as we shall argue in §11.C.) be amongst them.

As analysed and clarified by Paul M. Livingston in [Liv10]:

> "More generally, at the heart of Wittgenstein's critical remarks about Gödel's proof is his skepticism that there is such a well-defined notion of "mathematical truth" that can be held in common between a system such as *Principia Mathematica* and the English "translations" of various of its notions, and so can license the usual interpretation of Gödel's result as showing that there are "truths" that cannot be proven in *Principia* (or any given system). In particular, if, as Wittgenstein suggests, there is indeed no *neutral* sense of "truth" that can be used to characterize both sentences in *PM* and their English translations, then there is no reason to suspect that Gödel's proof indeed shows what it has most often been taken to, that there is a "truth" that cannot be proven or disproven by *PM*. What we have, instead, is simply a particular sentence in *PM*, one that formulates a "perfectly ordinary" and undistinguished arithmetical claim, one that bears literally no implications for the powers or structure of the system as a whole.
>
> When Gödel's theorem and its broader philosophical implications are discussed, the usual framework of discussion is a *model-theoretic* conception of truth. That is, the truth of the Gödel sentence P is conceived as a matter of its holding for a (natural) model, where it is assumed furthermore that there is at least one model where all of the objects of which it holds are natural numbers. As we have just seen, even remaining within a model-theoretic conception of truth, this last assumption is disputable, and might indeed well be disputed if a proof of $\sim P$ were to be given. However, just as importantly, the model-theoretic conception of truth itself might be disputed. Wittgenstein himself never held such a conception, tending to suggest instead a disquotational or redundancy theory. [15] On such a theory, as he suggests in the passage on Gödel's proof itself, there is no language- or system-independent notion of truth, and so there is no absolute sense to the claim that the Gödel sentence P expresses a "mathematical truth." Instead, as Wittgenstein suggests, the only available sense of "true" that is evidently applicable to the Gödel sentence, conceived as a sentence of PM, is the sense "proven in PM." Under the assumption that this is indeed

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[105]Expressed formally by Tarski's 1936 Theorem (cf. [Me64], Corollary 3.38, p151):

"The set Tr of Gödel-numbers of wfs of PA which are true in the standard model is not arithmetical, i.e. there is no wf $A(x)$ of PA such that Tr is the set of numbers k for which $A(x)$ is true in the standard model."

[106]We note that [An16] essentially addresses, and answers Query 8

the only relevant sense of "true," though, the Gödel sentence simply collapses to a version of the "Knower Paradox" (the sentence P that says: "P is known to be false") or the liar paradox: P iff it is not true that P.[16] (Here, we are still maintaining that $Prov(x)$ can be interpreted as a "Proof predicate" (and accordingly, under these assumptions, as a truth or knowledge predicate).)[17] This may again tend to suggest the inconsistency of PM, but crucially, it does not at all suggest that Gödel's proof bears witness to a substantial "truth" that is beyond the capacity of PM to prove."

... *Livingston: [Liv10], p.41-42*

Moreover, the persisting—and pervasive—influence of Gödel's various philosophical and meta-mathematical claims—widely debated on philosophical grounds and objected to, particularly, by Wittgenstein as incisively analysed and reviewed in [Liv10]—is reflected in Livingston's subsequent 'Realism and the Infinite' [Liv12].

Illustrating Melvyn B. Nathanson's thesis in [Na08] (as highlighted in §16.), Livingston seems—even 80 years after Gödel's [Go31]—to implicitly rely upon, in good faith, the essential soundness of the meta-mathematical conclusions (primarily as detailed in §11.G.i.(i) and §11.G.i.(ii)) that Gödel draws from his own formal argumentation in [Go31]:

"The significance of the investigation of formal systems for research into the structure of mathematical cognition and reality lies in the possibility it presents of rigorously posing various general questions about the capacities of such systems to solve mathematical problems or prove mathematical truths; for instance, one can pose as rigorous questions i) the question whether such a system is capable of proving all arithmetic truths about whole numbers; and ii) whether such a system is capable of proving a statement of its own consistency. Notoriously, Gödel's first and second incompleteness theorems, respectively, answer these two questions, for any consistent formal system capable of formulating the truths of arithmetic, in the negative: given any such system, it is possible to formulate an arithmetic sentence which can (intuitively) be seen to be true but cannot be proven by the system, and it is impossible for the system to prove a statement of its own consistency (unless it is in fact inconsistent)."

... *Livingston: [Liv12], p.3.*

We shall, however, seek to largely vindicate further Wittgenstein's philosophical perspective—particularly as articulated in his 'notorious' paragraph (see §9.A.)—by arguing next in Chapter 11. that, and why, the significance given in current paradigms to the various philosophical and meta-mathematical conclusions Gödel draws from his own formal reasoning in [Go31] ought not to be treated as definitive.

CHAPTER 11. *PHILOSOPHICAL FOUNDATIONS*

11.   The illusory significance of Gödel 1931

It is today's folklore that the 'incompleteness' theorems Kurt Gödel proved, in his seminal 1931 paper [Go31] on formally undecidable arithmetical propositions, have set absolute limits on the ability of the brain to express and communicate mental concepts verifiably; a perception falsified by [An16] (see §2.)[107] that has, however, meanwhile admitted a variety of misleading conclusions, misunderstandings and misrepresentations (for some of which we shall, to an extent, hold Gödel accountable in §11.B.) in disciplines ranging from mathematics, mathematics education, philosophy, and the natural sciences to social anthropology and theology. As Solomon Feferman notes:

> ". . . of the three major results that he obtained in mathematical logic in the 1930s, only the incompleteness theorem has registered on the general consciousness, and inevitably popularization has led to misunderstanding and misrepresentation. Actually there are two incompleteness theorems, and what people have in mind when they speak of Gödel's theorem is mainly the first of these. Like Heisenberg's Uncertainty Principle, it has captured the public imagination with the idea that there are absolute limits to what can be known. More specifically, it's said that Gödel's theorem tells us there are mathematical truths that can never be proved. Among postmodernists it's used to to support skepticism about objective truth; nothing can be known for sure. And in the Bibliography of Christianity and Mathematics (yes, there is such a publication!) it's asserted that "theologians can be comforted in their failure to systematize revealed truth because mathematicians cannot grasp all mathematical truths in their systems either." Not only that, the incompleteness theorem is held to imply the existence of God, since only He can decide all truths.

> Among those who know what the incompleteness theorems actually tell us, there are some interesting views about their wider significance for both mind and matter. In his 1960 Gibbs Lecture, Gödel himself drew the conclusion that "either mind infinitely surpasses any finite machine or there are absolutely unsolvable number theoretic problems." He evidently believed that mind can't be explained mechanically, but since he couldn't give an unassailable argument for that—in his typical style he formulated this in a more cautious way as a dichotomy. A lot has been written pro and con about the possible significance of Gödel's theorem for mechanical models of the mind by a number of logicians and philosophers; my own critique of Gödel's dichotomy is published in the July 2006 issue of the journal Philosophia Mathematica . . . One of the most prominent proponents of the claim that Gödel's theorem proves that mind is not mechanical is Roger Penrose (e.g. in Shadows of the Mind): "there must be more to human thinking than can ever be achieved by a computer". However, he thinks that there must be a scientific explanation of how the mind works, albeit in its non-mechanical way, and that ultimately must be given in physical terms, but that current physics is inadequate to do the job. As far as I know, Penrose does not say that Gödel's theorem puts any limits on what one may hope to arrive at in the search for those needed new laws of physics. But Stephen Hawking and Freeman Dyson, among others, have come to the conclusion that Gödel's theorem implies that there can't be a Theory of Everything. Both the supposed consequences of the incompleteness theorem for the nature of mind and the laws of the universe are quite interesting and should be examined on their own merits."
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> *. . . Feferman: [Fe06a], pp.2-4.*

Now, since [An16] has shown that PA is both *finitarily* consistent (see §2.C.a., Theorem 2.16) and *categorical* (see §2.E., Corollary 2.18), most deliberations based upon, and conclusions drawn from, the perceived existence of formally undecidable arithmetical propositions become infructuous, and cannot lay claim to significance unless validated independently of their falsifiable premise.

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[107]See also [An04].

For instance, from the *evidence-based* perspective of this investigation, which admits a finitary proof of consistency for the fiirst-order Peano Arithmetic PA ([An16], Theorem 6.8, p.41), by admitting rule-based, algorithmic assignments of satisfaction and truth to the formulas of PA (see [An16], §3, p.37; also §2.A.), the pervasive influence:

— of what we shall argue in this chapter as Gödel's conflation of his *un*falsifiable set-theoretical, and falsifiable arithmetical, argumentation in [Go31];

    – and the consequent misleading, and occasionally false, interpretations of, and conclusions drawn by Gödel from, his own formal reasoning in [Go31],

    – with respect to the common concepts *undecidability, incompleteness, truth/falsity, diagonalization, self-reference, syntax, semantics*;

        - and the question (see §11.G.h.) of when a putative formal arithmetical representation of a number-theoretic proposition,

        - necessarily *asserts/means*, under interpretation,

        - what the original number-theoretic relation proposition *asserts/means*;

— upon the usage and interpretation of parallel concepts in a generalised linguistic context from a deconstructive perspective;

is evidenced in Paul Livingston's perspective [Liv09], 'Derrida and Formal Logic: Formalising the Undecidable':

> "As early as 1970, Derrida suggested an analogy between what he calls the 'undecidable' and the incompleteness result discovered by Gödel and first announced in the article 'On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems published in 1931.[1] Derrida draws this connection in the course of a discussion in 'The Double Session' in which he juxtaposes an excerpt from Mallarmé's text *Mimique* with a passage from Plato's *Philebus*. The issue raised by both texts (but also, as Derrida argues, by the whole of the metaphysical tradition) is that of *mimesis*, and of the relationship between a representational text, image, or inscription and the 'original' that it represents. Mallarmé's text, Derrida argues, makes possible a thinking of *mimesis* whereby it is no longer understandable as the hierarchical relationship between a representation and a (present or deferred) original. Rather, Mallarmé's text gives us to think a 'play' of *mimesis* with no original, an order of mirroring defined by *allusion* rather than the hierarchical logic of truth and illusion:

>> In this perpetual allusion being performed in the background of the *entre* that has no ground, one can never know what the allusion alludes to, unless it is to itself in the process of alluding, weaving its hymen and manufacturing its text. Wherein allusion becomes a game conforming only to its own formal rules. As its name indicates, allusion *plays*. But that this play should in the last instance be independent of truth does not mean that it is false, an error, appearance, or illusion. Mallarmé writes 'allusion,' not 'illusion.' Allusion, or 'suggestion' as Mallarmé says elsewhere, is indeed that operation we are here *by analogy* calling undecidable. An undecidable proposition, as Gödel demonstrated in 1931, is a proposition which, given a system of axioms governing a multiplicity, is neither an analytical nor deductive consequence of those axioms, nor in contradiction with them, neither true nor false with respect to those axioms. *Tertium datur*, without synthesis.[2]

> Since 'undecidable' and 'undecidability' are terms that Derrida retains throughout his career, indeed putting them to a central use in his later analyses of such phenomena as hospitality and the gift, it is worth pausing over this analogy and asking what it shows us about the status

of deconstruction *vis a vis* formalism and formalization, of which Gödels result is a modern masterpiece.

As is familiar, the essence of Gödel's proof is to construct a sentence in the language of Russell and Whitehead's Principia Mathematica that is undecidable in the sense that, by way of the construction of a predicate that formalizes the rules of proof in that system, it 'asserts' of 'itself' that it cannot be proven or disproven within the system.[3] That is, the Gödel sentence GS for a particular system is defined in terms of a predicate that is intended to hold of all and only sentences in that system that are provable within it; in terms of this 'proof' predicate, the sentence GS 'says' of itself that it cannot be proven. Because of this, neither GS nor its negation can be assumed to be provable, on pain of contradiction. Such a sentence is thus 'undecidable' in the sense that the axioms and constitutive rules of the system do not *decide* it: that is, they themselves do not and cannot establish whether it is true or false. Gödel goes on to take this sentence as demonstrating the *incompleteness* of *Principia Mathematica* in the sense that (assuming *PM* is consistent) there is a truth that it cannot prove. For it is apparently possible to 'see' (by means, however, of an essentially informal argument) that the proposition asserted by the Gödel sentence is true (i.e. that it itself is indeed not provable within *PM*) although this cannot be proven within the system.[4] Moreover, although Gödel himself demonstrated the result only in the special case of *Principia Mathematica*, it can certainly be generalized. Though the specific 'Gödel sentence' for each system will be different, it can be proven that any formal system of a sufficient (relatively low) degree of complexity will have a Gödel sentence, and thus can be shown to be, necessarily, either inconsistent or incomplete. Thus the more general significance of Gödel's result is that it demonstrates undecidability as a *general* phenomenon of any moderately complex formal system, an inherent consequence of any attempt to formalize the total logic of a system of proof or the rule-governed establishment of truth.

In introducing the analogy to Gödel's result, Derrida is quick to emphasize that the undecidability that concerns him is not a matter of semantic ambiguity or polysemy:

> 'Undecidability' is not caused here by some enigmatic equivocality, some inexhaustible ambivalence of a world in a 'natural' language, and still less by some '*Gegensinn der Urworte*' (Abel). In dealing here with hymen, it is not a matter of what Hegel undertook to do with German words like *Aufhebung, Urteil, Meinen, Beispiel*, etc., marveling over that lucky accident that installs a natural language within the element of speculative dialectics. What counts here is not the lexical richness, the semantic infiniteness of a world or concept, its depth or breadth, the sedimentation that has produced inside it two contradictory layers of signification (continuity and discontinuity, inside and outside, identity and difference, etc.). What counts here is the formal or syntactical praxis that composes and decomposes it.[5]

That is, the undecidable, in the sense in which it concerns him, is not a matter of single terms having a multiplicity of non-equivalent or even mutually contradictory 'meanings.' Quite to the contrary, the undecidability that Derrida finds in Mallarmé's text is a consequence of a total structural relationship that is, in *this* case, figured in the undecidability (for instance between 'inside and outside,' 'continuity and discontinuity,') of the single term 'hymen.

One such term, employed in relation particularly to Saussures understanding of language as a 'system of differences without positive terms,' but bearing more general application as well, is the neologism '*différance*.' The term, as Derrida explains in the article of the same title, problematically expresses a kind of general condition for the possibility of presentation itself, given that linguistic presentation is possible only within a system of signs defined both by synchronic difference and diachronic deferring.[9] But the consequence of this is that *différance* itself cannot be presented or named, cannot be inscribed or marked by any positively meaningful representative sign. It is thus that '*différance* has no name in our language' or in any other language[10], and that accordingly '*différance* is neither a word or a concept.'[11] Rather, in its very incapacity to be named, it 'exceeds the order of truth at a certain precise point,' reserving itself or removing itself 'in regular fashion' from the systematic structural distinctions of truth or falsity or of presence and absence which it itself structures.[12]

This structure can, indeed, probably be extended to all of the key terms of deconstructive reading. *Différance, pharmakon, trace, supplement*, and (later) *chora* all indicate (without naming) the problematic point *within* a specific text at which the conditions of possibility of a total structural logic of presence and absence are figured and thereby undermined. All of these key terms, and the deconstructive operations they organize, figure the undecidable as the point at which the structural conditions for the possibility of the text fail in the movement of their own attempted figuration. It is in this sense that the regular, structural condition of possibility of the distinctions upon which these specific texts live and function is revealed as, simultaneously, the condition for their *im*possibility, the impossibility of completing the system of presence in a total description of its constitutive structural law.

Given this, it is possible to specify the analogy between Gödel's result and deconstructive undecidability on at least two significant points. First, both trade decisively on the capacity of a *total* system of signs, directed to the establishment of truth or the maintenance of presence, to represent its *own* constitutive conditions of possibility, to figure the basis of the central distinctions that organize the system itself. It is, in both cases, through the figuring of these conditions of possibility at a particular point—the Gödel sentence or Derrida's 'undecidable' terms—that these conditions are shown to undermine themselves, to be at the same time conditions of the *im*possibility of figuring the underlying logic of the system completely. In this way, in each case, the system achieves a kind of total *self-reference*, an 'encoding' at a single point of the total logic that governs the entirety of the system, as well as that point itself. This amounts to a demonstration of the essential *incompleteness* of the system in question, of the necessary existence of points or sentences (in particular, those that express the system's own conditions of possibility) that cannot be decided (as true or false, or as present or absent) in terms of it. In the case of the Gödel sentence itself, these conditions of possibility are represented in the 'provability predicate' for a particular system which encodes the systematic, rule-governed possibilities of proof within a particular system. In a similar fashion, Mallarmé's 'hymen' inscribes, within his text, the condition for the possibility of a structural distinction between original and representation in *mimesis*, and its inscription marks the point of the self-undermining of this distinction.

Second, in both cases the result can be *generalized*. Just as the existence of a Gödel sentence for *Principia Mathematica* points to the more general phenomenon of the incompleteness of any formal system of sufficient complexity, Derrida's *différance* encodes the *general* possibility of undecidability for any system of signs regularly governed by the opposition presence/absence. Thus, the phenomenon of undecidability demonstrated in the particular case of Mallarmé's text by the term 'hymen' is in no way limited to that particular text or term, but in fact can be extended to any mimetic system of signs whatsoever, wherever the general regular conditions of the possibility of representation are themselves represented. The more general terms *différance, trace*, and *supplement*, capture this, and are explicitly meant to intervene to destabilize the *totality* of the metaphysics of representation, or what Derrida elsewhere calls the 'metaphysics of presence' itself. They function wherever there is a field or structure of signs which presuppose, for their significative or truth-producing work, a distinction between presence and representation, or between truth and falsity—which is to say wherever there is a system of signs at all.

Thus, much as Gödel's result does not simply evince a contingent limitation of a particular system such as *Principia Mathematica*, but rather marks a fundamental problem for the possibility of formalization as *such* (one which may cause us to reconsider basic and otherwise plausible ideas about the nature of mathematical truth), Derrida's invocation of undecidability suggests a fundamental reconsideration of what is involved in *any* possible system of representation, and hence in any inscription or writing of any sign as such.[13] The possibility of both operations—the generation of the undecidable Gödel sentence, or the internal/external tracing of the undecidable in Derrida—is based on the way the general structural laws responsible for a system's capacity for presentation—the system's constitutive and determinative logic—can themselves be brought to problematic presentation within that system itself, and so can be seen to undermine themselves at the very point of their positive statement.

As we have seen, then, the undecidability long asserted by Derrida as an essential component of deconstruction, and that shown by Gödel's own meta-mathematical argument, are structurally

similar in at least two ways. First, both depend on a kind of 'self-referential' encoding whereby a system's total logic (the conditions for the possibility of its organizing distinctions) is formalized at a single point—the Gödel sentence or the 'undecidable term'—which in turn makes it possible to inscribe an 'undecidable.' Second, both suggest a *generalization* of this result to show that any system of sufficient complexity will allow the inscription of undecidables, and hence be 'incomplete' in a specific sense.

There is, moreover, a third, highly important point of analogy that, although it has been missed by most commentators, verifies this close connection and provides an essential clarification of the basis for any deconstructive strategy of reading. It is this: what Derrida calls the 'undecidable' always results from a semantical effect of *syntax* that cannot itself be excluded from any regular system of writing. Although this effect involves, as I shall argue, a kind of essential crossing or confusion between the internal, rule-governed structure of a system and its external 'meaning' or semantics, it essentially *cannot* be captured by any analysis that works on the level of semantic meaning alone.

This essential crossing of syntax and semantics suggests an important analogy with the general metalogical procedure of 'diagonalization,' which underlies Gödel's result, as well as several other key results of twentieth-century formal and metalogical reflection.[14] The best way to understand diagonalization, in general, is to think of an infinite number of elements of a system, the totality of which exhausts the system as a whole, or comprises the totality of elements with a certain property within it. For instance, we might take the (infinite) totality of sentences that are provable within a given formal system, or the (infinite) totality of sentences that are assertable as true within the logic of a particular text. Diagonalization, then, operates on this totality as a whole to produce another element which is both: i) *formally* a member of the totality in question (that is, it bears the right formal properties to be a member of the totality) and, at the same time ii) *demonstrably* not the same as *any* of the (infinite number of) elements that already comprise the totality.[15]

Thus, diagonalization allows the construction, given an infinite set, of an element which is formally part of that set but, as can be shown, differs from each of its (infinite number of) members, and so is not after all part of it. For instance, the Gödel sentence GS for a particular system *diagonalizes* the set of all decidable sentences of the system, in the following sense: given the arithmetical specification of the rules that decide provability (or provability of the negation) of any sentence, it can be shown that GS is not a member of this set; the sentence is itself generated by means of reasoning about what must escape these rules. Most, if not all, instances of diagonalization depend on just such an intervention on syntax.

Thus, the existence of the undecidable within a system depends, in each case, on a productive intervention on syntax, whereby the formal/syntactical rules governing the logic of the system as a whole are encoded at one specific point."

. . . Livingston: [Liv09], pp.2-6.

Whether the above interpretations, by both Livingston and Derrida, of Gödel's formal reasoning in [Go31] are appealed to for validating deconstructive argumentation—as, prima facie, the deliberately lengthy quote at times suggests—or for merely analogically illustrating such argumentation, is a question that lies beyond the scope and competence of this investigation:

"So far, I have argued, by way of the close analogy to Gödel's result and to diagonalization more generally, for three claims about Derrida's undecidables. First, I have argued that they are always the result of a reflective intervention on **syntax**, by means of which the constitutive rules governing a system are problematically 'represented' within that system itself. Second, I have argued that this possibility of inscribing the undecidable is always also the result of the possibility of encoding in ultimately syntactic rules some feature of the **totality** of the system itself, including in particular the totality of its 'decision procedures.' In this sense, the undecidable is an instance of **self-reference** or reflexivity: that is, an undecidable term functions by 'encoding' the logic of the totality of which it itself is a part. When this totality is the totality of language, or of the 'metaphysics of presence,' this reflexive moment means that the inscription of the

undecidable operates as a kind of writing of the very possibility of writing, or a formalization of formalism as such. Third and finally, I have suggested that the inscription of the undecidable, thus understood, gives critical thought access to a complex topology of the limit: neither the *closure* of the system by means of the drawing of a fixed and steady limit, nor its *openness* to the 'infinity' of a transcendent beyond, but what we can term, following Priest (but hyphenating to emphasize the derivation of the term), precisely *in-closure.* The system is closed only at the price of the inherent paradox of tracing its limits, and open just insofar as this paradoxical closure also operates as the diagonalization that generates a contradictory point that is both inside and outside. The work of criticism, or deconstruction, in relationship to the problematic totality of metaphysics, or language itself, is henceforth the inscription and tracing (which also takes the form of erasure) of this problematic limit."

. . . Livingston: [Liv09]*, p.10.*

11.A.   The significance of a shared misperception of Gödel 1931

However, from an *evidence-based* perspective, the one significant—and possibly least remarked upon—consequence of Gödel's reasoning in [Go31] has been that even conflicting—e.g., realist and constructivist—philosophic argumentations[108] in current mathematical and scientific paradigms *share* the common misperception that [Go31] entails Platonic, non-standard, models of the first-order Peano Arithmetic PA (see, e.g., [Ka91], [Ka11]).

The significance of admitting such, seemingly unavoidable, entailment is that PA forms the bedrock upon which all formal mathematical languages that admit rational and real numbers are founded (as detailed succinctly in Edmund Landau's classic exposition [La29] on the foundations of analysis).

Any mathematical representations of our observations of physical phenomena that appeal to real numbers must thus admit, and be circumscribed by, the mathematical and philosophical consequences of whether or not, and to what extent, we accept that a PA which admits Platonic, non-standard, models is consistent; an issue that characterises the perceived, seemingly irreconcilable, philosophical conflicts on the status, and nature, of not only the definitions that admit real numbers into scientific discourse, but also of what such definitions might unwittingly commit us to ontologically and epistemologically.

We therefore revisit Gödel's argumentation in [Go31] to ascertain, in hindsight, what it entails, and what it does not, from the evidence-based perspective of [An16] (see §2.).

11.B.   Can Gödel be held guilty of implicit obfuscation?

Given, then, the significance of [Go31] for current mathematical, philosophical, and scientific paradigms, and since [An16] challenges the broader conclusions—both philosophical and mathematical—that Kurt Gödel draws from his own formal reasoning in his seminal 1931 paper on formally undecidable arithmetical propositions, we consider the, not entirely unreasonable, question:

Query 9. *Can Gödel be held responsible for not distinguishing—in his seminal 1931 paper on formally undecidable propositions—between the implicit circularity in (masked by the non-constructive nature of) his set-theoretical proof of arithmetic undecidability in PM; and the lack of any circularity in his finitary proof of arithmetic undecidability in his Peano Arithmetic P?*

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[108]As evidenced in, for instance, [Ba16]; [Kr16].

The question of implicit obfuscation arises since Gödel's set-theoretical proof of arithmetic undecidability in PM is, in effect, a set-theoretical proof of arithmetic undecidability that is formally entailed by the axioms of a set-theory such as ZF, since all the propositions that Gödel assumes as true in PM correspond to propositions that are formally provable in ZF:

> "A formula of PM with exactly one free variable, which is of the type of the natural numbers (class of classes), will be called a class-expression. We think of the class-expressions ordered in a sequence in some manner[11], we denote the n-th by $R(n)$, and we note that the concept "class-expression" as well as the ordering relation can be defined in the system PM. Let α be an arbitrary class-expression; by $[\alpha; n]$ we denote the formula which arises from the class-expression α by substitution of the symbol for the natural number n for the free variable. The ternary relation $x = [y; z]$ also turns out to be definable within PM. We now define a class K of natural numbers in the following way:
>
> $$n \in K \equiv \overline{Bew} \, [R(n); n] \,^{11a} \qquad (1)$$
>
> (where Bew x means: x is a provable formula). Since the concepts occuring in the definiens are all definable in PM, so also is the concept K which is built up from them, i.e. there is a class-expression S^{12} such that the formula $[S; n]$, intuitively interpreted, says that the natural number n belongs to K. As a class-expression, S is identical with some definite $R(q)$, i.e
>
> $$S = R(q)$$
>
> holds for some definite natural number q. We now show that the proposition $[R(q); q]^{13}$ is undecidable in PM. For, if the proposition $[R(q); q]$ were assumed to be provable, then it would be true, i.e. according to what was said above, q would belong to K, i.e. according to (1), $\overline{Bew} \, [R(q); q]$ would hold, contradicting our assumption. On the other hand, if the negation of $[R(q); q]$ were provable, then $\overline{n \in K}$ would hold, i.e. $Bew \, [R(q); q]$ would be true. Hence, $[R(q); q]$ together with its negation would be provable, which is again impossible.
>
> The analogy of this result with Richard's antinomy is immediately evident; there is also a close relationship[14] with the Liar Paradox, for the undecidable proposition $[R(q); q]$ says that q belongs to K, i.e. according to (1), that $[R(q); q]$ is not provable. Thus we have a proposition before us which asserts its own unprovability[15].
>
> Fn11   Say, according to increasing sum of the terms, and lexicographically for equal sums.
>
> Fn11a   The bar above denotes negation.
>
> Fn12   Again there is not the slightest difficulty in actually writing down the formula S.
>
> Fn13   One should observe that "$[R(q); q]$" (or the synonymous "$[S; q]$") is merely a metamathematical description of the undecidable proposition. Nevertheless, as soon as one has obtained the formula S, one can, of course, also determine the number q, and therefore effectively write down the undecidable proposition itself.
>
> Fn14   Every epistemological antinomy can be used for a similar proof of undecidability.
>
> Fn15   Contrary to appearances, such a proposition is not circular, for, to begin with, it asserts the unprovability of a quite definite formula (namely, the q-th in the lexicographic ordering, after a certain substitution) and only subsequently (accidentally, as it were) does it turn out that this formula itself is precisely the one whose unprovability is expressed."
>
> . . . Gödel: [Go31], §{1}, pp.7-8.

The putative obfuscation arises due to Gödel's subsequent remarks in [Go31][109] that:

> "In the proof of Theorem VI no properties of the system P were used other than the following:
>
> 1. The class of axioms and the rules of inference (i.e. the relation "immediate consequence") are recursively definable (when the primitive symbols are replaced in some manner by natural numbers).
>
> 2. Every recursive relation is definable within the system P (in the sense of Theorem V).

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[109]Debatably misleading, since PA is not ω-consistent (see [An16], Corollary 8.4, p.42).

Hence, in every formal system which satisfies assumptions 1, 2 and is ω-consistent, there exist undecidable propositions of the form $(x)F(x)$, where F is a recursively definable property of natural numbers, and likewise in every extension of such a system by a recursively definable ω-consistent class of axioms. To the systems which satisfy assumptions 1, 2 belong, as one can easily confirm, the Zermelo-Fraenkel and the v. Neumann axiom systems for set theory,[47] and, in addition, the axiom system for number theory which consists of Peano's axioms, recursive definitions (according to schema (2)) and the logical rules.[48] Assumption 1 is fulfilled in general by every system whose rules of inference are the usual ones and whose axioms (as in P) result from substitution in finitely many schemata.[48a]

> Fn47   The proof of assumption 1 turns out to be even simpler here than in the case of the system P, since there is only one kind of primitive variable (resp. two in J. v. Neumann's system).
>
> Fn48   Cf. Problem III in D. Hilbert's address: "Probleme der Grundlegung der Mathematik", <u>Math</u>. <u>Ann</u>. 102.
>
> Fn48a The true reason for the incompleteness which attaches to all formal systems of mathematics lies, as will be shown in Part II of this paper, in the fact that the formation of higher and higher types can be continued into the transfinite (cf. D. Hilbert, "Über der Unendliche", <u>Math</u>. <u>Ann</u>. 95, p. 184), while, in every formal system, only countably many are available. Namely, one can show that the undecidable sentences which have been constructed here always become decidable through adjunction of sufficiently high types (e.g. of the type ω to the system P). A similar result holds for the axiom system of set theory."

> *. . . Gödel: [Go31], p.28.*

Consequently, since PA 'satisfies the assumptions $1, 2$', Gödel has, in fact—albeit unknowingly—illuminated Thoralf Skolem's cautionary remarks in [Sk22] (see §4.J.), about conflating interpretations across domains of different formal systems, by demonstrating in [Go31] that:

(a) There is a formally undecidable arithmetical proposition over the domain of the finite ordinals in ZF;

(b) There is a formally undecidable arithmetical proposition over the domain of the numerals in PA if, and only if[110], PA is assumed ω-consistent.

In other words, since PA is not ω-consistent by §2.F., Corollary 2.22 (see also §9.A.f., Theorem 9.5), Gödel's reasoning in [Go31] can be viewed as having proven that (see also §15., Corollary 15.2):

Lemma 11.1. *The structure of the finite ordinals under any putative interpretation of ZF is not isomorphic to the structure $\mathbb{N}$ of the natural numbers.*   □

Moreover, since PA is consistent (§2.C.a., Theorem 2.16), §11.B., Lemma 11.1 (see also §15., Corollary 15.2), raises the question of whether a set theory such as ZF—or ACA_0 (see §15.A.)—can reasonably be assumed, or even merely treated[111], as consistent; where conventional wisdom seems reflected in Colin J. Rittberg's comment:

> "Remark: it is unknown, in fact unprovable from the ZFC axioms, if ZFC is consistent. The same holds true for number theory, for example. However, experience with these axiom systems makes it reasonable to expect that these systems are in fact consistent. It is common practice in set theory to assume the consistency of ZFC, and Woodin does so as well: "There will be no discovery ever of an inconsistency in $ZF + AD$" (Woodin 2009b, p. 10) . . ."
> *. . . Rittberg: [Rit15], Footnote #3, p.127.*

We note that Rittberg's remark seeks to qualify a not uncommon perspective which can be traced back to that of Gödel in [Go31]—a perspective that is not only inconsistent with *evidence-based* reasoning (see §4.H.a.) and §11.B., Lemma 11.1, but one whose appeal to Gödel's Theorem XI in [Go31] is questionable (see §11.G.a.):

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[110]A consequence of §2.E., Corollary 2.18, that PA is categorical.

[111]The distinction seeks to reflect the perspective of §10.F.

"A large cardinal axiom is a statement that a very big set with certain properties exists. There are many different large cardinal axioms (also referred to as *axioms of strong infinity*, all proclaiming the existence of sets with certain properties. All such sets would be cardinals (hence the name), but the existence of such cardinals cannot be proved by our theory of sets, ZFC. One of the most illuminating examples of this is the large cardinal axiom, which states that there is an inaccessible cardinal (for the definitions of any large cardinal mentioned in this paper, see Sect. 5). If there is an inaccessible cardinal, then we have a model for ZFC. But having a model for ZFC is equivalent to knowing that there are no contradictions following from ZFC, i.e. knowing that ZFC is consistent. By Gödel's Second Incompleteness Theorem, we know that a theory can prove its own consistency if and only if it is inconsistent. Hence, if ZFC could prove that there is an inaccessible cardinal, then it could prove its own consistency and would hence be inconsistent."

<div align="right"><small>. . . Rittberg: [Rit15], p.127.</small></div>

11.C.   Is the 'Gödel' sentence unprovable but true?

One serious consequence of conflating Gödel's purportedly 'semantical' proof and his 'syntactical' proof in Theorem VI of [Go31], is the common misconception that, if we take the concept of truth in the integers for granted, i.e., as intuitively self-evident—so that quantified PA-formulas such as $[(\forall x)F(x)]$ and $[(\exists x)F(x)]$ are always intuitively decidable as either true or false under a *well-defined* interpretation that, however, is assumed to admit Aristotle's particularisation (see §4., Definition 12) over ℕ—then we must accept the false conclusion:

> *Gödel has constructed an arithmetical proposition $[(\forall x)R(x)]$ that is unprovable in any formal system of Peano Arithmetic, such as his formal system P, but true under any well-defined interpretation of the Arithmetic.*

Comment: The conclusion is false since, by §2.F., Corollary 2.20, $[\neg(\forall x)R(x)]$ is provable in PA, and therefore true under any well-defined interpretation of PA.

Comment: We note that, in their paper [FP00], Juliet Floyd and Hilary Putnam conclude by remarking that, as argued by Wittgenstein, such an assertion cannot claim to be mathematical; it is, at best, a metaphysical one:

> "That the Gödel theorem *shows* that (1) there is a well-defined notion of "mathematical truth" applicable to every formula of PM; and (2) that, if PM is consistent, then some "mathematical truths" in *that* sense are undecidable in PM, is *not* a mathematical result but a metaphysical claim. But that if P is provable in PM then PM is inconsistent and if $\neg P$ is provable in PM, then PM is ω-consistent is precisely the mathematical claim that Gödel proved. What Wittgenstein is criticizing is the philosophical naiveté involved in confusing the two, or thinking that the former follows from the latter. But not because Wittgenstein wants simply to *deny* the metaphysical claim; rather, he wants us to see how little sense we have succeeded in giving it."

<div align="right"><small>. . . Floyd/Putnam: [FP00], p.632.</small></div>

The seriousness of the misconception lies in the uncritical perspective it fosters not only amongst emerging logicians—especially when seeking to draw significant formal conclusions from Gödel's argumentation in [Go31]—but also amongst both non-logicians and logicians.

For instance, reportedly in his doctoral thesis where he seeks to argue that:

> "The proofs of Kleene, Chaitin and Boolos for Gödel's First Incompleteness Theorem are studied from the perspectives of constructivity and the Rosser property. A proof of the incompleteness theorem has the Rosser property when the independence of the true but unprovable sentence can be shown by assuming only the (simple) consistency of the theory. It is known that Gödels own

proof for his incompleteness theorem does not have the Rosser property, and we show that neither do Kleene's or Boolos' proofs. However, we show that a variant of Chaitin's proof can have the Rosser property. The proofs of Gödel, Rosser and Kleene are constructive in the sense that they explicitly construct, by algorithmic ways, the independent sentence(s) from the theory. We show that the proofs of Chaitin and Boolos are not constructive, and they prove only the mere existence of the independent sentences."

... *Salehi/Seraji: [SS18], Abstract.*

Payam Seraji uncritically states that:

"Gödel's original proof [6] for his incompleteness theorem is constructive, i.e., given a (finite) description of a consistent re theory (e.g. an input-free program which outputs the set of all the axioms of the theory) the proof exhibits, in an algorithmic way, a sentence which is true (in the standard model of natural numbers $\mathbb{N}$) but unprovable in the theory."

... *Salehi/Seraji: [SS18], §1, Introduction.*

Similarly, philosopher Joseph Vidal-Rosset bases his perspective—of whether or not Gödel's Incompleteness Theorem proves that truth transcends proof—on the premise that:

"Gödel's incompleteness theorem shows the existence of a statement (called 'Gödel sentence', or 'G sentence') true but undecidable in Peano arithmetic. Thus, at least in formal systems, 'somehow truth transcends proof'."

... *Joseph Vidal-Rosset: [Vid06], Introduction, p.51.*

Whilst critically reviewing Roger Penrose's Gödelian argument ([Pe90], [Pe94]), Martin Davis too argues that:

"... Gödel's incompleteness theorem (in a strengthened form based on work of J.B. Rosser as well as the solution of Hilbert's tenth problem) may be stated as follows: There is an algorithm which, given any consistent set of axioms, will output a polynomial equation $P = 0$ which in fact has no integer solutions, but such that this fact can not be deduced from the given axioms. Here then is the true but unprovable Gödel sentence on which Penrose relies and in a particularly simple form at that. Note that the sentence is provided by an algorithm. If insight is involved, it must be in convincing oneself that the given axioms are indeed consistent, since otherwise we will have no reason to believe that the Gödel sentence is true."

... *Davis: [Da95].*

Similarly, Panu Raatikainen begins his reflections on the philosophical relevance of Gödel's incompleteness theorems by asserting that:

"Gödel's first incompleteness theorem (as improved by Rosser (1936)) says that for any consistent formalized system F, which contains elementary arithmetic, there exists a sentence G_F of the language of the system which is true but unprovable in that system."

... *Raatikainen: [Raa05], The fate of Hilbert's program.*

In a talk on the nature and significance of Gödel's incompleteness theorems, Feferman too remarks:

"Taking the concept of truth in the integers for granted, we ... can now formulate one current precise version of Gödel's first incompleteness theorem as follows:

The first incompleteness theorem. If S is a formal system such that

   (i) the language of S contains the language of arithmetic,

   (ii) S includes PA, and

(iii) S is consistent

then there is an arithmetic sentence A which is true but not provable in S."
textit...Feferman: [Fe06a], pp.7-8.

In his highly accessible *Introduction to Gödel's Theorems* for a multi-disciplinary scholarship, philosopher Peter Smith too implicitly—albeit ambivalently—concludes that a language such as PA can have an interpretation[112] $\mathcal{I}_{PA}$ in which quantified PA-formulas—such as $[(\forall x)F(x)]$ and $[(\exists x)F(x)]$—are acceptable as always decidable as either true or false by appeal to some form of an intuitive—albeit mathematically grounded—reasoning:

> "Later, when we come to present Gödel's Theorems, we will describe how to take an arbitrary theory T built in the language L_A, and construct a sentence G_T which turns out to be true but unprovable-in-T. And while the sentence in question is a bit exotic, there is nothing in the least exotic about the notion of truth being applied to it here either: it is the same workaday notion we've just so simply explained. $\mathcal{I}_A$ explicitly defines what it takes for any L_A-sentence, however complex, to be true in this humdrum sense."
> ...Smith: [Smi13], p.40.

> "23.2 Some ways to argue that G_T is true
>
> The Gödelian arguments show that if a rich enough mathematical theory T is sound, or indeed if it is just consistent, then there will be a canonical Gödel sentence G_T which is unprovable in T, and—because it indirectly 'says' it is unprovable—G_T will then be true.[5]
>
> Believing on the basis of the Gödelian arguments that G_T really *is* true will therefore depend on believing that T is indeed sound, or at least is consistent. But note that our reasons for accepting G_T as true can be equally varied. Let's take some examples.
>
> ...
>
> Why note our different examples (1) to (4)? Two reasons. First, we want to drive home the message that, given a particular canonical Gödel sentence G_T, we might have *various* kinds of ground for believing it true, because we have varying grounds for believing T is sound or consistent. But second, our examples also reveal that while our grounds for accepting Gödel sentences may be various, the reasons we adduce in cases like (1) to (4) are—so to speak—perfectly ordinary mathematical reasons. When we initially met the idea of incompleteness at the very outset, we wondered whether we must have some special, rule-transcending, cognitive grasp of the numbers underlying our ability to recognize Gödel sentences as correct arithmetical propositions (see Section 1.4). That speculation should now perhaps begin to seem unnecessarily fanciful."
> ...Smith: [Smi13], p.173.

> "...The truths of basic arithmetic run beyond what is provable in any given formal system : even arithmetic is—so to speak—*inexhaustible.* Given any nice theory of arithmetic T which we accept as sound, we have to recognize that there are truths that T cannot prove (there's G_T for a start). So at least mathematicians are not going to run out of work, even at the level of arithmetic, as they develop ever richer formal frameworks in which to prove more truths."
> ...Smith: [Smi13], p.175.

The significance of an uncritical acceptance of Gödel's conclusions apropos his Theorems VI and XI in [Go31];

- and the perception that *any* assignment of 'truth' values to arithmetical propositions under a *well-defined* interpretation $\mathcal{I}_{PA}$ of PA,

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[112]Which, however, is assumed (see [Smi13], p.39(4)) to admit Aristotle's particularisation (§4., Definition 12) over $\mathbb{N}$.

- must yield arithmetical formulas that are PA-unprovable but true under $\mathcal{I}_{PA}$ in Gödel's dichotomous query: 'Are there absolutely undecidable statements?' (see [Fe06]);

is also highlighted by Hannes Leitgeb's concluding remarks in [Lei09]:

> "**5 Are there true but informally unprovable statements?**
>
> In this final section, we are not so much interested in how informal provability compares to formal provability extensionally, but rather how informal provability relates to truth extensionally. However, in order to say something non-trivial about the latter, it might be useful—and perhaps necessary—to invoke non-trivial insights into the former. For example, it follows from Gödel's First Incompleteness Theorem that if the set of informally provable statements is recursively enumerable, then truth exceeds informal provability extensionally. Or, by the Second Incompleteness Theorem, even if the set of informally provable statements is recursively enumerable, then it is not informally provable of a particular Turing machine that it enumerates all and only informally provable statements. (See Gödel (1951); Shapiro (1998) gives a modern reconstruction of these two corollaries to the Incompleteness Theorems.) Note that the latter result still does not rule out that it is informally provable that there exists a Turing machine which enumerates all and only informally provable statements. Benacerraf (1967) is aware of this, and Carlson (1984), (2000) proves the consistency of the informal provability of this existence claim with a formal system of epistemic arithmetic.[34] It is arguments and conclusions like these that we have in mind here.
>
> So what we are after is
>
> - In which ways does informal provability approximate truth extensionally?
>
> Or, equivalently, are there absolutely undecidable statements (cf. Gödel, 1951)?
>
> According to Hilberts famous *non ignorabimus* claim, the answer to the latter question is an emphatic "no!"; according to Cohen (2005, p. 2414) it is (a somewhat less emphatic) "yes!". Let us go with Cohen for the moment: how could we then argue in favour of the existence claim
>
> HG $\exists p(p \wedge \neg \Box p)$
>
> which we express in this case in operator terms, with a sentential operator $\Box$ for informal provability?
>
> ...
>
> Informal provability and truth coincide extensionally. Thus, instead of proving "merely" that HG is unprovable, we can even derive it is false (as (14) is logically equivalent to ¬HG). In light of the derivation, this is no longer so surprising, but maybe it is surprising at least at first glance since the underlying assumptions might have seemed to be pretty weak. Once again: Does this show that the status of HG has been settled now? The answer is still the same: No, as it is much more plausible to believe that the propositional epsilon calculus with a modal operator $\Box$ has been shown logically deficient, than thinking that the truth value of HG has been determined; instantiations of the epsilon axiom by expressions in which the propositional epsilon symbol is applied to a modal formula are simply not to be counted as logical truths and sometimes maybe not even as truths at all. So the Holy Grail in philosophy of mathematics—for this is what "HG" stands for—is still waiting to be found, as is further insight into the elusive but fundamental concept of informal provability."
>
> *. . . Leitgeb:* [Lei09], *pp.287-288.*

Now, if we read 'informally provable' as 'algorithmically *verifiable* as true', then HG is false!

Moreover, we note that any *unprovable-but-intuitively-true* conclusion (where we take the concept of truth in the integers for granted as in [Fe06a]) makes a *stronger* assumption than that in Gödel's claim (in [Go31], p.26(2)) for the formal 'undecidability' of his arithmetical formula $[(\forall x)R(x)]$—whose Gödel-number is $17Gen\ r$—in his Peano Arithmetic P.

Comment: Moreover, from an *evidence-based* perspective (see §2.E., Theorem 2.17[113]), such conflation can lead to misleading, if not patently false, conclusions:

> "The relation of Gödel's theorems to intuitionism is less straightforward. On the one hand, they seemed to confirm the intuitionists' misgivings about formalism. On the other hand, they underline the rather abstract nature of the intuitionistic notion of provability, with which intuitionists equate truth. For as a consequence of Gödel's theorems, truth cannot be equated with provability in any effectively axiomatizable theory."
>
> . . . Raatikainen: [Raa05], Intuitionism, truth and provability.

Stronger, since Gödel does not assume his proposition $[(\forall x)R(x)]$ to be *intuitively true*, but only demonstrates that though this formula is not provable in his Peano Arithmetic P if P is consistent, yet, for any specified P-numeral $[n]$, the P-formula $[R(n)]$ whose Gödel-number is $Sb\left(r \begin{array}{c} 17 \\ Z(n) \end{array}\right)$ *is* P-provable and, implicitly, *therefore meta-mathematically true* in any well-defined model of P.

He further demonstrates that if P is assumed ω-consistent, then the formula $[\neg(\forall x)R(x)]$ too is not provable in P.

Neither of these arguments entails that if P is ω-consistent, then $[(\forall x)R(x)]$ is a true arithmetical proposition in any well-defined model of P.

Reason: Since the negation $[\neg(\forall x)R(x)]$ of Gödel's proposition $[\forall x R(x)]$ is provable in P (see §2.F., Corollary 2.20), it follows that $[(\forall x)R(x)]$ cannot be true in any well-defined model of P.

Comment: We note that, by §2.F., Corollary 2.23, the classical standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA (though *well-defined* in the sense of §4.F., Definition 19) does not yield a *well-defined* model in the sense of §9., Definition 26.

The significance of this apparently paradoxical circumstance is seen—and resolved—when viewed from the broader *evidence-based* perspective of §4.C.; since what Gödel's argumentation in [Go31] actually establishes is that arithmetical 'truth' and 'arithmetical 'provability' are complementary. A complementarity (see §2.E., Theorem 2.17) which is best expressed in computational terms (see §2.F., Corollary 2.21) as:

> Under any *well-defined* interpretation of an arithmetic such as, say, the first-order Peano Arithmetic PA, Gödel's formula $[R(x)]$ translates as an arithmetical relation, say $R^*(x)$, such that, for any specified natural number n, the proposition $R^*(n)$ is algorithmically *verifiable* as true (see §2., Definition 2), but $R^*(n)$ is not algorithmically *computable* as true for any specified natural number n (see §2., Definition 3).

In other words, what neither Vidal-Rosset nor Raatikainen (both of whom can be argued as faithfully reflecting—even when disagreeing with—conventional wisdom) seem to consider in their argumentation and conclusions is the possible *fallibility* of Gödel's explicit assumption of ω-consistency for his Peano Arithmetic P; nor do they seem to feel a need to question Gödel's explicit claim in [Go31] (p.28) that we can 'easily confirm' such an assumption to be a characteristic of any first-order Peano Arithmetic such as PA.

This is a serious omission since not only is PA ω-*in*consistent (by §2.F., Corollary 2.22 and, independently, by §9.A.f., Theorem 9.5), but there are no formally undecidable propositions in PA (an immediate consequence of §2.F., Corollary 2.20; see also §14.).

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[113]**Provability Theorem for PA**: *"A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$."*

It is an omission, however, for which neither Vidal-Rosset nor Raatikainen, when wearing their philosopher's hats, should be held accountable (see [An07b], [An07c]) since they—like Wittgenstein, Lucas, Penrose and others of similar ilk—can, in this instance, be excused for depending upon fallible classical *and* intuitionistic wisdom to the effect that a Peano Arithmetic such as PA cannot be ω-*in*consistent:

> "...there is no equivocation. Either an adequate arithmetical logic is ω-inconsistent (in which case it is possible to prove false statements within it) or it has an unsolvable decision problem and is subject to the limitations of Gödel's incompleteness theorem."
>
> ... *Martin Davis: ([Da82], p.129(iii)).*

> "We don't want our system to be ω-inconsistent, even if it is consistent. In particular, if the simple consistency were provable metamathematically, then the formula $\neg\mathbf{A}_p(\mathbf{p})$ would under the interpretation express a proposition contradicting one that is true on finitary grounds; and in case $\neg\mathbf{A}_p(\mathbf{p})$ were provable, following Hilbert and Bernays (1939, p. 282) we should call the system *externally inconsistent*, i.e. inconsistent with respect to the finitary interpretation. Thus a proof of simple consistency alone would not secure the formalized mathematics against the possibility of establishing something intuitively false."
>
> ... *Kleene: [Kl52], p.212.*

It would not be unreasonable to conclude that such dependence and/or respect, for paradigms which reflect the wide acceptance of highly technical formal argumentation as definitive, might have rendered some of their philosophical misgivings 'dismissible' on the grounds of appeal to ambiguous interpretations of the formal argumentation.

For instance, as Paul M. Livingston argues in [Liv10], 'there are at least four ways, implicit in Wittgenstein's remarks, that we might resist the strong claim usually associated with Gödel's first incompleteness theorem (i.e. that it shows there is a "truth" that is beyond the capacity of PM to prove or disprove)' (see also §10.G.b.):

> "First, we might simply abstain from interpreting the Gödel sentence P in terms of truth, falsity, provability, or "self-reference" at all. On this option, the derivability of the Gödel sentence in PM simply shows that a "perfectly ordinary" and unremarkable arithmetical sentence of PM is derivable. There are then, quite simply, no further consequences for the nature or structure of PM at all. Second, while agreeing to interpret the Gödel sentence in terms of issues of truth and provability, we might refuse the model-theoretic conception of truth and opt for a disquotational notion. Then the Gödel sentence is just equivalent to the Liar paradox, and raises the same issues as does that paradox. These may (but do not obviously) include the implication that PM is inconsistent. [18] Third, we might *agree* to both the interpretation in terms of truth and falsity and the model-theoretic conception of truth, and still resist the interpretation of "Prov(x)" as a "provability predicate"; this is the interpretation suggested by Floyd and Putnam, according to which there is no admissible interpretation of PM whose models do not contain objects that are not natural numbers, and PM is accordingly ω-inconsistent (although not necessarily inconsistent outright); and fourth (and finally), we may, on any of the first three options or for other reasons, take the Gödel sentence to show PM to be (outright) inconsistent.

> On *any* of these four options, the Gödel sentence does not have the consequences of showing that "there is" a mathematical truth that can be neither proven nor disproven in PM. This is enough to underwrite Wittgenstein's marked suspicion about the result as it is usually presented, and to show that it would be over-hasty simply to concur with the metalogical interpretation that Gödel himself gives.

> Specifically, recall that Gödel's first theorem constructs a sentence P such that, as is provable in PM or a related system, $P \longleftrightarrow \sim \text{Prov}([P])$, where $Prov$ is a one-place "provability predicate" and enclosure in square brackets gives the Gödel number of the formula enclosed. Additionally, the

"provability predicate" itself is defined by means of the predicates NaturalNo(x), and Proof(x, t), where NaturalNo(x) is interpreted as "x is a natural number" and Proof(x, t) is interpreted as a relation supposed to hold between two numbers when x is the Gödel number of a proof whose last line has the Gödel number t.[12] (Here, t abbreviates an expression which calculates out to the Gödel number of P itself). All of these are, of course, interpretations, and might be resisted under the right circumstances. *In particular*, suppose we actually assume that $\sim P$ is proven in PM (or, one day, actually come across a proof of it). Then we are in a position, of course, also to prove Prov($[P]$). In this case, however, as Wittgenstein points out, we might well be justified in dropping the *interpretation* that holds that Prov($[P]$) is *in fact* a provability predicate. And if we drop this interpretation, there is no need to conclude that the Gödel sentence is indeed something that is "true", but unprovable in PM.""

... *Livingston:* [Liv10], *p.42-43*

11.D.   Is Gödel's proof self-referential?

A less serious consequence[114] of conflating Gödel's purportedly 'semantical' proof and his 'syntactical' proof in Theorem VI of [Go31] is the, not entirely unreasonable, argument which contends that:

Argument A: Gödel's set-theoretical definitions of '$[R(n); n]$' and '$\overline{Bew}\ [R(n); n]$' are:

(a) self-referential under interpretation—in the sense of the above quote from Gödel's Introduction in [Go31]—despite his denial of any circularity; and

(b) neither of the definitions can be verified by a deterministic Turing machine as yielding a valid formula of PM (or even of a first-order theory such as ZF).

Argument A is not entirely unjustified in its contentions. Such self-reference and non-constructiveness is a well-known, albeit philosophically debatable, characteristic of any set-theoretical system in which PM is interpretable.

Reason: Prima facie, the axioms of set-theoretical systems such as PM, ZF, etc. would, unlike PA (see §14.A.a., Theorem 14.1) all admit—under a well-defined interpretation, *if any*—infinite elements, in the putative domain of any such interpretation, which are not *well-definable* (by §4.F., Definition 19).

Moreover, by Gödel's self-confessed Platonism (apparent in his footnote #15 in the quote in §11.B.), we do not need to establish that his definitions of '$[R(n); n]$' and '$\overline{Bew}\ [R(n); n]$' need to be classically verifiable by a deterministic Turing machine to be valid formulas of PM (or of ZF).

However, *Argument A* needs to be distinguished from the unjustified *Argument B* with which it is often conflated:

Argument B: Gödel's formally undecidable P-formula, say $[(\forall x)R(x)]$—whose Gödel-number is defined as $17Gen\ r$ in Gödel's proof of his Theorem VI (in eqn (13) on p.25 of [Go31])—also cannot be verified by a deterministic Turing machine to be a valid formula of Gödel's Peano Arithmetic P (defined on pp.9-13 of [Go31]).

Though, to be fair to two generations of scholars who—apart from those who are able to comfortably wear the logician's hat—have laboured in attempts to distinguish between the two

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[114]Though a serious momentum-breaker for aspiring scholars!

Arguments, and to place the philosophical underpinnings of Gödel's reasoning (in [Go31]) in a coherent perspective (see §11.E.), Gödel must, to some extent, be held responsible—even if not culpable—for the lack of a clear-cut distinction between the non-constructivity implicit in his semantic proof of arithmetical undecidability in PM, and the finitarity that he explicitly ensures for his syntactic proof of arithmetical undecidability in his Peano Arithmetic P.

Reason: Neither in his title, nor elsewhere in his paper, does Gödel categorically state that his goal was:

(i) not only to incidentally demonstrate the existence of formally undecidable arithmetical propositions in PM, a system which admits non-finitary reasoning under any putative interpretation;

(ii) but, primarily, to prevent the admittance of non-finitary reasoning—precisely that which would admit conclusions such as *Argument B*—when demonstrating the existence of formally undecidable arithmetical propositions in 'related' systems such as his Peano Arithmetic P.

He merely hints at this by stating (see quote below from pp.6-9 of [Go31]) that his demonstration of (i) is a 'sketch'[115] that lacked the 'precision' (read 'finitism') which he intended to achieve in (ii) by:

(iii) weakening the implicit assumption, of the decidability of the semantic truth of PM-propositions under any well-defined interpretation of PM; the assumption which underlies his set-theoretical proof of the existence of formally undecidable arithmetical propositions in PM; and

(iv) insisting—in his proof of the existence of formally undecidable arithmetical propositions in his Peano Arithmetic P—upon the introduction of a methodology for constructively assigning unique truth values to only those (primitive recursive) quantified number-theoretic assertions (#1 to #45 on pp.17-22 of [Go31]) that are bounded when interpreted over the domain $\mathbb{N}$ of the natural numbers.

> "Before we go into details, let us first sketch the main ideas of the proof, naturally without any claim to rigor. The formulas of a formal system (we restrict ourselves here to the system PM) are, considered from the outside, finite sequences of primitive signs (variables, logical constants, and parentheses or dots) and one can make completely precise which sequences of primitive symbols are meaningful formulas and which are not . . . ."
>
> . . . *Gödel: [Go31], p.6.*

> "The method of proof which has just been explained can obviously be applied to every formal system which, first, possesses sufficient means of expression when interpreted according to its meaning to define the concepts (especially the concept "provable formula") occurring in the above argument; and, secondly, in which every provable formula is true. In the precise expression of the above proof, which now follows, we shall have the task (among others) of replacing the second of the assumptions just mentioned by a purely formal and much weaker assumption."
>
> . . . *Gödel: [Go31], p.9*

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[115]Albeit one that is easily seen to be formalisable in a first-order set-theory such as ZF without any semantic assumptions.

From the current perspective of [An16] (see §2.) one could, not entirely unreasonably, argue—as in the putative *Argument A* above—that Gödel is misleadingly suggesting in the above quote that his definitions of '$[R(n); n]$' and '$\overline{Bew}\ [R(n); n]$' may be treated as yielding 'meaningful' formulas of PM which are well-definable constructively (in the sense of being inductively definable by a deterministic Turing machine).

However, §11.E. details precisely why such an argument would be fragile, by showing how the introduction of the boundedness Gödel insisted upon in (iv) distinguishes:

(v) Gödel's semantic proof of the existence of formally undecidable set-theoretical propositions in PM, which admits *Argument A*;

from:

(vi) Gödel's syntactic proof of the existence of formally undecidable arithmetical propositions in the language of his Peano Arithmetic P, which does not admit *Argument B*.

Moreover, we note that:

(1) Whereas Gödel can legitimately claim[116] that his definition of '$Bew\ [R(n); n]$' yields a formally undecidable formula in PM[117], we cannot legitimately claim that his number-theoretic relation $Bew(x)$ is a formula in his Peano Arithmetic P.

(2) The latter is defined by Gödel in terms of his primitive recursive relation #45, 'xBy', as:

#46. $Bew(x) \equiv (\exists y)yBx$.

(3) In Gödel's terminology, '$Bew(x)$' translates under interpretation over the domain $\mathbb{N}$ of the natural numbers as:

'x is the Gödel-number of some provable formula $[F]$ of the Peano Arithmetic P'.

(4) However, unlike Gödel's primitive recursive functions and relations #1 to #45, both '$(\exists y)yBx$' and '$\neg(\exists y)yBx$' are number-theoretic relations which are not primitive recursive— whence they are not effectively decidable by a Turing machine under interpretation in $\mathbb{N}$.

(5) *Reason*: There is no bound on the quantifier '$(\exists y)$' in the definition of $Bew(x)$, whereas Gödel ensured such bounds in his definitions #1 to #45:

> "Everywhere in the following definitions where one of the expressions (x), (Ex), εx occurs it is followed by a bound for x. This bound serves merely to assure the recursive nature of the defined concept (cf. Theorem IV). On the other hand the extension of the defined concept would, in most cases, not be changed by omission of this bound."
> . . . *Gödel: [Go31], p.17, fn.34.*

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[116]Albeit non-constructively, and despite an implicit element of circularity arising from the platonic assumption of a pre-existing ontology under any well-defined interpretation, as implied in *Argument A*.

[117]Or in a set theory such as ZF by some axiom of separation corresponding to: "...for any set x and class Y, there is a set consisting of the common elements of x and Y." ([Me64], p.168).

Hence, by Turing's Halting Theorem (see §17.F.b., Corollary 17.5), we cannot claim—in the absence of specific proof to the contrary—that there must be some deterministic Turing machine which will determine whether or not, for any specified natural number m, the assertion $Bew(m)$ is true under interpretation in $\mathbb{N}$.

This is the crucial difference between Gödel's semantic proof of the existence of formally undecidable set-theoretical propositions in PM (which admits *Argument A*), and Gödel's syntactic proof of the existence of formally undecidable arithmetical propositions in the language of his Peano Arithmetic P (which does not admit *Argument B*).

(6) We cannot, therefore—in the absence of specific proof to the contrary—claim by Gödel's Theorems V or VII that there must be some P-formula, say $[\mathrm{Bew}_P(x)]$ (corresponding to the PM-formula $Bew[R(n):n]$), such that, for any specified natural number m:

   (a) If $Bew(m)$ is true under interpretation in $\mathbb{N}$, then $[\mathrm{Bew}_P(m)]$ is provable in P;

   (b) If $\neg Bew(m)$ is true under interpretation in $\mathbb{N}$, then $[\neg\, \mathrm{Bew}_P(m)]$ is provable in P;

where $[\mathrm{Bew}_P(x)]$ is a formula of P which is recursively definable strictly in terms of *only* the primitive symbols of P by Gödel's definition:

> "We define the class of formulas as the smallest class to which all elementary formulas belong and to which $\sim(a)$, $(a)\vee(b)$, $x\prod(a)$ (where x is an arbitrary variable) also belong whenever a and b belong."
>
> . . . Gödel: [Go31], p.11.

> **Comment**: We note that, by Gödel's stringent definition (which corresponds to the stipulation that the Peano Arithmetic PA is first-order), except Gödel's definitions #1-#3 ([Go31], pp.17-18), none of his primitive recursive functions and relations #4-#45 ([Go31], pp.18-22) can be treated as formulas of P[118], since they all contain the factorial symbol '!' which is not a primitive symbol of P:

>> "The primitive symbols of the system P are the following:
>>
>> I. Constants: "$\sim$" (not), "$\vee$" (or), "$\prod$" (for all), "0" (zero), "f" (the successor of), "(", ")" (parentheses).
>>
>> II. Variables of the first type (for individuals, i.e. natural numbers including 0): "x_1", "y_1", "z_1", . . .
>>
>> Variables of the second type (for classes of individuals): "x_2", "y_2", "z_2", . . .
>>
>> Variables of the third type (for classes of classes of individuals): "x_2", "y_2", "z_2", . . .
>>
>> — Etc., for every natural number as type."
>>
>> . . . Gödel: [Go31], p.10.

11.E.   Is Gödel's reasoning unverifiable?

Giving credence to §11.D., *Argument A*—which suggests that some scholars may yet harbour a lingering suspicion that Gödel's arithmetical definition of his formally undecidable arithmetical proposition $[(\forall x)R(x,p)]$[119] may also involve a latent contradiction, arising from a putative, implicit, circular self-reference that is masked by unverifiable, even if not patently invalid,

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[118]In other words, they can only be represented/expressed in P by *instantiationally* equivalent formulas of P as defined by Gödel's Theorem V ([Go31], p.22) and Theorem VII ([Go31], p.29).

[119]Defined by its Gödel number 17 *Gen r* in [Go31], p.25, eqn.13.

mathematical reasoning—the following proof of Gödel's Theorem VI of his 1931 paper [Go31] is intended to:

(a) strip away the usual mathematical jargon that shrouds proofs of Gödel's argument which make his—admittedly arcane—reasoning difficult for a non-logician to unravel;

and:

(b) show that, and why—unlike in the case of the paradoxical 'Liar' sentence: 'This sentence is a lie'—Gödel's proposition $[(\forall x)R(x, p)]$ does not involve any circular self-reference that could yield a Liar-like contradiction, either in a formal mathematical language, or when interpreted in any language of common discourse.

> **Comment**: If we give credence[120] to a thesis of this investigation, namely that mathematics *seeks* a language which must *serve* natural science and philosophy (see §10.C.), then:
>
> - it is incumbent upon mathematicians and logicians to pay adequate heed to, and help mitigate, by appropriate articulation,
> - the difficulties faced in understanding the strictures imposed by mathematical languages on the intellectual pursuits and perspectives of scholars[121] in philosophy and the natural sciences who, ultimately,
> - undertake the responsibility of providing to lay societies a coherent perspective of their observations and conceptions, of a common external reality,
> - in mathematical languages that can be relied upon for ensuring unambiguous and effective communication in practical endeavours, and situations,
> - that are subject to, and adjudicated within, systems of common law.

11.F.   The *implicit* Query in Gödel's reasoning

We begin by noting that:

(1) In his 1931 paper on formally 'undecidable' arithmetical propositions, Gödel shows that, given a well-defined system of Gödel-numbering, every formula of a first-order Peano Arithmetic such as PA can be Gödel-numbered by Gödel's primitive recursive relation #23, $Form(x)$, which is true if, and only if, x is the Gödel-number (GN) of a formula of PA.

> **Comment**: It is important to note that 'x' here is a variable over the natural numbers $0, 1, 2, \ldots$, and not to be confused with the variable $[x]$ over the numerals $[0], [0'], [0''], \ldots$ of PA.[122]
>
> Moreover, if we study its definition in terms of Gödel's 22 earlier primitive recursive functions and relations, we will find that $Form(x)$ is a number-theoretic relation which is true if, and only if, x is a natural number that can be decomposed by Gödel's method of Gödel-numbering to yield some unique string of symbols, say $[F]$, which a Turing machine can evidence as being a well-formed formula of PA (since the well-formed formulas of a first-order system such as PA are defined recursively).

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[120]With apologies to the shades of G. H. Hardy ([Har40])!

[121]Such as, for instance, [Cat07]; [Hej18]; [Per18]; [Per19].

[122]It may be pertinent to reiterate that we use square brackets to indicate the expression inside them is to be treated purely syntactically as a 'meaningless' string of symbols, which is not to be interpreted at any stage, but manipulated purely as per the rules of substitution allowed in its construction by the first-order requirements of the formal system.

(2) So, for any specified natural number n, (1) allows us to mechanically decompose n and effectively determine whether, or not, n is the GN of some PA formula.

> **Comment**: We note that such decomposition into PA formulas cannot be defined by the PA numerals within PA, but is only definable recursively in terms of the prime properties of natural numbers as per Gödel's primitive recursive relation #5, and the correspondence between the primitive symbols of PA and the natural numbers by Gödel's system of Gödel-numbering.

(3) Gödel also defines a primitive recursive relation #44, $Bw(x)$, which is true if, and only if, x is the GN of a finite sequence of formulas in PA, each of which is either an axiom, or an immediate consequence of two preceding formulas in the sequence.

(4) So, for any specified natural number n, (3) allows us to effectively determine whether, or not, the natural number n is the GN of a proof sequence in PA.

(5) Further, Gödel defines a primitive recursive relation #45, xBy, which is true if, and only if, x is the GN of a proof sequence in PA whose last formula has the GN y.

(6) Gödel then defines a primitive recursive relation, say $xBSUBy$[123], such that, for any m, n:

> $mBSUBn$ is true if, and only if, m happens to be a GN that can be decomposed into a proof sequence whose last member is some PA formula $[F(n)]$, and n happens to be a GN that decomposes into the PA-formula $[F(u)]$ with only the one variable $[u]$.

(7) The essence, and genius, of Gödel's reasoning in his Theorem VI lies in answering:

Query 10. *Is there any natural number n for which $mBSUBn$ is true?*

11.F.a. The essence, and genius, of Gödel's reasoning

Gödel's remarkable answer to Query 10 involved recognising that although—except for #1, #2 and #3—none of his 45 primitive recursive number-theoretic relations and functions (as also $xBSUBy$) are formulas of a first-order Peano Arithmetic such as PA[124], for any specified primitive recursive number-theoretic relation, say $f(x, y)$ (which is *not* definable recursively in the language of PA), it is possible (see Gödel's Theorems V and VII[125] in [Go31]) to define a PA-formula, say $[F(x, y)]$ (which *is* definable recursively in the language of PA) such that, for any natural numbers m, n:

If $f(m, n)$ is true, then $[F(m, n)]$ is provable in PA;

If $f(m, n)$ is false, then $[\neg F(m, n)]$ is provable in PA.

---

[123]Which corresponds to Gödel's $xB_\kappa \left[Sb \left(y \; \begin{matrix} 19 \\ Z(y) \end{matrix} \right) \right]$ in [Go31], eqn. (8.1), p.24.

[124]Since they are all defined in terms of the number-theoretic function 'n!', which is not definable in the restricted language of PA; see §23.(4), 'First-order language'.

[125]A standard representation theorem of arithmetic ([Me64], Proposition 3.23, p.131 and Corollary 3.24, p.134).

(8) Thus, $xBSUBy$ can be expressed in PA by some (formally well-defined) PA-formula, say $[\neg R(x,y)]$[126], such that, for any m, n:

   (a) If $mBSUBn$ is true, then $[\neg R(m,n)]$ is PA-provable;

   (b) If $\neg mBSUBn$ is true, then $[R(m,n)]$ is PA-provable.

(9) Further, by (6) and (8), for any m, n, if n is the GN of some PA-formula $[F(u)]$ with only one variable, say $[u]$, then:

   (a) If $mBSUBn$ is true, then $[\neg R(m,n)]$ is PA-provable; and m is the GN of a PA-proof of the PA-formula $[F(n)]$;

   (b) If $\neg mBSUBn$ is true, then $[R(m,n)]$ is PA-provable; and m is not the GN of a PA-proof of the PA-formula $[F(n)]$.

(10) In his Theorem VI, Gödel then argues as follows:

   (a) Let q be the GN of the PA-formula $[R(x,y)]$ defined in (8).

   (b) Let p be the GN of $[(\forall x)R(x,y)]$.

   (c) Let r be the GN of $[R(x,p)]$.

   (d) Let $17Gen\ r$ be the GN of $[(\forall x)R(x,p)]$.

(11) We note that all the above primitive recursive functions and relations are formally well-defined recursively by Gödel's Theorems I-IV ([Go31], pp.14-17); and all the PA-formulas—as well as their corresponding Gödel-numbers—are well-defined in the first-order Peano Arithmetic PA.

In other words, as Gödel emphasised in his paper, the 46—i.e., $45 + xBSUBy$—PRA functions and relations that he defines are all bounded, and therefore effectively decidable as true or false over the domain $\mathbb{N}$ of the natural numbers.

Moreover, the PA-formulas that he defines do not involve any reference—or self-reference—to either the meaning or the truth/falsity of any PA-formulas under an interpretation in $\mathbb{N}$, but only to their PA-provability which, he shows, is effectively decidable by his system of Gödel-numbering and his definition of the primitive recursive relation xBy (see (5) above).

(12) Since p is the GN of a formula, namely $[(\forall x)R(x,y)]$, with only the one variable $[y]$, we can substitute p for n, and therefore $[(\forall x)R(x,p)]$ for $[F(n)]$, in (9). We then have that:

   (i) If $mBSUBp$ is true, then $[\neg R(m,p)]$ is PA-provable; whence m is the GN of a PA-proof of $[(\forall x)R(x,p)]$, whose GN is p;

   (ii) If $\neg mBSUBp$ is true, then $[R(m,p)]$ is PA-provable; whence m is not the GN of any PA-proof of $[(\forall x)R(x,p)]$.

Hence $n = p$ answers Query 10 affirmatively.

Once we accept that the above argument is constructive from even an *evidence-based* perspective, the rest of Gödel's reasoning follows straightforwardly.

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[126]We have preferred to express this formula as $[\neg R(x,y)]$, rather than as $[R(x,y)]$, in order to mirror Gödel's reasoning (without any loss of generality).

11.F.b.   Theorem VI: Gödel's conclusions

(13) Gödel concludes that, if PA is consistent then:

 (a) By (12)(i), if $mSUBp$ is true for some m, then both $[\neg R(m,p)]$ and $[(\forall x)R(x,p)]$ are PA-provable—a contradiction since, by Generalisation in PA, the latter implies that $[R(m,p)]$ is provable in PA.

 (b) Hence $[(\forall x)R(x,p)]$, whose GN is $17Gen\ r$, is not provable in PA if PA is consistent.

(14) Moreover, if PA is assumed to be also ω-consistent (which means that we cannot have a PA-provable formula $[\neg(\forall x)F(x)]$ such that $[F(m)]$ is also provable in PA for any specified numeral $[m]$) then:

 (a) By (13), m is not the GN of any PA-proof of $[(\forall x)R(x,p)]$ for any specified m; whence $[R(m,p)]$ is PA-provable for any specified m by (12)(ii);

 (b) Hence $[\neg(\forall x)R(x,p)]$, whose GN is $Neg(17Genr)$, is not provable in PA.

Taken together with the [An16] proofs that PA is categorical (see §2.E., Corollary 2.18) and that PA is not ω-consistent (see §2.F., Corollary 2.22; proved independently in §9.A.f., Theorem 9.5), the above suggests that:

Thesis 5. *There can be no interpretation of Gödel's definition of his formally undecidable arithmetical proposition $[(\forall x)R(x,p)]$ over the domain $\mathbb{N}$ of the natural numbers—whether expressed mathematically or in any language of common discourse—that could lead to a contradiction.*

11.G.   Theorem XI: Where Gödel *can* be held guilty of obfuscation

Where Gödel *can* be held guilty of explicit obfuscation is in both his statement and interpretation of his Theorem XI ([Go31], pp35-38:

> "Theorem XI: Let κ be an arbitrary consistent class[63] of FORMULAS. Then the SENTENCE which asserts that κ is consistent is not κ-provable; in particular, the consistency of P is unprovable in P,[64] assuming that P is consistent (in the cone, of course, every statement is provable).
>
> Fn63 κ is consistent (abbreviated $Wid(\kappa)$) is defined as follows: $Wid(\kappa) \equiv (Ex)[Form(x)\ \&\ \overline{Bew}_\kappa(x)]$.
> Fn64 This follows when one substitutes for κ the empty class of FORMULAS."
> . . .   *Gödel: [Go31], p.36.*
>
> "We have limited ourselves in this paper essentially to the system P and have only indicated the applications to other systems. The results will be expressed and proved in full generality in a sequel to appear shortly. Also in that paper, the proof of Theorem XI, which has only be sketched here, will be presented in detail."
> . . . *Gödel: [Go31], p.38.*

Gödel's culpability lies in the fact that, by not providing the promised sequel in which he intended to prove his Theorem XI with the same rigour as his proof of Theorem VI, he not only overlooked a fatal flaw in his reasoning, but also emphasised in a footnote his earlier misleading conclusion (see §11.D.) that his proposition $[(\forall x)R(x,p)]$ "*asserts its own unprovability*":

> "The proof is (in outline) the following: Let κ be an arbitrary recursive class of FORMULAS (in the simplest case, the empty class) which, for the following considerations, is chosen once and for all. In the proof of the fact that $17\ Gen\ r$ is not κ-PROVABLE,[65] only the consistency of κ is used, as can be seen from 1. on page 25; that is, we have:

$$Wid(\kappa) \to \overline{Bew}_\kappa(17\ Gen\ r)$$

i.e., by virtue of (6.1):

$$Wid(\kappa) \to (x)\overline{xB_\kappa(17\ Gen\ r)}$$

By (13), $17\ Gen\ r = Sb(p\ {}^{19}_{Z(p)}\ )$ and therefore:

$$Wid(\kappa) \to (x)\overline{xB_\kappa Sb(p\ {}^{19}_{Z(p)}\ )}$$

i.e., by (8.1):

$$Wid(\kappa) \to (x)Q(x,p) \tag{24}$$

Now we establish the following: All the defined concepts (proved assertions) of Section 2 [66] and Section 4 are expressible (provable) in P. For, we have used throughout only the ordinary methods of definition and proof of classical mathematics, as they are formalized in the system P. In particular, κ (like every recursive class) is definable in P. Let ω be the SENTENCE by which $Wid(\kappa)$ is expressed in P. The relation $Q(x,y)$ is, according to (8.1), (9), (10), expressed by the PREDICATE q, and, consquently, $Q(x,p)$ by r (since, by (12), $r = Sb(q\ {}^{19}_{Z(p)}\ )$), and the sentence $(x)Q(x,p)$ by $17\ Gen\ r$.

Hence $\omega\ Imp\ (17\ Gen\ r)$ is, by virtue of (24), PROVABLE in P [67] (a fotiori, κ-PROVABLLE). Now, were ω to be κ-PROVABLE, then $17\ Gen\ r$ would also be κ-PROVABLE, whence, by (23), it would follow that κ is not consistent.

Fn65   Of course, r (as well as p) depends upon κ.

Fn66   From the definition of "recursive" on p. 14 until the proof of Theorem VI, inclusive.

Fn67   That the truth of $\omega\ Imp\ (17\ Gen\ r)$ can be deduced from (23) rests simply on the fact that the undecidable proposition $17\ Gen\ r$, as was remarked at the very beginning, asserts its own unprovability."

. . .   Gödel: [Go31], pp.36-37.

The fatal flaw in Gödel's reasoning lies in his implicit assumption that there must be a *well-defined P-formula* $[\omega]$ which would admit the presumption:

"Let ω be the SENTENCE by which $Wid(\kappa)$ is expressed in P".

Since $Wid(\kappa)$ is the number theoretic assertion $[(Ex)(Ey)[Form(x)\ \&\ y\ \overline{B}_\kappa x]$, it is—like the number-theoretic assertion $Bew_\kappa(x)$—not primitive recursive. Hence we cannot (see §11.D.(6)) conclude that:

(a) If $Wid(\kappa)$ is true under interpretation in $\mathbb{N}$, then $[\omega]$ is provable in P;

(b) If $\neg Wid(\kappa)$ is true under interpretation in $\mathbb{N}$, then $[\neg\omega]$ is provable in P.

In other words, for Gödel's claim in, and interpretation of, his Theorem XI to be justified, his reasoning must not only show that:

(i) The number-theoretic assertion '$[Wid(\kappa)]$' *can* be expressed by some formula of P as above;

but also that:

(ii) Under any *well-defined* interpretation of P, $[\omega]$ *must* be shown to interpret as an assertion which formally implies that κ is consistent.

Neither of these follows unambiguously from Gödel's outline of his proof of Theorem XI. As remarked by Mendelson:

> "One of the consequences of Feferman's work (1960) is the following precise version of Gödel's Second Theorem: Let K be a consistent extension of S. Let K_1 be any theory such that is an extension of K_1 and K_1 is an extension of Robinson's system Q. (In particular, k_1 may be S or K itself.) Let T_K be the set of Gödel numbers of theorems of K, and assume that $\mathcal{A}(x)$ is an RE-formula which expresses T_K in K_1. Then not-$\vdash_K Con_{\mathcal{A}}$ (The assumption that $\mathcal{A}(x)$ is an RE-formula is shown to be necessary by Feferman's proof ([1960], Corollary 5.10) that there is a wf $\mathcal{B}(x)$ which expresses T_S in S such that $Con_{\mathcal{A}}$)."
>
> . . . Mendelson: [Me64], p.149.

In other words, the necessity of establishing that $[\omega]$ is a *well-defined*, recursively enumerable, P-formula was highlighted in 1960 by Feferman[127]:

> "5.10. COROLLARY. *There is a π^* which bi-numerates P in $\mathcal{P}$ for which*
>
> $$\vdash_{\mathcal{P}} \mathrm{Con}_{\pi^*}^*.$$
>
> Rather than contradicting Gödel's second underivability theorem, 5.9 and 5.10 show the importance of a precise method of dealing with consistency statements, at any rate for theories with infinitely many axioms. We have maintained that insofar as a formula a expresses membership in A, the formula Pr_a^* expresses provability of $\mathcal{A}$ in $\mathcal{M}(\mathcal{P})$ and the sentence Con_a expresses the consistency of $\mathcal{A}$ in $\mathcal{M}$ and $\mathcal{P}$. Thus, one particular conclusion we can draw is that the formula a^*, although it extensionally corresponds to A, does not properly express membership in $\mathcal{A}$. Indeed, inspection of the proof of 5.9 reveals that it expresses membership in a certain subsystem of A which, independent of the consistency of $\mathcal{A}$, is always consistent.
>
> A first reaction following such realizations might be to restrict attention to a certain class of "natural" formulas a in problems of arithmetization; 5.6 suggests that the class of RE-formulas or some subclass of it might be appropriate for this purpose. However, we shall obtain in §6 results through the use of arbitrary formulas a which should be of interest even to those who would otherwise thus restrict attention. There is nothing "wrong" with the use of arbitrary formulas a; rather, the guiding consideration should be to investigate how different restrictions on the choice of a affect the results of the arithmetization.
>
> Let us return to the proof of 5.9. Although the arguments on the grounds of expressibility give us one way of seeing why the theorem is possible, we should also ask why it is not possible to carry through the proof of
>
> $$\mathrm{Con}_{a^*}^* \rightarrow v_{a^*},$$
>
> as in 5.6
>
> $$\sim v_{a^*} \rightarrow \mathrm{Pr}_{a^*}^*(\sim^* \overline{v_{a^*}}),$$
>
> which corresponds to the essential step in the proof of 5.6. Nevertheless, by the remark following 5.5, we shall be able to prove
>
> $$\sim v_a \rightarrow \mathrm{Pr}_{a^*}^*(\sim^* \overline{v_a}),$$
>
> when a is an RE-formula."
>
> . . . Feferman: [Fe60], Corollary 5.10, pp.59-60.

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[127]In the extract below we have replaced Feferman's symbolism of 'Con' with an asterisk * below by Con*; his 'Pr' with an asterisk * below by Pr*; and his '$\sim$' with an asterisk * below with $\sim^*$.

From the restrictions of *evidence-based* reasoning that underpin this investigation, Feferman's remark that:

"*There is nothing "wrong" with the use of arbitrary formulas a; rather, the guiding consideration should be to investigate how different restrictions on the choice of a affect the results of the arithmetization.*"

cautions about the possibility of unintended, non-constructive, consequences being entailed by admitting an arbitrary formula a into formal reasoning; as has been argued for the postulation of an *unspecified* value in classical, *faith-based* (see §4.B.), existential quantification—e.g., by appeal to Hilbert's ε-function (see §7.A.), or to Rosser's Rule C (see §5.G.)—without *evidencing* that such an *unspecified* value is *specifiable*—i.e., *well-defined* (see §4.F.; also Theorem 4.4)—by the rules that circumscribe the domain of the quantifier.

11.G.a. Why Gödel's interpretation of his Theorem XI is classically objectionable

Specifically, in his Theorem XI, Gödel constructs a formula $[\omega]$ in P and *assumes without proof* that $[\omega]$ translates—under some *well-defined* interpretation of P—as an arithmetical proposition that is true if, and only if, some *unspecified* formula of P is unprovable in P.

Now, even if there *were* such a P-formula, then its significance would be questionable since:

(a) If P were inconsistent then every well-formed formula of the system would be provable; whence a proof sequence for $[\omega]$ within P would have no significance.

(b) If P were consistent then whether or not $[\omega]$ is P-provable would be a trivial consequence, since the significance of such decidability under any well-defined interpretation of P would be conditional on P being consistent!

However, without establishing that there is, or can be, such a P-formula, Gödel lays unqualified claim to—and treats as significant—the conclusion that the consistency of any consistent formal system of Peano Arithmetic is not provable within the Arithmetic; a claim which immediately invites classical objection on the grounds of latent impredicativity in Gödel's underlying assumption.

11.G.b. Defining meta-propositions of P number-theoretically

To recap a little, Gödel first shows how 46 meta-propositions of P can be defined by means of primitive recursive functions and relations ([Go31], pp.17-22). These include:

(#23) A primitive recursive relation, $Form(x)$, which is true if, and only if, x is the Gödeldel-number of a formula of P;

(#45) A primitive recursive relation, xBy, which is true if, and only if, x is the Gödeldel-number of a proof sequence of P whose last formula has the Gödel-number y.

Gödel assures the constructive nature of the first 45 definitions by specifying:

"Everywhere in the following definitions where one of the expressions '$\forall x$', '$\exists x$', 'ϵx (There is a unique x)' occurs it is followed by a bound for x. This bound serves only to assure the recursive nature of the defined concept."
. . . Gödel: [Go31], p.17, footnote 34.

Gödel then defines a meta-mathematical proposition that is not primitive recursive:

(#46) A proposition, $Bew(x)$, which is true if, and only if, $(\exists y)yBx$ is true.

Thus $Bew(x)$ is true if, and only if, x is the Gödel-number of a provable formula of P.

11.G.c. Expressing primitive recursive functions and relations in P

Now, by Gödel's Theorem VII ([Go31], p.29), any primitive recursive number-theoretic relation, say $Q(x)$, can be represented in P by some arithmetical formula, say $[R(x)]$, such that, for any natural number n:

> If $Q(n)$ is true, then $[R(n)]$ is P-provable;

> If $Q(n)$ is false, then $[\neg R(n)]$ is P-provable.

However, Gödel's reasoning in the first half of his Theorem VI ([Go31], p.25(1)) establishes that the above representation does not extend to the closure of a recursive relation, in the sense that we cannot assume:

> If $(\forall x)Q(x)$ is true (i.e, $Q(n)$ is true for any specified natural number), then $[(\forall x)R(x)]$ is P-provable.

In other words, we cannot assume that, even though the recursive relation $Q(x)$ is instantiationally equivalent to a well-defined interpretation of the P-formula $[R(x)]$, the number-theoretic proposition $(\forall x)Q(x)$ must, necessarily, be logically equivalent to the well-defined interpretation of the P-formula $[(\forall x)R(x)]$.

Reason: In recursive arithmetic, the expression '$(\exists x)F(x)$' is an abbreviation for the assertion:

(*) There is some (at least one) natural number n such that $F(n)$ holds.

In a formal Peano Arithmetic such as P, however, the formula '$[(\exists x)F(x)]$' is simply an abbreviation for '$[\neg(\forall x)\neg F(x)]$' which, under a well-defined finitary interpretation of the Arithmetic, translates as:

(**) The relation $\neg F(x)$ is not algorithmically *computable* as always true.

Moreover, Gödel's Theorem VI establishes that we cannot conclude (*) from (**) without risking inconsistency.

Consequently, although a primitive recursive relation may be instantiationally equivalent to a well-defined interpretation of a P-formula, we cannot assume that the existential closure of the relation must have the same meaning as the interpretation of the existential closure of the corresponding P-formula.

However this, precisely, is the presumption made by Gödel in the proof of his Theorem XI, from which he concludes that the consistency of P *can* be expressed in P, but is not P-provable.

11.G.d. Ambiguity in the interpreted 'meaning' of formal mathematical expressions

The ambiguity in the 'meaning' of formal mathematical expressions containing unrestricted universal and existential closure under an interpretation (see also §10.D.b.) was emphasised by Wittgenstein:

> "Do I understand the proposition "There is . . . " when I have no possibility of finding where it exists? And here there are two points of view: as an English sentence for example I understand it, so far, that is, as I can explain it (and note how far my explanation goes). But what can I do with it? Well, not what I can do with a constructive proof. And in so far as what I can do with the proposition is the criterion of understanding it, thus far it is not clear in advance whether and to what extent I understand it."
>
> . . . Wittgenstein: [Wi74], p.299.

To see the significance of Wittgenstein's rhetorical protestation, we note that Gödel defines the notion of "P is consistent" classically as follows:

P is consistent if, and only if, Wid_P is true

where Wid_P is expressed symbolically as the number-theoretic assertion:

$(\exists x)(Form(x) \wedge \neg Bew_P(x))$

This translates as:

There is a natural number n which is the Gödel-number of some *unspecified* formula of P, and this formula is not P-provable.

Thus, Wid_P is true if, and only if, P is consistent.

11.G.e.   Gödel's presumption: "P is consistent" is always expressible in P

However, Gödel, then, presumes without proof that:

(i) Wid_P can be represented by some formula $[\omega]$ of P such that the assertion "$[\omega]$ is true" (under a well-defined interpretation of P) is logically equivalent to the assertion that "Wid_P is true";

(ii) if the recursive relation, $Q(x,p)$ ([Go31], p24(8.1)), is represented by the P-formula $[R(x,p)]$, then the proposition "$[(\forall x)R(x,p)]$ is true" (under a well-defined interpretation of P) is logically equivalent to (i.e., has the same meaning as) the number-theoretic assertion "$(\forall x)Q(x,p)$ is true".

11.G.f.   The loophole in Gödel's presumption in his Theorem XI

Now, although, §11.G.e.(ii):

(a) does follow if the P-formula "$[(\forall x)R(x,p)]$ is true" interprets, under some *well-defined* interpretation of P, as:

"$R(x,p)$ is algorithmically *computable* as always true",

(b) it does not if "$[(\forall x)R(x,p)]$ is true" interprets as:

"$R(x,p)$ is algorithmically *verifiable* as true for any specified natural number n, but it is not algorithmically *computable* as true for any specified natural number n".

So *IF*, and only *IF*, $[\omega]$, too, interprets as an arithmetical proposition that is algorithmically *verifiable* as true, but not algorithmically *computable* as true, then the consistency of *P MIGHT* be provable instantiationally in *P*.

Hence, at best, Gödel's reasoning can only be taken to establish that the consistency of *P* is not expressible as a *P*-formula that is algorithmically *computable* as true under some well-defined interpretation of *P*.

Gödel's broader conclusion only follows if *P* purports to prove its own consistency by means of a *P* formula that is algorithmically *computable* as true under some *well-defined* interpretation of *P*.

However, Gödel's particular argument, based on his definition of Wid_P, does not support this claim.

11.G.g.   The significance of Wittgenstein's insistence on interpreting Gödel's claims algorithmically

The significance of interpreting Gödel's claims:

(a) of having finitarily constructed an *unprovable* PA formula $[(\forall x)R(x)]$ that interprets under any well-defined interpretation of PA as a *true* arithmetical proposition which *asserts* that 'The PA-formula $[(\forall x)R(x)]$ is PA-unprovable' (see §11.C.); and

(b) of having finitarily constructed an *unprovable* PA formula $[w]$ that interprets under any well-defined interpretation of PA as a *true* arithmetical proposition which *asserts* that 'PA is consistent' (see §11.G.);

in terms of algorithmic *verifiability*, and algorithmic *computability*, is that it vindicates what Timm Lampert attributes in [Lam19] as:

 (i) 'Wittgenstein's algorithmic conception of proof in terms of a finite transformation of the problem into a representation in some notation that allows one to decide the initial question based on properties of the resulting expressions'; and

(ii) Wittgenstein's contention that 'representation of a formal property by a propositional function within the formal language itself', as in undecidability proofs, reduce 'the possibility to represent provability as a propositional function to absurdity, not the assumption of a decision procedure that is independent of such a representation'.

"Gödels undecidability proof proves that there exists at least one formula G in the language of PA (henceforth denoted by L_A) such that neither G nor $\neg G$ is provable from the axioms of PA. If Gödel had proven this result by providing a decision method for provability in PA, this would be in line with Wittgenstein's own proof conception. His paradigm for acceptable, algorithmic proofs of unprovability is manifested in the algebraic proofs of the unsolvability of certain problems within Euclidean geometry, such as the problem of angle trisection with a straightedge and compass (cf. RFM I, appendix I, §14). Such proofs of unprovability are part of a decision procedure that distinguishes between possible and impossible constructions on the basis of their algebraic representations: the angles that can be constructed with a straightedge and compass are those and only those that are representable by algebraic equations that can be solved with nested square roots.[8] This fits with Wittgenstein's algorithmic conception of proof in terms of a finite transformation of the problem into a representation in some notation that allows one to decide the initial question based on properties of the resulting expressions.

However, Gödel's proof is not of this sort. Instead, it rests on the representation of a formal property, namely, PA-provability, in L_A, i.e., a language that is based on FOL supplemented with constants for numbers and arithmetic functions. This means that provability is expressed by a certain open formula (abbreviated by $\exists y B y x$, according to Gödel's definition 46) in L_A iff, for all Gödel numbers n of L_A-propositions, [*the L_A-proposition with Gödel number*] n is provable iff $\exists y B y \overline{n}$ is true according to the intended interpretation of L_A.[9] According to Wittgenstein's proof conception, any intent to represent a formal property, such as provability, by an open formula (propositional function) must be founded on confusion between material and formal properties, which is the fundamental mistake of mathematical logic. In contrast to Gödel, Wittgenstein claimed that formal properties can only be "shown", i.e., identified through a decision procedure; they cannot be "said", i.e., expressed within the formal language to which they apply.

Wittgenstein rejected the application of the axiomatic method in Gödel's undecidability proof of his formula G. He did not do so by referring to the relevant proof of the representability of recursive functions within LA (cf. theorems V and VII in [Gödel (1931)], p. 186; theorem 13.4 in [Smith (2007)], p. 109; and [Lampert (2018b)] for detailed discussions). Instead, he was aware that he was instead "bypass[ing]" (RFM V, §17, last sentence) Gödel's proof since he was discussing not the details of the proof but rather what could be taken as a "forcible reason for giving up the search for a proof" (RFM I, appendix I, §14). For Wittgenstein, this was a question of what counts as a "criterion of (un)provability" (cf. RFM I, appendix I, §14-16, and V, §18f.). According to his algorithmic proof conception, a criterion for a formal property must be a decision criterion in terms of some property of ideal symbols. This is why the proof of the impossibility of trisecting an angle with a straightedge and compass counts as a criterion for giving up the search for such a construction (RFM I, appendix I, §14). By contrast, the criterion for a "forcible reason" to give up the search for a decision procedure is not satisfied by meta-mathematical undecidability proofs since they are based on the representation of a formal property by a propositional function within the formal language itself. According to Wittgenstein, undecidability proofs reduce the possibility to represent provability as a propositional function to absurdity, not the assumption of a decision procedure that is independent of such a representation. Indeed, the verdict regarding the representation of formal properties by propositional functions had lain at the heart of Wittgensteins critique of mathematical logic since the beginning (cf. TLP 4.126)."

... Lampert: [Lam19], §4.1 Wittgenstein's reaction to Gödel's undecidability proof.

Lambert analyses further how Wittgenstein's belief 'that formal properties are not representable by propositional functions' could justify 'why Wittgenstein could not accept Gödel's undecidability proof as a proof of incompleteness':

"One reason why Wittgenstein thought that formal properties are not representable by propositional functions is that he rejected the possibility of self-referential representations within a formalism based on FOL (cf. TLP 3.332f). He distinguished operations from functions and considered that it is only with operations that self-application comes into play (TLP 5.25f). However, the application of operations is a part of symbolic manipulation and is not something that is expressible by functions within a logical symbolism. Undecidability proofs, meanwhile, rest on diagonalization and, thus, on a formula that is intended to represent that the formula itself does (not) have a certain property. Gödel's formula G, for example, is intended to represent the property of unprovability of the formula G itself. On this basis, he proved that G cannot be captured in PA.[10] This proof method gives priority to semantics (representation) over syntax (capturing). It is only this priority that makes it possible to prove meta-mathematically that an algorithmic proof conception is limited. Such reasoning cannot convince an advocate of the algorithmic proof conception since such an advocate instead places priority on syntax. In the case of conflict, said advocate would deny the definability of the formal property in question. Thus, given G were provable from the axioms of PA, the diagonal case would simply turn out to be such a case of conflict. Therefore, Wittgenstein would not infer that PA is inconsistent but instead would deny that G, in fact, represents its own unprovability (RFM I, appendix I, §8, 10). This is also why Wittgenstein could not accept Gödel's undecidability proof as an proof of incompleteness."

... Lampert: [Lam19], §4.1 Wittgenstein's reaction to Gödel's undecidability proof.

Moreover, according to Lampert in [Lam19], Wittgenstein did not view such 'conflicts' as posing serious linguistic or philosophical concerns, since he argued that—from a rule-based perspective of constructive mathematical reasoning (compare §16.)—they were merely 'the outcome of the fundamental mistake of mathematical logic, namely, the assertion that formal properties of mathematics and meta-mathematics can be expressed by propositional functions':

> "Wittgenstein analysed undecidability proofs as proofs by contradiction (cf. RFM I, appendix I, §14, and cf. PI §125 below). In the case of Gödels undecidability proof, he mainly considered the contradiction as one between a supposed proof of G and the fact that G represents its own unprovability (RFMI, appendix I, §8, 10, 11). However, his rejection also applies to the so-called syntactic version of Gödel's proof since this version also relies on the assumption that the formal property of provability can be represented within L_A, which involves self-referential interpretations in the diagonal case. No proof of contradiction can be a compelling reason to give up the search for a decision procedure since an advocate of the algorithmic proof conception questions the assumption of representability for the formal property in question. Wittgenstein compared the contradiction arising in an undecidability proof to a paradox (RFM I, appendix I, §12f, §19). According to Wittgenstein's analysis, so-called semantic paradoxes, such as the Liar paradox, as well as paradoxes of mathematical logic, such as Russell's paradox, rely on the representation of formal properties by propositional functions (cf. TLP 3.33-3.334; WVC, p. 121; and PR, p. 207f.). The problem lies not with the specific properties (semantic properties vs. set-theoretical properties) but with the analysis of self-reference as something that is expressible by propositional functions and thus capable of being represented in a symbolism based on FOL. The distinction between meta- and object-language is not sufficient to prevent paradoxes, according to Wittgenstein's analysis. Instead, it is the distinction between formal and material properties that must be considered. This distinction comprises both semantic paradoxes and the paradoxes of mathematical logic. It even applies to arithmetic properties and their meta-mathematical correlates. For Wittgenstein, the arithmetic and meta-mathematical interpretations in the language of L_A were not an "absolutely uncontroversial part of mathematics" ([Wang (1987)], p. 49; however, cf. also [Gödel (1931)], p. 149, footnote 14) but rather the outcome of the fundamental mistake of mathematical logic, namely, the assertion that formal properties of mathematics and meta-mathematics can be expressed by propositional functions. Wittgenstein's algorithmic proof conception rules out such a possibility since it maintains that formal properties can be expressed only by symbolic properties of a proper notation. Wittgenstein believed in an algorithmic proof conception as the standard for a rigorous proof that can never be affected by any underlying intended interpretations of a logical symbolism to represent any properties, since such an interpretation necessarily extends beyond the realm of mere symbolic manipulations."
>
> *. . . Lampert: [Lam19], §4.1 Wittgenstein's reaction to Gödel's undecidability proof.*

11.G.h.   Gödel's reasoning ironically justifies Wittgenstein's 'notorious' paragraph

Ironically, we note that the *evidence-based* reasoning in [An16]:

— not only justifies Wittgenstein's insistence that the *meaning* (interpretation) of meta-mathematical propositions must be rule-based,

— but also Wittgenstein's 'notorious' objection, in [Wi78], to the meta-mathematical conclusions that Gödel draws from his own reasoning in [Go31],

— which Lampert describes in [Lam19] as objecting to a 'fundamental mistake of mathematical logic', namely:

– 'the assertion that formal properties of mathematics and meta-mathematics can be expressed by propositional functions';

since Wittgenstein's objection can now be seen to be entailed by Gödel's own proof, in [Go31] (p.25), that '17 Gen r is not $\kappa - \mathrm{PROVABLE}$'!

For, this proof entails that if, say, a number-theoretic assertion over the domain $\mathbb{N}$ of the natural numbers can be symbolically denoted by '$(\forall x)R^*(x)$', where the number-theoretic relation $R^*(x)$ can be formally expressed in PA by some PA-formula $[R(x)]$[128] such that, for any given natural number n (and corresponding PA-numeral $[\overline{n}]$):

(1) if $R^*(n)$ holds in $\mathbb{N}$, then $[R(\overline{n})]$ is provable in PA;

(2) if $\neg R^*(n)$ holds in $\mathbb{N}$, then $[\neg R(\overline{n})]$ is provable in PA;

we cannot claim, in the absence of a rule-based (*evidence-based*) proof that:

— if, under a *well-defined* interpretation of PA over $\mathbb{N}$, the PA-formula $[R(x)]$ interprets as an arithmetical relation over $\mathbb{N}$ denoted symbolically by $R'(x)$,

– so that the PA-formula $[(\forall x)R(x)]$ interprets as the arithmetical assertion over $\mathbb{N}$ denoted symbolically by $(\forall x)R'(x)$,

— then the arithmetical assertion $(\forall x)R'(x)$ must *assert/mean* in $\mathbb{N}$ what the number-theoretic assertion symbolically denoted by $(\forall x)R^*(x)$ *asserts/means* in $\mathbb{N}$;

even though, for any natural number n:

(1) $R^*(n)$ holds in $\mathbb{N}$ if, and only if, $R'(n)$ holds in $\mathbb{N}$;

(2) $\neg R^*(n)$ holds in $\mathbb{N}$ if, and only if, $\neg R'(n)$ holds in $\mathbb{N}$.

In other words, from the *evidence-based* perspective of this investigation:

— quantification of the formal arithmetical representation $[R(x)]$ of a number-theoretic relation $R^*(x)$ does not necessarily *assert/mean*, under interpretation, what

— quantification of the number-theoretic relation $R^*(x)$ *asserts/means*;

since, by [An16], Corollary 8.3 (p.42):

— the number-theoretic relation $R^*(x)$ can be algorithmically *computable*;

— whilst $[R(x)]$ interprets:

– under Tarski's inductive definitions of the 'satisfiability' and 'truth' of the formulas of a formal language under a *well-defined* interpretation,

– as an arithmetical relation, say $R'(x)$,

– which can be algorithmically *verifiable*, but not algorithmically *computable*.

---

[128]We use square brackets to distinguish between an expression that denotes a PA-formula, and an expression that denotes a natural number or a number-theoretic proposition/relation/function in/over the domain $\mathbb{N}$ of the natural numbers.

11.G.i.   A phenomenological critique of Gödel's interpretation of his own formal argumentation

We consider Gödel's interpretation of his own formal argumentation in [Go31] from another perspective where even if we do—as Stathis Livadas apparently does in his phenomenological critique [Lvd16] of Gödel's argumentation—admit Gödel's claims:

(i) of having finitarily constructed an *unprovable* PA formula $[(\forall x)R(x)]$ that interprets under any well-defined interpretation of PA as a *true* arithmetical proposition which *asserts* that 'The PA-formula $[(\forall x)R(x)]$ is PA-unprovable' (see §11.C.); and

(ii) of having finitarily constructed an *unprovable* PA formula $[w]$ that interprets under any well-defined interpretation of PA as a *true* arithmetical proposition which *asserts* that 'PA is consistent' (see §11.G.);

as prima facie reasonable—and presumed formally defensible—Livadas argues that, nevertheless, since Gödel's incompleteness results appeal critically to interpretations of 'quantification over an indefinite horizon within the realms of arithmetic', it would follow that 'both Gödel's incompleteness results in the various forms of their proof can be seen from a certain angle as essentially due to the insufficiency of finitistic arithmetic means to represent meta-mathematical statements incorporating a non-rigorous finitistic content':

> "Concerning Gödel's incompleteness results, quantification over an indefinite horizon within the realms of arithmetic is a critical factor in the generation of both incompleteness theorems by formally representing, through the application of universal quantifiers, the non-finitistic content of meta-mathematical statements within arithmetical calculus.[8] In the general view of this article the non-finitistic meta-mathematical content of certain expressions and properties in formal arithmetical calculus can be associated with the kind of actual infinity freely generated through the continuous unity of temporal consciousness and presented as an objective whole in acts of reflection.
>
> As it is known as main pillars in achieving Gödel's incompleteness results stand: (a) Gödel's complete arithmetization of formal (predicate) calculus (b) the complete arithmetization of meta-mathematical statements referring to expressions in the formal calculus and (c) the notion of the mapping of sets of meta-mathematical statements turned to expressions of the formal calculus onto arithmetical ones.
>
> Accordingly, the formula $[(\forall x)\neg \mathrm{Dem}(x, y)]$ is the arithmetical representation of the meta-mathematical statement 'for every x the sequence of formulas with Gödel number x is not a proof of the formula with Gödel number y'. By an ingenious technique Gödel constructed a universally quantified arithmetical formula (in S. C. Kleene's notation $A_p(\mathbf{p})$) which asserts of itself that it is not demonstrable (even though it is true) and corresponds to the meta-mathematical statement: 'For every x the sequence of formulas with Gödel number x is not a proof of the formula whose Gödel number is the Gödel number of the formula which is obtained by substituting in the place of numerical variable y the Gödel number of the formula $(\forall x)\neg \mathrm{Dem}(x, \mathrm{sub}\ (y, 13, y))$'. This latter represents in turn the meta-mathematical statement: 'The formula with Gödel number sub $(y, 13, y)$ is not demonstrable'.[9]
>
> In a definite sense by relying on the mapping of meta-mathematical statements onto arithmetical ones, in other words by arithmetizing a 'non-rigorous' discussion about mathematical objects Gödel essentially transposed meta-mathematical 'pathologies' of a non-finitistic content (those whose range of application is an indefinite horizon) onto arithmetical ones by means of a universal quantification over variables x with x being a Gödel number belonging to a certain (infinite) subset of N.

I note that in Gödel's original presentation it was proved that if the formal arithmetical system is (simply) consistent then $A_p(\mathbf{p})$ is not demonstrable and if the system is ω-consistent[10] then $\neg A_p(\mathbf{p})$ is not demonstrable (ω-consistency implies simple consistency). Consequently, if the arithmetical system is ω-consistent then it is incomplete with $A_p(\mathbf{p})$ an example of an undecidable formula (Kleene 1980, pp. 207-208). At this point it is noteworthy that the notion of ω-consistency points indirectly to the views presented in earlier sections, namely those bearing to the fact that mathematical objects or relations in general possess an 'inner' horizon which is open to new insights, new possibilities of intuitive elaboration, even to a mental reconfiguration of apprehended objects with regard to all existing and possible interrelations referring in a significant part to the capacities of a subject's categorical intuition. Technically this has to do here with the fact that a system may be ω-inconsistent without being inconsistent. This means that while formula $(\exists x)P(x)$ and [any] member of the infinite set of $\neg P(0), \neg P(1), \neg P(2),\dots$ should be both demonstrable by ω-inconsistency definition, the formula $(\forall x)\neg P(x)$ may nonetheless not be demonstrable in which case the system in question is not inconsistent since in that case $(\exists x)P(x)$ and $(\forall x)\neg P(x)$ should be both demonstrable (Nagel and Newman 1958, p.91). This formal result clearly shows that even though we may have an infinitely proceeding series of identical formulas 'indexed' by corresponding values of variables (these formulas being demonstrable), yet a universal quantification over these values may not yield a demonstrable formula.

In fact, one can hardly interpret this paradoxical situation at the subjective meta-theoretical level than by admitting to some infinity factor underlying universal quantification over an indefinite horizon which is non-eliminable by a discrete 'stepwise' approximation. Moreover, one can hardly proceed to an objectivity of understanding such as $(\forall x)\neg P(x)$ through a generation of objectivities like $\neg P(0), \neg P(1), \neg P(2),\dots$ which correspond to 'real-world' or immanently induced apprehensions, than by admitting some kind of temporal unity that makes up for the deficiency between the temporal moments of objectifying acts $\neg P(0), \neg P(1), \neg P(2),\dots$ going on ideally ad infinitum and the temporal moment in which the expression $(\forall x)\neg P(x)$ becomes an objectivity of understanding in immediate presentation.

As a matter of fact, both Gödel's incompleteness results in the various forms of their proof can be seen from a certain angle as essentially due to the insufficiency of finitistic arithmetical means to represent meta-mathematical statements incorporating a non-rigorous finitistic content. As meta-mathematical statements are mapped **onto** corresponding arithmetical ones a possible means to formally express the non-finitistic meta-mathematical content is by the application of universal quantifiers with an indefinite scope in the intermediate stage of predicate calculus. In my approach, any universal quantification of an indefinite scope even one concerning the set of natural numbers in its entirety may be taken as ultimately conditioned on the assumption of an actual infinity in the present now independently of any spatiotemporal constraints and at the same time as conditional on a stepwise enactment of mathematical intuitions (concerning formal individuals or generally 'concrete' mathematical objects) progressing ideally *ad infinitum.* It is thanks to these subjectively founded conditions that there exists a possibility of extending indefinitely the scope of concrete mathematical act in preserving the essential invariability of corresponding mathematical objects. On these grounds, for instance, we can construct the undecidable formula $A_p(\mathbf{p})$ in a way that the (universally quantified) variable b does not stand with $\mathbf{p}$ in the arithmetical relation $A(\mathbf{p},b)$, where $\mathbf{p}$ is the Gödel number of the formula $(\forall b)\neg(a,b)$. In the particular case this formal possibility is implemented by applying Cantor's diagonal method which is known to presuppose a meta-theoretical notion of an 'infinite' objective whole in presentational immediacy."

. . . Livadas: [Lvd16], §4 The Question of Universal-Existential Quantification . . . , pp.22-24.

From the *evidence-based* perspective of this investigation, Livadas is apparently making the point here that:

(a) Asserting the *truth/objectivity* of the arithmetical formula $[(\forall x)\neg P(x)]$—under any *well-defined* interpretation of the arithmetic—formally represents, through the application of a universal quantifier, the non-finitistic content of a meta-mathematical statement, within arithmetical calculus. This can then be associated with the kind of actual infinity freely

generated through the continuous unity of temporal consciousness, and presented as an objective whole in acts of reflection.

(b) Hence, asserting the *truth/objectivity* of the arithmetical formula $[(\forall x)\neg P(x)]$—under any *well-defined* interpretation of the arithmetic—entails, and is entailed by, asserting the *truth/objectivity* of the non-terminating sequence of formulas $[\neg P(0)], [\neg P(1)], [\neg P(2)], \ldots$ *jointly* (i.e., formally as algorithmically *computable* in the sense of §4.C., Definition 16; and informally in the sense of $[\neg P(0)]$ & $[\neg P(1)]$ & $[\neg P(2)], \ldots$) under the interpretation;

(c) However, it follows from Gödel's Theorem VI in [Go31] that asserting the *truth/objectivity* of each of the non-terminating sequence of arithmetical formulas $[\neg P(0)], [\neg P(1)], [\neg P(2)], \ldots$ *severally* (i.e., as algorithmically *verifiable* in the sense of §4.C., Definition 13)—under any putative interpretation of the arithmetic—would not entail the *truth/objectivity* of the formula $[(\forall x)\neg P(x)]$;

(d) It thus follows that asserting the *truth/objectivity* of each of the non-terminating sequence of arithmetical formulas $[\neg P(0)], [\neg P(1)], [\neg P(2)], \ldots$ *severally* (i.e., as algorithmically *verifiable* in the sense of §4.C., Definition 13)—under a *well-defined* interpretation of the arithmetic—does not entail the *truth/objectivity* of the sequence *jointly* (i.e., as algorithmically *computable* in the sense of §4.C., Definition 16) under the interpretation;

(e) Moreover, the provability of the arithmetical formula $[(\exists x)P(x)]$—which is merely an abbreviation for the arithmetical formula $[\neg(\forall x)\neg P(x)]$—is consistent with the provability of each of the formulas $[\neg P(0)], [\neg P(1)], [\neg P(2)], \ldots$ *severally* (i.e., as algorithmically *verifiable* in the sense of §4.C., Definition 13) in the arithmetic;

If so, Livadas seems to conclude that, from a phenomenological perspective, Gödel cannot therefore justifiably claim that his 'formally undecidable' arithmetical proposition is *unprovable* but *true* under any *well-defined*—hence *finitary*—interpretation of the arithmetic since:

> "...one can hardly interpret this paradoxical situation at the subjective meta-theoretical level than by admitting to some infinity factor underlying universal quantification over an indefinite horizon which is non-eliminable by a discrete 'stepwise' approximation".
>
> ...*Livadas: [Lvd16], §4 The Question of Universal-Existential Quantification ..., p.23.*

Livadas' 'paradoxical situation' refers to:

> "...the views presented in earlier sections, namely those bearing to the fact that mathematical objects or relations in general possess an 'inner' horizon which is open to new insights, new possibilities of intuitive elaboration, even to a mental reconfiguration of apprehended objects with regard to all existing and possible interrelations referring in a significant part to the capacities of a subject's categorical intuition. Technically this has to do here with the fact that a system may be ω-inconsistent without being inconsistent. This means that while formula $(\exists x)P(x)$ and [any] member of the infinite set of $\neg P(0), \neg P(1), \neg P(2), \ldots$ should be both demonstrable by ω-inconsistency definition, the formula $(\forall x)\neg P(x)$ may nonetheless not be demonstrable ...".
>
> ...*Livadas: [Lvd16], §4 The Question of Universal-Existential Quantification ..., p.23.*

The significance of Livadas' 'paradoxical situation' in the above phenomenological argument emerges if we interpret the above para, from the *evidence-based* perspective of this investigation, as suggesting that:

— despite having admitted both of Gödel's claims §11.G.i.(1) and §11.G.i.(2); and

— although Gödel's assumption of ω-consistency

    – in the second part ([Go31], p.26, '2. Neg(17 Gen r) is not κ-provable') of his metamathematical proof of the existence of a 'formally undecidable' arithmetical proposition;

— entails interpreting the universal quantifier non-finitarily as Aristotle's particularisation (see §4., Definition 12; and §5.D., Theorem 5.12),

— denial of such an assumption, *and* the non-finitarity that it entails (see §5.D., Corollary 5.10),

— could, thereby, admit instead ω-*in*consistency for arithmetic;

    – by Corollary 8.4 in [An16] (see also §2.F.: Corollary 2.20, Corollary 2.21, and Corollary 2.22); and, independently,

    – by §9.A.f., Theorem 9.5;

— an admittance, moreover, which does not entail non-finitary consequences.

Part IV
WHERE *EVIDENCE-BASED* REASONING CHALLENGES CURRENT PARADIGMS

We conclude this investigation by analysing the arguments where *evidence-based* reasoning challenges current paradigms in respect of:

- Cantor's Continuum Hypothesis (§12.);

- Gödel's and Rosser's proofs of 'undecidability' (§13.);

- Non-standard models of PA (§14.);

- Goodstein's argument (§15.);

- The logical and semantic paradoxes (§16.);

    - The mythical 'set-theoretical' limits of fractal constructions (§16.C.);
    - The mythical completability of metric spaces (§16.D.b.);

- Lucas' and Penrose's Gödelian Arguments (§17.);

    - A Definitive Turing Test (§17.E.);

- Prime divisibility and integer factorisation (§18.);

    - Estimating primes in an arithmetical progression (§18.C.);
    - Estimating twin primes (§18.D.);

- The *EPR* paradox (§19.A.);

    - The Bohr-Einstein debate (§19.B.);
    - Dimensionless constants (§19.D.);
    - Conjugate properties (§19.D.f.);
    - Entangled particles (§19.D.g.);
    - Schrödinger's cat 'paradox' (§19.D.h.).

- The philosophy of mathematics education (§20.);

- The significance of *evidence-based* reasoning and the Church-Turing Thesis for quantum computing (§21.);

- The significance of *evidence-based* reasoning for SETI (§22.).

CHAPTER 12. *PARADIGM CHALLENGING CONSEQUENCES*

12. Cantor's Continuum Hypothesis

The set-theoretical perspective on Cantor's Continuum Hypothesis CH[129] is well-known.

- Kurt Gödel showed in 1939[130] that CH is consistent with the usual Zermelo-Fraenkel (ZF) axioms for set theory *if* ZF is consistent. On this assumption, he then defined a putative model of ZF in which both the Axiom of Choice (AC) and CH would hold.

- Paul Cohen showed in 1963[131] that the negations of AC and CH are also consistent with ZF; in particular, he defined a putative model of ZF in which CH would fail whilst AC would hold *if* ZF is consistent.

We now argue that—since PA is finitarily consistent (see §2.C., 2.16), and since there can be no *evidence-based* assignment of a *truth* value to the ZF axiom of infinity[132]—there is a preferential *evidence-based* arithmetical perspective of CH which is based on distinguishing between algorithmically *verifiable* number-theoretic functions and algorithmically *computable* number-theoretic functions[133].

We shall appeal to Gödel's β-function, which uniquely corresponds each real number to an algorithmically *verifiable* arithmetical function.

Since PA is first-order, both the algorithmically *verifiable* arithmetical functions and the algorithmically *computable* arithmetical function are denumerable. We conclude that both the natural numbers and the reals are also denumerable; thereby illustrating the (Downwards) Löwenheim-Skolem Theorem (see §4.J.).

From a more practical perspective, the significance of an *evidence-based* interpretation of the Continuum Hypothesis for the computer sciences, and the development of mechanical intelligences, is highlighted by the, seemingly illusory, constraining influences of current paradigms.

For instance, Shai Ben-David, Pavel Hrubeš, Shay Moran, Amir Shpilka and Amir Yehudayoff argue that current paradigms suggest 'Learnability can be undecidable':

> "The mathematical foundations of machine learning play a key role in the development of the field. They improve our understanding and provide tools for designing new learning paradigms. The advantages of mathematics, however, sometimes come with a cost. Gödel and Cohen showed, in a nutshell, that not everything is provable. Here we show that machine learning shares this fate. We describe simple scenarios where learnability cannot be proved nor refuted using the standard axioms of mathematics. Our proof is based on the fact the continuum hypothesis cannot be proved nor refuted. We show that, in some cases, a solution to the 'estimating the maximum' problem is equivalent to the continuum hypothesis. The main idea is to prove an equivalence between

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[129]There is no set whose cardinality is strictly between the cardinality $\aleph_0$ of the integers and the cardinality $2^{\aleph_0}$ of the real numbers.

[130][Go40].

[131][Co66].

[132]Which asserts the existence of a completed infinity in every model of ZF.

[133]The distinction was introduced—and its significance highlighted—in [An16]. Since set-theoretic functions are defined extensionally, it is not obvious how—or even whether—this distinction can be reflected within ZF.

learnability and compression."

...*Ben-David et al: [Dvd19], Abstract*

"Identifying the learnable is a fundamental goal of machine learning. To achieve this goal, one should first choose a mathematical framework that allows a formal treatment of learnability. This framework should be rich enough to capture a wide variety of learning problems. Then, one should find concrete ways to characterize learnability within this framework. This paradigm has been successfully applied in many contexts of machine learning. In this work, however, we show that this paradigm fails in a well studied learning model. We exhibit a simple problem where learnability cannot be decided using the standard axioms of mathematics (that is, of ZermeloFraenkel set theory with the axiom of choice, or ZFC set theory). We deduce that there is no dimension-like quantity that characterizes learnability in full generality."

...*Ben-David et al: [Dvd19], Opening paragraph*

12.A. Gödel's β-function

We note that Gödel's β-function is defined as ([Me64], p.131):

$$\beta(x_1, x_2, x_3) = rm(1 + (x_3 + 1) \star x_2, x_1)$$

where $rm(x_1, x_2)$ denotes the remainder obtained on dividing x_2 by x_1.

We also note that:

Lemma 12.1. *For any non-terminating sequence of values $f(0), f(1), \ldots,$ we can construct natural numbers b_k, c_k such that:*

(i) $j_k = max(k, f(0), f(1), \ldots, f(k))$;

(ii) $c_k = j_k!$;

(iii) $\beta(b_k, c_k, i) = f(i)$ *for* $0 \le i \le k$.

Proof This is a standard result ([Me64], p.131, Proposition 3.22). □

Now we have the standard definition ([Me64], p.118):

Definition 31. *A number-theoretic function $f(x_1, \ldots, x_n)$ is said to be representable in the first order Peano Arithmetic PA if, and only if, there is a PA formula $[F(x_1, \ldots, x_{n+1})]$ with the free variables $[x_1, \ldots, x_{n+1}]$, such that, for any specified natural numbers $k_1, \ldots, k_{n+1}$:*

(i) *if $f(k_1, \ldots, k_n) = k_{n+1}$ then PA proves: $[F(k_1, \ldots, k_n, k_{n+1})]$;*

(ii) *PA proves: $[(\exists_1 x_{n+1}) F(k_1, \ldots, k_n, x_{n+1})]$.*

The function $f(x_1, \ldots, x_n)$ is said to be strongly representable in PA if we further have that:

(iii) *PA proves: $[(\exists_1 x_{n+1}) F(x_1, \ldots, x_n, x_{n+1})]$.* □

We also have that:

Lemma 12.2. *$\beta(x_1, x_2, x_3)$ is strongly represented in PA by $[Bt(x_1, x_2, x_3, x_4)]$, which is defined as follows:*

$$[(\exists w)(x_1 = ((1 + (x_3 + 1) \star x_2) \star w + x_4) \wedge (x_4 < 1 + (x_3 + 1) \star x_2))].$$

Proof This is a standard result ([Me64], p.131, proposition 3.21). □

12.B.   An *evidence-based* arithmetical perspective on Cantor's Continuum Hypothesis

From an *evidence-based* perspective every real number is algorithmically *verifiable* by definition, hence *well-defined* by §4.F., Definition 18.

Further, by §4.G., Theorem 4.2, there are *well-defined* real numbers that are algorithmically *verifiable* but not algorithmically *computable*.

Moreover, by §4.H., Theorem 4.4, every real number is *specifiable* in PA.

> **Comment**: It is not at all obvious how, or even whether, an algorithmically *verifiable* real number can be defined, or even merely treated, as *unspecifiable* within a theory, and *yet* have properties that can be entailed by the theory.
>
> In other words, unless postulation of an *unspecified* element within a theory can be shown not to invite contradiction, the putative existence of any such element, either within the theory or in the domain of any putative interpretation of the theory (as in the case of Aristotle's particularisation in §4., Definition 12), would, for instance, be no different in principle from postulation of the existence of a figure corresponding to the definition of a square circle in geometry.

We now show that:

Theorem 12.3. *The cardinality $2^{\aleph_0}$ of the real numbers cannot exceed the cardinality $\aleph_0$ of the integers.*

Proof. Let $\{r(n)\}$ be the denumerable sequence defined by the denumerable sequence of digits in the binary decimal expansion $\sum_{n=1}^{\infty} r(n).10^{-n}$ of a putatively *specifiable* real number $\mathbb{R}$ in the interval $0 < \mathbb{R} \leq 1$ where, for any specified i, $r(i)$ is either 0 or 1.

By convention, we denote the cardinality of the natural numbers by $\aleph_0$ and, since each $r(i)$ can have two possible values, the cardinality of the putative set $\{r(n)\}$ of *specifiable* reals by $2^{\aleph_0}$.

By §12.A., Lemma 12.1, for any *specified* natural number k, we can define natural numbers b_k, c_k such that, for any $1 \leq n \leq k$:

$\beta(b_k, c_k, n) = r(n)$.

By §12.A., Lemma 12.2, $\beta(b_k, c_k, n)$ is uniquely represented in the first order Peano Arithmetic PA by $[Bt(b_k, c_k, n, x)]$ such that, for any $1 \leq n \leq k$:

If $\beta(b_k, c_k, n) = r(n)$ then PA proves $[Bt(b_k, c_k, n, r(n))]$.

We now define the arithmetical formula $[R(b_k, c_k, n)]$ for any $1 \leq n \leq k$ by:

$[R(b_k, c_k, n) = r(n)]$ if, and only if, PA proves $[Bt(b_k, c_k, n, r(n))]$.

Hence every putatively specifiable real number $\mathbb{R}$ in the interval $0 < \mathbb{R} \leq 1$ can be uniquely corresponded to an algorithmically *verifiable* arithmetical formula $[R(x)]$ since:

> For any k, the primitive recursivity of $\beta(b_k, c_k, n)$ yields an algorithm $\mathrm{AL}_{(\beta, \mathbb{R}, k)}$ that provides objective evidence for deciding the unique value of each formula in the finite sequence $\{[R(1), R(2), \ldots, R(k)]\}$ by evidencing the truth under a sound interpretation of PA for:

$$[R(1) = R(b_k, c_k, 1)]$$
$$[R(b_k, c_k, 1) = r(1)]$$
$$[R(2) = R(b_k, c_k, 2)]$$
$$[R(b_k, c_k, 2) = r(2)]$$
$$\ldots$$
$$[R(k) = R(b_k, c_k, k)]$$
$$[R(b_k, c_k, k) = r(k)].$$

The correspondence is unique because, if $\mathbb{R}$ and $\mathbb{S}$ are two different, putatively *specifiable*, reals in the interval $0 < \mathbb{R}$, $\mathbb{S} \leq 1$, then there is always some m for which:

$$r(m) \neq s(m).$$

Hence we can always find corresponding arithmetical functions $[R(n)]$ and $[S(n)]$ such that:

$[R(n) = r(n)]$ for all $1 \leq n \leq m$.

$[S(n) = s(n)]$ for all $1 \leq n \leq m$.

$[R(m) \neq S(m)]$.

Since PA is first order, the cardinality of the reals cannot, therefore, exceed that of the integers.

The theorem follows.   □

We conclude, seemingly paradoxically, that—although the Continuum Hypothesis is independent of the axioms of ZF if ZF is consistent—when interpreted over the domain $\mathbb{R}$ of the real numbers, CH follows from the axioms of PA (which is finitarily provable as consistent by §2.C.a., Theorem 2.16):

Corollary 12.4. $\aleph_0 \longleftrightarrow 2^{\aleph_0}$ □

Now, one possible way of resolving such a paradoxical conclusion could be to heed the cautionary remarks by Thoraf Skolem (see §4.J.)—about unrestrictedly corresponding, meta-mathematically, putatively *specifiable* mathematical entities across domains of different axiom systems—in his 1922 address delivered in Helsinki before the Fifth Congress of Scandinavian Mathematicians, where Skolem improved upon both the argument and statement of Löwenheim's 1915 theorem ([Lo15], p.235, Theorem 2)—subsequently labelled as the (downwards) Löwenheim-Skolem Theorem ([Sk22], p.293).

Another perspective, that of Wittgenstein, is elaborated upon by Livingston in [Liv10]:

> "Now, it is familiar that Wittgenstein held, in general, a dim view of the purported *results* of various forms of the "diagonal procedure," including both Cantor's multiple infinites and the truth of Gödel's "self-referential" sentence. Do these doubts, expressed prominently in the *Remarks on the Foundations of Mathematics*, imply that there is not a very similar concern about the relationship of finite symbolism to infinitary techniques operative in Wittgenstein's own thoughts about rules and symbols? I think not, for the following reasons. In his critical remarks about the Gödel sentence as well as about Cantor's multiple infinities, Wittgenstein emphasizes that the existence of a procedure—even one with no fixed end, like the procedure of writing down numbers in Arabic numerals—does not imply the existence of a superlative *object*, either a "huge number" or a completed list of decimal expansions that itself contains "infinitely many" members. To a certain extent at least, these suspicions extend to the 'diagonal procedure" itself. Though Cantor can, with some justice, say how one *can* generate a decimal expansion that. as one can show, does not appear anywhere on an "infinite list" of expansions, he has not *in fact* generated

it; diagonalization is always in fact the "outcome" of an infinite procedure and cannot be said to have finished. However, Wittgenstein does not deny that there *is* such a procedure, and even that we can speak of it, with some justice, as one that shows (by giving sense to the proposition) that there is, for any set of decimal expansions, one that is not in the set (*RFM* II-29). Cantor has given us a procedure that allows us to say: *given* any series of numerical symbols, we *can* (i.e. we have a method that lets us) generate a different one. However, in understanding the possibility and implications of this procedure, we must also keep in mind that there is a difference between series of numerical *symbols* and series of *numbers* in the mathematical sense. *A series in the mathematical sense is not a sequence of signs but a method for generating sequences of signs.*[9] There are analogies between the two uses, but they are different; and given the difference, Wittgenstein suggests, the existence of a sign ("$\aleph_0$") that expresses the unlimited possibility—the unlimitedness of the method—of generating sequences of signs does not by itself ground a further calculus with this sign, for instance one relating it to "other" infinities or other sizes of infinity. Nevertheless, as we have seen, it is just this ambiguity between sequences of signs and methods for generating sequences of signs upon which the claim of diagonalization to establish "positive" results depends. Diagonalization intervenes upon what are in fact sequences of signs (series in the non-mathematical sense) to produce a new number, a new sequence of signs which may itself be unlimited. What operates in this ambiguity, and creates the "crossing" at infinity (real or illusory) between procedures and their symbolization that is essential to diagonalization, is our presumed *infinitary* capacity to produce symbols according to well-defined rules.

In adducing these distinctions and casting doubt on the positive results of diagonalization, Wittgenstein's point is emphatically not, however, to show the nonexistence or invalidity of diagonalization as an (infinitary) *technique*. Rather, it is to emphasize the extent to which this procedure or technique, as infinitary as it is, has a place within a human life, and does not derive its meaning or sense from any other source than this life itself. Much later, in *RFM*, Wittgenstein comes back to this point:

> The concept of the rule for the formation of an infinite decimal is—of course—not a specifically mathematical one. It is a concept connected with a rigidly determined *activity* in human life. The concept of this rule is not more mathematical than that of: following the rule. Or again: this latter is not less sharply defined than the concept of such a rule itself.—For the expression of the rule and its sense is only part of the language-game: following the rule. (*RFM* VII-42, p.409)

Again, Wittgenstein is not here denying that there is a valid concept of the rule for the formation of something that is indeed infinite. He is, rather, affirming that this formation—even in its strictness and rigidity—necessarily takes place as part of a human life, and gains its meaning and sense from this life. As it is capable of such infinite results, it would not, it seems, be quite right to call such a life, or the practice of following a rule within it (the language-game) that brings these about, "finite." Rather, the practice is *precisely* a technique: something of which beings with a finite spatiotemporal extent are capable, but whose *extension* is in principle without limit. It is thus neither the finitude of language nor the infinitude of meaning that makes possible its effect, but rather the gulf between them, in which Wittgenstein recognizes the openness of a human life.

There are, I think, two preliminary conclusions that can be drawn so far. The first is exegetical: Wittgenstein was certainly not in 1939, and probably never was, a finitist. That is, he *never* held that the finite character of language implied the non-existence or non-reality of infinite procedures. Rather, his focus is uniformly on the problem of the *grammar* of the infinite procedure: that is, just *how it is* that finite signs handled by finite beings gain the sense of infinity. This is none other than the radically posed question of the later Wittgenstein's thought: the question of the nature of a technique or practice. And it leads to the second conclusion, which is not exegetical but philosophical: that the infinity of technique is not an extension or intensification of the finite; nor is it a superlative or transcendent object that lies "beyond" all finite procedures. The infinity of technique enters a human life, rather, at the point of what might seem at first a radical paradox: that of its capture in finite signs, the crossing of syntax and semantics wherever the infinite rule is thought and symbolized as finite."

. . . Livingston: [*Liv10*]*, pp.35-37*

CHAPTER 13. *PARADIGM CHALLENGING CONSEQUENCES*

13. Analysing Gödel's and Rosser's proofs of 'undecidability'

We note that, in his seminal 1931 paper, Gödel constructively defined a Peano Arithmetic P, and a P-formula $[R(x)]$ (in his argument, Gödel refers to this formula only by its 'Gödel' number 'r'; [Go31], p.25, Eqn.(12)), such that ([Go31], Theorem VI, p.24, p.25(1) & p.26(2)):

Lemma 13.1. *If P is ω-consistent, both $[(\forall x)R(x)]$ and $[\neg(\forall x)R(x)]$ are not P-provable.* □

Of course, since every ω-consistent system is necessarily simply consistent, Gödel's conclusion is significant only if there is an ω-consistent language that seeks to formally express all our true propositions about the natural numbers.

The issue of whether there is, or can be, a finitarily provable ω-consistent system of Arithmetic at all, appears to have been widely (an exception being [Kl52], pp.212-213) treated as inconsequential[134] following J. Barkley Rosser's 1936 paper ([Ro36]), in which he claimed that Gödel's reasoning can be 'extended' to arrive at Gödel's intended result (i.e., construction of a formally undecidable arithmetical proposition in P) by assuming only that P is simply consistent (i.e., without assuming that P is ω-consistent).

For instance, in a relatively recent paper [SS17], Saeed Salehi and Payam Seraji argue that (see also §5.G., Corollary 5.21):

> "... Gödel's original first incompleteness theorem did not assume the soundness of the theory in question, and he introduced the notion of ω-consistency for that purpose. Later it was found out that the weaker notion of l-consistency suffices for the theorem (see e.g. [3] or [12]). By generalizing this equivalent notion to higher degrees (Π_n in general) we will prove some generalizations of Gödel's first incompleteness theorem for definable theories below. Finally, Rosser's Trick proves Gödel's result without assuming the l-consistency of the theory. So, Gödel-Rosser's Incompleteness Theorem, assuming only the consistency of the theory, states that *any consistent and* `RE` *theory containing* `PA` *is incomplete.* It is tempting to weaken the condition of recursive enumerability of the theory in this theorem; but we will see below that this is not possible. We can thus argue that Gödel-Rosser's theorem is optimal in a sense."
>
> ... *Salehi and Seraji:* [SS17]*, 1 Introduction and Preliminaries.*

However, we now analyse various expositions of Rosser's argument (vis à vis Gödel's reasoning), and show that they either implicitly appeal to Rosser's Rule C (which entails Aristotle's particularisation; see §5.G., Lemma 5.20), or tacitly to the weaker assumption (see §5.D.) that P is ω-consistent.

13.A. Rosser and formally undecidable arithmetical propositions

Although both Gödel's proof and Rosser's argument are complex, and not easy to unravel, the former has been extensively analysed, and its formal arguments validated[135], in a number of

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[134]See, for instance, [Be59], p.595; [Wa63], p.19 (Theorem 3) & p.25; [Me64], p.144; [Sh67], p.132 (Incompleteness Theorem); [EC89], p.215; [BBJ03], p.224 (Gödel's first incompleteness theorem).

[135]Possibly because Gödel's remarkably self-contained 1931 paper—it neither contained, nor needed, any formal citations—remains unsurpassed in mathematical literature for thoroughness, clarity, transparency and soundness of exposition (despite the critique in §11., concerning mainly Gödel's interpretation of his own formal reasoning in [Go31]).

expositions of Gödel's number-theoretic reasoning (e.g., [Kl52], [Me64], p.143; [EC89], p.210-211).

In sharp contrast, Rosser's widely cited argument does not appear to have received the same critical scrutiny, and its number-theoretic expositions generally remain either implicit or sketchy[136] (an exception being [Me64], Proposition 3.32, pp.145-146; see also §5.H.).

13.B.   Wang's outline of Rosser's argument

Wang, for instance, states that ([Wa63], p.337) from the formal provability of:

(i) $\neg(x)(B(x,\overline{q}) \supset (Ey)(y \leq x \ \& \ B(y, n(\overline{q}))))$

in his formal system of first-order Peano Arithmetic Z, we may infer the formal provability of:

(ii) $(Ex)(B(x,\overline{q}) \ \& \ \neg(Ey)(y \leq x \ \& \ B(y, n(\overline{q}))))$

> **Comment**: We note that although Wang does not explicitly define the interpretation of the formal Z-formula '$(Ex)F(x)$' as 'There is some x such that $F(x)$', this interpretation appears implicit in his discussion and definition of '$(Ev)A(v)$' in terms of Hilbert's ε-function ([Wa63], p.315(2.31); see also p.10 & pp.443-445) as a property of the underlying logic of Wang's Peano Arithmetic Z, and is obvious in the above argument. In other words Wang implicitly implies that the interpretation of existential quantification cannot be specific to any particular interpretation of a formal mathematical language, but must necessarily be determined by the predicate calculus that is to be applied uniformly to all the mathematical languages in question.

However, the inference (ii) from (i) appears to assume that the following deduction is valid for some *unspecified* $\overline{j}$:

$\neg(x)(B(x,\overline{q}) \supset (Ey)(y \leq x \ \& \ B(y, n(\overline{q}))))$

• $(Ex)\neg(B(x,\overline{q}) \supset (Ey)(y \leq x \ \& \ B(y, n(\overline{q}))))$

⋆ $\neg(B(\overline{j},\overline{q}) \supset (Ey)(y \leq \overline{j} \ \& \ B(y, n(\overline{q}))))$

$B(\overline{j},\overline{q}) \ \& \ \neg(Ey)(y \leq \overline{j} \ \& \ B(y, n(\overline{q})))$

$(Ex)(B(x,\overline{q}) \ \& \ \neg(Ey)(y \leq x \ \& \ B(y, n(\overline{q}))))$

Thus, Wang's conclusion appears to implicitly *assume* both Aristotle's particularisation (•) and Rosser's Rule C (⋆); entailing, ipso facto, that Z is ω-consistent (see §5.G.).

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[136]See, for instance, [Kl52], pp.212-213; [Be59], pp.593-595 (which focuses on Rosser's argument, and treats Gödel's proof of his Theorem VI ([Go31], p.24) as a, secondary, weaker result); [Wa63], p.337; [Sh67], p.232 (curiously, this introductory text contains *no* reference to Gödel or to his 1931 paper!); [Rg87], p.98; [EC89], p.215 and p.217, Ex.2; [Sm92], p.81; [BBJ03], p.226 (this introductory text, too, focuses on Rosser's argument, and treats Gödel's argument as more of a historical curiosity!).

13.C. Beth's outline of Rosser's argument

Similarly, in his outline of a formalisation of Rosser's argument, Beth implicitly concludes ([Be59], p.594 (ij)) that from the formal provability of:

(i) $\neg(q)[G_1(m^0, q, m^0) \rightarrow (s)\{B(s,q) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(q,r) \ \& \ B(t,r)\}]\}]$

in his formal system of first-order Peano Arithmetic P, we may infer the formal provability of:

(ii) $(Eq)[G_1(m^0, q, m^0) \ \& \ (s)\{B(s,q) \ \& \ (t)[t \leq s \rightarrow (r)\{H(q,r) \rightarrow \overline{B(t,r)}\}]\}]$

> **Comment**: We note that, in this case, Beth explicitly defines the interpretation of the formal P-formula '(Ex)' as 'There is a value of x such that' ([Be59], p.178). Thus Beth, too, implies that the interpretation of existential quantification in formalised axiomatics cannot be specific to any particular interpretation of a formal mathematical language, but must necessarily be determined by the predicate calculus that is to be applied uniformly to all the mathematical languages in question.

However, again, the inference (ii) from (i) appears to assume that the following deduction is valid for some *unspecified* $\overline{j}$:

$\neg(q)[G_1(m^0, q, m^0) \rightarrow (s)\{B(s,q) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(q,r) \ \& \ B(t,r)\}]\}]$

• $(Eq)\neg[G_1(m^0, q, m^0) \rightarrow (s)\{B(s,q) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(q,r) \ \& \ B(t,r)\}]\}]$

⋆ $\neg[G_1(m^0, \overline{j}, m^0) \rightarrow (s)\{B(s,\overline{j}) \rightarrow (Et)[t \leq s \ \& \ (Er)\{H(\overline{j},r) \ \& \ B(t,r)\}]\}]$

$G_1(m^0, \overline{j}, m^0) \ \& \ (s)\{B(s,\overline{j}) \ \& \ (t)[t \leq s \rightarrow (r)\{H(\overline{j},r) \rightarrow \overline{B(t,r)}\}]\}$

$(Eq)[G_1(m^0, q, m^0) \ \& \ (s)\{B(s,q) \ \& \ (t)[t \leq s \rightarrow (r)\{H(q,r) \rightarrow \overline{B(t,r)}\}]\}]$

Thus, Beth's conclusion, too, appears to implicitly *assume* both Aristotle's particularisation (•) and Rosser's Rule C (⋆); entailing, ipso facto, that Z is ω-consistent (see §5.G.).

13.D. Rosser's original argument *implicitly* presumes ω-consistency

Now, Rosser's claim in his 'extension' ([Ro36]) of Gödel's argument ([Go31]) is that, whereas Gödel's argument assumes that his Peano Arithmetic, P, is ω-consistent, Rosser's assumes only that P is simply consistent.

However, Rosser's original argument (also a sketch) appears to *implicitly* presume that the system of Peano Arithmetic in question is ω-consistent.

For instance, Rosser defines a P-formula $R(x,y)$ and concludes ([Ro36], p.234) that:

(i) If, for any specified natural number n, the formula $[\neg R(n,a)]$ in Gödel's Peano Arithmetic P whose Gödel-number is:

$$Neg(Sb(r \ \overset{u}{Z(n)} \ \overset{v}{Z(a)} \ ))$$

is P_κ-provable under the given premises;

> **Comment**: Gödel defines 'P_κ-provable' to mean provable from the axioms of P and an arbitrary class, κ, of P-formulas—including the case where κ is empty—by the rules of deduction of P.

(ii) Then, if P is simply consistent, the P-formula $[(\forall u)\neg R(u, a)]$ whose Gödel-number is:

$$uGen(Neg(Sb(r\ {}_{Z(a)}^{\ v}\ )))$$

is P_κ-provable;

(iii) Since:

> "... the formal analogue of $(z)[z = 0 \vee z = 1 \vee \ldots \vee z = x \vee (Ew)[z = x + w]]$
>
> is provable in P and hence in P_κ, and so $Bew_\kappa(uGen(Neg(Sb(r\ {}_{Z(a)}^{\ v}\ ))))$".

However, we note that Rosser's argument in (iii) above would need to assume Rosser's Rule C (as we highlight in §5.H.) in any proof sequence in P that involves an existentially quantified P-formula such as '$(Ew)[z = x + w]$', and which yields his conclusion (ii).

By §5.G., this would imply, however, that P is ω-consistent.

CHAPTER 14. *PARADIGM CHALLENGING CONSEQUENCES*

14.   The case against non-standard models of PA

Once we accept as logically sound the set-theoretically based meta-argument[137] that a first-order Peano Arithmetic PA (e.g., the theory S defined in [Me64], pp.102-103) can be forced—by appeal to the Compactness Theorem—into admitting non-standard models which contain an 'infinite' integer, then the set-theoretical properties of the algebraic and arithmetical structures of such putative models should perhaps follow without serious foundational reservation (as argued, for instance, in [Ka91]; [Bov00]; [BBJ03], ch.25, p.302; [KS06]; [Ka11]).

> **Compactness Theorem**: "If every finite subset of a set of sentences has a model, then the whole set has a model."
>
> . . . *Boolos, Burgess & Jeffrey: [BBJ03]. p.147*

From an *evidence-based* perspective, however, there is a cogent case *against* concluding from such arguments the existence of non-standard models of PA.

14.A.   The ambiguity in admitting an 'infinite' constant

To see this we need to first highlight, and eliminate, an ambiguity in the classical argument postulating the existence of non-standard models of arithmetic as is usually found in standard texts[138]:

> "**Corollary**. There is a non-standard model of **P** with domain the natural numbers in which the denotation of every nonlogical symbol is an arithmetical relation or function.
>
> *Proof.* As in the proof of the existence of nonstandard models of arithmetic, add a constant ∞ to the language of arithmetic and apply the Compactness Theorem to the theory
>
> $$\mathbf{P} \cup \{\infty \neq \mathbf{n}: n = 0, 1, 2, \ldots\}$$
>
> to conclude that it has a model (necessarily infinite, since all models of **P** are). The denotations of ∞ in any such model will be a non-standard element, guaranteeing that the model is non-standard. Then apply the arithmetical Löwenheim-Skolem theorem to conclude that the model may be taken to have domain the natural numbers, and the denotations of all nonlogical symbols arithmetical."
>
> . . . *Boolos, Burgess & Jeffrey: [BBJ03], p.306, Corollary 25.3.*

14.A.a.   We cannot force PA to admit a transfinite ordinal

The ambiguity lies in a possible interpretation of the symbol ∞ as a 'completed' infinity (such as Cantor's first transfinite ordinal ω) in the context of non-standard models of PA. To eliminate this possibility we establish trivially that, and briefly examine why:

Theorem 14.1. *No model of PA can admit a transfinite ordinal under the standard interpretation of the first-order logic FOL[139].*

---

[137]By which we mean arguments such as in [Ka91] or [Cho18], where the meta-theory is taken to be a set-theory such as ZF or ZFC, and the logical consistency of the meta-theory is not considered relevant to the argumentation.

[138]cf. [HP98], p.13, §0.29; [Me64], p.112, Ex. 2.

[139]For purposes of this investigation we may take FOL to be the first order predicate calculus K as defined in [Me64], p.57.

Proof. Let $[G(x)]$ denote the PA-formula:

$$[x = 0 \lor \neg(\forall y)\neg(x = Sy)]$$

This translates in every model of PA, as the algorithmically *verifiable* assertion:

> If x denotes an element in the domain of a model of PA, then either x is 0, or x is a 'successor'.

>> **Comment**: We note that $[\neg(\forall y)\neg(x = Sy)]$ is algorithmically *verifiable* as always true under any interpretation of PA; i.e., 'It is false that, for any specified natural number $n > 0$, there is a deterministic algorithm which will *evidence* that n is not the successor of $n - 1$' is a true statement since, 'For any specified natural number $n > 0$, there is a deterministic algorithm which will *evidence* that n is the successor of $n - 1$' is a true statement.

Further, in every model of PA, if $G^*(x)$ denotes the interpretation of $[G(x)]$:

> (a) $G^*(0)$ is true;
> (b) If $G^*(x)$ is true, then $G^*(Sx)$ is true.

Hence, by Gödel's completeness theorem:

> (c) PA proves $[G(0)]$;
> (d) PA proves $[G(x) \rightarrow G(Sx)]$.

>> **Gödel's Completeness Theorem**: In any first-order predicate calculus, the theorems are precisely the logically valid well-formed formulas (*i. e. those that are true in every model of the calculus*).

Further, by Generalisation:

> (e) PA proves $[(\forall x)(G(x) \rightarrow G(Sx))]$;

>> **Generalisation in PA**: $[(\forall x)A]$ follows from $[A]$.

Hence, by Induction:

> (f) $[(\forall x)G(x)]$ is provable in PA.

>> **Induction Axiom Schema of PA**: For any formula $[F(x)]$ of PA:
>> $[F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(Sx)) \rightarrow (\forall x)F(x))]$

Thus, except 0, every element in the domain of any model of PA is a 'successor'. Further, the standard PA axioms ensure that x can only be a 'successor' of a unique element in any model of PA.

Since Cantor's first limit ordinal ω is not the 'successor' of any ordinal in the sense required by the PA axioms, and since there are no infinitely descending sequences of ordinals[140] in a model—if any—of a first order set theory such as ZF, the theorem follows.   □

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[140]cf. [Me64], p.261.

14.A.b.   Why we cannot force PA to admit a transfinite ordinal

Theorem 14.1 reflects the fact that we can define the usual order relation '$<$' in PA so that every instance of the PA Axiom Schema of Finite Induction, such as, say:

$$(i) \; [F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(Sx)) \rightarrow (\forall x)F(x))]$$

yields the weaker PA theorem:

$$(ii) \; [F(0) \rightarrow ((\forall x)((\forall y)(y < x \rightarrow F(y)) \rightarrow F(x)) \rightarrow (\forall x)F(x))]$$

Now, if we interpret PA without relativisation in ZF[141]— i.e., numerals as finite ordinals, $[Sx]$ as $[x \cup \{x\}]$, etc.— then (ii) always translates in ZF as a theorem (*Principle of Transfinite Induction*):

$$(iii) \; [F(0) \rightarrow ((\forall x)((\forall y)(y \in x \rightarrow F(y)) \rightarrow F(x)) \rightarrow (\forall x)F(x))]$$

However, (i) does not always translate similarly as a ZF-theorem, since the following is not necessarily provable in ZF:

$$(iv) \; [F(0) \rightarrow ((\forall x)(F(x) \rightarrow F(x \cup \{x\})) \rightarrow (\forall x)F(x))]$$

Example: Define $[F(x)]$ as '$[x \in \omega]$'.

We conclude that, whereas the language of ZF admits as a constant the first limit ordinal ω which would interpret in any putative model of ZF as the ('completed' infinite) set ω of all finite ordinals:

Corollary 14.2. *The language of PA admits of no constant that interprets in any well-defined model of PA as the set N of all natural numbers.* □

We note that it is the non-logical Axiom Schema of Finite Induction of PA which does not allow us to introduce—contrary to what is suggested by standard texts[142]—an 'actual' (*or 'completed'*) infinity disguised as an arbitrary constant (usually denoted by c or ∞) into either the language, or any *well-defined* model, of PA.

> **Comment**: For instance, in his *A Course on Mathematical Logic*, Shashi Mohan Srivastava uncritically follows current paradigms when—contradicting Corollary 14.2 (as also §2.E., Corollary 2.18; §15.A., Corollary 15.3 and Theorem 15.4)—he notes:
>
> > "...some interesting results without proof.
> >
> > **Theorem 5.3.9** *Peano arithmetic PA has an interpretation in an extension by definitions of ZF. [6]*
> >
> > **Theorem 5.3.10** *Each of Peano arithmetic PA and ZF-Infinity has a faithful interpretation in an extension by definitions of the other. In particular, PA is consistent if and only if ZF-Infinity is consistent. [6, Exercise 30, p.149].*"
> > ...*Srivastava*: [Sri08], p.74.

---

[141]In the sense indicated by Feferman [Fe92].

[142]eg. [HP98], p.13, §0.29; [Ka91], p.11 & p.12, fig.1; [BBJ03]. p.306, Corollary 25.3; [Me64], p.112, Ex. 2; [Sri08], p.74.

14.B.   Forcing PA to admit denumerable descending dense sequences

The significance of §14.A.b., Theorem 14.1, is seen in the next two arguments, which attempt to implicitly bypass the Theorem's constraint by appeal to the Compactness Theorem for forcing a non-standard model onto PA[143].

However, we argue in both cases that applying the Compactness Theorem constructively—even from a classical perspective—does not logically yield a non-standard model for PA with an 'infinite' integer as claimed[144].

14.C.   An argument for a non-standard model of PA

The first is Laureano Luna's argument[145] that we can define a non-standard model of PA with an infinite descending chain of successors, where the only non-successor is the null element 0:

1. Let $<\mathbb{N}$ (*the set of natural numbers*); $=$ (*equality*); S (*the successor function*); $+$ (*the addition function*); $*$ (*the product function*); 0 (*the null element*)$>$ be the structure that serves to define a model of PA, say M.

2. Let T$[M]$ be the set of PA-formulas that are satisfied or true in M.

3. The PA-provable formulas form a subset of T$[M]$.

4. Let Γ be the countable set of all PA-formulas of the form $[c_n = Sc_{n+1}]$, where the index n is a natural number.

5. Let T be the union of Γ and T$[M]$.

6. T$[M]$ plus any finite set of members of Γ has a model, e.g., M itself, since M is a model of any finite descending chain of successors.

7. Consequently, by Compactness, T has a model; call it M'.

8. M' has an infinite descending sequence with respect to S because it is a model of Γ.

9. Since PA is a subset of T, M' is a non-standard model of PA.

14.C.a.   Why the argument in §14.C. is logically fragile

However if—as claimed in §14.C.(6) above—M is a model of T$[M]$ plus any finite set of members of Γ, and the PA term $[c_n]$ is well-defined for any specified natural number n, then:

1. All PA-formulas of the form $[c_n = Sc_{n+1}]$ are PA-provable,

2. Γ is a proper sub-set of the PA-provable formulas, and

3. T is identically T$[M]$.

---

[143]eg. [Ln08]; [Ka91], pp.10-11, p.74 & p.75, Theorem 6.4.

[144]And as suggested also by standard texts in such cases; eg. [BBJ03]. p.306, Corollary 25.3; [Me64], p.112, Ex. 2.

[145][Ln08].

Reason: The argument cannot be that some PA-formula of the form $[c_n = Sc_{n+1}]$ is true in M, but not PA-provable, as this would imply that if PA is consistent then PA+$[\neg(c_n = Sc_{n+1})]$ has a model other than M; in other words, it would presume that which is sought to be proved, namely that PA has a non-standard model!

Consequently, the postulated model M' of T in §14.C.(7) by 'Compactness' is the model M that defines T$[M]$. However, M has no infinite descending sequence with respect to S, even though it is a model of Γ.

Hence the argument does not establish the existence of a non-standard model of PA with an infinite descending sequence with respect to the successor function S.

14.D.   A formal argument for a non-standard model of PA

The second is Richard Kaye's more formal argument[146]:

> "Let $Th(\mathbb{N})$ denote the complete $\mathcal{L}_A$-theory of the standard model, i.e. $Th(\mathbb{N})$ is the collection of all true $\mathcal{L}_A$-sentences. For each $n \in \mathbb{N}$ we let $\underline{n}$ be the closed term $(\dots(((1+1)+1)+\dots+1)))(n \; 1s)$ of $\mathcal{L}_A$; $\underline{0}$ is just the constant symbol 0. We now expand our language $\mathcal{L}_A$ by adding to it a new constant symbol c, obtaining the new language $\mathcal{L}_c$, and consider the following $\mathcal{L}_c$-theory with axioms
>
> ρ (for each $\rho \in Th(\mathbb{N})$)
>
> and
>
> $c > \underline{n}$ (for each $n \in \mathbb{N}$)
>
> This theory is consistent, for each finite fragment of it is contained in
>
> $T_k = Th(\mathbb{N}) \cup \{c > \underline{n} \mid n < k\}$
>
> for some $k \in \mathbb{N}$, and clearly the $\mathcal{L}_c$-structure $(\mathbb{N}, \; k)$ with domain $\mathbb{N}$, 0, 1, $+$, $\cdot$ and $<$ interpreted naturally, and c interpreted by the integer k, satisfies T_k. Thus by the compactness theorem $\cup_{k \in \mathbb{N}} T_k$ is consistent and has a model M_c. The first thing to note about M_c is that
>
> $M_c \models c > \underline{n}$
>
> for all $n \in \mathbb{N}$, and hence it contains an 'infinite' integer."
>
> ...*Kayes: [Ka91], pp.10-11.*

14.D.a.   Why Kaye's argument too is logically fragile

We note again that, from an arithmetical perspective, any application of the Compactness Theorem to PA cannot, first, ignore currently accepted computationalist doctrines of objectivity (see, for instance, [Mu91], §1 Introduction; [Lob59], p.165) that underlie the *evidence-based* reasoning introduced in [An16] (see §4.C.) and, second, contradict the constructive assignment of satisfaction and truth to the atomic formulas of PA (therefore to the compound formulas under Tarski's inductive definitions) in terms of either algorithmical *verifiability* or algorithmic *computability*.

---

[146]Attributed by Kaye as essentially Skolem's argument in [Sk34].

Accordingly, from an arithmetical perspective we can only conclude by the Compactness Theorem that if $Th(\mathbb{N})$ is the $\mathcal{L}_A$-theory of the standard model (interpretation), then we may consistently add to it the following as an additional—not necessarily independent—axiom:

$(\exists y)(y > x).$

In other words, even from the *evidence-based* arithmetical perspective of [An16] (see §2.) anchored *strictly within* the framework of classical logic[147], we *can* conclude incontrovertibly by the Compactness Theorem that (cf., [Ka91], p.10-11):

Lemma 14.3. *If the collection $Th(\mathbb{N})$ of all true $\mathcal{L}_A$-sentences is the $\mathcal{L}_A$-theory of the standard model of Arithmetic, then we may consistently add to it the following as an additional—not necessarily independent—axiom:*

$(\exists y)(y > x).$

Proof. By §2., Definition 3, $(\exists y)(y > x)$ is algorithmically *computable* as always true in the standard model of Arithmetic considered above—whence all of its instances are in $Th(\mathbb{N})$. □

However, we cannot conclude (contrary to the argument, for instance, in [Ka91], p.10-11) by the Compactness Theorem that $\cup_{k\in\mathbb{N}}T_k$ is consistent and has a model M_c which contains an 'infinite' integer:

Theorem 14.4. *The Compactness Theorem does not entail that:*

(*) $\cup_{k\in\mathbb{N}}\{Th(\mathbb{N}) \cup \{c > \underline{n} \mid n < k\}\}$

is consistent and has a model M_c which contains an 'infinite' integer.

Proof. The condition '$k \in \mathbb{N}$' in (*) above requires, first of all, that we must be able to extend $Th(\mathbb{N})$ by the addition of a 'relativised' axiom (cf. [Fe92]; [Me64], p.192), such as:

$(\exists y)((x \in \mathbb{N}) \rightarrow (y > x)).$

Only then may we conclude that if a model M_c of:

$\{Th(\mathbb{N}) \cup (\exists y)((x \in \mathbb{N}) \rightarrow (y > x))\}$

exists, then it must have an 'infinite' integer c such that:

$M_c \models c > \underline{n}$

for all $n \in \mathbb{N}$.

However, even this would not yield a model for $Th(\mathbb{N})$, since every model of $Th(\mathbb{N})$ is by definition a model of (the provable formulas of) PA and, by the categoricity of PA by §2.E., Theorem 2.18—as also independently by §14.A.a., Theorem 14.1—we cannot introduce a 'completed' infinity such as c into either PA or any model of PA. □

---

[147]**Classical logic**: By 'classical logic' we mean the standard first-order predicate calculus FOL where the Law of the Excluded Middle is a theorem, but we do not assume that FOL is ω-consistent; i.e., we do not assume that Aristotle's particularisation (see §4., Definition 12) must hold under any interpretation of the logic.

We note that, as Kaye's argument stands, it seeks to violate finitarity by adding a new constant c to the language $\mathcal{L}_A$ of PA that is not definable in $\mathcal{L}_A$ and, ipso facto, adding an atomic formula $[c = x]$ to PA whose satisfaction under any interpretation of PA is not algorithmically *verifiable*!

Since the atomic formulas of PA are algorithmically *verifiable* under the standard interpretation[148], the above conclusion too postulates that which it seeks to prove!

Moreover, the postulation would be false if $Th(\mathbb{N})$ were categorical.

---

[148]See §2.B., Theorem 2.2.

CHAPTER 15. *PARADIGM CHALLENGING CONSEQUENCES*

15. Goodstein's argument over the natural numbers

The significance of Skolem's cautionary remarks in [Sk22] (see §4.I.) is further seen if note that, for any natural number m, R. L. Goodstein ([Gd44]) uses the properties of the hereditary representation of m to construct a sequence $G(m) \equiv \{g_1(m),\ g_2(m),\ \ldots\}$ of natural numbers by an unusual, but valid, algorithm.

> **Hereditary representation**: The representation of a number as a sum of powers of a base b, followed by expression of each of the exponents as a sum of powers of b, etc., until the process stops. For example, we may express the hereditary representations of 266 in base 2 and base 3 as follows:
>
> $$266_{[2]} \equiv 2^{8_{[2]}} + 2^{3_{[2]}} + 2 \equiv 2^{2^{(2^{2^0}+2^0)}} + 2^{2^{2^0}+2^0} + 2^{2^0}$$
>
> $$266_{[3]} \equiv 2.3^{4_{[3]}} + 2.3^{3_{[3]}} + 3^{2_{[3]}} + 1 \equiv 2.3^{(3^{3^0}+3^0)} + 2.3^{3^{3^0}} + 3^{2.3^0} + 3^0$$

We shall ignore here the peculiar manner of constructing the individual members of the Goodstein sequence, since these are not germane to understanding the essence of Goodstein's argument. We need simply accept for now that $G(m)$ is *well-defined* over the structure $\mathbb{N}$ of the natural numbers, and has the following properties:

(i) For any specified natural number $k > 0$ we can construct a hereditary representation—denoted[149] by $g_k(m)_{[k+1]}$—of $g_k(m)$ in the base $[k+1]$;

> *Example*: The hereditary representations of the first two terms $g_1(266) = 266$ and $g_2(266) = (3^{81} + 83)$ of $G(266)$ are[150]:
>
> $$g_1(266)_{[2]} \equiv 2^{2^{2+1}} + 2^{2+1} + 2$$
> $$g_2(266)_{[3]} \equiv 3^{3^{3+1}} + 3^{3+1} + 2$$

(ii) We can also *well-define* a Goodstein Functional Sequence:

> $$G(m)_{[x]} \equiv \{g_k(m)_{[(k+1)\,\hookrightarrow\,x]} : k > 0\} \text{ over } \mathbb{N}$$

by replacing the base $[k+1]$ in $g_k(m)_{[k+1]}$ with the variable x for each $k > 0$[151].

> *Example*: The first two terms of $G(266)_{[x]}$ are thus:
>
> $$g_1(266)_{[2\,\hookrightarrow\,x]} \equiv x^{x^{x+1}} + x^{x+1} + x$$
> $$g_2(266)_{[3\,\hookrightarrow\,x]} \equiv x^{x^{x+1}} + x^{x+1} + 2$$

---

[149]From a pedantic perspective the denotation should, of course, be: $(g_k(m))_{[k+1]}$.

[150]*Notation*: For ease of expression, we shall henceforth express 'a^0' as '1', and 'a^{b^0}' as 'a' unless indicated to the contrary.

[151]*Notation*: We prefer the notation $\hookrightarrow$ to that of the usual 'base bumping' function (cf. [Cai07]) as it seems to make the argument slightly more transparent.

(iii) We can show that some member of Goodstein's sequence $G(m)$ evaluates to 0 if, and only if, there is some natural number z such that for any specified natural number $k > 0$:

- If $g_k(m)_{[(k+1) \,\hookrightarrow\, z]} > 0$ in $G(m)_{[z]}$,

- Then $g_k(m)_{[(k+1) \,\hookrightarrow\, z]} > g_{k+1}(m)_{[(k+2) \,\hookrightarrow\, z]}$.

The proof of (iii)—which depends, of course, on the peculiar nature of Goodstein's algorithm—is tedious, but fairly straightforward (see §15.B.). The main point to note is that the proof is *finitary* and appeals only to the *arithmetical* properties of the natural numbers.

The question arises:

Query 11. *Are we free to postulate the existence of such a natural number z, and conclude that some member of $G(m)$ must evaluate to 0 in $\mathbb{N}$?*

> **Goodstein's Theorem**: Every Goodstein sequence defined over the natural numbers terminates in 0.

Though it appears absurd, the following theorem (compare with §11.B., Lemma 11.1) shows that this is precisely the freedom to which the, *transfinite*, ordinal-based argument for Goodstein's Theorem *curiously* lays claim (albeit implicitly)!

Theorem 15.1. *Goodstein's sequence $G_o(m_o)$[152] over the finite ordinals in any putative model $\mathbb{M}$ of ACA_0 terminates with respect to the ordinal inequality '$>_o$' even if Goodstein's sequence $G(m)$ over the natural numbers does not terminate with respect to the natural number inequality '$>$' in $\mathbb{M}$.*

Proof. Assume that Goodstein's sequence $G(m) \equiv \{g_k(m)_{[(k+1)]} : k > 0\}$ of natural numbers *does not terminate* with respect to the natural number inequality '$>$' in any putative model $\mathbb{M}$ of ACA_0.

Let n_{max} be the largest term amongst the first n terms of $G(m)$. It is tedious but straightforward to show that, by our assumption, n_{max} is a monotonically increasing sequence. Hence there is *no* natural number z such that:

$$g_k(m)_{[(k+1) \,\hookrightarrow\, z]} > g_{k+1}(m)_{[(k+2) \,\hookrightarrow\, z]} \text{ for all } k > 0.$$

Consider next Goodstein's ordinal number sequence $G_o(m_o) \equiv \{g_k(m_o) : k > 0\}$ over the finite ordinals.

Goodstein shows that, in the arithmetic of *transfinite* ordinals, the axiomatically postulated *transfinite* ordinal ω is such that:

$$g_k(m_o)_{[(k+1) \,\hookrightarrow\, \omega]} >_o g_{k+1}(m_o)_{[(k+2) \,\hookrightarrow\, \omega]} \text{ for all } k > 0.$$

---

[152]**Notation**: For convenience of expression, we shall henceforth denote by m_o the ordinal (set) in $\mathbb{M}$ corresponding to the natural number m in $\mathbb{M}$; by '$+_o$' and '$>_o$' the function/relation letters relating to ordinals in $\mathbb{M}$ that correspond to the function/relation letters '$+$' and '$>$' that correspond to the natural numbers in $\mathbb{M}$, etc.

Since there are no infinite descending sequences of ordinals with respect to the *transfinitely defined* ordinal relation '$>_o$', Goodstein's ordinal number sequence $G_o(m_o)$ must terminate finitely with respect to the *transfinite* ordinal relation '$>_o$' in any putative model $\mathbb{M}$ of ACA$_0$. $\square$

> **Comment**: In other words, accepting a Platonic 'existence' for the proofs of mathematics entails that, if ZF is consistent then, since PA is *finitarily* consistent (see §2.C.a., Theorem 2.16):
>
> - The axioms of ZF interpret as Platonically *true*, whilst Goodstein's Theorem over the natural numbers can interpret as Platonically *true*; *and*
>
> - The axioms of ZF interpret as Platonically *true*, whilst Goodstein's Theorem over the natural numbers can interpret as Platonically *false*.
>
> We conclude that there can be no mathematical or meta-mathematical proof of consistency for ZF.

Moreover, since the finite ordinals can be meta-mathematically put into a 1-1 correspondence with the natural numbers, it follows that:

Corollary 15.2. *The relationship of terminating finitely with respect to the transfinitely defined ordinal relation '$>_o$' over the set of finite ordinals does not entail the relationship of terminating finitely with respect to the finitarily defined natural number relation '$>$' over the set of natural numbers.* $\square$

15.A.   The subsystem ACA$_0$

We note that ACA$_0$ is defined (see [Fe97], pp.12-13) as an 'extension' of PA with the PA variables, say $[m], [n], \ldots$, ranging now over the ACA$_0$ numerals; with additional set variables $[X], [Y], [Z], \ldots$] ranging over ACA$_0$ sets; and with an additional arithmetical Comprehension Axiom schema where, if $[\varphi(n)]$ is a formula with a free numeral variable $[n]$—and possibly other free variables such as, say, $[m]$ and $[X]$, but not the set variable $[Z]$—the Comprehension Axiom for $[\varphi]$ is the formula that defines sets in ACA$_0$ by:

$$[(\forall m)(\forall X)(\exists Z)(\forall n)(n \in Z \leftrightarrow \varphi(n))]$$

It now immediately follows from §15., Corollary 15.2, that:

Corollary 15.3. *The subsystem ACA$_0$ of second-order arithmetic is not a conservative extension of PA.*

We note that the *curious* conclusion of Goodstein's Theorem reflects the circumstance that the 'truth' of the Comprehension Axioms of ACA$_0$ under an interpretation is not *well-definable*, since they contain an existential quantifier that is intended to admit Aristotle's particularisation under any interpretation.

We thus again conclude, independently of §15., Corollary 15.3, that:

Theorem 15.4. *The subsystem ACA$_0$ of second-order arithmetic is not a conservative extension of PA.*

Proof. By §2.C.a., Theorem 2.16, PA is consistent and has a model. If ACA_0 is a conservative extension of PA, then it too is consistent[153] and has a model which admits Aristotle's particularisation, and which is also a model of PA. However, by §5.D., Corollary 5.11, Aristotle's particularisation cannot hold in any model of PA. The theorem follows. $\square$

We note that §15.A., Theorem 15.4, contradicts conventional paradigms:

> "In other words, ACA_0 is a *conservative extension* of first order arithmetic. This may also be expressed by saying that Z_1, or equivalently PA, is the *first order part* of ACA_0."
>
> ...*Simpson: [Sim06], §I.3, REMARK I.3.3, p.8.*

> "As a logical footnote to that, the system $\mathbf{ACA}_0$, which I described here, is a conservative extension of Peano Arithmetic, even though it employs second order concepts."
>
> ...*Feferman: [Fe97], p.18.*

> "It is a classical result of proof theory that the system $\mathbf{ACA}_0$ is a conservative extension of the first-order system of Peano Arithmetic $\mathbf{PA}$."
>
> ...*Feferman: [Fe02], p.18.*

15.B.   Goodstein's sequence

For any given natural number m we can express Goodstein's sequence $G(m)$ so that each term is expressed in it's hereditary representation:

$$G(m) \;\equiv\; \big\{g_1(m)_{[2]}, \; g_2(m)_{[3]}, \; g_3(m)_{[4]}, \ldots\big\} \tag{15.1}$$

where the first term $g_1(m)_{[2]}$ denotes the unique hereditary representation of the natural number m in the natural number base $[2]$:

$$\text{e.g., } g_1(9)_{[2]} \equiv 1.2^{(1.2^{1.2^0}+1.2^0)} + 0.2^{(1.2^{1.2^0}+0.2^0)} + 0.2^{1.2^0} + 1.2^0$$

and if $n > 1$ then $g_{(n)}(m)_{[n+1]}$ is defined recursively from $g_{(n-1)}(m)_{[n]}$ as below.

15.B.a.   The recursive definition of Goodstein's Sequence

For $n > 1$ let the $(n-1)^{th}$ term $g_{(n-1)}(m)$ of the Goodstein sequence $G(m)$ be expressed syntactically by its hereditary representation as:

$$g_{(n-1)}(m)_{[n]} \;\equiv\; \sum_{i=0}^{l} a_i.n^{i_{[n]}} \tag{15.2}$$

where:

(a) $0 \le a_i < n$ over $0 \le i \le l$;

(b) $a_l \ne 0$;

(c) for each $0 \le i \le l$ the exponent i too is expressed syntactically by its hereditary representation $i_{[n]}$ in the base $[n]$; as also are all of its exponents and, in turn, all of their exponents, etc.

We then define the nth term of $G(m)$ as:

$$g_n(m) \;=\; \sum_{i=0}^{l} (a_i.(n+1)^{i_{[n \,\hookrightarrow\, (n+1)]}}) - 1 \tag{15.3}$$

---

[153]"If T' is a conservative extension of T, then T' is consistent iff T is consistent." ...*Shoenfield: [Sh67], p.42.*

15.B.b.   The hereditary representation of $g_n(m)$

Now we note that:

(a) if $a_0 \neq 0$ then the hereditary representation of $g_n(m)$ is:

$$g_n(m)_{[n+1]} \equiv \sum_{i=1}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) + (a_0 - 1) \tag{15.4}$$

(b) whilst if $a_i = 0$ for all $0 \leq i < k$, then the hereditary representation of $g_n(m)$ is:

$$g_n(m)_{[n+1]} \equiv \sum_{i=k+1}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) + c_{k[n+1]} \tag{15.5}$$

where:

$$\begin{aligned}
c_k &= a_k.(n+1)^{k_{[n \hookrightarrow (n+1)]}} - 1 \\
&= (a_k - 1).(n+1)^{k_{[n \hookrightarrow (n+1)]}} + \left\{(n+1)^{k_{[n \hookrightarrow (n+1)]}} - 1\right\} \\
&= (a_k - 1).(n+1)^{k_{[n \hookrightarrow (n+1)]}} + n\left\{(n+1)^{k_{[n \hookrightarrow (n+1)]}-1} + (n+1)^{k_{[n \hookrightarrow (n+1)]}-2}\ldots+1\right\}
\end{aligned}$$

and so its hereditary representation in the base $(n+1)$ is given by:

$$c_{k[n+1]} \equiv (a_k - 1).(n+1)^{k_{1[n+1]}} + n\left\{(n+1)^{k_{2[n+1]}} + (n+1)^{k_{3[n+1]}}\ldots+1\right\}$$

where $k_{1[n+1]} \equiv k_{[n \hookrightarrow (n+1)]}$ and $k_1 > k_2 > k_3 > \ldots \geq 1$.

15.B.c.   Goodstein's argument in arithmetic

For $n > 1$ we consider the difference:

$$d_{(n-1)} = \left\{g_{(n-1)}(m)_{[n]} - g_n(m)_{[n+1]}\right\}$$

Now:

(a) if $a_0 \neq 0$ we have:

$$d_{(n-1)} = \sum_{i=0}^{l}(a_i.n^{i_{[n]}}) - \sum_{i=1}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) - (a_0 - 1) \tag{15.6}$$

(b) whilst if $a_i = 0$ for all $0 \leq i < k$ we have:

$$\begin{aligned}
d_{(n-1)} = &\sum_{i=k}^{l}(a_i.n^{i_{[n]}}) - \sum_{i=(k+1)}^{l}(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}) - \\
&(a_k - 1).(n+1)^{k_{1[n+1]}} - \\
&n\left\{(n+1)^{k_{2[n+1]}} + (n+1)^{k_{3[n+1]}}\ldots+1\right\}
\end{aligned} \tag{15.7}$$

Further:

(c) if in equation 15.6 we replace the base $[n]$ by the variable $[z]$ in each term of:

$$\sum_{i=0}^{l} a_i.n^{i_{[n]}} \tag{15.8}$$

and, similarly, the base $[n+1]$ also by the variable $[z]$ in each term of:

$$\sum_{i=k+1}^{l} \left(a_i.(n+1)^{i_{[n \hookrightarrow (n+1)]}}\right) + (a_0 - 1) \tag{15.9}$$

then we have:

$$
\begin{aligned}
d'_{(n-1)} &= \sum_{i=0}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - \sum_{i=1}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - (a_0 - 1) \\
&= 1
\end{aligned}
\tag{15.10}
$$

since $\left(i_{[n \hookrightarrow (n+1)]}\right)_{[(n+1) \hookrightarrow z]} \equiv i_{[n \hookrightarrow z]}$;

(d) whilst if in equation 15.7 we replace the bases similarly, then we have:

$$
\begin{aligned}
d'_{(n-1)} &= \sum_{i=k}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - \sum_{i=(k+1)}^{l}(a_i.z^{i_{[n \hookrightarrow z]}}) - \\
&\quad (a_k - 1).z^{k_{1[(n+1) \hookrightarrow z]}} - n\left\{z^{k_{2[(n+1) \hookrightarrow z]}} + z^{k_{3[(n+1) \hookrightarrow z]}} \dots + 1\right\} \\
&= a_k.z^{k_{[n \hookrightarrow z]}} - (a_k - 1).z^{k_{1[(n+1) \hookrightarrow z]}}) - n(z^{k_{2[(n+1) \hookrightarrow z]}} + z^{k_{3[(n+1) \hookrightarrow z]}} \dots + 1)
\end{aligned}
$$

$$= z^{k_{1[(n+1) \hookrightarrow z]}} - n(z^{k_{2[(n+1) \hookrightarrow z]}} + z^{k_{3[(n+1) \hookrightarrow z]}} \dots + 1) \tag{15.11}$$

where $k_{1[(n+1) \hookrightarrow z]} \equiv k_{[n \hookrightarrow z]}$, and $k_{1[(n+1) \hookrightarrow z]} > k_{2[(n+1) \hookrightarrow z]} > k_{3[(n+1) \hookrightarrow z]} > \dots \geq 1$.

We consider now the sequence:

$$G(m)_{[z]} \equiv (g_1(m)_{[2 \hookrightarrow z]}, \; g_2(m)_{[3 \hookrightarrow z]}, \; g_3(m)_{[4 \hookrightarrow z]}, \dots)$$

obtained from Goodstein's sequence by replacing the base $[n+1]$ in each of the terms $g_n(m)_{[n+1]}$ by the base $[z]$ for all $n \geq 1$.

Clearly if $z > n$ for all non-zero terms of the Goodstein sequence, then $d'_{(n-1)} > 0$ in each of the cases—equation 15.10 and equation 15.11—since we have in equation 15.11:

$$d'_{(n-1)} \geq (z^k - (z-1)(z^{(k-1)} + z^{(k-2)} + z^{(k-3)} + \dots + 1)) = 1$$

The sequence $G(m)_{[z]}$ is then a descending sequence of natural numbers, and must terminate finitely in $\mathbb{N}$, if $z > n$.

Since $g_n(m)_{[(n+1) \hookrightarrow z]} \geq g_n(m)_{[n+1]}$ if $z > n$, Goodstein's sequence $G(m)$ too must terminate finitely in $\mathbb{N}$ if $z > n$.

Obviously, since we can always find a $z > n$ for all non-zero terms of the Goodstein sequence if it terminates finitely in $\mathbb{N}$, the condition that we can always find some $z > n$ for all non-zero

terms of any Goodstein sequence is equivalent to the assumption that any Goodstein sequence terminates finitely in $\mathbb{N}$.

From an *evidence-based* perspective, Goodstein's (vacuously true) argument is that, assuming the arithmetic of the finite ordinals is a conservative extension of the arithmetic of the natural numbers, we can always find some $z > n$—with respect to the natural number relation $>$—for all non-zero terms of any Goodstein sequence over the natural numbers since, if n_o is the finite ordinal corresponding to the natural number n, then the transfinite ordinal ω is such that $\omega >_o n_o$ with respect to the ordinal relation $>_o$ for all non-zero terms of any Goodstein sequence when expressed similarly over the finite ordinals!

CHAPTER 16. *PARADIGM CHALLENGING CONSEQUENCES*

16. The significance of *evidence-based* reasoning for the philosophy of science

We briefly consider, from an *evidence-based* perspective, the significance for the physical sciences of the semantic and logical paradoxes[154] which involve—either implicitly or explicitly—quantification over an infinitude.

Where such quantification is not, or cannot be, explicitly defined in formal logical terms—e.g., the classical expression of the Liar paradox as 'This sentence is a lie'[155]—the paradoxes per se cannot be considered as posing serious linguistic or philosophical concerns from an *evidence-based* perspective of constructive mathematics.

The practical significance of the semantic and logical paradoxes is, of course, that they illustrate the absurd extent to which languages of common discourse need to tolerate ambiguity; both for ease of expression and for practical—even if not theoretically unambiguous and effective—communication in non-critical cases amongst intelligences capable of a lingua franca.

Such absurdity is highlighted by the universal appreciation of Charles Dickens' Mr. Bumble's retort that 'The law is an ass'; a quote oft used to refer to the absurdities which sometimes surface[156] in cases when judicial pronouncements attempt to resolve an ambiguity by subjective fiat that appeals to the powers—and duties—bestowed upon the judicial authority for the practical resolution of precisely such an ambiguity, even when the ambiguity may be theoretically irresolvable!

In a thought-provoking Opinion piece, 'Desperately Seeking Mathematical Truth', in the August 2008 Notices of the American Mathematical Society, Melvyn B. Nathanson seeks to highlight the significance for the mathematical sciences when similar authority is vested by society—albeit tacitly—upon academic 'bosses' (a reference, presumably, to the collective of reputed—and respected—experts in any field of human endeavour):

> "... many great and important theorems don't actually have proofs. They have sketches of proofs, outlines of arguments, hints and intuitions that were obvious to the author (at least, at the time of writing) and that, hopefully, are understood and believed by some part of the mathematical community.
>
> But the community itself is tiny. In most fields of mathematics there are few experts. Indeed, there are very few active research mathematicians in the world, and many important problems, so the ratio of the number of mathematicians to the number of problems is small. In every field, there are "bosses" who proclaim the correctness or incorrectness of a new result, and its importance or unimportance.
>
> Sometimes they disagree, like gang leaders fighting over turf. In any case, there is a web of semi-proved theorems throughout mathematics. Our knowledge of the truth of a theorem depends on the correctness of its proof and on the correctness of all of the theorems used in its proof. It is

---

[154]Although commonly referred to as the paradoxes of 'self-reference' (see [Kau87] for an unusual perspective on such 'self-reference'), not all of them involve self-reference (e.g., the paradox constructed by Stephen Yablo [Ya93]).

[155]Or Lundgren's 'information liar paradox': "This is not semantic information", in [Lun17], §3, p.5.

[156]See www.shazbot.com/lawass/.

a shaky foundation."
. . . Nathanson: [Na08].

Nathanson's comments are intriguing, because addressing such ambiguity in critical cases—such as communication between mechanical artefacts, or a putative communication between terrestrial and extra-terrestrial intelligences—is the very raison d'être of mathematical activity!

Of course, it *would* be a matter of serious concern if the word 'This' in the English language sentence, 'This sentence is a lie', could be validly viewed as implicitly implying that:

(a) there is a *well-defined* infinite enumeration of English language sentences;

(b) to each of which a truth-value can be assigned *finitarily* by the rules of a two-valued logic; and,

(c) in which 'This' refers uniquely to a particular sentence in the enumeration.

In 1931, Kurt Gödel used the above perspective in his seminal paper on 'formally undecidable' arithmetical propositions:

(i) to show how the infinitude of formulas, in a formally defined Peano Arithmetic P ([Go31], pp.9-13), could be *finitarily* enumerated and referenced uniquely by natural numbers ([Go31], p.13-14);

(ii) to show how P-provability values could be *finitarily* assigned to P-formulas by the rules of a two-valued logic ([Go31], p.13); and,

(iii) to *finitarily* construct a P-formula which interprets as an arithmetical proposition that could, debatably (see §11.D.), be viewed—under the standard interpretation of the Peano Arithmetic P—as expressing the sentence, 'This P-sentence is P-unprovable' ([Go31], p.37, footnote 67), *without* inviting a 'Liar' type of contradiction.

We note that where the quantification can be made explicit—e.g., Russell's paradox or Yablo's paradox—the significance of the question whether such quantication is *well-defined* or not is immediately obvious.

Russell's paradox: Define the set S by $\{All\ x : x \in S$ iff $x \notin x\}$; then $S \in S$ iff $S \notin S$.

Yablo's paradox: Defining the sentence S_i for all $i \geq 0$ as 'For all $j > i$, S_j is not true' seems to lead to a contradiction ([Ya93]).

For instance, in Russell's case it could be cogently argued from an *evidence-based* perspective that the contradiction itself establishes that S cannot be *well-defined* over the range of the quantifier.

In Yablo's case it could, as cogently, be argued that truth values cannot be *finitarily* assigned to any sentence covered by the quantification since, in order to decide whether or not S_i can be assigned the value 'true' for any specified $i \geq 0$, we first need to decide whether or not S_{i+1} has already been assigned the value 'true'!

There are two issues involved here—not necessarily independent—highlighted by Timothy Gowers as follows:

> "If you ask a philosopher what the main problems are in the philosophy of mathematics, then the following two are likely to come up: what is the status of mathematical truth, and what is the nature of mathematical objects? That is, what gives mathematical statements their aura of infallibility, and what on earth are these statements about?"
>
> *. . . Gowers: [Gow02].*

16.A.   Is quantification currently interpreted constructively?

The first issue is whether the currently accepted interpretations of formal quantification—essentially as defined by Hilbert ([Hi27]; see also §7.A.) in his formalisation of Aristotle's logic of predicates in terms of his ε-function—can be treated as *well-defined* over an infinite domain.

Now, Brouwer ([Br08]) had emphatically—and justifiably so far as number theory was concerned (see §7.B.)—objected to such subjectivity, and asserted that Hilbert's interpretations of formal quantification were non-constructive (hence not *well-defined* by §4.F., Definition 19).

Although Hilbert's formalisation of the quantifiers (an integral part of his formalisation of Aristotle's logic of predicates) appeared adequate, Brouwer rejected Hilbert's interpretations of them on the grounds that the interpretations were open to ambiguity, and could not, therefore, be accepted as admitting categorical communication.

However, Brouwer's rejection of the Law of the Excluded Middle LEM as a resolution of the objection was seen—also justifiably (see §8.B.a.)—as unconvincingly rejecting a comfortable interpretation that—despite its Platonic overtones—appeared intuitively plausible to the larger body of academics that was increasingly attracted to, and influenced by, the remarkably expressive powers provided by Cantor-inspired set theories, such as ZF, for expressing our conceptual mathematical metaphors unambiguously (the significance of which is highlighted in §10.E.).

Since Hilbert's and Brouwer's seminal works preceded that of Alan Turing, neither was able to offer his critics an alternative—and intuitively convincing—constructive definition of quantification based on the view—gaining currency today—that a simple functional language can be used for specifying *evidence* for propositions in a constructive logic (see [Mu91]).

Moreover, since Brouwer's objections did not gain much currency amongst mainstream logicians, they were unable to influence Turing who, it is our contention, could perhaps have provided the necessary *well-defined* interpretations (introduced in [An12]) sought by Hilbert for number theory, had Turing not been influenced by Gödel's powerful presentation—and Gödel's persuasive Platonic, albeit (*contrary to accepted dogma*) logically rooted[157], interpretation of his own formal reasoning in [Go31].

Thus, in his 1939 paper ([Tu39]) on ordinal-based logics, Turing applied his computational method—which he had developed in his 1936 paper ([Tu36])—in seeking partial completeness in interpretations of Cantor's ordinal arithmetic (as defined in a set theory such as ZF)—rather than in seeking a categorical interpretation of PA. Turing perhaps viewed his 1936 paper as complementing and extending Gödel's and Cantor's reasoning:

---

[157]Although meriting a more complete discussion than is appropriate to the intent of this paper, it is worth noting that the rooting of Gödel's Platonism can be cogently argued as lying—contrary to generally held opinions—purely in a logical, rather than philosophical, presumption: more specifically in Gödel's belief that Peano Arithmetic is ω-consistent ([Go31], p.28). The belief seems unwittingly shared universally even by those who (cf. [Pas95], [Fe02]) accept Gödel's formal arguments in [Go31] but claim to reject Gödel's 'Platonic' interpretations of them.

> "The well-known theorem of Gödel shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence of $L, L_1 = L', L_2 = L_1', \ldots$ each more complete than the preceding. . . .
>
> Proceeding in this way we can associate a system of logic with any constructive ordinal. It may be asked whether a sequence of logics of this kind is complete in the sense that to any problem A there corresponds an ordinal α such that A is solvable by means of the logic L_α. I propose to investigate this question in a more general case, and to give some other examples of ways in which systems of logic may be associated with constructive ordinals."
>
> . . . *Turing: [Tu39], pp.155-156*.

Perhaps Turing also did not see any cogent reason then to question the validity of Gödel's belief that systems of Arithmetic such as PA are ω-consistent (as hinted at in [Go31], p.28) and, consequently, Gödel's interpretation of his argument in [Go31] as having meta-mathematically *proven* that systems of Arithmetic such as PA are essentially incomplete!

Turing thus overlooked the fact that his 1936 paper ([Tu36]) conflicted with Gödel's and Cantor's interpretations of their own, formal, reasoning; and that [Tu36] admits a definition of satisfaction that yields the finitary interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ of PA (see §2.C.a., Theorem 2.16).

As a result, classical paradigms continued to essentially follow Hilbert's Platonically-influenced (hence, subjective) definitions and interpretations of the quantifiers (based on accepting Aristotle's particularisation as valid) when defining them under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA.

Now, the latter definitions and interpretations (e.g., [Me64], pp.49-53) are, in turn, founded upon Tarski's analysis of the inductive definability of the truth of compound expressions of a symbolic language under an interpretation in terms of the satisfaction of the atomic expressions of the language under the interpretation ([Ta35]).

Tarski defines there the formal sentence P as True if and only if p—where p is the proposition expressed by P. In other words, the sentence 'Snow is white' is True if, and only if, it is *subjectively* true in all cases; and it is *subjectively* true in a particular case if, and only if, it expresses the *subjectively verifiable* fact that snow *is* white in that particular case. Thus, for Tarski the commonality of the satisfaction of the atomic formulas of a language under an interpretation is axiomatic (cf. [Me64], p.51(i)).

In this investigation we have highlighted the limitations of such subjectivity (see §4.C.) and, in the case of the 'standard' interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of the Peano Arithmetic PA, seen how to avoid violation of such constraints (see §2.B.) by requiring that the axioms of PA, and its rules of inference, be interpretable as algorithmically *verifiable* propositions.

16.B.   When is the concept of a completed infinity consistent?

The second issue is when, and whether, the concept of a completed infinity is consistent with the interpretation of a formal language.

Clearly, the consistency of the concept would follow immediately in any putative, *well-defined*, interpretation of the axioms (and rules of inference) of a set theory such as the Zermelo-Fraenkel (see [BF58]) first-order theory ZF (whether such an interpretation exists at all is, of course, another question).

In view of the perceived power of ZFC as an unsurpassed language of rich and adequate expression of mathematically expressible abstract concepts precisely, it is not surprising that many of the semantic and logical paradoxes depend on the implicit assumption that the domain over which the paradox quantifies can always be treated as a *well-defined* mathematical object that can be formalised in ZFC, even if this domain is not explicitly defined set-theoretically.

This assumption is rooted in the questionable[158] belief, expressed unapologetically by set-theorist Saharon Shelah (see also [Ma18] and [Ma18a]), that ZF can essentially express all mathematical 'truths':

> "My feeling is that ZFC exhausts our intuition except for things like consistency statements, so a proof means a proof in ZFC."
>
> . . . *Shelah: [She91], p.3*

From this it is but a short step to non-constructive perspectives—such as Gödel's Platonic interpretation of his own formal reasoning in his 1931 paper ([Go31])—which unjustifiably conclude (see §14.) that PA must have non-standard models.

However, it is our contention that both of the above foundational issues need to be reviewed carefully, and that we need to recognize explicitly (see §10.E.) the limitations on the ability of a highly expressive mathematical language such as ZF to communicate effectively; and the limitations on the ability of a categorical mathematical language such as PA to adequately express abstract concepts—such as those involving Cantor's first limit ordinal ω (see §14.A.a.).

Prima facie, the semantic and logical paradoxes—as also the seeming paradoxes associated with 'fractal' constructions such as the Cantor ternary set, and the constructions described below—seem to arise out of a blurring of this distinction, and an attempt to ask of a language more than it is designed to deliver.

16.C. Mythical 'set-theoretical' limits of fractal constructions

For instance, consider the claim (e.g., [Bar88], p.37, Theorem 1) that fractal 'constructions'—such as the Cantor ternary set, which is defined classically as a 'putative' set-theoretical limit (see [Ru53], p34; [Bar88], pp.44-45) of an iterative process in the 'putative' completion of a metric space—yield valid mathematical objects (sets) in the 'limit' (*presumably in some Platonic mathematical model*).

Now, the Cantor Set T_∞ is defined as the putative 'fractal' limit of the set of points obtained by taking the closed interval $T_0 = [0, 1]$), and:

(a) removing the open middle third to yield the set $T_1 = \{[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]\}$,

(b) then removing the middle third of each of the remaining closed intervals to yield the set $T_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$,

(c) repeating the process ad infinitum.

To see why such a limit needs to be treated as 'putative' from an *evidence-based* perspective[159], consider an equilateral triangle BAC of height h and side s (see Figs.1-3):

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[158]'Questionable' since we see how—in the case of Goodstein's Theorem—such a belief leads to a curious conclusion (see §15., Theorem 15.1).

[159]Compare with Lakoff and Núñez's analysis of a similar 'length paradox' in [LR00], p.325-333.

(i) Divide the base BC in half and construct two isosceles triangles of height $h.d$ and base $s/2$ on BC, where $1 \geq d > 0$.

(ii) Iterate the construction on each constructed triangle ad infinitum.

(iii) Thus, the height of each of the 2^n triangles on the base BC at the n'th construction is $h.d^n$, and the base of each triangle $s/2^n$.

(iv) Hence, the total area of all these triangles subtended by the base BC is $s.h.d^n/2$.

(v) Now, if $d = 1$, the total area of all the constructed triangles after each iteration remains constant at $s.h/2$, although the total length of all the sides opposing the base BC increases monotonically.

(vi) However, if $1 > d > 0$ (see Fig.1-Fig.3), it would appear that, geometrically, the base BC of the original equilateral triangle will always be the 'limiting' configuration of the sides opposing the base BC.

This is indeed so if $0 < d < 1/2$ (Fig.1), since the total length of all the sides opposing the base BC at the n'th iteration—say l_n—yields a Cauchy sequence whose limiting value is, indeed, the length s of the base BC.

Fig.1: $l_n \to s$ if $0 < d < 1/2$
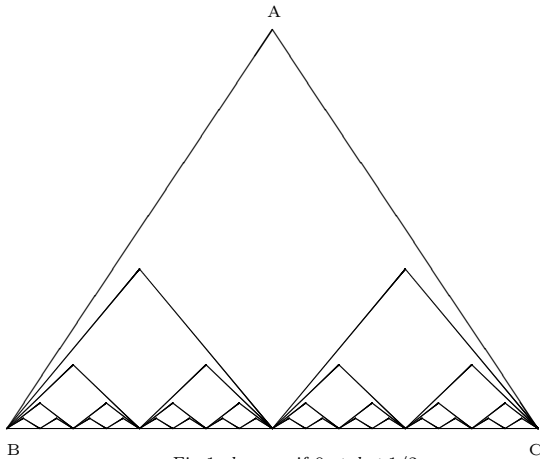
However, if $d = 1/2$ (Fig.2), the total length of all the sides opposing their base on BC is always $2s$; which, by definition, also yields a Cauchy sequence whose limiting value is $2s$.

Fig.2: $l_n = 2s$ if $d = 1/2$
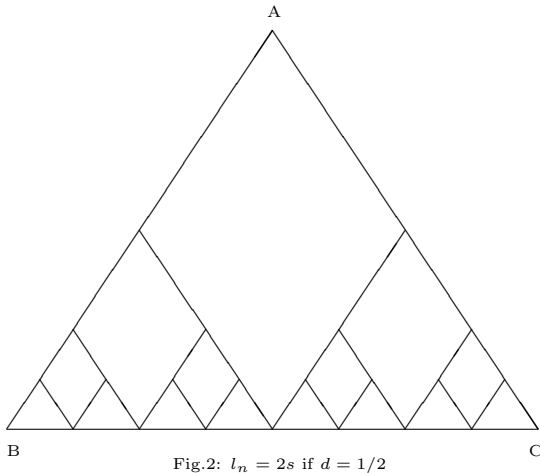
Finally, if $1 > d > 1/2$ (Fig. 3), the total length of all the sides opposing their base on BC is a monotonically increasing value.
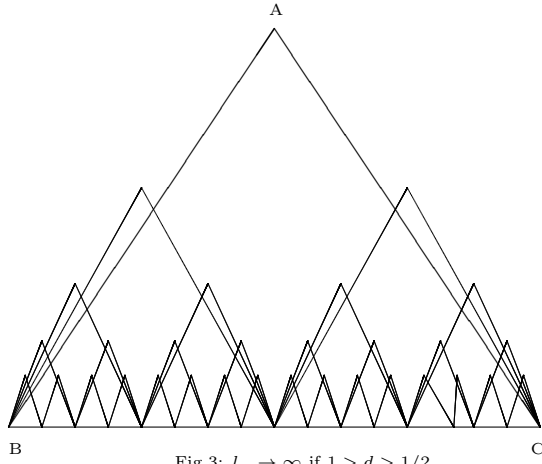
Fig.3: $l_n \to \infty$ if $1 > d > 1/2$

Consider now:

16.C.a.   Case 1: Interpretation as a virus cluster

Let the area BAC denote the population size of a virus cluster, where each virus cell has a 'virulence' measure h/s.

Let each triangle at the n'th iteration denote a virus cluster—with a virulence factor $h.d^n/(s/2^n)$—that reacts to the next generation anti-virus by splitting into two smaller clusters with inherited virulence $h.d^{n+1}/(s/2^{n+1})$.

We then have that:

   (a) If $d < 1/2$, the effects of the virus can—in a sense—be contained and eventually 'eliminated', since both the total population of the virus, and its virulence in each cluster, decrease monotonically;

   (b) If $d = 1/2$, the effects of the virus can be 'contained', but never 'eliminated' since, even though the total population of the virus decreases monotonically, its virulence in each cluster remains constant, albeit at a containable level, until the virus suffers a sudden, dinosaur-type, extinction at the 'limiting' point as $n \to \infty$;

   (c) However, if $d > 1/2$, the effects of the virus can neither be 'contained' nor 'eliminated' since, even though the total population of the virus decreases monotonically, its virulence in each cluster resists containment by increasing monotonically until, again, the virus suffers a sudden, dinosaur-type, extinction at the 'limiting' point as $n \to \infty$.

16.C.b.   Case 2: Interpretation as an elastic string

Let the base BC denote an elastic string, stretched iteratively into the above configurations. We then have that:

   (a) If $d < 1/2$, the elastic will, in principle, eventually return to its original state;

   (b) If $d > 1/2$, then the elastic must break at some point, into two pieces of elastic that is apparently normal, and invites no untoward curiosity, since it forms part of our everyday experience;

(c) However, what if $d = 1/2$? If such an experiment could be done in a lab feasibly then, at some point, the molecular structure of the elastic must break down and, as in the case of the splitting of an atom, the molecules of the elastic would transform into some other substance that *cannot* be taken to correspond to the putative limit of its associated Cauchy sequence.

In other words, we are confronted with a two-dimensional version of Zeno's arguments ([Rus37], pp.347-353), one way of resolving which is by admitting the possibility that such an elastic 'length' undergoes a 'steam-to-water-like' phase change in the 'limit' that need not correspond (see §4.H.a.) to the putative limit of its associated Cauchy sequence; where we note that, by definition, the sequence $\{a_0, a_1, a_2, \ldots\}$ where $a_0 = 1$ and $a_i = 3$ for all $i \geq 1$, is a Cauchy sequence whose mathematical limit is 3!

16.C.c. Case 3: Interpretation as a quantum chimera
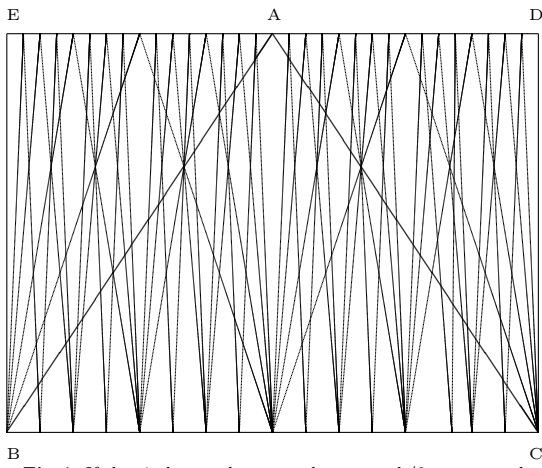
Fig.4: If $d = 1$ then as $l_n \to \infty$ the area $s.h/2 \to$ area $s.h$

To see the extent of divergence between postulated set-theoretical, and perceived geometrical, limits of fractal constructions, we consider the square $BCDE$, and the set-theoretically 'space-filling' (compare [Bar88], pp.240-247) curves defined by the re-iterative splitting of the triangle ABC if $d = 1$ ((see §16.C.(v) and Fig.4).

Now although, after each iteration, the total area of all the upright triangles on the base BC, and of all the inverted triangles on the base DE, remains constant at $s.h/2$ in each case, geometrically the 'limiting' configuration of the disjoint areas 'subtended' by the 'space-filling curve' on each of the bases BC and DE, respectively, will still be $s.h/2$, the area 'occupied' by the 'space-filling curve' on each of the bases BC and DE, respectively, will be the area $s.h$.

In other words since, from an *evidence-based* perspective:

- any *well-defined* point (in the sense of §4.F., Definition 18) of $BCDE$ must fall into some iterated triangle by §12.B., Theorem 12.3;

- and the base $h/(2^n)$ of each triangle at the nth iteration $\to 0$ as $n \to 0$;

- the putative 'space-filling' limit of the defined geometric construction consists of two 'self-adjoining' (i.e., self-contacting without self-crossing) curves which abut, but do not cross, each other at any point; and each of which (by definition in the technical sense) intersects itself everywhere and 'clearly' covers the square $ABCD$. Hence each is surjective onto $BCDE$.

We note, moreover, that admitting $BCDE$ as the putative, 'space-filling', fractal limit of the geometric construction in Fig.4 entails that the points of $BCDE$ can be well-ordered!

From an *evidence-based* perspective, we thus have a fractal construction where, as in the Banach-Tarski Theorem, the area $BCDE$ maps in the putative limit into two equal, disjoint, areas $BCDE/2$ each of which, however, 'occupies' the same metric space as $BCDE$.

> **Banach-Tarski Theorem**: Given a solid ball in 3-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball.

Similarly, from an *evidence-based* perspective of quantum phenomena (see §19.C.; also §19.B.d.), such a putative limit would interpret as a situation where the mathematical representations of two physical objects—for instance, two 'fluids' of massless particles with opposing polarity, corresponding to the disjoint areas sub-tended on the two bases BC and DE respectively—ultimately come to occupy the same physical space during a putative interaction/separation!

The question arises:

Query 12. *In what evidence-based mathematical sense of a metric 'space' can Fig.4 be described mathematically as defining a space-filling curve?*

16.C.d. Case 4: Interpretation as a political revolution

The significance of Query 12 for the mathematical modelling of real-life based gedanken is seen in the following:

Let the area BAC denote the contiguous population size $s.h/2$ of a rebellious movement R in a country with total population denoted by $BCDE$ in Fig.4; where R splits into two identical, disjoint, cells, each of reduced population $s.h/4$, when it is persecuted beyond a level $L/2$ by the population surrounding it, where we assume that the persecution level from the surrounding population in the country drops temporarily to below $L/4$ after the split.

Let each triangle at the n'th iteration denote a rebellious cell of size $s.h/2^{n-1}$, which reacts when the persecution level from the surrounding population crosses $L/2^{n-1}$, by again splitting identically into two disjoint cells.

We then have that any reactive persecution of R, administered as above, will eventually result in a revolution where R takes over the entire country, since the total size of the, increasingly dispersed, rebelling population remains constant at $s.h/2$, but the persecution level tends to 0!

16.D. Algorithmically *verifiable* but not algorithmically *computable* Cauchy sequences

We note that §4.G., Theorem 4.2 shows that Cauchy sequences which are defined as algorithmically *verifiable*, but not algorithmically *computable*, can correspond to 'essentially incompletable' real numbers (such as, for instance, the fundamental dimensionless constants considered in [An15], §4) whose Cauchy sequences cannot, in a sense, be known 'completely' even to Laplace's 'intellect' (see §4.H.a.).

The above example now show further that—and why—the numerical values of *even* some algorithmically *computable* Cauchy sequences may also need to be treated as formally specifiable, first-order, non-terminating processes:

(a) which are 'eternal work-in-progress' (see §4.H.a.), and

(b) which cannot be uniquely identified by a putative 'Cauchy limit' without limiting the ability of such sequences to model phase-changing physical phenomena faithfully!

> **Comment**: We note that the set-theoretically defined Specker sequences, too, are algorithmically *computable*, monotonically increasing, bounded sequences of rational numbers, whose supremum is taken to Platonically define putative, *uncomputable*, real numbers—even though the sequences themselves are not Cauchy sequences in the constructive (algorithmic) sense. They too, thus, need to be treated as formally specifiable, first-order, non-terminating processes.

> **Specker sequence**: Let A be any recursively enumerable set of natural numbers that is not decidable, and let (a_i) be an algorithmically *computable* enumeration of A without repetition. Define a sequence (q_n) of rational numbers with the rule:
>
> $q_n = \sum_{i=0}^{n} 2^{-a_i - 1}$.

In view of §4.H., Theorem 4.4, the gedanken in §16.C.a. to §16.C.d. highlight the disquieting issue sought to be raised, for instance, by Lakoff and Núñez in [LR00] (p.325-333), Simpson in [Sim88] and, most forcefully, by Krajewski in [Kr16]:

> "Examples of possible theological influences upon the development of mathematics are indicated. The best known connection can be found in the realm of infinite sets treated by us as known or graspable, which constitutes a divine-like approach. Also the move to treat infinite processes as if they were one finished object that can be identified with its limits is routine in mathematicians, but refers to seemingly super-human power."
>
> *. . . Krajewski: [Kr16].*

The disquiet can be expressed as (see also §10.; [Fe98]):

Query 13. *Since the raison d'être of a mathematical language is—or ideally should be—to express our abstractions of natural phenomena precisely, and communicate them unequivocally, in what sense can we sensibly admit an interpretation of a mathematical language that constrains all the above cases by 'limiting' configurations in a putative, set-theoretical, 'completion' of Euclidean Space?*

16.D.a.   Mathematical intuition vs mathematical monsters

The essence of the issue sought to be raised in §16.D., Query 13 (and also in §10., and in §20.) is the pedagogical challenge (see also §20.A.) in differentiating between the ontological (and epistemological) status of:

- what we express symbolically as our primary conceptual mathematical metaphors, and

- what we express symbolically as the secondary conceptual mathematical metaphors that arise from our subsequent perception of the symbolic expression/s of our primary conceptual mathematical metaphors,

The challenges faced in explicating such differentiation were implicitly addressed by Solomon Feferman in his *Mathematical Intuition Vs. Mathematical Monsters*, where he seeks to distinguish between counter-intuitive interpretations of terms and their properties in formal systems that do not admit a finitary interpretation, and counter-intuitive interpretations of terms and their properties in formal systems that are finitarily interpretable:

"The appearance of monsters was a direct result of the nineteenth century program for the rigorous foundation of analysis and its arithmetization, i.e. for the triumph of number over geometry, at the hands most notably of Bolzano, Cauchy, Weierstrass, Dedekind and Cantor. That program grew in response to the increasing uncertainty as to what it was legitimate to do and say in mathematics, and especially in analysis. One could no longer rely on calculations that looked right, or depend on physical applications to justify the mathematics. The completed program of arithmetization substituted the real number system for the measurement line and "ε, δ" definitions and proofs for limit concepts and arguments. The central notions which then emerged for functions were those of continuity and differentiability (both at a point or in a region) and integrability. In those terms, the notion of a curve in n dimensions was defined simply as a continuous map f on a closed interval $[a, b]$ to n-dimensional space R^n, and the tangent to such a curve at a point was then defined in terms of the derivatives of the components of f, when those exist. Use of these precise explications sufficed to verify rigorously many of the intuitively evident properties of continuous functions and curves in the prior informal sense, e.g. that a continuous f from $[a, b]$ to the real numbers R takes on a maximum and minimum on that interval, and that for differentiable f, such extrema can be located among the points where the tangent to its curve is horizontal. Of course, it was familiar and expected that reasonable functions could have isolated points of discontinuity and that a continuous function could have isolated points where there is no tangent to its graph. It was thus a surprise when Weierstrass produced an example of a function which is everywhere continuous and nowhere differentiable. Then Peano produced an example of a space-filling curve, i.e. a continuous function from the closed interval $[0,1]$ to R^2 whose range is the unit square $[0, 1]x[0, 1]$, thus violating the intuition that a curve is a one-dimensional object. Moreover, there is no reasonable assignment of length as a measure to Peano's curve. It was to such objects that Poincaré was reacting as "monsters".

By contrast to Poincaré, the mathematician Hans Hahn (one of the principals in the Vienna Circle and the teacher of Kurt Gödel) argued against intuition in mathematics in a famous 1933 essay, "The crisis in intuition".[8] Asserting its complete unreliability, he made use of a number of mathematical monsters to support his critique. Hahn's main target was the Kantian view of space as one of the forms of pure intuition.[9] Besides presenting simplified examples of a continuous curve without a tangent at any point and of a space-filling curve (in a form due to Hilbert), Hahn also described examples challenging intuitive topological concepts. One, due to Brouwer, is that of a map of three "countries" which meet each other at every point of their boundaries. Another, due to Sierpinski, produces a curve which intersects itself at every point. Typically, these objects are constructed as limits of reasonably well-behaved functions. For example, the Peano-Hilbert space-filling "curve" is a limit of curves that first go through every quadrant of the unit square, then more quickly through every sub-quadrant, and so. The Sierpinski "curve" is obtained by successively deleting the interior of an inscribed equilateral triangle within an initial such triangle; it is the skeleton of what's left in the limit.[10]

Hahn draws the following conclusion from such examples in his essay:

> Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more sceptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics. That is to say, every new mathematical concept was to be introduced through a purely logical definition; every mathematical proof was to be carried through by strictly logical means. ... The task of completely formalizing mathematics, of reducing it entirely to logic, was arduous and difficult; it meant nothing less than a reform in root and branch. Propositions that had formerly been accepted as intuitively evident had to be painstakingly proved.[11]

As to this last, Hahn cited the example of the Jordan curve theorem, according to which every simple closed curve in the plane is the boundary of two open connected regions, one (the "interior")

being bounded, the other (the "exterior") unbounded. It had been pointed out by Camille Jordan that it is necessary to formulate explicitly this bit of tacit intuitively obvious knowledge for the proper development of complex analysis, but it turned out to be devilishly difficult to prove even for reasonably well-behaved simple closed curves, namely those with polygonal boundary; after several faulty attempts by Jordan and others, it was finally proved in general for continuous boundaries in 1905 by Oswald Veblen. Note, however, that the problem with intuition in this case was not due to a challenge by a monster, but rather the apparent necessity to use complicated rigorous methods even for intuitively simple results."

...Feferman: [Fe98], §2: Geometrical and topological monsters.

Feferman forcefully argues that:

- without denying the conceptual utility

- of 'the geometrical and topological monsters that are supposed to demonstrate the unreliability of intuition',

- these ought, actually, to be interpreted additionally as

- serving 'counterexamples to intuitively expected results when certain notions are used as explications which serve various purposes well enough but which do not have all expected properties'

- that we are seeking to adequately express (and, ideally, effectively communicate) symbolically:

"Without in the least bit denying the necessity of developing mathematics—in particular analysis and topology—in a rigorous manner, evidently (in view of my remarks in section 1) I disagree with those who, like Hahn and others, believe that intuition has no value and that it must be expelled from mathematics. What, then, is one to say about the geometrical and topological monsters that are supposed to demonstrate the unreliability of intuition? The answer is simply that these serve as counterexamples to intuitively expected results when certain notions are used as explications which serve various purposes well enough but which do not have all expected properties. Unless one thinks that curves, for example, are laid up in a Platonic heaven as continuous functions from an interval to R^n, the arithmetized notion of curve must be treated as a model of an intuitive concept which itself isolates and describes in an idealized form certain aspects of experience. An explication that is closer to most ordinary experience requires of a curve that it is at least piece-wise differentiable. That less-stringent definitions of this notion may be valuable in modelling unusual parts of experience such as Brownian motion or fractal geometry (see ftn. 10) is not thereby denied; no one explication need be assumed to fit its purpose in all theories. Similarly, while the use of homeomorphism as the mathematical definition of the conceptual rubber sheet stretching of a sphere and other familiar surfaces (such as tori, etc.) serves to verify many expected properties (e.g. forms of connectedness, "hole"-iness, etc.), it does not model fully the informal concept. Thus one does not meet the kind of pathology represented by the Alexander horned sphere in the restriction to differentiable manifolds and diffeomorphisms between them. Of course, special applications of topology in scientific modeling may require more delicate distinctions, as, for example, René Thom's "catastrophe" theory required a central focus on singularities of differentiable mappings.

Though it is understandable for the time, given the continuing deep influence of Kant's views through the work of the neo-Kantians in philosophy, it seems to me that Hahn's focus on the Kantian account of geometric intuition is misplaced so far as mathematics is concerned, and that the examples brought forward against the unreliability of intuition serve a quite different and more general purpose. Namely, it is standard mathematical practice to seek best possible results of an expected kind, and one way to achieve such is to make weakest possible assumptions on the given data. In this respect the mathematical monsters serve simply to provide counter-examples to further possible improvements."

...Feferman: [Fe98], §2: Geometrical and topological monsters.

Feferman's conclusion reflects a thesis of this investigation (see §10.E.) that:

- although a first-order set theory such as ZFC might be acceptable as an adequate foundation for symbolic expression of all the conceptual mathematical metaphors that we can *conceive*,

- it is only a Peano Arithmetic such as the first-order PA upon which we can rely to isolate those of our conceptual mathematical metaphors that can be communicated effectively:

> "Simply put, the conflict between common-sense geometrical intuition and the Banach-Tarski paradox seems so egregious that it may force one to question the very basic intuitions about arbitrary sets which lead one to accept the principles lying behind the paradox, namely the principles of Zermelo-Fraenkel Set Theory together with the Axiom of Choice—or, if not that, then at least the relevance of those principles to applicable mathematics.
>
> If common-sense and set-theoretical intuitions are in actual conflict, then one or the other must be rejected (but see the Appendix below). Few would argue for the rejection of the set-theoretical position, on the grounds that it is the best current foundation of mathematics we have and it thereby accounts in a systematic and coherent way for all the mathematics that is used in physical applications. The supporter of set theory may argue that even though non-measurable sets dont actually arise in such applications it is not reasonable to exclude AC just on that account, since its manifold uses otherwise to obtain results in accord with everyday mathematical intuition justify it pragmatically.[20] This way of defending set theory, including AC, is a version of the Quine-Putnam indispensability arguments. Against that, I have made the case that all, or almost all, of scientifically applicable mathematics can be formalized in a system W conservative over Peano Arithmetic and thus do not require the assumption of any essentially set-theoretical notions and principles at all.[21] The cases of applications that are not at present covered involve highly speculative models in quantum theory. So one can come down on the side of common-sense intuition in a full rejection of set theory, while saving the mathematics needed for scientific applications. No doubt, the silent majority will not opt for either extreme, but will continue to accept, at least tacitly, the set-theoretical way of thinking in everyday mathematics while ignoring its bizarre consequences."
>
> *. . . Feferman: [Fe98], §3: Paradoxical decomposition of sets.*

> "To conclude, I return to the question raised in the introduction: to what extent do the challenges raised by monsters to the reliability of intuition undermine its uses in its everyday roles in research, teaching and the development of mathematics? I have argued that intuition is essential for all of these, but that intuition is not enough. In the end, to be sure, everything must be defined carefully and statements must be proved. And one service that the monsters lurking around the corners provide is forcing us to don such armor for our own protection. But if the proofs themselves produce such monsters, then the significance of *what* is proved requires closer attention, and that has to be dealt with on a case-by-case basis."
>
> *. . . Feferman: [Fe98], §4: Conclusion.*

16.D.b. The mythical completability of metric spaces

As also remarked upon vividly by Jon Awbrey:

> "Our thoughts live in natural and artificial languages the way fish swim in natural and artificial bodies of water.
>
> One of the lessons most strikingly impressed on me by my first year physics course and the mass of collateral reading I did at the time was to guard against the errors that arise from "projecting the properties and structures of any language or symbol system on the external world". This was mentioned especially often in discussions of quantum mechanics—it was a common observation that our difficulties grasping wave-particle duality might be due to our prior conditioning to see the world through the lenses of our subject-predicate languages and logics. Soon after, I learned

about the Sapir-Whorf hypothesis, and today I lump all these cautionary tales under the heading of GRAM ("Grammar Recycled As Metaphysics")."

... *Awbrey*: [Aw18].

From the *evidence-based* perspective of §16., we can now hypothesise:

Thesis 6. *There are no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena.*

Thesis 7. *If:*

    (a) *a physical process is representable by a Cauchy sequence (as in the above cases §16.C.a., §16.C.b.);*

and:

    (b) *we accept that there can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;*

then:

    (c) *in the absence of an extraneous, evidence-based, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit, the physical process must tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.*

The *significance* of such insistence on *evidence-based* reasoning for the physical sciences is that we may then be prohibited from claiming legitimacy for a mathematical theory which seeks to represent a physical process based on the assumption that the limiting behaviour of *every* physical process which can be described by a Cauchy sequence in the theory must *necessarily* correspond to—and so be constrained by—the behaviour of the Cauchy limit of the corresponding sequence.

> **Comment**: The argument that 'every Cauchy sequence of rational numbers cannot be postulated as defining a real number by the usual set-theoretical arguments' is implicit in physicist Nicolas Gisin's unusual, intuitionistic, interpretation of 'real numbers' as 'the hidden variables of classical physics' in [Gi19]; where he argues (see also [Gi20]) that 'real numbers should not be considered as "physical real" and classical mechanics, like quantum physics, is indeterministic' (see also §4.H.a.).

For instance the existence of Hawking radiation in cosmology is posited on the assumption that '*the consistent extension of this local thermal bath has a finite temperature at infinity*':

> "Hawking radiation is required by the Unruh effect and the equivalence principle applied to black hole horizons. Close to the event horizon of a black hole, a local observer must accelerate to keep from falling in. An accelerating observer sees a thermal bath of particles that pop out of the local acceleration horizon, turn around, and free-fall back in. The condition of local thermal equilibrium implies that the consistent extension of of this local thermal bath has a finite temperature at infinity, which imples that some of these particles emitted by the horizon are not reabsorbed and become outgoing Hawking radiation."
>
> ... *https://en.wikipedia.org/wiki/Hawking_radiation.* (Accessed 04/06/2018, 08:00 IST.)

As we have demonstrated in Fig. 2 (§16.C.) and §16.C.b., Case 2(c), *the consistent extension of the state of a stretched elastic string—as defined in Fig. 2—does not have a limiting mathematical value at infinity* which can be taken to correspond to its putatively limiting physical state.

The gedanken in §16.D.c. further illustrates that a mathematical singularity need not constrain a physical theory from positing a well-definable value for a limiting state of a physical process, contrary to what conventional wisdom accepts in the limiting cases of Einstein's equations for General Relativity:

> "The Big Bang is probably the most famous feature of standard cosmology. But it is also an undesirable one. That's because the classical model of the universe, described by Einstein's equations, breaks down in the conditions of the Big Bang, which include an infinite density and temperature, or what physicists call a singularity."
>
> . . . *Padmanabhan:* [Pd17].

Moreover, we shall argue (in §16.D.c.) that introduction of a, normally *weak*, anti-gravitational field whose strength can, however, accept quantum states that cause a universe to explode and implode in a predictable way at their corresponding 'mathematical' singularities, yields a mathematical model of a universe:

(1) That recycles endlessly from Big Bang to Ultimate Implosion;

(2) Which is time-reversal invariant; and

(3) In which the existence of 'dark energy' is intuitively unobjectionable.

Whether or not such features can be made to apply to the physical universe we inhabit is a separate issue that lies beyond the focus of the *evidence-based* perspective of this investigation.

However, it is worthwhile noting some of the barriers that mathematical 'singularities' are perceived as imposing upon our ability to faithfully comprehend, and mathematically represent, the laws of nature.

For instance, as queried by Thanu Padmanabhan in [Pd17]:

> "But what if there was no singularity? Since the 1960s, physicists have been working on describing the universe without a Big Bang by attempting to unify gravitational theory and quantum theory into something called quantum gravity. Physicists John Wheeler and Bryce deWitt were the first to apply these ideas to a hypothetical pre-geometric phase of the universe, in which notions of space and time have not yet-emerged from some as-yet unknown structure. This heralded the study of quantum cosmology, in which physicists attempted to describe the dynamics of simple toy models of the universe in quantum language. Needless to say, several different, but related, ideas for the description of the pre-geometric phase mushroomed over the decades. The unifying theme of these models is that the classical universe arises, without any singularity, through a transition from a pre-geometric phase to one in which spacetime is described by Einstein's equations. The main difficulty in constructing such a description is that we do not have a complete theory of quantum gravity, which would allow us to model the pre-geometric phase in detail."
>
> . . . *Padmanabhan:* [Pd17].

The issue is highlighted further by Padmanabhan in [Pd17a]:

"I will now raise a question which, at the outset, may sound somewhat strange. Why does the universe expand and, thereby, give us an arrow of time? To appreciate the significance of this question, recall that Eq. (9) is invariant under time reversal $t \to -t$. (After all, Einstein's equations themselves are time reversal invariant.) To match the observations, we have to choose a solution with $\grave{a} > 0$ at some fiducial time $t = t_{fid} > 0$ (say, at the current epoch), thereby breaking the time-reversal invariance of the system. This, by itself, is not an issue for a laboratory system. We know that a particular solution to the dynamical equations describing the system need not respect all the symmetries of the equations. But, for the universe, this is indeed an issue.

To see why, let us first discuss the case of $(\rho + 3p) > 0$ for all t. The choice $\dot{a} > 0$, at any instant of time, implies that we are postulating that the universe is expanding at that instant. Then Eq. (9) tells us that the universe will expand at all times in the past and will have a singularity ($a = 0$) at some finite time in the past (which we can take to be $t = 0$ without loss of generality). The structure of Eq. (9) prevents us from specifying the initial conditions at $t = 0$. So, if you insist on specifying the initial conditions and integrating the equations forward in time, you are forced to take $\dot{a} > 0$ at some time $t = \epsilon > 0$, thereby breaking the time reversal symmetry. The universe expands at present 'because' we chose it to expand at some instant in the past. This expansion, in turn, gives us an arrow of time [where] either t or a can be used as a time coordinate. But why do we have to choose the solution with $\dot{a} > 0$ at some instant? This is the essence of the so called expansion problem [6]. An alternative way of posing the same question is the following: How come a cosmological arrow of time emerges from the equations of motion which are time-reversal invariant?

In a laboratory, we can usually take another copy of the system we are studying and explore it with a time-reversal choice of initial conditions, because the time can be specified by degrees of freedom external to the system. We cannot do it for the universe because we do not have extra copies of it handy and—equally importantly—there is nothing external to it to specify the time. So the problem, as described, is specific to cosmology.

So far we assumed that $(\rho + 3p) > 0$, thereby leading to a singularity. Since meaningful theories must be nonsingular, we certainly expect a future theory of gravity—possibly a model for quantum gravity—to eliminate the singularity [effectively leading to $(\rho + 3p) < 0$. Can such a theory solve the problem of the arrow of time? This seems unlikely. To see this, let us ask what kind of dynamics we would expect in such a 'final' theory. The classical dynamics will certainly get modified at the Planck epoch, to govern the evolution of an (effective) expansion factor. The solutions could, for example, have a contracting phase (followed by a bounce) or could start from a Planck-size universe at $t = -\infty$, just to give two nn-singular possibilities. While we do not know these equations or their solutions, we can be confident that they will still be time-reversal invariant because quantum theory, as we know it, is time-reversal invariant.

So except through a choice for initial conditions (now possibly at $t = -\infty$), we still cannot explain how the cosmological arrow of time emerges. Since quantum gravity is unlikely to produce an arrow of time, it is a worthwhile pursuit to try and understand this problem in the (semi) classical context."

. . . Padmanabhan: [Pd17a].

In other words, the arguments in §16.C.a. and §16.C.b. suggest that:

Thesis 8. *The perceived barriers that inhibit mathematical modelling of a cyclic universe— one which admits broken symmetries, dark energy, and an ever-expanding multiverse—in a mathematical language that admits unambiguous communication are illusory; they arise out of an attempt to ask of the language selected for such representation more than the language is designed to deliver.*

16.D.c.   Case 5: Modelling the states of the total energy in a universe that recycles

"Both general relativity and Newtonian gravity appear to predict that negative mass would produce a repulsive gravitational field."

. . . Anti-gravity: *https://en.wikipedia.org/wiki/Anti-gravity; accessed 08/06/2018, 10:13:00.*

To illustrate why an *evidence-based* perspective—towards interpreting the propositions of a mathematical model realistically—would view such barriers as illusory, we consider the following gedanken.

Case 5: We treat Fig.2 in §16.C. as a mathematical representation of the 'confinement parameter' that determines the state of the total energy s, in a *finite* universe $\mathcal{U}$, which is subject to two constantly unequal and opposing, assumed additive, forces due to:

1. A *strong* confinement field G (induced by matter), whose state is determined by a single discrete dimensionless constant, defined as an Einsteinian gravitational, or confinement strength, constant (gsp), which is always $\frac{1}{2}$; and

2. A *weak* anti-confinement field R (induced by anti-matter), whose state is determined by discrete dimensionless values, defined as the Einsteinian repulsive, or anti-confinement strength, 'cosmological constants' (asp), where:

    (a) $asp = 1 > gsp$ when $\mathcal{U}$ is in an *exploding* state *at* event e_0;
    (b) $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{n+1}) > gsp$ when $\mathcal{U}$ is in an *imploding* state *at* event e_n for $n \geq 1$;
    (c) $asp = \frac{1}{3} < gsp$ when $\mathcal{U}$ is in a *steady* state:

        i. during which events, denoted by $e'_n, e''_n, \ldots$,
        ii. occur *between* events e_n and e_{n+1};
        iii. where $e'_n < e_m$ is an abbreviation for 'event e'_n occurs causally before event e_m'.

and where the following are assumed to hold:

(3) Classical laws of nature (see §19.D.c., Thesis 16) determine the nature and behaviour of all those properties of the physical world that are both *determinate* and *predictable*, and are therefore mathematically describable at any event $e(n)$ by *algorithmically computable* functions from a putatively specifiable initial state at event $e(0)$;

(4) Neo-classical (quantum) laws of nature (see §19.D.d., Thesis 17) determine the nature and behaviour of those properties of the physical world that are *determinate* but *not predictable*, and are therefore mathematically describable at any event $e(n)$ only by functions that are *algorithmically verifiable* but not *algorithmically computable* from any putatively specifiable initial state at event $e(0)$;

(5) There can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;

(6) All laws of nature are subject to *evidence-based* accountability as follows (see §16.D.b., Thesis 6):

    (a) If a physical process is representable by a Cauchy sequence (as in the above cases in §16.C.a. and §16.C.b.), then:

        (i) in the absence of an extraneous, *evidence-based*, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit;

(ii) the physical process *must* be taken to tend to a discontinuity (singularity) due to 'hidden' variables (in the sense of §19.B.a.) which have *not* been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.

A: We then define:

(1) The total, say s, units of energy of the universe $\mathcal{U}$ is:

    (a) in an *exploding* state *at* event e_0;

    (b) in a *steady* state *between* events e_n and e_{n+1} for $n \geq 1$;

    (c) in an *imploding* state *at* events e_n for $n \geq 1$.

(2) The state of the anti-confinement field in $\mathcal{U}$ at an event is defined with reference to Fig.2 as follows:

    (a) Initially at the Big Bang event e_0, where the energy s is in an unstable *exploding* state, the anti-confinement field strength:

        (i) is determined by the ratio $asp = \frac{s}{s} = 1 > gsp$ of the absolute value of the total energy s of the universe, and the absolute value of a confinement parameter represented by the length BC where, for convenience, we define the length BC as s;

        (ii) which also corresponds to the limiting case of the confinement parameter as $n \to \infty$ in Fig.2.

    (b) Between events e_n and e_{n+1} for $n > 0$, where the energy s is in a *steady* state, the anti-confinement field strength:

        (i) is determined by the ratio $asp = \frac{s}{l_n} = \frac{1}{3} < gsp$,

        (ii) where the confinement parameter $l_n = 3s$ is represented by the cumulative perimeter lengths of all the triangles on their common base BC in Fig.2.

    (c) At event e_n for $n \geq 1$, where the energy s is in an unstable *imploding* state, the anti-confinement field strength:

        (i) is determined by $asp = \frac{s}{l_n} + \frac{2}{3}(1 - \frac{1}{n+1}) > gsp > \frac{1}{3}$;

        (ii) where $\frac{2}{3}(1 - \frac{1}{n+1}) > \frac{1}{3}$ is defined as the implosion constant at event e_n.

B: We further define:

(1) At event e_0 the universe $\mathcal{U}$ explodes and expands 'instantaneously'—in a water-to-steam like phase change—to a *steady* state termed as event e'_0 where:

    (a) The strength of the confinement field, $gsp = \frac{1}{2}$,

    is now greater than:

    (b) The strength of the anti-confinement field, $asp = \frac{s}{3s} = \frac{1}{3}$.

(2) At any event e_0' the total energy s of the universe $\mathcal{U}$—which we assume can neither be created nor destroyed—is subjected to a confinement field due to gravitational effects that gradually concentrates:

   (a) some energy to form isolated matter;

   (b) some isolated matter to form stars;

   (c) some stars to form supernovas;

   (d) some supernovas to form 'black holes';

   (e) some 'black hole' to form the first 'critical black hole':

   (i) which we define as event e_0'' where $e_0'' \geq e_0$;

   (ii) during which matter is gradually drawn into the 'black hole',

   (iii) until, at event e_1, a 'critical' proportion of the total energy s of the parent universe corresponding to the state BAC has been drawn into the 'critical black hole':

   (a) which proportion, without loss of generality, we may take as $\frac{1}{2}$ in this example;

   (b) where we treat event e_1 as a singularity corresponding to the mid-point of BC;

   (c) such that this energy ($\frac{s}{2}$) has now been 'confined' into an imploding state with $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{2}) = \frac{2}{3} > gsp$;

   (i) and is extinguished in an 'instantaneous' implosion, defined as the event $e_1 \geq e_0''$,

   (ii) which forms an electromagnetically disconnected, independent, universe;

   (iii) which, without loss of generality, we treat as the splitting of the energy s of the parent universe $\mathcal{U}$ into two disconnected, isomorphic but not identical, twin sub-universes corresponding to the states $BAC_{1,1}$ and $BAC_{1,2}$ in Fig.2,

   (iv) that are situated in common, universal, confinement and anti-confinement fields G and R;

   (v) and which, without loss of generality, we assume obey identical laws of nature;

   (vi) where the total energy s is now divided equally between the twin states $BAC_{1,1}$ and $BAC_{1,2}$;

   (vii) where, without loss of generality, we may assume that the distribution of particles and their anti-particles between the twin states $BAC_{1,2}$ and $BAC_{1,1}$ is not necessarily symmetrical.

(3) Whence it follows that:

   (1) The total of any Hawking—or other, similarly putative[160]—energy radiated back into the 'observable' universe $\mathcal{U}$ corresponding to the state BAC during the period,

---

[160]'Putative' since the existence of such energy may be only on the basis of the debatable—see §16.D.b.—mathematical assumption that the limit of the mathematical representations of a sequence of physical phenomena must necessarily correspond to the putative behaviour of the physical phenomena in the putative limiting state.

defined as event e_0'', between the creation of the 'critical black hole' and its eventual extinction at event e_1 (corresponding to the mid-point of BC):

(a) is *not* $s/2$ (as conventional wisdom would expect in such a model);

(b) but, if at all, only a tiny fraction of the total energy—which is now $s/2$—of each sub-universe;

(c) although each sub-universe:

   (i) unaware of its isomorphic sibling,

   (ii) and under the illusion that it is still the entire parent universe,

   (iii) with merely 'black hole' concentrates of energy within it,

   (iv) which it believes will gradually extinguish once all the energy has seeped back into its domain as a result of a putative Hawking, or similar, radiation,

   (v) continues to lay claim to the energy of its extinguished sibling as 'dark energy',

   (vi) by an 'unknowably' misapplied appeal to the law of preservation of the total energy s of the original universe corresponding to the state BAC;

(2) Although the universe $\mathcal{U}$ is time-reversal invariant, each of the twin (isomorphic but not identical) sub-universes corresponding to the states $BAC_{1,1}$ and $BAC_{1,2}$ need not be time-reversal invariant, since the ratio of particles to their anti-particles in each of the twin sub-universes may no longer be symmetrical;

(3) Each sub-universe in turn forms the *next* 'critical black hole' singularity;

(a) that implodes similarly at—assumed without loss of generality as a common— event e_2,

(b) into two, isomorphic but electro-magnetically disconnected, twin sub-universes with equal, but asymmetrical, division of energy;

(4) The universe at event e_2 is a 'multiverse' of mutually disconnected 2^2 sub-universes corresponding to the states $\{BAC_{2,1}, BAC_{2,2}, BAC_{2,3}, BAC_{2,4}\}$;

(a) and so on ad infinitum.

C: In other words, the nth implosion at event e_n, for $n > 1$, is when the universe $\mathcal{U}$ is confined into the imploding state with a monotonically increased imploding anti-confinement strength $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{n+1}) > \frac{1}{3}$; and its energy divides further—corresponding to each of the 2^n triangles $BAC_{n,i}$ on the base BC, where $1 \leq i \leq 2^n$, dividing further into two similar sub-triangles—where:

(1) The total energy corresponding to each of the 2^n triangles *after* the event e_n is $s/2^{n-1}$ for $n > 0$;

(2) The strength of the anti-confinement field within each sub-universe remains constant at $asp = 1/3$ between events e_n and e_{n+1}, which is below the minimum imploding $asp = \frac{2}{3}$ of event e_1.

D: We thus have a mathematical model of an exploding and then imploding universe:

(1) That can be viewed as recycling endlessly in *either* direction of time;

(2) Whose state—exploding, steady, or imploding—at any event e is determined by the strength of an anti-confinement field that—in the direction of time chosen in this example—regularly impels $\mathcal{U}$ to split itself into a monotonically increasing number of isomorphic, but electromagnetically disconnected, sub-universes, all situated in a common confinement/anti-confinement field:

(a) where the laws of nature remain unchanged;

(b) where, for $n > 0$, the total energy within each sub-universe at event e_n has decreased monotonically to $s/2^{n-1}$ due to persisting imploding effects of assumed gravitational/anti-gravitational forces;

(c) that will further split each sub-universe into two at event e_{n+1} as illustrated in Fig.2 if the strength of the anti-confinement field is in the state $1 > asp > \frac{1}{3}$;

(3) Where the energy within each sub-universe during the steady state between events e_n and e_{n+1} appears as 'dark' to its siblings:

(a) since it is disconnected from, and disappears forever beyond, their event-horizon at an implosion;

(b) and because each sub-universe, unaware of its siblings, assumes that—since energy can neither be created nor destroyed—the total energy s of the universe must remain constant within their illusory 'universe', either as visible or as 'dark' energy;

(c) where the distribution of matter outside the critical black hole within each sub-universe may be perceived at any instant by an observer within the sub-universe as accelerating away from the observer in an apparently expanding 'universe' whose boundary is quantified by an ever-increasing value which also tends to a discontinuity as $n \to \infty$, corresponding to the *virulence* of the virus cluster considered in §16.C.a., Case 1(c), Fig.3;

(d) where any two, isomorphic but electro-magnetically disconnected, twin sub-universes have equal, but asymmetrical, division of energy;

(4) Where each sub-universe during the steady state between events e_n and e_{n+1} is expanding at an accelerating rate since the 'cosmological constant' $asp = \frac{1}{3} > 0$;

(5) The energy within each sub-universe at the limiting Zeno-type phase-change point—describable mathematically as '$n \to \infty$'—implodes finally to a 'dark point' in BC;

(6) Where the energy within the universe as a whole experiences a steam-to-water phase-changing collapse into the original Big Bang configuration represented by an exploding anti-gravitational state $asp = 1$ denoted by BC;

(a) thus triggering the next cycle of its rebirth (in the chosen time direction of this example);

16.D.d.   Asking of a language more than it is designed to deliver

To summarise briefly, in this investigation we have argued for the plausibility of the thesis (§16.D.b., Thesis 7) that if:

(a) a physical process is representable by a Cauchy sequence; and

(b) we accept that there can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;

then:

(c) in the absence of an extraneous, *evidence-based*, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit;

(d) the physical process *must* tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.

We have highlighted the practical significance of our thesis for the physical sciences by defining an, *in principle verifiable*, mathematical model in §16.C., Fig.2 that can be interpreted as describing the putative behaviour under a well-defined iteration of:

(1) a virus cluster (§16.C.a.);

(2) an elastic string (§16.C.b.);

(3) a quantum chimera (§16.C.c.);

(4) a political revolution (§16.C.d.);

where the physical process in each case can be 'seen' to tend to an 'ultimate' discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the process.

We have then highlighted the theoretical significance of our thesis for a realistic philosophy of science by showing that Fig.2 can also be interpreted as representing the, *essentially unverifiable*, state of the total energy of:

(5) a finite Universe $\mathcal{U}$ (§16.D.c.):

    (a) that recycles endlessly from Big Bang to Ultimate Implosion; and

    (b) in which the existence of 'dark energy' is mathematically and intuitionistically unobjectionable.

Moreover, the only assumptions we have made are that $\mathcal{U}$ obeys Einstein's equations and classical quantum theory, and that:

Thesis 9. *The anti-matter in $\mathcal{U}$ produces a repulsive, anti-gravitational, field:*

(a) that is consistent with both general relativity and Newtonian gravity;

(b) whose state at any instant is either exploding, steady, or imploding;

(c) whose 'energy anti-confinement' strength at any instant is determined by an anti-gravitational dimensionless 'cosmological constant' asp that can assume any of three values $asp = 1$ (exploding at the instant of the Big Bang), $asp = \frac{1}{3}$ (steady between an explosion and an implosion) or $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{n+1})$ (imploding at the instant of the extinguishing of the nth 'critical black hole' for all $n \geq 1$);

(d) which constantly opposes the 'energy confinement' strength of the Newtonian gravitational field whose state is determined at any instant by only one dimensionless gravitational constant[161] $gsp = \frac{1}{2}$.

Since it is conventional wisdom (see [BCST], [Vi11], [Chr97], [NG91]) that the existence of anti-matter which could produce a repulsive, anti-gravitational, field is admitted by both general relativty and Newtonian gravity, we conclude from §16.D.b., Thesis 7, and §16.D.d., Thesis 9, that the commonly perceived barriers to modelling the behaviour of such a universe $\mathcal{U}$ unambiguously in a mathematical language may be illusory, and reflect merely an attempt to ask of the language selected for such representation more than it is designed to deliver unequivocally.

More specifically, from the perspective of the *evidence-based* reasoning introduced in [An16] (see §2.), it can reasonably be argued that the commonly perceived barriers to modelling the behaviour of such a universe $\mathcal{U}$ realistically in a mathematical language may reflect the fact that:

(i) since the real numbers are defined by conventional wisdom in set-theoretical terms as the postulated limits of Cauchy sequences in a second-order *dichotomous*[162] arithmetic such as ACA_0,

(ii) the prevailing language of choice for representing physical phenomena and their associated abstractions (conceptual metaphors) mathematically is generally some language of Set Theory,

(iii) which admits axioms—such as an axiom of infinity—whose veridicality cannot be *evidence-based* under any putative, *well-defined*, interpretation of the theory,

(iv) and in which the *dichotomy* highlighted in ACA_0 could admit a contradiction under any putative, *well-defined*, interpretation of the theory.

16.D.e.   Have we reached the limits of what *can* be physically *evidenced*?

We note that Fig.2 in §16.C. is not a unique model for the 'confinement' properties of the universe $\mathcal{U}$. For instance, we could have started essentially similar iterations with a square $ABCD$ of side s.

Moreover, it is not necessary that each 'black hole' create isomorphic sub-universes; an assumption intended only to illustrate that an event such as an Ultimate Implosion is well-definable mathematically.

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[161]Which could be viewed as corresponding to the gravitational constant, denoted by G, common to both Newton's law of universal gravitation and Einstein's general theory of relativity; whose value in Planck units is *defined* as 1, and whose measured value is expressed in the *International System of Units* as approximately $6.674 \; x \; 10^{-11} \, N.kg^{-2}.m^2$.

[162]Since we show how—in the case of Goodstein's Theorem—such a belief leads to a dichotomous conclusion in §15., Theorem 15.1.

However, since the Ultimate Implosion is defined as corresponding to a mathematical limit as $n \to \infty$, and we postulate that there are no infinite processes in physical phenomena, it follows that the law determining such an Ultimate Implosion (as also the point of implosion of a 'black hole') may be of an essentially 'unknowable' quantum nature; in which case we cannot even assume in principle that a universe such as $\mathcal{U}$ can be shown to *actually* exist on the basis of *evidence-based* reasoning, nor whether or not it would recycle identically each time (in either direction).

It may thus be worth considering further, by the principle of Ockham's razor, whether the above simplistic mathematical model of the properties of a universe $\mathcal{U}$—which, defined as obeying Einstein's equations and quantum theory, seems to fit our known experimental observations—can be taken to suggest that, as implicitly argued by physicist Sabine Hossenfelder, we may have reached the foundations of physics beyond which the laws of nature are essentially 'unknowable':

> "So you want to know what holds the world together, how the universe was made, and what rules our existence goes by? The closest you will get to an answer is following the trail of facts down into the basement of science. Folow it until facts get sparse and your onward journey is blocked by theoreticians arguing whose theory is prettier. That's when you know you've reached the foundations.
>
> The foundations of physics are those ingredients of our theories that cannot, for all we presently know, be derived from anything simpler. At this bottommost level we presently have space, time, and twenty-five particles, together with the equations that encode their behaviour. . . .
>
> In the foundations of physics we deal only with particles that cannot be further decomposed; we call them "elementary particles." For all we presently know, they have no substructure. But the elementary particles can combine to make up atoms, molecules, proteins—and thereby create the enormous variety of structures we see around us. It's these twenty-five particles that you, I, and everything else in the universe are made of.
>
> But the particles themselves aren't all that interesting. What is interesting are the relations between them, the principles that determine their interaction, the structure of the laws that gave birth to the universe and enabled our existence. In our game, it's the rules we care about, not the pieces. And the most important lesson we have learned is that nature plays by the rules of mathematics."
>
> *. . . Hossenfelder:* [Hos18a]*, p.6.*

From the broader, multi-disciplinary, *evidence-based* perspective of this investigation, we view Hossenfelder as essentially arguing further—in [Hos18]—that committing intellectual and physical resources to seeking experimental verification for the putative existence of physical objects, or of a 'Theory', should:

(a) only follow if such putative objects, or the putative elements of the 'Theory', can be theoretically defined—even if only in principle—in a categorical mathematical language, such as the first-order Peano Arithmetic, which has a finitary *evidence-based* interpretation that can, conceivably, admit unambiguous communication between any two intelligences—whether human or mechanistic;

(b) and not merely on the basis that they can be conceptualised metaphorically and represented in a set-theoretical language such as ZF which, even though first-order, has no *evidence-based* interpretation that would admit unambiguous communication.

16.D.f. The crisis in physics is not only about physics

The issue, Hossenfelder passionately argues in her blog, is not only about physics.

"**The crisis in physics is not only about physics**

In the foundations of physics, we have not seen progress since the mid 1970s when the standard model of particle physics was completed. Ever since then, the theories we use to describe observations have remained unchanged. Sure, some aspects of these theories have only been experimentally confirmed later. The last to-be-confirmed particle was the Higgs-boson, predicted in the 1960s, measured in 2012. But all shortcomings of these theories—the lacking quantization of gravity, dark matter, the quantum measurement problem, and more—have been known for more than 80 years. And they are as unsolved today as they were then.

The major cause of this stagnation is that physics has changed, but physicists have not changed their methods. As physics has progressed, the foundations have become increasingly harder to probe by experiment. Technological advances have not kept size and expenses manageable. This is why, in physics today we have collaborations of thousands of people operating machines that cost billions of dollars.

With fewer experiments, serendipitous discoveries become increasingly unlikely. And lacking those discoveries, the technological progress that would be needed to keep experiments economically viable never materializes. It's a vicious cycle: Costly experiments result in lack of progress. Lack of progress increases the costs of further experiment. This cycle must eventually lead into a dead end when experiments become simply too expensive to remain affordable. A \$40 billion particle collider is such a dead end.

The only way to avoid being sucked into this vicious cycle is to choose carefully which hypothesis to put to the test. But physicists still operate by the "just look" idea like this was the 19th century. They do not think about which hypotheses are promising because their education has not taught them to do so. Such self-reflection would require knowledge of the philosophy and sociology of science, and those are subjects physicists merely make dismissive jokes about. They believe they are too intelligent to have to think about what they are doing.

The consequence has been that experiments in the foundations of physics past the 1970s have only confirmed the already existing theories. None found evidence of anything beyond what we already know.

But theoretical physicists did not learn the lesson and still ignore the philosophy and sociology of science. I encounter this dismissive behavior personally pretty much every time I try to explain to a cosmologist or particle physicists that we need smarter ways to share information and make decisions in large, like-minded communities. If they react at all, they are insulted if I point out that social reinforcement—aka group-think—befalls us all, unless we actively take measures to prevent it.

Instead of examining the way that they propose hypotheses and revising their methods, theoretical physicists have developed a habit of putting forward entirely baseless speculations. Over and over again I have heard them justifying their mindless production of mathematical fiction as "healthy speculation"—entirely ignoring that this type of speculation has demonstrably not worked for decades and continues to not work. There is nothing healthy about this. It's sick science. And, embarrassingly enough, that's plain to see for everyone who does not work in the field.

This behavior is based on the hopelessly naïve, not to mention ill-informed, belief that science always progresses somehow, and that sooner or later certainly someone will stumble over something interesting. But even if that happened—even if someone found a piece of the puzzle—at this point we wouldn't notice, because today any drop of genuine theoretical progress would drown in an ocean of "healthy speculation".

And so, what we have here in the foundation of physics is a plain failure of the scientific method. All these wrong predictions should have taught physicists that just because they can write down

equations for something does not mean this math is a scientifically promising hypothesis. String theory, supersymmetry, multiverses. There's math for it, alright. Pretty math, even. But that doesn't mean this math describes reality.

Physicists need new methods. Better methods. Methods that are appropriate to the present century.

...I have said many times that looking at the history of physics teaches us that resolving inconsistencies has been a reliable path to breakthroughs, so that's what we should focus on. I may be on the wrong track with this, of course. ...

Why don't physicists have a hard look at their history and learn from their failure? Because the existing scientific system does not encourage learning. Physicists today can happily make career by writing papers about things no one has ever observed, and never will observe. This continues to go on because there is nothing and no one that can stop it."
...Hossenfelder: [Hos19].

Hossenfelder's concerns were shared at a workshop in Munich in December 2015: "Why Trust a Theory? Reconsidering Scientic Methodology in Light of Modern Physics":

"In a 2014 letter to Nature entitled "Scientific method: Defend the integrity of physics," the noted cosmologists George Ellis and Joseph Silk expressed criticism of what they described as a dangerous tendency to soften principles of scientific reasoning in contemporary fundamental physics. This letter spurred the organization of a workshop in Munich in December 2015 entitled "Why Trust a Theory? Reconsidering Scientific Methodology in Light of Modern Physics." The workshop brought together leading physicists, historians, and philosophers of science to discuss and debate a range of pressing epistemological issues that confront contemporary fundamental physics. The majority of the contributions to this book are based on talks delivered at this meeting. As such, what the reader will find is the fruits of a sustained and constructive critical engagement between the various contributors that has taken place both in print and in person. Additional contributions were solicited by the editors with the aim of ensuring as full and balanced presentation as possible of the various positions in the debate."
...Dardashti, Dawid and Thébault: [DDT19], Preface.

According to Helge Kragh, one concern is 'whether certain recent developments in theoretical physics belong to science proper':

"In May 2008, there appeared in *New Scientist* an article with the provocative question: "Do we need to change the definition of science?" (Matthews, 2008). Six years later, *Nature* published the article "Scientific method: Defend the integrity of physics" (Ellis & Silk, 2014). Both articles discussed essentially the same question, namely whether certain recent developments in theoretical physics belong to science proper. For more than a decade there has been an ongoing and often heated dispute in the physics community, and also in some corners of the philosophical community, concerning the scientific status of theories such as superstring physics and multiverse hypotheses. These theories are cultivated by a fairly large number of professional physicists and, by ordinary sociological standards, are undoubtedly to be counted as scientific. But are they also scientific from an epistemic point of view, or does their status as branches of physics require an extension or revision of the traditional meaning of science?

The classical demarcation problem between science and non-science (which may or may not include pseudoscience) has taken a new turn with the appearance of fundamental and highly mathematical theories that may not be experimentally testable in the ordinary sense. So why believe in them? According to the philosopher Dudley Shapere (2000, pp. 15961), "physics is in fact approaching, or perhaps has reached, the stage where we can proceed without the need to subject our further theories to empirical test." He asks, "Could empirical enquiry, which has guided upto a certain point science in its history, lead at that point to a new stage wherein empiricism itself is transcended, outgrown, at least in a particular domain?" More than a few physicists would

presently respond affirmatively to Shapere's question. It should be noted that the demarcation problem and the traditional criteria of falsifiability and empirical testability are discussed not only by physicists but also in some other branches of science. For example, biologists have questioned these criteria and suggested, in striking analogy to the debate concerning multiverse physics, that methodological norms of what constitutes good science are not only irrelevant but actually detrimental to the progress of their science (Yang, 2008).

What it is all about can be summarized in the notion of "epistemic shifts," meaning claims that the basic methodological and epistemological rules of science are in need of revision (Kragh, 2011). These rules may be appropriate for most science and have been appropriate for all science until recently, but in some areas of modern physics they are no longer adequate and should therefore be replaced by other norms for the evaluation of theories. A proposed shift in epistemic standards may be of such a drastic nature that it challenges the very meaning of science as traditionally understood. In this case it effectively implies a new demarcation line separating what counts as science and what does not. This is what Steven Weinberg (2007) alluded to when he, referring to the string-based multiverse, said that "we may be at a new turning point, a radical change in what we accept as a legitimate foundation for a physical theory."

Another way of illustrating the notion of an epistemic shift is to compare it to Thomas Kuhn's idea of revolutions separated by different paradigms. Richard Dawid (2013, p. 124) speaks of the debate in the physics community as "a paradigm shift regarding the understanding of scientific theory assessment." According to the original version of Kuhn's philosophy of science, paradigm shifts include different criteria for what counts as acceptable science and also for evaluating theories. Rival paradigms carry with them rival conceptions of science; for this reason alone they are incommensurable. In principle, no rational argument can decide whether one paradigm is superior to a competing paradigm. The rhetoric of epistemic shifts has become part of modern physics. "We are in the middle of a remarkable paradigm shift in particle physics," asserts one physicist, referring to the anthropic string landscape (Schellekens, 2008, p. 1). According to another physicist, the multiverse promises "a deep change of paradigm that revolutionizes our understanding of nature" (Barrau, 2007, p. 16).

The purpose of this chapter is not to reexamine the recent debate concerning string theory and multiverse cosmology, but rather to look at it through the sharp lenses of the history of science. Although knowledge of the history of the physical sciences is of no direct relevance to the ongoing debate, it is of some indirect relevance. It may serve the purpose of correcting various mistakes and placing the subject in a broader historical perspective. Physicists may think that super-strings and the multiverse have ushered in a uniquely new situation in the history of science, but they are mistaken. There have been several cases in the past of a somewhat similar nature, if not of quite the same scale. I modestly suggest that modern fundamental physics can in some sense learn from its past. Before turning to this past I shall briefly review what is generally and for good reasons considered the most important of the traditional standards of theory evaluation—namely that a theory must be testable."

. . . Kragh: [Kra19], pp.13-15

Kragh ruefully notes that, despite 'what is generally and for good reasons considered the most important of the traditional standards of theory evaluation—namely that a theory must be testable', and everyone agreeing 'that actual and present testability, involving present instrument technologies or those of a foreseeable future, is preferable', that is where the agreement ends:

> "To speak of the "definition" of science is problematic. There simply is no trustworthy methodological formulation that encapsulates in a few sentences the essence of science and is valid across all periods and all disciplines. Nonetheless, some criteria of science and theory choice are relatively stable, enjoy general acceptance, and have been agreed upon since the early days of the scientific revolution (Kuhn, 1977). Almost all scientists subscribe to the belief that testability is more than just a desideratum that scientists have happened to agree upon and that suited science at a certain stage of development. They consider it a *sine qua non* for a theory being scientific that it must be possible to derive from it certain consequences that can be proved right or false by means of

observation or experiment. If there are no such consequences, the theory does not belong to the domain of science. In other words, although empirical testability is not a sufficient criterion for a theory being scientific, it is a necessary one. Einstein was a great believer in rationalism and mathematical simplicity and yet he was convinced that "Experience alone can decide on truth" (Einstein, 1950, p. 17). He is followed in this belief by the large majority of modern physicists, who often go to great lengths to argue that their theories, however speculative and mathematical they may appear to be, do connect with empirical reality. Lee Smolin (2004, p. 70) echoed Einstein when he concluded about the opposing views of string theory and loop quantum gravity that, "Because this is science, in the end experiment will decide."

Physicists working with string theory, multiverse cosmology, or related areas of fundamental physics are routinely accused of disregarding empirical testability and of replacing this criterion with mathematical arguments. These accusations are not quite fair (Johannson & Matsubaru, 2009; Dawid, 2013, p. 154). On the one hand, most physicists in these fields readily accept the importance of testability, admitting that empirical means of assessment have a higher epistemic status than non-empirical means. On the other hand, they stress the value of the latter methods, which sometimes may be the only ones available. At the same time they maintain that their theories have—or in the near future will have—consequences that at least indirectly can be tested experimentally. They have not really abandoned the commonly accepted view of experiment as the final arbiter of physical theory. "The acid test of a theory comes when it is confronted with experiments," two string theorists say (Burgess & Quevedo, 2007, p. 33). Unfortunately, the necessary experiments are in most cases unrealistic for the time being, but what matters to them is that predictions from the theories are not beyond empirical testability in principle.

Although one can identify a consensus view concerning testability, it is to some extent rhetorical and of limited practical consequence. It is one thing to agree that theories of physics must be testable, but another thing to determine the meaning of the concept of testability where there is no corresponding consensus. Everyone agrees that actual and present testability, involving present instrument technologies or those of a foreseeable future, is preferable, but that is where the agreement ends."

. . . Kragh: [Kra19]*, pp.15-16*

Kragh traces the roots of the present disquietitude to the dichotomy between the demands of 'empirical testability' and 'the principle of plenitude, which essentially states that whatever is conceived as possible must also have physical reality':

"As mentioned, empirical testability is an almost universally accepted criterion of science. But even with respect to this most sacred of the defining features of science, we find in the history of science a few exceptions. It is, after all, not a necessary ingredient of science. Dawid (2013, p. 97) argues that the role played by non-empirical theory assessment in modern fundamental physics is a continuation of earlier tendencies to be found in postWorld War II particle physics. This is undoubtedly correct—think of the development of S-matrix or "bootstrap" theory in the 1950s and 1960s—but in my view there is no reason to restrict the historical perspective to the era of quantum and relativity physics. There are also inspiration and instruction to be found in other and earlier examples from the history of physics.

During the early decades of the nineteenth century Romantic natural philosophy (known as Naturphilosophie) made a great impact on physics and the other sciences in Northern Europe (Cunningham & Jardine, 1990; Kragh, 2011, pp. 2634). In this grand attempt to revolutionize science and base it on an entirely new foundation, speculations and aesthetic sentiments were not just considered legitimate parts of science, they were necessary parts and even more fundamental than empirical investigations. The philosopher Friedrich Schelling, the spiritual leader of the Naturphilosophie movement, even founded a Journal of Speculative Physics as a means of promoting and communicating the new science. At the time the term "speculation" did not have the pejorative meaning it typically has today but rather was largely synonymous with "intuition." It was a fundamental assumption of the new speculative physics that mind and nature coexisted as a unity, such that one was unable to exist without the other. Schelling and those who followed his

thinking were not necessarily against experiments, but they thought that measuring the properties of objects and phenomena was of no great importance since it provided no understanding of the inner working of nature. In some cases natural philosophers went so far as to completely deny that observation and experiment could lead to any real insights into nature's secrets. The sort of nature that could be empirically investigated was regarded as a dull wrapping that contained and obscured the real, non-objective nature. The only way to recognize the latter was by taking the route of speculative physics—that is, to be guided by the intuitive mind of the genius. The laws of nature were thought to coincide with the laws of reason; they were true a priori and for this reason it made no sense to test them by means of experiment. Before dismissing Romantic natural philosophy as nothing but pseudo scientific and metaphysical nonsense, it should be recalled that some of the greatest physicists of the time were much influenced by this movement. Examples include H. C. Ørsted and Michael Faraday, the two celebrated pioneers of electromagnetism; another example is J. Ritter, the discoverer of ultraviolet radiation. Nonetheless, one cannot conclude from the case that good physics can flourish in the absence of experimental testing of theories. Neither Ørsted, nor Faraday, nor Ritter subscribed to Schelling's more extreme ideas and especially not to his disrespect of experiment. Ørsted's belief in a unity of electric and magnetic forces was rooted in the Romantic philosophy, but it was only when he verified it experimentally in 1820 that he turned it into a scientific discovery.

More than a century later we meet a very different version of rationalistic physics in the context of "cosmophysics," an ambitious attempt to formulate a complete and final theory of the universe and all what is in it. The leading cosmophysicists of the 1930s were two of Britain's most reputed scientists, Arthur Eddington and E. Arthur Milne. Although their world systems were quite different, both aimed at reconstructing the very foundation of physics; they did so by basing physics on a priori principles from which the laws of nature could be uniquely deduced by pure reason. Experimental tests played but an insignificant role, being subordinated logical and mathematical arguments. Milne seriously believed that when his system of world physics (as he called it) was completed there would be no contingent elements at all in the laws of nature; it would then turn out that the laws were no more arbitrary than the theorems of geometry. A mathematician knows whether a theorem is true or not without consulting nature. Likewise, Milne (1948, p. 10) wrote that "it is sufficient that the structure [of world physics] is self-consistent and free from contradiction." Eddington's idiosyncratic fundamental theory promised a way to deduce unambiguously all the laws and constants of nature from epistemic and mathematical considerations. In his bold attempt to unify cosmology and the quantum world, mathematics played a role no less elevated than in Milne's theory (Eddington, 1936, p. 3; Durham, 2006):

> It should be possible to judge whether the mathematical treatment and solutions are correct, without turning up the answer in the book of nature. My task is to show that our theoretical resources are sufficient and our methods powerful enough to calculate [of nature] the constants exactly—so that the observational test will be the same kind of perfunctory verification that we apply to theorems in geometry.

Of course, neither Milne nor Eddington could afford the luxury of disregarding experiments altogether. But they argued that experiments did not reveal the true laws of nature and consequently could not be used to test the laws. Eddington famously calculated the precise values of many of the constants of nature such as the fine-structure constant, the proton-to-electron mass ratio, and the cosmological constant. When experiments failed to agree with the predicted values he arrogantly maintained that the theory was correct; any discrepancy between theory and experiment must lie with the latter. The theories of Milne, Eddington, and their few followers shared the same fate as the revolutionary Romantic natural philosophers: they were unproductive mistakes and are today relegated to the long list of grand failures in the history of science. All the same they are of some relevance in that aspects of the same aspirations and rationalist methods can still be found in modern physics. The most extreme version is probably the Platonic "mathematical universe hypothesis" proposed by Max Tegmark (2014). Likewise, the history of string theory includes examples that show at least some similarity to the earlier ideas of cosmophysics. Referring to the theory of superstrings, John Schwarz (1998, p. 2) wrote, "I believe that we have found the unique

mathematical structure that consistently combines quantum mechanics and general relativity. So it must almost certainly be correct."

Unfortunately the prediction of supersymmetric particles remained unverified, but this did not worry Schwarz too much: "For this reason, even though I do expect supersymmetry to be found, I would not abandon this theory if supersymmetry turns out to be absent" (p. 2). Thus one can conclude from the history of physics that fundamental theories, to be successful from a physical (and not merely mathematical) point of view, must have some connection to empirical reality. The historical record of such theories suggests that empirical testability is a necessary condition for progress. But this is as far as the historical argument can go. Because one can observe some regularity in the past—say, that all physically progressive theories have been actually testable—there is no guarantee that the regularity will continue in the future. Many of the arguments in string theory and multiverse physics rely implicitly on two philosophical principles that can be traced back to Leibniz in the late seventeenth century. One is the doctrine of a pre-established harmony between the mathematical and physical sciences, making pure mathematics the royal road to progress and unification in physics (Kragh, 2015). The other is the principle of plenitude, which essentially states that whatever is conceived as possible must also have physical reality. The plenitude principle is a metaphysical claim that translates potential existence into real existence. In its more modern formulation it is often taken to mean that theoretical entities exist in nature insofar that they are consistent with the fundamental laws of physics. Since numerous universes other than ours are consistent with the equations of string theory they must presumably exist (Susskind, 2006, p. 268). The ontological plenitude principle has played a most important role in the history of science and ideas, including modern theoretical physics from Dirac's positron to Higgs's boson. Although in many cases it has been dramatically fruitful, it cannot be justified by reference to its historical record. For every example of success, there is one of failure. If the former are better known than the latter, it is because history is written by the victors. In this case, as in many others, the history of science is ambiguous. It does not speak unequivocally in favor of either the principle of plenitude or a pre-established relationship between mathematics and physics; nor does it speak unequivocally against the doctrines."

. . . Kragh: [*Kra19*], *pp.20-23*

From the *evidence-based* perspective of the Complementarity Thesis (§1., Thesis 1) of this investigation the deeper, foundational, issue intuited here by Hossenfelder—which needs to be addressed from a philosophical perspective concerning cognition that verily transcends that of physics—is that of a mathematical education (see §20.) which ignores the need for *evidence* when claiming that the provable propositions of a formal mathematical theory that admits—unarguably unambiguous—symbolic representation of our primary and secondary conceptual metaphors in the language of the theory, *must* interpret as true in the structure, say *ExternalReality*, which gave birth to the primary conceptual metaphors that the theory was initially intended, and designed, to represent symbolically—*even* when there is no *evidence-based* interpretation of our secondary conceptual metaphors in the *ExternalReality*!

Consequently, as Hossenfelder emphasises and rhetorically queries:

- ". . . what we have here in the foundation of physics is a plain failure of the scientific method. All these wrong predictions should have taught physicists that just because they can write down equations for something does not mean this math is a scientifically promising hypothesis. String theory, supersymmetry, multiverses. There's math for it, alright. Pretty math, even. But that doesn't mean this math describes reality."

- "Why don't physicists have a hard look at their history and learn from their failure? Because the existing scientific system does not encourage learning. Physicists today can happily make career by writing papers about things no one has ever observed, and never will observe."

However, from an *evidence-based* perspective, the failures which Hossenfelder ascribes to individual limitations—in transcending the scientific method of one's education—seem more

systemic; they seem symptomatic of a mathematical education in which postulation of putative mathematical limits—such as that, say, of Hawking's radiation, or of Einstein's equations of General Relativity at the putative 'Big Bang'—might mistakenly be taught, and accepted, as describing a plausible reality.

Mistakenly since the mathematical states which such mathematical/Cauchy limits postulate are purely platonic; and, in the absence of experiential evidence to the contrary, the actual physical phenomena that corresponds to what the extrapolated mathematical limits purport to describe might, sometimes (see §16.C.a. to §16.D.c.), be discontinuities corresponding to phase changes of the system that are not reflected in (and, conceivably, not expressible in), and therefore not entailed by, the postulates of the theory.

So, might the systemic failure be not of the scientific method, but of *fragile* (see §4.B.) philosophic underpinnings of a classical mathematical education (see also §20.) that we might—innocently and unquestioningly—have committed to in our formative years?

CHAPTER 17. *PARADIGM CHALLENGING CONSEQUENCES*

17.   The significance of *evidence-based* reasoning for Cognition

In his seminal 1931 paper [Go31] on formally undecidable arithmetical propositions, Kurt Gödel informally argues that, in a formal language as basic as Peano Arithmetic—which is considered as the foundation for all, significant, formal mathematical languages—there are well-defined formulas which can be recognised as intuitively true under a classical interpretation of the Arithmetic, but which are not formally provable within the Arithmetic.

The question immediately arises:

Query 14. *Does recognition of the 'truth' of Gödel's formally undecidable arithmetical proposition under the classical standard interpretation of a Peano Arithmetic imply that such recognition cannot be duplicated in any artificially constructed mechanism or organism whose design is based on classical logic?*

Query 14 suggests further that:

Thesis 10. (Gödelian Thesis) *Gödel's construction of an arithmetical proposition that is not provable in Peano Arithmetic, but true under interpretation if the Arithmetic is consistent, entails that there can be no mechanistic model of human reasoning.*

We note that Lucas and Penrose essentially address Query 14 in their Gödelian Arguments for their respective theses (§17.A., Thesis 11, and §17.B., Thesis 12).

However, both Arguments—as considered by their respective authors—have been criticized widely (see, for instance, [FG12], [Pic03])—and, as we shall show, not unreasonably—by mathematicians, computer scientists, and philosophers; the *only* consensus among experts in these fields being that Lucas' and Penrose's specific argumentations fail to sustain their respective claims.

As remarked by John Burgess:

> "...the consensus view of logicians today seems to be that the Lucas—Penrose argument is fallacious, though as I have said elsewhere, there is at least this much to be said for Lucas and Penrose, that logicians are not unanimously agreed as to where precisely the fallacy in their argument lies. There are at least three points at which the argument may be attacked."
>
> *...Burgess: [Bur10], p.131-132*

Hannes Leitgeb too remarks that both Lucas' and Penrose's arguments for their respective Mechanistic Theses are yet to be accepted as sound:

> "In a nutshell: what mathematicians mean when they speak of proof and provability, and what they do when they actually decide whether something is a proof, seems to differ substantially from what we called derivation and derivability in a formal system. "Real" provability does not seem to be relativized to any formal system but is rather absolute and informal (as was emphasized by Gödel himself, and later again by John Myhill in his "Some Remarks on the Notion of Proof"—see Myhill 1960).[17] So how exactly does absolute/informal proof and provability relate to proof and provability in formal systems?

Surprisingly, some information about this can be derived from the Incompleteness Theorems again. On their basis, it turns out to be possible to prove significant statements even about computability, the human mind, and the in-principle proving capabilities of human mathematicians. Indeed, this is a point at which artificial intelligence and cognitive science meet logic and the philosophy of mathematics. And logical methods are found to throw some light even on provability in the sense of mathematical practice (or at least on something close to that).[18]

Let

- T be the set of *true arithmetical* statements,
- K be the set of *humanly knowable arithmetical* statements,
- S_e be the set of all arithmetical statements *enumerated by the computer (Turing machine) e* according to the program of that computer,
- K' be the set of *humanly knowable* statements.

. . .

Here are two arguments for theses that are philosophically important, where each of the arguments relies on one of the Incompleteness Theorems:

. . .

Thesis 1: Mechanistic Thesis $\rightarrow K \neq T$

. . .

Thesis 2: Mechanistic Thesis (and provability conditions) $\rightarrow$ '$K = S_e$' is not a member of K'.

. . .

Both thesis 1 and thesis 2 are material implications. By classical propositional logic, they can be reformulated in terms of the following disjunctions:

The Mechanistic Thesis is false or $K \neq T$.

and

The Mechanistic Thesis is false (or the provability conditions are false) or '$K = S_e$' is not a member of K'.

The former thesis says: Either what we can know in principle about arithmetic surpasses the powers of any Turing machine, or there are arithmetical statements A and $\neg A$ for which we are for principled reasons unable to decide whether A is true or $\neg A$ is true. This is Kurt Gödel's famous dichotomy which he himself derived from his Incompleteness Theorems in his Gödel(1995).[19] The other thesis amounts to, if we ignore the part on the provability conditions (which one would need to make much more precise anyway): Either what we can know in principle about arithmetic surpasses the powers of any Turing machine, or for principled reasons we cannot know which Turing machine enumerates all and only those arithmetical truths that we can know to be true.

Is it perhaps possible to do better than these theses? That is: Is it possible to argue on the basis of the Incompleteness Theorems in favour of one of the disjuncts rather than "merely" in favour of the disjunctions from above? John Lucas (1961) and Roger Penrose (1989) thought so, when they tried to argue in such a manner just for the falsity of the Mechanistic Thesis, but careful philosophical and logical analysis of their arguments (which is still ongoing) indicates that none of their arguments is sound."

. . . *Leitgeb: [Lei10], §8.*

We shall show that such a lack of unanimity reflects the curious situation that not only Lucas' and Penrose's Arguments against reductionism, but also critiques of the Arguments, are equally fragile because they base their argumentations on the same assumption; they all appeal uncritically to classically inherited paradigms in which it is assumed—albeit implicitly—that there is a 'humanly knowable' arithmetical truth which is 'absolute' in some sense, and not well-definable under any Tarskian interpretation of the first-order Peano Arithmetic PA.

In order to, therefore, highlight the fragility of the implicit assumptions underlying Lucas' and Penrose's Gödelian Arguments, as well as those of their critiques, we shall treat the following as informal summaries of the substance of Lucas' (see §17.A.) and Penrose's (see §17.B.) respective arguments, and those of their critiques:

(a) *Lucas*: If we can treat intuitive arithmetical truth as well-defined meta-mathematically, then Gödel's reasoning entails that there can be no mechanistic model of human reasoning.

   Critique: Since intuitive arithmetical truth is not well-definable meta-mathematically, we cannot conclude formally from Gödel's reasoning that there can be no mechanistic model of human reasoning.

(b) *Penrose*: Gödel's construction of a formally unprovable, but Platonically/intuitively true, arithmetical proposition entails that, unlike human reasoning, no mechanistic reasoning can assign unique Tarskian satisfiability and truth values recursively to the propositions of any Peano Arithmetic under its standard interpretation.

   Critique: If we can only assign unique Tarskian satisfiability and truth values Platonically/intuitively to the propositions of a Peano Arithmetic under its standard interpretation, then we cannot conclude formally from Gödel's reasoning that there can be no mechanistic model of human reasoning.

However, since the relatively recent paper [An16] shows that, from an *evidence-based* perspective (see §2.), arithmetical truth is well-definable both meta-mathematically and mathematically (in two, hitherto unsuspected, ways—see §2.(1a) and §2.(1b)), we shall argue the thesis that Lucas's Gödelian Argument (see §17.A.) is essentially validated (see §17.D.).

Moreover, we shall argue that if we replace 'Platonically/intuitively' by 'non-recursively', then Penrose's Gödelian Argument (see §17.B.) too can be treated as essentially validated by the evidence-based paradigm within which the arguments of [An16] are situated.

17.A.   Lucas' perspective on the Gödelian Argument

An affirmative anti-mechanist argument for Query 14 was originally offered by Lucas in his 1961 paper [Lu61], where he essentially argued—purely on non-formal, philosophical, grounds—the thesis that Gödel's seminal 1931 paper [Go31] on formally undecidable arithmetical propositions entails:

Thesis 11. (Lucas' Gödelian Thesis) *There can be no mechanistic model of human reasoning.*

The best perspective of what Lucas' Gödelian argument is, and what it is not, is provided by Lucas himself in his 1996 revisiting of the Argument in [Lu96], where we note in particular his conclusion:

> "Thus, though the Gödelian formula is not a very interesting formula to enunciate, the Gödelian argument argues strongly for creativity, first in ruling out any reductionist account of the mind that would show us to be, au fond, necessarily unoriginal automata, and secondly by proving that the conceptual space exists in which it (is) intelligible to speak of someone's being creative, without having to hold that he must be either acting at random or else in accordance with an antecedently specifiable rule".
>
> . . . Lucas: [Lu96].

One reason Lucas' Gödelian Thesis 11 has not received the consideration it deserves from purists could be that Lucas', essentially informal, perspective uncritically assumes, in his following argument, that finitary arithmetical provability entails a *unique*, preferred, arithmetical truth under interpretation over the domain ℕ of the natural numbers:

> "...in the case of First-order Peano Arithmetic there are Gödelian formulae (many, in fact infinitely many, one for each system of coding) which are not assigned truth-values by the rules of the system, and which could therefore be assigned either TRUE or FALSE, each such assignment yielding a logically possible, consistent system. These systems are random variants, all satisfying the core description of Peano Arithmetic. But among them there is one, the one that assigns TRUE to all the Gödelian formulae which is reasonable, characterizing standard arithmetic, although not more in accordance with the specification of Peano Arithmetic than any of the others. So there is some sort of reasonableness, picking out this one instantiation of the specification in preference to all the others which is reasonable and right, though not any more in accordance with the antecedently formulated rules than any other instantiation."
>
> . . . Lucas: [Lu96].

> **Comment**: Lucas' argument was logically debatable even at the time, as noted by Burgess in [Bur10] (pp.131-132). It is now falsified by the *evidence-based* paradigm introduced in [An16] (see §2.), which shows that PA has, in fact, *two* constructively defined interpretations over the domain ℕ of the natural numbers (see §2.(1a) and §2.(1b)), and which further establishes PA as categorical (see §2.E., Corollary 2.18).

17.A.a.   The cognitive significance of Lucas' Gödelian Argument

The cognitive significance of Lucas' original Gödelian argument is highlighted by Jason Megill:

> "One reason Lucas's argument has received so much attention is that if the argument succeeds, the widely influential Computational Theory of Mind is false. Likewise, if the argument succeeds, then "strong artificial intelligence" is false; it is impossible to construct a machine that can perfectly mimic our cognitive abilities. But there are further implications; for example, a view in philosophy of mind known as Turing machine functionalism claims that the human mind is a Turing machine, and of course, if Lucas is right, this form of functionalism is false. (For more on Turing machine functionalism, see Putnam (1960)). So clearly there is much at stake."
>
> . . . Megill: [Meg]

The ramifications of Lucas' thesis (§17.A., Thesis 11)—against a reductionist account of the mind—also raise issues for the natural sciences that lie, and deserve consideration, beyond the formal argumentation of this investigation.

Validating such a thesis could, conceivably, also have broader economic significance globally, particularly in areas relating to the development of strategic and infra-structural products, facilities, and services that are based on the proposed replication of human intelligence by artificial mechanisms or organisms.

However, we shall restrict ourselves here to only the immediate, formal, consequences of addressing the Mechanist's counter-argument against the Gödelian Thesis 10 (see also [An07a], [An07b] and [An07c]):

"The Mechanist claims to have a model of the mind. We ask him whether it is consistent: if he cannot vouch for its consistency, it fails at the first examination; it just does not qualify as a plausible representation, since it does not distinguish those propositions it should affirm from those that it should deny, but is prepared to affirm both undiscriminatingly. We take the Mechanist seriously only if he will warrant that his purported model of the mind is consistent. In that case it passes the First Public Examination, but comes down at the Second, because knowing that it is consistent, we know that its Gödelian formula is true, which it cannot itself produce as true. More succinctly, we can, if a Mechanist presents us with a system that he claims is a model of the mind, ask him simply whether or not it can prove its Gödelian formula (according to some system of Gödel numbering). If he says it can, we know that it is inconsistent, and would be equally able to prove that 2 and 2 make 5, or that $0 = 1$, and we waste little time on examining it. If, however, he acknowledges that the system cannot prove its Gödelian formula, then we know it is consistent, since it cannot prove every well-formed formula, and knowing that it is consistent, know also that its Gödelian formula is true.

In this formulation we have, essentially, a dialogue between the Mechanist and the Mentalist, as we may call him, with the Mechanist claiming to be able to produce a mechanist model of the Mentalist's mind, and the Mentalist being able to refute each particular instance offered."

. . . Lucas: [Lu96].

Comment: We note that Lucas implicitly argues here that if the first-order Peano Arithmetic PA is consistent, then we can treat arithmetical truth as well-definable meta-mathematically under the *weak*, standard, interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ (see §2.B.) of PA.

This has attracted the valid criticism that since the *weak*, standard, interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA is not finitary, arithmetical truth cannot be treated as well-definable meta-mathematically under the standard interpretation.

However, since [An16] also (see §2.(1a)-(1b)) *well-defines* arithmetical truth finitarily such that the axioms of PA interpret as true under a *strong*, finitary, interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA, and the rules of inference of PA preserve such truth under the interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ (see §2.C.a., Theorem 2.15)—whence PA is finitarily consistent (see §2.C.a., Theorem 2.16)—we can strengthen Lucas' argument (see §10(a)) by eliminating its reliance upon the definition of arithmetical truth under the standard, non-finitary, interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA.

The Mechanist's argument can now be viewed, and refuted, from a broader perspective (than that of [An07a], [An07b] and [An07c]) which admits an *evidence-based*, Wittgensteinian, distinction between:

(1) meta-mathematically proven, i.e., algorithmically *verifiable* (see §2., Definition 2), 'truth'; and

(2) mathematically proven, i.e., algorithmically *computable* (see §2., Definition 3), 'truth';

to which §17.D., Theorem 17.1 appeals.

In other words, the importance of differentiating (as detailed in [An16]; see also §2.) between:

(i) the *strong*, algorithmically *computable*, 'truth'—of the provable formulas of a formal mathematical language L—definable by *finitary mathematical* reasoning from the axioms and rules of inference of L under a *strong evidence-based* interpretation; and

(ii) the *weak*, algorithmically *verifiable*, 'truth'—of the provable formulas of L—definable *only* by *non-finitary meta-mathematical* reasoning from the axioms and rules of inference of L under a *weak evidence-based* interpretation;

is implicit in, and an essential component of, Timm Lampert's interpretation of Ludwig Wittgenstein's objection (in the latter's 'notorious' paragraph in [Wi78]; see §9.A.) to the conclusions that Gödel drew from his undifferentiated *mathematical* and *meta-mathematical* reasoning in his 'undecidability' paper [Go31]:

> "The most crucial aspect of any comparison of two different types of unprovability proofs is the question of what serves as the "criterion of unprovability" (I, §15). According to Wittgenstein, such a criterion should be a purely syntactic criteria independent of any meta-mathematical interpretation of formulas. It is algorithmic proofs relying on nothing but syntactic criteria that serve as a measure for assessing meta-mathematical interpretations, not vice-versa."
>
> ...Lampert: [Lam17].

The distinction formalises the intent Lampert ascribes to Wittgenstein's objection—towards conflating mathematical and meta-mathematical entailments—by showing that:

(a) whereas the Mechanist correctly argues (see [An07a]) that, for any *specified* numeral [n], a Turing machine can always mathematically prove that Gödel's arithmetic formula [R(n)] (as defined in §17.D., Theorem 17.1) is a theorem in the classical first-order Peano Arithmetic PA;

(b) *no* Turing machine can mathematically prove that the arithmetical formula [R(x)] with Gödel number r (as defined in §17.D., Theorem 17.1)—or, equivalently by Generalisation, the PA-formula [(∀x)R(x)] with Gödel number 17*Gen* r—is a theorem in PA; and, ipso facto, meta-mathematically conclude—as a human intelligence can—that a mechanical intelligence too *could*, for any *unspecified* numeral [n], prove that Gödel's arithmetic formula [R(n)] is a theorem in PA.

Moreover, the distinction now admits:

- A formal validation (see §17.D., Theorem 17.1) of the Gödelian Argument (Thesis §17., 10); and

- A definitive Turing Test (see §17.E., Query 17) that seeks to affirmatively answer Alan Turing's earlier, but related, query—in his 1950 paper on computing machinery and intelligence (see [Tu50])—as to whether it is possible to definitively differentiate between a human intelligence and a mechanical intelligence.

17.B.   Penrose's perspective on the Gödelian Argument

The broader scientific, and philosophical, dimensions of Lucas' perspective on the Gödelian Argument—which can be viewed as seeking to answer §17., Query 14, affirmatively—were reviewed, and sought to be addressed more extensively, by Penrose (in [Pe90] and [Pe94]) from a more formal mathematical perspective, where he addresses the Gödelian Argument (Thesis §17., 10) from the perspective of Turing's seminal 1936 paper [Tu36] on computable numbers, by essentially addressing the question:

Query 15. *Are the concepts 'non-algorithmic' and 'non-constructive' necessarily synonymous in classical logic and mathematics?*

Penrose's perspective is based on a strongly Platonist thesis that sensory perceptions simply mirror aspects of an 'ideal' universe that exists, and will continue to exist, independent of any observer (see [Pe90], pp.123-128, pp.146-151).

On this view, individual consciousness would be a discovery of what there is, or can conceivably be, in such a universe; and be independent of the language in which such discovery is expressed. It follows that recognition of intuitive truth would be individually asserted—and, implicitly, fallible—correlations between the unverifiable—and, ipso facto, infallible—intuitive experiences of an individual consciousness, and the formal expressions of a communicable language.

The issue, then, is whether classical logic can adequately formalise intuitive truth, to make it infallible, or whether such recognition is essentially fallible.

Penrose apparently believes in a Platonic mathematical reality which manifests itself, first, in thought as conceptual metaphors which originate in the human mind consequent to sensory experience; and, second, in the representation of *only some* such metaphors in a *well-defined* language of *effective* communication.

From an *evidence-based* perspective, we could express this as:

Thesis 12. (Penrose's Gödelian Thesis) *Any formal system of logic cannot completely assign unique Tarskian truth values of satisfiability and truth to the propositions of any language that seeks to adequately represent, and unequivocally communicate, an individual's perception of a Platonic mathematical reality.*

Penrose seems to support such a view—when highlighting the 'ethereal' presence, and non-verifiable properties, of 'non-algorithmic' (see [Pe90], p.168), and implicitly non-constructive mathematical metaphors (such as those for four-dimensional manifolds)—by appeal to the view that such metaphors correspond to objects of our mathematical intuition which have an 'objective reality' that has been accepted in our formal languages as *essential* to classical mathematics (see [Pe90], p.123-8).

For instance, as argued forcefully by Gödel:

> "As far as the epistemological situation is concerned, it is to be said that by a proof of undecidability a question loses its meaning only if the system of axioms under consideration is interpreted as a hypothetico-deductive system; i.e., if the meanings of the primitive terms are left undetermined. In geometry, e.g., the question as to whether Euclid's fifth postulate is true retains its meaning if the primitive terms are taken in a definite sense, i.e., as referring to the behaviour of rigid bodies, rays of light, etc. The situation in set theory is similar, the difference is only that, in geometry, the meaning usually adopted today refers to physics rather than to mathematical intuition and that, therefore, a decision falls outside the range of mathematics. On the other hand, the objects of transfinite set theory, conceived in the manner explained on p. 262 and in footnote 14, clearly do not belong to the physical world and even their indirect connection with physical experiences is very loose (owing primarily to the fact that set-theoretical concepts play only a minor role in the physical theories of today).
>
> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical

paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics. That new mathematical intuition leading to a decision of such problems as Cantor's continuum hypothesis are perfectly possible was pointed out earlier (pp. 264-5).

It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas.[40] It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensation, their presence in us may be due to another kind of relationship between ourselves and reality.

However, the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. What, however, perhaps more than anything else, justifies the acceptance of this criterion of truth in set theory is the fact that continued appeals to mathematical intuition are necessary not only for obtaining unambiguous answers to the questions of transfinite set theory, but also for the solution of the problems of finitary number theory[41] (of the type of Goldbach's conjecture),[42] where the meaningfulness and unambiguity of the concepts entering into them can hardly be doubted. This follows from the fact that for every axiomatic system there are infinitely many undecidable propositions of this type."

. . . Gödel: [?], p.271.

Although Penrose's (and Gödel's) arguments represent only one, and perhaps an arguably (see, for instance, [Psy95]) extreme, point of view, they serve to emphasise that classical mathematics does not adequately legitimise the acceptance into a theory of even formally definable mathematical, 'computer-independent', *completed* structures such as the Mandelbrot *set* (cf. [Pe90], p.147); as distinct from those, such as Cantor's transfinite ordinals, which are unarguably and essentially non-constructive.

Reason: Such concepts can be argued as expressing what may be termed as *secondary*, 'abstract', conceptual metaphors which are rooted *only* in—unarguably physical—sensory perceptions by a human intelligence of the symbolic expressions of a *particular* mathematical language; rather than in any *primary* conceptual metaphors of putative objects, in our commonly accepted external reality, corresponding to which such symbolic expressions might be unarguably claimed as referring.

Comment: The following analysis by cognitive scientists George Lakoff and Rafael Núñez offers an appropriate perspective for viewing our primary conceptual metaphors as the source domain, and our secondary conceptual metaphors as referring to postulated, abstract, subjective visualisations in the target domain:

"We hypothesize that the idea of actual infinity in mathematics is metaphorical, that the various instances of actual infinity make use of the ultimate metaphorical *result* of a process without end. Literally, there is no such thing as the result of an endless

process: If a process has no end, there can be no "ultimate result." But the mechanism of metaphor allows us to conceptualize the "result" of an infinite process—in the only way we have for conceptualizing the result of a process—that is, in terms of a process that does not end.

We hypothesize that all cases of actual infinity—infinite sets, points at infinity, limits of infinite series, infinite intersections, least upper bounds—are special cases of a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result. We call this metaphor the *Basic Metaphor of Infinity*, or the BMI for short. The target domain of the BMI is the domain of processes without end—that is, what linguists call imperfective processes. The effect of the BMI is to add a metaphorical completion to the ongoing process so that it is seen as having a result—an infinite thing.

The source domain of the BMI consists of an ordinary iterative process with an indefinite (though finite) number of iterations with a completion and resultant state. The source and target domains are alike in certain ways:

- Both have an initial state.
- Both have an iterative process with an unspecified number of iterations.
- Both have a resultant state after each iteration.

In the metaphor, the initial state, the iterative process, and the result after each iteration are mapped onto the corresponding elements of the target domain. But the crucial effect of the metaphor is *to add to the target domain the completion of the process and its resulting state.* This metaphorical addition is indicated in boldface in the statement of the metaphor that follows. It is this last part of the metaphor that allows us to conceptualize the ongoing process in terms of a completed process—and so to produce the concept of actual infinity."

<div align="right"><small>. . . Lakoff and Núñez: [LR00], p.158.</small></div>

Now, we note that Penrose appears to base his thesis on, amongst others, a classical consequence of Gödel's reasoning and conclusions; namely Tarski's Theorem that we cannot express Tarskian definitions (see §2.A.) of the 'satisfiability', and 'truth', of the formulas of a first-order Peano Arithmetic under its standard interpretation recursively (see [Pe90], p.159).

> **Tarski's Theorem** (see [Me64], p.151): The set Tr of Gödel-numbers of the formal expressions of a first order Peano Arithmetic that are true in the standard model is not arithmetical.

He concludes from this that, although we may follow a common, intuitive, process for discovering common, mathematically expressible, aspects of the universe, not *all* our mathematically expressible discoveries are definable recursively in terms of classical algorithms (see [Pe90], p.533, p.548).

Moreover, Penrose's arguments also appear to imply further, albeit implicitly, that our recognition of intuitive 'arithmetical truth'—even when this is accepted as being adequately formalised by the classical Tarskian definitions of the 'satisfiability' and 'truth' of formal expressions under the *standard* interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ (see §2.B.) of a Peano Arithmetic such as PA—is 'absolutely' non-constructive (cf. [Pe90], p.145-6).

Thus, Penrose not only does not question the classical expression of Church's Thesis (see [Pe90], p.64-65; also §17.F.) as a *strong* identity which, essentially, postulates that every effectively computable number-theoretic function is algorithmically *computable* (compare with §4.F., Definition 18, Definition 21, and Theorem §4.G.a., 4.3), he seems to conclude from his arguments, concerning the inadequacy of classical logic, that there are 'non-algorithmic', 'non-constructive', ways of acquiring mathematical insight and knowledge (see [Pe90], pp.538-541); which we, however, can now see as appealing to *weakly* constructive algorithmic *verifiability*.

Church-Turing Thesis (see [Me64], p.147, p.227): A number-theoretic function is effectively computable (partially) if, and only if, it is (partially) recursive.

In other words, as is evidenced in his discussion of Lucas' Gödelian argument (see [Pe90], p.539), Penrose does not appear to entertain the possibility (evidenced in [An16]; see §2.) that what he terms as 'non-algorithmic' could be algorithmically *verifiable* in the sense of §2., Defintion 2 and, therefore, intuitionistically acceptable as constructive; his arguments seem to, implicitly, treat the terms 'non-algorithmic' and 'non-constructive' as synonymous.

We note that Martin Davis critically reviews this particular aspect of Penrose's argument, by essentially arguing that there is a polynomial equation $P = 0$ which—by the Provability Theorem for PA (see §2.E., Theorem 2.17)—is algorithmically *verifiable* (see §2., Definition 2) as always false, but not algorithmically *computable* (see §2., Definition 3) as always false:

> "... Gödel's incompleteness theorem (in a strengthened form based on work of J. B. Rosser as well as the solution of Hilbert's tenth problem) may be stated as follows: There is an algorithm which, given any consistent set of axioms, will output a polynomial equation $P = 0$ which in fact has no integer solutions, but such that this fact can not be deduced from the given axioms. Here then is the true but unprovable Gödel sentence on which Penrose relies and in a particularly simple form at that. Note that the sentence is provided by an algorithm. If insight is involved, it must be in convincing oneself that the given axioms are indeed consistent, since otherwise we will have no reason to believe that the Gödel sentence is true."
>
> . . . Davis: [Da95].

To be fair to Penrose, though (see [An07b]; [An07c]), the real 'Wittgensteinian' issue—which we seek to highlight in this paper—is not whether there is an algorithm that outputs a polynomial equation $[P(x_1, x_2 \ldots, x_n) \neq 0]$ which is unprovable in PA, even though $[P(a_1, a_2 \ldots, a_n) \neq 0]$ is PA-provable for any *given* set of numerals $[a_1, a_2 \ldots, a_n]$, but whether, for any given set of natural number values for its free variables, the fact that $P = 0$ has no integer solutions can be determined in a classically 'non-algorithmic', yet 'constructive' in the sense of algorithmically *verifiable* (see §2., Definition 2), way.

17.C.   Distinguishing between quantification over *specified* and *unspecified* values of a variable

From the *evidence-based* perspective (see §2.) of this investigation, we argue that what Penrose views as the essentially 'non-algorithmic' and 'non-constructive' aspects of mathematical concepts may simply be manifestations of a removable ambiguity in the classical Tarskian definitions of the satisfiability, and truth, of the formulas of a formal language under an interpretation (see, for instance, §2.A.b.) which do not distinguish between quantification over *specified* and *unspecified* values of a variable.

Moreover, eliminating the ambiguity yields an alternative to the classically accepted—albeit misleading in view of §2.E., Corollary 2.18, that PA is categorical—postulation of Platonic non-standard elements, in Platonic set-theoretical models of the first-order Peano Arithmetic PA (in the mistaken belief that they are entailments of Gödel's formal reasoning in his seminal 1931 paper [Go31]); an alternative with putatively far-reaching implications for the foundations of philosophy, logic, mathematics, computability and the natural sciences (see, for instance, the argumentation in [An15a]).

Thus we note that, in the proof of Theorem VI of his 1931 paper (see [Go31], p.24), Gödel argues that, in any consistent system P which formalises Peano's Arithmetic, we can well-define

an expression of the system, say $[R(x)]$, such that $[R(n)]$ is P-provable for any *specified* numeral $[n]$, but $[R(x)]$—and ipso facto $[(\forall x)R(x)]$ by Generalisation—are not P-provable.

The classical interpretation of this is that although $[(\forall x)R(x)]$ is not P-provable, it is true under its standard interpretation by Tarski's definitions.

We argue, however, that by implications which are implicit in Tarski's definitions (see §2.A.), $[R(n)]$ may be viewed alternatively as an expression—definable as algorithmically *verifiable* (see §2., Definition 2)—whose standard interpretation $R^*(n)$ can only be asserted as holding for *any specified* natural number n, but $R^*(n)$ cannot be asserted as holding for *any unspecified* natural number n; i.e., $[R(n)]$ cannot be defined as algorithmically *computable* (see §2., Definition 3).

In other words, we admit the possibility in *evidence-based* reasoning that, for instance (see §4.C.), if the PA formula $[(\forall x)F(x)]$[163] is intended to be read *weakly* under an interpretation as: *'For any specified x, $F^*(x)$ is decidable'*, where the formula $[F(x)]$ interprets as the arithmetical relation $F^*(x)$, then it must be consistently interpreted *weakly* in terms of algorithmic *verifiability*.

We note that, from an *evidence-based* perspective, §17.B., Query 15, is now expressible as:

Query 16. *Are the concepts 'algorithmically verifiable' and 'non-constructive' necessarily synonymous in classical logic and mathematics?*

> **Comment**: We note that if we treat the informal concept of 'constructivity' as expressible formally by 'well-definedness' (see §4.F., Definition 18), then §17.C., Query 16, admits a negative answer from an *evidence-based* perspective of 'effective computability' (see §4.G.a., Definition 21).

17.D.   *Evidence-based* validation of Lucas' and Penrose's Gödelian Theses

We note that §17., Query 14, also raises the question of whether—as implicitly implied by the Gödelian Thesis (§17., Thesis 10)—there is a distinction between our ability to express our *evidence-based* abstractions of natural phenomena precisely, and our *evidence-based* ability to communicate them unequivocally.

We consider, therefore, how *evidence-based* reasoning validates the common, anti-reductionist, core of Lucas' and Penrose's Gödelian Theses.

We reiterate that the introduction of *evidence-based* reasoning into the, conflicting, classical (Hilbert's) and intuitionistic (Brouwer's) interpretations of quantification yields two—hitherto unsuspected and essentially different—well-defined interpretations of the first-order Peano Arithmetic PA, over the structure N of the natural numbers, which are complementary, and not contradictory (see [An15], §1.1 and §2.1).

The former yields the *weak* standard interpretation $\mathcal{I}_{PA(N,SV)}$ of PA over $\mathbb{N}$ (see §2.B.; also the corresponding definition in [Me64], p.107 and p.49), which is *non-finitarily* defined *relative* to the assignment of *weak*, algorithmically *verifiable*, Tarskian truth values (see §2.A.a., Definitions 5 to 9) to the compound formulas of PA under $\mathcal{I}_{PA(N,SV)}$ (see §2.B.a., Theorem 2.7). However, though *well-defined*, the interpretation $\mathcal{I}_{PA(N,SV)}$ does not *well-define* a model of PA (see §2.F., Corollary 2.23).

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[163]For ease of exposition we consider, without loss of generality, only the case of a PA-formula with a single variable.

The latter yields the *strong* finitary interpretation $\mathcal{I}_{PA(N,SC)}$ of PA over N, which is *finitarily* well-defined *relative* to the assignment of *strong*, algorithmically *computable*, Tarskian truth values to the compound formulas of PA under $\mathcal{I}_{PA(N,SC)}$ (see §2.C.a., Theorem 2.15). The *well-definedness* of $\mathcal{I}_{PA(N,SC)}$ follows from the *finitary* proof of consistency for PA in [An16] (see §2.C.a., Theorem 2.16).

The complementarity can also now be viewed as validating the common anti-reductionist core of Lucas' and Penrose's Gödelian Arguments, if we treat the Gödelian Thesis (i.e., §17., Thesis 10) as essentially the claim that (see [An16], Thesis 1):

Theorem 17.1. *There can be no mechanist model of human reasoning if the standard interpretation $\mathcal{I}_{PA(N,SV)}$ of the first-order Peano Arithmetic PA can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions, and the finitary interpretation $\mathcal{I}_{PA(N,SC)}$ of PA can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.*

Proof. We note that Kurt Gödel has shown *meta-mathematically* how to construct an arithmetical formula with a single variable, say $[R(x)]$—Gödel refers to this formula only by its Gödel number r in [Go31] (p.25(12))—such that:

- $[R(x)]$ is not PA-provable; but

- $[R(n)]$ is PA-provable for any *specified* PA numeral $[n]$.

Hence, for any *specified* numeral $[n]$, Gödel's primitive recursive relation $xB\lceil[R(n)]\rceil$ must hold for some algorithmically *computable* natural number m:

- where xBy denotes Gödel's primitive recursive relation (see [Go31], p. 22(45)):

  'x is the Gödel-number of a proof sequence in PA whose last term is the PA formula with Gödel-number y';

- and $\lceil[R(n)]\rceil$ denotes the Gödel-number of $[R(n)]$;

We also note (see §4.G., Theorem 4.2), that we cannot conclude finitarily from Tarski's definitions (see §2.A.) whether, or not, a quantified PA formula $[(\forall x)F(x)]$ is algorithmically *verifiable* as always true, under $I_{PA(N,SV)}$, if $[F(x)]$ is algorithmically *verifiable* (see §2., Definition 2) under $\mathcal{I}_{PA(N,SV)}$, but not algorithmically *computable* (see §2., Definition 3) under $\mathcal{I}_{PA(N,SC)}$. Now:

(i) Since Gödel has shown *meta-mathematically* that the PA-formula $[R(n)]$ is PA-provable for any *specified* PA-numeral $[n]$, it follows that:

    (a) For any *specified* natural number n, there is always a deterministic algorithm which will provide evidence that the interpretation $R^*(n)$ of $[R(n)]$ under $\mathcal{I}_{PA(N,SV)}$ is an algorithmically *verifiable* true arithmetical proposition.

Moreover:

(ii) By §2.F., Corollary 2.20, the formula $[\neg(\forall x)R(x)]$ is provable in PA. Hence, since PA is *finitarily* consistent (see §2.C.a., Theorem 2.16), we can *mathematically* conclude, under $\mathcal{I}_{PA(N,SC)}$, that:

   (a) There is no deterministic algorithm which, for any *specified* numeral $[n]$, will provide evidence that the interpretation $R^*(n)$ of $[R(n)]$ under $\mathcal{I}_{PA(N,SC)}$ is an algorithmically *computable* true arithmetical proposition.

However:

(iii) Since PA is also *non-finitarily* consistent (see §2.B.a., Theorem 2.8), we cannot contradict (i)(a) by *non-finitarily* interpreting quantification under $\mathcal{I}_{PA(N,SV)}$ and *mathematically* concluding from the PA-provability of the formula $[\neg(\forall x)R(x)]$ either that:

   (a) For some *unspecified* natural number n, there is a deterministic algorithm which provides evidence that the interpretation $R^*(n)$ of $[R(n)]$ under $\mathcal{I}_{PA(N,SV)}$ is not an algorithmically *verifiable* true arithmetical proposition.

   or that:

   (b) For some *unspecified* natural number n, there is no deterministic algorithm which will provide evidence that the interpretation $R^*(n)$ of $[R(n)]$ under $\mathcal{I}_{PA(N,SV)}$ is an algorithmically *verifiable* true arithmetical proposition.

Instead:

(iv) By §2.F., Corollary 2.21, we can only conclude, under $\mathcal{I}_{PA(N,SV)}$, that the PA-provability of the formula $[\neg(\forall x)R(x)]$ implies that we cannot *mathematically* conclude from the axioms and rules of inference of PA that:

   (a) For any *specified* natural number n, there is always a deterministic algorithm which will provide evidence that the interpretation $R^*(n)$ of $[R(n)]$ under $\mathcal{I}_{PA(N,SV)}$ is an algorithmically *verifiable* true arithmetical proposition.

If we now assume that the *strong*, finitary, interpretation $\mathcal{I}_{PA(N,SC)}$ of PA (see §2.C.) can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions, whence any mechanical witness can only reason *mathematically*—i.e., *finitarily* from the PA axioms and rules of inference, as in (iv) (see §2.E., Theorem 2.17)—then although, for any *specified* numeral $[n]$, a mechanical witness can give evidence under the *finitary* interpretation $\mathcal{I}_{PA(N,SC)}$ that the PA formula $[R(n)]$ holds in N, no mechanical witness can conclude *finitarily* under the *finitary* interpretation $\mathcal{I}_{PA(N,SC)}$ of PA that, for any *unspecified* numeral $[n]$, the PA formula $[R(n)]$ holds in N since, by §2.F., Corollary 2.21, the formula $[\neg(\forall x)R(x)]$ is provable in PA.

Whereas, if we assume that the *weak*, standard, interpretation $\mathcal{I}_{PA(N,SV)}$ of PA (see §2.B.) can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions—so a human witness can also reason *meta-mathematically*, i.e., *non-finitarily*, from

the PA axioms and rules of inference, as in (i)—then a human witness can conclude under the *non-finitary* standard interpretation $\mathcal{I}_{PA(N,SV)}$ of PA that, for any *unspecified* numeral $[n]$, the PA formula $[R(n)]$ must hold in N.

The theorem follows. □

We conclude this investigation into some significant consequences of the *evidence-based* paradigm introduced in [An16] (see §2.) by noting that §17.D., Theorem 17.1, entails:

Corollary 17.2. *Although no mechanical intelligence can evidence that Gödel's formula $[(\forall x)R(x)]$ is algorithmically verifiable, a human intelligence can evidence that $[(\forall x)R(x)]$ is algorithmically verifiable.*

Proof. Since $[\neg(\forall x)R(x)]$ is provable in PA (see §2.F., Corollary 2.20), it is therefore 'true' under both the interpretations $\mathcal{I}_{PA(N,SV)}$ and $\mathcal{I}_{PA(N,SC)}$ of PA in the following sense:

(a) Under $\mathcal{I}_{PA(N,SV)}$ the formula $[\neg(\forall x)R(x)]$ interprets as:

 − There is no algorithm which will evidence that $[(\forall x)R(x)]$ is algorithmically *verifiable* as a true arithmetical proposition;

(b) Under $\mathcal{I}_{PA(N,SC)}$ the formula $[\neg(\forall x)R(x)]$ interprets as:

 − There is no algorithm which will evidence that $[(\forall x)R(x)]$ is algorithmically *computable* as a true arithmetical proposition.

The corollary follows since Gödel has shown *meta-mathematically* that $[(\forall x)R(x)]$ is algorithmically *verifiable*. □

17.E. Are you a man or a machine: A Definitive Turing Test

In a philosophically prophetic 1950 paper, '*Computing Machinery and Intelligence*' [Tu50], Alan Turing posed, and broadly analysed, the question: 'Can machines think'. He proposed a test—his '*Imitation Game*', subsequently labelled as a Turing Test (TT)—which, Turing argued, could serve as a basis for designing tests that could effectively differentiate between a human being and any mechanical artefact designed to mimic a 'human computer', such as the discrete-state machines defined in Turing's seminal 1936 paper on computable numbers [Tu36].

> 1. *The Imitation Game.*
>
> It is played with three people, a man (A), a woman (B), and an interrogator (C) . . . The object of the game for the interrogator is to determine which of the other two is the man and which is the woman. . . .
>
> We now ask the question, 'What will happen when a machine takes the part of A in this game?'
> . . . *Turing: [Tu50], p.433.*

Turing's intent here seems very clear. He is not interested in distinguishing—either qualitatively or quantitatively—between whether a woman is as, or more, intelligent than a man, or any discrete-state machine; that would be like putting the cart before the horse.

Rather, Turing's interest is in any features—in this case only verbally expressed 'intelligent' responses—that would enable an interrogator to logically differentiate between a man and a woman, or a discrete-state machine and a woman.

> **Comment**: We note the legal definition of 'brain death' has made Turing's query 'Can machines think?' irrelevant. Since brain activity has—albeit under controlled circumstances—been shown capable of predicting human intent to act in response to stimuli before the intended action (see [GDGGR], [GDGGP]), both humans and machines *can* today be defined as 'thinking' in Turing's intended sense—as distinct from being 'conscious', or being 'aware', in a cognitive sense—so long as their respective 'brains' are able to show some measurable processing activity: neuronic (nerve-cell based) in one case, and what might be defined as 'meuronic' (micro-processor based) in the other.

In other words, Turing's implicit intent in [Tu50] can be viewed as seeking to distinguish between the logical reasoning abilities of a human being and an android (i.e., an 'intelligence' whose brain functions are circumscribed by the limitations of Turing's discrete state machines) even if the two—when functioning as intended by evolutionary or by mechanical design, respectively—may be indistinguishable on the basis of physical features (whether external or internal), or social interactions/behaviour.

> **Comment**: We note that by divorcing the 'Turing Test' from Turing's intent when describing the '*Imitation Game*', and conflating ([Mil14], for one, being an exception) a test designed to recognise intelligence/intelligent behaviour with a test designed to recognise differentiable 'intelligences', subsequent deliberations on TT (see, for instance, [Frn00], [SCA00], [Pic03][164], [Coh05], [Gr17]) have come to be considered—not without reason—as amongst the most disputed topics in artificial intelligence, philosophy of mind, and the cognitive sciences.

Reflecting Turing's intent, §17.D., Corollary 17.2, can now be interpreted as yielding a Yes/No Turing Test (Query 17) that seeks to distinguish between a logician, treated as representative of human intelligence, and a Turing machine, treated as representative of mechanical intelligence.

Query 17. (Turing Test) *Can you prove that, for any well-defined numeral* $[n]$*, Gödel's arithmetic formula* $[R(n)]$ *is a theorem in the first-order Peano Arithmetic PA, where* $[R(x)]$ *is defined by its Gödel number* r *in eqn.12, and* $[(\forall x)R(x)]$ *is defined by its Gödel number* $17Gen\ r$ *in eqn.13, on p.25 of [Go31]? Answer only either 'Yes' or 'No'.*

> *Logician*: Yes.

> *Reason*: By Gödel's *meta-mathematical* reasoning on p.26(2) of [Go31], a logician can conclude that, if a numeral $[n]$ is well-defined, then the formula $[R(n)]$ is a theorem in PA; even though the formula $[(\forall x)R(x)]$ is *not* a theorem in PA.

> *Turing Machine*: No.

> *Reason*: By §2.F., Corollary 2.20, the formula $[\neg(\forall x)R(x)]$ is provable in PA and so, by the Provability Theorem for PA (§2.E., Theorem 2.17), *no* Turing machine can *prove* that the formula $[(\forall x)R(x)]$ with Gödel number $17Gen\ r$ is a theorem in PA and, ipso facto, conclude that, for any well-definable numeral $[n]$, Gödel's arithmetic formula $[R(n)]$ is a theorem in PA.

We note that, as remarked by Marcin Miłkowski, conventional wisdom—apropos differentiating between a human mind and a TM—mostly seeks to compare their respective abilities to *compute* number-theoretic functions by positing various *computational models* of the mind:

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[164] "For Turing, the Turing test was not an "operational definition of 'thinking' or 'intelligence' or 'consciousness'" (as sometimes maintained, e.g. by Hodges, 1983, p. 415)—the test only gave a sufficient condition for a machine to be considered intelligent, or thinking (Turing, 1950, p. 435). "Intelligence" and "thinking" were used interchangeably by Turing." . . . *Piccinini*: [Pic03]

"2. Two senses of "is", and some senses of "mind"

In philosophy of mathematics, and in most early discussions in philosophy of mind, only one sense of the claim that the mind is (or is not) a TM was taken into account. Namely, it was held that all that is required for the mind to be a TM is that the set of functions computable by the mind is exactly the same as the set of TM-computable functions. For this reason, it is more apt to talk of *computable function equivalence* rather than of strict identity. Note also that computable function equivalence does not imply that the mind is an abstract entity, which would be implied by the literal strict identity claim that the mind is a TM.

It is the computable function equivalence that is presupposed in the discussions over the arguments in favour or against the claim that in contradistinction to the TM, the human mind is capable of, say, ascertaining the truth of Gödel sentences, as argued by Lucas [23]. In other words, what is relevant in this discussion is whether mathematicians are able to compute all TM-computable functions, and whether the TM is capable of computing of all human-computable functions.

Interestingly enough, the equivalence of computable functions was also presupposed in early functionalist discussions in the philosophy of mind [40]. Let us suppose that there exists a computational model M that explains the operation of the human mind H. If M is equivalent with respect to the functions computed by H, it is weakly equivalent in the sense defined by Jerry Fodor [16]: namely, input / output relationships of M will be isomorphic to input / output relationships of H. Computable functions equivalence is not enough to make sense of the third objection I mentioned at the outset. Namely, the objection that focuses on the architecture of the human as dramatically different from that of a TM would obviously miss the point, as computable function equivalent machines simply need not be architecturally equivalent.[1] To understand the last objection, which cannot be dismissed as simply nonsensical, we need a stronger rendering of "is" in the thesis that the human mind is a TM.

Following Fodor again, we can talk of strong equivalence: the model M is strongly equivalent to its target T if and only if (a) M is weakly-equivalent to T; and (b) the way the input is transformed into the output is the same in M and T. The second criterion is admittedly quite vague and the distinction proposed by Fodor was immediately criticized by philosophers who tried to show that strong equivalence collapses to weak equivalence [44]. Instead of discussing their criticism, in what follows, I will propose another way of understanding strong equivalence, which is in line with the current theory of mechanistic explanation that has gained considerable attention in the philosophy of science [24]. I will namely define strong equivalence as the equivalence of the computational models being implemented, where implementation is rendered in a mechanistic way. This the focus of section 3 below.

The mechanistic sense of "is", i.e., the implementation equivalence, is what is presupposed in the architecture objection.[2] It is important to note that only strongly-equivalent models are explanatory of the human mental processes in a way relevant for cognitive science. All that the weak equivalence requires is that the set of functions is the same, and for that reason, the weakly-equivalent computational model is not explanatory of some of the psychological properties of the human mind. It may be used to describe some part of the functioning of the human mind but it cannot be used to predict some of its activity that is of interest for cognitive research."

... Miłkowski: [Mil14], §2. Two senses of "is", and some senses of "mind".

From the *evidence-based* perspective of this investigation, however, Query 17 suggests that what we should treat as relevant in the above discussion is:

- whether mathematicians are able to correctly *identify* the *algorithmically computable* truth values assigned by a TM to arithmetical sentences, and whether the TM is capable of correctly *identifying* the *algorithmically verifiable* truth values assigned by a human mind to arithmetical sentences;

and not whether or not the respective abilities of a human mind and a TM to *compute* number-theoretic functions are equivalent.

17.E.a.   Does Query 17 yield a definitive Turing Test?

The question arises:

Query 18. *Does Query 17 yield a definitive Turing Test?*

The issue here is that, in order to establish the above test as definitive, we might need to make allowance for the fact that:

— a Turing machine can verify the PA-provability of the formula $[R(n)]$ for any *specified* PA-numeral in a finite time by means of Gödel's primitive recursive formula xBy ([Go31], Definition 45, p.22);

— even though it cannot do so for all the PA-numerals in a finite time (since $[R(x)]$ is not PA-provable).

The underlying problem was treated as an unsurmountable barrier by Lucas:

> "If the only thing that will budge the Mechanists is a rule-governed inference which cannot be resisted on pain of inconsistency, then they cannot be made to see the general applicability of Gödelian arguments. All that can be done is to refute each and every particular claim they put forward."
>
> . . . *Lucas*: [Lu96].

We entertain the possibility, however, that §17.E., Query 17, might be capable of a refinement that avoids semantic ambiguities by forcing the response 'No' formally from the Turing Machine in a finite time.

We thus address the question:

Query 19. *Can a Turing machine be forced to reply in a finite time whether or not a partial recursive function $F(x_1, \ldots, x_n)$ is effectively computable as total?*

17.F.   Turing's Halting problem

We note that classical theory holds:

(a) Every Turing-computable function $F(x_1, \ldots, x_n)$ is partial recursive, and, if $F(x_1, \ldots, x_n)$ is total, then $F(x_1, \ldots, x_n)$ is recursive (see [Me64], p.233, Corollary 5.13).

(b) Every partial recursive function $F(x_1, \ldots, x_n)$ is Turing-computable (see [Me64], p.237, Corollary 5.15).

From this, classical theory concludes that the following, essentially unverifiable (since it treats the notion of 'effective computability' as intuitive, and not definable formally) but refutable, theses (informally referred to as the Church-Turing Thesis CT) are equivalent (see [Me64], p.237):

Church's Thesis: A number-theoretic function $F(x_1, \ldots, x_n)$ is effectively computable if, and only if, $F(x_1, \ldots, x_n)$ is recursive (see [Me64], p.227).

Turing's Thesis: A number-theoretic function $F(x_1, \ldots, x_n)$ is effectively computable if, and only if, $F(x_1, \ldots, x_n)$ is Turing-computable (see [BBJ03], p.33).

We note however that, even classically, the above equivalence does not hold strictly, and needs further qualification. The following argument highlights this, where $F(x_1, \ldots, x_n)$ is any number-theoretic function:

(i) Assume Church's Thesis. Then:

- If $F(x_1, \ldots, x_n)$ is Turing-computable then it is partial recursive. *If $F(x_1, \ldots, x_n)$ is total*, then it is both recursive (see [Me64], p.227) and, by our assumption, effectively computable.

- If $F(x_1, \ldots, x_n)$ is effectively computable then, by our assumption, it is recursive. Hence, by definition, $F(x_1, \ldots, x_n)$ is partial recursive and Turing-computable.

(ii) Assume Turing's Thesis. Then:

- If $F(x_1, \ldots, x_n)$ is recursive, it is partial recursive and Turing-computable. Hence, by our assumption, $F(x_1, \ldots, x_n)$ is effectively computable.

- If $F(x_1, \ldots, x_n)$ is effectively computable then, by our assumption, it is Turing-computable. Hence it is partial recursive and, *if $F(x_1, \ldots, x_n)$ is total*, then it is recursive.

The question arises:

Query 20. (Turing's Halting problem) *Is it always decidable by a Turing machine whether or not a specified partial recursive function $F(x_1, \ldots, x_n)$ is total?*

Turing addressed this issue in his seminal paper on computable numbers (see [Tu36]), where he showed ([Tu36], pp.133-134) by contradiction that the Halting problem (§17.F., Query 20) is unsolvable by a Turing machine, in the sense that whether or not a partial recursive function is total is not always decidable by a Turing machine:

> "Let us suppose that there is such a process; that is to say, that we can invent a machine $\mathcal{D}$ which, when supplied with the S.D of any computing machine $\mathcal{M}$ will test this S.D and if $\mathcal{M}$ is circular will mark the S.D with the symbol "u" and if it is circle-free will mark it with "s". By combining the machines $\mathcal{D}$ and $\mathcal{U}$ we could construct a machine $\mathcal{H}$ to compute the sequence β'. The machine $\mathcal{D}$ may require a tape. We may suppose that it uses the E-squares beyond all symbols on F-squares, and that when it has reached its verdict all the rough work done by $\mathcal{D}$ is erased.
>
> The machine $\mathcal{H}$ has its motion divided into sections. In the first $N - 1$ sections, among other things, the integers 1, 2, $\ldots$, $N - 1$ have been written down and tested by the machine $\mathcal{D}$. A certain number, say $R(N - 1)$, of them have been found to be the D.N's of circle-free machines. In the N-th section the machine $\mathcal{D}$ tests the number N. If N is satisfactory, *i.e.*, if it is the D.N of a circle-free machine, then $R(N) = 1 + R(N - 1)$ and the first $R(N)$ figures of the sequence of which a D.N is N are calculated. The $R(N)$-th figure of this sequence is written down as one of the figures of the sequence β' computed by $\mathcal{H}$. If N is not satisfactory, then $R(N) = R(N - 1)$ and the machine goes on to the $(N + 1)$-th section of its motion.
>
> From the construction of $\mathcal{H}$ we can see that $\mathcal{H}$ is circle-free. Each section of the motion of $\mathcal{H}$ comes to an end after a finite number of steps. For, by our assumption about $\mathcal{D}$, the decision as to whether N is satisfactory is reached in a finite number of steps. If N is not satisfactory, then the N-th section is finished. If N is satisfactory, this means that the machine $\mathcal{M}(N)$ whose D.N is N is circle-free, and therefore its $R(N)$-th figure can be calculated in a finite number of steps. When

this figure has been calculated and written down as the $R(N)$-th figure of β', the N-th section is finished. Hence $\mathcal{H}$ is circle-free.

Now let K be the D.N of $\mathcal{H}$. What does $\mathcal{H}$ do in the K-th section of its motion? It must test whether K is satisfactory, giving a verdict "s" or "u". Since K is the D.N of $\mathcal{H}$ and since $\mathcal{H}$ is circle-free, the verdict cannot be "u". On the other hand the verdict cannot be "s". For if it were, then in the K-th section of its motion $\mathcal{H}$ would be bound to compute the first $R(K-1)+1 = R(K)$ figures of the sequence computed by $\mathcal{H}$. The computation of the first $R(K-1)$ figures would be carried out all right, but the instructions for calculating the $R(K)$-th would amount to "calculate the first $R(K)$ figures computed by $[\mathcal{H}]$ and write down the $R(K)$-th". This $R(K)$-th figure would never be found. *I.e.*, $\mathcal{H}$ is circular, contrary both to what we have found in the last paragraph and to the verdict "s". Thus both verdicts are impossible and we conclude that there can be no machine $\mathcal{D}$."

... *Turing:* [Tu36]*, p.133.*

"We can show further that *there can be no machine $\mathcal{E}$ which, when supplied with the S.D of an arbitrary machine $\mathcal{M}$, will determine whether $\mathcal{M}$ ever prints a given symbol* (0 *say*)."

... *Turing:* [Tu36]*, p.134.*

Since a number-theoretic function is Turing-computable if, and only if, it is partially Markov-computable (see [Me64], p.233, Corollary 5.13 & p.237, Corollary 5.15), Turing's argument for his Halting Theorem asserts that it is essentially unverifiable algorithmically whether, or not, a Turing machine that computes a *specified n*-ary number-theoretic function $F(x_1, \ldots, x_n)$ will always yield a computation for every *specified n*-ary sequence of natural numbers (for which it is defined) as input.

In other words, there may be some n-ary number-theoretic function $F(x_1, \ldots, x_n)$ such that, for some *unspecified n*-ary sequence of natural numbers as input, any Turing machine that computes $F(x_1, \ldots, x_n)$ will go into a non-terminating loop (which would cause the circular motion of the putative machine $\mathcal{H}$ in the K-th section in the previous quote from [Tu36], p.133), where:

Definition 32. (Non-terminating loop) *A non-terminating loop is any repetition of the instantaneous tape description of a Turing machine during a computation.*

"An instantaneous tape description describes the condition of the machine and the tape at a given moment. When read from left to right, the tape symbols in the description represent the symbols on the tape at the moment. The internal state q_s in the description is the internal state of the machine at the moment, and the tape symbol occurring immediately to the right of q_s in the tape description represents the symbol being scanned by the machine at the moment."

... *Mendelson:* ([Me64]*, p.230, footnote 1).*

In other words, a non-terminating loop can be described as a 'consecutive repeating state cycle in a Turing machine':

"A consecutive repeating state cycle in a Turing machine occurs when a finite sequence of standard machine instructions $\{I_i\}$ is executed by the Turing machine two consecutive times: $I_1 \rightarrow I_2 \rightarrow \ldots I_k \rightarrow I_1 \rightarrow I_2 \ldots I_k$ and the machine configuration before the first instruction I_1 is executed equals the machine configuration after the instruction I_k has completed its execution a second time."

... *Fiske:* [Fi19]*, ¶ Two Research Problems.*

17.F.a.   The Halting-decidability problem for T

We shall now show why—challenging current paradigms—the Halting problem (§17.F., Query 20) cannot be conflated with (see [Me64], p.256) the problem of whether or not it is *effectively* decidable that a *specified* computation of a partial recursive function $F(x_1, \ldots, x_n)$ is *effectively* decidable:

Query 21. (Halting-decidability problem for T) *Given a Turing machine T, can one effectively decide, given any instantaneous description α, whether or not there is a computation of T beginning with α?*

which can also be expressed as:

Query 22. (Halting-decidability problem for T) *Is it always possible to effectively determine whether a Turing machine T will halt or not when computing a specified partial recursive function $F(x_1, \ldots, x_n)$?*

We shall now answer Query 22 affirmatively (see §17.F.b., Theorem 17.3) by showing—as a consequence of the Provability Theorem for PA (§2.E., Theorem 2.17)—that whether or not a Turing machine will always halt when computing a partial recursive function $F(x_1, \ldots, x_n)$ *is* effectively decidable by a trio $(T_1 \; // \; T_2 \; // \; T_3)$ of Turing machines operating in parallel.

Now, we note that any Turing machine T can be provided with an auxiliary infinite tape (see [Rg87], p.130) to effectively recognise a non-terminating looping situation at the register, i.e. tape description, level; it simply records every instantaneous tape description at the execution of each machine instruction on the auxiliary tape, and compares the current instantaneous tape description with the record.

Moreover, T can be meta-programmed to abort the impending non-terminating loop *if* an instantaneous tape description is repeated, and to return a meta-symbol indicating self-termination.

> **Comment**: It is convenient to visualise the tape of such a Turing machine as that of a two-dimensional virtual-teleprinter, which maintains a copy of every instantaneous tape description in a random-access memory during a computation.

17.F.b.   Forcing a Turing machine to decide effective computability

It now follows from the Provability Theorem for PA (§2.E., Theorem 2.17) that:

Theorem 17.3. (Forced Halting Theorem) *It is always possible to effectively determine whether a Turing machine will halt or not when computing any partial recursive function $F(x_1, \ldots, x_n)$.*

Proof. We assume that the partial recursive function $F(x_1, \ldots, x_n)$ is obtained from a recursive function $G(x_1, \ldots, x_n)$ by means of the unrestricted μ-operator[165]; in other words, that (see [Me64], p.214):

$$F(x_1, \ldots, x_n) = \mu y(G(x_1, \ldots, x_n, y) = 0).$$

---

[165]Where 'μy' interprets as 'The least y such that $\ldots$'.

If $[H(x_1, \ldots, x_n, y)]$ expresses $\neg(G(x_1, \ldots, x_n, y) = 0)$ in PA we have, by definition, that any interpretation $H^*(x_1, \ldots, x_n, y)$ of $[H(x_1, \ldots, x_n, y)]$ in $\mathbb{N}$ is instantiationally equivalent to $\neg(G(x_1, \ldots, x_n, y) = 0)$ (see [Me64], §2, p.117).

We now consider the PA-provability and Turing computability of the arithmetical formula $[H(x_1, \ldots, x_n, y)]$ by a Turing machine T that inputs every *specified* sequence of numerals $\{[a_1], \ldots, [a_n]\}$ of PA simultaneously into the parallel trio $(T_1 \; // \; T_2 \; // \; T_3)$ of Turing machines, as below:

(a) Let Q_1 be the meta-assertion that the PA-formula $[H(a_1, \ldots, a_n, y)]$ is not algorithmically *verifiable* as always true under interpretation in $\mathbb{N}$.

It follows that there is some *unspecified* finite k such that $H^*(a_1, \ldots, a_n, k)$ does not hold in $\mathbb{N}$; and so $G(a_1, \ldots, a_n, k)$ holds.

Since $G(a_1, \ldots, a_n, y)$ is recursive, any Turing machine T_1 that computes $G(a_1, \ldots, a_n, y)$ will halt and return the value 0 at $y = k$.

(b) Let Q_2 be the meta-assertion that the PA-formula $[H(a_1, \ldots, a_n, y)]$ is algorithmically *verifiable* as always true, but not algorithmically *computable* as always true, under interpretation in $\mathbb{N}$.

Hence, for any *specified* $[k]$, the formula $[H(a_1, \ldots, a_n, k)]$ interprets as true in $\mathbb{N}$, but there is no Turing machine that, for any *specified* $[k]$, computes the formula $[H(a_1, \ldots, a_n, k)]$ as 'true' under interpretation in $\mathbb{N}$.

Now it follows from the Provability Theorem for PA (§2.E., Theorem 2.17) that, since every *specified* instantiation $[H(a_1, \ldots, a_n, k)]$ of the PA-formula $[H(a_1, \ldots, a_n, y)]$ is algorithmically *computable* as true under interpretation in $\mathbb{N}$, the formula $[H(a_1, \ldots, a_n, k)]$ is PA-provable.

However, since $[H(a_1, \ldots, a_n, y)]$ is not algorithmically *computable* as always true under interpretation in $\mathbb{N}$, any Turing machine T_2 that computes the value of $[y]$ at which $[H(a_1, \ldots, a_n, y)]$ is true cannot return the value 'true' for all *specified* values of $[y]$.

Hence T_2 must necessarily initiate a non-terminating loop at some *unspecified* $[y = k']$ and halt, since its auxiliary tape will return the symbol for self-termination at $[y = k']$.

(c) Finally, let Q_3 be the meta-assertion that the PA-formula $[H(a_1, \ldots, a_n, y)]$ is algorithmically *computable* as always true under interpretation in $\mathbb{N}$.

Hence the Turing machine T_2 will return the value 'true' on any *specified* input for $[y]$.

By the Provability Theorem for PA (§2.E., Theorem 2.17), the formula $[H(a_1, \ldots, a_n, y)]$ is therefore PA-provable.

Let h be the Gödel-number of $[H(a_1, \ldots, a_n, y)]$. We consider, then, Gödel's primitive recursive number-theoretic relation xBy (see [Go31], p.22, definition 45), which holds if, and only if, x is the Gödel-number of a proof sequence in PA for the PA-formula whose Gödel-number is y. It follows that there is some *unspecified* finite k'' such that any Turing

machine T_3, which computes the characteristic function of xBh, will halt and return the value 0 ('true') for $x = k''$.

Since Q_1, Q_2 and Q_3 are mutually exclusive and exhaustive, it follows that, when run simultaneously over the sequence $1, 2, 3, \ldots$ of values for y, one of the parallel trio $(T_1 \; // \; T_2 \; // \; T_3)$ of Turing machines will always halt for some *unspecified* finite value of y. Moreover:

- If T_1 halts, then a Turing machine will halt when computing the partial recursive function $F(x_1, \ldots, x_n)$, and T_1 will report that $F(x_1, \ldots, x_n)$ is algorithmically *decidable* as *not* a total function.

- If T_2 halts, then a Turing machine will not halt when computing the partial recursive function $F(x_1, \ldots, x_n)$, and T_2 will report that it is not algorithmically *decidable* whether or not $F(x_1, \ldots, x_n)$ is a total function.

  > **Comment**: We note that a human intelligence would, however, *know* that $F(x_1, \ldots, x_n)$ is algorithmically *verifiable* as total by specification in this case, but not algorithmically *computable*—hence not algorithmically *decidable* as total.

- If T_3 halts, then a Turing machine will not halt when computing the partial recursive function $F(x_1, \ldots, x_n)$, and T_3 will report that $F(x_1, \ldots, x_n)$ is algorithmically *decidable* as a total function.

The theorem follows. □

We note that Theorem 17.3 immediately yields a more insightful proof of Turing's Halting Theorem by answering §17.F.a., Query 21 (Halting-decidability problem for T) affirmatively:

Corollary 17.4. (Halting-decidability Theorem for T) *Given a Turing machine T, one can effectively decide, given any instantaneous description α, whether or not there is a computation of T beginning with α.* □

In other words, treating $(T_1 \; // \; T_2 \; // \; T_3)$ as an Oracle machine, we now have an *evidence-based* proof of Turing's Halting Theorem (see §17.F., Query 20):

Corollary 17.5. (Turing's Halting Theorem) *Whether or not a specified partial recursive function $F(x_1, \ldots, x_n)$ is total is not always decidable by a Turing machine.* □

17.F.c. Why some algorithically *verifiable* functions are *essentially* uncomputable

We note that the Forced Halting Theorem (§17.F.b., Theorem 17.3) entails that some algorithically *verifiable* functions are *essentially* uncomputable (even by what are termed as 'quantum' computers).

Theorem 17.6. (Essential Uncomputability Theorem) *Some functions that are algorithmically verifiable, but not algorithmically computable, are essentially uncomputable.*

Proof. We note that:

(i) By Turing's Halting Theorem (§17.F.b., Corollary 17.5), there is some partial recursive function, say $F(x_1, \ldots, x_n) = \mu y(G(x_1, \ldots, x_n, y) = 0)$, *all* of whose instantiations are *not* algorithmically *computable* by any Turing machine.

By the Forced Halting Theorem (§17.F.b., Theorem 17.3), we can define a trio $(T_1 \;//\; T_2 \;//\; T_3)$ of Turing machines such that:

- If T_1 halts, then a Turing machine will halt when computing the partial recursive function $F(x_1, \ldots, x_n)$, and T_1 will report that $F(x_1, \ldots, x_n)$ is algorithmically *decidable* as *not* a total function.

- If T_2 halts, then a Turing machine will not halt when computing the partial recursive function $F(x_1, \ldots, x_n)$, and T_2 will report that it is not algorithmically *decidable* whether or not $F(x_1, \ldots, x_n)$ is a total function.

- If T_3 halts, then a Turing machine will not halt when computing the partial recursive function $F(x_1, \ldots, x_n)$, and T_3 will report that $F(x_1, \ldots, x_n)$ is algorithmically *decidable* as a total function.

(ii) We consider the case where $F(x_1, \ldots, x_n)$ is algorithmically *verifiable*, but *not* algorithmically *computable*, as total; whence only T_2 will halt.

(iii) Now, by [Go31], Theorem VII, we can define a Turing machine T_3 that monitors some primitive recursive Gödel β-function (see §12.A.), say $F_3(x_1, \ldots, x_n)$, whose values are identical to those of $F(x_1, \ldots, x_n)$ *before* T_2 reports its first halting state.

(iv) Since the values of $F(x_1, \ldots, x_n)$ are algorithmically *verifiable*, it is total, and we assume that $F_3(x_1, \ldots, x_n)$ can also correctly compute some values of $F(x_1, \ldots, x_n)$ *beyond* the first halting state of T_2.

(v) However, since the values of $F(x_1, \ldots, x_n)$ are *not* algorithmically *computable*, it follows that T_3 too must halt for some input beyond the halting point of T_2 and report that $F(x_1, \ldots, x_n)$ is not computable by $F_3(x_1, \ldots, x_n)$ at that input; at which point we can consider extending T_3 by some T_4, just as we replaced T_2 with T_3.

(vi) Since the values of $F(x_1, \ldots, x_n)$ are not algorithmically *computable*, it follows that the above path—which would correspond to a putative, algorithmically *verifiable*, computation of $F(x_1, \ldots, x_n)$—is *essentially* non-terminating, since it cannot terminate at some node which defines a Gödel β-function whose values can be *claimed* (compare with the argument in §4.H.a., Theorem 4.5) as *always* identical to those of $F(x_1, \ldots, x_n)$ *beyond* that node.

The theorem follows.   □

17.G.   On the Collatz conjecture and similar, open, arithmetical problems

The practical significance[166] of the distinction between Turing's Halting Theorem (§17.F.b., Corollary 17.5) and the Halting-decidability Theorem for T (§17.F.b., Corollary 17.4) is seen

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[166]For the computer sciences in general, and the development of self-modified learning by artificial intelligences in particular (see, for instance, [AAB19] from the perspective of §21.).

in the paper presented (see [Fi19a]) at the Logic Colloquium 2019, where Michael Stephen Fiske essentially seeks an answer to the Halting-decidability Problem for T (see §17.F.a., Query 21/Query 22) by considering a class of *evolving* 'quantum, self-modifiable machines':

> "Among the fundamental questions in computer science, at least two have a deep impact on mathematics. What can computation compute? How many steps does a computation require to solve an instance of the 3-SAT problem? Our work addresses the first question, by introducing a new model called an *ex-machine*. The ex-machine executes Turing machine instructions and two special types of instructions. *Quantum random instructions* are physically realizable with a quantum random number generator. *Meta instructions* can add new states and add new instructions to the ex-machine. A countable set of ex-machines is constructed, each with a finite number of states and instructions; each machine can compute a Turing incomputable language, whenever the quantum randomness measurements behave like unbiased Bernoulli trials. In 1963, Alan Turing posed the halting problem for Turing machines and proved that his problem is unsolvable for Turing machines. Consider an enumeration $\mathcal{E}_\alpha(i) = (\mathcal{M}_i, T_i)$ of all Turing machines $\mathcal{M}_i$ and initial tapes T_i. Does there exist an ex-machine $\mathcal{X}$ that has at least one evolutionary path $\mathcal{X} \to \mathcal{X}_1 \to \mathcal{X}_2 \to \dots \mathcal{X}_m$, so at the mth stage ex-machine $\mathcal{X}_m$ can correctly determine for $0 \le i \le m$ whether $\mathcal{M}_i$'s execution on tape T_i eventually halts?"
>
> *. . . Fiske: [Fi19], Abstract.*

Fiske begins by introducing what he terms as a '*quantum random, self-modifiable machine*':

> "**2 The Ex-Machine**
>
> We introduce a *quantum random, self-modifiable machine* that adds two special instructions to the Turing machine [97]. Before the quantum random and meta instructions are defined, we present some preliminary notation, the standard instructions, and a Collatz machine example.
>
> $\mathbb{Z}$ denotes the integers. $\mathbb{N}$ and $\mathbb{N}^+$ are the non-negative and positive integers, respectively. The finite set $Q = \{0, 1, 2, \dots, n-1\}\} \subset \mathbb{N}$ represents the ex-machine states. This representation of the ex-machine states helps specify how new states are added to Q when a meta instruction is executed. Let $\mathfrak{U} = \{a_1, \dots, a_n\}$, where each a_i represents a distinct symbol. The set $A = \{\texttt{0, 1, \#}\} \cup \mathfrak{U}$ consists of alphabet (tape) symbols, where # is the blank symbol and $\{\texttt{0, 1, \#}\} \cap \mathfrak{U}$ is the empty set. In some ex-machines, $A = \{\texttt{0, 1, \#, Y, N, a}\}$, where $a_i = \texttt{Y}, a_2 = \texttt{N}, a_3 = \texttt{a}$. In some ex-machines, $A = \{\texttt{0, 1, \#}\}$, where $\mathfrak{U}$ is the empty set. The alphabet symbols are read from and written on the tape. The ex-machine tape T is a function of $T : \mathbb{Z} \to A$ with an initial condition: before the ex-machine starts executing, there exists an $N > 0$ so that $T(k) = \texttt{\#}$ when $|k| > N$. In other words, before the ex-machine starts executing, all tape squares contain blank symbols, except for a finite number of tape squares. When this initial condition holds for tape T, we say that tape T is *finitely bounded*.
>
> **2.1 Standard Instructions**
>
> **Definition 2.1.** Execution of Standard Instructions
>
> The standard ex-machine instructions $\mathcal{S}$ satisfy $\mathcal{S} \subset Q \times A \times Q \times A \times \{-1, 0, 1\}$ and a uniqueness condition: If $(q_1, \alpha_1, r_1, a_1, y_1) \in \mathcal{S}$ and $(q_2, \alpha_2, r_2, a_2, y_2) \in \mathcal{S}$ and $(q_!, \alpha_1, r_1, a_1, y_1) \ne (q_2, \alpha_2, r_2, a_2, y_2)$, then $q_1 \ne q_2$ or $\alpha_1 \ne \alpha_2$. A standard instruction $I = (q, a, r, \alpha, y)$ is similar to a Turing machine tuple [30, 75, 97]. When the ex-machine is in state q and the tape head is scanning alphabet symbol $a = T(k)$ at tape square k, instruction I is executed as follows:
>
> - The ex-machine state moves from state q to state r.
> - The ex-machine replaces alphabet symbol a with alphabet symbol α so that $T(k) = \alpha$. The rest of the tape remains unchanged.
> - If $y = -1$, the ex-machine moves its tape head one square to the left on the tape and is subsequently scanning the alphabet symbol $T(k-1)$ in tape square $k-1$.

- If $y = +1$, the ex-machine moves its tape head one square to the right on the tape and is subsequently scanning the alphabet symbol $T(k + 1)$ in tape square $k + 1$.

- If $y = 0$, the ex-machine does not move its tape head and is subsequently scanning the alphabet symbol $T(k) = \alpha$ in tape square k.

Remark 2.1. A Turing machine [97] has a finite set of states Q, a finite alphabet A, a finitely bounded tape, and a finite set of standard ex-machine [sic] instructions that are executed according to definition 2.1. In other words,, an ex-machine that uses only standard instructions is computationally equivalent to a Turing machine. Hence, an ex-machine with only standard instructions will be called a standard machine or a Turing machine."

. . . Fiske: [Fi19], 2.1 Standard Instructions.

Fiske then defines the Collatz function, Collatz conjecture, and a Collatz (Turing) machine that computes the Collatz 'orbit' (sequence):

"The Collatz conjecture has an interesting relationship to Turing's halting problem, which will be discussed further in section 7. Furthermore, there is a generalisation of the Collatz function that is unsolvable foe a standard machine [25].

Definition 2.2 Collatz Conjecture

Define the Collatz function $f : \mathbb{N}^+ \to \mathbb{N}^+$, where $f(n) = \frac{n}{2}$ when n is even and $f(n) = 3n + 1$ when n is odd. Zero iterations of f is $f^0(n) = n$. k iterations of f is represented as $f^k(n)$. The orbit of n with respect to f is $\mathcal{O}(f, n) = \{f^k(n) : k \in \mathbb{N}\}$. Observe that $f(5) = 16, f^2 = 8, f^3(5) = 4, f^4(5) = 2, f^5(5) = 1$, so $\mathcal{O}(f, 5) = \{5, 16, 8, 4, 2, 1\}$. The Collatz conjecture states that for any positive integer n, $\mathcal{O}(f, n)$ contains 1."

. . . Fiske: [Fi19], 2.1 Standard Instructions.

Fiske further defines a set of *Machine Instructions* 1 for a Collatz (Turing) machine that, for each specified n, computes the orbit $\mathcal{O}(f, n)$:

"We specify a Turing machine that for each n computes the orbit $\mathcal{O}(f, n)$. The standard machine halts if the orbit $\mathcal{O}(f, n)$ contains 1. Set $A = \{$`0`, `1`, `#`, `E`$\}$. Set $Q = \{$`a`, `b`, `c`, `d`, `e`, `f`, `g`, `h`, `i`, `j`, `k`, `l`, `m`, `n`, `p`, `q`$\}$ where `a = 0`, `b = 1`, `c = 2`, `...`, `n = 13`, `p = 14`, and `q = 15`.

Machine instructions 1 shows a list of standard instructions that compute $\mathcal{O}(f, n)$. The initial tape is `#` `#1`^{n}`#`, where it is understood that the remaining tape squares, beyond the leftmost `#` and rightmost `#`, contain only blank symbols. The space means the tape head is scanning the `#` adjacent to the leftmost 1 [sic]. The initial state is `q`.

Machine Instructions 1. Collatz Machine

. . .

With input `#` `#1`^{n}`#`, the execution of the Collatz machine halts (i.e., moves to the halting state h) if the orbit $\mathcal{O}(f, n)$ reaches 1. Below shows the Collatz machine executing the first ten instructions with initial tape `#` `#11111#` and initial state `q`. Each row shows the current tape and machine state after the instruction in that row has been executed. The complete xecution of the Collatz machine is shown in the appendix 8. It computes $\mathcal{O}(f, 5)$."

. . . Fiske: [Fi19], 2.1 Standard Instructions.

Fiske then introduces his concept of '*quantum random instructions*', by identifying select machine states that appeal to an oracle (as in the case of his *Random Walk* ex-machine) for an essentially unpredictable value—from the computing ex-machine's perspective—that is to be substituted in the currently scanned cell:

"**2.2 Quantum Random Instructions**

Repeated trials are called *quantum random Bernoulli trials* [37] if there are only two possible outcomes for each trial (i.e., quantum random measurement) and the probability of each outcome remains constant for all trials. *Unbiased* means the probability of both outcomes is the same. Below are the formal definitions.

Axiom 1. *Unbiased Trials.*

Consider the bit sequence $(x_1 x_2 \ldots)$ in the infinite product space $\{0, 1\}^{\mathbb{N}}$. A single outcome x_i of a bit sequence $(x_1 x_2 \ldots)$ generated by quantum randomness is unbiased. The probability of measuring a 0 or a 1 are equal: $P(x_1 = 1) = p(x_i = 0) = \frac{1}{2}$.

Axiom 2. *Stochastic Independence.*

History has no effect on the next quantum random measurement. Each outcome x_i is independent of the history. No correlation exists between previous or future outcomes. This is expressed in terms of the conditional probabilities: $P(x_i = 1 \mid x_1 = b_1, \ldots, x_{i-1} = b_{i-1}) = \frac{1}{2}$ and $P(x_i = 0 \mid x_1 = b_1, \ldots, x_{i-1} = b_{i-1}) = \frac{1}{2}$ for each $b_i \in \{0, 1\}$.

...

The quantum random instructions $\mathcal{R}$ are subsets of $Q \times A \times Q \times \{-1, 0, 1\} = \{(q, a, r, y) : q, r \text{ are in } Q \text{ and } a \text{ in } A \text{ and } y \text{ in } \{-1, 0, 1\}\}$ that satisfy a uniqueness condition defined below.

Definition 2.3. Execution of Quantum Random Instructions

The quantum random instructions $\mathcal{R}$ satisfy $\mathcal{R} \subset Q \times A \times Q \times \{-1, 0, 1\}$ and the following uniqueness condition: If $(q_1, \alpha_1, r_1, y_1) \in \mathcal{R}$ and $(q_2, \alpha_2, r_2, y_2) \in \mathcal{R}$ and $(q_1, \alpha_1, r_1, y_1) \neq (q_2, \alpha_2, r_2, y_2)$, then $q_1 \neq q_2$ or $\alpha_1 \neq \alpha_2$. When the tape head is scanning alphabet symbol a and the ex-machine is in state q, the quantum random instruction (q, a, r, y) executes as follows:

- The ex-machine measures a quantum random source that returns a random bit $b \in \{0, 1\}$. (It is assumed that the quantum measurements satisfy unbiased Bernoulli trial axioms 1 and 2.)

- On the tape, alphabet symbol a is replaced with random bit b. (This is why A always contains both symbols 0 and 1.)

- The ex-machine state changes to state r.

- The ex-machine moves its tape head left if $y = -1$, right if $y = +1$, or the tape head does not move if $y = 0$.

Machine instructions 2 lists a random walk machine that has only standard instructions and quantum random instructions. Alphabet $A = \{0, 1, \#, E\}$. The states are $Q = \{0, 1, 2, 3, 4, 5, 6, h\}$, where the halting state $h = 7$. A valid initial tape contains only blank symbols; that is, # ##. The valid initial state is 0.

There are three quantum random instructions: (0, #, 0, 0), (1, #, 1, 0) and (4, #, 4, 0). The random instruction (0, #, 0, 0) is executed first. If the quantum random source measures a 1, the machine jumps to state 4 and the tape head moves to the right of tape square 0. If the quantum random source measures a 0, the machine jumps to state i and the tape head moves to the left of tape square 0. Instructions containing alphabet symbol E provide error checking for an invalid initial tape or initial state; in this case, the machine halts with an error.

Machine Instructions 2. Random Walk

...

Below are 31 computational steps of the ex-machine's first execution. This random walk machine never halts when the initial tape is blank and the initial state is 0. The first quantum random instruction executed is (0, #, 0, 0). The quantum random source measured a 0, so the execution

of this instruction is shown as (0, #, 0, 0_qr, 0). The second quantum random intruction executed is (1, #, 1, 0). The quantum random source measured a 1, so the execution of instruction (1, #, 1, 0) is shown as (1, #, 1, 1_qr, 0).

2nd Execution of Random Walk Machine. Computational Steps 1-31.

. . .

Below are the 31 steps of the ex-machine's second execution. The first quantum random instruction executed is (0, #, 0, 0). The quantum random bit measured was 1, so the result of this instruction is shown as (0, #, 0, 1_qr, 0). The second quantum random instruction executed is (1, #, 1, 0), which measured a 0, so the result of this instruction is shown as (1, #, 1, 0_qr, 0).

1st Execution of Random Walk Machine. Computational Steps 1-31.

. . .

The first and second executions of the random walk ex-machine verify our statement in the introduction: in contrast with the Turing machine, the execution behaviour of the same ex-machine may be distinct at two different instances, even though each instance of the ex-machine starts its execution with the same input on the tape, the same initial states and same initial instructions. Hence, the ex-machine is a discrete, non-autonomous dynamical system."

. . . *Fiske: [Fi19], 2.2 Quantum Random Instructions.*

It is not, however, obvious whether the appeal to an oracle for an essentially unpredictable value—from the computing ex-machine's perspective—that is to be substituted in the currently scanned cell in the above argumentation is to be made at:

- the occurrence of the pre-defined halting state of the Collatz machine;

- an 'abnormal' halt at other than the defined halting state of the Collatz machine; or

- 'a consecutive repeating state cycle' that does not reach a halting point (compare with §17.F., Definition 32, '*non-terminating loop*'), where:

  > "A consecutive repeating state cycle in a Turing machine occurs when a finite sequence of standard machine instructions $\{I_i\}$ is executed by the Turing machine two consecutive times: $I_1 \to I_2 \to \ldots I_k \to I_1 \to I_2 \ldots I_k$ and the machine configuration before the first instruction I_1 is executed equals the machine configuration after the instruction I_k has completed its execution a second time."
  >
  > . . . *Fiske: [Fi19], 7 Two Research Problems.*

The necessity—for Fiske's ex-machines—of an equal probability for each outcome in Fiske's Axiom 1 (Unbiased Trials) and Axiom 2 (Stochastic Independence) is also not obvious.

In other words, his description of a *Random Walk* ex-machine as 'a discrete, non-autonomous dynamical system' might still apply if we were to relax Axiom 1 and Axiom 2, and treat his 'quantum random source' alternatively as, for instance:

1. an idealised, eternally computing, Turing machine,

2. situated in a stabilised, globally accepted, International Standards locale,

3. that outputs only the 0's and 1's encountered in the non-terminating computation of the binary representation of an irrational number, say, π,

4. which are then sequentially released on demand on-line,

5. as a π-based—albeit *seemingly* random—source of the bits 0 and 1 to intending ex-machines.

However, even if we were to relax Fiske's stipulation—that his *quantum random* source must be constricted by an equal probability for each outcome in his Axiom 1 (Unbiased Trials) and Axiom 2 (Stochastic Independence)—a source of π-based, *seemingly* random (compare §4.G., Definition 20), bits as in 1-5 above would still not eliminate (since π is algorithmically *computable*) the perceived practical barrier that:

> "Overall, the ex-machine uses quantum randomness as a computational tool. Hence, part of our goal was to use axioms 1 and 2 for our quantum random instructions, because the axioms are supported by the empirical evidence of various quantum random number generators [1, 5, 61, 64, 77, 93, 101, 102]. In practice, however, a physical implementation of a quantum random number generator can only generate a finite amount of data and only a finite number of statistical tests can be performed on the data. Due to these limitations, one goal of quantum random theory [72, 14, 15, 16, 96], besides general understanding, is to certify the mathematical properties, assumed about actual quantum random number generators, and assure that the theory is a reasonable extension of quantum mechanics [8, 9, 10, 51, 52, 53, 84, 85, 36, 4, 24, 59]."
>
> *. . . Fiske: [Fi19], 3 Quantum Randomness.*

In other words, to obtain a number that could qualify as a 'quantum random number', we would have to assume, for instance, that a digital clock monitors the emission of radio-active particles from, say, an eternally replenished uranium source $\mathcal{U}$, where each tick of the clock is represented by a 0 if there is no emission since the previous tick, and a 1 if there is such an emission; and let $0 < \rho_1 < 1$ be the putative real number defined by the emissions of $\mathcal{U}$.

From the *evidence-based* perspective of this investigation, the sequence ρ_1 could then be treated as *relatively* random (see §4.G., Definition 20) vis á vis the ex-machine, since experimental observations confirm that such emission (compare with Bell's Inequality Theorem in [Bl64]; see also §19.B.):

- *cannot* be defined as obeying the laws of *classical* (see §19.D.c., Thesis 16) mechanics—by which ρ_1 would be both algorithmically *verifiable* (hence its value would be *deterministic at* each measurement), *and* algorithmically *computable* (hence its value would also be *predictable before* each measurement);

- but *could* be defined as obeying the laws of *neo-classical* (see §19.D.d., Thesis 17) quantum mechanics—in which case ρ_1 could be algorithmically *verifiable* (hence its value would be *deterministic at* each measurement[167]), but not algorithmically *computable* (hence its value would be *unpredictable before* each measurement, albeit constrained by a determinate probability distribution).

> **Comment**: In other words, a *relatively* random number-theoretic sequence can be defined (see §4.G., Definition 20) as algorithmically *verifiable*, but not algorithmically *computable* (see §4.G., Theorem 4.2). It cannot thus be represented—nor defined—as a partial recursive function; since the latter always defines, and is defined by, some deterministic Turing machine (see [Me64], p.237; [Kl52], p.373). However—complementing the Provability Theorem for PA which models algorithmically *computable* functions arithmetically (see §2.E.a.)—any *relatively* random sequence—such as, prima facie, that defined by one of Chaitin's Ω constants (see §4.G., Theorem 12.3)—*can* be defined within PA in terms of Gödel's β-functions (see §12.B.).

---

[167]Reflected by the value of the corresponding, 'collapsed', Schrödinger wave-function.

We note that Fiske's development of his ex-machine as 'a discrete, non-autonomous dynamical system' is apparently to further introduce special meta instructions that would enable 'the ex-machine to self-modify its instructions':

Comment: It is, again, not obvious here whether such self-modification is to be made at:

- the occurrence of the pre-defined halting state of the ex-machine;

- an 'abnormal' halt at other than the defined halting state of the ex-machine; or

- 'a consecutive repeating state cycle' that does not reach a halting point (compare with §17.F., Definition 32, *non-terminating loop*').

"2.3 Meta Instructions

Meta instructions are the second type of special instructions. The execution of a meta-instruction enables the ex-machine to self-modify its instructions. This means that an ex-machine's meta-instructions can add new states, add new instructions or replace instructions. Formally, the meta-instructions $\mathcal{M}$ satisfy $\{\mathcal{M} \subset \{(q, a, r, \alpha, y, J)\} : q \in Q \text{ and } r \in R \cup \{|Q|\} \text{ and } a, \alpha \in A \text{ and instruction } J \in \mathcal{S} \cup \mathcal{R}\}$.

Define $\mathcal{I} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{M}$, as the set of standard, quantum random, and meta instructions. To help describe how a meta instruction modifies $\mathcal{I}$, the *unique state, scanning symbol condition* is defined: for any two distinct instructions chosen from $\mathcal{I}$ at least one of the first two coordinates must differ. More precisely, all 6 of the following uniqueness conditions must hold:

1. If $(q_1, \alpha_1, r_1, \beta_1, y_1)$ and $(q_2, \alpha_2, r_2, \beta_2, y_2)$ are both in $\mathcal{S}$, then $q_1 \neq q_2$ or $\alpha_1 \neq \alpha_2$.

2. If $(q_1, \alpha_1, r_1, \beta_1, y_1) \in \mathcal{S}$ and $(q_2, \alpha_2, r_2, y_2) \in \mathcal{R}$ or vice versa, then $q_1 \neq q_2$ or $\alpha_1 \neq \alpha_2$.

3. If $(q_1, \alpha_1, r_1, y_1)$ and $(q_2, \alpha_2, r_2, y_2)$ are both in $\mathcal{R}$, then $q_1 \neq q_2$ or $\alpha_1 \neq \alpha_2$.

4. If $(q_1, \alpha_1, r_1, y_1) \in \mathcal{R}$ and $(q_2, \alpha_2, r_2, a_2, y_2, J_2) \in \mathcal{M}$ or vice versa, then $q_1 \neq q_2$ or $\alpha_1 \neq \alpha_2$.

5. If $(q_1, \alpha_1, r_1, \beta_1, y_1) \in \mathcal{S}$ and $(q_2, \alpha_2, r_2, a_2, y_2, J_2) \in \mathcal{M}$ or vice versa, then $q_1 \neq q_2$ or $\alpha_1 \neq \alpha_2$.

6. If $(q_1, \alpha_1, r_1, a_1, y_1, J_1)$ and $(q_2, \alpha_2, r_2, a_2, y_2, J_2)$ are both in $\mathcal{M}$, then $q_1 \neq q_2$ or $\alpha_1 \neq \alpha_2$.

Before a valid machine instruction starts, it is assumed that the standard, quantum random and meta instructions $\mathcal{S} \cup \mathcal{R} \cup \mathcal{M}$ always satisfy the unique state, scanning symbol condition. This condition assures that there is no ambiguity on what instruction should be executed when the machine is in state q and is scanning tape symbol a. Furthermore, the execution of a meta-instruction preserves this uniqueness condition.

Definition 2.4. Execution of Meta Instructions

A meta instruction (q, a, r, α, y, J) in $\mathcal{M}$ is executed as follows.

- The first five coordinates (q, a, r, α, y) are executed as a standard instruction according to definition 2.1 with one caveat. State q may be expressed as $|Q| - c_1$ and state r may be expressed as $|Q|$ or $|Q| - c_2$, where $0 < c_1, c_2 \leq |Q|$. When (q, a, r, α, y) is executed, if q is expressed as $|Q| - c_1$, the value of q is instantiated to the current value of $|Q|$ minus c_1. Similarly, if r is expressed as $|Q|$ or $|Q| - c_2$, the value of state r is instantiated to the current value of $|Q|$ or $|Q|$ minus c_2, respectively.

- Subsequently, instruction J modifies $\mathcal{I}$, where instruction J has one of two forms: $J = (q, a, r, \alpha, y)$ or $J = (q, a, r, y)$.

- For both forms, if $\mathcal{I} \cup \{J\}$ still satisfies the unique state, scanning symbol condition, then $\mathcal{I}$ is updated to $\mathcal{I} \cup \{J\}$.

- Otherwise, there is an instruction I in $\mathcal{I}$ whose first two coordinates q, a, are equal to instruction J's first two coordinates. In this case, instruction J replaces instruction I in $\mathcal{I}$. That is, $\mathcal{I}$ is updated to $\mathcal{I} \cup \{J\} - \{I\}$."

  . . . Fiske: [Fi19], 2.3 Meta Instructions

Fiske illustrates *how* his 'new states are instantiated and added' to the existing states of an ex-machine $\mathcal{X}$ during a computation—still without explicitly specifying *when*, and *why*, such additions are to be made—in a process that he describes as invoking a 'simple type of *self-reflection*' that 'poses no obstacles in physical realizations':

"In regard to definition 2.4, example 1 shows how instruction I is added to $\mathcal{I}$ and how new states are instantiated and added to Q.

Example 1. Adding New States

Consider the meta instruction $(q, a_1, |Q| - 1, \alpha 1, y_1, J)$, where $J = (|Q| - 1, a_2, |Q|, \alpha_2, y_2)$. After the standard instruction $(q, a_1, |Q| - 1, \alpha_1, y_1)$ is executed, this meta instruction adds one new state $|Q|$ to the machine states Q and also adds the instruction J, instantiated with the current value of $|Q|$.

. . .

Let $\mathcal{X}$ be an ex-machine. The instantiation of $|Q| - 1$ and $|Q|$ in a meta instruction I invokes *self-reflection* about $\mathcal{X}$s current number of states, at the moment when $\mathcal{X}$ executes I. This simple type of self-reflection poses no obstacles in physical realizations. In particular, a LISP implementation [70] along with quantum random bits measured from [101] simulates all executions of the ex-machines provided therein."

. . . Fiske: [Fi19], 2.3 Meta Instructions

However, Fiske does implicitly suggest his intent in his remark that if an 'ex-machine $\mathcal{X}_0$ with tape T_0' does self-modify its instructions during an execution, *and halt*, then it can be said to have *evolved* to an 'ex-machine $\mathcal{X}_1$ with tape S_1':

"**Definition 2.7.** Evolving an ex-machine

Let $T_0, T_1, T_2 \ldots T_{i-1}$ each be a finitely bounded tape. Consider ex-machine $\mathcal{X}_0$ with finite initial conditions. $\mathcal{X}_0$ starts executing with tape T_0 and evolves to ex-machine $\mathcal{X}_1$ with tape S_1. Subsequently, $\mathcal{X}_1$ starts executing with tape T_1 and evolves to $\mathcal{X}_2$ with tape S_2. This means that when ex-machine $\mathcal{X}_1$ starts executing on tape T_1, its instructions are preserved after the halt with tape S_1. The ex-machine evolution continues until $\mathcal{X}_{i-1}$ starts executing with tape T_{i-1} and evolves to ex-machine $\mathcal{X}_i$ with tape S_i. One says that ex-machine $\mathcal{X}_0$ with finitely bounded tapes $T_0, T_1, T_2 \ldots T_{i-1}$ evolves to ex-machine $\mathcal{X}_i$ after i halts."

. . . Fiske: [Fi19], 2.3 Meta Instructions

Now, if we treat Fiske's ex-machine $\mathcal{X}_0$ as defining the partial recursive function, say $(\mu x)(F_0(x) = a_0)$ then, *for some k_0, defined by the instruction set for $\mathcal{X}_0$, and such that $F_0(k_0) \neq a_0$*, the above process constructs an ex-machine $\mathcal{X}_1$ that defines some '*random*' partial recursive function, say $(\mu x)(F_1(x) = a_1)$ such that $\mathcal{X}_1$ will first compute $\mathcal{X}_0$, *halt* and return the value k_0, *before* computing $\mathcal{X}_1$ by the modified instruction set for $\mathcal{X}_1$.

Iterating the process, we obtain the non-terminating sequence (Fiske's '*evolutionary path*' of $\mathcal{X}_0$) of, '*randomly*' *defined*, non-deterministic ex-machines, $\mathcal{X}_0 \to \mathcal{X}_1 \to \mathcal{X}_0 \ldots \to \mathcal{X}_n \ldots$, *such that* each $i \geq 0$ defines a partial recursive function $(\mu x)F_i(x) = a_i$ *and* some 'random' k_i for which $F_i(k_i) \neq a_i$, and at which the instruction set of $\mathcal{X}_i$ is modified to that of $\mathcal{X}_{i+1}$.
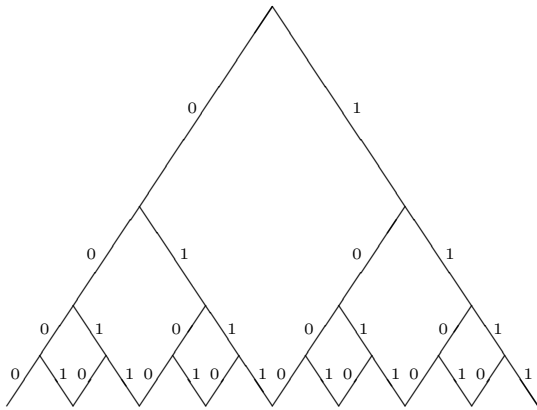
Fig.5: Infinite binary tree. A graphical representation of $\{0, 1\}^{\mathbb{N}}$

Fiske notes that, if his '*random*' modifications to the existing machine states and instructions at each halt in the above process appeal to an oracle (such as a 'random' walk machine) that supplies only binary values, then the result of running $\mathcal{X}_0$ repeatedly on tape T_0 would yield an infinite binary tree structure such as Fig. 5 above (corresponding to Figure 3 in [Fi19]) where, at each k_i, $\mathcal{X}_i$ would halt and, essentially, define two different partial recursive functions to be computed by two distinctly different self-modified ex-machines, say $\mathcal{X}_{(i+1)_a}$ and $\mathcal{X}_{(i+1)_b}$.

> **Comment**: We note that there seems no imperative that the *random* values supplied by the oracle (such as Fiske's random walk machine) be binary and, prima facie, Fiske's construction should hold even if the *random* values supplied by the oracle were n-ary instead of binary.

However, Fiske notes further that, for a particular string in the language $L \in \mathcal{L} = \bigcup_{L \subset \mathtt{a}^*}\{L\}$, where $\mathtt{a}^* = \{\mathtt{a}^n : n \in \mathbb{N}\}$, some ex-machine could, conceivably, oscillate indefinitely—even periodically—between halting states that accept the string, and those that do not:

> "In order to define the halting syntax for the language in $\mathcal{L}$ that an ex-machine computes, choose alphabet set $A = \{\mathtt{\#, 0, 1, N, Y, a}\}$.
>
> **Definition 4.2.** *Language L in $\mathcal{L}$* that ex-machine $\mathcal{X}$ computes
>
> Let $\mathcal{X}$ be an ex-machine. The language L in $\mathcal{L}$ that $\mathcal{X}$ computes is defined as follows. A valid initial tape has the form $\mathtt{\#\ \#a}^n\mathtt{\#}$. The valid initial tape $\mathtt{\#\ \#\#}$ represents the empty string. After machine $\mathcal{X}$ starts executing with initial tape $\mathtt{\#\ \#a}^n\mathtt{\#}$, string $\mathtt{a}^n$ is in $\mathcal{X}$s language if ex-machine $\mathcal{X}$ halts with tape $\mathtt{\#a}^n\mathtt{\#\ Y\#}$. String $\mathtt{a}^n$ is not in $\mathcal{X}$s language if $\mathcal{X}$ halts with tape $\mathtt{\#a}^n\mathtt{\#\ N\#}$.
>
> The use of special alphabet symbols $\mathtt{Y}$ and $\mathtt{N}$—to decide whether $\mathtt{a}^n$ is in the language or not in the language—follows [63].
>
> For a particular string $\mathtt{\#\ \#m}^m\mathtt{\#}$, some ex-machine $\mathcal{X}$ could first halt with $\mathtt{\#a}^m\mathtt{\#\ N\#}$ and in a second computation with input $\mathtt{\#\ \#a}^m\mathtt{\#}$ could halt with $\mathtt{\#a}^m\mathtt{\#\ Y\#}$. This oscillation of halting outputs could continue indefinitely and in some cases the oscillation can be periodic. In this case, $\mathcal{X}$s language would not be well-defined according to definition 4.2."
>
> ...*Fiske: [Fi19], 4 Computing Ex-Machine Languages*

To eliminate such ex-machines whose halting output never stabilises, Fiske defines a class of ex-machines as evolutions of a fundamental ex-machine $\mathfrak{D}$, with 15 specified initial instructions, that compute languages L that are subsets of $\{\mathtt{a}^* = \{\mathtt{a}^n : n \in \mathbb{N}\}\}$; where the expression $\mathtt{a}^n$ represents a string of n consecutive $\mathtt{a}$'s. He then notes that:

> "There is a subtle difference between $\mathfrak{D}$ and an ex-machine $\mathcal{X}$ whose halting output never stabilises. In contrast to the Turing machine, two different instances of the ex-machine $\mathfrak{D}$ can evolve two to

two different machines and compute distinct languages according to definition 4.2. However, after $\mathfrak{D}$ has evolved to a new machine $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$ as a result of a prior execution with input tape # #$\mathtt{a}^m$#, then for each i with $0 \le i \le m$, machine $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$ always halts with the same output when presented with input tape # #$\mathtt{a}^i$#. In other words, $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$'s halting output stabilises on all input strings $\mathtt{a}^i$ where $0 \le i \le m$. Furthermore, it is the ability of $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$ to exploit the non-autonomous behaviour of its two quantum random instructions that enables an evolution of $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$ to compute languages that are Turing incomputable."

. . . Fiske: [Fi19] *4 Computing Ex-Machine Languages.*

Fiske then formally concludes that:

"**Lemma 4.1.** *Whenever i satisfies $0 \le i \le m$, string $\mathtt{a}^i$ is in $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$'s language if $a_i = 1$; string $\mathtt{a}^i$ is not in $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$'s language if $a_i = 0$. Whenever $n > m$, it has not yet been determined whether string $\mathtt{a}^n$ is in $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$'s language or not in its language.*

Definition 4.3. Define $\mathcal{U}$ as the union of $\mathfrak{D}(x)$ and all ex-machines $\mathfrak{D}(a_0 \ldots a_m \ x)$ for each $m \in \mathbb{N}$ and for each $a_0 \ldots a_m$ in $\{0,1\}^{m+1}$. In other words,

$$\mathcal{U} = \{\mathfrak{D}(x)\} \bigcup \bigcup_{m=0}^{\infty} \bigcup_{a_0 \ldots a_m \in \{0,1\}^{m+1}} \{\mathfrak{D}(a_0 a_1 \ldots a_m \ x)\}$$

Theorem 4.2. *Each language L_f in $\mathcal{L}$ can be computed by the evolving sequence of ex-machines $\mathfrak{D}(x), \mathfrak{D}(f(0)x), \mathfrak{D}(f(0)f(1) \ x), \ldots, \mathfrak{D}(f(0)f(1) \ldots f(n) \ x), \ldots$*

Corollary 4.3. *Given function $f : \mathbb{N} \to \{0,1\}$, for any arbitrary large n, the evolving sequence of ex-machines $\mathfrak{D}(f(0)f(1) \ldots f(n) \ x), \mathfrak{D}(f(0)f(1) \ldots f(n)f(n+1) \ x), \ldots$ computes language L_f.*

Corollary 4.4. *Moreover, for each n, all ex-machines $\mathfrak{D}(x), \mathfrak{D}(f(0)x), \mathfrak{D}(f(0)f(1) \ x), \ldots, \mathfrak{D}(f(0)f(1) \ldots f(n) \ x)$ combined have used only a finite amount of tape, finite number of states, finite number of instructions, finite number of executions of instructions and only a finite amount of quantum random information measured by the quantum random instructions.*"

. . . Fiske: [Fi19] *4 Computing Ex-Machine Languages.*

Fiske argues further that:

"Since the set of all Turing machines is countable and each Turing machine only recognizes a single language *most (in the sense of Cantor's hierarchy of infinities [19]) languages L_f are not computable with a Turing machine.* More precisely, the set of languages L_f computable with a Turing machine is a computable set, while the set of all languages $\mathcal{L}$ is an uncountable set.

For each non-negative integer n, define the language tree $\mathcal{L}(a_0 a_1 \ldots a_n) = \{L_f : f \in \{0,1\}^{\mathbb{N}} \ \mathtt{and} \ f(i) = a_i \ \mathtt{for} \ i \ \mathtt{satisfying} \ 0 \le i \le n\}$. Let Ψ denote this 1-to-1 correspondence, where $\mathcal{L} \overset{\Psi}{\leftrightarrow} \{0,1\}^{\mathbb{N}}$ and $\mathcal{L}(a_0 a_1 \ldots a_n) \overset{\Psi}{\leftrightarrow} \mathcal{S}(a_0 a_1 \ldots a_n)$.

Since the two random axioms 1 and 2 are satisfied, each finite path $f_0 f_1 \ldots f_n$ is equally likely and there are 2^{n+1} of these paths. Thus, each path of length $n+1$ has probability $2^{-(n+1)}$. These uniform probabilities on finite strings of the same length can be extended to the Lebesgue measure μ on probability space $\{0,1\}^{\mathbb{N}}$ [37, 38]. Hence, each subset $\mathcal{S}(a_0 a_1 \ldots a_n)$ has measure $2^{-(n+1)}$. That is, $\mu(\mathcal{S}(a_0 a_1 \ldots a_n)) = 2^{-(n+1)}$ and $\mu(\{0,1\}^{\mathbb{N}}) = 1$. Via the Ψ correspondence between each language tree $\mathcal{L}(a_0 a_1 \ldots a_n)$ and subset $\mathcal{S}(a_0 a_1 \ldots a_n)$, uniform probability measure μ induces a uniform probability measure ν on $\mathcal{L}$, where $\nu(\mathcal{L}(a_0 a_1 \ldots a_n)) = 2^{-(n+1)}$ and $\nu(\mathcal{L}) = 1$.

Theorem 4.5. *For functions $f : \mathbb{N} \to \{0,1\}$, the probability that language L_f is Turing incomputable has measure 1 in $(\nu, \mathcal{L})$.*"

. . . Fiske: [Fi19] *4 Computing Ex-Machine Languages.*

Corollary 4.6. $\mathfrak{D}(x)$ *is not a Turing machine. Each ex-machine $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$ in $\mathcal{U}$ is not a Turing machine.*"

. . . Fiske: [Fi19] *4 Computing Ex-Machine Languages.*

17.G.a. Representing fundamental physical constants by Fiske's ex-machines $\mathfrak{D}$

Now, one of the challenging issues in physics is to mathematically define some fundamental dimensionless constants whose values cannot be calculated theoretically, but can only be determined experimentally since, from the *evidence-based* perspective of this investigation (see §19.D., Thesis 13), such constants are only representable mathematically by real numbers that are algorithmically *verifiable*, but not algorithmically *computable*.

Fiske's Theorem 4.2 and Corollary 4.3 now suggest (compare §4.H.; also §12.B.) that, from a computational perspective, each such dimensionless constant could be viewed as the putative computation of an appropriate ex-machine $\mathfrak{D}$, since every real number could, presumably, be computed by some ex-machine $\mathfrak{D}$. As remarked by Fiske:

> "At first glance, the results from the prior section may seem paradoxical. Even though there are only a computable number of initial ex-machines in $\mathcal{U}$, the ex-machines evolving from $\mathfrak{D}(x)$ can compute languages L_f where each $f : \mathbb{N} \to \{0,1\}$ corresponds to a particular instance selected from an uncountable number of infinite paths in the infinite binary tree (i.e, $\{0,1\}^{\mathbb{N}}$ is uncountable [18]). With initial state 0 and initial tape `# #a`^{n}`#`, for every n and m with $n > m$, each ex-machine $\mathfrak{D}(a_0 a_1 \ldots a_m \ x)$ has an uncountably infinite number of possible execution behaviors. On the other hand, a Turing machine with the same initial state 0 and initial tape `# #a`^{n}`#` always has exactly one execution behavior. Hence, a Turing machine can only have a countable number of execution behaviors for all initial tapes `# #a`^{n}`#`, where $n > m$."
>
> *. . . Fiske: [Fi19] 5 Some $\mathfrak{D}(x)$ Observations Based on Cantor and Gödel.*

Fiske remarks upon the peculiarity, 'that the countable set $\mathcal{U}$ of ex-machines can evolve to compute an uncountable number of languages L_f', by drawing an analogy to the definition of real numbers in elementary analysis (see, for instance, [La29]), whereby 'each real number can be realized as a sequence of rational numbers, even though the real numbers are uncountable. Furthermore, each rational number in that sequence is representable with a finite amount of information (bits)'.

However, from the *evidence-based* perspective of this investigation, the analogy that Fiske seeks to draw can be viewed as essentially the argument in §4.H., Theorem 4.4, that every real number is specifiable in PA; and, moreover, that (see §4.H.a.) *evidence-based* reasoning does not admit Cantor's theorem.

In other words, from an *evidence-based* perspective, §4.H., Theorem 4.4 implies that real numbers do not exist in some Platonic, set-theoretic, universe of points that constitute a line, but are arithmetical constructs identifiable as specific number-theoretic definitions that are algorithmically *verifiable* (hence *well-defined*), but not necessarily algorithmically *computable*.

As illustrated by the gedanken in §16.C., they assume significance (which can, debatably, be termed as 'existence') mathematically only when such a definition is made explicit formally in an argumentation (compare with Brouwer's parallel perspective cited in §4.K.).

Thus, if the intent is *not* to postulate the existence of a notional 'halting' state—i.e., a *completed evolutionary path* whose limiting state can be treated as *the* real number '*computed/realized*' by a 'quantum algorithm' (a postulation that could violate §4.H.a., Theorem 4.5)—corresponding to the Cauchy limit of the Cauchy sequence defined by a quantum algorithm for an evolving sequence of ex-machines, then Fiske is justified in asserting that his formal conclusions entail:

> "…each language L_f can be computed (i.e., realized) by the evolving sequence of ex-machines $\mathfrak{D}(x), \mathfrak{D}(f(0)x), \mathfrak{D}(f(0)f(1) \ x), \ldots, \mathfrak{D}(f(0)f(1) \ldots f(n) \ x), \ldots$, where for each n, all ex-machines

$\mathfrak{D}(x), \mathfrak{D}(f(0)x), \mathfrak{D}(f(0)f(1) \ x), \dots, \mathfrak{D}(f(0)f(1) \dots f(n) \ x)$ have used only a finite amount of tape, finite number of states, finite number of instructions, finite number of executions of instructions and a finite amount of quantum random information has been measured."

... *Fiske: [Fi19] 5 Some $\mathfrak{D}(x)$ Observations Based on Cantor and Gödel.*

since such 'computations' can be treated as trivially instantiating Gödel's primitive recursive 'β-function' Lemma 1 in [Go31] (compare with §17.F.c., Theorem 17.6):

"If f is an arbitrary sequence of natural numbers and k is an arbitrary natural number, then there exists a pair of natural numbers n, d such that $f^{(n,d)}$ and f coincide in their first k terms."

... *Gödel: [Go31], Lemma 1, p.31*

This suggests that the 'evolving' value of an 'essentially incompletable' (see §4.H.a.) dimensionless 'quantum' constant could, conceivably, be viewed as being determined somehow by an n-ary, weighted, 'random' probability function that, at each evolution of the sequence of ex-machines $\mathfrak{D}(x), \mathfrak{D}(f(0)x), \mathfrak{D}(f(0)f(1) \ x), \dots$, determines the 'Heisenberg' probability of the 'halting' value at the next evolution of an ex-machine in the sequence.

17.G.b.   An *evidence-based* perspective of the 'decidability' of the Collatz conjecture

Finally, apropos the decidability of the Collatz conjecture, which apparently motivated him to consider the development of *quantum random, self-modifiable machines* such as his *evolving* ex-machines $\mathcal{X}$ and $\mathfrak{D}$, Fiske concludes:

"In [42], the main theorem shows that the consecutive repeating state cycles characterize the periodic points of a Turing machine. A periodic point that does not reach a halting state indicates that the Turing machine execution is immortal (i.e., never halts). Can this consecutive repeating state cycle theorem or an extension of this theorem be used to help an ex-machine find a proof? If the standard $\sqrt{2}$ machine writes symbols on the tape in a periodic sequence, this indicates that $\sqrt{2}$ is rational. If an ex-machine can construct rules which prove that the standard $\sqrt{2}$ machine never halt, then these ex-machine rules provide a proof that the $\sqrt{2}$ is irrational.

. . .

Transform Collatz machine 1's execution of each individual orbit $\mathcal{O}(f, n)$ into a single ex-machine computation that collectively makes a determination about all individual orbits. That is, find an ex-machine computation that evolves to a decision whether 1 is in $\mathcal{O}(f, n)$ for all $n \in \mathbb{N}$. Is it possible to accomplish this with an ex-machine computation? If it is impossible, why?

Consider the augmentation of Collatz machine 1 to an enumerated Collatz machine $\mathcal{E}$. The standard machine $\mathcal{E}$ iterates over the odd numbers $3, 5, 7, \dots$. $\mathcal{E}$ first writes `# # 111#` on the input tape and hands this computation over to Collatz machine 1. After Collatz machine 1 halts at 1, then $\mathcal{E}$ updates the input tape to `# # 11111#`, representing 5, and hands this to the Collatz machine again. After the Collatz machine halts at 1, then $\mathcal{E}$ updates the input tape to `# # 1111111#`, and so on. If the Collatz conjecture is true, this execution of $\mathcal{E}$ never halts and $\mathcal{E}$ iterates over every odd number."

... *Fiske: [Fi19] 7 Two Research Problems.*

Fiske observes that:

At least part of the challenge with machine $\mathcal{E}$ seems to be that there could exist some n such that n's Collatz orbit reaches a periodic attractor that does not conain 1. Another possibility is that there exists some u whose Collatz orbit aperiodically oscillates and never reaches 1. In this case, u's orbit does not have an upper bound. That is, $\sup \mathcal{O}(f, u) = \infty$. In both cases, the orbit of n and the orbit of u do not halt at 1. If the conjecture is true, how does one distinguish these two

different types of immortal orbits from the enumerated Collatz machine that halts at 1 for each odd output, but is also immortal?

Is it possible to transform (either by human ingenuity or by ex-machine evolution or a combination) this enumerated Collatz machine $\mathcal{E}$ into a non-vacuous, explicit Turing machine so that an immortal proves or disproves that the Collatz conjecture is true? If this transformation exists, does there exist an ex-machine that can construct this transformation?"

...*Fiske: [Fi19] 7 Two Research Problems.*

However, from the *evidence-based* perspective of this investigation—which admits the Provability Theorem for PA (see §2.E., Theorem 2.17) and identifies the algorithmically *computable* number-theoretic functions as precisely the PA-provable formulas—the putative 'challenge with machine $\mathcal{E}$' faced by Fiske in differentiating between:

(i) the case where 'some n such that n's Collatz orbit reaches a periodic attractor that does not contain 1'; and

(ii) the case where there exists some u whose Collatz orbit aperiodically oscillates and never reaches 1. In this case, u's orbit does not have an upper bound. That is, $\mathrm{sup}\ \mathcal{O}(f, u) = \infty$.

dissolves if we note that (see also the observations in §2.E.a.):

(ia) the case (i) corresponds to the operation of the Turing machine T_2 in the proof of Theorem 17.3 in §17.F.b.; whilst

(iia) the case (ii) corresponds to the operation of the Turing machine T_3 in the proof of Theorem 17.3 in §17.F.b..

We can then argue that:

1. The Collatz function $f(n) = n/2$ if n is even, and $f(n) = 3n + 1$ if n is odd.

  1a. For each $n \geq 1$, we then have the non-terminating, recursive, Collatz sequence:
  $$C(n) = c(n, 1), c(n, 2), c(n, 3), \ldots$$
  where $c(n, 1) = f(n)$ and, for $i > 1$:
  $$c(n, i + 1) = c(n, i)/2 \text{ if } c(n, i) \text{ is even;}$$
  $$c(n, i + 1) = 3c(n, i) + 1 \text{ if } c(n, i) \text{ is odd.}$$

2. Define $D(n)$ as the least i such that $c(n, i) = 1$.

  2a. Thus $D(n)$ is a partial recursive function, which is Turing-computable (but not necessarily Turing-determinate for all n).

  2c. *Reason*: Turing's Halting Theorem (§17.F.b., Corollary 17.5) states that it is not always decidable by a Turing machine whether or not a specified partial recursive function is *total*.

  2d. In other words, there are partial recursive functions $F(n)$ such that, for some value m, no Turing machine T that computes $F(n)$ will output a value for $F(m)$.

3. The Collatz conjecture is then that any Turing machine T which accepts $D(n)$ will, for any m, output a value of $D(m)$ and halt.

4. Now, the Halting-decidability Theorem for T (§17.F.b., Corollary 17.4) entails that, given a Turing machine T, one can effectively decide, given any instantaneous description α, whether or not there is a computation of T beginning with α.

    4a. In other words, it is always possible to effectively determine whether a Turing machine T will halt or not when computing a specified partial recursive function.

5. So, if it is determined by the Halting-decidability Theorem for T (§17.F.b., Corollary 17.4) that T accepts $D(n)$, and halts for all n (corresponding to case (c) in §17.F.b., Theorem 17.3), then the Collatz conjecture is proved.

    5a. However, if it is determined that T accepts $D(n)$, but does not halt for some m, then the Collatz conjecture is either decidable as false (corresponding to case (a) in §17.F.b., Theorem 17.3) or computationally undecidable (corresponding to case (b) in §17.F.b., Theorem 17.3).

6. If the Collatz conjecture were proved to be undecidable (corresponding to case (b) in §17.F.b.), the reason would be that, for any specified m, there would be some, m-specific, Turing machine T_m which would compute $D(m)$ and halt; so $D(n)$ would be *algorithmically verifiable*.

    6a. However, there would be no Turing machine T that accepts $D(n)$ and halts for all n; so $D(n)$ would not be *algorithmically computable*.

17.H.   An *evidence-based* perspective of Deutsch's Church-Turing principle

The theoretical significance of the distinction between Turing's Halting Theorem (§17.F.b., Corollary 17.5) and the Halting-decidability Theorem for T (§17.F.b., Corollary 17.4) is seen in a seminal paper [Deu85] by David Deutsch, *Quantum theory, the Church-Turing principle and the universal quantum computer*; where he articulates a 'Church-Turing principle' that essentially seeks to differentiate between classical Turing-computability, and what he terms as a 'class of model computing machines that is the quantum generalization of the class of Turing machines' which 'could, in principle, be built and would have many remarkable properties not reproducible by any Turing machine' ([Deu85], Abstract).

From the *evidence-based* perspective of this investigation, Deutsch's perspective can be viewed as essentially that:

- The Church-Turing hypothesis is a mathematical assertion;

- The Church-Turing principle is a physical principle (law);

- Classical physics seeks to model physical processes in terms of continuous functions;

- Turing machines seek to model physical processes in terms of discrete number-theoretic functions;

- Both classical physics and Turing machines do not obey the strong form of the Church-Turing principle;

- Both quantum theory and a class of model computing machines that is the quantum generalization of the class of Turing machines are compatible with the physical Church-Turing principle;

- Quantum computing machines do not claim to compute non-recursive functions;

- Quantum computers can model 'quantum parallelism';

- Quantum computers can perform some probabilistic tasks faster than any classical computer (Turing machine);

- The intuitive explanation of the properties that distinguish quantum computers from classical computers is consistent with Everett's 'many-world' interpretation of quantum theory;

- Quantum complexity theory allows a physically more reasonable description of the 'complexity' or 'knowledge' in a physical system than does classical complexity theory.

For instance, Deutsch argues that:

> "... Intuitively, a computing machine is any physical system whose dynamical evolution takes it from one of a set of 'input' states to one of a set of 'output' states. The states are labelled in some canonical way, the machine is prepared in a state with a given input label and then, following some motion, the output state is measured. For a classical deterministic system the measured output label is a definite function f of the prepared input label; moreover the value of that label can in principle be measured by an outside observer (the '*user*') and the machine is said to '*compute*' the function f.
>
> Two classical deterministic computing machines are '*computationally equivalent*' under given labellings of their input and output states if they compute the same function under those labellings. But quantum computing machines, and indeed classical stochastic computing machines, do not 'compute functions' in the above sense: the output state of a stochastic machine is random with only the probability distribution function for the possible outputs depending on the input state. The output state of a of a quantum machine, although fully determined by the input state is not an observable and so cannot in general discover its label. Nevertheless, the notion of computational equivalence can be generalized to apply to such machines also."
>
> ... *Deutsch: [Deu85], §1 Computing machines and the Church-Turing principle, p.2.*

However, from the *evidence-based* perspective of this investigation, the outputs of both a stochastic and a quantum machine would be viewed as 'relatively random' (see §4.G., Definition 20) if, and only if:

- they were determined completely by their inputs

- which, in both cases, were mathematically expressible in terms of functions, and relations,

- at least one of which is, necessarily, algorithmically *verifiable* (hence determinate), but not algorithmically *computable* (hence unpredictable).

The significance of such an *evidence-based* perspective is seen in Deutsch's further reasoning that:

> "Again we define computational equivalence *under given labellings*, but it is now necessary to specify more precisely what is to be labelled. As far as the input is concerned, labels must be given for each of the possible ways of preparing the machine, which correspond, by definition, to all the possible input states. This is identical with the classical deterministic case. However, there is an asymmetry between input and output because there is an asymmetry between preparation and measurement: whereas a quantum system can be prepared in any desired permitted input state, measurement cannot in general determine its output state; instead one must measure the value of some observable. (Throughout this paper I shall be using the Schrödinger picture, in which the quantum state is a function of time but observables are constant operators.) Thus what must be labelled is the set of ordered pairs consisting of an output observable and a possible measured value of that observable (in quantum theory, a Hermitian operator and one of its eigenvalues). Such an ordered pair contains, in effect, the specification of a possible experiment that could be made on the output, together with the possible result of that experiment.
>
> Two computing machines are computationally equivalent under given labellings if in any possible experiment or sequence of experiments in which their inputs were prepared equivalently under the input labellings, and observables corresponding to each other under the output labellings were measured, the measured values of these observables for the two machines would be statistically indistinguishable. That is, the probability distribution functions for the outputs of the two machines would be identical."
>
> . . . Deutsch: [Deu85], §1 Computing machines and the Church-Turing principle, p.2.

since the output of both machines would be viewed as corresponding to:

- an algorithmically *verifiable* (hence determinate), but not algorithmically *computable* (hence unpredictable), 'observable';

  - where the Hermitian operator which is determined by the input (program);

  - determines the permissible, algorithmically *computable*, values into which the output observable 'collapses' upon interpretation (corresponding to a physical measurement);

  - with the precise, algorithmically *computable*, 'observed' value depending deterministically upon the precision (comparable to the length of a specified finite sequence of non-negative integers in the determination of the sequence's unique Gödel β-function, as detailed in §12.A.);

  - admitted by the 'observer' at the 'measurement'.

We note that Deutsch seeks to distinguish here between classical, deterministic, Turing machines, and seemingly 'non-deterministic' quantum machines, purely on the basis of how their outputs are to be interpreted and presented by the machine's program.

In other words, both classical Turing machines and quantum machines are deterministic in the sense that:

- whereas the former computes the 'output' values of partial recursive functions whose 'input' arguments are natural numbers,

- the latter computes the 'output' quantum states of functions whose 'input' arguments are ordered pairs corresponding to the quantum state of an 'observable' before a measurement.

However, whereas any output computed by a classical Turing machine for a given input is a unique natural number, the output computed by a quantum machine is a unique ordered pair

(quantum state) that, when interpreted by the machine as a 'measurement', does not yield a unique value but a probability distribution for a range of values into which the 'quantum state' of the 'observable' can 'collapse' upon 'measurement'.

Since both classical Turing machines and quantum machines have finite alphabets and finite programs—i.e., the set of permissible machine states that uniquely define the operations of the machine—the latter can be uniquely coded by Gödel-numbers as detailed in [Go31], pp.13-14 (see §23.B.), to yield a number-theoretic relation $g(n)$ that holds if, and only if, n is the Gödel number of a computing machine.

Moreover, such Gödel-numbering entails that $g(n)$ is computable by a classical Turing machine, say $\mathcal{G}$, so as to yield the set $C(\mathcal{T})$ of permissible machine states—say '$\mathcal{T}$-computable functions'—that define the operations of the computing machine—whether classical or quantum— Gödel-numbered by n that are computable by Turing's universal computing machine $\mathcal{T}$.

As expressed by Deutsch, a 'function f is $\mathcal{M}$-computable if $\mathcal{M}$ can compute f when prepared with some program'; where he further notes that 'for functions from the integers $\mathbb{Z}$ to themselves' the set $C(\mathcal{T})$ 'also known as the set of recursive functions, is denumerable and therefore infinitely smaller than the set of all functions from $\mathbb{Z}$ to $\mathbb{M}$'.

However, he concludes from this that 'although logic does not forbid the physical computation of arbitrary functions, it seems that physics does':

> "In the sense just described, a given computing machine $\mathcal{M}$ computes at most one function. However, there ought to be no fundamental difference between altering the input state in which $\mathcal{M}$ is prepared, and altering systematically the constitution of $\mathcal{M}$ so that it becomes a different machine $\mathcal{M}'$ computing a different function. To formalize such operations, it is often useful to consider machines with two inputs, the preparation of one constituting a 'program' determining which function of the other is to be computed. To each such machine $\mathcal{M}$ there corresponds a set $C(\mathcal{M})$ of '$\mathcal{M}$-computable functions'. A function f is $\mathcal{M}$-computable if $\mathcal{M}$ can compute f when prepared with some program.
>
> The set $C(\mathcal{M})$ can be enlarged by enlarging the set of changes in the constitution of $\mathcal{M}$ that are labelled as possible $\mathcal{M}$-programs. Given two machines $\mathcal{M}$ and $\mathcal{M}'$ it is possible to construct a composite machine whose set of computable functions contains the union of $C(\mathcal{M})$ and $C(\mathcal{M}')$.
>
> There is no purely logical reason why one could not go on *ad infinitum* building more powerful computing machines, nor why there should exist any function that is outside the computable set of every physically possible machine. Yet although logic does not forbid the physical computation of arbitrary functions, it seems that physics does. As is well known, when designing computing machines one rapidly reaches a point when adding additional hardware does not alter the machine's set of computable functions (under the idealization that the memory capacity is in effect unlimited); moreover, for functions from the integers $\mathbb{Z}$ to themselves the set $C(\mathcal{M})$ is always contained in $C(\mathcal{T})$, where $\mathcal{T}$ is Turing's universal computing machine (Turing 1936). $C(\mathcal{T})$ itself, also known as the set of recursive functions, is denumerable and therefore infinitely smaller than the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$."
>
> . . . Deutsch: [Deu85], §1 Computing machines and the Church-Turing principle, pp.2-3.

From the *evidence-based* perspective of this investigation, Deutsch's conclusion that Turing's universal computing machine entails:

- 'when designing computing machines one rapidly reaches a point when adding additional hardware does not alter the machine's set of computable functions (under the idealization that the memory capacity is in effect unlimited)',

- even though there 'is no purely logical reason why one could not go on *ad infinitum* building more powerful computing machines, nor why there should exist any function that is outside the computable set of every physically possible machine'

suggests that the 'arbitrary' functions contemplated by Deutsch might be what we have termed as 'quantum' functions (see §19.D., Thesis 13) that are algorithmically *verifiable* but not algorithmically *computable* (by $\mathcal{T}$); and which Deutsch further refers to as 'functions which may in principle be computed by a real physical system':

> "Church (1936) and Turing (1936) conjectured that these limitations on what can be computed are not imposed by the state-of-the-art in designing computing machines, nor by our ingenuity in constructing models for computation, but are universal. This is called the 'Church-Turing hypothesis'; according to Turing,
>
> > Every 'function which would naturally be regarded as computable' can be computed by the universal Turing machine. (1.1)
>
> The conventional, non-physical view of (1.1) interprets it as the quasi-mathematical conjecture that all possible formalizations of the intuitive mathematical notion of 'algorithm' or 'computation' are equivalent to each other. But we shall see that it can also be regarded as asserting a new physical principle, which I shall call the Church-Turing *principle* to distinguish it from other implications and connotations of the conjecture (1.1).
>
> Hypothesis (1.1) and other formulations that exist in the literature (see Hofstadter (1979) for an interesting discussion of various versions) are very vague by comparison with physical principles such as the laws of thermodynamics or the gravitational equivalence principle. But it will be seen below that my statement of the Church-Turing principle (1.2) is manifestly physical, and unambiguous. I shall now show that it has the same epistemological status as other physical principles.
>
> I propose to reinterpret Turing's 'functions which would naturally be regarded as computable' as the functions which may in principle be computed by a real physical system. For it would surely be hard to regard a function 'naturally' as computable if it could not be computed in Nature, and conversely. To this end I shall define the notion of '*perfect simulation*'. A computing machine $\mathcal{M}$ is capable of perfectly simulating a physical system $\mathcal{S}$, under a given labelling of their inputs and outputs, if there exists a program $\pi(\mathcal{S})$ for $\mathcal{M}$ that renders $\mathcal{M}$ computationally equivalent to $\mathcal{S}$ under that labelling. In other words, $\pi(\mathcal{S})$ converts $\mathcal{M}$ into a 'black box' functionally indistinguishable from $\mathcal{S}$."
>
> *. . . Deutsch: [Deu85], §1 Computing machines and the Church-Turing principle, p.3.*

Such an interpretation of Deutsch's intent seems, prima facie, consistent with Deutsch's postulation of a Church-Turing principle which 'is so strong that it is *not* satisfied by Turing's machine in classical physics':

> "I can now state the physical version of the Church-Turing principle:
>
> > '*Every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means.*' (1.2)
>
> This formulation is both better defined and more physical than Turing's own way of expressing it (1.1), because it refers exclusively to objective concepts such as 'measurement', 'preparation' and 'physical system', which are already present in measure theory. It avoids terminology like 'would naturally be regarded', which does not fit well into the existing structure of physics.
>
> The 'finitely realizable physical systems' referred to in (1.2) must include any physical object upon which experimentation is possible. The 'universal computing machine' on the other hand,

need only be an idealized (but theoretically permitted) finitely specifiable model. The labelling implicitly referred to in (1.2) must also be finitely realizable.

The reference in (1.1) to a specific universal computing machine (Turing's) has of necessity been replaced in (1.2) by the more general requirement that this machine operate by 'finite means'. 'Finite means' can be defined axiomatically, without restricting assumptions about the form of physical laws (cf. Gandy 1980). If we think of a computing machine as proceeding in a sequence of steps whose duration has a non-zero lower bound, then it operates by 'finite means' if (i) only a finite subsystem (though not always the same one) is in motion during any one step, and (ii) the motion depends only on the state of a finite subsystem, and (iii) the rule that specifies the motion can be given finitely in the mathematical sense (for example as an integer). Turing machines satisfy these conditions, and so does the universal quantum computer $\mathcal{Q}$ (see §2).

The statement of the Church-Turing principle (1.2) is stronger than what is strictly necessitated by (1.1). Indeed it is so strong that it is *not* satisfied by Turing's machine in classical physics. Owing to the continuity of classical dynamics, the possible states of a classical system necessarily form a continuum. Yet there are only countably many ways of preparing a finite input for $\mathcal{T}$. Consequently $\mathcal{T}$ cannot perfectly simulate any classical dynamical system. (The well studied theory of the 'simulation' of continuous systems by $\mathcal{T}$ concerns itself not with perfect simulation in my sense but with successive discrete approximation.)"

. . . Deutsch: [Deu85], §1 Computing machines and the Church-Turing principle, pp.3-4.

However, Deutsch's belief that owing 'to the continuity of classical dynamics, the possible states of a classical system necessarily form a continuum', whence the universal Turing machine '$\mathcal{T}$ cannot perfectly simulate any classical dynamical system', needs qualification.

For instance, if we accept that any physical measurement of an 'unpredictable' observable whose values range over the real continuum $\mathbb{R}$ can only be made to a precision whose level is indexed by, say, a finite natural number $p_\alpha \geq 0$ that specifies the physical limitation level of the observer's sensory apparatus α, then:

Comment: It seems pertinent to note that, as argued by Diederik Aerts et al in [Aetal] (see also §19.B.d., para (3)), reference to an 'observable' in this context could implicitly entail an extraneous, and avoidable, ontological commitment to the putative existence of a (platonic) 'potential entity' in the universe we inhabit, say U, with misleading philosophical and mathematical consequences which, in any *well-defined* mathematical model M that seeks to represent U faithfully, may not be falsifiable either in M, or under any *well-defined* interpretation of M in U.

- although the value of any such observable *before* measurement can, in principle, be assumed mathematically as a putative, algorithmically *verifiable* but not algorithmically *computable*, real number, say r,

  – such that $|r| = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{10^i}$, where $a_0 \geq 0$ is an integer and a_i is either 0 or 1,

- any physically measured value of r can—even in principle—only be assumed upto a maximum number of terms in the above sequence (the latter can, thus, be treated as defining p_α);

  – whence $|r| = \sum_{i=1}^{p_\alpha} \frac{a_i}{10^i} + \sum_{i=p_\alpha+1}^{\infty} \frac{a_i}{10^i}$,
  – where $\sum_{i=1}^{p_\alpha} \frac{a_i}{10^i}$ denotes the measurable part of r to the precision p_α.

Now, we can define a unique, primitive recursive, Gödel β-function $\beta_{p_\alpha}(n)$ such that $\beta_{p_\alpha}(i) = a_i$ for all $1 \leq i \leq p_\alpha$.

Moreover, $\beta_{p_\alpha}(n)$ defines a Turing machine, say $T_{\beta_{p_\alpha}}$ which can be uniquely Gödel-numbered by an integer, say $\lceil T_{\beta_{p_\alpha}} \rceil$.

Thus, under the assumption—made, for instance, in the definition of a putative quantum computer—that the instantaneous tape description of a Turing machine admits of an infinite tape:

- the non-terminating binary sequence $a_0, a_1, \ldots$ defining any algorithmically *computable* real number r,

  – such that $|r| = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{10^i}$, where $a_0 \geq 0$ is an integer and a_i is either 0 or 1,

  can be referenced by the Gödel number, say $\lceil T \rceil$, of the Turing machine T that computes it, and admitted as an input of a universal Turing machine;

- whilst, for any specified natural number $p_\alpha > 0$, any algorithmically *verifiable*, but not algorithmically *computable*, real number r,

  – such that $|r| = \sum_{i=1}^{p_\alpha} \frac{a_i}{10^i} + \sum_{i=p_\alpha+1}^{\infty} \frac{a_i}{10^i}$,
  – where $\sum_{i=1}^{p_\alpha} \frac{a_i}{10^i}$ denotes the measurable part of r,
    * to the integral precision level p_α
    * to which an observer's sensory apparatus α
    * which measures r is calibrated,

  can be referenced by the Gödel number, say $\lceil T_{\beta_{p_\alpha}} \rceil$, of the unique Turing machine $T_{\beta_{p_\alpha}}$ that computes the unique Gódel β-function for the sequence $a_0, a_1, \ldots, a_{p_\alpha}$; and the non-terminating sequence defining the β-function admitted as an input.

In other words, we can treat the real number computed by $T_{\beta_{p_\alpha}}$ as a p_α-approximation value into which the value r of an 'unpredictable' observable 'collapses' upon a measurement with precision index p_α.

What is also problematic, from an *evidence-based* perspective, is Deutsch's further postulation that 'every real (dissipative) finite physical system can be perfectly simulated' (see, for instance, Diederik Aerts' argument—in §19.B.c. and §19.B.d.—regarding the physical impossibility of 'exact replicability' of spatially separated experiments) by a universal quantum computer $\mathcal{Q}$':

> "...In §3, I shall show that it is consistent with our present knowledge of the interactions present in Nature that every real (dissipative) finite physical system can be perfectly simulated by the universal quantum computer $\mathcal{Q}$. Thus quantum theory is compatible with the strong form (1.2) of the Church-Turing principle."
>
> ...Deutsch: [Deu85], §1 Computing machines and the Church-Turing principle, p.4.

We note that Deutsch seeks to validate his postulation of the physical Church-Turing principle by arguing 'that (1.2) is an empirical assertion', on the grounds that 'since the number of possible programs for a universal computer is infinite, no experiment could in general verify that none of them can simulate a system that is thought to be a counter-example to (1.2)':

"I now return to my argument that (1.2) is an empirical assertion. The usual criterion for the empirical status of a theory is that it be experimentally falsifiable (Popper 1959), i.e. that there exist potential observations that would contradict it. However, since the deeper theories we call 'principles' make reference to experiment only *via* other theories, the criterion of falsifiability must be applied indirectly in their case. The principle of conservation of energy, for example, is not in itself contradicted by any conceivable observation because it contains no specification of how to measure energy. The third law of thermodynamics whose form

> '*No finite process can reduce the entropy or temperature of a finitely realizable physical system to zero*'   (1.3)

bears a certain resemblance to that of the Church-Turing principle, is likewise not directly refutable: no temperature measurement of finite accuracy could distinguish absolute zero from an arbitrarily small positive temperature. Similarly, since the number of possible programs for a universal computer is infinite, no experiment could in general verify that none of them can simulate a system that is thought to be a counter-example to (1.2)."

. . . *Deutsch:* [Deu85], *§1 Computing machines and the Church-Turing principle, p.4.*

where he further argues that 'there is no *a priori* reason why physical laws should respect the limitations of the mathematical processes we call 'algorithms' (i.e. the functions C($\mathcal{T}$))' and that 'there is nothing paradoxical or inconsistent in postulating physical systems which compute functions not in C(T)':

"But all this does not place 'principles' outside the realm of empirical science. On the contrary, they are essential frameworks within which directly testable theories are formulated. Whether or not a given physical theory contradicts a principle is first determined by logic alone. Then, if the directly testable theory survives crucial tests but contradicts the principle, that principle is deemed refuted, albeit indirectly. If all known experimentally corroborated theories satisfy a restrictive principle, then that principle is corroborated and becomes, on the one hand, a guide in the construction of new theories, and on the other, a means of understanding more deeply the content of existing theories.

It is often claimed that every 'reasonable' *physical* (as opposed to mathematical) model for computation, at least for the deterministic computation of functions from $\mathbb{Z}$ to $\mathbb{Z}$, is equivalent to Turing's. But this is not so; there is no *a priori* reason why physical laws should respect the limitations of the mathematical processes we call 'algorithms' (i.e. the functions C($\mathcal{T}$)). Although I shall not in this paper find it necessary to do so, there is nothing paradoxical or inconsistent in postulating physical systems which compute functions not in C(T). There could be experimentally testable theories to that effect: e.g. consider any recursively enumerable non-recursive set (such as the set of integers representing programs for terminating algorithms on a given Turing machine). In principle, a physical theory might have among its implications that a certain physical device $\mathcal{F}$ could compute in a specified time whether or not an arbitrary integer in its input belonged to that set. This theory would be experimentally refuted if a more pedestrian Turing-type computer, programmed to enumerate the set, ever disagreed with $\mathcal{F}$. (Of course the theory would have to make other predictions as well, otherwise it could never be non-trivially *corroborated*, and its structure would have to be such that its exotic predictions about $\mathcal{F}$ could not naturally be severed from its other physical content. All this is logically possible.)"

. . . *Deutsch:* [Deu85], *§1 Computing machines and the Church-Turing principle, pp.4-5.*

Now, from an *evidence-based* perspective, the only functions that are not in C($\mathcal{T}$) are number-theoretic functions that, by definition, are algorithmically *verifiable* but not algorithmically *computable*.

Moreover, since the 'value' of a measurable physical observable must be *evidence-based*, every function that seeks to mathematically represent a measurable physical observable must

be algorithmically *verifiable* for it be defined as 'effectively computable' (see §4.G.a., Definition 21).

From such a perspective, Deutsch's speculation that 'there is nothing paradoxical or inconsistent in postulating physical systems which compute functions not in C(T)' might only be viewable as 'logically possible' in the absence of an *evidence-based* definition of 'effective computability'.

For it is only the absence of such definition which seemingly allows him to argue that 'the laws of physics 'happen to' permit the existence of physical models for the operations of arithmetic such as addition, subtraction and multiplication'; and that the 'reason why we are confident that the machines we call calculators do indeed compute the arithmetic functions they claim to compute is not that we can 'check' their answers', but that the 'theory, including its assertion that the abstract functions of arithmetic are realized in Nature, is empirical':

> Nor, conversely, is it obvious *a priori* that any of the familiar recursive functions is in reality computable. The reason we find it possible to construct, say, electronic calculators, and indeed why we can perform mental arithmetic, cannot be found in mathematics or logic. *The reason is that the laws of physics 'happen to' permit the existence of physical models for the operations of arithmetic such* as addition, subtraction and multiplication. If they did not, these familiar operations would be non-computable functions. We might still know *of* them and invoke them in mathematical proofs (which would presumably be called 'non-constructive') but we could not perform the
>
> If the dynamics of some physical system did depend on a function not in C($\mathcal{T}$), then that system could in principle be used to compute the function. Chaitin (1977) has shown how the truth values of all 'interesting' non-Turing decidable propositions of a given formal system might be tabulated very efficiently in the first few significant digits of a single physical constant.
>
> But if they were, it might be argued, we could never know because we could not check the accuracy of the 'table' provided by Nature. This is a fallacy. The reason why we are confident that the machines we call calculators do indeed compute the arithmetic functions they claim to compute is not that we can 'check' their answers, for this is ultimately a futile process of comparing one machine with another: *Quis custodiet ipsos custodes?* The real reason is that we believe the detailed physical theory that was used in their design. That theory, including its assertion that the abstract functions of arithmetic are realized in Nature, is empirical."
>
> . . . Deutsch: [*Deu85*], §1, *Computing machines and the Church-Turing principle, p.5.*

However, from the *evidence-based* perspective of the Complementarity Thesis (see §1., Thesis 1) underlying this investigation:

- it is not that 'the laws of physics 'happen to' permit the existence of physical models for the operations of arithmetic such as addition, subtraction and multiplication';

- it is that the 'operations of arithmetic such as addition, subtraction and multiplication', amongst other *well-definable* mathematically expressions, allow us to:

    - faithfully express, and categorically communicate,
    - without any ontological entailment (see §19.B.d.),
    - conceptual metaphors (as described by Lakoff and Núñez in [LR00]),
    - corresponding to our physical observations/measurements,
    - in a mathematical language that admits, both:

∗ functions that are algorithmically *computable* (hence *deterministic* and *pre-dictable*); as well as

∗ functions that are algorithmically *verifiable* but not algorithmically *computable* (hence *deterministic* but *essentially unpredictable*).

Moreover, from the *evidence-based* perspective of this investigation, although Deutsch *can* justifiably claim 'that classical physics and the classical universal Turing machine do not obey the Church-Turing principle in the strong physical form', he *cannot*, in the absence of a plausible, alternative, *evidence-based* definition of '*effective computability*', justifiably claim without qualification that every 'existing general model of computation is effectively classical', nor 'that classical physics is false', without contradicting §4.G.a., Definition 21:

> "Every existing general model of computation is effectively classical. That is, a full specification of its state at any instant is equivalent to the specification of a set of numbers, all of which are in principle measurable. Yet according to quantum theory there exist no physical systems with this property. The fact that classical physics and the classical universal Turing machine do not obey the Church-Turing principle in the strong physical form (1.2) is one motivation for seeking a truly quantum model. The more urgent motivation is, of course, that classical physics is false.
>
> . . . *Deutsch: [Deu85], 2 Quantum computers.*

17.I. Kalmár's perspective on the Church-Turing Thesis

We note that the reasoning in the proof of §17.F.b., Theorem 17.3, further entails that (for a more direct proof see §4.G.a., Theorem 4.3):

Corollary 17.7. *The classical Church-Turing thesis is false in any interpretation of the first-order Peano Arithmetic PA that admits evidence-based reasoning.*

Moreover, it is essentially similar to Kalmár's argument in [Km59] (p.74).

In other words—excepting that it would always calculate the function $g(n)$ constructively (even in the absence of a uniform procedure) within a fixed postulate system—the reasoning used in §4.G.a., Theorem 17.3 is, essentially, the same as Selmer Bringsjord's concise expression, of Kalmár's argument, in the former's narrational case against Church's Thesis:

> "First, he draws our attention to a function g that isn't Turing-computable, given that f is[168]:
>
> $g(x) = \mu y(f(x, y) = 0) = $ the least y such that $f(x, y) = 0$ if y exists; and 0 if there is no such y
>
> Kalmár proceeds to point out that for any n in $\mathbb{N}$ for which a natural number y with $f(n, y) = 0$ exists, 'an obvious method for the calculation of the least such y ... can be given,' namely, calculate in succession the values $f(n, 0), f(n, 1), f(n, 2), \ldots$ (which, by hypothesis, is something a computist or TM can do) until we hit a natural number m such that $f(n, m) = 0$, and set $y = m$.
>
> On the other hand, for any natural number n for which we can prove, not in the frame of some fixed postulate system but by means of arbitrary—of course, correct—arguments that no natural number y with $f(n, y) = 0$ exists, we have also a method to calculate the value $g(n)$ in a finite number of steps.
>
> Kalmár goes on to argue as follows. The definition of g itself implies the tertium non datur, and from it and CT we can infer the existence of a natural number p which is such that

---

[168]Bringsjord notes that the original proof can be found on page 741 of Kleene [Kl36].

(*) there is no natural number y such that $f(p, y) = 0$; and

(**) this cannot be proved by any correct means.

Kalmár claims that (*) and (**) are very strange, and that therefore CT is at the very least implausible."

...*Bringsjord: [Bri93].*

Kalmár himself argues further to the effect that the proposition stating that, for this p, there is a natural number y such that $f(p, y) = 0$, would then be *absolutely* undecidable in the sense that:

"...the problem if this proposition holds or not, would be unsolvable, not in Gödel's sense of a proposition neither provable nor disprovable in the frame of a fixed postulate system, nor in Church's sense of a problem with a parameter for which no general recursive method exists to decide, for any given value of the parameter in a finite number of steps, which is the correct answer to the corresponding particular case of the problem, "yes" or "no". As a matter of fact, the problem, if the proposition in question holds or not, does not contain any parameter and, supposing Church's thesis, *the proposition itself can be neither proved nor disproved*, not only in the frame of a fixed postulate system, but *even admitting any correct means.* It cannot be proved for it is false and it cannot be disproved for its negation cannot be proved. According to my knowledge, this consequence of Church's thesis, viz. the existence of a proposition (without a parameter) which is undecidable in this, *really absolute sense*, has not been remarked so far.

However, this "absolutely undecidable proposition" has a defect of beauty: we can decide it, for we know, it is false. Hence, *Church's thesis implies the existence of an absolutely undecidable proposition which can be decided viz., it is false, or*, in another formulation, *the existence of an absolutely unsolvable problem with a known definite solution*, a very strange consequence indeed."

...*Kalmár: [Km59], p.75.*

17.J. Is the brain a Turing machine?

We note that the proposed Turing Test in §17.E. (Query 17) only seeks to differentiate between human and mechanistic *intelligences*. It does not address the more fundamental question of whether an organic brain is, or is not, essentially a Turing machine.

The distinction is not always obvious, or respected. For instance, as remarked upon in a 2014 BBC interview by Stephen Hawking (see also [Sha06]):

"I believe there is no deep difference between what can be achieved by a biological brain and what can be achieved by a computer. It therefore follows that computers can, in theory, emulate human intelligence—and exceed it."

...*Hawking: Stephen Hawking—Will AI kill or save humankind?*

Although consideration of the issue lies beyond the scope and competence of this investigation, nevertheless, if we posit that all outputs of sensory organs can only be received/perceived and/or transmitted as digital pulses to/by the brain then, from the *evidence-based* perspective of this investigation, one *could* speculate that an organic brain *can* be modeled by a Turing machine, and *strongly* hypothesise that:

Hypothesis 1. *Whilst an organic brain can evidence that an arithmetical proposition is algorithmically computable as true under an interpretation, only the sensory organs (such as those of sight, smell, hearing, taste and touch) can evidence that an arithmetical proposition is algorithmically verifiable as true under an interpretation.*

In other words, whilst the brain functions can be treated as essentially digital, and representable completely by a Turing machine, the functions of the sensory organs could be treated as essentially analog, and representable only by geometrical models that cannot always be represented completely in their limiting cases by a Turing machine.

Nevertheless, the *evidence-based* perspective of this investigations suggests the possibility that—assuming the intuitive meaning of 'the state of a physical process'—we can tentatively define what it means for a process to be 'analog':

Definition 33. (Analog process) *A physical process is analog if, and only if, it's states can be represented mathematically by a number-theoretic function that is algorithmically verifiable.*

Definition 34. (Digital process) *A physical process is digital if, and only if, it's states can be represented mathematically by a number-theoretic function that is algorithmically computable.*

> **Comment**: We note, however, that the above definitions may be at variance with the term 'analog' as used, for example, by Piccinini in [Pic15] where he notes that:
>
>> "...the notion of analog modeling is a semantic notion—it defines 'analog' in terms of a representational relation between the model and the target system. True, analog computers (like other analog models) are typically used to model other systems. Also true, the history of analog modeling is intertwined with the history of analog computing properly so called (Care 2010). But, as we saw in Chapter 3, computation does not require representation."
>>
>> *...Piccinini: [Pic15], §1, Disambiguating 'Analog'.*

Thus, as suggested by §16.C., Cases 16.C.a. to 16.C.d., and §16.D., Case 16.D.c., some geometric models of physical phenomena can only be defined in terms of algorithmically *verifiable*, but not algorithmically *computable*, functions (compare with the putative mathematical representation of quantum phenomena posited in §19.A.).

> **Comment**: Such a perspective can be viewed as implicit in Marcin Miłkowski's query (compare §17.E.) in [Mil14]: *Is the mind a Turing machine? How could we tell?*, which addresses—within the broader issue of Church's Thesis (compare §4.G.a., Theorem 4.3)—whether the mind:
>
> - treated as the source of the conceptual metaphors
>   - that correspond, initially, to 'analog' processes sought to be 'captured' by organic/mechanical sensory perceptions/perceptors,
>   - and subsequently sought to be represented symbolically
>   - in an appropriate language of adequate expression
>   - and, ideally, of categorical communication,
> - is, or functions as, an 'analog' computer;
>
> where we note the similarity between the above issue, and that of whether a sports television broadcasting system:
>
> - treated as the source of the images and sound projected digitally on a home TV,
>   - that correspond, initially, to 'analog' processes sought to be 'captured' by mechanical sensors (cameras/microphones) on a playing field,
>   - and subsequently represented symbolically
>   - in an appropriate language of adequate expression
>   - and categorical communication,

- is, or functions as, an 'analog' computer.

"I. Introduction

In philosophical discussions, it is often assumed that the computational theory of mind implies that it is a Universal Turing Machine (UTM). The reason why the UTM has been proposed as a model of the mind is that it is *the* standard model of computation, and that is a *universal* machine—i.e., any other digital computer may be simulated by the UTM. Were the mind unable to compute everything that the UTM can, the UTM would still be able to simulate it. Of course, this makes the truth of the Church-Turing thesis vital for the explanatory value of computational modelling of cognition. Or so it seems.

In recent years, several criticisms of the proposed identity of the mind and the UTM have been raised. First, it was argued that the UTM requires an infinite tape, which is physically impossible (or impossible for a physical part of the brain). Second, some have claimed that there are physical systems capable of hyper-Turing computation, so the UTM might not be the strongest model of computation available. Some theorists claim, for example, that brains are hypercomputational analogue machines that need to be modelled as using genuine reals (as opposed to TM-computable reals). Third, it was argued that the UTM is not a good candidate for explaining the intricacies of the human mind as it has a completely different architecture: so, while the set of functions computed by the machine could be the same as the one computed by the mind, it would differ dramatically in terms of speed and space requirements. A related objection is that brains are analogue computers that cannot be modelled directly as the UTM.

Are these strong objections? Do they undermine the general claim that the mind is a Turing Machine (TM)? To answer these two questions, one needs to understand what it is for a system to be a TM (due to limitations of space, in what follows I will set the universality of the machine aside). Otherwise, there is a danger of serious equivocation in arguments in favour of certain theories of the mind [14].

It is important to distinguish two senses in which one could say that a physical system "is" a TM:

(1) when a physical system has a function that could be simulated with a TM (functional sense);
(2) when a physical system is a mechanism best described as a TM (mechanistic sense).

In this paper, I enumerate conditions that must be fulfilled to qualify a physical system to be a TM in both senses and show differences between the two. I then argue that the first two objections are relevant for the functional sense of "is" (which is also implicated by the mechanistic sense), and the last one can be understood only in the mechanistic sense.

I briefly review how these objections would be elaborated (though the space prohibits any further discussion in detail). I argue that the first two are not really relevant for computationalism in the theory of cognition, and that the last one is plausible: I justify the view that the mind does not seem to be a UTM in the mechanistic sense at all because of its architecture; and proper computational explanations in cognitive science require that the architecture be matched strictly. It transpires that the philosophically popular UTM is not a good candidate for a scientific model of the mind, even if we accept the standard computational theory of mind. To defeat computationalism, one should look beyond the TMs."

... *Miłkowski: [Mil14].*

We note that Hypothesis 1 could, conceivably, offer in support the experimental observations

and conclusions by Humberto Maturana, Jerome Lettvin, Warren McCulloch, and Walter Pitts in [LMMP], where they investigate the behaviour of a frog:

> ## "I. INTRODUCTION
>
> ### *A. Behavior of a Frog*
>
> A frog hunts on land by vision. He escapes enemies mainly by seeing them. His eyes do not move, as do ours, to follow prey, attend suspicious events, or search for things of interest. If his body changes its position with respect to gravity or the whole visual world is rotated about him, then he shows compensatory eye movements. These movements enter his hunting and evading habits only, e.g., as he sits on a rocking lily pad. Thus his eyes are actively stabilized. He has no fovea, or region of greatest acuity in vision, upon which he must center a part of the image. He has only a single visual system, retina to colliculus, not a double one such as ours where the retina sends fibers not only to colliculus but to the lateral geniculate body which relays to cerebral cortex. Thus, we chose to work on the frog because of the uniformity of his retina, the normal lack of eye and head movements except for those which stabilize the retinal image, and the relative simplicity of the connection of his eye to his brain.
>
> The frog does not seem to see or, at any rate, is not concerned with the detail of stationary parts of the world around him. He will starve to death surrounded by food if it is not moving. His choice of food is determined only by size and movement. He will leap to capture any object the size of an insect or worm, providing it moves like one. He can be fooled easily not only by a bit of dangled meat but by any moving small object. His sex life is conducted by sound and touch. His choice of paths in escaping enemies does not seem to be governed by anything more devious than leaping to where it is darker. Since he is equally at home in water and on land, why should it matter where he lights after jumping or what particular direction he takes? He does remember a moving thing providing it stays within his field of vision and he is not distracted."
>
> *. . . Maturana, Lettvin, McCulloch, Pitts: [LMMP].*

Maturana, Lettvin, McCulloch, and Pitts sought to verify/challenge the assumption that 'the eye mainly senses light, whose local distribution is transmitted to the brain in a kind of copy by a mosaic of impulses':

> ### "*D. Initial Argument*
>
> The assumption has always been that the eye mainly senses light, whose local distribution is transmitted to the brain in a kind of copy by a mosaic of impulses. Suppose we held otherwise, that the nervous apparatus in the eye is itself devoted to detecting certain patterns of light and their changes, corresponding to particular relations in the visible world. If this should be the case, the laws found by using small spots of light in the retina may be true and yet, in a sense, be misleading. Consider, for example, a bright spot appearing in a receptive field. Its actual and sensible properties include not only intensity, but the shape of the edge, its size, curvature, contrast, etc.
>
> We decided then how we ought to work. First, we should find a way of recording from single myelinated and unmyelinated fibers in the intact optic nerve. Second, we should present the frog with as wide a range of visible stimuli as we could, not only spots of light but things he would be disposed to eat, other things from which he would flee, sundry geometrical figures, stationary and moving about, etc. From the variety of stimuli we should then try to discover what common features were abstracted by whatever groups of fibers we could find in the optic nerve. Third, we should seek the anatomical basis for the grouping."
>
> *. . . Maturana, Lettvin, McCulloch, Pitts: [LMMP].*

Their experiments showed 'that the eye speaks to the brain in a language already highly organized and interpreted, instead of transmitting some more or less accurate copy of the distribution of light on the receptors':

"VII. GENERAL DISCUSSION

What are the consequences of this work? Fundamentally, it shows that the eye speaks to the brain in a language already highly organized and interpreted, instead of transmitting some more or less accurate copy of the distribution of light on the receptors. As a crude analogy, suppose we have a man watching the clouds and reporting them to a weather station. If he is using a code, and one can see his portion of the sky too, then it is not difficult to find out what he is saying. It is certainly true that he is watching a distribution of light; nevertheless, local variations of light are not the terms in which he speaks nor the terms in which he is best understood. Indeed, if his vocabulary is restricted to types of things that he sees in the sky, trying to find his language by using flashes of light as stimuli will certainly fail. Now, since the purpose of a frog's vision is to get him food and allow him to evade predators no matter how bright or dim it is about him, it is not enough to know the reaction of his visual system to points of light. To get useful records from individual receptors (the rods and cones), assuming that they operate independently and under no reflex control, this stimulus may be adequate. But when one inspects responses that are a few nervous transformations removed from the receptors, as in the optic nerve, that same choice of stimulus is difficult to defend. It is equivalent to assuming that all of the interpretation is done further on in the nervous system. But, as we have seen, this is false."

> . . . *Maturana, Lettvin, McCulloch, Pitts:* [LMMP].

They concluded that, apropos the operations of the eye (ergo, conceivably, of other sensory organs), 'the language in which they are best described is the language of complex abstractions from the visual image':

"VIII. CONCLUSION

The output from the retina of the frog is a set of four distributed operations of the visual image. These operations are independent of the level of general illumination and express the image in terms of 1) local sharp edges and contrast, 2) the curvature of edge of a dark object, 3) the movement of edges, and 4) the local dimming produced by movement or rapid general darkening. Each group of fibers serving one operation maps the retina continuously in a single sheet of endings in the frog's brain. There are four such sheets in the brain, corresponding to the four operations, and their maps are in registration. When all axonal connections between eye and brain are broken and the fibers grow back, they reconstitute the original retinal maps and also arrange themselves in depth in the original order with no mistakes. If there is any randomness in the connections of this system, it must be at a very fine level indeed. In this, we consider Sperry (14) completely right.

We have described each of the operations on the retinal image in terms of what common factors in a large variety of stimuli cause response and what common factors have no effect. What, then, does a particular fiber in the optic nerve measure? We have considered it to be how much there is in a stimulus of that quality which excites the fiber maximally, naming that quality.

The operations thus have much more the flavor of perception than of sensation, if that distinction has any meaning now. That is to say that the language in which they are best described is the language of complex abstractions from the visual image. We have been tempted, for example, to call the convexity detectors "bugperceivers." Such a fiber (operation 2) responds best when a dark object, smaller than a receptive field, enters that field, stops, and moves about intermittently thereafter. The response is not affected if the lighting changes or if the background (say a picture of grass and flowers) is moving, and is not there if only the background, moving or still, is in the field. Could one better describe a system for detecting an accessible bug?"

> . . . *Maturana, Lettvin, McCulloch, Pitts:* [LMMP].

Following their work, the accepted paradigm today is, apparently, that Maturana, Lettvin, McCulloch and Pitts have:

"...conclusively demonstrated that "analog processes in the eye were doing at least part of the interpretive work" in image processing as opposed to "the brain computing information digital neuron by digital neuron using the exacting implement of mathematical logic", ..."

...Wikipedia: https://en.wikipedia.org/wiki/Walter_Pitt, downloaded 25/11/2019.

As reported by science writer Amanda Gefter for a lay reader, in a tribute to Walter Pitts in her *Nautilus* blogpage *The Man Who Tried to Redeem the World with Logic*:

"At the time, biologists believed that the eye was like a photographic plate that passively recorded dots of light and sent them, dot for dot, to the brain, which did the heavy lifting of interpretation. Lettvin decided to put the idea to the test, opening up the frog's skulls and attaching electrodes to single fibers in their optic nerves.

Together with Pitts, McCulloch and the Chilean biologist and philosopher Humberto Maturana, he subjected the frogs to various visual experiences—brightening and dimming the lights, showing them color photographs of their natural habitat, magnetically dangling artificial flies—and recorded what the eye measured before it sent the information off to the brain. To everyone's surprise, it didn't merely record what it saw, but filtered and analyzed information about visual features like contrast, curvature, and movement. "The eye speaks to the brain in a language already highly organized and interpreted," they reported in the now-seminal paper "What the Frog's Eye Tells the Frog's Brain," published in 1959.

The results shook Pitts' worldview to its core. Instead of the brain computing information digital neuron by digital neuron using the exacting implement of mathematical logic, messy, analog processes in the eye were doing at least part of the interpretive work."

...Gefter: Nautilus, February 5, 2015, http://nautil.us/issue/21/information/the-man-who-tried-to-redeem-the-world-with-logic

17.J.a.   Awareness is *not* an emergent *state*

Moreover:

- apropos the mind-brain dualism articulated by René Descartes in the 17th century,

- reflected implicitly in *the* putative, non-material, 'I' in the expression 'Je pense, donc je suis',

the above hypothesis (§17.J., Hypothesis 1) suggests that we could, then, further qualify Descartes' well-known aphorism:

- by the anti-emergent hypothesis (Hypothesis 2),

- that awareness is *not* an emergent *state* of an intelligence,

- but an *ability* to express and communicate sensory perceptions (not necessarily symbolically).

Hypothesis 2. (Awareness) *Awareness is the primary conceptual metaphor that corresponds to the ability of an intelligence to—not necessarily symbolically—express sensory perceptions in the first person as 'I sense'.*

and, further, that:

Hypothesis 3. (Self-awareness) *Self-awareness is the secondary conceptual metaphor that corresponds to the ability of an intelligence to symbolically postulate the existence of an id that can be subjectively identified as aware, and which is implicit in the expression 'I sense, therefore I am'.*

In other words, Hypotheses 2 and 3 suggest that:

- Intelligences which can pro-actively protect themselves, their habitats, and/or their species from life-threatening situations can be treated as being aware; whilst

- Intelligences which can answer the Turing Test (see §17.E., Query 17) affirmatively could be treated as being self-aware.

CHAPTER 18. *PARADIGM CHALLENGING CONSEQUENCES*

18. The significance of *evidence-based* reasoning for the Theory of Numbers and Computational Complexity

The significance of *evidence-based* reasoning for distinguishing between algorithmically *verifiable* and algorithmically *computable* number-theoretic functions, as in §2., Definitions 2 and 3 is that, without extending the language in which number-theoretic functions and relations are expressed, it provides a formal foundation for placing in perspective, and complementing, an uncomfortably counter-intuitive entailment in number theory—namely §18.A., Theorem 18.3—which has been treated by conventional wisdom (see §18.A.a.) as sufficient for concluding that the prime divisors of an integer *cannot* be proven to be mutually independent.

However, we shall show that such informally perceived barriers are, in this instance, illusory (§18.A.b.); and that admitting the above distinction illustrates:

(a) *Why* the prime divisors of an integer *are* mutually independent (see §18.A.c., Theorem 18.11);

(b) *Why* determining whether the signature (Definition 35) yielded by simultaneous spins of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig.4)—is that of a prime, or not, can be done deterministically in polynomial time $O(log_e n)$ (Theorem 18.13); as compared to the time $\ddot{O}(log_e^{15/2}n)$ given by Agrawal et al in [AKS04], and improved to $\ddot{O}(log_e^6 n)$ by Lenstra and Pomerance in [LP11], for determining whether a specified integer n is a prime or not.

(c) *Why* it can be cogently argued that determining a factor of a *specified* integer deterministically cannot be polynomial time (see §18.A.f., Theorem 18.15).

Definition 35. (Signature of a natural number) *The*[169] *signature of a specified integer n is the sequence of residues $< a_{n,i} >$ where $n + a_{n,i} \equiv 0 \mod (p_i)$ for all primes p_i such that $1 \leq i \leq \pi(\sqrt{n})$.*

Definition 36. (Value of a natural number) *The value of a specified integer n is any well-defined interpretation—over the domain of the natural numbers—of the (unique) numeral [n] that represents n in the first-order Peano Arithmetic PA.*

18.A. Are the prime divisors of an integer mutually independent?

We begin by addressing the query:

Query 23. *Are the prime divisors of an integer n mutually independent?*

Definition 37. *Two events are independent if the occurrence of one event does not influence (and is not influenced by) the occurrence of the other.*

---

[169]Unique since, if $p^2_{\pi(\sqrt{m})+1} > m \geq p^2_{\pi(\sqrt{m})}$ and $p^2_{\pi(\sqrt{n})+1} > n \geq p^2_{\pi(\sqrt{n})}$ have the same signature, then $|m - n| = c_1 . \prod_{i=1}^{\pi(\sqrt{m})} p_i = c_2 . \prod_{i=1}^{\pi(\sqrt{n})} p_i$; whence $c_1 = c_2 = 0$ since $\prod_{i=1}^{k} p_i > (\prod_{i=2}^{k-2} p_i).p_k^2 > p_{k+1}^2$ for $k > 4$ by appeal to Bertrand's Postulate $2.p_k > p_{k+1}$; and the uniqueness is easily verified for $k \leq 4$.

Intuitively, the prime divisors of an integer *seem* to be mutually independent by virtue of the Fundamental Theorem of Arithmetic:

Theorem 18.1. (Fundamental Theorem of Arithmetic) *Every positive integer $n > 1$ can be represented in exactly one way as a product of prime powers:*

$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} = \prod_{i=1}^{k} p_i^{n_i}$

where $p_1 < p_2 < \ldots < p_k$ are primes and the n_i are positive integers (including 0).

Moreover, the prime divisors of n can also be *seen* to be mutually independent in the usual, linearly displayed, Sieve of Eratosthenes, where whether an integer n is crossed out as a multiple of a prime p is *obviously* independent (in the sense of Definition 37) of whether it is also crossed out as a multiple of a prime $q \neq p$:

$E(\cancel{1}), \; E(2), \; E(3), \; E(\cancel{4}), \; E(5), \; E(\cancel{6}), \; E(7), \; E(\cancel{8}), \; E(\cancel{9}), \; E(\cancel{10}), \; E(11), \; \ldots$

Despite such compelling evidence, conventional wisdom appears to accept as definitive the counter-intuitive conclusion (addressed in §18.A.a.) that although we can *see* it as true, we cannot mathematically prove the following proposition as *true*:

Proposition 18.2. *Whether or not a prime p divides an integer n is independent of whether or not a prime $q \neq p$ divides the integer n.*

> **Comment** (see also §11.C.): We note that such an *unprovable-but-intuitively-true* conclusion makes a *stronger* assumption than that in Gödel's similar claim for his arithmetical formula $[(\forall x)R(x)]$—whose Gödel-number is $17Gen\ r$—in [Go31], p.26(2). *Stronger*, since Gödel does not assume his proposition to be *intuitively true*, but shows that though the arithmetical formula with Gödel-number $17Gen\ r$ is not provable in his Peano Arithmetic P yet, for any P-numeral $[n]$, the formula $[R(n)]$ whose Gödel-number is $Sb\left(r \; \begin{matrix} 17 \\ Z(n) \end{matrix} \right)$ *is P-provable, and therefore meta-mathematically* true under any well-defined Tarskian interpretation of P (see §2.A.).
>
> Expressed in computational terms (see §2.F., Corollary 2.21), under any well-defined interpretation of P, Gödel's formula $[R(x)]$ translates as an arithmetical relation, say $R'(x)$, such that $R'(n)$ is algorithmically *verifiable*, but not algorithmically *computable*, as always true over N, since $[\neg(\forall x)R(x)]$ is P-provable (see §2.F., Corollary 2.20).

We, however, argue that a perspective which denies Proposition 18.2 is based on perceived barriers that reflect, and are peculiar to, *only* the argument that:

Theorem 18.3. *There is no deterministic algorithm that, for any specified n, and any specified prime $p \geq 2$, will evidence that the probability $\mathbb{P}(p \mid n)$ that p divides n is $\frac{1}{p}$, and the probability $\mathbb{P}(p \nmid n)$ that p does not divide n is $1 - \frac{1}{p}$.*

Proof. By a standard result in the Theory of Numbers ([Ste02], Chapter 2, p.9, Theorem 2.1[170]), we cannot define a probability function for the probability that a random n is prime over the probability space $(1, 2, 3, \ldots,)$. $\qquad\square$

In other words, treating Theorem 18.3 as an absolute barrier does not admit the possibility—which has consequences for the resolution of outstanding problems in both the theory of numbers and computational complexity—that Proposition 18.2 is algorithmically *verifiable*, but not algorithmically *computable*, as *true*, since we shall show that:

---

[170]Compare with the informal argument in [HL23], pp.36-37; also with those in §18.3(i).

Theorem 18.4. *For any specified n, there is a deterministic algorithm that, for any specified prime $p \geq 2$, will evidence that the probability $\mathbb{P}(p \mid n)$ that p divides n is $\frac{1}{p}$, and the probability $\mathbb{P}(p \nmid n)$ that p does not divide n is $1 - \frac{1}{p}$.*

Proof. Take i as p in §18.A.c., Corollary 18.6 and Corollary 18.7. □

18.A.a.   Current paradigms

We note that the justification for current paradigms denying §18.A., Proposition 18.2 appears to be more *faith-based* than *evidence-based*, since it is expressed:

(i) either explicitly, but without formal proof:

    (a) "Here is the code of the algorithm. ...the input x is a product of two prime numbers, ϕ is a polynomial in just one variable, and *gcd* refers to the greatest-common-divisor algorithm expounded by Euclid around 300 B.C. ...

        * Repeat until exit:

        * $a :=$ a random number in $1, \ldots, x - 1$;

        * if $gcd(b, x) > 1$ then exit.

    Exiting enables carrying out the two prime factors of x...

    How many iterations must one expect to make through this maze before exit? How and when can the choice of the polynomial ϕ speed up the exploration? ...

    Note that we cannot consider the events $b \equiv 0 \ mod(p)$ and $b \equiv 0 \ mod(q)$ to be independent, even though p and q are prime, because $b = \phi a$ and ϕ may introduce bias."
    ...Regan: [Re16]*.*

    (b) "...the probabilities are not independent. ...The probability that a number n is divisible by a prime p is $1/p$, if concerning n we know only that it is large compared with p. If we know that n is *near N^2* and *not divisible by any prime smaller than p*, then the probability that n is divisible by p is not $1/p$, but f/p."
    ...Furry: [Fu42]*.*

    (c) "Prof. E. M. Wright, some months ago, sent me privately a proof on somewhat similar lines that that the probabilities could not be independent for primes greater than $n^{0.76}$."
    ...Cherwell: [Che42]*.*

    (d) "*Find the probability that x, a large integer chosen at random, is a prime number.* ...If the integer x is not divisible by any prime p which does not exceed $x^{1/2}$, x itself must be a prime—and so divisibility by primes exceeding $x^{1/2}$ is, in fact, *not* independent of the smaller primes."[171]
    ...Pólya: [Pol59]*.*

(ii) or implicitly, by arguing—as, for instance, in [Ste02], Chapter 2, p.9, Theorem 2.1—that a proof to the contrary must entail a contradiction.

---

[171]It is not obvious whether Pólya's—rather curious—perspective is unconsidered, or whether it falls within the criteria of information that we *hold* to be *true*—short of Platonic *belief*—since it *can* be treated as *self-evident*.

18.A.b.   Illusory barriers

However, we shall show in §18.A.c. that the barriers faced by current paradigms in addressing Query 23 unequivocally are illusory; they dissolve if we differentiate between the following probabilities:

(i) The probability $P_1(n \in \phi)$ of *selecting* an integer that has the property ϕ from a *specified* set S of integers;

> *Example 1: If N is the domain of the natural numbers, what is the probability of selecting an integer $n \in N$ that has the property of being a prime?*

> We note that since we cannot define a precise ratio of primes to composites in N, but only an order of magnitude such as $O(\frac{1}{log_e n})$, the probability $P_1(p) \equiv P_1(n \in N$ *is a prime*$)$ of *selecting* an integer that has the property of being a prime obviously cannot be defined in N.

(ii) The probability $P_2(n \in \phi)$ that an *unspecified* integer, in a *specified* set S of integers, *has* the property ϕ;

> *Example 2: If N^+ is the set of positive integers, what is the probability that an unspecified integer $n \in N^+$ secreted in a black box is even?*

> We note that since any $n \in N^+$ is either odd or even, the probability $P_2(p) \equiv P_2(n \in N^+$ *is even*$)$ that the *unspecified* integer $n \in N^+$ secreted in the black box *has* the property of being even must be $\frac{1}{2}$.

> We note that the probability $P_2(p) \equiv P_2(n \in N^+$ *is even*$)$ cannot *depend* upon the probability $P_1(p) \equiv P_1(n \in N^+$ *is even*$)$ of *selecting* an integer $n \in N^+$ that has the property of being even, as the latter would require[172] that $\sum_{i=1}^{\infty} P_2(i \in N^+$ *is even*$) = 1$, which is not the case in this example.

>> **Comment**: Such *dependence* would also appear to eerily echo the *curious* argument (see §19.B.)—preferred by the Copenhagen interpretation of quantum theory—that whether or not the putative cat *is* alive—and not just *known* to be alive—at any moment in Schrödinger's famous *gedanken*, would depend ultimately open whether or not we were to open the *box* at *that* moment!

(iii) The probability $P_3(n \in \phi)$ of *determining* that a *specified* integer n has the property ϕ.

> *Example 3: I give you a 5-digit combination lock along with a 10-digit integer n. The lock only opens if you set the combination to a proper factor of n which is greater than 1. What is the probability that a specified combination will open the lock.*

> We note that this is the basis for RSA encryption, which provides the cryptosystem used by many banks for securing their communications.

> It is the basis we shall use to illustrate that the probability $P_3(p|n)$ of *determining* that a prime p divides a *specified* integer n is $\frac{1}{p}$, and is independent of whether or not a prime $q \neq p$ divides n.

---

[172]See Steuding [Ste02], Chapter 2, p.9, Theorem 2.1.

18.A.c.   Why the prime divisors of an integer are mutually independent

We define the probability $P_3(p|n)$ of *determining* (in the sense detailed in §18.A.b.(iii)), by the spin of a modified Bazeries Cylinder, that a prime p divides a *specified* integer n, and show it is independent of whether or not a prime $q \neq p$ divides n.

Fig.1. An i-faced Bazeries wheel B_i and a j-faced Bazeries wheel B_j.

Definition 38. (Bazeries Cylinder) *A modified Bazeries Cylinder is a set of polygonal wheels—not necessarily identical (such as B_i and B_j in Fig. 1)—rotating independently on a common spindle, whose faces are coded with, say, integer values 0 to $i-1$ and 0 to $j-1$ respectively, where the event $B_i(u)$ (Fig 2) is the value $0 \leq u \leq i-1$ yielded by a spin of a single i-faced Bazeries wheel B_i, and the event $B_{ij}(u,v)$ (Fig, 3) is the value (u,v)—where $0 \leq u \leq i-1$ and $0 \leq v \leq j-1$—yielded by simultaneous, but independent, spins of an i-faced Bazeries wheel B_i and a j-faced Bazeries wheel B_j.*

Hypothesis 4. *The event yielded by the simultaneous spins of a set of Bazeries wheels is random.*

(1) We consider first, for any *specified* $n > i > 1$, the probability $P_3(B_i(u))$—over the probability space $(0, 1, 2, \ldots, i-1)$ (see also §18.B.a.)—of *determining* that the spin of the Bazeries wheel B_i—with faces numbered $0, 1, 2, \ldots, i-1$—yields the event $B_i(u)$.
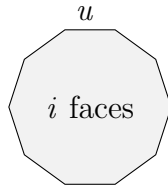
Fig.2. The event $B_i(u)$ for a single i-faced Bazeries wheel B_i

We conclude by Hypothesis 4 that, for any $0 \leq u \leq i-1$:

Lemma 18.5. $P_3(B_i(u)) = \frac{1}{i}$.

Now, if $n \equiv u \ (mod \ i)$ where $i > u \geq 0$, then i divides n if, and only if, $u = 0$. The probability $P_3(i|n)$ of *determining* by the spin of a Bazeries wheel whether the event i divides n is thus:

Corollary 18.6. $P_3(i|n) = P_3(B_i(0)) = \frac{1}{i}$.

Hence the probability $P_3(i \nmid n)$ of similarly *determining* that the event i does not divide n is:

Corollary 18.7. $P_3(i \nmid n) = 1 - \frac{1}{i}$.

(2) We consider next, for any *specified* $n > i$, $j > 1$ where $i \neq j$, the joint probability $P_3(B_{ij}(u,v))$ of *determining* whether the simultaneous, but independent, spins of the pair of Bazerian wheels (Fig.3) B_i—with faces numbered $0, 1, 2, \ldots, i-1$—and B_j—with faces numbered $0, 1, 2, \ldots, j-1$—yields the event $B_{ij}(u,v)$.

$$u \qquad\qquad\qquad\qquad v$$

i faces $\qquad\qquad\qquad\qquad$ j faces

Fig.3. The event $B_{ij}(u,v)$ for a set of two Bazeries wheels B_i and B_j.

Since the two events $B_i(u)$ and $B_j(v)$ are mutually independent by definition, we conclude by Hypothesis 4 that[173]:

Lemma 18.8. $P_3(B_{ij}(u,v)) = P_3(B_i(u)).P_3(B_j(v)) = \frac{1}{ij}$.

(3) We conclude further by Hypothesis 4, Lemma 18.5, Corollary 18.6, and Lemma 18.8, that:

Lemma 18.9. $P_3(i|n \ \& \ j|n) = P_3(i|n).P_3(j|n)$ *if, and only if, $n > i, j > 1$ and i, j are co-prime.*

Proof. We note that:

(a) The assumption that i, j be co-prime is sufficient. Thus, if i, j are co-prime, and:

$$n \equiv u \ (mod \ i), \ n \equiv v \ (mod \ j), \ n \equiv w \ (mod \ ij)$$

where $i > u \geq 0$, $j > v \geq 0$, $ij > w \geq 0$, then the ij integers $v.i + u.j$ are all incongruent and form a complete system of residues[174].

Hence $i|n$ and $j|n$ if, and only if, $u = v = 0$.

It follows that $P_3(i|n \ \& \ j|n) = P_3(B_{ij}(0,0))$.

By Corollary 18.6, $P_3(i|n) = P_3(B_i(0)) = \frac{1}{i}$ and $P_3(j|n) = P_3(B_j(0)) = \frac{1}{j}$.

By Lemma 18.8, $P_3(B_{ij}(0,0)) = \frac{1}{ij}$.

Hence, if i, j are co-prime, then $P_3(i|n \ \& \ j|n) = P_3(i|n).P_3(j|n)$.

(b) The assumption that i, j be co-prime is necessary.

For instance, if $j = 2i$, then $i|n$ and $j|n$ if, and only if, $v = 0$.

Hence $P_3(i|n \ \& \ j|n) = P_3(B_j(0))$

By Corollary 18.6, $P_3(i|n) = P_3(B_i(0)) = \frac{1}{i}$ and $P_3(j|n) = P_3(B_j(0)) = \frac{1}{j}$.

Hence $P_3(i|n \ \& \ j|n) \neq P_3(i|n).P_3(j|n)$.

---

[173]Grinstead and Snell [GS97], Chapter 4, §4.1, Definition 4.2, p.141.
[174]Hardy and Wright [HW60], p.52, Theorem 59.

The lemma follows.   □

(4) We thus conclude from Lemma 18.9 that:

Corollary 18.10. *If p and q are two unequal primes, $P_3(p|n \ \& \ q|n) = P_3(p|n).P_3(q|n)$.*   □

We thus have a model-theoretic argument that (for a proof-theoretic argument see §18.B.b., Corollary 18.24):

Theorem 18.11. *The prime divisors of an integer are mutually independent.*   □

18.A.d.   On the probability of *determining* that a *specified* integer n is a prime

We consider the compound event where $B_i(0)$ does not occur for any of a set of $\pi(\sqrt{n})$ Bazeries wheels (Fig.4).
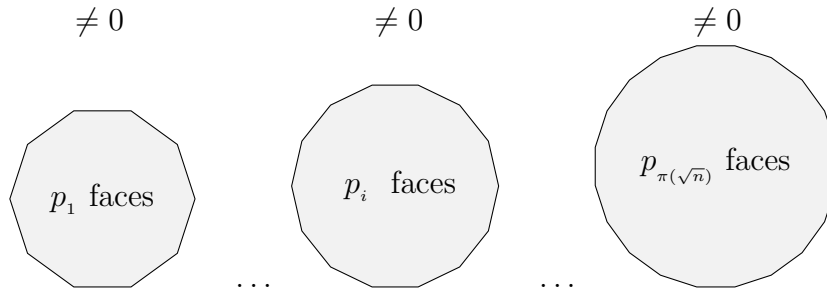
Fig.4. The event where $B_i(0)$ does not occur for any of a set of $\pi(\sqrt{n})$ Bazeries wheels.

Now, even though we cannot define the probability $P_1(n \ is \ a \ prime)$ of *selecting* an integer n from the set N of all natural numbers that has the property of being prime[175], since we have by Corollary 18.7 that the probability $P_3(i \nmid n)$ of *determining* by the spin of a Bazeries wheel that a prime $p < n$ does not divide a *specified* n is $1 - \frac{1}{p}$, it follows from Theorem 18.11 that:

Theorem 18.12. *The probability $P_3(n \ is \ co-prime \ to \ p \leq \sqrt{n})$[176] of determining that n is not divisible by any prime $p \leq \sqrt{n}$ is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.*

Proof. By Definition 38, Hypothesis 4, and Lemma 18.8, the probability that $B_i(0)$ does not occur for any i in a simultaneous spin of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig. 4)—is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.

If k is such that $k \not\equiv 0 \ (mod \ p)$ for any prime $p \leq \sqrt{n}$, then the probability $P_3(k \ is \ co-prime \ to \ p \leq \sqrt{n})$ of *determining* by the simultaneous spin of the above $\pi(\sqrt{n})$ Bazeries wheels that k is not divisible by any prime $p \leq \sqrt{n}$ is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$.

In the particular case where n is such that $n \not\equiv 0 \ (mod \ p)$ for any prime $p \leq \sqrt{n}$, the probability $P_3(n \ is \ co-prime \ to \ p \leq \sqrt{n})$ of *determining* by the simultaneous spin of the above $\pi(\sqrt{n})$ Bazeries wheels that n is not divisible by any prime $p \leq \sqrt{n}$ is $\prod_{i=1}^{\pi(\sqrt{n})}(1 - \frac{1}{p_i})$. The theorem follows.   □

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[175]See §18.A.a. (2)(i).
[176]See §18.A.a. (2)(iii).

18.A.e. Why determining primality is deterministically 'polynomial time'

We now have that:

Theorem 18.13. *The minimum number of events needed for determining that the signature yielded by simultaneous spins of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig.4)—is that of a number not divisible by any prime $p \leq \sqrt{n}$ is of order $O(\log_e n)$.*

Proof. By Theorem 18.12, the expected number of events which *determine* that a *specified* n is not divisible by any prime $p \leq \sqrt{n}$ in a set of k simultaneous spins of the $\pi(\sqrt{n})$ Bazeries wheels[177]—where p_i is the i'th prime and B_i has p_i faces (Fig.4)—is $k \cdot \prod_{i=1}^{\pi(\sqrt{n})}(1-\frac{1}{p_i})$; which—by Mertens' Theorem[178] $\prod_{p \leq x}(1-\frac{1}{p}) \sim \frac{e^{-\lambda}}{\log_e x}$—is ≥ 1 if $k \geq \frac{e^{\lambda}}{2}.\log_e n$. The lemma follows by Definition 35 for minimum k. $\square$

We note the standard definition:

Definition 39. *A deterministic algorithm computes a number-theoretical function $f(n)$ in polynomial-time[179] if there exists k such that, for all inputs n, the algorithm computes $f(n)$ in $\leq (\log_e n)^k + k$ steps.*

By Definition 39, we further conclude that:

Corollary 18.14. *Determining whether the signature yielded by simultaneous spins of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig.4)—is that of a prime, or not, can be simulated by a deterministic algorithm in polynomial time $O(\log_e n)$.[180]*

Proof. Let $r(n)$ be the ratio $\pi(n)/\pi_{coprime}(n)$ of the number $\pi(n)$ of primes $\leq n$ and the number $\pi_{coprime}(n)$ of integers $\leq n$ and co-prime to n. By the Prime Number Theorem and Theorem 18.12, $Lt_{n\to\infty} r(n) = e^{-\lambda}$. Hence $r(n)$ is bounded by some constant c. The corollary follows by Theorem 18.13. $\square$

18.A.f. *Why Integer Factorising* cannot be deterministically 'polynomial-time'

Given that n is composite, §18.A.c., Theorem 18.11, now suggests the thesis:

Theorem 18.15. ($DPTvNDPT$ **Thesis**) *There are integers n such that no deterministic algorithm can compute a prime factor of n in polynomial-time.*

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[177]We note that this is *not* equivalent to the throws of a $\prod_{i=1}^{\pi(\sqrt{n})} p_{\pi(\sqrt{i})}$-sided die, each of whose faces is equally possible as a key to the code in question, since such throws do not use the fact—Theorem 18.11—that the prime divisors of n are mutually independent.

[178]Hardy and Wright [HW60], p. 351, Theorem 22.8; where $\lambda = 0.57722\ldots$ is the Euler-Mascheroni constant and $\frac{e^{\lambda}}{2} = 0.89053\ldots$.

[179]cf. Cook [Cook], p.1; also Brent [Brn00], p.1, fn.1: "For a polynomial-time algorithm the expected running time should be a polynomial in the length of the input, i.e. $O((\log N)^c)$ for some constant c".

[180]We note that, in a seminal paper 'PRIMES is in P', Agrawal et al [AKS04] have shown that deciding whether an integer n is a prime or not can be done in polynomial time $\ddot{O}(\log_e^{15/2} n)$; improved to $\ddot{O}(\log_e^6 n)$ by Lenstra and Pomerance in [LP11].

Proof. By the Prime Number Theorem and Mertens' Theorem, the expected number of primes $\leq \sqrt{n}$ is $O(\frac{\sqrt{n}}{log_e \sqrt{n}})$. Moreover, any computational process that successfully identifies a prime divisor of n must necessarily appeal to at least one logical operation for identifying such a factor.

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, it follows that if, for instance, $n = p^k$ for some $k \geq 2$, then determining p may require at least one logical operation for algorithmically testing each prime $\leq \sqrt{n}$ deterministically if, for some n, the prime p is the one that is tested last in the particular method of testing the primes $\leq \sqrt{n}$.

Since any algorithmically deterministic method of testing the primes $\leq \sqrt{n}$ must be independent of n, and always have some prime p that is tested last for any *specified* n, the algorithm cannot be guaranteed to always determine in polynomial time that p is a prime factor of n if $n = p^k$ for some $k \geq 2$.

Since the primes to be tested if p is tested last, and $n = p^k$, are of order $O(\sqrt{n}/log_e n)$, the number of computations required by any deterministic algorithm that always computes a prime factor of n cannot be polynomial-time—i.e. of order $O((log_e n)^c)$ for any c—in the length of the input n. $\square$

18.B.   Formal properties of the residues $r_i(n)$.

We give formal proofs of the foregoing considerations by defining the residues $r_i(n)$ for all $n \geq 2$ and all $i \geq 2$ as below:

Definition 40. $n + r_i(n) \equiv 0 \ (mod \ i) \ where \ i > r_i(n) \geq 0.$

Since each residue $r_i(n)$ cycles over the i values $(i - 1, i - 2, \ldots, 0)$, these values are all incongruent and form a complete system of residues[181] *mod i*.

It immediately follows that:

Lemma 18.16. $r_i(n) = 0$ *if, and only if, i is a divisor of n.* $\square$

18.B.a.   The probability model $\mathbb{M}_i = \{(0, 1, 2, \ldots, i - 1), \ r_i(n), \frac{1}{i}\}$

By the standard definition of the probability $\mathbb{P}(e)$ of an event e[182], we have by §18.B., Lemma 18.16, that:

Lemma 18.17. *For any $n \geq 2$, $i \geq 2$ and any specified integer $i > u \geq 0$:*

- *the probability $\mathbb{P}(r_i(n) = u)$ that $r_i(n) = u$ is $\frac{1}{i}$;*

- $\sum_{u=0}^{u=i-1} \mathbb{P}(r_i(n) = u) = 1;$

- *and the probability $\mathbb{P}(r_i(n) \neq u)$ that $r_i(n) \neq u$ is $1 - \frac{1}{i}$.* $\square$

By the standard definition of a probability model, we conclude that:

Theorem 18.18. *For any $i \geq 2$, $\mathbb{M}_i = \{(0, 1, 2, \ldots, i - 1), \ r_i(n), \frac{1}{i}\}$ yields a probability model for each of the values of $r_i(n)$.* $\square$

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[181][HW60], p.49.
[182]See [Ko56], Chapter I, §1, Axiom III, pg.2.

Corollary 18.19. *For any specified n, i and u such that $r_i(n) = u$, the probability that the roll of an i-sided cylindrical die will yield the value u is $\frac{1}{i}$ by the probability model defined in Theorem 18.18 over the probability space $(0, 1, 2, \ldots, i-1)$.* $\qquad\square$

Corollary 18.20. *For any $n \geq 2$ and any prime $p \geq 2$, the probability $\mathbb{P}(r_p(n) = 0)$ that $r_p(n) = 0$, and that p divides n, is $\frac{1}{p}$; and the probability $\mathbb{P}(r_p(n) \neq 0)$ that $r_p(n) \neq 0$, and that p does not divide n, is $1 - \frac{1}{p}$.* $\qquad\square$

We also note the standard definition[183]:

Definition 41. *Two events e_i and e_j are mutually independent for $i \neq j$ if, and only if, $\mathbb{P}(e_i \cap e_j) = \mathbb{P}(e_i).\mathbb{P}(e_j)$.*

18.B.b. The prime divisors of any integer n are mutually independent

We further note that:

Lemma 18.21. *If $n \geq 2$ and $n > i, j > 1$, where $i \neq j$, then:*

$$\mathbb{P}((r_i(n) = u) \cap (r_j(n) = v)) = \mathbb{P}(r_i(n) = u).\mathbb{P}(r_j(n) = v)$$

where $i > u \geq 0$ and $j > v \geq 0$.

Proof. We note that:

(i) If $n \geq 2$ and $n > i, j > 1$, where $i \neq j$, then we can always determine a unique pair of residues $r_i(n) = u$ and $r_j(n) = v$, where $i > u \geq 0$, $j > v \geq 0$, i divides $n + u$, and j divides $n + v$.

(ii) There are $i.j$ pairs (u, v) such that $i > u \geq 0$ and $j > v \geq 0$.

(iii) The compound probability that the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die will yield the values u and v, respectively, is thus $\frac{1}{i.j}$ by the probability model for such a simultaneous event as defined over the probability space $\{(u, v) : i > u \geq 0, j > v \geq 0\}$, where we note that:

   – the probability $\mathbb{P}((r_i(n) = u) \cap (r_j(n) = v))$ that $r_i(n) = u$ and $r_j(n) = v$ is $\frac{1}{i.j}$;

   – $\sum_{All\ (u,v):\ i>u\geq0,\ j>v\geq0} \mathbb{P}((r_i(n) = u) \cap (r_j(n) = v)) = 1$;

(iv) By §18.B.a., Lemma 18.17, the product of the probability $\frac{1}{i}$ that the roll of an i-sided cylindrical die will yield the value u, and the probability $\frac{1}{j}$ that the roll of a j-sided cylindrical die will yield the value v, is $\frac{1}{i.j}$.[184]

(v) It follows that:

---

[183]See [Ko56], Chapter VI, §1, Definition 1, pg.57 and §2, pg.58.

[184]In other words, the compound probability of determining u and v correctly from the simultaneous roll of one i-sided cylindrical die and one j-sided cylindrical die, is the product of the probability of determining u correctly from the roll of an i-sided cylindrical die, and the probability of determining v correctly from the roll of a j-sided cylindrical die.

$$\mathbb{P}((r_i(n) = u) \cap (r_j(n) = v)) = \tfrac{1}{i.j}$$
$$\mathbb{P}(r_i(n) = u).\mathbb{P}(r_j(n) = v) = (\tfrac{1}{i})(\tfrac{1}{j}).$$

The lemma follows.

Corollary 18.22. $\mathbb{P}((r_i(n) = 0) \cap (r_j(n) = 0)) = \mathbb{P}(r_i(n) = 0).\mathbb{P}(r_j(n) = 0).$ □

Since, by §18.B., Lemma 18.16, $r_i(n) = 0$ if, and only if, i is a divisor of n, it follows from Corollary 18.22 that:

Theorem 18.23. *If i and j are co-prime and $i \neq j$, then whether, or not, i divides any specified natural number n is independent of whether, or not, j divides n.*

Proof. We note that

(i) By Corollary 18.21, we have that:

$$\mathbb{P}((r_i(n) = 0) \cap (r_j(n) = 0)) = \tfrac{1}{i.j}$$
$$\mathbb{P}(r_i(n) = 0).\mathbb{P}(r_j(n) = 0) = (\tfrac{1}{i})(\tfrac{1}{j}).$$

(ii) Further, if i and j are co-prime, and $n + r_{i.j}(n) \equiv 0 \ (mod \ i.j)$, then the $i.j$ integers $r_j(n).i + r_i(n).j$ are all incongruent and form a complete system of residues. It follows that $n = a.i$—whence i divides n—and also $n = b.j$—whence j divides n—if, and only if, $r_i(n) = r_j(n) = r_{i.j}(n) = 0$.

The lemma follows. □

We thus have a formal proof of the model-theoretic argument §18.A.c., Theorem 18.11, that:

Corollary 18.24. *The prime divisors of any integer n are mutually independent.* □

18.B.c.   Density of integers not divisible by primes $Q = \{q_1, q_2, \ldots, q_k\}$

We conclude further from §18.B.b., Corollary 18.24 that:

Lemma 18.25. *The asymptotic density of the set of all integers that are not divisible by any of a specified set of primes $Q = \{q_1, q_2, \ldots, q_k\}$ is:*

$$\prod_{q \in Q}(1 - 1/q).$$ □

It follows that:

Lemma 18.26. *The expected number of integers in any interval (a,b) that are not divisible by any of a specified set of primes $Q = \{q_1, q_2, \ldots, q_k\}$ is:*

$$(b - a) \prod_{q \in Q}(1 - 1/q).$$ □

18.C. Primes in an arithmetic progression

The significance of the distinction between algorthmic *verifiability* and algorithmic *computability*, which yields §18.B.b., Corollary 18.24, is further seen in the following, elementary, proof of Dirichlet's Theorem—which is the assertion that if a and d are co-prime and $1 \leq a < d$, then the arithmetic progression $a + m.d$, where $m \geq 1$, contains an infinitude of (Dirichlet) primes.

We first note that, by §18.B.b., Lemma 18.21:

Lemma 18.27. *If p_i and p_j are two primes where $i \neq j$ then, for any $n \geq 2$, $\alpha, \beta \geq 1$, we have:*

$$\mathbb{P}((r_{p_i^\alpha}(n) = u) \cap (r_{p_j^\beta}(n) = v)) = \mathbb{P}(r_{p_i^\alpha}(n) = u).\mathbb{P}(r_{p_j^\beta}(n) = v)$$

where $p_i^\alpha > u \geq 0$ and $p_j^\beta > v \geq 0$.   □

Now, the $p_i^\alpha.p_j^\beta$ numbers $d.p_i^\alpha + c.p_j^\beta$, where $p_i^\alpha > c \geq 0$ and $p_j^\beta > d \geq 0$, are all incongruent and form a complete system of residues[185] $mod\ (p_i^\alpha.p_j^\beta)$. It follows that $n = a.p_i^\alpha$—whence p_i^α divides n—and also $n = b.p_i^\beta$—whence p_i^β divides n—if, and only if $r_{p_i^\alpha}(n) = r_{p_j^\beta}(n) = 0$.

If $u = 0$ and $v = 0$ in Lemma 18.27, so that both p_i and p_j are prime divisors of n, we immediately conclude that:

$$\mathbb{P}((r_{p_i^\alpha}(n) = 0) \cap (r_{p_j^\beta}(n) = 0)) = \frac{1}{p_i^\alpha.p_j^\beta}$$

$$\mathbb{P}(r_{p_i^\alpha}(n) = 0).\mathbb{P}(r_{p_j^\beta}(n) = 0) = (\frac{1}{p_i^\alpha})(\frac{1}{p_j^\beta}).$$

Corollary 18.28. $\mathbb{P}((r_{p_i^\alpha}(n) = 0) \cap (r_{p_j^\beta}(n) = 0)) = \mathbb{P}(r_{p_i^\alpha}(n) = 0).\mathbb{P}(r_{p_j\beta}(n) = 0).$ □

It also immediately follows that §18.B.b., Corollary 18.24, can be extended to prime powers in general:

Theorem 18.29. *For any two primes $p \neq q$ and natural numbers $n, \alpha, \beta \geq 1$, whether or not p^α divides n is independent of whether or not q^β divides n.*   □

18.C.a. The asymptotic density of Dirichlet integers

We note next that:

Lemma 18.30. *For any co-prime natural numbers $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ where:*

$q_1 < q_2 < \ldots < q_k$ *are primes and $\alpha_1, \alpha_2 \ldots \alpha_k \geq 1$ are natural numbers;*

the natural number n is of the form $a + m.d$ for some natural number $m \geq 1$ if, and only if:

$a + r_{q_i^{\alpha_i}}(n) \equiv 0\ (mod\ q_i^{\alpha_i})$ *for all $1 \leq i \leq k$*

where $0 \leq r_i(n) < i$ is defined for all $i > 1$ by:

---

[185][HW60], p.52, Theorem 59.

$$n + r_i(n) \equiv 0 \ (mod \ i) \ .$$

Proof. First, if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_k}$, then:

$$
\begin{aligned}
n &\equiv a \ (mod \ d) \\
and: \ n + r_{q_i^{\alpha_i}}(n) &\equiv 0 \ (mod \ q_i^{\alpha_i}) \quad for \ all \ 1 \leq i \leq k \\
whence: \ a + r_{q_i^{\alpha_i}}(n) &\equiv 0 \ (mod \ q_i^{\alpha_i}) \quad for \ all \ 1 \leq i \leq k
\end{aligned}
$$

Second:

$$
\begin{aligned}
If: \ a + r_{q_i^{\alpha_i}}(n) &\equiv 0 \ (mod \ q_i^{\alpha_i}) \quad for \ all \ 1 \leq i \leq k \\
and: \ n + r_{q_i^{\alpha_i}}(n) &\equiv 0 \ (mod \ q_i^{\alpha_i}) \quad for \ all \ 1 \leq i \leq k \\
then: \ n - a &\equiv 0 \ (mod \ q_i^{\alpha_i}) \quad for \ all \ 1 \leq i \leq k \\
whence: \ n &\equiv a \ (mod \ d)
\end{aligned}
$$

The Lemma follows. $\square$

By §18.B.a., Lemma 18.17, it immediately follows that:

Corollary 18.31. *The probability that $a + r_{q_i^{\alpha_i}}(n) \equiv 0 \ (mod \ q_i^{\alpha_i})$ for any $1 \leq i \leq k$ is $\frac{1}{q_i^{\alpha_i}}$.* $\square$

By §18.C., Lemma 18.27, and §18.C., Theorem 18.29, it further follows that:

Corollary 18.32. *The joint probability that $a + r_{q_i^{\alpha_i}}(n) \equiv 0 \ (mod \ q_i^{\alpha_i})$ for all $1 \leq i \leq k$ is* $\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}}$. $\square$

We conclude by Lemma 18.30 that:

Corollary 18.33. *The asymptotic density of Dirichlet integers, defined as numbers of the form $a + m.d$ for some natural number $m \geq 1$ which are not divisible by any specified set of primes $\mathbb{R} = \{r_1, r_2, \ldots, r_l\}$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_l}$ is:*

$$\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{r \in \mathbb{R} \ \& \ r \neq q_i} (1 - \tfrac{1}{r}).$$

Proof. Since a, d are co-prime, we have by Lemma 18.30 that if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_k}$, we have that:

$$
\begin{aligned}
n &\equiv a \ (mod \ q_i) \quad for \ all \ 1 \leq i \leq k \\
whilst: \ n + r_i(n) &\equiv 0 \ (mod \ i) \quad for \ all \ 1 \leq i \\
whence: \ a + r_{q_i}(n) &\equiv 0 \ (mod \ q_i) \quad for \ all \ 1 \leq i \leq k \\
r_{q_i}(n) &\neq 0 \quad for \ all \ 1 \leq i \leq k \\
and: \ q_i &\nmid n \quad for \ all \ 1 \leq i \leq k
\end{aligned}
$$

Hence, if n is of the form $a + m.d$ for some natural number $m \geq 1$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_k}$ and $(a, d) = 1$, the probability that $q_i \nmid n$ for all $1 \leq i \leq k$ is 1.

By §18.B.a., Lemma 18.25, §18.B.a., Lemma 18.26 and §18.C., Theorem 18.29, the asymptotic density of Dirichlet numbers of the form $a + m.d$ which are not divisible by any specified set of primes $\mathbb{R} = \{r_1, r_2, \ldots, r_l\}$ is thus:

$$\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{r \in \mathbb{R} \ \& \ r \neq q_i} (1 - \frac{1}{r})$$

The Corollary follows.   □

Corollary 18.34. *The expected number of Dirichlet integers in any interval* (a, b) *is:*

$$(b - a) \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} \cdot \prod_{r \in \mathbb{R}} (1 - \frac{1}{r}).$$ □

18.C.b.   An elementary proof of Dirichlet's Theorem

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, it follows that the number $\pi_{(a,d)}(n)$ of Dirichlet primes, of the form $a + m.d$ for some natural number $m \geq 1$ and $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, that are less than or equal to any $n \geq q_k^2$ is cumulatively approximated by the Dirichlet prime counting function:

Definition 42. $\pi_D(n) = \sum_{l=1}^{n} (\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} \cdot \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j})).$

We conclude that:

Lemma 18.35. $\pi_{(a,d)}(n) \approx \pi_D(n) \to \infty$ *as* $n \to \infty$.

Proof. If a, d are co-prime and $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we have for any $n \geq q_k^2$:

$$
\begin{aligned}
\pi_D(n) &= \sum_{l=1}^{n} (\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} \cdot \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j})) \\
&= \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} \cdot \sum_{l=1}^{n} \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j}) \\
&\geq \prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} . n . \prod_{j=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_j})
\end{aligned}
$$

Since, by Mertens' Theorem, $\prod_{p \leq x} (1 - \frac{1}{p}) \sim \frac{e^{-\lambda}}{log_e x}$, we have that:

$$n . \prod_{j=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_j}) \sim \frac{2e^{-\gamma} n}{log_e(n)} \to \infty \text{ as } n \to \infty.$$

the lemma follows.   □

Since $p_{n+1}^2 - p_n^2 \to \infty$ as $n \to \infty$, we conclude further that:

Theorem 18.36. (Dirichlet's Theorem) *There are an infinity of primes in any arithmetic progression* $a + m.d$ *where* $(a, d) = 1$[186]*.*   □

---

[186]Compare [HW60], p.13, Theorem 15*.

18.D.   An elementary proof that there are infinite twin-primes

The significance of §18.B.b., Corollary 18.24, and of *evidence-based* reasoning, for the theory of numbers is further highlighted by the following, elementary, proof that there are an infinitude of twin primes.

We define $\pi_2(n)$ as the number of integers $p \leq n$ such that both p and $p + 2$ are prime. In order to estimate $\pi_2(n)$, we first define:

Definition 43. *An integer n is a $\mathbb{TW}(k)$ integer if, and only if, $r_{p_i}(n) \neq 0$ and $r_{p_i}(n) \neq 2$ for all $1 \leq i \leq k$, where $0 \leq r_i(n) < i$ is defined for all $i > 1$ by:*

$$n + r_i(n) \equiv 0 \ (mod \ i).$$

We note that:

Lemma 18.37. *If n is a $\mathbb{TW}(k)$ integer, then both n and $n + 2$ are not divisible by any of the first k primes $\{p_1, p_2, \ldots, p_k\}$.*

Proof. The lemma follows immediately from Definition 43 and Lemma 18.16.   □

Since each residue $r_i(n)$ cycles over the i values $(i - 1, i - 2, \ldots, 0)$, these values are all incongruent and form a complete system of residues *mod i*.

It thus follows from Definition 43 that the asymptotic density of $\mathbb{TW}(k)$ integers over the set of natural numbers is:

Lemma 18.38. $\mathbb{D}(\mathbb{TW}(k)) = \prod_{i=2}^{k}(1 - \frac{2}{p_i})$.   □

We also have that:

Lemma 18.39. *If $p_k^2 \leq n \leq p_{k+1}^2$ is a $\mathbb{TW}(k)$ integer, then n is a prime and either $n + 2$ is also a prime, or $n + 2 = p_{k+1}^2$.*

Proof. By Definition 43 and §18.B., Definition 40:

$$r_{p_i}(n) \ \neq \ 2 \ \text{for all } 1 \leq i \leq k$$
$$n + 2 \ \neq \ \lambda.p_i \ \text{for all } 1 \leq i \leq k, \ \lambda \geq 1$$

Hence n is prime; and either $n + 2$ is divisible by p_{k+1}, in which case $n + 2 = p_{k+1}^2$, or it is a prime.   □

If we define $\pi_{\mathbb{TW}(k)}(n)$ as the number of $\mathbb{TW}(k)$ integers $\leq n$, by Lemma 18.38 the expected number of $\mathbb{TW}(k)$ integers in any interval (a, b) is given by:

Lemma 18.40. $\pi_{\mathbb{TW}(k)}(b) - \pi_{\mathbb{TW}(k)}(a) \approx (b - a) \prod_{i=2}^{k}(1 - \frac{2}{p_i})$.   □

Since n is a prime if, and only if, it is not divisible by any prime $p \leq \sqrt{n}$, it follows from Lemma 18.39 that $\pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2)$ is at most one less than the number of twin-primes in the interval $(p_{k+1}^2 - p_k^2)$.

Lemma 18.41. $\pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2) + 1 \geq \pi_2(p_{k+1}^2) - \pi_2(p_{k)}^2) \geq \pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2)$ □

Now, by Lemma 18.40 the expected number of $\mathbb{TW}(k)$ integers in the interval $(p_{k+1}^2 - p_k^2)$ is given by:

Lemma 18.42. $\pi_{\mathbb{TW}(k)}(p_{k+1}^2) - \pi_{\mathbb{TW}(k)}(p_k^2) \approx (p_{k+1}^2 - p_k^2) \prod_{i=2}^{k}(1 - \frac{2}{p_i})$. $\qquad\square$

We conclude that the number $\pi_2(p_{k+1}^2)$ of twin primes $\leq p_{k+1}^2$ is given by the cumulative, non-heuristic, approximation:

Lemma 18.43. $\sum_{j=1}^{k}(\pi_2(p_{j+1}^2) - \pi_2(p_j^2)) = \pi_2(p_{k+1}^2) \approx \sum_{j=1}^{k}(p_{j+1}^2 - p_j^2) \prod_{i=2}^{j}(1 - \frac{2}{p_i})$. $\qquad\square$

We further conclude that:

Theorem 18.44. (Twin Prime Theorem) $\pi_2(n) \to \infty$ *as* $n \to \infty$.

Proof. We have that, for $k \geq 2$:

$$\sum_{j=1}^{k}(p_{j+1}^2 - p_j^2) \prod_{i=2}^{j}(1 - \frac{2}{p_i}) = \sum_{j=9}^{p_{k+1}^2} \prod_{i=2}^{\pi(\sqrt{j})-1}(1 - \frac{2}{p_i})$$

$$\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \frac{2}{p_i})$$

$$\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \frac{1}{p_i})(1 - \frac{1}{(p_i - 1)})$$

$$\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \frac{1}{p_i})(1 - \frac{1}{p_{i-1}})$$

$$\geq (p_{k+1}^2 - 9).\prod_{i=2}^{k}(1 - \frac{1}{p_{i-1}})^2$$

$$\geq (p_{k+1}^2 - 9).\prod_{i=1}^{k}(1 - \frac{1}{p_i})^2$$

Now, by Mertens' Theorem, we have that:

$$(p_{k+1}^2 - 9).\prod_{i=1}^{k}(1 - \frac{1}{p_i})^2 \quad \sim \quad (p_{k+1}^2 - 9).(\frac{e^{-\gamma}}{log_e k})^2$$

$$\to \quad \infty \ as \ n \to \infty$$

The theorem follows by Lemma 18.43. $\qquad\square$

18.D.a. The Generalised Prime Counting Function: $\sum_{j=1}^{n} \prod_{i=a}^{\pi(\sqrt{j})}(1 - \frac{b}{p_i})$

We note that the argument of §18.D., Theorem 18.44, is a special case of the behaviour as $n \to \infty$ of the Generalised Prime Counting Function $\sum_{j=1}^{n} \prod_{i=a}^{\pi(\sqrt{j})}(1 - \frac{b}{p_i})$, which estimates the number of integers $\leq n$ such that there are b values that cannot occur amongst the residues $r_{p_i}(n)$ for $a \leq i \leq \pi(\sqrt{j})$[187]:

Theorem 18.45. (Generalised Prime Counting Theorem) $\sum_{j=1}^{n} \prod_{i=a}^{\pi(\sqrt{j})}(1 - \frac{b}{p_i}) \to \infty$ *as* $n \to \infty$ *if* $p_a > b \geq 1$.

---

[187] Thus $b = 1$ yields a non-heuristic estimate for the number $\pi(n)$ of primes $\leq n$, and $b = 2$ an estimate for the number of TW primes (Definition 43) $\leq n$. We note that the Prime Number Theorem $\pi(n) \sim n/log_e n$ suggests that $\pi(n)$ can have various non-heuristic estimations for finite values of n.

Proof. For $p_a > b \geq 1$, we have that:

$$\sum_{j=1}^{n} \prod_{i=a}^{\pi(\sqrt{j})} (1 - \tfrac{b}{p_i}) \;\geq\; \sum_{j=p_a^2}^{n} \prod_{i=a}^{\pi(\sqrt{j})} (1 - \tfrac{b}{p_i})$$

$$\geq\; \sum_{j=p_a^2}^{n} \prod_{i=a}^{\pi(\sqrt{n})} (1 - \tfrac{b}{p_i})$$

$$\geq\; (n - p_a^2) . \prod_{i=a}^{\pi(\sqrt{n})} (1 - \tfrac{b}{p_i})$$

$$\geq\; (n - p_a^2) . \prod_{i=a}^{n} (1 - \tfrac{b}{p_i})$$

The theorem follows if:

$$log_e (n - p_a^2) + \sum_{i=a}^{n} log_e (1 - \tfrac{b}{p_i}) \to \infty$$

(i) We note first the standard result for $|x| < 1$ that:

$$log_e (1 - x) = - \sum_{m=1}^{\infty} \tfrac{x^m}{m}$$

For any $p_i > b \geq 1$, we thus have:

$$log_e (1 - \tfrac{b}{p_i}) = - \sum_{m=1}^{\infty} \tfrac{(b/p_i)^m}{m} = -\tfrac{b}{p_i} - \sum_{m=2}^{\infty} \tfrac{(b/p_i)^m}{m}$$

Hence:

$$\sum_{i=a}^{n} log_e (1 - \tfrac{b}{p_i}) = - \sum_{i=a}^{n} (\tfrac{b}{p_i}) - \sum_{i=a}^{n} (\sum_{m=2}^{\infty} \tfrac{(b/p_i)^m}{m})$$

(ii) We note next that, for all $i \geq a$:

$$c < (1 - \tfrac{b}{p_a}) \to c < (1 - \tfrac{b}{p_i})$$

It follows for any such c that:

$$\sum_{m=2}^{\infty} \tfrac{(b/p_i)^m}{m} \leq \sum_{m=2}^{\infty} (\tfrac{b}{p_i})^m = \tfrac{(b/p_i)^2}{1 - b/p_i} \leq \tfrac{b^2}{c.p_i^2}$$

Since:

$$\sum_{i=1}^{\infty} \tfrac{1}{p_i^2} = O(1)$$

it further follows that:

$$\sum_{i=a}^{n} (\sum_{m=2}^{\infty} \tfrac{(b/p_i)^m}{m}) \leq \sum_{i=a}^{n} (\tfrac{b^2}{c.p_i^2}) = O(1)$$

(iii) From the standard result[188]:

$$\sum_{p \leq x} \tfrac{1}{p} = log_e log_e x + O(1) + o(1)$$

it then follows that:

---

[188] [HW60], p.351, Theorem 427.

$$\sum_{i=a}^{n} log_e\left(1 - \frac{b}{p_i}\right) \geq -\sum_{i=a}^{n}\left(\frac{b}{p_i}\right) - O(1)$$

$$\geq -b.(log_e log_e n + O(1) + o(1)) - O(1)$$

The theorem follows since:

$$log_e(n - p_a^2) - b.(log_e log_e n + O(1) + o(1)) - O(1) \to \infty$$

and so:

$$log_e(n - p_a^2) + \sum_{i=a}^{n} log_e\left(1 - \frac{b}{p_i}\right) \to \infty \qquad \square$$

CHAPTER 19. *PARADIGM CHALLENGING CONSEQUENCES*

19.   The significance of *evidence-based* reasoning for the physical sciences

The significance of *evidence-based* reasoning for the physical sciences is that it admits, for instance, the hypothesis that the paradoxical element which surfaced as a result of the EPR argument, due to the perceived conflict (as implied by Bell's inequality in [Bl64]) between:

— the, seemingly essential, non-locality required by current interpretations of Quantum Mechanics; and

— the essential locality required by current interpretations of Classical Mechanics;

merely reflects an ambiguity in current paradigms which, when iterpreting the meaning and truth of a formal mathematical argument, fail to recognise that some determinate physical phenomena may not be expressible recursively.

In other words (see §1., Thesis 1), the perceived conflict between our mathematical representations—and subsequent interpretations—of classical and quantum phenomena may simply reflect that some determinate physical phenomena may not be expressible by recursive functions, even though the observed probabilities of the occurrence of such phenomena may be expressible mathematically by recursive functions such as, for instance, Schrödinger's wave function or Heisenberg's matrices.

19.A.   Some determinate physical phenomena may not be express-ible recursively

In particular, we show how some perceived anomalies—such as the EPR paradox—in our mathematical descriptions of quantum behaviour may dissolve if a physicist could cogently argue that:

(i) All properties of physical reality are deterministic, but not necessarily mathematically predictable—in the sense that any physical property could have one, and only one, value at any time $t(n)$, where the value is completely determined by some natural law which need not, however, be representable by algorithmically *computable*—and ipso facto predictable—number-theoretic functions.

(ii) There are elements of such a physical reality whose properties at any time $t(n)$ are determined completely in terms of their putative properties at some earlier time $t(0)$.

   (a) Such properties are predictable mathematically since they are representable by algorithmically *computable* functions.

   (b) The values of any two such functions with respect to their variables are, by the definition of algorithmic *computability* (see §2., Definition 3), independent of each other and must, therefore, obey Bell's inequality, which applies *only* to classical, algorithmically *computable* functions and relations.

(c) The Laws of Classical Mechanics determine the nature and behaviour of such physical reality only, and circumscribe the limits of reasoning and cognition in any emergent mechanical intelligence which, by current definitions (see §17.E., Query 17 (Turing Test)), may be treated as essentially Turing machines.

(iii) There could be elements of such a physical reality whose properties at any time $t(n)$ cannot be theoretically determined completely from their putative properties at some earlier time $t(0)$.

(a) Such properties are unpredictable mathematically since they are only representable mathematically by algorithmically *verifiable*, but not algorithmically *computable*, functions.

(b) The values of any two such functions with respect to their variables could, by the definition of algorithmic *verifiability* (see §2., Definition 2), admit dependence upon each other; in which case they need not obey Bell's inequality.

(c) The Laws of Quantum Mechanics determine the nature and behaviour of such physical reality, and circumscribe the limits of reasoning and cognition in any emergent humanlike (i.e., organically evolved) intelligence.

19.A.a.   Describable but not predictable

We note that functions which are algorithmically *verifiable*, but not algorithmically *computable*, would fall into the category of 'strange' and 'incomputable' functions considered by scientist Jonathan Bartlett in his *Describable but Not Predictable: Mathematical Modeling and Non-Naturalistic Causation*.

Bartlett's criteria (which he proposes in §2, p.115 of [Brt18], and defends elsewhere) for distinguishing between naturalistic (presumably physically grounded) abstract concepts (which may be taken to correspond to Lakoff and Núñez's *primary* conceptual metaphors in [LR00]) and non-naturalistic (presumably non-physically grounded) abstract concepts (which may be taken to correspond to Lakoff and Núñez's *secondary* conceptual metaphors in [LR00]) is that of "computability", since:

> "Under this rubric, physical processes are those whose results can, at least in principle, be calculated by computational systems, while non-physical processes are those which cannot."
>
> . . . Bartlett: [Brt18], §2, p.115.

Bartlett argues that the cited functions defined by him as 'strange'—vis à vis the usual 'well-behaved' functions of classical theory—might "be important to helping researchers think about causality"; and that whilst "strange functions should not be introduced lightly, there is no reason to avoid them in models. The preference for well-behaved functions is just that—a preference":

> "**5 Using Incomputable Functions in Modeling**
>
> As we have seen, by expanding our view of mathematics beyond the typical well-behaved functions, we can incorporate models of non-mechanical (i.e., non-naturalistic) causes and modes of operation into mathematical descriptions. Such mathematical descriptions would enable better integration between naturalistic and non-naturalistic causes at work in a system. For a system to be mathematical does not mean that it must be predictive or even computational. Many functions

in mathematics are not computable, and in fact, computability may only cover a very small proportion of them.

One may wonder, why bother with mathematics at all? Mathematics is just a formalization of logic. Using mathematics requires that a person distill their ideas into the most rigorous and abstract form. Additionally, because mathematics as a discipline is well-studied, boiling ideas down into mathematical forms, even if incalculable, allows mathematical tools to be used to analyze and reason from these ideas. It also makes it easier to combine different ideas. If each idea is expressed mathematically, then the combination of these ideas can be more readily determined.

One of the advantages of our well-behaved functions is that their long-term behavior can be arbitrarily extrapolated from limited observations. Additionally, with a finite set of observations, it is difficult to distinguish a strange function from a more well-behaved nearby function, especially within the limited data set. A major issue with using strange functions in modeling is determining whether or not such a function is what is being observed or if its more well-behaved nearby function is at work.

Therefore, what is required for establishing a strange function as the basis for an observed effect is a *logical* reason for preferring the function. That is, there must be something in the nature of the causal relationships that would indicate the usage of a strange function in the model. One other thing that may indicate that a strange function is indeed required is the need to continually change the model with more and more data. This may indicate that a strange function is at play, and the causal relationships should be investigated to see if a strange function may properly model what is happening.

Since this paper only presents a small smattering of the known strange functions, more and more modeling power will be available by understanding more and more strange functions and how they relate to causality. The ones presented here were picked because the author could see ways in which they may be important to helping researchers think about causality. Additionally, more functions may be available by simple creative construction. Once the strictures of well-behaved functions are removed, and a person gets acquainted with the nature of strange functions, the ability to construct definitions of new strange functions to match the causality in question will be increased.

While strange functions should not be introduced lightly, there is no reason to avoid them in models. The preference for well-behaved functions is just that—a preference. There is no reason why reality must conform to our preferences. Linear components in models are better-behaved than non-linear components, but that doesn't mean our models must always contain linear components. The goal of modeling for science is to provide a deeper understanding of the nature of the subject under investigation. Other goals (such as using models for engineering) may substitute similar well-behaved functions in order to simplify calculations, but science, as an attempt to learn more about the true nature of realty, should in most cases prefer whatever function is the truest model of reality."

. . . Bartlett: [Brt18]*, §5, pp.123-124.*

Bartlett further argues that even "though strange functions are not always calculable or predictive, they do lend themselves to reasoning about relationships":

"6 Testing Models that Use Strange Functions

One of the key features of science is testability. As pointed out in Bartlett (2016), the two main features of methodological naturalism that made it successful were that it defined the scope of inquiry and it provided a system of justification. With well-behaved functions, the system of justification is fairly straightforward. The model will predict how a system will behave for tests that have not been performed yet; the experimenter will then perform the tests and see if the results match the model within a margin of error. With strange functions, however, the models do not always predict behavior. Therefore, the system of justification used will have to be modified in order to accommodate strange functions.

However, before we look at how we can test our new models, we should think about why it is that we test models in the first place. The goal of testing is to allow reality to push back on our ideas. That is, we have ideas about the nature of reality, but our ideas must *conform* to external reality, not the other way round. Testing is done to make sure that reality has a chance to give us feedback on the truth of our ideas.

We should recognize that testing is not an absolute truth-teller. It is more a sanity check than a rigid determiner of truth. For a finite set of data points, there are infinitely many functions that would be within the margin of error for those data points (Kukla, 1996). So how might someone decide between two empirically-equivalent theories? As it stands, our scientific ideas do not emanate entirely from empricism. If they did,, then this would be a problem. Instead, empricism provides the dataset that we use to establish rationalistic models, and it provides additional data to validate such models. But the models themselves are based on logical relationships between entities under investigation.

Therefore the key to testing is not that the data points must be uniquely determined by the theory, but rather that the theory must flow feom a proposed logical relationship between entities and the data must be consistent with it within a margin of error.

Even though strange functions are not always calculable or predictive, they do lend themselves to reasoning about relationships, otherwise they would not be considered functions. Therefore, it is possible to find patterns that are true with a strange function that may be tested for, even if the strange function itself is not directly testable. In the next section, we shall look at a specific example."

...Bartlett: [Brt18], §6, pp.124-125.

Bartlett concludes by observing that 'randomness', which can be treated as a 'strange' property of an infinite sequence, has been "used to determine if a mutation is in *response* to a selective pressure or if the mutation preceded the selection":

7 Randomness as an Examplar Strange Function

While strange functions generally have not been given much scientific weight, one in particular has been used regularly—randomness. While randomness is not well-behaved like most of the functions within science, the willingness of scientists operating under methodological naturalism to use it probably stems from the fact that it does not appear on the surface to imply teleology (though see Bartlett (2008) for an alternative view). Therefore, its adoption in the scientific community as a viable model allows us to demonstrate the utility of strange functions in scientific modeling.

Randomness is actually a property of an infinite sequence. Therefore, no finite sequence of events can prove that the sequence is random. Additionally, randomness is not predictive. Therefore, including randomness in the model does not help to predict any specific outcome. Thus, randomness matches what we have called strange functions.

Since randomness doesn't predict a specific outcome and cannot be tested directly, how was it included in scientific theories? Basically, if a model of an event has multiple possible outcomes, and the outcomes proceed in an order that is statistically stable but does not point toward any other structure, then the suggestion of randomness is quite appropriate.

For instance, in a Poisson distribution, the mean is equal to the variance. Therefore one can "test" for such a distribution by checking the means against the variance. If they are close, then the suggestion that the process is a random process following a Poisson distribution can be maintained. There are many different ways that the mean can equal the variance, but if ur formal reasoning leads us to expect such a distribution, and the distribution's characteristic features match our expectations, then the test can be considered confirmed. This is used, for instance, in the Luria-Delbrück experiment where the test for randomness is used to determine if a mutation is in *response* to a selective pressure or if the mutation preceded the selection."

...Bartlett: [Brt18], §7, p.125.

However, unlike that of the functions sought to be described by Bartlett in [Brt18], the 'strange' *relative randomness* (see §4.G., Definition 20) of functions that are algorithmically *verifiable*, but not algorithmically *computable*, is not—and does not describe the behaviour of non-naturalistic mathematical entities—entailed by the mathematical theory in which they are defined, but by the behaviour of the naturalistic physical phenomena that such functions seek to represent mathematically:

In an unusual, 'intuitionistic', interpretation of 'real numbers' as 'the hidden variables of classical physics', physicist Nicolas Gisin argues in [Gi19] (see also [Gi20]) that 'real numbers, are not all given at once, but are processes that develop in time':

> "We argued that classical physics theory can quite naturally and intuitively be considered as an indeterministic theory supplemented with additional variables and that these additional variables are nothing but our familiar real numbers. This is interesting from several points of view.
>
> First, it allows one to discuss additional variables outside the framework of quantum mechanics. Next, it shows that the infamous quantum measurement problem is not restricted to quantum theory, but is actually present in all indeterministic theories [2]. Third, it illustrates possible classical limits of knowledge.
>
> Finally, our discussion illustrates the important role played by classical mathematics. Indeed, in classical mathematics, formalized following Hilbert's huge influence at the beginning of last century, the digits of all real numbers are assumed to be all given at once. This translates in classical physics, in particular in chaotic classical dynamical systems, by the assumption that all the future is given at once, i.e., encoded in the real-valued initial conditions. However, classical (Platonistic) mathematics is not the only form of mathematics. There is also intuitionistic mathematics, a form of mathematics much less known than classical mathematics, but a quite well-developed mathematics in which numbers, in particular real numbers, are not all given at once, but are processes that develop in time. Brouwer, the father of intuitionism, named these processes choice sequences [3, 5, 6, 16]. Clearly, if physics is expressed in the language of intuitionistic mathematics, then one concludes very naturally that classical physics is indeterministic. The fact is that, historically, Hilbert won his debate with Brouwer (and Einstein won his debate with Bergson), hence physicists use classical/Platonistic mathematics and time was expulsed from physics. But there is no logical nor empirical necessity for this state of affair. It illustrates what everyone speaking more than one language knows, namely that different languages make certain thoughts easier to express in one language than in another, like determinism is easier to express in the classical mathematics language and indeterminism easier in intuitionistic mathematics [4].
>
> Finally, let us come back to the question raised in the introduction: is it scientific to believe that scientific theories limit human knowledge? Almost all colleagues would answer in the negative. However, most may conclude that scientific theories should be deterministic, as, if not, the future would be intrinsically unknowable. Here we argued on the contrary for indeterminism, as, if not, past real-valued initial conditions would be intrinsically unknowable. For me, there is nothing unscientific in claiming that the future is open, hence that there are limits to our knowledge of the future, but actual physical parameters of our scientific theories should be, at least in principle, knowable."
>
> *. . . Gisin: [Gi19], Conclusion.*

We note that, if we do not limit Gisin's reasoning to the self-imposed, and unnecessary, constraints of intuitionistic mathematics (highlighted in §8.B.), but to those imposed by *evidence-based* reasoning (highlighted in §8.C.), we can view Gisin as implicitly positing that all our knowledge of physical phenomena must, necessarily, be representable mathematically by algorithmically *verifiable* (in the sense of §2., Definition 2; and therefore *determinate*) functions which can be viewed as 'evolving' in time, even if some such functions are algorithmically *uncomputable* (in the sense of §2., Definition 2; and therefore *unpredictable*).

In this investigation we seek to formally define the common language and logic (in the sense of §1., Definition 1) of such functions. We then suggest a perspective from which to view the anomalous philosophical issues underlying some current concepts of quantum phenomena such as:

- EPR paradox;

- Bell's inequalities;

- Indeterminacy;

- Uncertainty;

- Fundamental dimensionless constants;

- Conjugate properties;

- Entanglement;

- Schrödinger's cat paradox.

19.B. *EPR*, Bohm's *hidden variables*, and Bell's equations

Now, amongst the philosophically disturbing features of the standard Copenhagen interpretation of Quantum Theory are its essential indeterminateness, and its essential separation of the world into 'system' and 'observer'.

For instance, the following remarks by Sheldon Goldstein et al in 2011 upon an accepted implication of 'orthodox formulations of quantum theory' suggest that the implication is yet philosophically disturbing:

> "It is a general principle of orthodox formulations of quantum theory that measurements of physical quantities do not simply reveal pre-existing or predetermined values, the way they do in classical theories. Instead, the particular outcome of the measurement somehow "emerges" from the dynamical interaction of the system being measured with the measuring device, so that even someone who was omniscient about the states of the system and device prior to the interaction couldn't have predicted in advance which outcome would be realized."
>
> . . . Goldstein et al: [Sh+11].

As highlighed in 1935 by Erwin Schrödinger's famously caustic observation, regarding the philosophical consequences of the proposed mathematical interpretation of the ψ-function if taken to imply (however, see §19.D.h. for an alternative, *evidence-based*, interpretation) that the objective state of nature is essentially probabilistic:

> "One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The ψ-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts."
>
> . . . Schrödinger: [Sc35], §5.

Schrödinger's gedanken was essentially an acknowledgement of a philosophical consequence of the, seemingly paradoxical, mathematical consequence of such an interpretation that was highlighted earlier in 1935 by Albert Einstein, Boris Podolsky and Nathan Rosen:

"The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one such way, whenever the conditions set down in it occur. Regarded not as a necessary, but merely as a sufficient, condition of reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality.

To illustrate the ideas involved let us consider the quantum-mechanical description of the behavior of a particle having a single degree of freedom. The fundamental concept of the theory is the concept of state, which is supposed to be completely characterized by the wave function ψ, which is a function of the variables chosen to describe the particle's behavior. Corresponding to each physically observable quantity A there is an operator, which may be designated by the same letter.

If ψ is an eigenfunction of the operator A, that is, if

$$\psi' \equiv A\psi = a\psi \tag{1}$$

where a is a number, then the physical quantity A has with certainty the value a whenever the particle is in the state given by ψ. In accordance with our criterion of reality, for a particle in the state given by ψ for which Eq. (1) holds, there is an element of physical reality corresponding to the physical quantity A. Let, for example,

$$\psi = e^{(2\pi i/h)p_0 x}, \tag{2}$$

where h is Planck's constant, p_0 is some constant number, and x the independent variable. Since the operator corresponding to the momentum of the particle is

$$p = (h/2\pi i)\partial/\partial x, \tag{3}$$

we obtain

$$\psi' = p\psi = (h/2\pi i)\partial\psi/\partial x = p_0\psi. \tag{4}$$

Thus, in the state given by Eq. (2), the momentum has certainly the value p_0. It thus has meaning to say that the momentum of the particle in the state given by Eq. (2) is real.

On the other hand if Eq. (1) does not hold, we can no longer speak of the physical quantity A having a particular value. This is the case, for example, with the coordinate of the particle. The operator corresponding to it, say q, is the operator of multiplication by the independent variable.

Thus,

$$q\psi = x\psi \neq a\psi. \tag{5}$$

In accordance with quantum mechanics we can only say that the relative probability that a measurement of the coordinate will give a result lying between a and b is

$$P(a,b) = \int_a^b \overline{\psi}\psi dx = \int_a^b dx = b - a. \tag{6}$$

Since this probability is independent of a, but depends only upon the difference $b - a$, we see that all values of the coordinate are equally probable.

A definite value of the coordinate, for a particle in the state given by Eq. (2), is thus not predictable, but may be obtained only by a direct measurement. Such a measurement however disturbs the particle and thus alters its state. After the coordinate is determined, the article will no longer be in the state given by Eq. (2). The usual conclusion from this in quantum mechanics is that *when the momentum of a particle is known, its coordinate has no physical reality.*

More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if $AB \neq BA$, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first.

From this follows that either (1) *the quantum-mechanical description of reality given by the wave function is not complete* or (2) *when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.* For if both of them had simultaneous reality—and thus definite values—these values would enter into the complete description, according to the condition of completeness. If then the wave function provided such a complete description of reality, it would contain these values; these would then be predictable. This not being the case, we are left with the alternatives stated.

In quantum mechanics it is usually assumed that the wave function *does* contain a complete description of the physical reality of the system in the state to which it corresponds. At first sight this assumption is entirely reasonable, for the information obtainable from a wave function seems to correspond exactly to what can be measured without altering the state of the system. We shall show, however, that this assumption, together with the criterion of reality given above, leads to a contradiction."

. . . Einstein, Podolsky and Rosen: [EPR35], p.777-778.

In other words, Albert Einstein, Boris Podolsky and Nathan Rosen argue that accepting Quantum Theory, but denying these features of the Copenhagen interpretation, logically entails accepting:

- Either that the world is non-local (thus contradicting Special Relativity);

    Comment: "Non-local' . . . means that there exist interactions between events that are too far apart in space and too close together in time for the events to be connected even by signals moving at the speed of light."
    . . . Shelson et al: [Sh+11].

- Or that Quantum Theory is not a complete theory, thus implicitly implying that there could be 'hidden variables' which would eliminate the need for accepting these features as necesary to any well-defined interpretation of Quantum Theory.

    Comment: "Traditionally, the phrase 'hidden variables' is used to characterize any elements supplementing the wave function of orthodox quantum theory."
    . . . Sheldon et al: [Sh+11].

"In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave-function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the

problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to a result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete."

<div align="right">. . . Einstein, Podolsky and Rosen: [EPR35], p.777</div>

In his 1949 '*Discussions with Einstein on Epistemological Problems in Atomic Physics*', Niels Bohr describes his immediate response to [EPR35], quoting from an article that he had published a few months thereafter:

> "Due to the lucidity and apparently incontestable character of the argument, the paper of Einstein, Podolsky and Rosen created a stir among physicists and has played a large role in general philosophical discussion. Certainly the issue is of a very subtle character and suited to emphasise how far, in quantum theory, we are beyond the reach of pictorial visualisation. It will be seen, however, that we are here dealing with problems of just the same kind as those raised by Einstein in previous discussions, and, in an article which appeared a few months later, I tried to show that from the point of view of complementarity the apparent inconsistencies were completely removed. The trend of the argumentation was in substance the same as that exposed in the foregoing pages, but the aim of recalling the way in which the situation was discussed at that time may be an apology for citing passages from my article.

> Thus, after referring to the conclusions derived by Einstein, Podolsky and Rosen on the basis of their criterion, I wrote:

>> Such an argumentation, how ever, would hardly seem suited to affect the soundness of quantum-mechanical description, which is based on a coherent mathematical formalism covering automatically any procedure of measurement like that indicated. The apparent contradiction in fact discloses only an essential inadequacy of the customary viewpoint of natural philosophy for a rational account of physical phenomena of the type with which we are concerned in quantum mechanics. Indeed the *finite interaction between object and measuring agencies* conditioned by the very existence of the quantum of action entails - because of the impossibility of controlling the reaction of the object on the measuring instruments, if these are to serve their purpose - the necessity of a final renunciation of the classical ideal of causality and a radical revision of our attitude towards the problem of physical realty. In fact, as we shall see, a criterion of reality like that proposed by the named authors contains - however cautious its formulation may appear - an essential ambiguity when it is applied to the actual problems with which we are here concerned.

> As regards the special problem treated by Einstein, Podolsky and Rosen, it was next shown that the consequences of the formalism as regards the representation of the state of a system consisting of two interacting atomic objects correspond to the simple arguments mentioned in the preceding in connection with the discussion of the experimental arrangements suited for the study of complementary phenomena. In fact, although any pair q and p, of conjugate space and momentum variables obeys the rule of non-commutative multiplication expressed by (2), and can thus only be fixed with reciprocal latitudes given by (3), the difference $q_1 - q_2$ between two space-co-ordinates referring to the constituents of the system will commute with the sum $p_1 + p_2$ of the corresponding momentum components, as follows directly from the commutability of q_1 with p_2 and q_2 with p_1. Both $q_1 - q_2$ and $p_1 + p_2$ can, therefore, be accurately fixed in a state of the complex system and, consequently, we can predict the values of either q_1 or p_1 if either q_2 or p_2 respectively, are determined by direct measurements. If, for two parts of the system, we take a particle and a diaphragm, like that sketched in Fig. 5, we see that the possibilities of specifying the state of the particle by measurements on the diaphragm just correspond to the situation described above, where it was mentioned that, after the particle has passed through the diaphragm, we have in principle the choice of measuring either the position of the diaphragm or its momentum and, in each case, to make predictions as to subsequent observations pertaining to

the particle. As repeatedly stressed, the principal point is here that such measurements demand mutually exclusive experimental arrangements.

The argumentation of the article was summarised in the following passage:

> From our point of [view] we now see that the wording of the above-mentioned criteria of physical reality proposed by Einstein, Podolsky, and Rosen contains an ambiguity as regards the meaning of the expression 'without in any way disturbing a system.' Of course there is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of *an influence on the very conditions which define the possible types of predictions regarding the future behaviour of the system.* Since these conditions constitute an inherent element of the description of any phenomena to which the term "physical reality" can be properly attached, we see that the argumentation of the mentioned authors does not justify their conclusion that quantum-mechanical description is essentially incomplete. On the contrary, this description, as appears from the preceding discussion, may be characterised as a rational utilisation of all possibilities of unambiguous interpretation of measurements, compatible with the finite and uncontrollable interaction between the objects and the measuring instruments in the field of quantum theory. In fact, it is only the mutual exclusion of any two experimental procedures, permitting the unambiguous definition of complementary physical quantities, which provide room for new physical laws, the coexistence of which might at first sight appear irreconcilable with the basic principles of science. It is just this entirely new situation as regards the description of physical phenomena that the notion of *complementarity* aims at characterising.

Regarding these passages, I am deeply aware of the inefficiency of expression which must have made it very difficult to appreciate the trend of the argumentation aiming to bring out the essential ambiguity involved in a reference to physical attributes of objects when dealing with phenomena where no sharp distinction can be made between the behaviour of the objects themselves and their interaction with the measuring instruments. I hope, however, that the present account of the discussion with Einstein in the foregoing years, which contributed so greatly to make us familiar with the situation in quantum physics, may give a clearer impression of the necessity of a radical revision of basic principles for physical explanation in order to restore logical order in this field of experience.

Einstein's own views at that time are presented in an article *Physics and Reality*, published in 1936 in the *Journal of the Franklin Institute*. Starting from a most illuminating exposition of the gradual development of the fundamental principles in the theories of classical physics and their relation to the problem of physical reality, Einstein here argues that the quantum-mechanical description is to be considered merely a means of accounting for the average behaviour of a large number of atomic systems and his attitude to the belief that it should offer an exhaustive description of the individual phenomena is expressed in the following words: "To believe this is logically possible without contradiction; but it is so very contrary to my scientific instinct that I cannot [forgo] the search for a more complete conception."

Even if such an attitude might seem well-balanced in itself, it nevertheless implies a rejection of the whole argumentation exposed in the preceding, aiming to show that, in quantum mechanics, we are not dealing with an arbitrary renumciation of a more detailed analysis of atomic phenomena, but with a recognition that such an analysis is *in principle* excluded. The peculiar individuality of the quantum effects presents us, as regards the comprehension of well-defined evidence, with a novel situation unforeseen in classical physics and irreconcilable with conventional ideas suited for our orientation and adjustment to ordinary experience. It is in this respect that quantum theory has called for a renewed revision of the foundation for the unambiguous use of elementary concepts, as a further step in the development which, since the advent of relativity theory, has been so characteristic of modern science."

. . . Bohr: [Boh49].

Although Bohr does not here identify precisely what classical paradigm Einstein is defending, which is being challenged by a '*novel situation unforeseen in classical physics and irreconcilable with conventional ideas suited for our orientation and adjustment to ordinary experience*', the fundamental point of his difference with Einstein's epistemological perspective which emerges from the above is apparently an implicit perception that:

— Any mathematical representation of Einstein's 'more complete conception' of a putative universe that admits all of our conceptual metaphors which faithfully reflect observations of the elements of physical reality would necessarily admit only classical algorithmically *computable* functions and relations (which are necessarily both *determinate* and *predictable* by §2.E., Theorem 2.17);

— Any mathematical representation of a putative universe that admits all of our conceptual metaphors which faithfully reflect observations of the elements of physical reality must admit functions and relations corresponding to experimental observations of quantum phenomena that are essentially *unpredictable* (as experimentally verified subsequently by Bell's inequalities).

In 1952 David Bohm proposed ([Bo52]) an alternative mathematical development of the existing Quantum Theory:

> "The usual interpretation of the quantum theory is self-consistent, but it involves an assumption that cannot be tested experimentally, viz., that the most complete possible specification of an individual system is in terms of a wave function that determines only probable results of actual measurement processes. The only way of investigating the truth of this assumption is by trying to find some other interpretation of the quantum theory in terms of at present "hidden" variables, which in principle determine the precise behavior of an individual system, but which are in practice averaged over in measurements of the types that can now be carried out. In this paper and in a subsequent paper, an interpretation of the quantum theory in terms of just such "hidden" variables is suggested. It is shown that as long as the mathematical theory retains its present general form, this suggested interpretation leads to precisely the same results for all physical processes as does the usual interpretation. Nevertheless, the suggested interpretation provides a broader conceptual framework than the usual interpretation, because it makes possible a precise and continuous description of all processes, even at the quantum level. This broader conceptual framework allows more general mathematical formulations of the theory than those allowed by the usual interpretation. Now, the usual mathematical formulation seems to lead to insoluble difficulties when it is extrapolated into the domain of distances of the order of 10-13 cm or less. It is therefore entirely possible that the interpretation suggested here may be needed for the resolution of these difficulties. In any case, the mere possibility of such an interpretation proves that it is not necessary for us to give up a precise, rational, and objective description of individual systems at a quantum level of accuracy."
>
> *. . . Bohm: [Bo52], Abstract.*

We note that by stating '*the usual mathematical formulation seems to lead to insoluble difficulties when it is extrapolated into the domain of distances of the order of 10-13 cm or less*', Bohm is implicitly implying here that:

(a) Current interpretations of Quantum Theory reflect a conflict between our mathematical representation of the behaviour of physical phenomena in limiting cases, and their actual behaviour in such cases (as addressed in §16.D. by Query 13); and that

(b) A resolution of the conflict requires us to seek a mathematical representation of the behaviour of physical phenomena in limiting cases other than that presently circumscribed by both the tacit, and explicit, assumptions which underlie current mathematical formulations of Quantum Theory.

If so, this is a crucial—possibly less appreciated, and perhaps even unsuspected—observation that is also implicit in [EPR35], and one whose critically limiting consequences—for the theoretical sciences in achieving a coherent philosophical perspective of our commonly accepted external reality—we highlight in §16.C.a. (Case 1) to §16.C.d. (Case 4).

Moreover, from the *evidence-based* perspective of [An16] (see §2.), we suggest that the resolution suggested by Bohm might be achievable by:

— insisting upon *evidence-based* reasoning for the interpretation of our mathematical formulations of physical phenomena (see §9.); and

— distinguishing between phenomena which might only be mathematically representable by algorithmically *verifiable*, but not algorithmically *computable*, functions and relations (see §4.G.).

Historically, even though Bohm's interpretation eliminated the need for indeterminism and the separation of the world into 'system' and 'observer', it appealed unappealingly to:

— hidden variables[189]; and, presumably,

— hidden natural laws that—we may reasonably presume further as reflecting the mathematical paradigms of the day—were implicitly assumed by Bohm to be representable only by well-defined classical algorithmically *computable* functions which could, in principle, be considered as having pre-existing or predetermined mathematical values over the domain over which the functions are well-defined.

Comment: We note an objection of current paradigms to Bohm's interpretation is that:

"...any hidden-variable theory faces the bijection problem $\aleph_0 \nleftrightarrow 2^{\aleph_0}$ for quantum states and their measurements, a paradox that affects Everett's multiverse interpretation as well as any hidden-variable theory in QM, relativistic and non-relativistic alike."

...Gauthier: [Ga15], Ch.5, §5.17.1, Arithmetical Foundations for Physical Theories, p.114.

We show in §12., though, that the objection dissolves once we recognise the denumerability of both algorithmically verifiable number-theoretic functions and algorithmically computable number-theoretic functions.

Moreover, in 1964 John Stewart Bell proved theoretically ([Bl64]) that:

(i) Any interpretation of Quantum Theory which appeals to local hidden variables in the above, classical, algorithmically *computable*, sense must necessarily be non-local—i.e., admit information between two spatially separated locations to travel faster than the speed of light—which would conflict with the deterministic mathematical representation of the laws of Special Relativity;

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[189] "This terminology is, however, particularly unfortunate in the case of the de Broglie-Bohm theory, where it is in: the supplementary variables—definite particle positions—that one finds an image of the manifest world of ordinary experience". [Sh+11].

(ii) It can be experimentally verified whether our physical reality admits local hidden variables in the above sense.

I. Introduction "The paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions."

. . . Bell: [Bl64], pp.195-199.

II. Formulation "With the example advocated by Bohm and Aharanov [6], the EPR argument is the following. Consider a pair of spin one-half particles formed somehow in the singlet spin state and moving freely in opposite directions. Measurements can be made, say, by Stern-Gerlach magnets, on selected components of the spins $\overrightarrow{\sigma}_1$ and $\overrightarrow{\sigma}_2$. If measurement of the component $\overrightarrow{\sigma}_1 \cdot \overrightarrow{a}$, where $\overrightarrow{a}$ is some unit vector, yields the value $+1$ then, according to quantum mechanics, measurement of $\overrightarrow{\sigma}_2 \cdot \overrightarrow{a}$ must yield the value -1 and vice versa. Now we make the hypothesis [2], and it seems one at least worth considering, that if the two measurements are made at places remote from one another the orientation of one magnet does not influence the result obtained with the other. Since we can predict in advance the result of measuring any chosen component of $\overrightarrow{\sigma}_2$, by previously measuring the same component of $\overline{\sigma}_1$, it follows that the result of any such measurement must actually be predetermined. Since the initial quantum mechanical wave function does *not* determine the result of an individual measurement, this predetermination implies the possibility of a more complete specification of the state.

Let this more complete specification be effected by means of parameters λ. It is a matter of indifference in the following whether λ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous. However we write as if λ were a single continuous parameter. The result A of measuring $\overrightarrow{\sigma}_1 \cdot \overrightarrow{a}$ is then determined by $\overrightarrow{a}$ and λ, and the result B of measuring $\overrightarrow{\sigma}_2 \cdot \overrightarrow{b}$ in the same distance is determined by $\overrightarrow{b}$ and λ and:

$$A(\overrightarrow{a}, \lambda) = \pm 1, B(\overrightarrow{b}, \lambda) = \pm 1 \tag{1}$$

The vital assumption [2] is that the result B for particle 2 does not depend on the setting $\overrightarrow{a}$, of the magnet for particle 1, nor A on $\overrightarrow{b}$.

If $\rho(\lambda)$ is the probability distribution of λ then the expectation value of the product of the two components $\overrightarrow{\sigma}_1.\overrightarrow{a}$ and $\overrightarrow{\sigma}_2.\overrightarrow{b}$ is

$$P(\overrightarrow{a}.\overrightarrow{b}) = \int d\lambda \rho(\lambda) A(\overrightarrow{a}, \lambda) B(\overrightarrow{b}, \lambda) \tag{2}$$

This should equal the quantum mechanical expectation value, which for the singlet state is

$$< \overrightarrow{\sigma}_1 \cdot \overrightarrow{a} \ \ \overrightarrow{\sigma}_2 \cdot \overrightarrow{b} >= -\overrightarrow{a} \cdot \overrightarrow{b}. \tag{3}$$

But it will be shown that this is not possible.

Some might prefer a formulation in which the hidden variables fall into two sets, with A dependent on one and B on the other; this possibility is contained in the above, since λ stands for any number of variables and the dependences thereon of A and B are unrestricted. In a complete physical

theory of the type envisaged by Einstein, the hidden variables would have dynamical significance and laws of motion; our λ can then be thought of as an initial values of these variables at some suitable instant."

...Bell: [Bl64], pp.195-199.

"**VI. Conclusion** In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant."

...Bell: [Bl64], pp.195-199.

The significance of Bell's argumentation for the EPR paradox—and Bohr's interpretation of it—lay in the fact that Bell's argument did not refer to any particular theory of local hidden variables.

However, what it showed conclusively was that nature violates the most general assumptions behind *any* mathematical theory that admits *only* classical, algorithmically *computable* functions and relations—which are both *determinate* and *predictable*—for faithfully representing our conceptual metaphors of experimental observations.

In other words, no combination of local deterministic and local 'random' hidden variables—classically assumed as mutually independent—can reproduce in such a theory the phenomena predicted by quantum mechanics and repeatedly observed in experiments.

Moreover, Bell defined inequalities that could distinguish between the expected behaviour of entangled particles under the hypothesis of a quantum theory that appeals to local hidden variables in the above sense, and their expected behaviour under a quantum theory that is not constrained by such local, algorithmically *computable*, hidden variables.

When put to the test, the overwhelming results of various experiments designed to study the actual behaviour of entangled particles has subsequently been accepted as conclusively showing that quantum theory cannot admit any local, algorithmically *computable*, hidden variables.

In a 1985 BBC radio interview, Bell apparently sought to place his argumentation in a broader philosophical perspective by reportedly remarking that the EPR argument might yet be salvaged by appeal to non-local hidden variables which—presumably reflecting the mathematical paradigms of the day and age—could also be taken to define only algorithmically *computable* functions and relations that, again, are both *determinate* and *predictable*.

Such a solution however, he argued, might demand a philosophically unaffordable, and psychologically inhuman, price: *superdeterminism*. Thus:

"There is a way to escape the inference of superluminal speeds and spooky action at a distance. But it involves absolute determinism in the universe, the complete absence of free will. Suppose the world is super-deterministic, with not just inanimate nature running on behind-the-scenes clockwork, but with our behavior, including our belief that we are free to choose to do one experiment rather than another, absolutely predetermined, including the 'decision' by the experimenter to carry out one set of measurements rather than another, the difficulty disappears. There is no need for a faster-than-light signal to tell particle A what measurement has been carried out on particle B, because the universe, including particle A, already 'knows' what that measurement, and its outcome, will be."

...Bell: Quotation from [DB93], pp.45-46; reportedly an adaptation from the edited transcript of a 1985 BBC radio interview with John Stewart Bell.

From an *evidence-based* perspective (see 4.C.), we shall now argue that it is essentially the illusory barrier of *superdeterminism* that has prevented classical mechanics and quantum mechanics from being seen as two, complementary and not contradictory, pieces of the key that would allow us to correspond—without appealing to non-locality—every element of physical reality to a unique, algorithmically *verifiable* but not necessarily algorithmically *computable* (see §4.G.), physical quantity in some putative mathematical representation of a putative universe which could admit all of our conceptual metaphors that faithfully reflect observations of the elements of physical reality (in the sense of §10.C.).

19.B.a.   An *evidence-based* perspective of EPR, Bohm's *hidden variables*, and Bell's equations

In other words, our foundational investigations into the (apparently unrelated) area of evidence-based and finitary interpretations of the first order Peano Arithmetic PA ([An16]) (see §2.) now suggest that:

- If our above presumption concerning an implicit consideration by Bohm and Bell—of *only* functions that are implicitly assumed to be classically *computable* algorithmically—is correct;

  > **Comment**: The following quote from [EPR35] supports our presumption that Einstein's— and presumably Bohm's—'more complete conception' of reality than that admitted by quantum theory demanded that all laws of nature—whether accessible to a mathematical representation explicitly, or implicitly as associated hidden variables—-must be both *determinate* and *predictable*; thus implying that the functions and relations representing these laws mathematically are necessarily algorithmically *computable*:
  >
  > > "A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*"
  > > *. . . op. cit.*

- and Bell's λ (in [Bl64], # II, Formulation) is essentially a Tarskian interpretation that corresponds the algorithmically *computable* functions and relations of one mathematical language into putative algorithmically *computable* functions and relations of another, such that the latter can admit the probability distribution in Bell's equation (2) (in [Bl64], pp.196-199) which, however, appeals to an integration that is definable—albeit implicitly—*only* over algorithmically *computable* functions in classical real analysis;

- then the hidden variables in the Bohm-de Broglie interpretation of Quantum Theory could as well be presumed to involve natural laws which are mathematically representable *only* by functions that are algorithmically *verifiable* (see §2., Definition 2), but not algorithmically *computable* (see §2., Definition 3)—hence mathematically determinate but unpredictable even by Laplace's *vast intelligence* (see §4.H.a.);

  > **Comment**: In other words, if Bell's parameter λ in [Bl64] (# II, Formulation) associates the result A of measuring $\overrightarrow{\sigma}_1 \cdot \overrightarrow{a}$ with putative, algorithmically *verifiable* but not algorithmically *computable*, functions:
  >
  > $$A(\overrightarrow{a}, \lambda) = \pm 1, B(\overrightarrow{b}, \lambda) = \pm 1$$

then the probability distribution ρ of λ cannot be assumed to yield the expectation value of the product of the two components $\overrightarrow{\sigma}_1 . \overrightarrow{a}$ and $\overrightarrow{\sigma}_2 . \overrightarrow{b}$ as defined in [Bl64], equation (2).

- in which case Bohm's interpretation might avoid being held as admitting 'non-locality' by Bell's reasoning.

From the *evidence-basd* perspective of [An16] (see §2.), we thus conclude from EPR's and Bell's argumentations that:

— If any putative, well-defined, mathematical model $M(U)$ of a putative universe U (such as, for instance, the putative universe defined mathematically in §16.D.c.) which seeks to faithfully describe some states of the universe U at any time t—starting from a putative time t_0—admits interactions with elements that are not defined in the model $M(U)$—such as the the measurement $\overrightarrow{\sigma}_1 \cdot \overrightarrow{a}$;

> **Comment**: We note that admitting such extraneous interactions into the mathematical model U of the putative universe being considered is prohibited in [EPR35], whose argument is predicated on the stipulation that in any complete mathematical representation of our observations of physical phenomena "we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity".

— then, by definition, it is incomplete and must admit a conservative extension U_1 of U with a well-defined mathematical model $M_1(U)$ that admits the extraneous measurement $\overrightarrow{\sigma}_1 \cdot \overrightarrow{a}$;

> **Conservative extension**: A theory T_2 is a (proof theoretic) conservative extension of a theory T_1 if the language of T_2 extends the language of T_2; that is, every theorem of T_1 is a theorem of T_2, and any theorem of T_2 in the language of T_1 is already a theorem of T_1.

— and, moreover, U_1 must similarly admit a conservative extension U_2 of U_1 with a well-defined mathematical model $M_2(U)$ that admits the extraneous measurement $\overrightarrow{\sigma}_2 \cdot \overrightarrow{b}$.

> **Comment**: That the criteria of what can be, and what cannot be, treated as a conservative extension of a formal mathematical theory is misleading from an *evidence-based* perspective, and not *well-defined* in current paradigms of classical mathematics, is highlighted by §15., Corollary 15.3 and, independently, by §15.A., Theorem 15.4.

Suffice to say that neither [EPR35] nor Bell's [Bl64] address the question—implicit in Bell's formal expression of the EPR paradox as above—of whether the interpretation λ admits algorithmically *verifiable* functions and relations, or whether it—and EPR—implicitly assume that the mathematical models of U, U_1, and U_2 must admit *only* algorithmically *computable* functions and relations.

Nor do they seem to admit the perspective that any mathematical theory—intended to unambiguously represent the conceptual metaphors (in the sense of [LR00]) that seek to faithfully describe our observations of physical phenomena—must:

— not only have a well-defined *proof-theoretic* logic (in the sense of §9., Definition 24) that unambiguously assigns *evidence-based* values—of formal *provability* from the axiomatic assumptions of the theory—to the formal propositions of the theory;

— but also admit a well-defined *model-theoretic* logic (in the sense of §9., Definition 25) that unambiguously assigns *evidence-based* values—of *truth* under an interpretation—to the formal propositions of the theory.

From an *evidence-based* perspective, the significance of Bell's argumentation is that it highlights the fact that the EPR paradox, and other paradoxes of quantum theory, are only *local* mathematical paradoxes (as highlighted further in §16.) in the mathematical language in which the theory seeks to faithfully represent the conceptual metaphors that reflect our sensory perceptions of an external reality; they should not be conflated with any philosophical considerations of the nature and substance of our sensory perceptions of the external reality.

The fact that we can admit functions and relations which are algorithmically *verifiable* but not algorithmically *computable* (see §4.G.)—hence interpretable as 'deterministic' but not 'predictable'—into a well-defined, categorical, mathematical theory such as the first-order Peano Arithmetic PA illustrates this point.

In other words, from an *evidence-based* perspective, the EPR paradox simply shows that any mathematical theory which seeks to faithfully represent the conceptual metaphors that reflect our sensory perceptions of an external reality is incomplete in the sense that it cannot predict—in the sense of formal provability from the logic and axiomatic assumptions of the theory—the consequences of a relationship between the terms of the theory, and elements that are not axiomatically defined as terms of the theory, but which might be capable of being introduced post-facto as terms into a conservative extension of the theory (as is suggested in the *evidence-based* mathematical interpretation of Schrödinger's gedanken in §19.D.h.).

19.B.b.   Dissolving the Bohr-Einstein debate

The significance of *evidence-based* reasoning is also highlighted in the case of the Bohr-Einstein debate (see [Boh49]; [Ei36]; [Ei36a], §5, p.371), on whether or not every element of physical reality can be corresponded—without appealing to non-locality—to a unique physical quantity in some putative mathematical representation of a putative universe which admits all of our conceptual metaphors that faithfully reflect observations of the elements of physical reality.

It turns out that the debate dissolves (compare with the dissolution of the, foundationally parallel, Poincaré-Hilbert debate in §2.D.; see also §10.F.(i)-(iii)) since:

(a) **Einstein's contention** is validated if we accept that every conceivable state of energy distribution must occur at some instant in the mathematical model, say $M(\mathcal{U})$, considered in §16.D.c. of a putative finite Universe $\mathcal{U}$ that recycles endlessly from Big Bang to Ultimate Implosion, and which admits:

— all of our conceptual metaphors which faithfully reflect observations of the elements of physical reality;

— Einstein's equations;

— classical quantum theory;

— neo-classical functions and relations that are algorithmically *verifiable* but not algorithmically *computable* (whence the universe is *determinate* but not *predictable*, and therefore no cycle of states from a Big Bang to an Ultimate Implosion can ever be repeated recursively);

— locality in all interactions (since there is no separation of 'system' and 'observer' *within* the putative—essentially *determinate* but *unpredictable*—universe $\mathcal{U}$, mathematically represented by $M(\mathcal{U})$, that would require some values to be determinate only at some future instant of 'observation' *after* an 'interaction');

whilst:

(b) **Bohr's contention**, supported by the experimental evidence provided by Bell's inequalities, is also validated if we accept that:

- no mathematical representation, say L,

- of a putative universe, say U, that admits all of our conceptual metaphors (see [LR00]) which faithfully reflect observations of the elements of physical reality in U,

- can admit only classical algorithmically *computable* functions and relations (which are necessarily both *determinate* and *predictable* by §2.E., Theorem 2.17), since:

  – every observation (which we can informally view as corresponding to Carnap's *explicandum* in [Ca62a]) of a physical process in U,

  – that is sought to be formally represented (in the sense of §10.C.) by a classical—presumably first-order, categorical, and therefore determinate and predictable—mathematical representation in L (which we can informally view as corresponding to Carnap's *explicatum* in [Ca62a]),

  – such that any well-defined model M_L of L can be said to *empirically* model U,

  – is an *empirical* meta-statement about L, and ipso fact about M_L,

  – that cannot be unrestrictedly assumed to be well-definable within L, or within M_L (see §2.E.a.; also §11.G.),

  – without risking the paradoxes of impredicativity (see §16.C.).

    Comment: In other words, the question of whether, and to what extent, any Model M_L of L succeeds in faithfully representing the structure U mathematically (in the sense of Carnap's *explicatum* and *explicandum* in [Ca62a]), is an *empirical* question for the cognitive sciences (cf. [LR00]) that cannot be assumed to be theoretically decidable mathematically (compare also §1., Definition 1).

As argued in §16.C., Cases §16.C.a. to §16.D.c.:

- any classical representation of some physical processes of U in a first-order mathematical language L,

- could entail Cauchy sequences and associated Cauchy limits within L,

- at points corresponding to observations in U,

- which might constitute discontinuities in the state of the process within U,

- but which are not discontinuities within the mathematical representation L of the system U, or within any of the well-defined models M_L of L,

- since any observations by an observer in U cannot be assumed well-definable within the intended mathematical representation L of U, or within any of the well-defined models M_L of L,

- without risking the paradoxes of impredicativity (see §16.C.).

Thus, one could conceivably argue that the consequences of any interaction between an observer in U with that which is sought to be observed in U:

- can only be subject to deterministic probabilistic laws,

- that entail only the probable states of the physical process after an observational interaction whose consequences are not well-definable—hence predictable—within any mathematical representation L of U and/or any of the well-defined models M_L of L.

Moreover, any observation of a physical process in U is an interaction which, essentially, adds (as implicitly argued in [EPR35]) new—hence unpredictable—elements to an existing quantum-mathematical description L of U; elements which would thus be determinate, but unpredictable, in any conservatively extended, quantum-mechanical, representation L_1 of L, to even a notional, omnipotent, 'external' observer in U of the model M_{L_1} of L_1 (as argued in §19.D.h., Thesis 20).

19.B.c.   Aerts' 'overlooked solution' to the Einstein-Podolsky-Rosen paradox

Prima facie, the above argumentation (see also §21.) is essentially that outlined by Massimiliano Sassoli de Bianchi in [Bia18] and [Bia19], if we treat processes that are representable mathematically only by algorithmically *verifiable*, but not by algorithmically *computable*, functions and relations, as corresponding to those reportedly considered by Diederik Aerts in [Ae82]:

- where spatial separation in a physical process/experiment need not necessarily imply disconnection in the corresponding mathematical representation/model of the process/experiment;

and

- where, 'while describing the situation of bipartite systems formed by separate quantum entities, he was able to prove, this time in a perfectly constructive way, that quantum mechanics is structurally unable to describe these situations':

  > "To explain Aerts' solution, it is important to emphasize that EPR's reasoning is an *ex absurdum* one, that is, a reasoning which starts from certain premises and reaches a contradiction. What EPR have shown is that if their premises are assumed to be correct, then quantum theory has to be considered incomplete, as unable to describe all elements of reality of a physical system. Those who have taken seriously this conclusion thus tried to find remedies, for instance by supplementing the theory with additional variables for the quantum states, to allow position and velocity to have simultaneous definite values and escape the limitations of Heisenberg's uncertainty relations. This hidden variables program, however, subsequently met the obstacle of so called *no-go theorems*, drastically limiting the class of admissible hidden-variable theories.[31−36]

  > The premise that was part of the EPR reasoning, as we explained, is that for two quantum entities that have interacted and flown apart, it was natural to expect that their spatial separation was equivalent to an experimental separation. In addition to that, EPR applied the quantum formalism to describe the situation, which means they implicitly also assumed that quantum mechanics is able to describe a system formed by *separate physical entities*. But since this produced a contradiction, one is forced to conclude that the assumption is incorrect, that is, that quantum mechanics is unable to describe separate entities.

Now, one may object that this is too strong a conclusion, in the sense that the only mistake committed by EPR was to expect that spatial separation would necessarily imply disconnection. This expectation, as we know today has been overruled by numerous experiments, showing that by making sufficient efforts and taking all necessary precautions, experimental situations can indeed be created where microscopic entities, after having interacted, can remain interconnected, even when arbitrarily large spatial distances separate them. The mistake of EPR was therefore to think about a situation where there is no experimental separation between two entities, as a situation of actual experimental separation.

So, apparently problem solved: EPR-like experiments, like those performed by the group of Alain Aspect, have precisely shown that in the situation considered by EPR[,] quantum mechanics does actually provide the correct description of two quantum entities flying apart, since Bell's inequalities are violated, in accordance with the quantum predictions. Thus, one would be tempted to conclude that EPR reasoning is not valid. Well, yes and no. Yes, because at their time the possibility of producing these non-local/non-spatial states was a truly remarkable and totally unexpected possibility, based on classical prejudices, so the EPR *ex absurdum* reasoning was indeed applied to a wrong experimental situation, if such situation is considered to be correctly described by an entangled state. No, because the possibility of producing and preserving entangled states has very little to do with EPR's reasoning per se. Indeed, one can in principle also assume that experiments could be performed where instead of making efforts to preserve the quantum connectedness of the two flying apart entities, an effort is made instead to obtain the opposite situation of two flying apart entities eventually becoming perfectly disconnected, i.e., separated.

Experiments of this kind have never been worked out consciously, but these would indeed correspond to situations leading to the EPR paradox. In other words, the incompleteness of quantum mechanics is not revealed in the physical situation of quantum entities flying apart and remaining non-separate, as these are the situations which are perfectly well described by the quantum formalism (as the violation of Bell's inequalities proves), and there is no contradiction/paradox in this case, but by the experimental situations that can provide a disconnection, and which in the setting of EPR-like experiments would be interpreted as "badly performed experiments". These are precisely the situations that quantum mechanics would be unable to describe, certainly not by means of entangled states, as[,] if we assume it can, then we reach a contradiction.

Having clarified that the logical reasoning of EPR is not directly affected by the experimental discovery of entangled states, the question thus remains about the completeness of the quantum formalism, in relation to its ability to describe separate physical entities. It is here that Aerts' work join[s] the game. Indeed, among the topics of his doctoral research there was that of elaborating a mathematical framework for the general description of separate mathematical quantities. Aerts approached the issue using Piron's axiomatic approach to quantum mechanics, a very general formalism which was precisely:[37] "obtained by taking seriously the realistic point of view of Einstein and describing a physical system in terms of 'elements of reality." This allowed him to view the EPR work from a completely new angle. Indeed, while describing the situation of bipartite systems formed by separate quantum entities, he was able to prove, this time in a perfectly constructive way, that quantum mechanics is structurally unable to describe these situations."

. . . de Bianchi: [Bia18], §IV, The Solution, p.7-9.

Thus, from the *evidence-based* perspective of this investigation, Aerts' argument is that:

- the 'shortcoming of quantum theory in describing separate systems cannot be detected at the level of states'

    - which are 'perfectly well described in quantum mechanics by so-called product states, that is, states of the tensor product $\psi \otimes \phi$',

        * which are, presumably, algorithmically *computable*;

- 'but at the level of the properties,

  – which in the quantum formalism are described by orthogonal projection operators',

  ∗ which, too, are implicitly presumed algorithmically *computable*

  – if 'certain properties of a bipartite system formed by separate components cannot be represented by algorithmically *computable* orthogonal projection operators',

  ∗ since such properties might be representable mathematically only in terms of some function or relation that is algorithmically *verifiable*, but not algorithmically *computable*.

As de Bianchi notes further:

> "EPR were right about the incompleteness of quantum mechanics, but not for the reason they believed: quantum mechanics is incomplete because [it is] unable to describe separate physical systems. Of course, depending on the viewpoint adopted, this can be seen as a weak or strong trait of the theory. If separate systems exist in nature, then it is a weak trait, if they don't, then it is a strong trait. We will come back on that in the conclusive section, but let us now sketch the content of Aerts' constructive proof, which is actually quite simple.
>
> Note that despite the simplicity of the proof, it usually comes as a surprise that quantum mechanics would have this sort of shortcoming. Indeed, the first reaction I usually get, when discussing Aerts' result with colleagues, is that this cannot be true, as separate systems are perfectly well described in quantum mechanics by so-called product states, that is, states of the tensor product $\psi \otimes \phi$, where $\psi \in \mathcal{H}_A$ and $\phi \in \mathcal{H}_B$, with $\mathcal{H}_A$ the Hilbert (state) space of entity A and $\mathcal{H}_B$ that of entity B, the Hilbert space $\mathcal{H}$ of the bipartite system formed by A and B being isomorphic to $\mathcal{H}_A \otimes \mathcal{H}_B$. This is correct, and in fact the shortcoming of quantum theory in describing separate systems cannot be detected at the level of the states, as in a sense there is an overabundance of them, but at the level of the properties, which in the quantum formalism are described by orthogonal projection operators. In fact, it is precisely this overabundance of states that produces a deficiency of properties, in the sense that certain properties of a bipartite system formed by separate components be represented by orthogonal projection operators.
>
> Technically speaking, the only difficulty of Aerts' proof is that one needs to work it out in all generality, independently of specific representations, like the tensorial one, so that one can be certain that its conclusions are inescapable.[11−14,16] Without entering into all details, the demonstration goes as follows. First, one has to define what it means for two entities A and B to be experimentally separate. As we mentioned already, this means that measurements individually performed on them do not influence each other. In other words, separate entities are such that their measurements are *separate measurements*. More precisely, two measurements $\mathcal{M}_A$ and $\mathcal{M}_B$ are separate if they can be performed together without influencing each others [measurement]. This means that: (1) the execution of $\mathcal{M}_{AB}$ on the bipartite entity formed by A and B corresponds to the execution of $\mathcal{A}$ on A and of $\mathcal{M}_B$ on B, and (2) the outcomes of $\mathcal{M}_{AB}$ are given by all possible couples of outcomes obtained from $\mathcal{M}_A$ and $\mathcal{M}_B$.
>
> What Aerts then shows is that there is no self-adjoint operator O_{AB} that can represent such measurement $\mathcal{M}_{AB}$."
>
> *. . . de Bianchi: [Bia18], §V, Aerts' Proof, p.9-10.*

We note that, in his more technical 'proof' in [Ae82], based on his original doctoral thesis, Aerts explicitly restricts the *properties* attributed to physical phenomena to what 'we know at this moment and we hope that these properties will remain interesting properties in the future'; and seeks to distinguish between the '*actual*' and '*potential*' properties of an '*entity*':

> "We have to remark that from a logical point of view there are two possible conclusions. Either quantum mechanics is considered to be wrong and then it should be replaced by a theory that makes it possible to describe separated entities (for example the theory proposed in this article) or

otherwise we believe that quantum mechanics can describe all entities that appear in nature, but then we have to conclude that separated entities do not exist.

Our opinion is that this choice is not a metaphysical one, because we define separated entities by means of separated questions. The definition of separated questions is experimentally verifiable, and it is very easy to find examples of separated questions. It is these separated question[s] that cannot be described by quantum mechanics. One can argue further and say that perhaps separated questions do exist but no separated entities. We can remark then that although in this paper we only treat the case of two separated entities, it is really the separated questions that cannot be described by quantum mechanics (see Ref. 9). In Section 6.1 we also show that the concept of separated entities is not an idealization, once we agree that the concept of entity is a good idealization.

2. THE CONCEPT OF ENTITY

Physics describes the phenomena that take place. The universe is what we call the collection of all these phenomena. One way to understand this universe is to concentrate on certain phenomena that "strike the eye," certain phenomena that we can experience without being forced to experience all the other phenomena that are taking place. We give names to these phenomena. For example: rain, tree, electron, ... Since we can experience such a phenomena apart from all other phenomena, it makes sense to attribute *properties* to the phenomena and to study these properties.

We shall study the set of properties that we want to attribute in this way to a certain phenomena. The "thing" described by this set of properties, and which is an idealization of the phenomena in the sense explained above, we shall call an "*entity.*"

By the set of properties of a[n] entity, we do not mean the set of all possible properties that we could attribute to the phenomena defining the entity. No, we restrict ourselves to those properties that we know at this moment and we hope that these properties will remain interesting properties in the future. It seems to us that we cannot do more. Hence the set of properties of an entity is a well-defined set which is complete in the sense that it contains really all that we know about the phenomenon. Among the properties of an entity, some are "*actual,*" the entity has them "*in acto,*" and others are "*potential,*" the entity has the possibility of obtaining them. The *evolution* of the entity is the changing of actual properties into potential properties and potential properties into actual properties. In physics we say that the state of the entity changes. Hence the *state* of the entity is the collection of all actual properties."

. . . Aerts: [Ae82], §1, Introduction and §2, The Concept of Entity, p.1134-1135.

From the *evidence-based* perspective of this investigation, Aerts' distinction between:

- *actual* properties of an entity; and

- *potential* properties of an entity;

could be viewed as reflecting the distinction between:

- (*actual*) properties that are mathematically expressible by algorithmically *computable* functions and relations; and

- (*potential*) properties that are mathematically expressible only by algorithmically *verifiable* functions and relations, but not by algorithmically *computable* functions and relations.

In a subsequent 2014 paper [AdB14], Aerts and de Bianchi review their earlier work, clarifying and explaining—in an illuminating, inter-disciplinary, *Introduction,* excerpted at length below—how, and why, their perspective ought to be be viewed as:

- complementing, and not contradicting;

    Comment: Compare with the argument seeking dissolution of the, foundationally parallel, Poincaré-Hilbert debate in §2.D.; and with the argument in §8. that Hilbert's and Brouwer's interpretations of quantification are complementary and not contradictory (see also §10.F.(i)-(iii)).

- current paradigms concerning the possibility of a cognitive grounding for the quantum mechanical formalisations of quantum phenomena that, by admitting:

    - a lack of knowledge which 'is not to be understood in a subjective sense, as it results from an *objective* condition of *lack of control* regarding the way a potential interaction is actualized during a measurement',

    - and 'the possibility of describing degenerate observables',

    - could 'render useless the idea that quantum probabilities would necessarily have an ontological nature':

"One of the major problems of quantum mechanics, since its inception, has been that of explaining the origin of the statistical regularities predicted by its formalism. Simplistically, we could say that two diametrically opposite approaches to this problem stand out: the *instrumentalist* and the *realist*. According to the former, the solution of the problem is equivalent to its elimination: quantum probabilities are not required to be further explained, as what really matters in a physical theory is its predictive power, expressed by means of a *rule of correspondence* between the formalism of the theory and the results of the measurements, performed in the laboratories; and quantum mechanics is equipped with an extremely effective rule of this kind: the so-called *Born rule*, first stated by Max Born in the context of scattering theory [1].

While for the instrumentalist (by virtue of necessity and because of the difficulty of finding a coherent picture) it is unnecessary, if not wrong, to explain the predictive power of the Born rule, for the realist explanation must precede prediction, and one cannot settle for simply checking that the Born rule makes excellent correspondences: one also has to explain the reason of such success, possibly deriving the rule from first principles, even if this is at the price of having to postulate the existence of new *elements of reality*, which so far have remained *hidden* to our direct observation, in accordance with Chattons anti-razor principle: "no less than is necessary" [2].

The main way to do this, is to create a *model*, in which the different terms of the quantum formalism possibly find a correspondence, receiving in this way a better interpretation and explanation; and if the additional explanations contained in the model are able to produce new predictions, the model can also become a candidate for an upgraded version of the theory, providing a more refined correspondence with the experiments, through which in turn the model can be tested and possibly refuted.

Among the major obstacles that have prevented the development of new explicative models for quantum mechanics, and more specifically for quantum probabilities, there are the famous *no-go theorems* about hidden-variables, which restrict the permissible hidden-variable models explaining the origin of quantum randomness [3], [4], [5], [6], [7], [8]. So much so that, over time, this has led many physicists to believe that the nature of quantum probabilities would be *ontological*, and not *epistemic*, that is, that they would be quantities not explainable as a condition of lack of knowledge about an objective deeper reality.

The no-go theorems, which all draw their inspiration from von Neumanns original proof [3], affirm that quantum probabilities cannot reflect a lack of knowledge about "better defined states" of a quantum entity, so that quantum observables would be interpretable as averages over the physical quantities deterministically associated with these hypothetical better defined states (much in the spirit of classical statistical mechanics). As a consequence, if quantum probabilities are explainable

as a lack of knowledge about an underlying reality, such reality cannot be associated with an improved specification of the actual states of the quantum entities.

Therefore, to bypass the obstacle of the no-go theorems, one must think of the hidden-variables not as elements of reality that would make a quantum mechanical state a more "dispersion free" state, but as something describing a different aspect of the reality of a quantum entity interacting with its environment, and in particular with a measuring system. This possibility was explored by one of us, in the eighties of the last century, by showing that if hidden-variables are associated, rather than with the state of the quantum entity, with its interaction with the measuring system, one can easily derive the Born rule of correspondence and render useless the idea that quantum probabilities would necessarily have an ontological nature [9].

This preliminary 1986 study has generated over the years a number of works (see [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] and the references cited therein) further exploring the explicative power contained in this approach to the measurement problem, today known as the *hidden-measurement approach*, or *hidden-measurement interpretation*. More precisely, the very natural idea that was brought forward at that time, and subsequently developed, is that in a typical quantum measurement the experimenter is in a situation of *lack of knowledge* regarding the specific measurement interaction which is selected at each run of the measurement. And since these different potential measurement interactions would not in general be equivalent, as to the change they induce on the state of the measured entity, they can produce different outcomes, although each individual interaction can be considered to act deterministically (or almost deterministically, and we will specify in the following in detail what we mean by 'almost deterministically').

We emphasize that this condition of lack of knowledge is not to be understood in a subjective sense, as it results from an *objective* condition of *lack of control* regarding the way a potential interaction is actualized during a measurement, as a consequence of the irreducible fluctuations inherent to the experimental context, and of the fact that the operational definition of the measured physical quantity does not allow the experimental protocol to be altered, in order to reduce them [22].

The purpose of the present article is to put forward, for the first time, a complete self-consistent hidden-measurement modelization of a quantum measurement process, valid for arbitrary N-dimensional quantum entities, which will fully highlight the explicative power contained in the hidden-measurement interpretation. But to fully appreciate the novel aspects contained in this work, it will be useful to first recall what has been proven in the past, and what are the points that still needed to be clarified and elaborated.

What was initially proved in [9], [23], is that hidden-measurement models could in principle be constructed for arbitrary quantum mechanical entities of finite dimension, and the possibility of constructing hidden-measurement models for infinite-dimensional entities was afterwards demonstrated by Coecke [12]. However, these proofs, although general, were only about that aspect of a measurement that we may call the "naked measurement", corresponding to the description of the pure "potentiality region" of contact between the states of the entity under investigation and those describing the measuring apparatus. A measurement, however, is known to contain much more structure than just that associated with such "potentiality region".

What we are here referring to is the *structure of the set of states* of the measured entity (which is Hilbertian for quantum entities, but could be non-Hilbertian for entities of a more general nature [24], [25]), and how these states relate, *geometrically*, to those describing the measuring system. This is what in the standard Hilbertian formalism is described by means of the so-called (Dirac) *transformation theory*, which allows to calculate, for a given state, not only the probabilities associated with a single observable, but also those associated with all possible observables one may choose to measure. And of course, to obtain a complete description of a measurement process, also this additional geometric information, associated with the "generalized rotations in Hilbert space", needed to be taken into account, and incorporated in the mathematical modelization.

This, however, was only possible to do (until the present work) in the special situation of two-dimensional entities, like spin-$\frac{1}{2}$ entities, and for higher-dimensional entities it was not at all

obvious to understand how to transform the state relative to a given measurement context (defined by a given observable), when a different measurement context (defined by a different observable) was considered.

This "transformationally complete" two-dimensional model has been extensively studied over the years, and is today known by different names. One of these names is *spin quantum-machine*, with the term "machine" referring to the fact that the model is not just an abstract construct, but also the description of a macroscopic object that can be in principle constructed in reality, thus allowing to fully visualize how quantum and quantum-like probabilities arise. Another name for the model is ϵ-model [17], [19], [26], where the ϵ refers to a parameter in the model that can be continuously varied, describing the transition between quantum and classical measurements, passing through measurement situations which are neither quantum nor classical, but truly intermediary. A third name is *sphere-model* [16], where the term "sphere" refers to the *Bloch sphere*, the well known geometrical representation of the state space of a two-dimensional quantum entity (qubit).

In fact, the possibility of representing the full measurement process (not just its "naked part") of two-dimensional entities, in terms of hidden-measurement interactions, is related to the existence of a complete representation of the complex quantum states (the vectors in the two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$) in a real two-dimensional unit sphere, or in a three-dimensional unit ball, if also density operators are considered. Such representation was not available for higher dimensional entities, and this was the reason why a complete representation for the full measurement process was still lacking.

In retrospect, we can say that this technical difficulty did not favor the spread of the hidden-measurement ideas, and possibly promoted a certain suspicion about the true reach of this interpretation, as a candidate to solve the measurement problem. In this regard, we can mention the fact that when presenting the spin machine-model to an audience, the objection was sometimes raised that this kind of models could only be conceived for two-dimensional quantum entities, because of Gleason's theorem [5] and an article by Kochen and Specker [7]. Indeed, Gleason's theorem is only valid for a Hilbert space with more than two dimensions, hence not for the two-dimensional complex Hilbert space that is used in quantum mechanics to describe the spin of a spin-$\frac{1}{2}$ entity. And in addition to that, Kochen and Specker constructed in the above mentioned work a spin model for the spin of a spin-$\frac{1}{2}$ entity, proposing also a real macroscopic realization for it, but also pointing out, on different occasions, that such a real model could only be constructed for a quantum entity with a Hilbert space of dimension not larger than two.

Afterwards, some effort was given to clarify this dimensionality issue, and counter act the prejudice about the impossibility of a hidden-measurement model beyond the two-dimensional situation. In [27], for example, a mechanistic model was proposed for a macroscopic physical entity whose measurements give rise to a description in a three-dimensional (real) Hilbert space, a situation where Gleason's theorem is already fully applicable. However, although certainly sufficient to make the point of the *non sequitur* of the no-go theorems in a simple and explicit example, the model was admittedly not particularly elegant, and a bit *ad hoc*, and this may have prevented a full recognition of its consequences, as to the status of the hidden-measurement interpretation.

In the same period, Coecke also proposed a more general approach, showing that a complete representation of the measurement process, and not just of its "naked" part, was possible also for a general N-dimensional quantum entity [11]. This was undoubtedly an important progress, as for the first time it was possible to affirmatively answer the question about the existence of a generalization of the two-dimensional sphere-model to an arbitrary number of dimensions. However, although Coecke could successfully show that an Euclidean real representation of the complex states of a quantum entity was possible, and that in such representation the hidden-measurements could also be incorporated, the number of dimensions he used to do this was not optimal. Indeed, he represented a N-dimensional complex Hilbert space in a N^2-dimensional real Euclidean space, and for the $N = 2$ case this gave an Euclidean representation in $\mathbb{R}^4$, whereas the Bloch sphere lives in $\mathbb{R}^3$. So, strictly speaking, Coecke's model was not the natural generalization of the sphere-model, but a different model whose mathematics was less immediate and the physics less transparent.

To complete this short overview, a more recent work of Sven Aerts [21] should also be mentioned, in which the author successfully formalized the hidden-measurement approach within the general ambit of an *interactive probability model*, showing how to characterize, in a complex Hilbert space, the hidden-measurement scheme, deriving the Born rule from a *principle of consistent interaction*, used to partition the apparatus' states.

Now, for those physicists who from the beginning evaluated in a positive way the explicative power contained in the hidden-measurement interpretation, all the mentioned results incontrovertibly showed that there was a way to go to find more advanced models. But we can also observe that the approach remained difficult to evaluate by those who were less involved in these developments, mainly for the lack of a natural higher-dimensional generalization of the $N = 2$ sphere-model representation, and the fact that it was known that the two-dimensional situation was, in a sense, a "degenerate" one, as it excluded the possibility of sub-measurements, and Gleason's theorem did not apply.

This situation started to change recently. Indeed, in the ambit of so-called *quantum models of cognition and decision* (an emerging transdisciplinary field of research where quantum mechanics is intensively used and investigated [28], [29]) we could provide a very general mechanistic-like modelization of the "naked part" of a measurement process, including the possibility of describing degenerate observables, which is something that was not done in the past [24], [25]. In that context, we also succeeded to show that the uniform average over the measurement interactions, from which the Born rule was derived, could be replaced by a much ampler averaging process, describing a much more general condition of lack of knowledge in a measurement, in what was called a *universal measurements*. In other terms, what we could prove is that quantum measurements are interpretable as universal measurements having a Hilbertian structure, which in part could explain the great success of the quantum statistics in the description of a large class of phenomena (like for instance those associated with human cognition [29], [28], [24], [25]).

Once we completed this more detailed analysis of the "potentiality region" of a measurement process (which, as a side benefit, allowed us to propose a solution to the longstanding Bertrand's paradox [30]), we became aware of the existence of some very interesting mathematical results, exploiting the generators of $SU(N)$ (the special unitary group of degree N) to generalize the Bloch representation of the states of a quantum entity to an arbitrary number N of dimensions [31], [32], [33], [34], [35], [36]. This was precisely the missing piece of the puzzle that we needed in order to complete the modelization of a quantum measurement process, by also including the entire structure of the state space. Contrary to the model proposed by Coecke, this generalized Bloch representation was carried out in a $(N^2 - 1)$-dimensional real Euclidean space, that is, a space with an optimal number of dimensions, which reduces exactly to the standard Bloch sphere (or ball) when $N = 2$. In other terms, it is the natural generalization of the two-dimensional Bloch sphere representation.

Bringing together our recent results regarding the modelization of the "naked part" of a measurement process [24], [25], with the new mathematical results on the generalized Bloch representation [31], [32], [33], [34], [35], [36], we are in a position to present, in this article, what we think is the natural N-dimensional generalization of the sphere-model, providing a self-consistent and complete modelization of a general finite-dimensional quantum measurement, also incorporating the full Hilbertian structure of the set of states, and the description of how the quantum entity enters into contact with the "potentiality region" of the measuring system, and subsequently remerges from it, thus producing an outcome. To our opinion, the modelization has now reached a very clear physical and mathematical expression, describing what possibly happens—"behind the macroscopic scene"—during a quantum measurement process, thus offering a challenging solution to the central (measurement) problem of quantum theory.

Before describing how the article is organized, a last remark is in order. The hidden-measurement interpretation can certainly be understood as a hidden-variable theory. However, it should not be understood as a tentative to resurrect classical physics. Quantum mechanics is here to stay, and cannot be replaced by classical mechanics. However, we also think that there are aspects of the theory which can, and need to, be demystified, and that only when this is done the truly deeper

aspects of what the theory reveals to us, about our physical reality, can be fully appreciated. When hidden-measurements are used to explain how probabilities enter quantum mechanics, the measurement problem can be solved in a convincing way, and an explanation is given for that part of quantum physics. This, however, requires us to accept that quantum observations cannot be understood only as processes of pure discovery, and that the non-locality of elementary quantum entities is in fact a manifestation of a more general condition of *non-spatiality* . . . ."

. . . *Aerts/de Bianchi:* [AdB14], §1, Introduction.

The *evidence-based* perspective of this investigation suggests that the continuously varying parameter 'ϵ' in the above description could be taken to reflect our 'ignorance' of the state—of the putative 'entity' sought to be observed in a measurement—that would yield its finally measured value, where:

- 'lack of knowledge is not to be understood in a subjective sense, as it results from an *objective* condition of *lack of control* regarding the way a potential interaction is actualized during a measurement, as a consequence of the irreducible fluctuations inherent to the experimental context, and of the fact that the operational definition of the measured physical quantity does not allow the experimental protocol to be altered, in order to reduce them';

- 'the ϵ refers to a parameter in the model that can be continuously varied, describing the transition between quantum and classical measurements, passing through measurement situations which are neither quantum nor classical, but truly intermediary';

- 'the quantum entity enters into contact with the "potentiality region" of the measuring system, and subsequently remerges from it, thus producing an outcome';

- 'hidden-measurements are used to explain how probabilities enter quantum mechanics, the measurement problem can be solved in a convincing way, and an explanation is given for that part of quantum physics';

and where ϵ could, therefore, be interpreted as a function that is algorithmically *verifiable* (hence deterministic), but not algorithmically *computable* (hence unpredictable):

- whence its value at any instant before measurement could not be represented mathematically by the experimenter, even in principle, in terms of an algorithmically *computable* real number (see §4.H.a.);

- since such representability could entail knowledge that would attract the paradoxes of impredicativity (as highlighted in §19.B.b.);

- and, moreover, since the limiting state represented by such a function could correspond (compare §16.C.b.(c)) to a mathematical discontinuity at the measurement, where the value/state of the function 'collapses' (as determined by the corresponding ψ-function) into a state that is mathematically representable in terms of algorithmically *computable* real numbers.

Prima facie, the above interpretation could be viewed as implicit in [AdB14]:

"It is worth emphasizing that the above description of a measurement, although formulated in Hilbert space and not explicitly mentioning the hidden-interactions, is fully compatible with the logic of the hidden-measurement interpretation. Indeed, a measurement context, associated with a given observable, can be understood as a collection of potential interactions, which once selected (actualized) can bring a given initial state into a predetermined final state, corresponding to the outcome of the measurement. In other terms, the hidden-interactions are those elements of reality producing the quantum transition, so that, in a sense, we can say that the standard Hilbert space formulation of quantum mechanics already contains, in embryo, the hidden-measurement modelization."

. . . Aerts/de Bianchi: [AdB14], §2, Operator-states and Lüders-von Neumann formula.

Informally, the underlying perspective here would be that:

(i) If U is the universe we inhabit:

- and T is, say, the Taj Mahal,
- whilst D is a set of categorical engineering specifications,
- for building a physical model M of T in which we can perform repeated destructive experiments;

(ii) Then, as observers that exist in U:

- whilst we can talk categorically about the properties of T representable in M;
- we can only conjecture/postulate;
    - on the basis of statistical correlations,
    - between what we observe in U,
    - and what we measure in M;
- to what extent M is a representative model;
    - not of T;
    - but of *that* part of T;
- which we can observe in U;
- and well-define in D;
- without destroying T.

Thus:

(a) If there are *hidden* features H of T that are not observable in U, but which essentially influence any measurements that we make concerning T in U, then such features, by definition, are not represented/representable in D or M.

(b) Since it is axiomatic that the only 'true' model of T is T itself then, as Aerts has—albeit implicitly—argued above, repeated 'destructive' measurements in M are essentially separated spatially, and cannot, even in principle, yield deterministic definitions of all such properties H that can be built into a more complete model M' of T, even though T itself might exist in a deterministic universe.

(c) We conclude that even if such a universe can be said to obey deterministic physical laws which can be postulated as independent of us, since we can reflect them mathematically in D by algorithmically *computable* (hence deterministic and predictable) functions and relations, the property of being *predictable* is only well-defined with respect to M, and makes no sense in U in cases where, as Aerts has argued, it is not falsifiable in M since it involves spatial separation in destructive experiments upon M.

(d) Moreover, the most that we can claim from our observations is that there might be deterministic physical laws that are not representable in D by algorithmically *computable* (hence stochastically *deterministic* and *predictable* in repeated destructive experiments on M) functions and relations but, as shown in [An16] (Theorem 2.1, p.37), such laws might be representable in D by functions and relations that are algorithmically *verifiable* (hence stochastically *deterministic* by repeated destructive experiments on M), but (see also Bohr's argument in §19.B.) *not* algorithmically *computable* (hence stochastically *unpredictable* by repeated destructive experiments on M) as would be required by Bell's inequalities.

19.B.d.   An *evidence-based* 'ontological' perspective of Aerts' 'overlooked solution' to EPR

We recall that this investigation essentially seeks to consider both formal, and informal, consequences of introducing the concepts of 'evidence-based' (in the sense of, for instance, Chetan Murthy and Martin Löb as detailed in §2.) and 'well-definedness' (in the sense of §4.F., Definition 19) as necessary, and sufficient, standards of rigour for finitary mathematical reasoning in mathematics, philosophy, and the natural sciences (whose respective domains we seek to differentiate in §10. and §10.C.).

From such a perspective, an issue of immediate significance that could benefit from clarification is that of ontological commitment in Aerts' 'overlooked solution' to EPR.

1. For instance, we would view any ontological commitments—in this context—as concerning the *putative* existence of what we informally refer to as 'entities'/'properties'/'laws' in the *actual* universe we inhabit, vis a vis the *well-definability* of corresponding 'entities'/'properties'/'laws' in a *well-defined* mathematical model of that which we seek to observe and/or measure in the *actual* universe we inhabit.

2. Thus, the most important aspect of Aerts' argumentation would be the insight that, since no experiment or observation in the *actual* universe we inhabit can be exactly replicated physically, all measurements of physical phenomena that we observe and/or record can only be of a statistical nature; albeit with some acceptable variance—from any theoretical prediction—that would allow two separate/distinct instances of the physical phenomena—sought to be studied by *actual* experiment and/or observation—to be treated as theoretically identical.

Such would be the case, both, whether premised:

(i) on the philosophical assumption in classical mechanics that the *actual* universe we inhabit can be consistently treated as merely an/one instantiation, of an *intuitively potential* universe, which obeys *putative* laws—in the *intuitively potential* universe—that can always be *intuitively* treated as both deterministic and predictable;

(ii) or on the negation of this assumption in quantum mechanics.

3. From Aerts' argumentation, we conclude that such an *intuitively potential* universe—as envisaged in 2(i)—is essentially problematic and/or impossible.

In other words, we need to be very explicit—at each step in any argumentation—whether we are referring:

(i) to observable and/or measurable properties of *putative* entities (observables) in the *actual* universe we inhabit (on the basis of our *actual* observations/measurements);

> **Comment**: We note that, in [Aetal], Aerts et al propose a *conceptuality interpretation* that seeks to 'explain the strange behavior of quantum and relativistic entities' by suggesting a cognitive description of such *putative* entities as 'concepts' that are not grounded in any *well-definable* mathematical representation of some corresponding, pre-existing, reality; but which can, nevertheless, be assumed to correspond to what is physically 'sensed' by any sensory 'apparatus'—whether of an organic or inorganic nature—at the moment of a physical observation/measurement.

or:

(ii) to *well-definable* entities/properties/laws in *well-definable potential* universes in a *well-defined* mathematical model which, under interpretation, would be consistent with what we *actually* can, or do, observe and/or measure in our *actual* universe.

4. Prima facie, no matter which philosophical assumption, 2(i) or 2(ii), we prefer as our premise, any *well-defined* mathematical representation/model of the conceptual metaphors that we can correspond faithfully (in some sense of 'faithfully') to physical phenomena—which we observe and record in our *actual* universe—ideally demands that:

(i) The *well-defined* mathematical representation/model must admit gedanken which can be *well-defined* as 'exactly replicated' in the representation/model;

(ii) The *well-defined* mathematical representation of *putative* classical laws—which can reliably be assumed to be governing the *actual* universe we inhabit at, say, the 'macro' scale, and which are suggested by what we observe and/or measure in the *actual* universe we inhabit—should be both *well-definable* as *deterministic* and *predictable* within the representation/model.

Moreover, they must yield results in a classical gedanken (within the representation/model) that are consistent with both Bell's inequalities (within the representation/model), and the *actual* statistical results of any corresponding physical experiments (in our *actual* universe).

(iii) The *well-defined* mathematical representation of *putative* quantum laws—which can reliably be assumed to be governing the actual universe we inhabit at, say, the 'micro' scale, and which are suggested by what we observe and/or measure in the *actual* universe we inhabit—should be *well-definable* as *essentially unpredictable* within the representation/model.

Moreover, these too must yield results in a quantum gedanken (within the representation/model) that are consistent with both Bell's inequalities (within the representation/model), and the *actual* statistical results of the corresponding physical experiments (in our *actual* universe).

(iv) The *well-defined* mathematical representation/model must establish that the assumption that classical laws govern physical phenomena at the 'macro' scale in our *actual* universe *complements*, and does not *contradict*, the assumption that quantum laws govern physical phenomena at the 'micro' scale in our *actual* universe.

5. Now, post-Turing, our observations of physical phenomena in our *actual* universe has established that:

   (i) algorithmically *computable* number theoretic functions, which are *finitarily* both *determinate* and *predictable*, *can* be taken to *well-define* and mathematically represent the *putative* laws of classical mechanics (which we postulate as obeyed in the *actual* universe we inhabit at both the 'macro' and 'micro' scale) faithfully in the sense of 4(ii);

   (ii) algorithmically *computable* number theoretic functions, which are both *determinate* and *predictable*, *cannot* be taken to *well-define* and mathematically represent the, similarly *putative*, laws of quantum mechanics faithfully in the sense of 4(iii).

6. The challenge, then, has been to determine whether we can:

   (i) *well-define* number theoretic functions that are further *well-definable* as *essentially unpredictable* (in a *well-defined* mathematical representation/model), yet which can be taken to mathematically represent the *putative* laws of quantum mechanics faithfully in the sense of 4(iii); and

   (ii) *well-define* a mathematical representation/model such that it can *well-define* 'exact replicability' of theoretical experiments in the representation/model.

7. Challenge 6(i) is met by Schroedinger's ψ function; the standard Copenhagen interpretation of which, following Bohr, we take to postulate that:

   (i) the *well-defined* ψ-function yields a probability distribution for the values—in any *well-defined* representation/model—which can be taken to correspond to the physical observations and/or measurement of *actual* experiments that a quantum gedanken is intended to represent ideally in the representation/model;

   (ii) the *well-defined* probability distribution for the outcome of a repeated—but not necessarily 'exactly replicable'—quantum gedanken in any *well-definable* mathematical model is *all* that is theoretically 'knowable' about that which we seek to observe and/or measure in the gedanken (in the *well-defined* mathematical representation/model).

8. Treating postulation 7(ii) as necessary, however, requires us to commit—on philosophical grounds—to the thesis (which admits uncomfortable 'non-locality' as essential) that:

   (i) any *well-defined* mathematical representations of the *putative* quantum laws determining some of our *actual* observations and/or measurements are only *well-definable* as *essentially probabilistic*, and therefore *essentially non-deterministic*, in the representation/model; whence

(ii) admitting 6(ii) in the representation/model would contradict the applicable Bell's inequalities.

9. However, based on mathematical and logical considerations of a foundational nature in [An16] (see §2.), such a postulation, though sufficient to its immediate intent, is not necessary; since it follows from [An16] that:

   (i) The first-order Peano Arithmetic PA admits *well-defined* formal arithmetical functions that are algorithmically *computable* under interpretation over the domain of the natural numbers.

      Comment: By virtue of the Provability Theorem for PA (§2.E., Theorem 2.17), we shall treat as extrapolatable (albeit, without formal proof) that such functions are capable of yielding *well-defined* real functions that are, further, *well-definable* as, both, *determinate* and *predictable*, in any *well-defined* mathematical theory of the real numbers based on PA.

   (ii) The first-order Peano Arithmetic PA also admits—hitherto unsuspected—*well-defined* formal arithmetical functions that are algorithmically *verifiable*, but not algorithmically *computable*, under interpretation over the domain of the natural numbers.

      Comment: By virtue of the Provability Theorem for PA (§2.E., Theorem 2.17), we shall treat as extrapolatable (albeit, without formal proof) that such functions are capable of yielding *well-defined* real functions that are, further, *well-definable* as *determinate*, but *essentially unpredictable*, in any *well-defined* mathematical theory of the real numbers based on PA—thereby admitting both 6(i) and 6(ii).

   (iii) Contradicting current paradigms, the first-order Peano Arithmetic PA is categorical.

      Comment: It thus both uniquely defines, and is uniquely defined by, the structure of the natural numbers (which—as argued in §4.H. and §4.H.a.—yields *well-defined* real numbers that can be treated as corresponding to what we *actually* observe and/or measure physically in the universe we inhabit).

10. Since PA serves as a foundation for the mathematical representation of *well-defined* real numbers (see [La29]), we take it as admitting the argument (in §19.B.b.) that the debate between Albert Einstein and Niels Bohr dissolves from the perspective where we treat:

   (i) Einstein as justifiably asserting that all natural laws must be capable of being treated philosophically as *deterministic* in the *actual* universe we inhabit—even if they do not yield *predictable* results in physical experiments—since they *can* be mathematically represented by *well-defined* algorithmically *verifiable* functions.

      Hence we *can* consistently ascribe *putative* spatial existence to postulated *potential* entities, in our *actual* universe, that correspond to *mathematical* entities (i.e., 'terms' of a formal theory) which are definable by algorithmically *verifiable* functions in any mathematical representation/model of a *potential* universe.

   (ii) Bohr as justifiably asserting that some natural laws in the *actual* universe we inhabit yield *essentially unpredictable* results in physical experiments; since they *cannot* be mathematically represented by *well-defined* algorithmically *computable* functions.

      Hence we *cannot* consistently ascribe *putative* spatial existence to postulated *potential* entities, in our *actual* universe, that correspond to *mathematical* entities (i.e., 'terms'

of a formal theory) which are definable by algorithmically *computable* functions in any mathematical representation/model of a *potential* universe.

11. Moreover, since all assertions about the representation of physical phenomena in a *well-defined* mathematical model must be capable of *evidence-based* verification (within the model) that can be corresponded faithfully to statistical verification in what the mathematical assertions represent in our *actual* universe, we cannot admit elements in the model that are not *well-defined* and which cannot, consequently, be taken to correspond to some *putative*, and not merely *intuitively potential*, 'observable' entity whose existence we can consistently postulate in our *actual* universe.

> **Comment**: In other words—unlike any *putative* model in the language of set theory, which can have no *well-defined* model if it contains an axiom of infinity—a *well-defined* arithmetical—hence computational by virtue of the Provability Theorem for PA (§2.E., Theorem 2.17)—model of our *actual* universe does not admit reference to (i.e, cannot commit us ontologically to) elements in the model that are not *well-definable* in the language of arithmetic and/or computability (as argued in §2.E.a., §4.H., and §4.H.a.).
>
> The ontological significance of admitting an axiom of infinity in formal languages that seek to represent our conceptual metaphors—corresponding to our observations/measurements of physical phenomena in the universe we inhabit—is highlighted by Feferman in [Fe02] on p.6 (Poincaré vs. the logicists and the Cantorians: from paradoxes to practice), on p.9 (Weyl's predicative development of analysis), on p.11 (Predicativity sidelined: 1920-1950), and on p.23 (Rethinking predicativity II: 1970-1996).

19.C.   How *evidence-based* reasoning dissolves the EPR 'paradox'

The underlying perspective of §19. is thus that:

(1) Classical physics assumes that all the observable laws of nature can be mathematically represented in terms of *well-defined* functions that are algorithmically *computable* (see §2., Definition 3).

   (a) Since the functions are well-defined, their values are pre-existing and predetermined as mappings that are capable of being known in their infinite totalities to an omniscient intelligence such as, for instance, Laplace's *vast intelligence*.

(2) However, the overwhelming experimental verification of the mathematical predictions of Quantum Theory suggests that the actual behavior of the real world cannot be assumed as pre-existing and predetermined in this sense.

   (a) In other words, the consequences of some experimental interactions are theoretically incapable of being completely known in advance even to an omniscient intelligence, such as Laplace's *vast intelligence*.

   (b) All observable laws of nature cannot, thus, be represented mathematically in terms of functions that are algorithmically *computable* (Definition 3).

(3) It follows that:

   (a) Either there is no way of representing all the observable laws of nature mathematically in a deterministic model;

    (b) Or all the observable laws of nature *can* be represented mathematically in a deterministic model—but in terms of functions that are 'computable' in the *weak* sense of being algorithmically *verifiable* (Definition 2) (hence well-defined by Definition 18) but not necessarily 'computable' in the classically *strong* sense of being algorithmically *computable* (see §2., Definition 3).

(4) The Copenhagen interpretation appears to favour option (3)(a); and hold that there is no way of representing all the observable laws of nature mathematically in a deterministic model.

    (a) In other words, the interpretation is not overly concerned with the seemingly essential non-locality of Quantum Theory, and its conflict with the deterministic mathematical representation of the laws of Special Relativity.

(5) The Bohm-de Broglie interpretation appears to reject option (3)(a); and to propose a way of representing all the observable laws of nature mathematically in a deterministic model and, presumably, in terms of functions that the interpretation—reflecting the paradigms of the times—implicitly presumes to be necessarily algorithmically *computable*.

    (a) However, the Bohm-de Broglie interpretation has not so far been viewed as being capable of mathematically avoiding the seemingly essential non-local feature of Quantum Theory implied by Bell's inequalities.

(6) In this paper we propose (3)(b); that the apparently non-local feature of Quantum Theory may actually be indicative of a non-constructive and 'counter intuitive-to-human-intelligence' phenomena in nature that could, however, be mathematically represented by functions that:

    (a) Are algorithmically *verifiable* (see §2., Definition 2);

    (b) But not algorithmically *computable* (see §2., Definition 3).

19.C.a.   *EPR* **paradox: An implicit mathematical ambiguity in interpreting quantification non-constructively**

We shall thus argue that if we avoid conflating a mathematical model M_L of a mathematical language L that seeks to admit representations of physical phenomena *before* a measurement with a, post-facto, mathematical model M_{L_1} of a conservative extension L_1 of L that seeks to represent the phenomena which *predicts* the measurement and its consequences (see §19.B.), then the EPR paradox is essentially a mathematical argument whose paradoxical conclusion reflects an implicit, foundational, mathematical ambiguity in interpreting quantification (see §4.C.; also [An15], [An15p]), and whose roots lie in the assumption of conventional Gödelian wisdom, which postulates that:

  (i) The 'true' sentences of the model M_L cannot be defined algorithmically,

  (ii) But are an essential feature of the structure $M_L = < A, \alpha >$,

  (iii) Which is defined by a non-empty domain A, and an algebra α defined over A.

However, we hold that such a non-constructive perspective implicitly implies (see §2.A.) that the concept of 'truth' must then be both 'absolute' and 'exist' Platonically, in the sense of needing to be discovered by some witness-dependent means—eerily akin to a spiritual, and necessarily subjective, 'revelation'—if the domain A is infinite.

19.C.b.   Truth-values must be a computational convention

In other words, we reject such a Platonic faith in an 'absolute' mathematical 'truth' that is accessible only subjectively, and propose instead the *evidence-based* perspective that (see §1., Thesis 1; also §4.C. and §8.C.):

(i) The 'true' sentences of M_L must be defined as objective assignments,

(ii) By a computational convention that is witness-independent,

(iii) In terms of the Tarskian 'satisfaction' and 'truth' of the corresponding formulas of L under a constructive interpretation over the domain A.

19.C.c.   A finitary perspective of the structure $\mathbb{N}$ of the natural numbers

The perspective we choose for addressing these issues is that of the structure $\mathbb{N}$, defined by:

- $\{N$ (*the set of natural numbers*);

- $=$ (*equality*);

- S (*the successor function*);

- $+$ (*the addition function*);

- $*$ (*the product function*);

- 0 (*the null element*)$\}$

which serves for a definition of the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of the first-order Peano Arithmetic PA.

Our reason for choosing PA as the basis for our perspective is that PA is a categorical mathematical language of both adequate expression (see §2.E., Corollary 2.18) and effective communication (see §2.E., Theorem 2.17)[190].

Moreover, PA forms the foundational bedrock (see, for instance, [La29]) on which all formal mathematical languages that admit rational and real numbers and, ultimately, all of geometry, are grounded; as noted by Richard Zach in his review of '*Hilbert's Program Then and Now*':

"2 HILBERT'S PROGRAM THEN

2.1 Hilberts early work on foundations

Hilbert's work on the foundations of mathematics can be traced to his work on geometry of the 1890s which resulted in his influential textbook *Foundations of Geometry* [1899]. One philosophical advance of this work was the development of Hilbert's conception of the axiomatic method. Hilbert believed that the proper way to develop any scientific subject rigorously required an axiomatic

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[190]In contrast to a first-order set theory such as ZF, which can only be termed as a language of adequate expression, but not a language of effective communication (see §10.E.).

approach. In providing an axiomatic treatment, the theory would be developed independently of any need for intuition, and it would facilitate an analysis of the logical relationships between the basic concepts and the axioms. Of basic importance for an axiomatic treatment are, so Hilbert [*believed?*], investigation of the independence and, above all, of the consistency of the axioms. In his 1902 lectures on the foundations of geometry, he puts it thus:

> Every science takes its starting point from a sufficiently coherent body of facts is given. It takes form, however, only by *organizing* this body of facts. This organization takes place through the *axiomatic method*, i.e., one constructs a *logical structure of concepts* so that the relationships between the concepts correspond to relationships between the facts to be organized.

> There is arbitrariness in the construction of such a structure of concepts; we, however, *demand* of it:

> 1) completeness, 2) independence, 3) consistency. [Hilbert, 2004, 540]

From the time of his work on geometry forward, the last consideration, consistency, was of special importance in Hilbert's conception of the axiomatic method in general and the foundations of mathematics in particular. Hilbert was heavily influenced by the foundational views of late-19th century mathematicians, in particular, Cantor, Dedekind, and Kronecker. He shared with Dedekind and Cantor the view that mathematical activity should be free of constraints, which led to his view, highlighted in his correspondence with Frege, that consistency of an axiomatic theory guarantees the existence of the structure described, and is in this sense sufficient to justify the use of the theory. And he shared with Kronecker a recognition that elementary arithmetic has a privileged role in mathematics, although he was of course opposed to the converse espoused by Kronecker, viz., that the natural numbers, and constructions based on elementary arithmetic, exhaust legitimate mathematics. These two influences in Hilbert's thought are at the root of his investigations of consistency.

Proofs of consistency for the axioms of geometry can be given by providing an interpretation of the system in the real plane, and thus the consistency of geometry is reduced to the consistency of analysis. Analysis, of course, itself requires justification. In [1900b], Hilbert approached the problem from the axiomatic standpoint by proposing an axiomatization of the real numbers. In order to show the consistency of this system, Hilbert expressly rejected the construction of a model, e.g., a construction based on Dedekind cuts of rationals, as an option. He considered the construction of the reals from the rationals and ultimately the natural numbers using the "genetic method" as insufficient: "Despite the high pedagogical and heuristic value of the genetic method, for the final presentation and the complete logical grounding of our knowledge the axiomatic method deserves the first rank" [Hilbert, 1900b, 1093]. Hilbert thus was after a direct consistency proof of analysis, i.e., one not based on reduction to another theory. He proposed the problem of finding such a proof as the second of his 23 mathematical problems in his address to the International Congress of Mathematicians in 1900 [1900a]."

...Zach: [Zac07], pp.412-413.

In other words the *strong*, finitary, consistency of PA (see §2.C.a., Theorem 2.16) can provide the sound foundation needed by any computational language in which mechanical artefacts record their observations—of a putative 'common' external world—that reflect and extend what is directly experienced, or conjectured as indirectly observable, by our sensory perceptions, and in which applied science attempts to mathematically model the putative laws of nature that our conceptual metaphors (see [LR00]) of such observations suggest.

19.C.d. Functions that are algorithmically *verifiable* but not algorithmically *computable*

From the point of view of a finitary mathematical philosophy (see §1., Thesis 1)—which is the constraint within which an applied science ought to ideally operate—the significant difference

between algorithmic *verifiability* and algorithmic *computability* could be expressed by saying that:

(a) We may treat the decimal representation of a real number as corresponding to a physically measurable limit[191]—and not only to a mathematically definable limit—if and only if such representation is definable by an algorithmically *computable* function (see §4.H.a.).

We note that although every algorithmically *computable* relation is algorithmically *verifiable*, the converse is not true (see §4.G., Theorem 4.2).

19.C.e. Some well-known functions that are algorithmically *verifiable* but not algorithmically *computable*

We further note that:

(i) All the mathematically defined functions known to, and used by, science are algorithmically *computable*, including those that define transcendental numbers such as π, e, etc. They can be *computed* algorithmically as they are all definable as the limit of some well-defined infinite series of rationals.

(ii) The existence of mathematical constants that are defined by functions which are algorithmically *verifiable* but not algorithmically *computable*—suggested most famously by Georg Cantor's diagonal argument—has been a philosophically debatable deduction.

Such existential deductions have been viewed with both suspicion and scepticism by scientists such as Henri Poincaré, L. E. J. Brouwer, etc., and disputed most vociferously on philosophical grounds by others such as Ludwig Wittgenstein ([Wi78]).

(iii) A constructive definition of an arithmetical Boolean function $[(\forall x)R(x)]$[192] that is true under any well-defined interpretation of his Peano Arithmetic P—hence algorithmically *verifiable*—but not provable in P—hence algorithmically not *computable* (see §2.F., Corollary 2.21)—was given by Kurt Gödel in his 1931 paper on formally undecidable arithmetical propositions ([Go31]).

(iv) The definition of a number-theoretic function that is algorithmically *verifiable* but not algorithmically *computable* was also given by Alan Turing in his 1936 paper on computable numbers ([Tu36]).

He defined a halting function, say $H(n)$, that is 0 if, and only if, the Turing machine with code number n halts on input n. Such a function is mathematically well-defined, but assuming that it defines an algorithmically *computable* real number leads to a contradiction, Turing concluded the mathematical existence of algorithmically *uncomputable* real numbers.

(v) A definition of a number-theoretic function that is algorithmically *verifiable* but not algorithmically *computable* was given by Gregory Chaitin ([Ct82]); he defined a class of constants—denoted by Ω—which is such that if $C(n)$ is the nth digit in the decimal expression of an Ω constant, then the function $C(x)$ is algorithmically *verifiable* but not algorithmically *computable*.

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[191]In the sense of a physically 'completable' infinite sequence (as needed to resolve Zeno's paradox).

[192]Gödel refers to this formula only by its Gödel-number '17*Gen r*' ([Go31], p.25, eqn.13).

19.D.   Are some physical constants algorithmically *uncomputable*?

The question arises: Are some physical constants representable by real numbers which are definable only by algorithmically *verifiable* but not algorithmically *computable* functions?

The possibility is suggested by the following perspective of one of the challenging issues in physics, which seeks to theoretically determine the magnitude of some fundamental dimensionless constants:

> "...the numerical values of dimensionless physical constants are independent of the units used. These constants cannot be eliminated by any choice of a system of units. Such constants include:
>
> - α, the fine structure constant, the coupling constant for the electromagnetic interaction ($\approx 1/137.036$). Also the square of the electron charge, expressed in Planck units. This defines the scale of charge of elementary particles with charge.
> - μ or β, the proton-to-electron mass ratio, the rest mass of the proton divided by that of the electron (≈ 1836.15). More generally, the rest masses of all elementary particles relative to that of the electron.
> - α_s, the coupling constant for the strong force (≈ 1)
> - αG, the gravitational coupling constant ($\approx 10^{-38}$) which is the square of the electron mass, expressed in Planck units. This defines the scale of the mass of elementary particles.
>
> At the present time, the values of the dimensionless physical constants cannot be calculated; they are determined only by physical measurement. This is one of the unsolved problems of physics. ...
>
> The list of fundamental dimensionless constants decreases when advances in physics show how some previously known constant can be computed in terms of others. A long-sought goal of theoretical physics is to find first principles from which all of the fundamental dimensionless constants can be calculated and compared to the measured values. A successful 'Theory of Everything' would allow such a calculation, but so far, this goal has remained elusive."
>
> *...Dimensionless physical constant - Wikipedia*

From the perspective of Section 19.C.c., we could now suggest that:

Thesis 13. (Dimensionless constants) *Some of the dimensionless physical constants are only representable in a mathematical language as 'unmeasurable' real numbers that are defined by quantum functions which are algorithmically verifiable, but not algorithmically computable.*

In other words, we cannot treat such constants as denoting—even in principle—a measurable limit, as we could a constant that is representable mathematically by a real number that is definable by algorithmically *computable* functions.

One reason why dimensionless physical constants may not be representable in a mathematical language L as 'measurable' real numbers might be that they are determined by the mathematical model M_L of the physical phenomena sought to be represented by the model.

If so, the value of a dimensionless constant in a mathematical model M_L of physical phenomena *before* a measurement must differ with the value of the dimensionless constant in a post-facto mathematical model M_{L_1} of the phenomena which *predicts* the measurement and its consequences.

In other words, if the values of some dimensionless constants are defined by 'quantum' functions which are algorithmically *verifiable*, but not algorithmically *computable*, then they are of an *evolving* nature (compare with Gisin's perspective in [Gi19] and [Gi20]); they *unfold* only as, and when, a measurement in a model M_L—of a mathematical theory L that seeks to faithfully represent observations of some physical phenomena—is integrated into a model M_{L_1} of a conservative extension L_1 of the theory L (in the sense of §19.B.a.).

19.D.a. Completed Infinities

From the point of view of mathematical philosophy, this distinction would be expressed by the theses (the significance of this is highlighted in §16.C.; see also §4.H.a., Theorem 4.5):

Thesis 14. (Unmeasurable constants) *Whilst a symbol for an 'unmeasurable' physical constant may be introduced into a physical theory as a primitive term without inviting inconsistency in the theory, the sequence of digits in the decimal representation of the 'measure' of an 'unmeasurable' physical constant cannot be treated in the mathematical language of the theory as a 'completed' infinite sequence whose 'measure' is the Cauchy limit of the sequence.*

Thesis 15. (Measurable constants) *The sequence corresponding to the decimal representation of the 'measure' of a 'measurable' physical constant, when introduced as a primitive term into a physical theory, can be treated as a 'completed' infinite sequence, whose 'measure' is the Cauchy limit of the sequence in the mathematical language of the theory, without inviting inconsistency.*

19.D.b. Zeno's arguments

We note that Zeno's paradoxical arguments ([Rus37], pp.347-353; as qualified, however, by §16.C.b.) highlight the philosophical and theological dichotomy between our essentially 'continuous' perception of the physical reality that we seek to capture with our measurements, and the essential 'discreteness' of any mathematical language of arithmetic in which we seek to express such measurements categorically.

The *evidence-based* distinction between algorithmically *verifiable* and algorithmically *computable* arithmetical functions could be seen as reflecting the dichotomy mathematically.

19.D.c. Classical laws of nature

The *evidence-based* distinction between algorithmic *verifiability* (see §2., Definition 2) and algorithmic *computability* (see §2., Definition 3) suggests that classical mechanics *could* be held as complete with respect to the algorithmically *computable* representation of our observations of physical phenomena, in the sense that:

Thesis 16. (Classical laws) *Classical laws of nature determine the nature and behaviour of all those properties of the physical world which are mathematically describable completely at any moment of time $t(n)$ by algorithmically computable functions from a given initial state at time $t(0)$.*

In other words, classical laws are characterised by the property that if a physical process is representable by a Cauchy sequence, then the limit of the sequence corresponds to a limiting state of the physical process.

19.D.d. Neo-classical laws of nature

Moreover, the distinction further suggests that:

Thesis 17. (Neo-classical laws) *Neo-classical laws of nature determine the nature and behaviour of those properties of the physical world which are describable completely at any moment of time $t(n)$ by algorithmically verifiable functions; however such properties are not completely describable by algorithmically computable functions from any given initial state at time $t(0)$.*

In other words, neo-classical laws are characterised by the property that if a physical process is representable by a Cauchy sequence, then the limit of the sequence need not correspond to a limiting state of the physical process; which may require an additional, conceivably probabilistic, law to *deterministically* govern the permissible states of the physical process at the limit.

Since such behaviour follows fixed laws and is thus determinate (even if not algorithmically predictable by classical laws, since their limiting states are revealed only as probabilities), the hypothetical universe considered in §16.D.c. suggests that Albert Einstein could have been justified in his belief (in the sense of §10.F.), reiterated in 1943 to William Hermanns:

> "As I have said so many times, God doesn't play dice with the world".
> *. . . Hermanns: [Her83], p.58.*

> **Comment**: 'Justified' from the *evidence-based* perspective of this investigation if, by 'play dice', Einstein's belief/remark could be interpreted:
>
> - as intending that the probabilities considered in current quantum-mechanical descriptions of physical processes *cannot* be taken to be defined over a probability space *globally* (compare §18.A., Theorem 18.3);
> - but allowing the possibility that such probabilities may be definable in terms of *locally* definable probability spaces—as argued in the case of defining the probability of an integer n being a prime (see §18.A., and §18.A.d.).
>
> The consequences of such a qualification for the physical sciences could, conceivably, be as far-reaching as those that are argued for computability theory (see §18.A.e. and §18.A.f.) and the theory of numbers (see §18.C.b.; also §18.D. and §18.D.a.).

A putative model for such behaviour is also considered by Frank Waaldjik in [Wl03]:

> "The second way to model our real world is to assume that it is deterministic. . . . It would be worthwhile to explore the consequences of a deterministic world with incomplete information (since under the assumption of determinancy in the author's eyes this comes closest to real life). That is a world in which each infinite sequence is given by an algorithm, which in most cases is completely unknown. We can model such a world by introducing two players, where player I picks algorithms and hands out the computed values of these algorithms to player II, one at a time. Sometimes player I discloses (partial) information about the algorithms themselves. Player II can of course construct her or his own algorithms, but still is confronted with recursive elements of player I about which she/he has incomplete information."
> *. . . Waaldjik: [Wl03], §1.5, p.5.*

where Waaldijk further emphasises the need for a universally common, constructive, foundation for the mathematical representation of elements of reality such as those considered in this investigation:

> "Our investigations lead us to consider the possibilities for 'reuniting the antipodes'. The antipodes being classical mathematics (CLASS) and intuitionism (INT). . . . It therefore seems worthwhile to explore the 'formal' common ground of classical and intuitionistic mathematics. If systematically developed, many intuitionistic results would be seen to hold classically as well, and thus offer a way to develop a strong constructive theory which is still consistent with the rest of classical mathematics. Such a constructive theory can form a conceptual framework for applied mathematics and information technology. These sciences now use an ad-hoc approach to reality since the classical framework is inadequate. . . . [*and can*] easily use the richness of ideas already present in classical mathematics, if classical mathematics were to be systematically developed along the common grounds before the unconstructive elements are brought in."
> *. . . Waaldjik: [Wl03], §1.6, p.5.*

"...we propose that Laplacian determinism be seen in the light of constructive mathematics and Church's Thesis. This means amongst other things that infinite sequences (of natural numbers; a real number is then given by such an infinite sequence) are never 'finished', instead we see them developing in the course of time. Now a very consequent, therefore elegant interpretation of Laplacian determinism runs as follows. Suppose that there is in the real world a developing-infinite sequence of natural numbers, say α. Then how to interpret the statement that this sequence is 'uniquely determined' by the state of the world at time zero? At time zero we can have at most finite information since, according to our constructive viewpoint, infinity is never attained. So this finite information about α supposedly enables us to 'uniquely determine' α in its course of time. It is now hard to see another interpretation of this last statement, than the one given by Church's Thesis, namely that this finite information must be a (Turing-)algorithm that we can use to compute $\alpha(n)$ for any $n \in (N)$.

With classical logic and omniscience, the previous can be stated thus: 'for every (potentially infinite) sequence of numbers $(a_n)_{n \in \mathbb{N}}$ taken from reality there is a recursive algorithm α such that $\alpha(n) = a_n$ for each $n \in \mathbb{N}$. This statement is sometimes denoted as '$\mathbf{CT}_{phys}$', ...this classical omniscient interpretation is easily seen to fail in real life. Therefore we adopt the constructive viewpoint. The statement 'the real world is deterministic' can then best be interpreted as: 'a (potentially infinite) sequence of numbers $(a_n)_{n \in \mathbb{N}}$ taken from reality cannot be apart from every recursive algorithm α (in symbols: $\neg \forall \alpha \in \sigma_{\omega REC} \exists n \in \mathbb{N} \, [\alpha(n) \neq a_n])$'."

...Waaldjik: [Wl03], §7.2, p.24.

19.D.e.   Incompleteness: Arithmetical analogy

The distinction also suggests that neither classical mechanics, nor neo-classical quantum mechanics, could be described as 'mathematically complete' with respect to the algorithmically *verifiable* behaviour of the physical world.

(a) The analogy here is that Gödel showed in 1931 ([Go31]) that any formal arithmetic is not mathematically complete with respect to the algorithmically *verifiable* properties of the natural numbers[193].

(b) However it can be shown that the first-order Peano Arithmetic PA *is* both complete and categorical ([An16], Corollary 7.2, p.41) with respect to the algorithmically *computable* properties of the natural numbers.

In this sense, the EPR paper may not be entirely wrong in concluding that:

"Returning now to the general case contemplated in Eqs. (7) and (8), we assume that ψ_k and ϕ_r are indeed eigenfunctions of some non-commuting operators P and Q, corresponding to the eigenvalues p_k and q_r respectively. Thus, by measuring either A or B we are in a position to predict with certainty, and without in any way disturbing the second system, either the value of the quantity P (that is p_k) or the value of the quantity Q (that is q_r). In accordance with our criterion of reality, in the first case we must consider the quantity P as being an element of reality, in the second case the quantity Q is an element of reality. But, as we have seen, both wave functions ψ_k and ϕ_r belong to the same reality.

Previously we proved that either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. Starting then with the assumption that the wave function does give a complete description of the physical reality, we arrived at the conclusion that two physical quantities, with non-commuting operators, can have

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[193]Which—as shown in [An12] and [An16]—are the ones sought to be captured by the standard interpretation of PA.

simultaneous reality. Thus the negation of (1) leads to the negation of the only other alternative (2). We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.

One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality *only when they can be simultaneously measured or predicted*. On this point of' view, since either one or the other, but not both simultaneously, of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this.

While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible."

. . . Einstein, Podolsky and Rosen: [EPR35]*, p.780*

19.D.f.   Conjugate properties

The above also suggests that:

Thesis 18. (Conjugate properties) *The nature and behaviour of two conjugate properties F_1 and F_2 of a particle P that are determined by neo-classical laws are described mathematically at any time $t(n)$ by two algorithmically verifiable, but not algorithmically computable, functions f_1 and f_2.*

(a) In other words, it is the very essence of the neo-classical laws determining the nature and behaviour of the particle that—at any time $t(n)$—we can only determine either $f_1(n)$ or $f_2(n)$, but not both.

(b) Hence measuring either one makes the other indeterminate as we cannot go back in time. This does not contradict the assumption that any property of an object must obey some deterministic natural law for any possible measurement that is made at any time.

19.D.g.   Entangled particles

The above similarly suggests that:

Thesis 19. (Entangled particles) *The nature and behaviour of an entangled property of two particles P and Q are determined by neo-classical laws, and are describable mathematically at any time $t(n)$ by two algorithmically verifiable—but not algorithmically computable—functions f_1 and g_1.*

(a) In other words, it is the very essence of the neo-classical laws determining the nature and behaviour of the entangled properties of two particles that—at any time $t(n)$—determining the state of one immediately gives the state of the other without measurement if the properties are entangled in a known manner.

(b) This does not contradict the assumption that any property of an object must obey some deterministic natural law for any possible measurement that is made at any time. Nor does it require any information to travel from one particle to another consequent to a measurement.

19.D.h.   Schrödinger's cat 'paradox'

If $[F(x)]$ is an algorithmically *verifiable* but not algorithmically *computable* Boolean function, we can take the query:

(i) Is $F(n) = 0$ for all natural numbers?

as corresponding to the Schrödinger question:

(ii) Is the cat dead or alive at any given time t?

We can then argue that there is no mathematical paradox involved in the assertion that the cat is both dead and alive at a select moment of time, if we take this to mean that:

Thesis 20. (Schrödinger's cat) *I may either assume the cat in Schrödinger's gedunken to be alive until a given time t (in the future), or assume the cat to be dead until the time t, without arriving at any logical contradiction in my existing Quantum description of nature.*

In other words:

(a) Once we accept Quantum Theory as a valid description of nature, then there is no paradox in stating that the theory essentially cannot predict the state of the cat at any moment of future time.

(b) The inability to predict such a state does not arise out of a lack of sufficient information about the laws of the system that Quantum theory is describing, but stems from the very nature of these laws.

The mathematical analogy for the above would be:

(c) Once we accept that Peano Arithmetic is *strongly* consistent (see §2.C.a., Theorem 2.16) and categorical (see §2.E., Corollary 2.18) then we cannot deduce from the axioms of PA whether $F(n) = 0$ for all natural numbers, or whether $F(n) = 1$ for some natural number.

CHAPTER 20. *PARADIGM CHALLENGING CONSEQUENCES*

20.   What should be a first philosophy of mathematics education

We highlight the foundational significance of *evidence-based* reasoning and §1., Thesis 1, for mathematics education, and the philosophy of mathematics education, by briefly suggesting an alternative paradigm to that of Paul Ernest in [Er12], albeit towards the common goal that we need to ground mathematics education on firmer, consensual, *evidence-based* (see §2.) rather than contentious, faith-based (see §4.B.), foundations.

We argue that Mathematics has no *intrinsic* content per se; it is merely a powerful[194] set of languages that serves the needs of the natural sciences and their philosophies[195] for:

— first, expressing those of our conceptual metaphors which can be *well-defined* in a mathematical language such as the first-order set theory, ZFC; and,

— second, identifying those conceptual metaphors which can be further communicated *categorically* in a mathematical language such as the first-order Peano Arithmetic PA which, by the Provability Theorem for PA (see §2.E., Theorem 2.17) models, and circumscribes, the reasoning ability of a Turing-machine based mechanical intelligence (see also §10.E.; §17.E., Query 17).

We argue, moreover, that:

— current set-theoretically founded mathematical paradigms appeal to faith-based reasoning (see §4.B.);

— which ascribes an illusory existence to what is sought to be defined unambiguously in mathematical symbolism (see §10.E.);

— only because they fail to admit such a distinction;

— primarily due to the misleading influence of three powerful, but false, dogmas inherited from Hilbert, Brouwer and Gödel (see §3.).

Classical paradigms about what mathematics *is*, and what it *is not*, thus obscure that:

— whereas a philosophy of mathematics is a specialised part of a philosophy of linguistics (see §10.);

— a philosophy of mathematics education is a specialised part of a philosophy of education (which lies beyond the scope and competence of this investigation).

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[194]The power of any such language being measured by the extent of its ability to *faithfully* and *unambiguously* express our abstract concepts symbolically, and communicate them *categorically*.

[195]Just as the various computer languages such as Ada, ALGOL, Basic, C, C^+, C^++, COBOL, Coq, Fortran, Java, Lisp, Pascal, Perl, Python etc. are interpreted in familiar day-to-day computational models that serve the needs of the emerging science of Artificial Intelligence.

The need for such distinction is visible, for instance[196], in the essay *What is our 'first philosophy' in mathematics education?*, where Paul Ernest queries:

> "What theoretical bases underpin research and practice in mathematics education? For most of the late 20th century, the theoretical underpinnings of mathematics education were mathematics and psychology. But in the past two decades other disciplines have grown in importance, including philosophy, sociology and linguistics, and have been used by a growing number of researchers to underpin their work. My own concern has been to draw on the insights and theories that philosophy offers. But this raises a question: "which of the branches of philosophy demands priority in mathematics education": what is our "first philosophy", if such exists?
>
> . . .
>
> Can mathematics education have a first philosophy? Is there a branch of philosophy that is a sine qua non for mathematics education research and possibly its practice as well? Are there philosophical assumptions that cannot be avoided in pursuing any inquiries whatsoever in our field? Can these assumptions be located in one branch of philosophy? In this article, I argue that much is presupposed when we embark on research in mathematics education, including philosophical assumptions. Identifying a first philosophy for mathematics education, if one exists, is a vital task, because any theories we use rest on assumptions, both overt and covert. These assumptions must be cognizant of and consistent with such a first philosophy. This condition is important because, as I argue in the conclusion, some recently popular theories, such as radical constructivism, fail such a test.
>
> Three candidates for a first philosophy for mathematics education research were mentioned above: ontology (representing metaphysics), epistemology and ethics. In addition, two further branches of philosophy are relevant, the philosophy of mathematics, which inquires into the nature of mathematics including its objects and knowledge, and critical theory, which considers the role of scientific and mathematical knowledge in society, as well as issues of social justice and social critique. I consider, below, the claims of each of these five candidates to be the first philosophy of mathematics education.
>
> . . .
>
> Perhaps the most frequently considered area of philosophy in mathematics education is the philosophy of mathematics. It is argued that understanding the nature of the subject of mathematics and its philosophical underpinnings is necessary, both for teaching the subject thoughtfully and for research in mathematics education.
>
> . . .
>
> The maverick tradition in the philosophy of mathematics has also challenged the traditional absolutist accounts of mathematical knowledge as certain, absolute, superhuman and incorrigible. The alternative fallibilist (Lakatos, 1976), humanist and social constructivist accounts view mathematical knowledge as fallible and humanly created. These accounts resonate with many of the most controversial theoretical developments in mathematics education, including radical constructivism, social constructivism, socio-cultural theory, postmodernism and critical mathematics education. Even the problem-solving and investigations movements in mathematical pedagogy have drawn on these newer philosophies of mathematics and their challenge to mathematical absolutism.
>
> However, it is not just these newer developments in the philosophy of mathematics that are claimed to underpin mathematics education. The educational relevance of the philosophy of mathematics as a whole has been argued more widely: "Whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics (Thom, 1973, p. 204). Thus, as Hersh (1979) argues:
>
>> The issue, then, is not, What is the best way to teach? but, What is mathematics really all about? [...] Controversies about [...] teaching cannot be resolved without confronting problems about the nature of mathematics. (p. 34)

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[196]See also *Yushua, Bokhari, Mji, Wessels*: [YBM04]; *Luitel*: [Lui19].

In discussing such philosophies of mathematics embedded in the mathematics curriculum I have shifted from referring to formal academic philosophies as discussed by professional philosophers (e.g., intuitionism, logicism, formalism), to discussing informal philosophies, perhaps better described as images of mathematics. There is an analogy here with Tall and Vinners (1981) distinction between concept definition (formal, explicit, publicly justifiable description) and concept image (visual and other representations and associations). Concept images represent a deep level of meaning, partly implicit, and may influence their holders dispositions and actions. Similarly, images of mathematics can include a wide range of representations and associations from sources including philosophy and accounts of the nature of mathematics, but also including representations from the media, classroom presentations and parent, peer and other narratives about mathematics. Personal images of mathematics can utilize mental pictures, including visual, verbal, and narrative representations, originating from past experiences, social talk, etc., and include cognitive, affective and behavioural dimensions, including beliefs. Clearly, with all this variety of representations, a personal philosophy or image of mathematics cannot be the same as a fully articulated academic philosophy of mathematics."

...*Ernest: [Er12], p.8 and p.9.*

Since a teacher's belief system may be responsible for implicitly influencing the belief system of a student—sometimes unjustifiably and permanently (see §4.B.; also *Friend, Molinini*: [FM15], pp.201-202)—the need for, and absence of, a holistic *evidence-based* perspective concerning the nature of mathematics as a whole has been a continuing concern of educationists. As highlighted further by Ernest:

"**CONCEPTION OF THE NATURE OF MATHEMATICS**

This is a teacher's belief system concerning the nature of mathematics as a whole. Such views form the basis of the philosophy of mathematics, although some of the views likely to be held by teachers may not have been elaborated into fully articulated philosophies. Teachers' conceptions of the nature of mathematics by no means have to be consciously held views; rather they may be implicitly held philosophies. The importance of such views of subject matter has been noted both across a range of subjects (Feinman-Nemser and Floden, 1986) and for mathematics in particular (Ernest 1985, 1987, 1988b; Lerman, 1983, 1986; Thom, 1973). Out of a number of possible variations, three philosophies of mathematics are distinguished because of their observed occurrence in the teaching of mathematics (Thompson, 1984), as well as for their significance in the academic study of the philosophy of mathematics (Benecerraf and Putnam, 1964; Davis and Hersh, 1980; Lakatos, 1976; Tymoczko, 1985). They are presented here in simplified form, and in practice teachers may combine elements from more than one of the views.

First of all, there is a dynamic, problem-driven view of mathematics as a continually expanding field of human enquiry. Mathematics is not a finished product, and its results remain open to revision (the problem solving view).

Secondly, there is the view of mathematics as a static but unified body of knowledge, consisting interconnecting structures and truths. Mathematics is a monolith, a static immutable product, which is discovered, not created (the Platonist view).

Thirdly, there is the view that mathematics is a useful but unrelated collection of facts, rules and skills (the instrumentalist view).

Teachers' views of the nature of mathematics may also be compounded with additional constructs, such as views of the relationship between different subject matter areas, for example. Is mathematics entirely distinct from other disciplines? Or are mathematics and other areas of knowledge interrelated or partly integrated, sharing concepts and methods of inquiry? The conception of knowledge as integrated is often associated with a problem solving view of mathematics, but there is no strict necessity behind this link.

The different philosophies of mathematics have practical classroom outcomes. For example, an active, problem solving view of mathematical knowledge can lead to the acceptance of children's

methods and approaches to tasks. In contrast, a static Platonist or instrumentalist view of mathematics can lead to the teacher's insistence on there being a single 'correct' method for solving each problem. Again, a teacher's view of knowledge as integrated can lead to teaching in which mathematics and other subject matter areas are interrelated. The opposite view can result in an insistence that questions of mathematics and geography, for example, are dealt with separately during mathematics lessons and geography lessons.

Some of the main curriculum reform movements in mathematics have been based on views of mathematics. The Modern Maths movement of the early 1960's can be likened to the Platonist view through its stress on structure, the laws of number, and central and unifying concepts of mathematics, such as sets and functions. A second parallel can be drawn between the instrumentalist view and that underlying the Back-to-Basics movement. This movement emphasises basic numeracy as knowledge of facts, rules and skills, without regard for meaningful connections within this knowledge. More recently, the problem solving view of mathematics has been reflected in the recommendations of official bodies (NCTM, 1980; Cockcroft, 1982; HMI, 1985; National Curriculum Mathematics Working Group, 1987). Namely, that the processes and strategies of mathematical activity are central, and that the main aim of mathematics teaching is to empower children to become creative and confident solvers of problems.

Teachers' views of mathematics evidently affect the extent to which such curriculum innovations or movements take hold, through the way mathematics is taught (Cooney, 1988; Thom, 1973). For beliefs about mathematics are reflected in teachers' models of the teaching and learning of mathematics, and hence in their practices (Thompson, 1984).

. . .

THE IMPACT OF BELIEFS ON TEACHING MATHEMATICS

The teacher's view of the nature of mathematics provides a basis for his or her mental models of the teaching and learning of mathematics. For views of the nature of mathematics are likely to correspond to views of its teaching and learning. Thus, for example, the instrumental view of mathematics is likely to be associated with a transmission model of teaching, and with the strict following of a text or scheme. It may also be associated with the child's compliant behaviour and mastery of skills model of learning. Similar links can be conjectured between other views and models, for example:

Mathematics as a Platonist unified body of knowledge corresponds to a view of the teacher as explainer, and learning as the reception of knowledge, although an emphasis on the child constructing a meaningful body of knowledge, is also consistent with this view;

Mathematics as problem solving corresponds to a view of the teacher as facilitator, and learning as autonomous problem posing and solving, perhaps also as the active construction of understanding."
. . . Ernest: [Er06].

The thesis of this investigation (see §1., Thesis 1) suggests that classical set-theoretically founded paradigms conflate mathematical education with the teaching of *only* the grammar of formal mathematical languages.

In other words, as is reflected in [Er06] above, classical set-theoretically founded paradigms:

— seem to focus only on the first-order construction of grammatically correct, contradiction-free, provable set-theoretical propositions of the language;

— without *any* attempt to assign *evidence-based* meanings, in terms of *primary* conceptual metaphors, to the grammatically correct (provable/unprovable) propositions of the language since, if we accept §16.D.b., Thesis 6, there can be no *evidence-based* interpretation of a theory that contains an axiom of infinity;

— so any interpretation of the set-theoretically defined formulas of a mathematical language must essentially admit *secondary* conceptual metaphors;

— which *exist* only symbolically in the very propositions of the language whose meaning is sought from the perspective of an 'understanding' that seeks to avoid impredicativity;

— by planting the seeds of an inescapable impredicativity that must implicitly harbour contradiction, as is highlighted by the well-known logical, semantic, and mathematical paradoxes (see §16.)!

If so, it is uncertain to what extent one could, for instance:

— sustain the interest of one unfamiliar with the English language by teaching only, say, English vocabulary and grammar;

— and/or expect appreciation of the expressive and communicative power of the language for both practical applications (such as asking for directions) and literary pursuits (such as reading Shakespeare).

Moreover, this would suggest explicitly highlighting—as early as possible in any mathematical education—that the grammar of a language that teaches the construction of valid narrative and declarative sentences of a language does not inform what the sentences narrate and/or declare; but only that the language can be used to narrate and declare what we intend to talk about, and communicate, by means of the language.

> **Comment**: Which would further suggest that Bertrand Russell's oft-quoted explanatory remark which, it may not be entirely unreasonable to claim, has significantly influenced the mathematical perspective, and philosophy, of mathematics teachers and students over the years:
>
> > "…mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."
> > …*Russell:* [Rus17], *p.74, paragraph 3.*
>
> might be more to the point if read today as:
>
> > "…mathematics may be defined as the subject in which we never talk about anything, nor claim that what we are saying is true."

20.A. Math Anxiety

We conclude by remarking that, from the *evidence-based* perspective of [An16] (see §2.), the challenges, labeled as 'math anxiety', that inhibit the teaching of mathematics as a 'live' discipline could also:

— be a reflection of implicit, unreasonable, expectations in current mathematical education paradigms;

— of a median student's innate ability, and motivation, to commit, in the absence of *evidence-based* and *explicit* interpretations that are encountered daily in the student's life;

— to *subjectively* seeking an appreciation of the significance, and meaningful association, of the grammar and vocabulary of an unfamiliar language;

— apropos *implicit* interpretations which might, or might not, be what the language is intended to express and communicate.

"WHAT IS MATH ANXIETY?

Math anxiety refers to feelings of fear, tension,, and apprehension that many people experience when engaging with math (Ashcraft, 2002). Math anxiety is thought to be a trait-level anxiety and is distinguished from both test anxiety (Kazelski et al, 2001) and state anxiety (Hembree, 1990). For a math anxious student, math creates more than a feeling of dislike or worry; it also affects physiological outcomes, such as heart rate, neural activation, and cortisol . . . .

The various accounts explaining why individuals might develop math anxiety have been helpful in generating important programs of research. One shortcoming of these accounts, however, is that they do not explain why poor math abilities or negative learning experiences necessarily lead to math anxiety. After all, many students receive lower grades in math or learn under the same teachers as anxious students, and yet these students do not end up developing math anxiety. Conversely, we also find that there are many students who are both high achievers in math and highly math anxious (Lee, 2009). To reconcile this, we propose a novel Interpretation Account of how math anxiety develops and demonstrate how this new framework allows us to resolve some seemingly contradictory findings in the literature.

We draw from a large and disparate body of research to argue that students' development of math anxiety is largely determined by how they interpret(i.e., appraise) previous math experiences and outcomes (rather than the outcomes themselves). That is, math anxiety derives not just from a students avoidance tendencies, reduced competency, or performance worries that shape the development of math anxiety but rather how individuals interpret their math-related experiences."
. . . *Ramirez, Shaw, Maloney*: [RSM18], *p.151*

> **Comment**: We note that issues of 'math anxiety—which include, but need to be differentiated from the more specific learning challenges due to Dyslexia and Dyscalculia (see *Chinn*: [Chn18])—seem to be of increasing concern to nations across the globe in societies increasingly dependent on the reliability of the mathematical languages that can provide the sound foundation (see, for instance, §2.E., Theorem 2.17) for the programming of mechanical intelligences in emerging products and services of critical economical, political, and military, interest to the state. See, for instance *Uusimaki, Nason*: [UN04], Australia; *Yushua, Bokhari, Mji, Wessels*: [YBM04], Saudi Arabia; *Adnan, Zakaria* [AZ10], Malaysia; *Bekdemir*: [Bkd10], Turkey; *Tungosiamu*: [Tng10], Ghana; *Boyd, Foster, Smith, Boyd*: [Byd14], Australia; *Reali, Jiménez-Leal, Maldonado-Carreño, Devine, Szücs*: [Rli16], Colombia; *Alam, Halder*: [AH18], India; *Orbach, Herzog, Fritz*: [OHF19], Germany; *Sorvo, Koponen, Viholainen, Räikkönen, Peura, Tolvanen, Aro*: [Srv19], Finland; *Zhang, Jhao, Kong*: [ZZK19], China.

In a 2010 paper [Blb10], educationist Shashidhar Belbase seeks to place the issue of 'maths anxiety' within a broader perspective of the epistemological and philosophical foundations (see also §10.)—such as *Realist*, *Intuitionistic. Constructivist*, *Nominalistic*, *Logicistic*, *Formalist*, *Absolutist*, etc.—which underpin individual and collective perceptions, and attitudes, towards the teaching and learning of mathematics:

"Epistemological and Philosophical Foundation

What is the nature of mathematics? How this nature is perceived by an individual? What are personal epistemologies and philosophies? How these epistemologies and philosophies impact in our understanding of mathematics? There are some questions we need to think in order discuss the relationship among images, anxieties and attitude toward mathematics.

The nature of mathematics can be viewed differently from different epistemological and philosophical perspectives. For *Realists*, mathematics is viewed as the science of numbers, sets, functions, etc.,

just as physical science is the study of ordinary physical objects, astronomical bodies, subatomic particles, and so on (Maddy, 1990). According to the *Realist* school, images of mathematics relate to the nature of mathematics as being fixed, epistemologically priori, and it is infallible. This epistemological perspective believes that mathematical knowledge is fixed, and it is out there that we have to discover it. Many mathematicians have this dominant image of mathematics, and their teaching and learning in the classroom is affected by such image influenced by their epistemological perspective and personal philosophy. *Intuitionism* is based on the idea that mathematics is a creation of the mind. The truth of a mathematical statement can only be conceived via mental construction that proves it to be true and the communication between mathematicians only serves as a means to create the same mental process in different minds. The image of mathematics from this epistemological perspective and philosophical lens is that mathematics is a mental creation; mathematical objects are created by the intuition of mind, irrespective of language in which one thinks (Iemhoff, 2008).

The epistemology and philosophy of *Constructivism* conceives that the learners actively construct their own knowledge, rather than passively receive it. *Constructivists* argue that the term knowledge is problematic because it evokes a static, rather than dynamic image of learning, and they prefer to talk about learning or knowing, interpreting and making sense of experiences. A popular conception of *Constructivism* claims that learners can only construct meaningful understanding in relation to their prior knowledge. The image of mathematics is viewed from this epistemological and philosophical lens as a co-construction of mathematical ideas through social critical discourse of various mathematical phenomena (Confrey & Kazak, 2006).

There exists a common feature of all the views just described, that is, that they all take mathematics to deal with abstract objects, whether one takes these to have an independent existence in their own right, or to be abstracted from our experience (Avigad, 2007). An alternative, as suggested by Avigad, is simply to deny such object's ontological status in the first place, and think of mathematics, instead, as a science governing the use of (relatively concrete) signs. The challenge then is to give an account of mathematical knowledge that explains what it is that gives certain manipulations of signs normative force and also explains the applicability of mathematics to the sciences. Positions that adopt such an approach fall under the rubric of *Nominalism* (Avigad, 2007). According to *Nominalism*, mathematical objects do not exist or, at least, they need not be taken to exist for us to make sense of mathematics (Bueno & Zalta, 2005).

Strawderman (2010) has proposed three domains to study mathematics anxiety: *social/motivational* domain, *intellectual/educational* domain, and *psychological/emotional* domain. Strawderman clarified that the *social/motivational* domain includes those forces that act upon a person through the agencies of family, friends, and society as a whole. The *intellectual/educational* domain is comprised of those influences that are cognitive in nature. These cognitive influences include but are not limited to, the knowledge and skills an individual has and or is expected to acquire and his or her perception of success or failure in them. The *psychological/emotional* domain is formed by the faculties that are affective in nature. It is largely comprised of the individual's emotional history, reactions to stimuli, and arousal states. Hence, the continuum associated with this domain is feelings. At either end of the feelings continuum lie anxiety and confidence. These domains of anxiety are related to attitude towards mathematics in terms of emotions, expectations and values (Hannula, 2002). These discussions lead us to conclude that images of mathematics are greatly shaped by the epistemological and philosophical perspectives of one who views mathematics either as priori or posteriori, absolute or relative, and concrete or nominal. These images are further associated with perceptions, feelings and anxiety of mathematics leading to negative or positive attitude toward mathematics."

. . . *Belbase:* [Blb10], *§Epistemological and Philosophical Foundations, pp.5-6.*

Belbase argues that treating an 'image of mathematics' as 'a representation of mathematics that is either *social* or *personal*', is important for understanding why, for many pupils, their image about mathematics becomes 'influenced by dichotomous thought as [to whether] mathematics is *absolute* or *fallible*', with 'negative feelings about mathematics' as being 'difficult, cold,

abstract, theoretical, and ultra-rational' as well as 'being *remote and inaccessible* to all but a few super-intelligent beings with 'mathematical minds'':

> "When I talk or think about images of mathematics, two things come into my mind: images as objects or images as abstraction. I think images as objects in relation to mathematics are related to symbols $(+, -, \%, [\,], \Delta, \pi, <, =, >, \sqrt{}, \sum, \int$ etc.) and images as abstraction are related to operations that go in our mind. The image as an object is static and it visualizes mathematics as a subject matter. The image as an abstraction is dynamic and it visualizes mathematics as a process or operation that goes on in our mind.

> Tall and Vinner (1981) define a concept image as all the cognitive structures, conscious or unconscious, associated with a concept, including mental images and words. A concept, such as an apple, must allow for variability. If we imagine an object shaped like an apple that is purple, we can still believe that it is an apple. We have the freedom to recombine familiar ideas in novel ways. But, since we have never seen a purple apple, it is unlikely that we would form an image of one, when hearing the word apple (Browne, 2009). McGinn (2004) asserts that images are part of our active nature, since they are subject to the will of the viewer. Percepts belong to the passive part of thinking and imagination. In other words, one must make an effort to form an image of something, while the same may not hold true for just looking. That is to say that something that we see may have different mental image than that appears to us. McGinn classifies images as a distinct mental category, separating them from percepts. In mathematics, images represent perception in terms of nature of mathematics as viewed by a person. Lakoff and Nunez (2000) have argued that the conceptual metaphor plays a fundamental role in mathematical understanding because it provides a means to map ideas in one conceptual domain to corresponding ideas in another conceptual domain. For example, it makes possible for us to understand difficult ideas such as infinity.

> Based on our experience, we all may develop different images in relation to mathematics and its nature. Specifically, we all have developed images of and about mathematics. Ernest (2008) argues that there are many components of learner attitudes and beliefs about mathematics. These attitudes and beliefs play an important role in problem solving and in learner participation in advanced mathematical studies and careers. According to Ernest, developing a positive image of mathematics leads a learner toward advancement and to the benefit of society.

> In *absolutist* perspective, images of mathematics are viewed as an objective, absolute, certain, and incorrigible body of knowledge, which rests on the firm foundations of deductive logic. Among twentieth century philosophies, *Logicism*, *Formalism*, and, to some extent, *Intuitionism* and *Platonism* may be said to be *Absolutist* in this way (Ernest, 1991). However, Ernest (2008) claimed that absolutist philosophies of mathematics are not concerned about describing mathematics or mathematical knowledge as they are practiced or applied in the world around us.

> Rensaa (2006) asserts that in the past few decades a new wave of epistemology and philosophy of mathematics have been gaining ground and these propose a non-absolutist account of mathematics. Kitcher and Aspray (1988) described this as the 'maverick' tradition that emphasizes the practice of, and human side of mathematics, and characterizes mathematical knowledge as historical, changing, and corrigible. Image of mathematics is viewed as falsifiable, contextual, and relative.

> According to Ernest (1994), one of the innovations associated with a fallibilist view of mathematics is a reconceptualized view of the nature of mathematics. It is no longer seen as a body of pure and abstract knowledge which exists in a superhuman, objective realm. The perfection of mathematics is ideal and, therefore, the false image of perfection of mathematics must be dropped (Davis, 1972).

> Before discussing the relationship of images of mathematics with values and epistemology it is necessary to indicate what we mean by an image of mathematics in this context. Taking an image of mathematics is a representation of mathematics that is either *social* or *personal*. *Social images* of mathematics are public representations encompassing mass media representations including films, cartoons, pictures, popular music; presentations and displays in school mathematics classrooms and the related learning experiences relating to them; parent, peer, or other narratives about

mathematics; and representations of mathematics utilizing any other semiotic education modes or means. These public images of mathematics may possibly have a significant impact in shaping children's attitudes toward mathematics (Ernest, 2008).

Ernest (2008) further claimed that *personal images* of mathematics are personal representations of mathematics utilizing some form of mental picture, visual, verbal, narrative or other personal representation, originating from past experiences of mathematics, or are from social talk or other representations of mathematics, which may potentially compromise cognitive affective and behavioral dimensions. To me, these *personal images* in relation to *social images* develop our perceptions, values, and attitudes towards mathematics. The conception of mathematics as it is represented in such images may vary across a range encompassing research mathematics and mathematicians, school mathematics, and mathematical applications in everyday or otherwise.

A widespread *public image* of mathematics in the West is that it is difficult, cold, abstract, theoretical, and ultra-rational, and, also important and largely masculine (Ernest, 2008). It also has the image of being *remote and inaccessible* to all but a few super-intelligent beings with 'mathematical minds' (Buerk, 1982; Buxton, 1981; Ernest, 1996; Lim & Ernest, 1998; Picker & Berry, 2000). For many people, this *negative image* of mathematics is also associated with anxiety and failure. When Brigid Sewell was gathering data on adult numeracy for the Cockcroft (1982) inquiry, she asked a sample of adults on the street if they would answer some questions. Half of them refused to answer further questions when they understood it was about mathematics, suggesting *negative attitudes.* Extremely *negative attitudes* such as '*mathephobia*' (Maxwell, 1989) probably only occur in a small minority in Western societies, and may not be significant at all in other countries. In fact, the world-wide consensus of mathematics educators is that school mathematics must counter that image, and offer instead something that is personally engaging, and useful, or motivating in some other way, if it is to fulfill its social functions (Howson & Wilson, 1986; NCTM, 1989; Skovsmose, 1994).

In a broad sense, images of mathematics as separated and connected to values that further lead us to formulation of a school mathematics either as *disconnected or connected* images of mathematics. Finally, teaching and learning of mathematics influenced by the *absolutist* epistemology and philosophy help students develop *separate (fragmented) image* of mathematics, while *fallibilist* epistemology and philosophy help students to develop a *connected (logically related) image.* However, for many pupils the image about mathematics become influenced by dichotomous thought as mathematics is *absolute* or *fallible*, they are gradually changing from *positive* to *negative* feelings through schools because of not being able to understand that they can create mathematical objects rather than imitate from others. Such negative feelings about mathematics, for instance, have been seen in reports from the United States, Austalia, or closer in Norway (Ernest, 2008). Renssa (2006) also claimed in the same line as Ernest that pupil's images of mathematics and mathematicians are derived as a result of social experiences, either through school, peers, parents, or mass media. In real life the picture is more complex as these influences interact each other.

Within the public society, adults and parents' images of mathematics are important when it comes to influence children's perceptions. As stated by Ernest (1996) there is no doubt about impact of adult's and parents' perception and attitude toward mathematics on children's attitude to the subject. It indicates that how important parental encouragement is to children's learning of mathematics. Ferry et al. (2000) found in their research on family background context variable, parental encouragement in mathematics and science significantly influence learning experiences. Learning experiences, in turn, were found to significantly influence self-efficacy and outcome expectations. These results support the role of family context in Lent et al.'s social cognitive career development model (Lent et al., 1994). Children not having this support may therefore have a drawback when it comes to continuing with *negative images* of mathematics."

. . . *Belbase:* [Blb10], *§Images of Mathematics, pp.8-12.*

Belbase thus argues that, within 'the public society, adults and parents' images of mathematics are important when it comes to influence children's perceptions', indicating 'how important parental encouragement is to children's learning of mathematics'.

He concludes that children 'not having this support may therefore have a drawback when it comes to continuing with *negative images* of mathematics' that could, conceivably, result in a mathematical anxiety syndrome where 'students under this anxiety have fear of mathematics class, homework, exam and any situation when comes with mathematics':

"When I think or talk about mathematics anxiety, two things come to my mind: one is anxiety as progressive thinking and the other is anxiety as regressive thinking. To me all anxieties are not bad things. Anxieties can be both good and bad. If it promotes for progressive thinking (like when I am puzzling in a mathematics problem for a few days and I am trying to solve it in a variety of ways), then certainly it is good thing. But anxiety is mostly taken as regressive thinking in which a person having anxiety tries to go away or get rid of problem simply by avoiding it and taking it negatively.

Mathematics anxiety is an anxious state in response to mathematics-related situations that are perceived as threatening to self-esteem. Cemen (1987) proposed a model of mathematics anxiety reaction consisting of environmental antecedents (e.g., negative mathematics experiences, lack of parental encouragement), dispositional antecedents (e.g., negative attitudes, lack of confidence), and situational antecedents (e.g., classroom factors, instructional format) are seen to interact to produce an anxious reaction with its physiological manifestations (e.g., perspiring, increased heart beat). Many researchers (e.g., Ma, 1997; Richardson & Suinn, 1972; Tobias & Weissbrod, 1980) have reported the consequences of being anxious toward mathematics, including the inability to do mathematics, the decline in mathematics achievement, the avoidance of mathematics courses, the limitation in selecting college majors and future careers, and the negative feelings of guilt and shame. Ma (1997) claimed that mathematics anxiety is usually associated with mathematics achievement individually. A student's level of mathematics anxiety can significantly predict his or her mathematics performance (Fennema & Sherman, 1977; Wigfield & Meece, 1988).

In their study Miller and Bichsel (2004) claimed that math anxiety appears to primarily impact visual working memory, contradicting previous findings that anxiety is primarily processed in verbal working memory and supporting the hypothesis that math anxiety does not function similarly to other types of anxiety. They referred to past researches which investigated the underlying cognitive processes that contribute to individual differences in math ability, the most investigated of which appears to be working memory. Many researchers in the past have shown that the processes involved in working memory, namely, temporary retrieval, processing, and storage, explain much of the variance in math ability (e.g., Adams & Hitch, 1998; Ashcraft,1995; Miller & Bichsel, 2004). The concluding remarks can be made based on these claims that individuals who are more efficient and adept in carrying out these processes are likely to perform better on tests of math ability.

Miller and Bichsel (2004) identified two general types of anxiety: trait and state. They clarified that individuals experiencing trait anxiety have a characteristic tendency to feel anxious across all types of situations. In contrast, individuals possessing state anxiety tend to experience it only in specific personally stressful or fearful situations. Trait anxiety is more related to wide range of situations to which one feels a kind of threat, unsecured, and challenge all the time. In mathematics, students under this anxiety have fear of mathematics class, homework, exam and any situation when comes with mathematics. According to Spielberger et al. (1970), state anxiety reflects a transitory emotional state or condition of the human organism that is characterized by subjective, consciously perceived feelings of tension and apprehension, and heightened autonomic nervous system activity. Several past studies demonstrated that both state and trait anxiety affect task performance (e.g., Leon & Revelle, 1985; MacLeod & Donnellan, 1993; Miller & Bichsel, 2004). Concluding the findings from these researches, Miller and Bichsel stated that individuals with high trait anxiety show poorer performance on various tasks than low trait anxiety individuals. This difference tends to be exacerbated in a high state anxiety condition. With reference to research on impact of gender on math anxiety, Hembree (1990) found math anxiety being more predictive of math performance in males than in females."

... *Belbase: [Blb10], pp.5-12*

In a study that investigated the causes underlying a sample of eighteen third-year Australian

pre-service primary teachers' negative beliefs and anxiety about mathematics, Liisas Uusimaki and Rod Nason give examples which suggest that the roots of 'math anxiety' can be traced more to *how* mathematics is taught than to *what* is taught.

Essentially, what they found was that most of the participants' maths-anxiety could be attributed to their primary school experiences in enforced learning of mathematical structures *without*, apparently, any *evidence-based* interpretations of such symbolic expressions that would assign meanings to them from familiar day-to-day experience:

"**Issue 1: Origins of negative beliefs and anxiety about mathematics**

The analysis of data revealed that 66% of the participants (n = 12) perceived that their negative beliefs and anxiety towards mathematics emerged in primary school. Linda for example, remembers "exactly" what year in primary school she learnt to dislike mathematics.

> *When I was in Grade 5 and we started doing division and I was away the very first day they introduced division and I came back the next day and I had no clues what everyone else in the class seemed to know really well. And my teacher never took the time to actually sit down and go through it with me so I was trying to play catch up and I feel like Ive been playing catch up every since . . .*

. . . Tina, for example, remembers the time in primary school as a time when,

> *I used to make lots of mistakes and I was always frightened I vividly remember, actually in Grade 1, getting into huge trouble because I couldn't fit a puzzle together. I vividly remember that. Just absolutely getting caned by this teacher.*

. . . Petra's comment about one of her secondary school mathematics teachers exemplified the type of comments made by these four participants about some of their secondary mathematics teachers.

> *I had a teacher called Mr O, a bit of a Hitler looking fellow but I just have visions of him throwing dusters at students you know to get their attentions and he just never explained anything . . . just wrote it on the board and then you just copied it and then you just had to really go home and try and work it out so I was pretty stressed about that cause I kept thinking you need to talk about it, you need to go through it together and ask whether you understand it.*

Only 11% of the participants (n = 2) identified tertiary education as the time when their negative beliefs and anxieties towards mathematics emerged. An important aspect of the comments made by these two participants was that their negative beliefs about mathematics was not traced back to how mathematics was taught but back to specific content of mathematics.

Issue 2: Situations causing most maths-anxiety

The participants felt most anxious about mathematics when they had to communicate their mathematical knowledge in some way (48%), for example, in test situations or verbal explanations. Also, causing a lot of anxiety was the teaching of mathematics in practicum situations (33%) due to insecure feelings of making mistakes or not being able to solve it correctly. For example, Rose explains that her most anxious moments are:

> *When I'm being called on to answer questions and I don't know the right language and I try to answer the question as best I can but you don't really get your meaning across because you don't understand the language and you don't know what language to use.*

> *Testing . . . Just when somebody tests my knowledge . . . It does and it makes me feel as if I dont know what I am talking about.*

Issue 3: Types of mathematics causing maths-anxiety

Two strands from the Queensland Studies Authority (2003) syllabus caused most anxiety: 'algebra and patterns' (33%) and 'space' (31%). Number operations especially division, was also a concern (21%). The anxiety caused by these strands was well exemplified by Ann's response to Question 4.

> *Long division! Couldn't ever do that. Dividing. Can't do that. Times tables. You know how they used to learn the times tables. I still can't do them because they sing that song. One, ones are one and all that and I never had a very good memory so I could never learn them. I'm making myself sound really bad ... And with addition and subtraction, I still use my fingers to count up things ... I used to do it under my desk so the teacher couldn't see 'cos you're supposed to know just what 6 plus 6 is without counting it on your fingers sort of thing.*

SUMMARY AND CONCLUSIONS

Most of the findings from this study regarding the causes of negative beliefs and anxieties about mathematics were consistent with the findings reported in the research literature. (e.g., Brown, McNamara, Hanley, & Jones, 1999; Carroll, 1998; Cornell, 1999; Nicol, Gooya, & Martin, 2002; Trujillo, & Hadfield, 1999). For example, this study found that the origin of maths-anxiety in most of these participants could be attributed to prior school experiences (cf., Levine, 1996; Martinez & Martinez, 1996). Whilst the literature suggests that negativity toward mathematics originates predominantly in secondary school (e.g., Brown, McNamara, Hanley, & Jones, 1999; Nicol, Gooya & Martin, 2002), data from this study suggests that negative experiences of the participants in this study most commonly originated in the early and middle primary school. The perceived reasons for these negative experiences are attributed to the teacher, particularly to primary school teachers (72%) rather than to specific mathematical content or to social factors such as family and peers.

Situations which caused most anxiety for the participants included communicating one's mathematical knowledge, whether in a test situation or in the teaching of mathematics such as that required on practicum. This is consistent with findings in the literature that suggests that maths-anxiety surface most dramatically when the subject is seen to be under evaluation (e.g., Tooke & Lindstrom, 1998). Specific mathematical concepts, such as algebra, followed by space and number sense, caused most concern amongst the participants.

Many of these findings have clear implications for the intervention program to follow this study. For example, the findings that many of the participants' maths-anxiety was teacher-caused indicate the need for the facilitator in the ensuing workshops to be warm, non-intimidating and supportive in nature. The findings also imply that the participants need to be provided with learning environments where they are able to: 1) freely explore and communicate about mathematics in a supportive group environment 2) explore and relearn basic mathematical concepts, and 3), apply this re-learnt knowledge in real-life and authentic situations. As evidenced by the latent themes in the participants' responses, it is also clear that isolation and evaluation anxieties will not be allayed via merely arming pre-service teachers with content knowledge. This would act to further problematise the individual and dismiss the fundamental importance of the individual feeling part of an emerging mathematics community in which they perceive themselves to be supported."

... *Uusimaki & Nason: [UN04], pp.372-375.*

What ought to be particularly striking about the reported responses in the above study is that the challenge faced by math-anxious students in successfully finding:

— on their own, and without a given, appropriate, *evidence-based*—hence *verifiable*—interpretation into terms already familiar to the student;

— a meaning to what is taught essentially in an unfamiliar symbolic language of intended precise expression;

— that could entail a feeling of understanding the significance of that which a mathematical expression symbolises;

do not, prima facie, appear to have been encountered in the learning of, and achieving fluency in, the— far from precise—students' mother tongues!

The pedagogical challenge in differentiating between the ontological (and epistemological) status of:

- what we express symbolically as our primary conceptual mathematical metaphors, and

- what we express symbolically as the secondary conceptual mathematical metaphors that arise from our subsequent perception of the symbolic expression/s of our primary conceptual mathematical metaphors,

is addressed by Feferman in his *Mathematical Intuition Vs. Mathematical Monsters*, where his conclusion that 'explication of the basic concepts can only be tested holistically by the degree to which these theories are successful', reflects the *evidence-based* perspective underlying the Complementarity Thesis (§1., Thesis 1) of this investigation:

> "In the teaching of mathematics, both geometrical and physical intuition are constantly called upon at all levels for motivation of notions and results, and even in some cases for proofs. As examples of the latter, no proof of Pythagoras' theorem can be more directly convincing than those involving dissection and rearrangement of figures, in some cases in combination with some elementary algebra. Given the geometrical and physical applications of the calculus, it is not surprising that the corresponding intuitions should be called on regularly in the teaching of that subject. But those same intuitions, suitably cultivated and extended, serve to carry one confidently into the study of analysis in higher dimensional spaces and then on into functional analysis. There too, as in linear algebra, geometrical intuition is frequently appealed to in the use of notions of vector addition, length, angle, projection, etc. And the near universal appearance of analogues of Pythagoras' theorem in analysis and higher geometry is a linchpin in the extension of one's intuition from familiar ground to the most diverse settings.
>
> Topology serves to cultivate its own distinctive intuitions as rubber sheet geometry. Closed orientable surfaces in three dimensions provide a playing ground where one can adapt those intuitions to the notions and techniques of combinatorial topology in order to deal with less visualizable manifolds. And, as a final pedagogical example, a good current course in axiomatic set theory will start with the intuitive conception of the cumulative hierarchy and appeal to it to justify the Zermelo-Fraenkel axioms and various plausible extensions. Moreover, one returns to that in modified forms in the constructible and relative constructible hierarchies employed in various consistency and independence results.
>
> Such examples can be multiplied a thousand-fold. The point here is not to enumerate them, but rather to recognize the ubiquity of intuition in the common experience of teaching and learning mathematics, and the reasons for that: it is essential for motivation of notions and results and to guide one's conceptions via tacit or explicit analogies in the transfer from familiar grounds to unfamiliar terrain. In sum, no less than the absorption of the techniques of systematic, rigorous, logically developed mathematics, intuition is necessary for the understanding of mathematics.
>
> Historically, and for the same reasons, it also played an essential role in the development of mathematics. The precise mathematical expression of various parts of our perceptual experience is mediated to begin with by intuitive concepts of point, line, curve, angle, tangent, length, area, volume, etc. These are not uniquely determined in some Platonic heaven. Mathematics models these concepts in more or less rigorous terms (sufficient unto the day), and then interweaves them to form more elaborate models or theories of physical experience as well as purely mathematical

theories. The adequacy of explication of the basic concepts can only be tested holistically by the degree to which these theories are successful."

...Feferman: [Fe98], §1.

So perhaps, once we accept the thesis of this investigation (see §1., Thesis 1; also §10.) that:

— current paradigms of mathematical education are remiss in treating mathematics as a discipline;

— where 'natural laws' of logic/mathematics are to be discovered as is the case with the natural sciences;

— rather than as a language that avoids the ambiguities inherent in languages of common discourse;

— in areas where unambiguity of expression and categorical communication are de jure;

it might not be entirely unreasonable to posit that the challenges labeled as 'math anxiety' can be illuminated—and placed in a more appropriate perspective—by the challenges being faced in the persisting efforts to promote the, far more intuitive albeit artificial, language *Esperanto* as an additional auxiliary language at both national and international levels:

> "**The Challenge of Esperanto**
>
> We like to think that language is mankind's greatest accomplishments, a defining quality that makes us human. Yet the languages we speak were not the result of a blueprint factory effort, or in any way a conscious contribution on our part. Whatever our first language is, we speak it spontaneously, and of course with the markers of its evolution over time.
>
> Not so with invented, artificial languages, Esperanto perhaps being the best known. Millions know it to one degree or another. Fewer are highly functional in it, a language that was born in 19th century Poland from the belief of Ludovic Zamenhoff that language differences were a major contributor to violent struggle between different ethnic groups. The theory was that too many languages prevented mutual understanding, that the world in fact would be more peaceful if a language were chosen that everybody could understand.
>
> . . .
>
> For over thirty years I've studied Esperanto in an "on-again, off-again" way. Its cerebral charm still holds. I've always liked the sound of Esperanto, especially the balanced melodic influences of several natural languages. Much of the vocabulary is derived from Latin roots common to English. Yes, I know that beauty lies in the eye of the beholder, but the poetic rhythm and the easy logic of Esperanto are to me a happy mix.
>
> The excessiveness of the dismissive humor or the cynical hostility that often greets the Esperantist is perplexing. The Esperanto movement has always been animated by the humanitarian impulse towards international and inter-ethnic friendship. Esperanto is often associated with the idealism of the world peace movement. Critics say that sperantists are "naïve simpletons."
>
> An easier criticism is that Esperanto, like any artificial language, makes communication too neutral, too antiseptic, even soulless. It doesn't help that there's no terra firma for Esperanto, the last effort was in 1908 in the tiny neutral state of Morsenet, the orphan of a border dispute between the Netherlands and Prussia, where an unsuccessful uprising included a call for Esperanto to be an official language of what was to be called the state of Amikejo (Friendship Place). But in the tense, nationalistic, atmosphere of pre-war Europe, there was no possibility for success.
>
> Critics of Esperanto say that there's a boring, sanitary neutrality that comes with speaking Esperanto. But Esperanto isn't as drab as a rainy day, or a concrete traffic barrier. Still, the color

and life of the natural languages, is, well, missing. I'm thinking of the exasperating subjunctive of Spanish, the convoluted word order of German, the tonal challenges of Chinese, or the way Russian famously stacks up three or four consonants on top of each other every chance it gets.

The resulting ambiguity, or fuzziness of meaning, according to Arika Orent, n *In the Land of Invented Languages*, is not a flaw of natural language but a feature that gives it flexibility and suits our minds and the way we think. "Likewise," she says, "the fact that languages depend on arbitrary convention or cultural habit is a feature that allows us to rein in the fuzziness by establishing agreed-upon meanings at different levels of precision." In other words, language needs its idiosyncracies in order to do the enormous range of things we use it for.

My conclusion about Esperanto's lack of widespread adoption is that learning any language, including Esperanto, takes some time and effort, and most people simply aren't interested in the hard work of learning something new without a clear pay-off. Ironically, the cultural neutrality that is part of the Esperanto movement's mantra serves to limit its growth because languages tend to spread along with the cultures that give rise to them."

. . . Tom Thompson: *The Challenge of Esperanto.* In 'Omniglot: the online encyclopedia of writing systems & languages'.

The similarity—from a student's perspective—between the challenges faced by a student in learning a purely symbolic language such as mathematics, and in learning a spoken language such as French, is also remarked upon by Richard Barwell in [Bwl]:

"Let me begin, however, by inviting you to work on a problem.

A problem of mathematics and language: la chèvre de M. Séguin.

Il était une fois un fermier qui s'appelait M. Séguin. M. Séguin avait une chèvre, une bien belle chevre. Chaque jour, M. Séguin se levait tôt le matin et amenait la chèvre au pré, pour qu'elle puisse brouter l'herbe. Elle avait faim ce chèvre! Dans le pré M. Séguin avait construit un abri qui protégait la chèvre contre les intempéries. Il attachait la chèvre au coin de l'abri avec une corde. Il la laissait brouter toute la journée. L'abri était sur roues afin que M. Séguin puisse le bouger lorsque la chèvre avait fini de manger l'herbe à sa portée. Votre problème est celui-ci: quelle superficie de pré la chèvre peut-elle manger?

How did you get on? Did you read the problem right through to the end? Perhaps you know a little French and were able to make some sense of it. Did you work out what the problem is asking you to do? Can you work on the problem and report your findings in French? I have used this problem with secondary PGCE mathematics students. I present the problem verbally, much as it is written above. When I later ask the students how they felt when I did this, they often talk of feeling frustrated, angry, incompetent, disempowered and of switching off. They get an insight into what learning mathematics through a language only partially understood might feel like."

. . . Barwell: [Bwl].

20.A.a.   Is mathematics a language?

The following excerpts are intended to briefly illustrate the dichotomy in current paradigms of mathematics education as to whether, and/or to what extent, mathematics ought to be (or even can be) treated—and taught—as a language (or as a second language).

". . . If you think that mathematics *is* a language, you might see learning English as less important: if children can learn to speak mathematics, their proficiency in English might not matter. Whilst the idea that mathematics is a language is a useful metaphor, it should not be allowed to obscure the complex role of language in mathematics (see Pimm, 1987). Children's English *does* matter, as the Framework for Teaching Mathematics (DfES, 1999) acknowledges English provides the means for children to think about mathematics, as well as to express that thinking. If children are not

supported to develop mathematical English, they are less likely to be able to participate fully in mathematics lessons, and so will have fewer opportunities to make progress in the subject."
. . . *Barwell:* [Bwl].

"A range of useful examples of activities that support language development in relation to different aspects of mathematical discourse within mathematics teaching can be found in both *Access and Engagement in Mathematics* (DfES, 2005), and in *Secondary Mathematics and English as an Additional Language* (Driver, 2005)

Bilingual education and mathematics

In the section on *bilingualism and second language acquisition* Charlotte Franson highlights 5 key research findings concerning the education of bilingual learners in mainstream classrooms. These points can be related more specifically to mathematics:

Point 1: The learner's first language plays a significant role in the learning of the second language in terms of cognitive, linguistic and socio-cultural influences. . . .

Point 2: Bilingual education can be very beneficial in the development of the second language. . . .

Point 3: Most EAL and bilingual learners will develop a functional level of English in the first two years of schooling in English but they will need continued support to develop the cognitive academic language proficiency necessary for academic success. . . .

Point 4: Learning a second language will not necessarily proceed in an orderly and systematic fashion. . . .

Point 5: Learning a language and becoming bilingual is also about learning and living in different societies and cultures. . . .

The assessment of mathematics and EAL

Assessment in mathematics covers a range of approaches, from nationally set standardised tests, to teachers' judgements of attainment based on systematic classroom observation. Assessment of the mathematical attainment of learners of EAL is, however, highly problematical, since any method of assessment relies to some extent on learners' use of English. In principle, for example, mathematics tests and examinations should not be proxy tests of English. In practice, this principle is difficult to implement. . . ."
. . . *Barwell:* [Bwl].

The significance of the Complementarity Thesis (see §1., Thesis 1) is reflected in Anne Marie Helmenstine's concluding paragraph in *Why Mathematics Is a Language*, where she notes that '*modern elementary-high school curricula uses techniques from language education for teaching mathematics*'.

From the *evident-based* perspective of the Complementarity Thesis, such techniques could, perhaps, be beneficially introduced also into the teaching of mathematics not only in higher school curricula, but also at under-graduate and graduate levels in universities, for alleviating 'math anxiety' (see §20.A.).

"Mathematics is called a language of science. Italian astronomer and physicist Galileo Galilei is attributed with the quote, "*Mathematics is the language in which God has written the unibverse.*" Most likely this quote is a summary of his statement in *Opere Il Saggiatore*:

> *[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometric figures, without which means it is humanly impossible to comprehend a single word.*

Yet, is mathematics truly a language, like English or Chinese? To answer the question, it helps to know what language is and how the vocabulary and grammar of mathematicians are used to construct sentences.

...

Key Takeaways: Why Math is a Language

- In order to be considered a language, a system of communication must have vocabulary, gramma, syntax, and people who use and understand it.
- Mathematics meets this definition of a language. Linguists who don't consider math a language cite its use as a written rather than spoken form of communication.
- Math is a universal language. The symbols and organization to form equations are the same in every country in the world.

...

What Is a Language?

There are multiple definitions of "language." A language may be a system of words or codes used within a discipline. Language may refer to a system of communication using symbols or sounds. Linguist Noam Chomsky defined language as a set of sentences constructed using a finite set of elements. Some linguists believe language should be able to represent events and abstract concepts.

Whatever definition is used, a language contains the following components:

- There must be a **vocabulary** of words or symbols.
- **Meaning** must be attached to the words or symbols.
- A language employs **grammar**, which is a set of rules that outline how vocabulary is used.
- A **syntax** organizes symbols into linear structures or propositions.
- A **narrative** or discourse consists of strings of syntactic propositions.
- There must be (or have been) a group of people who use and understand the symbols.

Mathematics meets all these requirements. The symbols, their meanings, syntax, and grammar are the same throughout the world. Mathematicians, scientists, and others use math to communicate concepts. Mathematics describes itself (a field called meta-mathematics), real-world phenomena, and abstract concepts.

...

The Argument Against Math as a Language

Not everyone agrees that mathematics is a language. Some definitions of "language" describe it as a spoken form of communication. Mathematics is a written form of communication. While it may be easy to read a simple addition statement aloud (e.g., $1 + 1 = 2$), it's much harder to read other equations aloud (e.g., Maxwell's equations). Also, the spoken statements would be rendered in the speaker's native language, not a universal language.

However, sign language would also be disqualified based on this criterion. Most linguists accept sign language as a true language. There are a handful of dead languages that no one alive knows how to pronounce or even read anymore.

A strong case for mathematics as a language is that modern elementary-high school curricula uses techniques from language education for teaching mathematics. Educational psychologisr Paul Riccomini and colleagues wrote that students learning mathematics require "a robust vocabulary knowledge base; flexibility; fluency; and proficiency with numbers, symbols, words, and diagrams; and comprehension skills."

. . . Helmenstine: [Hlm19].

CHAPTER 21. *PARADIGM CHALLENGING CONSEQUENCES*

21. The significance of *evidence-based* reasoning and the Church-Turing Thesis for quantum computing

We begin by noting that, prima facie, it is the *well-definability* (see §4.F., Definition 19) of a 'self-modifying' Turing machine—essentially such as in §2.E.a.(c)—that Michael Stephen Fiske considers in [Fi19] (see §17.G.).

Contrastingly—from the *evidence-based* perspective of the Provability Theorem for PA (§2.E., Theorem 2.17), and the proposed definition of the neo-classical laws determining 'quantum phenomena' based upon §19.D.c., Thesis 16, and §19.D.d., Thesis 17:

— the claim in [AAB19], by a 78-member team of researchers, to have successfully reached the threshold of quantum supremacy by building[197] at Google AI Quantum, Mountain View, California, USA, a:

> '... high-fidelity processor capable of running quantum algorithms in an exponentially large computational space ...'
> ... Arute, Arya, Babbush, et al: [AAB19], §The future.

— apparently *assumes* such *well-definability* and implicitly appeals—as their experiment seemingly suggests—to a putative computability of functions:

– that are algorithmically *verifiable*, but not algorithmically *computable* (see §4.G., Theorem 4.2; also §4.H.a., Theorem 4.5);

– and which violate both the classical Church-Turing Thesis (see §4.G.a., Theorem 4.3), as well as the 'physical version of the Church-Turing principle' (see §17.H.) defined by David Deutsch as (compare with Jack Copeland's *Thesis M* in [Cop04], p.106):

> "*Every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means*"
> ... Deutsch: [Deu85], p.99.

If so, the above claim in [AAB19] could prove fragile if:

— given the absence of a formal, mathematical, definition, of what constitutes a 'quantum process', which does not appeal to Deutsch's 'physical version of the Church-Turing principle',

it were to *implicitly* entail a mathematical model of a self-modifying Turing machine that can evolve (prima facie, as suggested by Deutsch in [Deu85], and by Fiske in [Fi19]) into a non-Turing machine that violates the classical Church-Turing Thesis, and which could claim (contradicting §17.F.c., Theorem 17.6) to achieve 'quantum supremacy' by computing an algorithmically *verifiable*, but not algorithmically *computable*, function (see §4.G.) in polynomial time:

---

[197]Structured, prima facie, essentially as in Deutsch, [Deu85] (see also Fiske, [Fi19]; §17.G.).

> "The extended Church-Turing thesis formulated by Bernstein and Vaziriani[55] asserts that any 'reasonable' model of computation can be efficiently simulated by a Turing machine. Our experiment suggests that a model of computation may now be available that violates this assertion. We have performed random quantum circuit sampling in polynomial time using a physically realizable quantum processor (with sufficiently low error rates), yet no efficient method is known to exist for classical computing machinery."
>
> . . . Arute, Arya, Babbush, et al: [AAB19], §The future.

In other words, from the *evidence-based* perspective of this investigation:

- which seeks to constrain the postulation of natural laws by the physical sciences,

- for their observations/measurements of physical phenomena,

- to that which can mathematically be both *well-defined* and categorically communicated (see §10.E.)

the argumentation in [AAB19] may need to recognise, and admit, that every *well-defined* mathematical function is, by definition, algorithmically *verifiable*.

Moreover:

(a) Classical laws governing natural phenomena (see §19.D.c., Thesis 16) could be characterised by the property that:

— if a physical process subject to such a law is representable by a Cauchy sequence,

— then the mathematical limit of the sequence can be treated as corresponding to a limiting state of the physical process (as illustrated by the various Zeno-type paradoxes).

(b) Neo-classical laws governing natural phenomena (see §19.D.d., Thesis 17) could be characterised by the property that:

— if a physical process subject to such a law is representable by a Cauchy sequence,

— then the mathematical limit of the sequence need not correspond similarly to a limiting state of the physical process; and

— any mathematical representation of such a limiting physical state may, for instance, require an additional, conceivably probabilistic, law to govern the permissible states of the physical process at the limit.

(c) Consequently, neo-classical behaviour could follow fixed laws and thus be *deterministic*, but not necessarily algorithmically *predictable* as in the case of classical laws since, for instance:

— the physical limiting states revealed to a witness (observer) W at a measurement,

— might only be the mathematical representations—in terms of probability 'waves'—reflecting physical laws that determine the expected quantum states at a discontinuity,

– such as, for instance, the prospective discontinuity considered in each of the gedanken §16.C., Cases 16.C.a. to 16.C.d.,

    – that is imminent at the measurement due to local factors which (contradicting Roger Colbeck and Renato Renner's thesis in [CR12]) cannot be reflected in the representation without inviting (see §16.) paradoxes of impredicativity.

(d) In which case, as argued by Matthew F. Pusey, Jonathan Barrett, and Terry Rudolph, in [PBR12]:

    — Schrödinger's wave function could mathematically represent a natural law,

    — grounded in the physical behaviour of a particle,

        – such as, say, at a measurement that affects the behaviour of the particle in a way that can be described as a discontinuous limit,

        – which could be termed as a phase change,

        – as detailed, for instance, in §16.C., Cases 16.C.a. to 16.C.d., and §16.D.b., Case 16.D.c.,

    — that could be viewed as corresponding to the collapse of the wave function,

    — which, consequently, cannot be interpreted as a statistical tool that reflects our *essential* ignorance of even a *putative* physical state of the particle sought to be measured.

    "Quantum states are the key mathematical objects in quantum theory. It is therefore surprising that physicists have been unable to agree on what a quantum state truly represents. One possibility is that a pure quantum state corresponds directly to reality. However, there is a long history of suggestions that a quantum state (even a pure state) represents only knowledge or information about some aspect of reality. Here we show that any model in which a quantum state represents mere information about an underlying physical state of the system, and in which systems that are prepared independently have independent physical states, must make predictions which contradict those of quantum theory."

    . . . *Pusey/Barrett/Rudolph: [PBR12], Abstract.*

From the *evidence-based* perspective of this investigation (see §1., Thesis 1), one could even seek to express this as:

(e) *even* if a universe U were to be completely determinate—such as, say, the hypothetical, time-reversible, universe considered in §16.D.c.—in the sense that all its laws were definable recursively;

(f) *and* its classical laws algorithmically *computable* in U, and therefore—by the Provability Theorem for PA (§2.E., Theorem 2.17)—*provable* as well-formed propositions in any formal language L that sought to represent U unambiguously, and to communicate its laws categorically;

(g) there would *still* be well-formed propositions of L that would be *unprovable* in L, but which a witness W of/within U would interpret as algorithmically *verifiable* quantum laws that are *true* in U, but are not necessarily definable in L as well-defined propositions;

(h) since any such definition might invite inconsistency through (see §16.) impredicativity;

(i) whence Bell's inequalities (see [Bl64]; also §19. and §19.B.), which are expressed in the language of L, *would be taken to apply only to properties that are definable within L.*

21.A. An *evidence-based* perspective of Deutsch's Church-Turing principle in his quantum computer

We therefore consider, from the foregoing perspective, the significance of David Deutsch's claim ([Deu85], §2, p.5; see also §17.H.) that every 'existing general model of computation is effectively classical'; where Deutsch is not-so-implicitly implying that classical physics is false:

> "It is argued that underlying the Church-Turing hypothesis there is an implicit physical assertion. Here, this assertion is presented explicitly as a physical principle: 'every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means'. Classical physics and and the universal Turing machine, because the former is continuous and the latter discrete, do not obey the principle, at least in the strong form above. A class of model computing machines that is the quantum generalization of the class of Turing machines is described, and it is shown that quantum theory and the 'universal quantum computer' are compatible with the principle. Computing machines resembling the universal quantum computer could, in principle, be built and would have many remarkable properties not reproducible by any Turing machine. These do not include the computation of non-recursive functions, but they do include 'quantum parallelism', a method by which certain probabilistic tasks can be performed faster by a universal quantum computer than by any classical restriction of it. The intuitive explanation of these properties places an intolerable strain on all interpretations of quantum theory other than Everett's. Some of the numerous connections between the quantum theory of computation and the rest of physics are explored. Quantum complexity theory allows a physically more reasonable description of the 'complexity' or 'knowledge' in a physical system than does classical complexity theory."
>
> *. . . Deutsch:* [Deu85], *Abstract.*

Deutsch argues further that:

> Albert (1983) has described a quantum mechanical measurement 'automaton' and has remarked that its properties on being set to measure itself have no analogue among classical automata. Albert's automata, though they are not general purpose computing machines, are true quantum computers, members of the general class that I shall study in this section.

> In this section I present a general, fully quantum model for computation. I then describe the universal quantum computer $\mathcal{Q}$, which is capable of perfectly simulating every finite, realizable physical system. It can simulate ideal closed (zero temperature) systems, including all other instances of quantum computers and quantum simulators, with arbitrarily high but not perfect accuracy. In computing strict functions from $\mathbb{Z}$ to $\mathbb{Z}$ it generates precisely the classical recursive functions $C(\mathcal{T})$ (a manifestation of the correspondence principle). Unlike $\mathcal{T}$, it can simulate any finite classical discrete stochastic process perfectly. Furthermore, as we shall see in §3, it [h]as many remarkable and potentially useful capabilities that have no classical analogues.

> Like a Turing machine, a model quantum computer $\mathcal{Q}$, consists of two components, a finite processor and an infinite memory, of which only a finite portion is ever used. The computation proceeds in steps of fixed duration T, and during each step only the processor and a finite part of the memory interact, the rest of the memory remaining static.

> The processor consists of M 2-state observables

$$\{\hat{n}_i\} \ (i \in \mathbb{Z}_M) \tag{2.1}$$

> where $\mathbb{Z}_M$ is the set of integers from 0 to $M - 1$. The memory consiste of an infinite sequence

$$\{\hat{m}_i\} \ (i \in \mathbb{Z}_M) \tag{2.2}$$

> of 2-state observables. This corresponds to the infinitely long memory 'tape' in a Turing machine. I shall refer to the $\{\hat{n}_i\}$ collectively as $\hat{\boldsymbol{n}}$, and to the $\{\hat{m}_i\}$ as $\hat{\boldsymbol{m}}$. Corresponding to Turing's 'tape

position' is another observable $\hat{x}$ which has the whole of $\mathbb{Z}$ as its spectrum. The observable $\hat{x}$ is the 'address' number of the currently scanned tape location. Since the 'tape' is infinitely long, but will be in motion during computations, it must not be rigid or it could not be made to move 'by finite means'. A mechanism that moved the tape according to signals transmitted at finite speed between adjacent segments only would satisfy the 'finite means' requirement and would be sufficient to implement what follows. Having satisfied ourselves that such a mechanism is possible, we shall not need to model it explicitly. Thus the state of $\mathcal{Q}$ is a unit vector in the space $\mathcal{H}$ spanned by the simultaneous eigenvectors:

$$|x; \boldsymbol{n}; \boldsymbol{m}\rangle \equiv |x; n_0, n_1 \ldots n_{M-1}; \ldots m_{-1}, m_0, m_1 \ldots\rangle \tag{2.3}$$

of $\hat{x}$, $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{m}}$, labelled by the corresponding eigenvalues x, $\boldsymbol{n}$ and $\boldsymbol{m}$. I call (2.3) the 'computational basis states'. It is convenient to take the spectrum of our 2-state observables to be $\mathbb{Z}_2$, i.e. the set $\{0, 1\}$, rather than $\{-\frac{1}{2}, +\frac{1}{2}\}$ as is customary in physics. An observable with spectrum $\{0, 1\}$ has a natural interpretation as a 'one-bit' memory element.

The dynamics of $\mathcal{Q}$ are summarized by a constant unitary operator $\mathbf{U}$ on $\mathcal{H}$. $\mathbf{U}$ specifies the evolution of any state $|\psi(t)\rangle \in \mathcal{H}$ (in the Schrödinger picture at time t) during a single computational step

$$|\psi(nT)\rangle = \mathbf{U}^n|\psi(0)\rangle \ (n \in \mathbb{Z}^+) \tag{2.4}$$

$$\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \hat{1}. \tag{2.5}$$

We shall not need to specify the state at times other than non-negative integer multiples of T. The computation begins at $t = 0$. At this time $\hat{x}$ and $\hat{\boldsymbol{n}}$ are prepared with the value zero, the state of a finite number of the $\hat{\boldsymbol{m}}$ is prepared as the 'program' and 'input' in the sense of §1 and the rest are set to zero. Thus

$$\left.\begin{array}{rcl} |\psi(0)\rangle & = & \Sigma_m \lambda |0; \boldsymbol{0}; \boldsymbol{m}\rangle, \\[2mm] \Sigma_m |\lambda_m|^2 & = & 1, \end{array}\right\} \tag{2.6}$$

where only a finite number of the λ_m are non-zero and λ_m vanishes whenever an infinite number of the $\boldsymbol{m}$ are non-zero.

To satisfy the requirement that $\mathcal{Q}$ operate 'by finite means', the matrix elements of $\mathbf{U}$ take the following form:

$$\langle x'; \boldsymbol{n}'; \boldsymbol{m}'|\mathbf{U}|x, \boldsymbol{n}; \boldsymbol{m}\rangle = [\delta_{x'}^{x+1}\mathbf{U}^+(\boldsymbol{n}', m_x'|\boldsymbol{n}, m_x) + \delta_{x'}^{x-1}\mathbf{U}^-(\boldsymbol{n}', m_x'|\boldsymbol{n}, m_x)] \prod_{y \neq x} \delta_{m_y}^{m_y} \tag{2.7}$$

The continued product on the right ensures that only one memory bit, the xth, participates in a single computational step. The terms $\delta_{x'}^{x\pm1}$ ensure that during each step the tape position x cannot change by more than one unit, forward or backwards, or both. The functions $U^\pm(\boldsymbol{n}', m'|\boldsymbol{n}, m)$, which represent a dynamical motion depending only on the 'local' observables $\hat{\boldsymbol{n}}$ and $\hat{m}_x$, are arbitrary except for the requirement (2.5) that $\mathbf{U}$ be unitary. Each choice defines a different quantum computer, $\mathcal{Q}[\mathbf{U}^+, \mathbf{U}^-]$.

Turing machines are said to 'halt', signalling the end of the computation, when two consecutive states are identical. A 'valid' program is one that causes the machine to halt after a finite number of steps. However, (2.4) shows that two consecutive states of a quantum computer $\mathcal{Q}$ can never be identical after a non-trivial computation. (This is true of any reversible computer.)

Moreover, $\mathcal{Q}$ must not be observed before the computation has ended since this would, in general, alter its relative state. Therefore, quantum computers need to signal actively that they have halted. One of the processor's internal bits, say $\hat{n}_0$, must be set aside for this purpose. Every valid $\mathcal{Q}$-program sets n_0 to 1 when it terminates but does not interact with $\hat{n}_0$ otherwise. The observable $\hat{n}_0$ can then be periodically observed from the outside without affecting the operation of $\mathcal{Q}$. The analogue of the classical condition for a program to be valid would be that the expectation value of $\hat{n}_0$ must go to one in a finite time. However, it is physically reasonable to allow a wider

class of $\mathcal{Q}$-programs. A $\mathcal{Q}$-program is valid if the expectation value of its *running time* is finite."
...Deutsch: [Deu85], 2 Quantum computers.

However, since Deutsch's mathematical model of his physical 'quantum' computer $\mathcal{Q}$ in [Deu85], by definition, can compute only algorithmically *computable* functions, his above argument—that '$\mathcal{Q}$ must not be observed before the computation has ended since this would, in general, alter its relative state'—entails (see Aerts' argument as detailed in §19.B.c. and §19.B.d.) the untenable conclusion that:

(i) The making, or not making, of a physical measurement/observation in the *actual* universe that we inhabit,

  − which can be treated as corresponding to that which is sought to be represented mathematically as the computation of a *well-defined*, algorithmically *computable*, mathematical function f in a well-defined mathematical model M of U,

(ii) by a physical 'quantum' computer $\mathcal{Q}$,

  − which admits *relatively* random (see §4.G., Definition 20) inputs/outputs,

(iii) can influence and/or determine the computational complexity of the *well-defined*, algorithmically *computable*, mathematical function f,

  − in the well-defined mathematical *model M of U*,
  − in which $\mathcal{Q}$ is not well-definable (since a *relatively* random output cannot be realised in M)!

'Untenable' since the underlying perspective here is that:

(iv) If U is the universe we inhabit:

  − and T is, say, the Taj Mahal,
  − whilst D is the set of categorical engineering specifications,
  − for building a physical model M of T in which we can perform repeated destructive experiments;

(v) Then, as observers that exist in U:

  − whilst we can talk categorically about the properties of T representable in M;
  − we can only conjecture/postulate;
      * on the basis of statistical correlations,
      * between what we observe in U,
      * and what we measure in M;
  − to what extent M is a representative model;
      * not of T;
      * but of *that* part of T;

       – which we can observe in U;

       – and well-define in D;

       – without destroying T.

(vi) We cannot, however, claim that any unpredictable structural change in T instantaneously causes a corresponding structural change in M!

In other words:

(a) If there are *hidden* features H of T that are not observable in U, but which essentially influence any measurements that we make concerning T in U, then such features, by definition, are not represented/representable in D or M.

(b) Since it is axiomatic that the only 'true' model of T is T itself then, as Diederik Aerts and Massimiliano Sassoli de Bianchi have, albeit implicitly, argued in [AdB14] (see §19.B.c.), repeated 'destructive' measurements in M are essentially separated spatially, and cannot, even in principle, yield deterministic definitions of all such properties H that can be built into a more complete model M' of T, even though T itself might exist in a deterministic universe.

(c) The conclusion being that even if such a universe can be said to obey deterministic physical laws that can be postulated as independent of us, since we can reflect them mathematically in D by algorithmically *computable* (hence deterministic and predictable) functions and relations, the property of being *predictable* is only well-defined with respect to M, and makes no sense in U in cases where, as Aerts and de Bianchi have argued, it is not falsifiable in M since it involves spatial separation in destructive experiments upon M.

(d) The most that we can claim from our observations is that there might be deterministic physical laws that are not representable in D by algorithmically *computable* (hence stochastically *deterministic* and *predictable* in repeated destructive experiments on M) functions and relations but, as shown in [An16] (Theorem 2.1, p.37), such laws might be representable in D by functions and relations that are algorithmically *verifiable* (hence stochastically *deterministic* by repeated destructive experiments on M), but (see also Bohr's argument in §19.B.) *not* algorithmically *computable* (hence stochastically *unpredictable* by repeated destructive experiments on M) as would be required by Bell's inequalities.

21.A.a. Deutsch's *oracle* argument appeals to algorithmically *verifiable*, but not algorithmically *computable*, functions

In a subsequent paper [DJ92], David Deutsch and Richard Jozsa sought to justify the argumentation of [Deu85] by introducing the concept of a mathematical 'oracle' that—from the *evidence-based* perspective of this investigation—can *apparently* carry out computations on algorithmically *verifiable*, but not algorithmically *computable* functions which, by definition, could not be computed by a Turing machine:

"A class of problems is described which can be solved more efficiently by quantum computation than by any classical or stochastic method. The quantum computation solves the problem with certainty in exponentially less time than any classical deterministic computation.

The operation of any computing machine is necessarily a physical process. Nevertheless, the standard mathematical theory which is used to study the possibilities and limitations of computing (e.g. based on Turing machines) disallows quantum mechanical effects, in particular the presence of coherent superpositions during the computational evolution. A suitable notion of a *quantum computer*, which, like the Turing machine, is idealized as functioning faultlessly and having an unlimited memory capacity, but which is able to exploit quantum effects in a programmable way, has been formulated by one of us (Deutsch 1985). Quantum computers cannot compute any function which is not turing-computable, but they do provide new modes of computation for many classes of problem. In this paper we demonstrate the importance of quantum processes for issues in computational complexity. We describe a problem which can be solved more efficiently by a quantum computer than by any classical computer. The quantum computer solves the problem with certainty in exponentially less time than the expected time of any classical stochastic computer.

Let U_f be a device that computes a function $f : Z_m \to Z_n$. Given an input i, U_f will, after some time, output the value of f_i. In general terms the class of computational task which we shall be considering involves being given U_f and then using it to determine some property $G[f]$ (that is, some function of the sequence $f(0), f(1), \ldots, f(m-1)$) in the least possible time.

In the analysis of this type of task, it is often an excellent approximation that the internal workings of U_f are inaccessible, in which case U_f is known as an *oracle* for f. The approximation would be nearly exact if U_f were a new type of physical object with an unknown law of motion.

If U_f were simply a program for evaluating f on our computer, making the approximation is tantamount to assuming that there is no faster method of obtaining $G[f]$ from the program U_f (e.g. by a textual analysis) than actually executing U_f to obtain sufficiently many values $f(i)$ to determine $G[f]$. It seems obvious that this is true for all properties G—obvious, but like $P \neq NP$, hard to prove.

If U_f were a ROM (read-only memory) containing a sequence of m integers from Z_n, the approximation is that there is no faster way of obtaining $G[f]$ from U_f than reading from the ROM sufficiently many values $f(i)$ to determine $G[f]$. This is clearly not true in general—there could be physical ways of measuring $G[f]$ directly, like measuring the total spin if the values of values $f(i)$ were stored as individual spin values—but it is a good description in many realistic situations.

It is useful to classify computational tasks into evaluations of *functions* and solutions of *problems*. In the case of functions, the task is to obtain the unique output that is the specified function of the input. For example, U_f, as we have defined it, evaluates the function f. In the case of solving problems the task is to obtain any one output that has a specified property. For example, to find a factor of a given composite number is a problem. Finding the least prime factor is a function evaluation.

When a classical deterministic (Turng) computer solves a problem, it always does so by evaluating a function. For example, a factorization program will always find the same factor of a given input. Which factor it finds could be specified by an additional constraint, narrowing the task to a function evaluation. Therefore when solving problems a classical computer cannot help performing a harder computational task than the one it was set.

A stochastic computer (i.e. one containing a hardware random number generator) need not always evaluate functions because the course of its computation, and therefore its output, need not be uniquely determined by the input. However, this gives a stochastic computer little advantage over a Turing one in solving problems, for if every possible output of a stochastic computation has the specified property that solves the problem, what is the purpose of choosing numbers randomly in

the course of the computation? One reason might be that there is a deterministic algorithm for solving the problem, which takes a parameter, and the running time depends on the parameter. If most values of the parameter give a short running time, but there are exceptional ones, which cannot easily be predicted, which give a long running time, it might be desirable to choose the parameter randomly if one wanted to reduce the expectation value of the running time.

A quantum computer (Deutsch 1985) is one in which quantum-mechanical interference can be harnessed to perform computations. Such a computation also need not necessarily evaluate functions when it is solving problems, because the state of its output might be a coherent superposition of states corresponding to different answers, each of which solves the problem. This allows quantum computers to solve problems by methods which are not available to any classical device.

Let us assume that, however U_f works, its operation is a coherent quantum mechanical process. Of course all physical processes conform to this assumption at some sufficiently complete level of description, possibly including their environment. But we mean that U_f can conveniently be made part of the coherent computation of a quantum computer.

Let $\mathcal{H}_{mn}$ be a Hilbert space of dimension mn and let

$$\{|i,j\rangle\}(i \in \mathbf{Z}_m, j \in \mathbf{Z}_n) \tag{1}$$

be a fixed orthonormal basis in $\mathcal{H}_{mn}$. Suppose that U_f operates by accepting input in any state $|k,0\rangle$ of the basis, representing the value k, and converting it to output in the state $|k,f(k)\rangle$, from which the value $f(k)$ can be read off with probability 1. More generally, we may suppose that U_f effects the unitary evolution

$$|i,j\rangle \overset{U_f}{\rightarrow} |i,j+f(i)\rangle, \tag{2}$$

where the addition in the expression $j + f(i)$ is performed modulo n. Then, by the linearity of quantum evolution, U_f will evolve the input state

$$m^{-\frac{1}{2}}(|0,0\rangle + \ldots + |m-1,0\rangle) \tag{3}$$

to the output state

$$m^{-\frac{1}{2}}(|0,f(0)\rangle + \ldots + |m-1,f(m-1)\rangle). \tag{4}$$

Thus, by running U_f only once, we have in some sense computed all m values of f, in superposition. Elementary quantum measurement theory shows that no quantum measurement applied to the system in the state (4) can be used to obtain more than one of the m values $f(0), \ldots, f(m-1)$. However, it is possible to extract some joint properties $G[f(0), \ldots, f(m-1)]$ of the m values, by measuring certain observables which are not diagonal in the basis (1). This is called the method of *computation* by *quantum parallelism* and is possible only with computers whose computations are coherent quantum quantum processes. For examples see Deutsch (1985) and Jozsa (1991)."

. . . Deutsch/Jozsa: [DJ92], *pp.553-555.*

From the *evidence-based* perspective of this investigation, we note the dichotomy in the above argument, which we treat as essentially postulating that, for an appropriate, algorithmically *computable*, function f:

(i) we can design a physical quantum computer with an associated oracle U_f such that, for any given natural number m, it can be in the physical states, say $(3')$ and $(4')$,

(ii) which are mathematically representable by the algorithmically *computable* formulas (3) and (4) respectively in Deutsch's above argument,

(iii) where the formula (3) mathematically entails that the algorithmically *computable* formula (4) *cannot* (as is argued for integer factorisation in §18.A.f., Theorem 18.15) be computed in deterministic polynomial time (number of discrete computational steps/operations mathematically) in m,

(iv) even though U_f claims, *without evidence*, to have evolved state (3′) to state (4′) in deterministic polynomial time (discrete computational steps/operations physically) in m since, as argued by Deutsch and Jozsa in [DJ92]:

- elementary quantum theory shows that no physical quantum measurement applied to the system in the putative physical state (4′) can be used to obtain more than one of the m values $f(0), \ldots, f(m-1)$,

- whence any physical measurement seeking to *evidence* the *actual* (in the sense of §19.B.d.) machine state (4′),

- would result in 'destroying the information about f' contained in the physically 'unmeasured' state (4′).

Deutsch and Jozsa seek to justify this dichotomy by differentiating between classical and quantum computing, and arguing further that there are problems for which 'quantum parallelism gives a solution with certainty in a given time, and it is absolutely more efficient than any classical or stochastic method':

> "To date, all known computational tasks which can be performed more efficiently by quantum parallelism than by any classical method have the following two properties. Firstly, the answer is not obtained with certainty in a given time; that is, there is a certain probability that the program will report that it has failed., destroying the information about f, so that in general it has to be run repeatedly before the answer is obtained. Secondly, although on some occasions it runs faster than any classical algorithm, the quantum algorithm is on average no more efficient than a classical one. It can be shown (Deutsch 1985) that the second property must hold for at least one choice of input in the quantum computation of any function.
>
> It is the purpose of this communication to describe a problem for which quantum parallelism gives a solution with certainty in a given time, and it is absolutely more efficient than any classical or stochastic method.
>
> The problem is as follows: Given a natural number N and an oracle U_f for a function $f : \boldsymbol{Z}_{2N} \to \boldsymbol{Z}_2$, find a true statement in the list:
>
> (A) f is not a constant function (at 0 or 1);
> (B) the sequence $f(0), \ldots, f(2N-1)$ of values of f does not contain exactly N zeros.
>
> Note that for any f, at least one of (A) or (B) is always true. It may be that both are true, in which case either (A) or (B) is an acceptable solution. That is why the solution of this problem is not necessarily tantamount to the computation of a function. A stochastic or quantum algorithm for solving it may have the property that when (A) and (B) are both true, it returns either answer, randomly. But when only one of them is true, the algorithm must return that one with certainty.
>
> Consider first the classical solution. We repeatedly run U_f to calculate values of f in some order, say $f(\boldsymbol{\Pi}(0)), f(\boldsymbol{\Pi}(1)), f(\boldsymbol{\Pi}(2)), \ldots$ where $\boldsymbol{\Pi}$ is a permutation on $\boldsymbol{Z}_{2N}$. This is always achieved in at most $N + 1$ invocations of U_f, though many functions f will require fewer invocations. Representing a function f by the $2N$-sequence $f(\boldsymbol{\Pi}(0)), \ldots, f(\boldsymbol{\Pi}(2N-1))$ of zeros and ones, we have the results of table 1.

Hence, given a large number of random fs, the average number of invocations of U_f required to solve the problem for each f is

$$\frac{N+1}{2^{N-1}} + \sum_{n=2}^{N} n(\tfrac{1}{2})^{n-1} = 3 - \frac{1}{2^{N-1}} \tag{5}$$

i.e. approximately three invocations for large N. If we are exceptionally unlucky, or if the fs are not presented randomly, but perversely by someone who knows what algorithm we are going to use, we shall require $N+1$ invocations. With a classical stochastic computer we can choose the permutations $\mathbf{\Pi}$ randomly, a process which requires $O(\text{In}(N))$ steps on average, and can thereby expect to solve the problem in approximately three invocations, though again in unlucky cases this may rise to $N+1$ invocations, plus an overhead of $O(N\text{In}(N))$ steps.

Now we present a method of solution using quantum parallelism. Let S be the unitary operation defined by

$$S|i,j\rangle = (-1)^j |i,j\rangle. \tag{6}$$

This operation can be performed by a quantum computer (cf. Deutsch 1985) fixed number of steps, independent of N and f. The state

$$|\phi\rangle = \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} |i,0\rangle \tag{7}$$

can be prepared, starting with the 'blank' input $|0,0\rangle$, in $O(\text{In}(N))$ steps, independently of f. For example, if $2N$ is a power of two, this could be done by applying the elementary one-bit transformation

$$|x\rangle \rightarrow \frac{1}{\sqrt{2}}(|x\rangle + (-1)^x |1-x\rangle) \quad (x \in \mathbf{Z}_2) \tag{8}$$

successively to each of the $\log_2(2N)$ bits that hold the value i in (7).

Given a quantum oracle U_f, apply the three operations U_f, S, U_f successively to the memory locations prepared in the state $|\phi\rangle$. Then, from (1), (6) and (7) the evolution is

$$
\begin{aligned}
|\phi\rangle &\overset{U_f}{\rightarrow} && \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} |i, f(i)\rangle \\
&\overset{S}{\rightarrow} && \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} (-1)^{f(i)} |i, f(i)\rangle \\
&\overset{U_f}{\rightarrow} && \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} (-1)^{f(i)} |i, 0\rangle \equiv |\psi\rangle.
\end{aligned}
\tag{9}
$$

The magnitude of the inner product

$$|\langle\phi|\psi\rangle| = \frac{1}{2N} \left| \sum_{i=0}^{2N-1} (-1)^{f(i)} \right| \tag{10}$$

is zero when statement (B) is false, and unity when statement (A) is false. Therefore if, after performing the operations in (9), we measure the projection observable $|\langle\phi|\psi\rangle|$, and the outcome is 0, we can be sure that $|\psi\rangle$ was not parallel to $|\phi\rangle$, and hence that (A) is true. And if the outcome is 1, we can be sure that $|\psi\rangle$ was not orthogonal to $\phi\rangle$, and hence that (B) is true. The outcome must be either 0 or 1, because those are the only eigenvalues of any projection observable. Therefore the procedure cannot fail to establish the truth of either (A) or (B).

The measurement of $|\phi\rangle\langle\phi|$ can be performed in $O(\text{In})$ steps, by first performing the inverse of the transformation which prepared $|\phi\rangle$ from a blank input $|0,0\rangle$, and then measuring the observable $|0,0\rangle\langle0,0|$, which is simply a matter of measuring each bit independently. The oracle U_f is invoked exactly twice in (9), and no other invocations are required. This is a clear improvement over the

worse case ($N + 1$ invocations) for either of those methods. Note that the problem is solved on
each occasion with certainty."

…*Deutsch/Jozsa: [DJ92], pp.555-557.*

However, we note that the physical operation of any quantum computer in the problem
considered above in [DJ92]—where f is, albeit implicitly, assumed to be an algorithmically
computable function—is essentially equivalent to:

(a) a finite (albeit within a potentially infinite) sequence of, say $2N$, masked light bulbs
$b_{i1}, b_{i2}, \ldots, b_{i2N}$—grouped in a combination of $\log_2(2N)$ qubits and/or qubit registers—
where each bulb is either on, or off, at any instant of time t_i; and

(b) a program that sequentially determines the state of the sequence of bulbs at any instant.

We note further that, since we are only in interested in the $2N$ values $f(0), f(1), \ldots, f(2N-1)$, the above argument can be modified to accommodate the more general case, in *evidence-based*
reasoning, where:

(c) the Church-Turing Thesis is false (see §4.G.a., Theorem 4.3); and

(d) Deutsch's Church-Turing principle is entailed by the definition §4.G.a., Definition 21, of
effective computability;

"Every finitely realizable physical system can be perfectly simulated by a universal model
computing machine operating by finite means."

…*Deutsch: [Deu85], Church-Turing principle, p.99.*

(e) since the number-theoretic function f can be an algorithmically *verifiable*, but not
necessarily algorithmically *computable*, function where:

— for any natural number n, we can consider instead of f the unique, primitive recursive,
algorithmically *computable*, Gödel β-function $g_{f,n}$ (see §12.A.) such that $f(i) = g_{f,n}(i)$
for all $0 \leq i \leq 2N \leq n$.

From the *evidence-based* perspective of this investigation, the assumption of an 'oracle' that,
for any requested natural number n, can provide the value of $f(n)$, would thus either involve
assuming prior knowledge of that which is to be computed if f is algorithmically *computable*,
or to the contradiction that this can be done in deterministic polynomial time otherwise (i.e., if
f is algorithmically *verifiable*, but not algorithmically *computable*).

21.A.b.   Why Deutsch's 'oracle' argument cannot appeal to 'information loss'

We note that, since every switching ON/OFF of a bulb necessarily generates some 'noise', we
may reasonably assume that, in principle, this 'noise' is detectable by an oracle which can,
moreover, record the bulbs' physical state $\ldots b_{i1} b_{i2} \ldots b_{i2N} \ldots$ at any time t_i as $r_{i1} r_{i2} \ldots r_{i2N}$,
where $b_{ij} = r_{ij}$ for $i \geq 0, 1 \leq j \leq 2N$.

Thus, given a quantum oracle U_f, consecutive application of the three operations U_f, S, U_f
successively to the memory locations prepared in the state $|\phi_{t_0}\rangle$—as described in the above
excerpt from [DJ92] as (9)—can be viewed as yielding the evolution:

$$|\phi_{t_0}\rangle \;=\; \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} |i,0\rangle \qquad\qquad Knowledge\ state\ at\ t_0$$

$$\ldots b_{01}b_{02}\ldots b_{02N}\ldots \qquad\qquad Bulbs'\ state\ at\ t_0$$

$$r_{01}r_{02}\ldots r_{02N} \qquad\qquad Recorded\ state\ at\ t_0$$

$$|\phi_{t_0}\rangle \;\overset{U_f}{\to}\; \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} |i,f(i)\rangle \qquad\qquad Knowledge\ state\ at\ t_1$$

$$\ldots b_{11}b_{12}\ldots b_{12N}\ldots \qquad\qquad Bulbs'\ state\ at\ t_1$$

$$r_{11}r_{12}\ldots r_{12N} \qquad\qquad Recorded\ state\ at\ t_1$$

$$\overset{S}{\to}\; \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} (-1)^{f(i)}|i,f(i)\rangle \qquad\qquad Knowledge\ state\ at\ t_2$$

$$\ldots b_{21}b_{22}\ldots b_{22N}\ldots \qquad\qquad Bulbs'\ state\ at\ t_2$$

$$r_{21}r_{22}\ldots r_{22N} \qquad\qquad Recorded\ state\ at\ t_2$$

$$\overset{U_f}{\to}\; \frac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} (-1)^{f(i)}|i,0\rangle \equiv |\psi\rangle. \qquad\qquad Knowledge\ state\ at\ t_3$$

$$\ldots b_{31}b_{32}\ldots b_{32N}\ldots \qquad\qquad Bulbs'\ state\ at\ t_3$$

$$r_{31}r_{32}\ldots r_{32N} \qquad\qquad Recorded\ state\ at\ t_3 \qquad\qquad (9')$$

In other words, even if the $2N$ bulbs are masked, and assuming that the status of each bulb can be either ON (defined mathematically as bulb-status 0 when the bulb is on) or OFF (defined mathematically as bulb status 1 when the bulb is off) with equal probability, we can reasonably assume that our knowledge of the initial physical 'state' $\ldots b_{01}b_{02}\ldots b_{02N}\ldots$ of the masked sequence of bulbs at time t_0 is given by:

$$|\phi_{t_0}\rangle = \tfrac{1}{\sqrt{(2N)}} \sum_{i=0}^{2N-1} |i,0\rangle$$

Clearly there is no loss in the information content of the recording $r_{i1}r_{i2}\ldots r_{i2N}$ which is created by, and cannot at any stage alter/affect, the bulbs' physical state $\ldots b_{i1}b_{i2}\ldots b_{i2N}\ldots$.

Moreover, since the recorded states in the evolution $(9')$ described above would effectively be measurements of the states of the quantum computer during computation, they would contradict Deutsch's stricture:

> "$\ldots \mathcal{Q}$ must not be observed before the computation has ended since this would, in general, alter its relative state."
>
> $\ldots$ *Deutsch*: [Deu85], *2 Quantum computers.*

From the *evidence-based* perspective of this investigation, we would further argue that, since all the operations of the quantum computer in the above example are determined by the instructions defined by U_f and S, we may conclude from the above perspective/gedanken that the computational complexity of the problem considered above in [DJ92] is also determined completely by well-defined theoretical functions, and is not dependent upon any physical features of the quantum computer that are not represented completely in the mathematical formulation of the problem.

21.A.c.   Why is quantum computing faster than classical computing?

The question arises: How would such a perspective account for the observed quantitative computational advantages of quantum computing—such as those reported in [AAB19]—over classical computing?

We note that current paradigms—such as that of Deutsch and Jozsa in [DJ92]—view such advantages as reflecting a significant qualitative differentiation in the theoretical foundations underlying the technology, rather than only incremental quantitive technological improvements in the implementation of an existing theory:

> "It is interesting to compare the computational complexity of this problem relative with classical and quantum computers. In the classical case, polynomial equivalence class complexity theory (Garey & Johnson 1979) is based on deterministic (`DTM`) and non-deterministic (`NDTM`) Turing machine models. We first note the result (referred to as (*)) that for any classical solution of our problem, using a `DTM`, there exists a function $f : \mathbf{Z}_{2N} \to \mathbf{Z}_2$ which requires at least $N + 1$ invocations of the oracle. To see this, suppose that a `DTM` can solve the problem for every f using only $M \ll N$ invocations. Let f_c be a constant function so that statement (A) is false and the machine must conclude that statement (B) is true. Then for any M invocations, for inputs chosen in any way whatsoever, there exists a function g which agrees with f_c at all M choices, and has exactly N zero values. Since, by assumption, the M values constitute the only information that the `DTM` has about the function, it cannot distinguish U_{f_c} from U_g, i.e. it cannot conclude that statement (B) is true. The same argument applies to `NDTM`s, showing that the decision problem of whether B is true or not, is not in the class NP (though the corresponding problem for A is in NP but not in P).
>
> To assess the complexity of the problem consider first an idealized situation in which the oracle is deemed to deliver its result in one computational step, and not to contribute to the size of the problem's input. Then the problem is specified by giving N, which has size $O(\mathrm{In}N)$. Hence by (*), *exponential* time is required for its solution. The quantum solution, requiring only two invocations, and a time of $O(\mathrm{In}N)$ to set up the input state, solves the problem in *polynomial* time. Thus the problem is in QP, the quantum analogue of the class P."
>
> *. . . Deutsch/Jozsa: [DJ92], pp.557-558.*

The question, as Krzysztof Wójtowicz addresses in [Wo09], is then:

> "Why is such an increase in speed possible? This is because in QCT we make use of some special features of the quantum world. In the classical model (Turing machine) we only transform 0-1 sequences in a mechanical way, not appealing to the laws of quantum mechanics.[16] We do not need quantum mechanics in order to explain, how the 0-1 strings of bits evolve within the Turing machine (or the classical computer). Things are quite different in the case of quantum algorithms."
>
> *. . . Wójtowicz; [Wo09], §1.2. Decidable problems, p.319.*

where:

> "A Turing machine could be made of wood or steel and powered by horses or by steam, as no quantum phenomena are in use."
>
> *. . . Wójtowicz; [Wo09], §1.2. Decidable problems, p.319, footnote 16.*

From the foregoing *evidence-based* perspective, we would argue, however, that the claimed polynomial-time computation attributed to 'only two invocations' of a *physical* oracle U_f in a quantum computer by supposing that:

> "$\ldots U_f$ operates by accepting input in any state $|k, 0\rangle$ of the basis, representing the value k, and converting it to output in the state $|k, f(k)\rangle$, from which the value $f(k)$ can be read off with probability 1."
>
> *. . . Deutsch/Jozsa: [DJ92], pp.554-555.*

is illusory; and that any appeal to a *notional* oracle—as described in the evolution of state (3) to state (4) in [DJ92]—cannot influence the *actual* computational complexity in physically determining which of (A) or (B) is true in the problem posed in [DJ92].

In other words, we posit that any reduction in computational time by a quantum computer—such as claimed in [DJ92], [Wo09], and [AAB19]—results:

(a) from only quantitative technological improvements in the quality and quantity of the sub-routines built into the underlying circuitry:

— that are independent of any particular value of the function being computed; and

— which appeal to proofs within a formal first-order theory of Peano arithmetic such as PA for the validity of each sub-routine; and

— which, by the Provability Theorem for PA (§2.E., Theorem 2.17), do not take into account the 'computational time' equivalent of each such proof;

(b) not from any qualitative physical reductions in computational time resulting from a physical 'quantum' phenomena that is, both, essential to the physical computation, and is representable mathematically only statistically (such as, for instance, by Schrödinger's ψ-function).

We would further argue from the *evidence-based* perspective of this investigation, as reflected in §19.D.c., Thesis 16 and 19.D.d., Thesis 17, that:

Thesis 21. (Computational equivalence) *Any polynomial-time computation by a quantum computing device can be executed in polynomial time by a classical Turing machine.*

where we note that the mathematical representations of stochastic processes, such as the result of tossing a coin interminably, cannot be verified/computed by a quantum computer since such processes are assumed—and their representations defined—to be both indeterminate (have no algorithmically *verifiable* 'past') and unpredictable (have no algorithmically *computable* 'future').

Moreover, we could then tentatively define a quantum computer functionally by, say:

Definition 44. (Quantum verifier) *A physical device is a quantum verifier if it verifies only the first m values of an algorithmically verifiable, but not algorithmically computable, number-theoretic function f whose unique Gödel beta-function for m is g_m, where $g_i = f_i$ for $0 \leq i \leq m$, and the value $f(i)$ is determined by a probability function governed by a function such as Schrödinger's ψ-function.*

Definition 45. (Knowledge verifier) *A physical device is a knowledge verifier if, for any given m, it can verify the first m values of an algorithmically computable, number-theoretic, function f whose unique Gödel β-function for m is g_m, where $g_i = f_i$ for $0 \leq i \leq m$.*

Definition 46. (Quantum computer) *A physical device is a quantum computer if, and only if, it is a quantum verifier.*

Lemma 21.1. *A classical computer is a knowledge verifier but not a quantum verifier.*

Lemma 21.2. *A quantum computer is a knowledge verifier and a quantum verifier.*

21.A.d.   The significance of *information loss* in quantum computing

The significance of the evolution §21.A.a.(9′), in the context of the putative phenomena of 'information loss' when measuring the computation of a quantum computer under current paradigms, is vividly illustrated/explained by Krzysztof Wójtowicz's lucid, albeit philosophically oriented, perspective of Deutsch's reasoning in [Wo09] and [Wo19], where his focus is primarily the question of '*how* a quantum computation might, conceivably, enable us to make use of the very special features of the quantum world' in quantum computation theory (QCT):

> "From the classical point of view, the basic unit of information is a bit, which is either 0 or 1. A full description of the state of the bit is therefore given by a single number: 0 or 1.
>
> In QCT, the basic unit of information is a qubit—the quantum counterpart of the bit. Qubits are much more complicated than bits, they cannot be described by the Boolean values 0 and 1, as they can occupy more states. The classical Boolean values have their counterparts—two distinguished (basic) states of the qubit, usually denoted by $|0\rangle$ and $|1\rangle$. But qubits can be also in a superposition of $|0\rangle$ and $|1\rangle$—i.e. they are (speaking in informal terms) somehow in both states at the same time. Such a superposition is described by the expression $a_0|0\rangle + a_1|1\rangle$, where the co-efficients a_0 and a_1 are two complex numbers such that $|a_0|^2 + |a_1|^2 = 1$. From the formal point of view, a qubit is a vector of length 1 in a two-dimensional complex Hilbert space.[17] A physical realization of a qubit is e.g. a photon (there are numerous other examples), but in this paper we are not interested in the "hardware", but rather in the theoretical foundations of QCT.[18]
>
> The classical computation consists of computational steps, transforming the initial $0 - 1$ sequence of bits.[19] We just put the initial dat[a] on the tape of the Turing machine, and start the process. After a finite number of steps (provided the machine indeed halts, but here we restrict our attention only to such cases) the machine enters the terminal state, and the tape contains the solution. Of corse, at any moment of the computation, the tape contains a certain finite $0 - 1$ string.
>
> The technical details are not of primary importance here and will often be omitted. What is really interesting is *how* a quantum computation enables us to make use of the very special features of the quantum world."
>
> . . . *Wójtowicz: [Wo09], pp.320-321.*

Wójtowicz outlines the basic notions of QCT, in relatively non-technical terms, thus:

> "In QCT we consider not only single qubits, but also strings of qubits—quantum registers. Such registers have some quite special properties, which have no counterpart in the classical world. Quantum algorithms make use of these properties, and this makes them (at least in some cases) extremely powerful in comparison with the available classical algorithms.
>
> A single qubit has the form $a_0|0\rangle + a_1|1\rangle$—i.e. we need 2 complex numbers to describe its state. How many complex parameters are needed to describe the state of a quantum register—e.g. a system of 10 photons? Consider first a system of 10 points on the plane, and imagine that we are interested in the positions of the points only. For each of the points, we need two parameters (x, y) in order to describe its position, so 20 parameters are sufficient to provide a complete description of the system of 10 points. This is quite obvious: every point has a position on its own, independent of the other points, and in order to describe the position of a single point we do not have to worry about the other points (why should we?). Therefore, one might be tempted to think, that we also need 20 parameters in order to describe the quantum system of 10 qubits (photons). But in the quantum world things are not always so simple: it can happen, that the qubits constituting the quantum register do not have a state on their own. That means, that the register of 10 qubits is in a certain state *as a whole*, but it does not make sense to speak of the states of the individual qubits. In such cases we speak of quantum entanglement (a notion, which will be defined in a more precise way later). Because of that fact, the description of the quantum system is much more complicated than a person trying to apply intuitions from classical physics might expect. The dimension of the system increases exponentially with the increase of the number of qubits: in the

general case, the description of the quantum register consisting of n qubits requires 2^n parameters (as this is the dimension of the Hilbert space needed to provide the description).[20] Of course, we could not even dream of writing down such a description for $n = 100$.

This is very different from the case of classical bits of information. To understand the underlying mechanism, consider the example of two qubits treated as one single quantum system. The states of the qubits are $a_0|0\rangle + a_1|1\rangle$ and $b_0|0\rangle + b_1|1\rangle$ correspondingly. The state of the quantum register can be written as a product (a tensor product) of these two states:

$$(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$$

(we will [henceforth] omit the symbol $\otimes$). If we treat this as an algebraic expression and perform the multiplication, we obtain:

$$a_0 b_0|0\rangle|0\rangle + a_0 b_1|0\rangle|1\rangle + a_1 b_0|1\rangle|0\rangle + a_1 b_1|1\rangle|1\rangle$$

To simplify the notation, we will write $|00\rangle$ instead of $|0\rangle|0\rangle$; $|01\rangle$ instead of $|0\rangle|1\rangle$ *etc.* The result is:

$$a_0 b_0|00\rangle + a_0 b_1|01\rangle + a_1 b_0|10\rangle + a_1 b_1|11\rangle.$$

The vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis for the 2-qubit quantum register. The vector $a_0 b_0|00\rangle + a_0 b_1|01\rangle + |a_1 b_0|10\rangle + a_1 b_1|11\rangle$ describes the state of the 2-qubit register, such that the first qubit is in the state $a_0|0\rangle + a_1|1\rangle$, and the second in the state $b_0|0\rangle + b_1|1\rangle$. In general, the state of any two-qubit register can be described as : $c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ (where $c_{00}, c_{01}, c_{10}, c_{11}$ are complex numbers satisfying the condition $|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$).

Of course, the state $a_0 b_0|00\rangle + a_0 b_1|01\rangle + |a_1 b_0|10\rangle + a_1 b_1|11\rangle$ is simply a product of the two states: $(a_0|0\rangle + a_1|1\rangle)(b_0|0\rangle + b_1|1\rangle)$. But not every state of the 2-qubit register can be represented as a tensor product: in most cases, the vector $c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ does not allow for such a factorization.[21] In such cases we say, that the two qubits are *entangled*. The presence of entanglement is a very special feature of quantum systems and has no classical counterpart.[22]

For three (and more) qubits, the situation is analogous: consider the system consisting of three qubits: $a_0|0\rangle + a_1|1\rangle$, $b_0|0\rangle + b_1|1\rangle$, $c_0|0\rangle + c_1|1\rangle$. Again, we multiply them just like algebraic equations, obtaining the product, which can be written down (using the obvious abbreviations, e.g. $|010\rangle$ instead of $|0\rangle|1\rangle|0\rangle$ *etc.*) as:

$$a_0 b_0 c_0|000\rangle + a_0 b_0 c_1|001\rangle + a_0 b_1 c_0|010\rangle + a_0 b_1 c_1|011\rangle +$$
$$a_1 b_0 c_0|100\rangle + a_1 b_0 c_1|101\rangle + a_1 b_1 c_0|110\rangle + a_1 b_1 c_1|111\rangle.$$

The eight vectors $|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$ form a base of a 3-qubit register, and in the general case, the state of such a register can be presented as:

$$a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle +$$
$$a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle$$

If such an expression cannot be presented as a tensor product of three qubits $a_0|0\rangle + a_1|1\rangle$, $b_0|0\rangle + b_1|1\rangle$, $c_0|0\rangle + c_1|1\rangle$ we are again confronted with quantum entanglement: the qubits forming the register do not have a state of their own.[23] In the general case of an entangled n-qubit register, we need 2^n components in order to describe the state of this register (the dimensions of the corresponding Hilbert space is 2^n).[24] That shows in particular, that the computer simulation of the evolution of a quantum system in an efficient way is not possible: to describe the evolution of a system of n qubits, we would have to store (and describe the evolution of) 2^n values at once. Of course, this is not possible in the case of e.g. 300 qubits."

. . . *Wójtowicz: [Wo09], pp.321-323.*

Wójtowicz notes that a classical computation 'consists of Boolean gates, and each gate transforms the initial string of bits into an output (which is another string of bits)', whereas

the 'quantum counterpart of such a Boolean gate is a quantum gate, which acts either on one qubit or on a quantum register':

> "We can view a classical computation as a action of a Boolean network on initial data.[25] Such a circuit consists of Boolean gates, and each gate transforms the initial string of bits into an output (which is another string of bits). The quantum counterpart of such a Boolean gate is a quantum gate, which acts either on one qubit or on a quantum register. The actions of a quantum gate on a qubit can be described as:
>
> $$V : \ a_0|0\rangle + a_1|1\rangle \to b_0|0\rangle + b_1|1\rangle,$$
>
> where $a_0|0\rangle + a_1|1\rangle$ is the input (the initial state of the qubit), and $b_0|0\rangle + b_1|1\rangle$ is the output, i.e. the final state of the qubit (remember that a_0, a_1, b_0, b_1 are complex coefficients). In the general case a quantum gate transforms an n-qubit register, giving another register as an output.
>
> A quantum computation consists of a sequence of such transitions, performed on a quantum register. These quantum gates manipulate the information stored in the qubit, or in the quantum register (the system of qubits). From the mathematical point of view, quantum gates are linear unitary operators, i.e. they preserve the norm of the quantum state (but these technical details are not important here).[26]
>
> Every quantum gate is a linear operator on the appropriate Hilbert space (the dimension of this space is 2^n, where n is the size of the register). Due to the linearity of the operator, it is sufficient to define the action of the operator on the basic states $|0\rangle$ and $|1\rangle$ of the particular qubits.[27] A simple example is the Hadamard gate H, which acts in the following way:
>
> $$|0\rangle \to 1/\sqrt{2}(|0\rangle + |1\rangle),$$
> $$|1\rangle \to 1/\sqrt{2}(|0\rangle - |1\rangle).[28]$$
>
> A straightforward computation shows, that $H^2 = \text{Id}$."
>
> *. . . Wójtowicz:* [Wo09], *pp.323-324.*

Noting that every 'quantum gate is a linear operator on the appropriate Hilbert space', Wójtowicz defines a Hadamard gate H mathematically; and further notes its 'strange property' that, whereas its application once as a classically random device—in a coin-tossing gedanken— gives us an output with the expected classical probability, 'performing it twice gives us a deterministic outcome':

> "Now consider the following thought experiment, which will give us some insight into the peculiarities of the quantum world and of quantum computation. Imagine a random device, which— independently of the input (which is 0 or 1) gives us as output 0 or 1 with the same probability 1/2 (e.g. a fair coin toss). If such an operation is performed twice (i.e. we simply concatenate two such devices M) it is quite obvious, that the output will likewise be random: we just feed the first device with the input 0 or 1, perform the random operation, observe the output, pass the output to the second device, perform the second random operation and observe the output. Of course the second output is random, regardless of the first outcome.
>
> This is how the classical coin works. However, in the quantum world, strange things happen: we can set up a random operation U, with the strange property, that performing it twice gives us a deterministic outcome. This is of course very counterintuitive, as our intuitions are modeled by the classical (i.e. macroscopic) world. A classical coin like this does not exist. But the "quantum coin" acts in the following way:
>
> $$U(0) = 0 \text{ or } 1 \text{ with probability } 1/2,$$
> $$U(1) = 0 \text{ or } 1 \text{ with probability } 1/2.$$

But(!)

$$U^2(0) = 1,$$
$$U^2(1) = 0.\text{''}$$

... Wójtowicz: [Wo09], p.325.

Wójtowicz then uses as an analogy Louis H. Kauffman's 'square root of negation' ([Kau87], p.72; [KC19]) to illustrate, and argue, that we can 'harness quantum mechanics in order to produce effective information processing procedures', so that the outcome of a 'measurement allows us to "extract" information about the state of the system before the measurement' in Deutsch's algorithm:

> "From the logical point of view, the operation U^2 is the negation. So U can be viewed as the square root of the negation. In classical logic, such a logical operation does not exist. But there is a quantum device, which acts exactly in this way—i.e. it can be (in a sense) viewed as an experimental realization of $\sqrt{\text{NOT}}$.
>
> What is its formal counterpart? Consider the operator U, defined as:
>
> $$U : \ |0\rangle \rightarrow \tfrac{1}{2}(1-i)|0\rangle + \tfrac{1}{2}(1+i)|1\rangle$$
> $$U : \ |0\rangle \rightarrow \tfrac{1}{2}(1+i)|0\rangle + \tfrac{1}{2}(1-i)|1\rangle$$
>
> A straightforward computation shows, that $UU|0\rangle = 1$, and $UU|1\rangle = 0$. $U^2(p) = \neg p$, so U is the square root of the negation.
>
> But why do we claim, that $\sqrt{\text{NOT}}$, acting e.g. on photons is a random device? There is no sign of randomness in the definition of the operator U. Indeed, U acts on the vectors in the Hilbert space in a purely deterministic way. But in order to "extract" the information from the quantum system we have to perform a measurement. A measurement is probabilistic in nature, as stated by one of the basic postulates of quantum mechanics (which we remind here in a simplified form, concerning only qubits):
>
> - If we perform the measurement on a qubit, being in the state $a_0|0\rangle + a_1|1\rangle$, there are two possible outcomes: 0 and 1. The probability of obtaining the result 0 equals $|a_0|^2$, and the probability of obtaining 1 equals $|a_1|^2$. After the measurement the state of the quantum system is projected onto one of the basic states: if the outcome was 0 (resp. 1), the state after the measurement is projected onto $|0\rangle$ (resp. $|1\rangle$).[29]
>
> In particular, we usually cannot learn from the outcome of the measurement, what was the state of the system before the measurement (we cannot tell whether the state was e.g. $\tfrac{1}{2}(1-i)|0\rangle + \tfrac{1}{2}(1+i)|1\rangle$ or rather $\tfrac{1}{2}i|0\rangle + \sqrt{\tfrac{3}{4}}|1\rangle$). For example, if the outcome of the measurement was 0, the only information about the state of the system *before* the measurement, is that $a_0 \neq 0$. After the measurement the state of the system collapses to $|0\rangle$ (and of course the outcome of the next measurement will be 0 with probability 1). So (with few exceptions), the act of measurement causes changes in the state of the system."
>
> *... Wójtowicz: [Wo09], pp.325-326.*

Ostensibly, the above argument seems to implicitly imply that the physical act of measurement of a quantum phenomena *cannot* be represented in any mathematical model of the phenomena in a way that would allow us to 'learn from the outcome of the measurement, what was the state of the system before the measurement'.

If so, this would entail that—contrary to what is claimed—the operator U is *not* the formal counterpart of: 'a quantum device, which acts exactly in this way—i.e. it can be (in a sense) viewed as an experimental realization of $\sqrt{\text{NOT}}$'.

Comment: We note that, from the *evidence-based* perspective of this investigation, a quantum phenomena *can*, however, be viewed (see §19.D.c., Thesis 16, and §19.D.d., Thesis 17) as represented mathematically by functions which are algorithmically *verifiable* (hence determinate), but not algorithmically *computable* (hence not predictable).

Moreover, the assumption that the physical act of measurement of a quantum phenomena cannot be represented in any mathematical model of the phenomena would not admit the significance of the central Complementarity Thesis (§1., Thesis 1) of this investigation; nor that of the distinction between the interdependent roles of natural scientists, philosophers, and mathematicians, as suggested in §10.C.

Presumably aware of what may be seen as the implicit implications of the above argumentation, Wójtowicz seeks to further explain the actions of the physical $\sqrt{\text{NOT}}$ operator thus:

"Let us turn back to $\sqrt{\text{NOT}}$. It transforms $|0\rangle$ to $\frac{1}{2}(1-i)|0\rangle + \frac{1}{2}(1+i)|1\rangle$. If we now perform the measurements, we will obtain 0 or 1 with equal probabilities $\frac{1}{2}$. That means, that the procedure consisting of:

1. preparing the quantum system in the stat$|0\rangle$,
2. applying the $\sqrt{\text{NOT}}$ operation to this system,
3. performing the measurement,

is a purely random procedure, just like tossing a random coin. But if we *do not* perform the measurement after the first application of $\sqrt{\text{NOT}}$, but apply $\sqrt{\text{NOT}}$ again, the whole procedure will transform 0 into 1 and 1 into 0 in a deterministic way. This looks strange, but remember, that we do not perform the measurement after the first application of the $\sqrt{\text{NOT}}$ gate, but we transfer the result to the second gate—and the measurement is performed *after* the second application of $\sqrt{\text{NOT}}$. Of course, if we measured the state of the quantum state after the first $\sqrt{\text{NOT}}$, the state of the quantum system would collapse (become either $|0\rangle$ or $|1\rangle$), so the input of the second gate would be either $|0\rangle$ or $|1\rangle$, and the second measurement would yield either 0 or 1 with the same probability.

As was already mentioned before, in the general case the measurement does not give any information about the state of the system *before* the measurement. So, if we perform a quantum computation and afterwards perform the measurement of the system, we usually will not be able to tell, what was the state of the system *before* the measurement. That means, that in the general case during the measurement we lose the information that was obtained in course of the computation. But in some interesting cases, some additional information about the evolution of the system will make it possible to deduce the final state of the system from the result of the measurement—and in these cases we will be able to harness quantum mechanics in order to produce effective information processing procedures."

... *Wójtowicz: [Wo09], pp.326-327.*

However, if we accept that the actual *physical* state—as well as the *physical* act of measurement—of a quantum phenomena *cannot* be represented in any classical mathematical model of the phenomena—i.e. a model which admits only algorithmically *computable* functions/relations (in the sense of §2., Definition 3)—then we note the anomaly in the claim that (see also §21.A.b.) *choosing* to remain ignorant of the outcome of a *physical* quantum measurement is equivalent to the outcome being *essentially unknowable*; a claim implicit in the above argumentation, which is essentially that:

(i) the *physical* (circuit) operator $\sqrt{\text{NOT}}$ transforms the *physical* state of a system S, *denoted* symbolically by $|0\rangle$, to a *metaphysical* (abstract) state k_0 of our *knowledge* of the system,

(ii) *represented* mathematically within a formal system L by $\frac{1}{2}(1-i)|0\rangle + \frac{1}{2}(1+i)|1\rangle$;

since:

(iii) if we choose then to perform a *physical* measurement m_0 on the *physical* system S,

(iv) we will obtain a new *physical* state of S, denoted by 0 or 1, each with equal probability $\frac{1}{2}$,

(v) on the basis of a single measurement,

whilst:

(vi) if we choose not to perform the *physical* measurement m_0 on the *physical* system S,

(vii) we can treat the new *physical* state of S as being denoted by $\frac{1}{2}(1-i)|0\rangle + \frac{1}{2}(1+i)|1\rangle$;

where:

(viii) we have prior (*metaphysical*) mathematical *knowledge* that the *physical* operation of $\sqrt{}$NOT transforms the *physical* state of the system, *denoted* symbolically by $|0\rangle$,

(ix) under a mathematically well-defined probability distribution within L,

(x) which is *postulated* as mathematically representing—within L—the *physical* measurements of repeated, non-terminating, *physical* application of the *physical* (circuit) operator $\sqrt{}$NOT to the *physical* state of the system S denoted by $|0\rangle$.

In other words, Wójtowicz seems to be essentially arguing that the *physical* outcome of a *physical* (circuit) operator $\sqrt{}$NOT *physically* transforming the *physical* state of a system S does not constitute a *measurement* even if—as argued in §21.A.b.—the *putative* transformed *physical* state of S can, in principle, be recorded faithfully.

> **Comment**: From the *evidence-based* perspective of this investigation, however:
>
> (a) *choosing to measure* the system S at the step denoted by §21.A.d.(iii) simply means that we have appealed to the oracle to transform our *ignorance* of the algorithmically *computable* state of the system at that point to *knowledge* of the state that is already known/available to the oracle (and, in principle, to us, since the state is assumed algorithmically *computable*), as we *need* this knowledge for completing the desired computation;
>
> (b) *choosing not to measure* the system S at the step denoted by §21.A.d.(vi) simply means that we have *not* appealed to the oracle to transform our *ignorance* of the algorithmically *computable* state of the system at that point to *knowledge* of the state that is already known/available to the oracle (and, in principle, to us, since the state is assumed algorithmically *computable*), as we *do not need* this knowledge for completing the desired computation;
>
> From a logical perspective, the difference here is between *choosing* to know '*the x*'—denoted symbolically by 'ιx' (see [Me64], p.85)—and *choosing* to know *only* that 'there exists *an x*'—denoted symbolically by '$\exists x$' (see [Me64], p.45).
>
> Obviously, any computation that only requires knowledge of the latter for completing a computation *could* be significantly faster than one where we, unnecessarily, compute the former.
>
> Moreover, use of a *physical* (circuit) operator $\sqrt{}$NOT *physically* for transforming the *physical* state of a system S is not essential to quantum computation per se, but is only needed where the algorithmically *computable* state, whose knowledge the oracle is called upon to furnish, appeals to the formal representation of the *physical* $\sqrt{}$NOT operation in the language of algorithmically *computable* functions.

Wójtowicz argues further that admitting such a distinction between the *physical* state of a system (i.e., what it *actually* is in Aerts' sense; see §19.B.d.), and *knowledge* of the *physical* state of a system, 'allows us to "extract" information about the state of the system before the measurement':

> "Two simple examples may illuminate the point:
>
> (a) Consider one qubit, which is known *in advance* to be in one of the two states from the computational basis (i.e. it is either $|0\rangle$ or $|1\rangle$, but we do not know which one). Of course, in that case the outcome of the measurement gives us information about the internal state.
>
> (b) Consider a 2-qubit quantum register, which is known *in advance* to be in one of the two states:
>
> $$\Phi_0: \ 1/\sqrt{2}(|00\rangle + |01\rangle),$$
>
> $$\Phi_1: \ 1/\sqrt{2}(|10\rangle + |11\rangle),$$
>
> If we perform the measurement on the *first* qubit, we will obtain the complete information about the state of the *whole* register, as we could obtian 0 only when the state of the register is Φ_0 (and analogously, *only* Φ_1 can result in obtaining 1). That means, that in some cases the outcome of the measurement allows us to "extract" information about the state of the system before the measurement. We can make use of this fact in quantum algorithms, the simplest of which is Deutsch's algorithm."
>
> *. . . Wójtowicz:* [Wo09], *pp.326-327.*

Wójtowicz then seeks to illustrate how the above distinction admits significant quantitative advantages of quantum computing over classical, Turing-machine based, computing (which could, arguably, be taken to account for the fewer computational steps evidenced in [AAB19]):

> "**2.4 Examples of quantum algorithms**[30]
>
> Stated in colloquial terms, we have to decide, whether a coin is a genuine coin or not (in which case it has two tails or two heads). Of course, classically we have to look at the coin twice. The mathematical counterpart is as follows: for a given function $f: \{0,1\} \to \{0,1\}$, we have to find out, whether the function is constant ($f(0) = f(1)$—this corresponds to the fake coin), or whether it is balanced ($f(0) \neq f(1)$—a genuine coin). How many function evaluations are necessary in order to find out, whether f is constant or balanced? Classically—of course two: we have to evaluate both the values $f(0)$ and $f(1)$, and then compare them. In the quantum world we can do this in one quantum step.
>
> Consider a quantum "black box" U_f, which acts on a two-qubit register in the following way:
>
> $$U_f: \ |x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle,$$
>
> where $\oplus$ is addition *modulo* 2 (i.e. $0 \oplus 0 = 0; 0 \oplus 1 = 1; 1 \oplus 0 = 1; 1 \oplus 1 = 0$).[32]
>
> The Deutsch algorithm proceeds as follows (to simplify matters I will omit all the coefficients like $1/\sqrt{2}, \frac{1}{2}$ *etc.*):
>
> 1. We prepare two qubits in initial states $|0\rangle$ and $|0\rangle - |1\rangle$, so the state of the register can be written down as the product $|0\rangle(|0\rangle - |1\rangle)$. We apply the Hadamard gate to the first qubit, and identity (i.e. do nothing) to the second qubit:[33]
>
> $$H: \ |0\rangle(|0\rangle - |1\rangle) \to (|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$$
>
> 2. Now we apply the quantum gate U_f. A straightforward computation shows, that:
>
> $$U_f: \ (|0\rangle + |1\rangle)(|0\rangle - |1\rangle) \to ((-1)^{f_0}|0\rangle + (-1)^{f_1}|1\rangle))(|0\rangle - |1\rangle).$$

The state of the first qubit is therefore

$$(-1)^{f_0}|0\rangle + (-1)^{f_1}|1\rangle)$$

(the state of the second qubit is $|0\rangle - |1\rangle$, but that is of no further relevance to our problem). There are two possibilities:

(i) If $f(0) = f(1)$, then the state of the first qubit is either $|0\rangle + |1\rangle$ (this happens when $f(0) = f(1) = 0$), or $-(|0\rangle + |1\rangle)$ (this happens when $f(0) = f(1) = 1$).

(ii) If $f(0) \neq f(1)$, then the state of the first qubit is either $|0\rangle - |1\rangle$, or $-(|0\rangle - |1\rangle)$.

Now we apply the Hadamard gate again to the first qubit. The result is:

- if (i) was the case: $|0\rangle$ or $-|1\rangle$,
- if (ii) was the case: $|1\rangle$ or $-|1\rangle$.

If we now perform the measurement on the first qubit, we obtain either 0 or 1, and in this particular case we are able to deduce, whether (i) or (ii) took place. This is because we knew in advance, that the qubit had to be in one of four particular states before the final measurement. Observe, that in this algorithm we evaluated the function f *only once*. The measurement is performed *after* the algorithm terminates—otherwise we would cause the measured qubit to collapse and therefore destroy the computation."

. . . *Wójtowicz: [Wo09], pp.328-329.*

Wójtowicz notes further that such quantitative advantages of quantum computing over classical, Turing-machine based, computing, could, conceivably, also translate into qualitative advantages of reducing non-deterministic polynomial time computations to deterministic polynomial-time:

"The Deutsch algorithm is interesting, but it seems somehow artificial, and the increase in speed (1 call of the function f instead of 2 calls) is not very spectacular. But it has an interesting generaliszation: the Deutsch-Jozsa algorithm. Here we have a function $f : \{0,1\}^n \to \{0,1\}$ (so any $0-1$ sequence of the length n is mapped either on 0 or 1). We know in advance, that f is either constant (all the values of f ae 0 or all the values are 1), or balanced (which means, that f takes as many times the value 0 as the value 1). In the classical algorithm we have to call f approximately 2^{n-1} times.[34] But the Deutsch-Jozsa algorithm solves the problem in polynomial time."

. . . *Wójtowicz: [Wo09], pp.329-330.*

The consequences, Wójtowicz notes, could provide 'a theoretical possibility of cracking the RSA code, the security of which rests on the assumption, that factoring is hard' (an assumption sought to be justified by the argumentation in §18.A.f.; as a consequence of the *evidence-based* proof—in §18.A.c., Theorem 18.11—that the prime divisors of an integer are mutually independent):

"The by far most impressive example of a quantum algorithm is Shor's algorithm,[35] which can be called one of the cornerstones of quantum computation theory. Shor's algorithm shows, how a problem believed to be hard becomes easy by referring to quantum mechanics.

The algorithm deals with the factorization problem, which belongs to the class **NP**: given a solution we can check within a polynomial time whether it in fact is the desired solution, but factoring a number into primes is a complicated task (try this with e.g. 1062347—which is not a very big number). Every known classical algorithm for factoring requires an exponentially increasing number of steps. This fact is exploited in cryptographic protocols: the security of these

> protocols relies on the assumption, that factoring is intractable. We will not go into the technical details of Shor's algorithm—it consists of a classical and a quantum part. In the classical part we exploit some number-theoretic results (concerning finding prime factors by determining a period of a certain function). The problem of factorization reduces to the problem of finding the period of a certain periodic function. This problem can be solved efficiently by Shor's quantum algorithm. In particular, Shor's algorithm provides a theoretical possibility of cracking the RSA code, the security of which rests on the assumption, that factoring is hard."
>
> ... *Wójtowicz:* [Wo09], *p.330.*

The significance of the *evidence-based* argumentation highlighting—and challenging as above—the implicit premises in current paradigms concerning the claimed (as in [AAB19]) quantitative, and qualitative, advantages of quantum computing over classical, Turing-machine based, computing lies in Wójtowicz concluding remarks (see also [Wo19]) that 'the area of quantum computing has a profound impact on our understanding of some classical philosophical and methodological notions' that could impact on 'the status of mathematical knowledge compatible with the advances in QCT':

> "Quantum computers could be very powerful—so why there are none of them available? Not going into technical details, quantum registers are extremely fragile. One of the most technical problems is to prevent the surrounding environment from interacting with the qubit registers. In a sense, we have to encapsulate the quantum computer and prevent it from losing the information in the environment (a decoherence, which would destroy the computation). That means, that there are formidable technical problem to be overcome before a quantum computer can be build.[36]
>
> But in spite of these practical problems, I think that the area of quantum computng has a profound impact on our understanding of some classical philosophical and methodological notions. These issues will be discussed in the subsequent paper. In particular, I will discuss the philosophical impact which this theory has on philosophy of mathematics, and—in particular—I will examine the thesis, that the best explanation of the status of mathematical knowledge compatible with the advances in QCT is offered by the quasi-empiricist stance (which incorporates mathematical knowledge into our "web of belief", including also scientific knowledge)."
>
> ... *Wójtowicz:* [Wo09], *pp.330-331.*

21.A.e.   A philosophical perspective of quantum computing

In a 2019 continuation [Wo19] of his earlier, more technical, paper [Wo09], Wójtowicz seeks to discuss:

> "...the philosophical significance of quantum computation theory for philosophy of mathematics, in particular for the following issues: the nature of mathematical proof; the potential empirical aspects of mathematics and the realism-antirealism debate (in the context of the indispensability argument)."
>
> ... *Wójtowicz:* [Wo19], *Introduction, p.173.*

where he remarks that:

> "The investigations within the paper concern mainly quantum computation, but they apply to the general problem of the relationship between mathematics and physics. I argue, that the possibility of quantum proofs present serious difficulties for the recently much discussed Enhanced Indispensabity Argument (EIA) for mathematical realism—and propose a way of solving the emerging problems. In particular, I claim that the best philosophical account is quasi-empiricism in Quine's manner. The paper therefore gives a support for the realistic account of mathematics."
>
> ... *Wójtowicz:* [Wo19], *Introduction, p.173.*

Wójtowicz also comments on his intent in [Wo19]:

> "One of the motivations for investigating quantum-computational models is the intractability of many computational (combinatorial, number-theoretical, graph-theoretical etc.) problems.[1] An important example of such a intractable problem is factorization, where no quick, (i.e. polynomial) classical algorithm is known—but there is a quick quantum algorithm [Shor, 1994]."
>
> *. . . Wójtowicz: [Wo19],§1. General remarks on quantum algorithms, p.174.*

From the *evidence-based* perspective of this investigation, the significance of Wójtowicz's intent lies in his recurring appeal to the premise that:

— Shor's algorithm is an instance of a quantum algorithm; and

— the class of problems decidable by quantum algorithms is exactly the class of (Turing) decidable problems;

more so since §18.A.f., Theorem 18.15, argues that factorization is not decidable by a deterministic Turing machine in polynomial time!

For instance, Wójtowicz, notes that:

> "A natural set of complex computational problems arises, when we consider simulating the behavior of quantum systems. Usually, the computer simulation of the evolution of a quantum system is impossible because we need exponentially many coefficients even to describe the quantum system in question.[2] So the computation corresponding to the evolution of the quantum system is extraordinarily complex. But this gives us the possibility to exploit the specific features of the quantum world in order to solve computational problems.
>
> The general idea here is—broadly speaking—to reverse the way we usually conceive the relationship between the physical system and the computer simulation: instead of providing a computer simulation of the physical system, we use the physical system to perform a physical simulation of the (mathematical) computational process.[3]
>
> So consider a computational problem P which corresponds (in some identifiable way) to the evolution of a quantum system $Q(P)$. In particular—the final state of the evolution of the quantum system $Q(P)$ corresponds to the result of the computation P. In such situations we could exploit the (quick) quantum evolution instead of the (slow) computationto solve the computational problem P. Trivially, such a correspondence obtains, when we start with a quantum system Q, and consider its computer simulation P_Q (then of course $Q(P_Q) = Q$). But this is not the point: the crucial question is, whether there are any MATHEMATICALLY MOTIVATED computational problems P (i.e. problem which arise within ordinary mathematics, and not for the purpose of describing quantum systems) for which such quantum systems $Q(P)$ exist. This is indeed the case—as demonstrated by the famous Shor's algorithm for factoring numbers."
>
> *. . . Wójtowicz: [Wo19],§1. General remarks on quantum algorithms, pp.174-175.*

Now, an explicit thesis of *evidence-based* argumentation is that we cannot 'reverse the way we usually conceive the relationship between the physical system and the computer simulation', so that 'instead of providing a computer simulation of the physical system' (i.e., appealing to *evidence-based* reasoning in the sense of [An16]; see also §4.C.), we use 'the physical system to perform a physical simulation of the (mathematical) computational process' (i.e., appeal to *faith-based* reasoning in the sense of §4.B.).

We could thus argue (see also §21.A.g.) that—reflecting inherited mathematical paradigms—Wójtowicz is here, implicitly and unwittingly, conflating a 'physical system' with the 'formal system of first-order Peano Arithmetic PA'.

An unsuspected, putative, conflation which could then allow Wójtowicz to further, curiously, suggest that an 'algorithm' can be conceived in some sense as the mathematical 'counterpart' of the motion of a 'system of photons':

> "Quantum algorithms are mathematical counterparts of certain quantum processes (such as for example a system of photons passing through a system of half-silvered mirrors).[4] They exploit the peculiarities of the quantum world (entanglement and superposition). The class of problems decidable by quantum algorithms is exactly the class of (Turing) decidable problems, so in particular—unsolvable problems remain unsolvable. But—at least in some cases—there can be an enormous increase in computational speed, and this makes them particularly attractive.[5]"
>
> . . . Wójtowicz: [Wo19],§1. General remarks on quantum algorithms, p.175.

We shall argue, however, that from the *evidence-based* perspective of this investigation, quantum algorithms in Deutsch and Wójtowicz's sense can be viewed as, essentially, admitting proof assistants during their computation that appeal—not to any physical system, but— only to the formal, first-order, system of Peano Arithmetic PA for validating algorithmically *computable* functions/relations by appeal to equivalent provable formulas of PA as admitted by the Provability Theorem for PA ([An16], Theorem 7.1, p.41; see also §2.E., Theorem 2.17):

> *Provability Theorem for PA*: A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically *computable* as always true in $\mathbb{N}$.

especially if, at some points, the computation only requires knowledge that 'there exists *an* x'—denoted symbolically by '$\exists x$' (see [Me64], p.45).[198] for completing a computation rather than knowledge of '*the* x'—denoted symbolically by 'ιx' (see [Me64], p.85).

An unsuspected, but possibly intuited, putative conflation as above could also account for Wójtowicz reservation:

> "However, there are no quantum computers available, because the technical problems to be overcome are formidable (due to the fragility of quantum states, which have to be isolated from their environment, i.e. the external world). It may well be the case, that even the impressive Shor's factoring algorithm remains just a purely theoretical possibility. There is also a perhaps deeper, conceptual problem: the class of known interesting quantum algorithms is limited. Factorization is not NP-complete, so even if we had a quantum computer available, Shor's algorithm would not give us a general method of solving NP-complete problems. It far from obvious, that a quantum algorithm for solving NP-complete problems (e.g. SAT) will ever be found.[6]"
>
> . . . Wójtowicz: [Wo19],§1. General remarks on quantum algorithms, pp.175-176.

Moreover, the appeal of a perspective that would make such a putative conflation natural, and psychologically difficult for philosophy and the natural sciences to resist unyieldingly in a Wittgensteinian sense—given the existing mathematical paradigms that do not insist upon *evidence-based* reasoning (as argued in [An16]; see also §1., Thesis 1)—could then be seen in Wójtowicz's remarks that:

> "The emergence of (applicable) quantum computers would certainly lead to major changes in science and technology. And even the theoretical possibility inspires us to reconsider philosophical questions concerning the nature of mathematical knowledge, the role of mathematical proofs and the relationships between mathematics and science. We face the question of the relationships between mathematics and the mathematical notion of computation on one hand—and the laws of physics and the "computational resources" of the universe on the other. And even if these considerations have the character of a thought experiment, they can shed new light on fundamental philosophical problems."
>
> . . . Wójtowicz: [Wo19],§1. General remarks on quantum algorithms, p.176.

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[198]We need to note the caveat here that, in *evidence-based* reasoning, the meaning of that which is sought to be represented formally as '$[\iota x]$' and '$[\exists x]$', by what is denoted symbolically as 'ιx' and '$\exists x$', need not be the meaning of any subsequent, *well-defined*, interpretation of the formulas '$[\iota x]$' and '$[\exists x]$' (see, for instance, §11.G.h.).

21.A.f.   What, precisely, is the role of proof assistants in quantum computing?

Although Wójtowicz emphasises the critical role of proof assistants in what he terms as 'quantum assisted proofs' (QAP), but—in the absence of an *evidence-based* perspective of the complementarity of mathematical proof, mathematical truth, and computability that is entailed by the Complementarity Thesis (§1., Thesis 1) and the Provability Theorem for PA ([An16], Theorem 7.1, p.41; see also §2.E., Theorem 2.17)—his concluding remarks in [Wo19] evidence his ambivalence as to from where, precisely, such proof assistants draw the assumed validity of the 'assistance' they offer during the course of a computation:

> "Quantum computation is a quickly developing area. However, there are no quantum computers, and there are also theoretical limitations to quantum algorithms. It might well happen, that Shor's factoring algorithm will remain the most spectacular theoretical achievement for a long time—and also that quantum computers of a practical importance will never be built.[36] Nevertheless, quantum computation theory is philosophically intriguing, also for philosophy of mathematics.
>
> The possibility of "oracle-like QAPs" poses some difficulties for the pro-realistic EIA. I have argued, that they are best explained within the holistic account, where traditional theorems, computer-assisted theorems, "quantum theorems" etc. are all integrated within one coherent system of knowledge.
>
> There has been an extensive discussion concerning the role of empirical procedures in mathematics [e.g. Baker, 2008]—and the hypothetical QAPs would constitute an important theoretical and philosophical novelty. I hope, that the thought experiment presented here contributes to the discussion concerning the empirical aspects of mathematics and the interplay between physical and mathematical knowledge."
>
> *. . . Wójtowicz: [Wo19], 5. Concluding remarks, pp.189-190.*

From the *evidence-based* perspective of this investigation, the value of Wójtowicz's philosophical analysis of quantum computing—in the following, copius, extracts from [Wo19]—thus lies as much in the putative role that he ascribes to proof assistants in the course of a computation, as in the challenges he describes that face his—what can reasonably be viewed, from an *evidence-based* perspective, as a contrived—attempt to force a 'quasi-empiricist' explanation for the putative effectiveness of such proof assistants in quantum computation within an inherited paradigm that appeals to *faith-based* reasoning (as sought to be distinguished in §4.B. and §4.C.):

> "The standard scheme of using mathematics as an explanatory tool is (more or less) as follows:
>
> 1. We are presented with a physical (biological, chemical etc.) phenomenon S.
> 2. We learn, that there is a theorem α (of standard mathematics M).
> 3. We see, that α (including its proof) helps us to explain the phenomenon S.
> 4. (And—being adherents of EIA—we consider this fact to be an important argument in the discussion).
>
> But what if α is proved via a QAP? A QAP certainly does not offer any understanding or explanation, attributed usually to traditional proofs. The only information we could get from a QAP is the fact, that a sentence α can be demonstrated, and nothing more. An "oracle proof" would not preserve the explanatory virtues of the theorem α, being a part of the mathematical theory. This would weaken the pro-realistic argument (as one of the premises of EIA would lose its fundaments).
>
> *. . . Wójtowicz: [Wo19], §4.2. "Quantum theorems" as empirical data, p.187.*

We note that Wójtowicz's subsequent argumentation in [Wo19] depends critically upon the—fragile, since lacking any apparent justification—assumption that a QAP *cannot* appeal to an oracle which can access the theorems of the first-order Peano Arithmetic PA, but *must* necessarily appeal to an "oracle proof" that 'would not preserve the explanatory virtues of the theorem α'.

For instance, it admits Wójtowicz's argument that a "quantumly proved theorem" such as α can be 'believed' as an *intuitive mathematical truth*, since our 'system of beliefs (including mathematical beliefs) has to fit the data, and these data include in particular the outcomes of the experiments, including (quantum) computer simulations':

> "To overcome this difficulty, I propose to view these new results (i.e. quantumly demonstrated propositions) not as full-fledged mathematical theorems, but rather as available empirical data, which have to be explained. So they would become rather a part of the *explanandum*, not the *explanans*.

> Let M be standard mathematics[31], and α a QAP-proved theorem. Accepting $M + \alpha$ better fits and explains the empirical data (including the quantum experiment yielding α) than other choices. Even if we refrain from accepting the quantum process as a legitimate mathematical proof, and even if we share the doubts resulting from the "explanatoriness postulate" (on which EIA rests), we are entitled to include α into our system of beliefs. This rests on the fact, that we equipped the physical experiment (performed for example on a system of photons flying around) with a semantics: we interpret the results of the experiment as information about the provability of α.[32] Of course, in this case $M + \alpha$ fits the empirical data well, and is a very natural rounding out of our knowledge. But its acceptance does not follow from the fact, that α is mathematically reliable (e.g. completes a theory in a mathematically natural way, fits the mathematical intuitions of the experts etc.), but rather from the fact, that $M + \alpha$ proves to be a good tool in physics.[33]

> To give a better feeling for this way of viewing "quantumly proved theorems", consider the case, where the status of a mathematical claim α is unknown (i.e. we have no proof and do not even know, whether it is consistent with M). It might be the case, that $M + \alpha$ suits the purposes of physics (it provides better methods of describing and explaining certain phenomena, it has a better predictive power etc.)—but as we do not know, whether it is consistent, we have an uneasy feeling about it.[34] Now, if we prove, that $M + \alpha$ is (relatively) consistent, the methodological obstacle is overcome. But what is the mathematical status of α? We haven't proved α (perhaps it is even independent from M, so unprovable). Should we accept α as a new mathematical axiom only because it suits the purposes of physics?

> Consider now a different situation, where we use a (relatively) consistent theory $M + \alpha$ in physics, and—later on—α happens to be proved by a QAP. This would give us perhaps an even stronger belief in the consistency of $M + \alpha$ (than just having the "old fashioned" metamathematical proof, e.g. by some exotic forcing or model-theoretic arguments).

> Consider the following two situations:

>   (i) We know, that $M + \alpha$ is relatively consistent (but do not know, whether α is provable within M)—and it fits the empirical data well.
>   (ii) We have a QAP of α.

> Is there—from the point of view of the EIA-realist—an important difference between (i) and (ii)? Anyway, we have already decided to make use of $M + \alpha$ as a tool in science, and—being EIA-realists—we included $M + \alpha$ into our system of beliefs (in particular, accepting its ontological commitments). $M + \alpha$ was accepted before the QAP of α, and its provability within M becomes a question of the internal logical structure of $M + \alpha$—not the question of accepting α.

> I claim, that this way of viewing quantumly proved theorems gives a better philosophical explanation of the status of $M + \alpha$ from the point of view of mathematical realism based on the indispensability

argument. In particular, it solves the problem of the lack of explanatoriness, which presents a difficulty for the EIA-realist. Our system of beliefs (including mathematical beliefs) has to fit the data, and these data include in particular the outcomes of the experiments, including (quantum) computer simulations. Ultimately—from the point of view of EIA-realism—what matters is the fact, that mathematical sentences gain the status of truths *via* the empirical theory they are part of—not via conceptual, *a priori* insights.[35]"

... *Wójtowicz: [Wo19], §4.2. "Quantum theorems" as empirical data, pp.187-189.*

However, the *evidence-based* perspective of this investigation demands that *intuitive mathematical truth* (such as that sought to be putatively ascribed to the axioms of a formal mathematical theory) be definable:

— as formal, *evidence-based*, assignments-by-convention under Tarski's definitions of the 'satisfaction', and 'truth', of mathematical formulas under a *well-defined* (i.e., *evidence-based*) interpretation; and

— as distinctly different in nature and substance from the empirical 'truths' putatively ascribed statistically on the basis of our observations to properties exhibited with reliable predictability in physical phenomena that can, subsequently, be expressed/represented in a formal mathematical language;

Prima facie, Wójtowicz's 'quasi-empiricism'—reflecting current mathematical paradigms—seems to—disconcertingly, since without apparent justification—equate the latter with *intuitive mathematical truth*.

If so, such an interpretation of 'quasi-empiricism' would, however, entail that *intuitive mathematical truth* is not only of a statistical nature, but falsifiable!

Given the putative constraints of an inherited *faith-based* mathematical paradigm, the justification for Wójtowicz's attempt to tolerate such a philosophically disquieting perspective into his argumentation can be viewed as lying in his perception that:

— not only is the possibility of quantum computation 'exciting' in itself for the peculiarities it compels us to accommodate, and place in a coherent perspective; but

— that we 'can even imagine, that one of the big mathematical open problems (say, Riemann's hypothesis or Goldbach's conjecture) is proved with the help of a QAP, which would surely be sensational':

"The (theoretical) possibility of quantum computation, and executing quantum assisted proofs (for which I will use the acronym QAP) is very exciting. In order to understand the peculiarities of the possible QAP, let us exhibit its most important features. In general, it would consist of the following steps.

1. The mathematical (conceptual) phase. Consider a computational problem P (e.g. factoring numbers). Our task is to define a quantum system $Q(P)$, which is connected to P in an explicit way—in particular there is a way of identifying the outcome of P from the outcome of $Q(P)$. Here we reverse the usual way of viewing the relationship between real-world situations and computer simulations. Instead of running a computer simulation to find out, what would happen in a physical situation (e.g. whether a bridge would break down), we perform the physical experiment in order to learn, what the outcome of the (perhaps extraordinarily long) computation would be. This problem becomes interesting, when P has a natural mathematical motivation, and arises within a natural mathematical context. In this case, finding a corresponding quantum system $Q(P)$ would allow us to solve P.

2. The experimental phase consists of:

(a) Preparing the quantum system $Q(P)$ in an appropriate initial state.

(b) Initiating the quantum process. The crucial feature of this process is that during this computation we cannot in any way interfere with it, as this would destroy the process (so we have to wait patiently for the outcome).

(c) Performing the final measurement. This means, roughly speaking, that we extract the available information from the quantum system.

The outcome of the experiment with the use of $Q(P)$ yields a solution of the problem P."

. . . Wójtowicz: [Wo19], §2. Quantum-assisted proofs (QAPs), pp.176-177.

Wójtowicz's seems to ascribe any unsettling aspects of his philosophical perspective to the 'peculiarities' of executing quantum assisted proofs (QAPs), which may be viewed as resulting from reversing 'the usual way of viewing the relationship between real-world situations and computer simulations'.

According to Wójtowicz's analysis, such a 'peculiarity' would, prima facie, be the claim of quantum computing that even if a computational problem 'P has a natural mathematical motivation, and arises within a natural mathematical context', such as 'factoring numbers', finding a corresponding physical representation—quantum system—$Q(P)$ 'would allow us to solve P' where, even in the absence of 'a computer simulation to find out, what would happen in a physical situation (e.g. whether a bridge would break down), we perform the physical experiment in order to learn, what the outcome of the (perhaps extraordinarily long) computation would be.'

If so, such a claim would—from the *evidence-based* perspective of this investigation—contradict the Provability Theorem for PA ([An16], Theorem 7.1, p.41; see also §2.E., Theorem 2.17):

Provability Theorem for PA: A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically *computable* as always true in $\mathbb{N}$.

since it apparently posits that we can have an algorithmic proof of an arithmetical theorem, by appeal to the physical properties of a quantum system (computer), even when the algorithm is not entailed by the formal proof of the theorem.

It also ignores Thoralf Skolem's cautionary remarks (see §4.J.) against inviting paradox[199] by conflating entailments of formal systems under different interpretations (as evidenced, for instance, in §2.D.), or over different domains.

Moreover, it admits the 'untenable' argument (see §21.A.d.) that 'measurements have a probabilistic nature, so in general we cannot identify the state of the quantum system before the measurement, and the information is lost':

"A natural question follows: measurements have a probabilistic nature, so in general we cannot identify the state of the quantum system before the measurement, and the information is lost.[7] This is true—but in some cases, it is possible to "extract" enough information from the quantum system. For example, if we knew in advance (i.e. before the measurement), that the qubit could have been only in one of the two basic states, then the measurement would give us complete information. A similar situation can happen with more complicated n-qubit registers: if we know in advance, that they are in one of few possible states, appropriate measurements will enable us to identify it.[8] This is crucial for quantum algorithms."

. . . Wójtowicz: [Wo19], §2. Quantum-assisted proofs (QAPs), p.177.

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[199]See, for instance, Goodstein's argument in §15., Theorem 15.1.

Wójtowicz notes further that, with respect to 'quantum-assisted proofs of some mathematical theorems' the 'outcome might differ from the outcome of the classical computation in one important respect: we would not be able to know which of the theoretically possible computational paths was the successful one':

> "Many mathematical problems involve a complex computational part, so a quick computational method might settle some of such problems (as it happens in the case of ordinary computer assisted proofs). In particular, we might get quantum-assisted proofs of some mathematical theorems. The outcome might differ from the outcome of the classical computation in one important respect: we would not be able to know which of the theoretically possible computational paths was the successful one. Indeed, there are quantum algorithms, which do not always exhibit CONCRETE solutions, but rather provide some general information about the problem. For example, the quantum algorithm presented in [Harrow et al., 2009] allows to get some information about systems of equations: the algorithm outputs a quantum state with certain properties, and not explicitly the solution.[9] In order to "extract" the solution from this state we would have to perform a large number of measurements. But sometimes we are interested not in the exact solution, but in some general property of it, which might be established by performing just few measurements.[10]"
>
> *. . . Wójtowicz: [Wo19], §2. Quantum-assisted proofs (QAPs), pp.177-178.*

We note that Wójtowicz is essentially arguing the point made in §21.A.e. that, at some points, a quantum computation may only require *logical* knowledge that 'there exists *an x*'—denoted symbolically by '∃*x*' (see [Me64], p.45), rather than knowledge of '*the x*'—denoted symbolically by '*ιx*' (see [Me64], p.85), if 'we are interested not in the exact solution', for which 'we would have to perform a large number of measurements', but in some general property of it, which might be established by performing just few measurements'.

However, Wójtowicz then seems to implicitly assume—apparently as an 'axiomatic' feature of quantum computations—that even if such knowledge yields computation times 'exponentially quicker [than] classical algorithms', it is not entailed by *logical* reasoning alone from the mathematical representation of the state of the quantum system being computed, but is the outcome of 'a kind of "quantum non-constructive existence argument"' for which we 'have to rely strongly [on] physical theories in order to treat these procedures as reliable':

> "In some cases, solving a computational problem is an essential part of a proof.[11] So we might get a QAP of a possibly important mathematical theorem. The situation becomes philosophically even more interesting, when we consider logical problems in their combinatorial (number-theoretic) formulation/disguise. Formal proofs can be encoded as numbers (via arithmetization of syntax), So—ultimately—the question whether there is a formal proof of a sentence α within a formal theory T becomes a computational problem. Usually this is not a decidable problem (and will not become "quantumly decidable" either[12]), but we can always check, whether a given string σ of symbols is a formal proof of α within T—and we can also check, whether there is a proof of α within a given finite set of strings S). A quick computational procedure would allow us to find answer to questions like: "Is there a proof of α within T of the length bounded by n?" (T being for example ZFC or PA or RCA$_0$ or any other formal theory of interest). At least in the case of some open problems, the answer would be positive.[13] But then, from our point of view, after the process terminates, only a big "YES!" is displayed on the screen . . . . Even if it happened only in one single case, i.e. even if one such a quantum demonstration of the existence of a formal proof succeeded only once, the question of the status of such knowledge would become philosophically intriguing.
>
> So, in general we might think of two possible scenarios:
>
> 1. A "direct" QAP: i.e. a computation, which solves a computational problem, yielding a proof of a mathematical theorem α.[14]

2. A "meta-QAP": the computation has a direct metamathematical interpretation, yielding a positive answer to the question "Is there a formal proof of α within T of length bounded by n?"[15]

Could either of these processes be considered a proof of α? We cannot even dream of reading out any details of this proof from the process, as measurements cannot be performed during the computation. In particular, in the "meta-QAP" case we would be confronted with a kind of "quantum non-constructive existence argument": we only learn, that such a formal proof (of the length $\leq n$) exists—and NOTHING MORE.

Observe the following crucial features of a potential QAP:

1. It is quick (it might even be exponentially quicker that the classical algorithms).

2. We have no insight into the process—we only can perform the final measurement.

3. We have to rely strongly of physical theories in order to treat these procedures as reliable.

We are therefore faced with a situation, where we exploit some physical processes (resources) in order to solve computationally difficult problems. The analyses given here apply to all cases, in which these three conditions are met.[16] In some cases such a computational support can lead to new important results. Indeed, this was exactly the case of computer-assisted proofs (CAPs). The most famous example is probably the proof of the four-color theorem (4CT).17 Its computer-assisted proof was presented in [Appel and Haken, 1977; Appel et al., 1997]. As the proof required the use of a computer (in its original form, they needed ca. 1200 hours), several methodological, conceptual and philosophical questions concerning the proof and the epistemological status of 4CT arose.[18] The fundamental question is whether this CAP REALLY is a mathematical proof, i.e. whether the four color HYPOTHESIS turned into a mathematical THEOREM.

The problem becomes more intricate in the case of QAP's. We can even imagine, that one of the big mathematical open problems (say, Riemann's hypothesis or Goldbach's conjecture) is proved with the help of a QAP, which would surely be sensational. But even if α is just an ordinary mathematical problem, the philosophical status of α remains to be examined."

. . . *Wójtowicz:* [*Wo19*], *§2. Quantum-assisted proofs (QAPs), pp.178-180.*

We note that Wójtowicz's remark that we 'are therefore faced with a situation, where we exploit some physical processes (resources) in order to solve computationally difficult problems' implicitly appeals to a putative 'intuitive-empirical' truth that, in turn, would admit a Platonic ontology in any *well-defined* interpretation of formal mathematical systems such as the first-order Peano Arithmetic PA.

However, such a perspective seems fragile, since by the minimalism of Ockham's Razor, the *evidence-based* perspective of this investigation—which views mathematics as, essentially, *evidenced-based* languages (see §9. and §10.)—is arguably shown as sufficient to its intended purpose of serving the minimal needs of philosophy and the natural sciences to:

— first, faithfully represent (in languages such as the first-order set theory ZF), conceptual metaphors corresponding to their observations of physical phenomena; and,

— second, categorically communicate (in languages such as the first-order Peano Arithmetic PA), some of these representations;

without recourse to a Platonic ontology.

Comment: We note that, from the *evidence-based* perspective of this investigation:

(a) Any first-order set theory, such as ZF, which includes an axiom of infinity cannot have a *well-defined* (in the sense of §4.F., Definition 19) assignment of 'satisfaction' and 'truth' to its formulas under any *well-defined* Tarskian interpretation, since.

— ZF can admit only algorithmically *computable* terms and provable/unprovable formulas;

— ZF cannot admit a *well-defined* interpretation that admits Platonic elements;

— Assuming that ZF is consistent only entails, by postulated definition, that, for any ZF-formula $[F]$, we cannot have that both of $[F]$ and $[\neg F]$ are ZF-provable;

— Rosser's argument in [Ro36] entails that ZF is essentially incomplete if it admits Rosser's Rule C (see §5.G.); in which case, however:

  — we *can* always extend ZF conservatively by further addition of axioms that entail additional terms in ZF which, under any *putative* interpretation of ZF, *could* then be claimed to correspond to only non-finitary, Platonically conceivable, mathematical objects in the putative domain of the interpretation.

(b) On the other hand, PA cannot admit an interpretation that admits non-finitary, Platonically conceived, elements, since:

— PA has a *well-defined* assignment of 'satisfaction' and 'truth' to PA-formulas under a *well-defined*, finitary, Tarskian interpretation over the domain $\mathbb{N}$ of the natural numbers (by [An16], Theorem 6.7, p.41; see also §2.C., Theorem 2.15);

— Any PA-numeral $[n]$ interprets in a putative domain $\mathbb{N}$ of the natural numbers as the natural number n;

— Any natural number n in the putative domain $\mathbb{N}$ can, in turn, be interpreted as an abbreviation for a putative heap of, say, n grains of sand:

  — if we assume, not unreasonably, that the grains of sand in a putative expanding universe—such as, say, the one we *actually* inhabit—is not finite;

— PA is categorical (by [An16], Corollary 7.2, p.41; see also §2.E., Corollary 2.18).

Moreover, from such a perspective, the fragility of Wójtowicz's philosophical perspective—as reflected in his remark that we 'can even imagine, that one of the big mathematical open problems (say, Riemann's hypothesis or Goldbach's conjecture) is proved with the help of a QAP'—is further seen in the *evidence-based* argument that:

— even without quantum computing as it is currently interpreted in [Deu85] and [Wo09], and [Wo19];

— for at least 'open' arithmetical problems such as Goldbach's conjecture and the Collatz conjecture (see §17.G.b.);

— the Halting-decidability Theorem for T (§17.F.b., Corollary 17.4),

  – as distinct from Turing's Halting Theorem (§17.F.b., Corollary 17.5);

— entails that, by the Forced Halting Theorem (§17.F.b., Theorem 17.3):

  – either the conjecture is *formally provable* in PA,

    - and therefore algorithmically *computable* as true by the Provability Theorem for PA ([An16], Theorem 7.1, p.41; see also §2.E., Theorem 2.17);

– or, if not so, the *decidability* of;

- whether the conjecture is algorithmically *computable* as *false* under any *well-defined* interpretation of PA,

- or, if not so, whether the conjecture is neither algorithmically *verifiable* as *true*, nor as *false*, under any *well-defined* interpretation of PA,

- and therefore *undecidable* on the basis of *evidence-based* reasoning (by Turing's Halting Theorem, §17.F.b., Corollary 17.5).

– is *decidable* on the basis of *evidence-based* reasoning (by the Halting-decidability Theorem for T, §17.F.b., Corollary 17.4).

The most unsettling philosophical challenge faced by current paradigms of the nature and role that proof assistants play in the course of a quantum computation is that of interpreting the significance of the admission, or denial, of 'access during the quantum process to the temporary state of the computation'.

As described by Wójtowicz's—no less challenged from the perspective of §21.A.d.—attempt to place it in a coherent philosophical perspective, the challenge reflects that:

> "... The differences between classical computers and quantum computers are much deeper, as we have no access during the quantum process to the temporary state of the computation. A classical computer performs the computations we could also perform. So in principle we could proceed with the proof in the traditional way. We can stop the computation at any stage, examine the temporary state and continue with the process—so in particular, we could analyze fragments of the computation, and reconstruct an ordinary proof (so the computer would serve as heuristic device). We could imagine a group of 1000 mathematicians examining a computer assisted proof, but in the case of QAPs, the situation is radically different. A quantum proof is a kind of black box—as there are no knowable intermediate states: regardless of the size of the quantum circuit, we only have access to the final outcome (through measurement). And quantum phenomena, like entanglement and interference are built into the procedure.
>
> From the epistemological point of view, the "minimal item" is the experiment conceived as a whole. QAPs are not even partially verifiable or acceptable in any way—we have to accept them as certain wholes, as "atomic procedures". A part (usually a significant part) of the information, which is present during the quantum computation, is definitely lost in the final step and cannot be retrieved in any way. The quantum system does not "remember" which of the computational paths involved (simulated in the experiment) corresponds to the successful proof. In a sense, we are presented with a kind of empirical oracle, which can answer some questions, leading to the acceptance of a sentence α. Maybe it can even answer questions of the kind "Does T formally prove α within n steps?"—but the answers can only be 'YES' and 'NO' (or perhaps: 'YES' and 'TRY A LARGER n')—without giving any hints concerning the structure and general ideas of the proof.[20]"

> ... Wójtowicz: [Wo19], §3. Are there empirical proofs?, p.182.

However, as we have argued in §21.A.e., the need for:

> 'a kind of empirical oracle, which can answer some questions, leading to the acceptance of a sentence α'

dissolves if we replace it with:

> 'a kind of formal oracle—appealing, for instance, to such as the first-order Peano Arithmetic PA—which can answer some questions, leading to the acceptance of a sentence α.'

21.A.g. Can we treat PA as Deutsch's and Wójtowicz's quantum oracle?

In other words, by the Provability Theorem for PA ([An16], Theorem 7.1, p.41; see also §2.E., Theorem 2.17), we can now reasonably posit that:

Thesis 22. (Quantum algorithm) *The quantum algorithm appealed to in Deutsch's quantum computer in [DJ92] is a provable formula of PA.*

Thesis 23. (Quantum oracle) *The quantum oracle appealed to in Deutsch's quantum computer in [DJ92] is the first-order Peano Arithmetic PA.*

Moreover:

(i) The making, or not making, of a physical measurement/observation in the *actual* universe that we inhabit,

— which can be treated as corresponding to that which is sought to be represented mathematically as the computation of a *well-defined*, algorithmically *computable*, mathematical function f in a well-defined mathematical model M of U,

(ii) by a physical 'quantum' computer $\mathcal{Q}$,

— which admits *relatively* random (see §4.G., Definition 20) inputs/outputs,

(iii) *cannot* influence and/or determine the computational complexity of the *well-defined*, algorithmically *computable*, mathematical function f,

— in the well-defined mathematical *model M* of U,

— in which $\mathcal{Q}$ is not well-definable (since a *relatively* random output cannot be realised in M);

since the underlying perspective here is that:

(iv) If U is the universe we inhabit:

— and T is, say, the Taj Mahal,

— whilst D is the set of categorical engineering specifications,

— for building a physical model M of T in which we can perform repeated destructive experiments;

(v) Then, as observers that exist in U:

— whilst we can talk categorically about the properties of T representable in M;

— we can only conjecture/postulate;

* on the basis of statistical correlations,

* between what we observe in U,

* and what we measure in M;

   – to what extent M is a representative model;

      ∗ not of T;

      ∗ but of *that* part of T;

   – which we can observe in U;

   – and well-define in D;

   – without destroying T.

(vi) We cannot, however, claim that any unpredictable structural change in T instantaneously causes a corresponding structural change in M!

This reflects the *evidence-based* perspective of this investigation which, we iterate, is that:

(a) *choosing to measure* the system S—at the step in Wójtowicz's argument denoted by §21.A.d.(iii)—simply means that we have appealed to the oracle to transform our *ignorance* of the algorithmically *computable* state of the system at that point to *knowledge* of the state that is already known/available to the oracle (and, in principle, to us, since the state is assumed algorithmically *computable*), as we *need* this knowledge for completing the desired computation;

(b) *choosing not to measure* the system S—at the step in Wójtowicz's argument denoted by §21.A.d.(vi)—simply means that we have *not* appealed to the oracle to transform our *ignorance* of the algorithmically *computable* state of the system at that point to *knowledge* of the state that is already known/available to the oracle (and, in principle, to us, since the state is assumed algorithmically *computable*), as we *do not need* this knowledge for completing the desired computation;

From a logical perspective, the difference here is between *choosing* to *compute/determine/know* '*the x*'—denoted symbolically by 'ιx' (see [Me64], p.85)—and *choosing* to *compute/determine/know only* that 'there exists *an x*'—denoted symbolically by '$\exists x$' (see [Me64], p.45).[200]

Obviously, any computation that only requires knowledge of the latter for completing a computation *could* be significantly faster than one where we, unnecessarily, compute the former.

For instance, determining that a specified integer n has more than one prime factor (primality) *can*—depending upon the specification—be done deterministically in polynomial time $O(log_e n)$ (see §18.A.e., Corollary 18.14); whilst, by the argument of §18.A.f., Theorem 18.15, determining a prime factor of n (factorisation) *cannot* be done deterministically in polynomial time $O(log_e n)$.

Moreover, use of a *physical* (circuit) operator $\sqrt{}$NOT *physically* for transforming the *physical* state of a system S is not essential to quantum computation per se, but is only needed where the algorithmically *computable* state, whose knowledge the oracle is called upon to furnish, appeals to the formal representation of the *physical* $\sqrt{}$NOT operation in the language of algorithmically *computable* functions.

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[200]We note that in *evidence-based* reasoning, the meaning of that which is sought to be represented formally as '[ιx]' and '[$\exists x$]' by what is denoted symbolically as 'ιx' and '$\exists x$' need not be the meaning of any subsequent, *well-defined*, interpretation of the formulas '[ιx]' and '[$\exists x$]' (see, for instance, §11.G.h.).

CHAPTER 22. *PARADIGM CHALLENGING CONSEQUENCES*

22.   The significance of *evidence-based* reasoning for METI

Work in progress as on 16/04/2020

We conclude this investigation by briefly speculating that the issue of whether, or not, there is a universal language capable of admitting effective, and unambiguous, communication with an extra-terrestrial intelligence is intimately linked with the question of whether or not the first-order Peano Arithmetic PA is categorical, and whether it has an algorithmically *computable* model. This issue lies at the heart of the 'constructivity' debate that seeks to distinguish the computer sciences from other mathematical disciplines.

We shall address the question from the perspective of seekers of extra-terrestrial intelligence who may, conceivably, be faced with a situation where a lay person—whose financial support is sought for METI/SETI—may reasonably require a reassuring response to the question:

Query 24. *Is there a rational danger to humankind in actively seeking an extra-terrestrial intelligence?*

22.A.   Is there a rational danger to humankind in actively seeking an extra-terrestrial intelligence?

The broader significance of this question was addressed—in an article originally written in September 2006—by scientist David Brin, who feared that *'SETI has Taken a Worrisome Turn Into Dangerous Territory'*, and noted that:

> "...In *The Third Chimpanzee*, Jared Diamond offers an essay on the risks of attempting to contact ETIs, based on the history of what happened on Earth whenever more advanced civilizations encountered less advanced ones ...or indeed, when the same thing happens during contact between species that evolved in differing ecosystems. The results are often not good: in inter-human relations slavery, colonialism, etc. Among contacting species: extinction."
>
> *...Brin:* [Bn08].

A decade later the issue is still alive, according to science writer Steven Johnson who, in a 2017 Feature article for *The New York Times Magazine*, dramatically titled *Greetings, E.T. (Please Don't Murder Us.)*, reports on the 1974 *Arecibo* message, and the profoundly unsettling issues that are reflected in a continuing ambivalence towards the dangers of a resolute exploration of space that may invite the attentions of any malevolent intelligence the cosmos might nurture in its depths:

> "That 168 seconds of noise, now known as the Arecibo message, was the brainchild of the astronomer Frank Drake, then director of the organization that oversaw the Arecibo facility. The broadcast marked the first time a human being had intentionally transmitted a message targeting another solar system."
>
> *...Johnson:* [Jo17].

The ambivalence became apparent immediately. As Johnson notes:

> "It seemed to most of the onlookers to be a hopeful act, if a largely symbolic one: a message in a bottle tossed into the deep sea of space. But within days, the Royal Astronomer of England, Martin Ryle, released a thunderous condemnation of Drake's stunt. By alerting the cosmos of our existence, Ryle wrote, we were risking catastrophe. Arguing that "any creatures out there [might be] malevolent or hungry," Ryle demanded that the International Astronomical Union denounce Drake's message and explicitly forbid any further communications. It was irresponsible, Ryle fumed, to tinker with interstellar outreach when such gestures, however noble their intentions, might lead to the destruction of all life on earth.
>
> Today, more than four decades later, we still do not know if Ryle's fears were warranted, because the Arecibo message is still eons away from its intended recipient, a cluster of roughly 300,000 stars known as M13."
>
> *. . . Johnson: [Jo17].*

Nor has the ambivalence diminished. As noted by Johnson, even though scientists have, meanwhile 'spent far more energy trying to look for signs that other life might exist than they have signaling the existence of our own':

> "Now this taciturn phase may be coming to an end, if a growing multidisciplinary group of scientists and amateur space enthusiasts have their own way. A newly formed group known as METI (Messaging Extra Terrestrial Intelligence), led by the former SETI scientist Douglas Vakoch, is planning an ongoing series of messages to begin in 2018."
>
> *. . . Johnson: [Jo17].*

However, as Johnson observes further:

> "...as messaging schemes proliferate, they have been met with resistance. The intellectual descendents of Martin Ryle include luminaries like Elon Musk and Stephen Hawking, and they caution that an assumption of interstellar friendship is the wrong way to approach the question of extraterrestrial life. They argue that an advanced alien civilization might well respond to our interstellar greetings with the same graciousness that Cortés showed the Aztecs, making silence the more prudent option."
>
> *. . . Johnson: [Jo17].*

22.B.   So what's new? Unprecedented categorical communication

We shall restrict ourselves here to considering, from the *evidence-based* perspective of this investigation, only one aspect of this complex issue that may need to be reassessed in view of the argumentation of the paper [An16] published in the December 2016 issue of *Cognitive Systems Research*, which suggests:

> *The possibility of categorical communication between humans and extra-terrestrials.*

Since the finitary proof of consistency for the first-order Peano Arithmetic PA ([An16], Theorem 6.8, p.41; see also §2.C.a., Theorem 2.16) entails that PA is a categorical language ([An16], Corollary 7.2, p.41; see also §2.E., Corollary 2.18), it raises the possibility of unambiguous and effective, essentially mechanical intelligence based, dialogue with an extra-terrestrial intelligence that, too, may be pro-actively seeking dialogue similarly.

The significance of this is that, historically:

> *There is no precedent where categorical communication between two spatially separated civilisations preceded first proximal contact;*

which, in the case of a first contact between humankind and an extra-terrestrial intelligencee, could be by the order of thousands of earthly years!

However, on the not unreasonable assumption that, to the best of our knowledge, the evolution of life anywhere in the cosmos can/ought to be treated as Darwinian, the issue raised by Ryle—of an implicit responsibility against engaging in any scientific endeavour that might jeopardise the survival of hunamkind—does need to be addressed anew, from the broadest multi-disciplinary perspective, *before* we attempt to investigate whether, and if so how, our ability to communicate categorically within our species can be extended to categorical communication between essentially different species.

In other words, it is not unreasonable to require that we proffer a definitive perspective on both the necessity, and the level, of the SETI/METI-associated responsibility for any implicit assumptions—such as, for instance, the various Theses 24 to 36—when posed the question:

Query 25. *Is fear of actively seeking an ETI merely paranoia, or does it have a rational component?*

raised by Marko Kovic's 2017 ZIPAR Policy Brief [Kov18], where his thesis is that '*METI (Messaging Extra Terrestrial Intelligence) is irrational and should be abandoned*' since:

> "No one will answer our METI calls. All METI does is create (existential) risk for humankind."
> *. . . Kovic: [Kov18], Abstract.*

22.C.   Is it rational to fear communicating with an extra-terrestrial intelligence?

Shorn of paranoiac overtones, this fear can be expressed as the query:

Query 26. *Can we responsibly seek communication with an extra-terrestrial intelligence actively (as in the 1974 Aricebo message) or is there a logically sound possibility that we may be initiating a process which could imperil humankind at a future date?*

To place the issue in a debatable perspective, we need to make some reasonable assumptions. For instance, we may reasonably assume that:

Thesis 24. *Any communication with an extra-terrestrial intelligence will involve periods of upto thousands of years between the sending of a message and receipt of a response.*

Thesis 25. *We can only communicate with an essentially different form of extra-terrestrial intelligence in a platform-independent language of a mechanistically reasoning artificial intelligence.*

Thesis 26. *Nature is not malicious and so, for an ETI to be malevolent towards us, they must perceive us as an essentially different form of intelligence that threatens their survival merely on the basis of our communications.*

22.D.   Recursive Arithmetic: The language of algorithms

Now, it is reasonable to assume that:

Thesis 27. *The language of algorithmically computable functions and relations is platform-independent.*

This is the algorithm-based machine-language defined by Gödel's recursive arithmetic ([Go31]), by Church's lambda calculus ([Ch36]), by Turing's computing machines ([Tu36]), and by Markov's theory of algorithms ([Mrk54]).

As Mandelbrot has shown ([Mn77]), the language appears sufficiently rich to model a number of complex natural phenomena observed by us ([Bar88], [BPS88], [PR86]), which earlier appeared intractable.

To simplify the issue within reason, we may thus assume that:

Thesis 28. *All natural phenomena which are observable by human intelligence, and which can be modelled by algorithms, are interpretable isomorphically by an extra-terrestrial intelligence.*

However, it is also reasonable to assume that:

Thesis 29. *There are innumerable, distinctly different, observable natural phenomena.*

In other words, the language of algorithms must admit—and require—denumerable primitive symbols for expressing natural phenomena.

Now, an extra-terrestrial intelligence which observes natural phenomena under an interpretation that—although structurally isomorphic to ours—uses different modes/means of observation, may not be able to recognise any of our symbolisms effectively. Hence:

Thesis 30. *A language of algorithms with a denumerable alphabet does not admit categorical communication with an ETI.*

22.E.   First-order Peano Arithmetic PA: A universal language of Arithmetic

However, in his seminal 1931 paper on 'formally undecidable' arithmetical sentences, Kurt Gödel showed, essentially (compare [Go31], Theorem VII, p.29; see also §10.D.a., and [Me64], Proposition 3.23, p.131), that:

Lemma 22.1. *Every algorithmically computable number-theoretic function can be formally represented by some formula of a first-order Peano Arithmetic, PA.*

PA is the language defined over the structure $\mathcal{N}$ of the natural numbers—namely, $\{N$ (*the set of natural numbers*); $=$ (*equality*); $'$ (*the successor function*); $+$ (*the addition function*); $*$ (*the product function*); 0 (*the null element*)$\}$.

The axioms and rules of inference of PA are:

PA$_1$: $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;
PA$_2$: $[(x_1 = x_2) \rightarrow (x'_1 = x'_2)]$;
PA$_3$: $[0 \neq x'_1]$;
PA$_4$: $[(x'_1 = x'_2) \rightarrow (x_1 = x_2)]$;
PA$_5$: $[(x_1 + 0) = x_1]$;
PA$_6$: $[(x_1 + x'_2) = (x_1 + x_2)']$;
PA$_7$: $[(x_1 \star 0) = 0]$;
PA$_8$: $[(x_1 \star x'_2) = ((x_1 \star x_2) + x_1)]$;
PA$_9$: For any well-formed formula $[F(x)]$ of PA:
$\quad\quad [(F(0) \rightarrow (\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x)]$.

Modus Ponens in PA: If $[A]$ and $[A \rightarrow B]$ are PA-provable, then so is $[B]$;

Generalisation in PA: If $[A]$ is PA-provable, then so is $[(\forall x)A]$.

PA is a good candidate for a language of universal communication with an ETI because—as Turing showed in 1936—its well-formed formulas can be expressed digitally, and—as Gödel showed in his 1931 paper—Peano Arithmetic has a finite alphabet with finitary rules for:

(i) the formation of well-formed formulas;
(ii) deciding whether a given formula is a well-formed formula;
(iii) deciding whether a given formula is an axiom;
(iv) deciding whether a finite sequence of formulas is a valid deduction/proof sequence;
(v) deciding whether a formula is a consequence of the axioms (a theorem).

22.F.   Communicating PA and its Theorems categorically

Moreover, since PA is categorical ([An16], Corollary 7.2, p.41; see also §2.E., Corollary 2.18) and can be expressed (as entailed by [An16], Theorem 7.1, Provability Theorem for PA, p.41; see also §2.E., Theorem 2.17) and beamed digitally, we may reasonably assume that:

Thesis 31. *Any advanced ETI can communicate with us by recognising that the signals we are beaming:*

(a) are not random;
(b) contain a primer that defines PA;
(c) contain theorems that describe common natural phenomena as observed by us.

Carl Sagan obliquely suggested such a strategy in his novel, 'Contact' ([Sa85]). Whether, and how, it can be implemented—as suggested above—is an issue beyond the scope of the limited point sought to be addressed in this paper.

22.G.   How we currently interpret PA

Currently, the *classical* standard interpretation $\mathcal{I}_{PA(\mathbb{N},S)}$ of PA ([Me64], p.107, *standard model*; see also §2.A.b.) is the one over the structure $\mathbb{N}$ of the natural numbers, where the logical constants have their standard Tarskian interpretation ([Me64], §2, p.49; see also §2.A.):

(a) the set of non-negative integers is the domain;

(b) the integer 0 is the interpretation of the symbol [0];

(c) the successor operation (addition of 1) is the interpretation of the ['] function;

(d) ordinary addition and multiplication are the interpretations of [+] and [∗];

(e) the interpretation of the predicate letter [=] is the identity relation.

22.H.   A malevolent ETI must interpret some of our true arithmetical propositions as false

Now, it is reasonable to assume that:

Thesis 32. *A malevolent ETI could perceive us as an essentially different form of intelligence that threatens their survival if they have an interpretation of PA that is essentially different to our standard interpretation $\mathcal{I}_{PA(\mathbb{N},S)}$ of PA.*

Thesis 33. *An ETI would perceive their interpretation $\mathcal{I}_{PA(\mathbb{N},ET)}$ of PA, over the structure $\mathbb{N}$ of the natural numbers, as essentially different to our standard interpretation $\mathcal{I}_{PA(\mathbb{N},S)}$ of PA if, and only if, there is a PA formula which—from our perspective—would either interpret as 'false' under $\mathcal{I}_{PA(\mathbb{N},ET)}$ and 'true' under $\mathcal{I}_{PA(\mathbb{N},S)}$, or as 'true' under $\mathcal{I}_{PA(\mathbb{N},ET)}$ and 'false' under $\mathcal{I}_{PA(\mathbb{N},S)}$.*

22.I.   The rational argument for not seeking an ETI pro-actively

In 1931 Gödel detailed an argument from which he concluded that:

- There is an 'undecidable' proposition in Peano Arithmetic;

- Two intelligences *can* logically interpret it and arrive at conflicting conclusions.

Once we accept the meta-mathematical and philosophical conclusions that Gödel draws from his formal reasoning in [Go31] as definitive, it *can* be argued that we must—on the basis of our experiences with conflicting belief-driven perspectives between different faiths and cultures on earth—be prepared—both politically and militarily—for the, admittedly far-fetched, possibility that an ETI may:

- Interpret Gödel's formally 'undecidable' proposition as what we would term as 'false';

- Believe that any intelligence which interprets the proposition as what we would term 'true' (as we do under Gödel's interpretation of his own formal reasoning in [Go31]; but see also §11.C.) could be a potential threat to it—hence one that must be exterminated!

22.J.   Does PA lend itself to essentially different interpretations?

So, the question is: Does PA lend itself to essentially different interpretations?

This question of whether there is a PA formula which can interpret as false under a putative 'non-standard' interpretation $\mathcal{I}_{PA(\mathbb{N},NS)}$ of PA, but true under the classical 'standard' interpretation $\mathcal{I}_{PA(\mathbb{N},S)}$ (see §2.A.b.), is—almost universally—believed to have been settled in the affirmative by Gödel in his seminal 1931 paper on formally 'undecidable' arithmetical propositions.

However, in this investigation we have shown, from an *evidence-based* (i.e., platform-independent) perspective, that—and why—this belief is misleading, and that we need to read the fine print of Gödels paper carefully (see §11.) to see why this belief is founded on an untenable assumption (see §14.) whose roots lie in the faith-based (see §4.B.) extrapolation of Aristotle's particularisation to infinite domains.

Moreover, since we have shown that PA is categorical (see §2.E., Corollary 2.18), it would follow that:

Thesis 34. *Any extra-terrestrial intelligence which is capable of learning the language PA would interpret the satisfaction, and truth, of the formulas of PA—under a well-defined interpretation of PA—precisely as would a human intelligence.*

Finally, if we accept Thesis 24, Thesis 25, and Thesis 26, then it is reasonable to posit further that:

Thesis 35. *Since thousands, if not millions, of years would elapse before a putative first, spatially separated, contact with any extra-terrestrial intelligence, and any putative subsequent, proximal, contact between the two species:*

(a) any species that survives over eons must evolve in harmony with its environment through intellectual evolution and respect for natural law;

(b) categorical communications between the two species in the interregnum would engender, and nurture, respect for life forms with:

— seemingly comparable respect for natural law; and

— the capacity for survival through intellectual evolution.

In other words, on both temporal and spiritual grounds, our thoughts and explorations ought not to be fearfully limited by the false trails human societies have often followed in the past, but should be fearlessly guided by the lessons that current societies evince—by the direction of their present states of evolution—as having been learnt, no matter how hesitatingly and imperfectly, from past experiences; a direction that would suggest placing faith in the belief that:

Thesis 36. *Any extra-terrestrial intelligence would not rationally perceive us as being an essentially different form of intelligence that would necessarily be inimical to their interests and/or survival.*

We could even argue that an extra-terrestrial intelligence would be so far advanced, and so secure in its own evolution, that it would not behave irrationally when faced with evidence of a parallel cosmic evolution; and, moreover that we have a moral responsibility towards cosmic intelligence to leave a 'time capsule' that might, conceivably, outlast our species:

> "Near the end of my conversation with Frank Drake, I came back to the question of our increasingly quiet planet: all those inefficient radio and television signals giving way to the undetectable transmissions of the internet age. Maybe that's the long-term argument for sending intentional messages, I suggested; even if it fails in our lifetime, we will have created a signal that might enable an interstellar connection thousands of years from now.

> Drake leaned forward, nodding. "It raises a very interesting, nonscientific question, which is: Are extraterrestrial civilizations altruistic? Do they recognize this problem and establish a beacon for the benefit of the other folks out there? My answer is: I think it's actually Darwinian; I think evolution favors altruistic societies. So my guess is yes. And that means there might be one powerful signal for each civilization." Given the transit time across the universe, that signal might well outlast us as a species, in which case it might ultimately serve as a memorial as much as a message, like an interstellar version of the Great Pyramids: proof that a technologically advanced organism evolved on this planet, whatever that organism's ultimate fate."
>
> *. . . Johnson: [Jo17].*

Nevertheless—taking into consideration what is historically evident in human societies—there can be no definitive *scientific* perspective on what, eventually, *must* remain a globally *inclusive* political assessment:

> "There is something about the METI question that forces the mind to stretch beyond its usual limits. You have to imagine some radically different form of intelligence, using only your human intelligence. You have to imagine time scales on which a decision made in 2017 might trigger momentous consequences 10,000 years from now. The sheer magnitude of those consequences challenges our usual measures of cause and effect. Whether you believe that the aliens are likely to be warriors or Zen masters, if you think that METI has a reasonable chance of making contact with another intelligent organism somewhere in the Milky Way, then you have to accept that this small group of astronomers and science-fiction authors and billionaire patrons debating semi-prime numbers and the ubiquity of visual intelligence may in fact be wrestling with a decision that could prove to be the most transformative one in the history of human civilization.
>
> All of which takes us back to a much more down-to-earth, but no less challenging, question: Who gets to decide? After many years of debate, the SETI community established an agreed-upon procedure that scientists and government agencies should follow in the event that the SETI searches actually stumble upon an intelligible signal from space. The protocols specifically ordain that "no response to a signal or other evidence of extraterrestrial intelligence should be sent until appropriate international consultations have taken place." But an equivalent set of guidelines does not yet exist to govern our own interstellar outreach.
>
> One of the most thoughtful participants in the METI debate, Kathryn Denning, an anthropologist at York University in Toronto, has argued that our decisions about extraterrestrial contact are ultimately more political than scientific. "If I had to take a position, I'd say that broad consultation regarding METI is essential, and so I greatly respect the efforts in that direction," Denning says. "But no matter how much consultation there is, it's inevitable that there will be significant disagreement about the advisability of transmitting, and I don't think this is the sort of thing where a simple majority vote or even supermajority should carry the day . . . so this keeps bringing us back to the same key question: Is it O.K. for some people to transmit messages at significant power when other people dont want them to?"
>
> In a sense, the METI debate runs parallel to other existential decisions that we will be confronting in the coming decades, as our technological and scientific powers increase. Should we create superintelligent machines that exceed our own intellectual capabilities by such a wide margin that we cease to understand how their intelligence works? Should we "cure" death, as many technologists are proposing? Like METI, these are potentially among the most momentous decisions human beings will ever make, and yet the number of people actively participating in those decisions—or even aware such decisions are being made—is minuscule."
>
> *. . . Johnson: [Jo17].*

Johnson's disquiet echoes a parallel uneasiness—albeit in a different discipline—expressed by mathematician Melvyn B. Nathanson who, in his 2008 essay 'Desperately Seeking Mathematical Truth' ([Na08]; see also §16.), cautioned about the consequences of treating as definitive debatable decisions in the very foundations of mathematics where, too, 'the number of people

actively participating in those decisions—or even aware such decisions are being made—is minuscule'. Similar disquiet is shared, as Johnson reports further, by Denning:

> "When I asked Denning where she stands on the METI issue, she told me: "I have to answer that question with a question: Why are you asking me? Why should my opinion matter more than that of a 6-year-old girl in Namibia? We both have exactly the same amount at stake, arguably, she more than I, since the odds of being dead before any consequences of transmission occur are probably a bit higher for me, assuming she has access to clean water and decent health care and isn't killed far too young in war." She continued: "I think the METI debate may be one of those rare topics where scientific knowledge is highly relevant to the discussion, but its connection to obvious policy is tenuous at best, because in the final analysis, it's all about how much risk the people of Earth are willing to tolerate. ... And why exactly should astronomers, cosmologists, physicists, anthropologists, psychologists, sociologists, biologists, sci-fi authors or anyone else (in no particular order), get to decide what those tolerances should be?""
>
> *... Johnson:* [Jo17].

Johnson concludes that:

> "Wrestling with the METI question suggests, to me at least, that the one invention human society needs is more conceptual than technological: We need to define a special class of decisions that potentially create extinction-level risk. New technologies (like superintelligent computers) or interventions (like METI) that pose even the slightest risk of causing human extinction would require some novel form of global oversight. And part of that process would entail establishing, as Denning suggests, some measure of risk tolerance on a planetary level. If we don't, then by default the gamblers will always set the agenda, and the rest of us will have to live with the consequences of their wagers.
>
> In 2017, the idea of global oversight on any issue, however existential the threat it poses, may sound naïve. It may also be that technologies have their own inevitability, and we can only rein them in for so long: If contact with aliens is technically possible, then someone, somewhere is going to do it soon enough. There is not a lot of historical precedent for humans voluntarily swearing off a new technological capability—or choosing not to make contact with another society—because of some threat that might not arrive for generations. But maybe it's time that humans learned how to make that kind of choice. This turns out to be one of the surprising gifts of the METI debate, whichever side you happen to take. Thinking hard about what kinds of civilization we might be able to talk to ends up making us think even harder about what kind of civilization we want to be ourselves."
>
> *... Johnson:* [Jo17].

Part V
APPENDICES

23. Appendix A: Some comments on standard definitions, notations, and concepts

We take Elliott Mendelson [Me64], George Boolos et al [BBJ03], and Hartley Rogers [Rg87], as representative—in the areas that they cover—of standard expositions of classical first order logic and of effective computability.

1. **Axioms and rules of inference of the first-order Peano Arithmetic PA**

    PA$_1$ $[(x_1 = x_2) \rightarrow ((x_1 = x_3) \rightarrow (x_2 = x_3))]$;

    PA$_2$ $[(x_1 = x_2) \rightarrow (x'_1 = x'_2)]$;

    PA$_3$ $[0 \neq x'_1]$;

    PA$_4$ $[(x'_1 = x'_2) \rightarrow (x_1 = x_2)]$;

    PA$_5$ $[(x_1 + 0) = x_1]$;

    PA$_6$ $[(x_1 + x'_2) = (x_1 + x_2)']$;

    PA$_7$ $[(x_1 \star 0) = 0]$;

    PA$_8$ $[(x_1 \star x'_2) = ((x_1 \star x_2) + x_1)]$;

    PA$_9$ For any well-formed formula $[F(x)]$ of PA:
    $[F(0) \rightarrow (((\forall x)(F(x) \rightarrow F(x'))) \rightarrow (\forall x)F(x))]$.

    Generalisation in PA If $[A]$ is PA-provable, then so is $[(\forall x)A]$.

    Modus Ponens in PA If $[A]$ and $[A \rightarrow B]$ are PA-provable, then so is $[B]$.

2. **Church-Turing Thesis** We note that classical theory holds that:

    (a) Every Turing-computable function F is partial recursive, and, if F is total, then F is recursive ([Me64], p.233, Corollary 5.13).

    (b) Every partial recursive function is Turing-computable ([Me64], p.237, Corollary 5.15).

    From this, classical theory concludes that the following, essentially unverifiable (since it treats the notion of 'effective computability' as intuitive, and not definable formally) but refutable, theses (informally referred to as CT) are equivalent ([Me64], p.237):

    Church's Thesis: A number-theoretic function is effectively computable if, and only if, it is recursive ([Me64], p.227).

    Turing's Thesis: A number-theoretic function is effectively computable if, and only if, it is Turing-computable ([BBJ03], p.33).

3. **Conservative extension**: A theory T_2 is a (proof theoretic) conservative extension of a theory T_1 if the language of T_2 extends the language of T_2; that is, every theorem of T_1 is a theorem of T_2, and any theorem of T_2 in the language of T_1 is already a theorem of T_1.

4. **First-order language** ([Me64], p.29): A first-order language L consists of:

   (1) A countable set of symbols. A finite sequence of symbols of L is called an *expression* of L;

   (2) There is a subset of the expressions of L called the set of *well-formed formulas* (abbreviated 'wffs') of L;

   (3) There is an effective procedure (based on *evidence-based* reasoning) to determine whether a specified expression of L is a wff of L.

   Moreover—reflecting the *evidence-based* perspective of this investigation as detailed in the proposed Definitions 24 to 26—we shall explicitly distinguish between a first-order language and:

   - any *first-order theory* that seeks—on the basis of *evidence-based* reasoning—to assign the values 'provable/unprovable' to the well-formed formulas of the language under a *proof-theoretic logic*;

   - any *first-order theory* that seeks—on the basis of *evidence-based* reasoning—to assign the values 'true/false' to the well-formed formulas of the language under a *model-theoretic logic*.

5. **First-order language with quantifiers** ([Me64], pp.56-57): A first-order language K with quantifiers is a first-order language whose alphabet consists of:

   (1) The propositional connectives '¬' and '→';

   (2) The punctuation marks '(', ')' and ',';

   (3) Denumerably many individual variables $x_1, x_2, \ldots,$;

   (4) A finite or denumerable non-empty set of predicate letters $A_j^n (n, j \geq 1)$;

   (5) A finite or denumerable, possibly empty, set of function letters $f_j^n (n, j \geq 1)$;

   (6) A finite or denumerable, possibly empty, set of individual constants $a_i (i \geq 1)$;

   where the function letters applied to the variables and individual constants generate the *terms* as follows:

   (a) Variables and individual constants are terms;

   (b) If f_i^n is a function letter, and $t_1, \ldots, t_n$ are terms, then $f_i^n(t_1, \ldots, t_n)$ is a term;

   (c) An expression of K is a term only if it can be shown (on the basis of *evidence-based* reasoning) to be a term on the basis of clauses (a) and (b).

   Further:

   (d) The predicate letters applied to terms yield the *atomic formulas*, i.e., if A_i^n is a predicate letter and $t_1, \ldots, t_n$ are terms, then $A_i^n(t_1, \ldots, t_n)$ is an atomic formula.

and:

(e) The well-formed formulas (wffs) of K are defined as follows:

  (i) Every atomic formula is a wff;

  (ii) If $\mathcal{A}$ and $\mathcal{B}$ are wffs and y is a variable, then '$\neg\mathcal{A}$', '$\mathcal{A} \rightarrow \mathcal{B}$' and '$(\forall y)\mathcal{A}$' are wffs;

  (iii) An expression of K is a wff of K only if it can be shown (on the basis of *evidence-based* reasoning) to be a wff on the basis of clauses (i) and (ii).

Moreover, we follow the convention that defines:

(f) '$\mathcal{A} \wedge \mathcal{B}$' as an abbreviation for '$\neg(\mathcal{A} \rightarrow \mathcal{B})$';

(g) '$\mathcal{A} \vee \mathcal{B}$' as an abbreviation for '$(\neg\mathcal{A}) \rightarrow \mathcal{B}$';

(h) '$\mathcal{A} \equiv \mathcal{B}$' as an abbreviation for '$(\mathcal{A} \rightarrow \mathcal{B}) \wedge (\mathcal{B} \rightarrow \mathcal{A})$';

(i) '$(\exists x)\mathcal{A}$' as an abbreviation for '$\neg((\forall x)\neg\mathcal{A})$'.

6. **First-order theory with quantifiers** ([Me64], pp.56-57): A first-order theory S with quantifiers is a first-order language with quantifiers plus a set of rules—which we define as the *proof-theoretic logic* of S—that assigns *evidence-based* 'provability' values to the wffs of S by means of logical axioms, proper axioms, and rules of inference as follows:

  I: If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are wffs of S, then the following logical axioms are designated as provable wffs of S:

  (1) $\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{A})$;

  (2) $(\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})) \rightarrow ((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C}))$;

  (3) $(\neg\mathcal{B} \rightarrow \neg\mathcal{A}) \rightarrow ((\neg\mathcal{B} \rightarrow \mathcal{A}) \rightarrow \mathcal{B})$;

  (4) $(\forall x_i)\mathcal{A}(x_i) \rightarrow \mathcal{A}(t)$ if $\mathcal{A}(x_i)$ is a wff of S and t is a term of S free for x_i in $\mathcal{A}(x_i)$;

  (5) $(\forall x_i)(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow (\forall x_i)\mathcal{B})$ if $\mathcal{A}$ is a wff of S containing no free occurences of x_i.

  II: The proper axioms of S which are to be designated as provable wffs of S vary from theory to theory.

  A first-order theory in which there are no proper axioms is called the first-order logic FOL.

  III: The rules of inference of any first-order theory are:

  (i) Modus ponens: If $\mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{B}$ are provable wffs of S, then $\mathcal{B}$ is a provable formula of S;

  (ii) Generalisation: If $\mathcal{A}$ is a provable wff of S, then $(\forall x_i)\mathcal{A}$ is a provable wff of S.

  IV: A wff $\mathcal{A}$ of S is provable if, and only if:

    – $\mathcal{A}$ is a logical axiom of S; or

    – $\mathcal{A}$ is a proper axiom of S; or

    – $\mathcal{A}$ is the final wff of a finite sequence of wffs of S such that each formula of the sequence is:

- either an axiom of S,

- or is a provable formula of S by application of the rules of inference of S to the formulas preceding it in the sequence.

Moreover, we define a first-order theory S with quantifiers as *well-defined model-theoretically* if, and only if, it has a *well-defined model* in the sense of the proposed Definitions 24 to 26.

7. **Hilbert's Second Problem:** In this investigation, we treat Hilbert's intent[201] behind the enunciation of his Second Problem as essentially seeking a finitary proof for the consistency of arithmetic when formalised in a language such as the first-order Peano Arithmetic PA.

> "When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. . . . But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results. In geometry, the proof of the compatibility of the axioms can be effected by constructing a suitable field of numbers, such that analogous relations between the numbers of this field correspond to the geometrical axioms. . . . On the other hand a direct method is needed for the proof of the compatibility of the arithmetical axioms."
>
> *. . . Excerpted from Maby Winton Newson's English translation [Nw02] of Hilbert's address [Hi00] at the International Congress of Mathematicians in Paris in 1900.*

8. **Interpretation** ([Me64], p.49): An *interpretation* of the:

- predicate letters;

- function letters; and

- individual and logical constants;

of a formal system S consists of:

- a non-empty set D, called the *domain* of the interpretation;

and an *evidence-based* assignment:

- to each predicate letter A_j^n of an n-place relation in D;

- to each function letter f_j^n of an n-place operation in D (i.e., a function from D into D); and

- to each individual constant a_i of some fixed element of D.

Given such an interpretation, variables are thought of as ranging over the set D, and $\neg, \rightarrow$, and quantifiers are given their usual meaning.

Moreover, we define an interpretation as *well-defined* if, and only if, all the above assignments are *well-defined* in the sense of the proposed Definitions 24 to 26.

---

[201] Compare Curtis Franks' thesis in [Fr09] that Hilbert's intent behind the enunciation of his Second Problem was essentially to establish the autonomy of arithmetical truth without appeal to any debatable philosophical considerations.

9. **Law of the Excluded Middle LEM** (cf., [Me64], p.4): For any well-formed formula P of a formal system S, $P \ v \ \neg P$ is a theorem of S.

10. **Model** ([Me64], p.49): An interpretation $\mathcal{I}$ defines a *model* of a formal system S if, and only if, there is a set of rules—which we define as the *model-theoretic logic* of S—that assign *evidence-based* truth values of 'satisfaction', 'truth', and 'falsity' to the formulas of S under $\mathcal{I}$ such that the axioms of S interpret as 'true' under $\mathcal{I}$, and the rules of inference of S preserve such 'truth' under $\mathcal{I}$.

    Moreover, we define a model as *well-defined* if, and only if, it is defined by a *well-defined* interpretation in the sense of the proposed Definitions 24 to 26.

11. **Non-standard model**: A non-standard model of a formal system S is a model of S that admits elements in the domain D of the interpretation defining the model if, and only if, there is some element in D which is not definable in S.

12. **ω-consistency:** A formal system S is ω-consistent if, and only if, there is no S-formula $[F(x)]$ for which, first, $[\neg(\forall x)F(x)]$ is S-provable and, second, $[F(a)]$ is S-provable for any specified S-term $[a]$.

13. **Partial recursive**: Classically, a partial function F of n arguments is called partial recursive if, and only if, F can be obtained from the initial functions (zero function), projection functions, and successor function (of classical recursive function theory) by means of substitution, recursion and the classical, unrestricted, μ-operator.

    F is said to come from G by means of the unrestricted μ-operator, where $G(x_1, \ldots, x_n, y)$ is recursive, if, and only if, $F(x_1, \ldots, x_n) = \mu y(G(x_1, \ldots, x_n, y) = 0)$, where $\mu y(G(x_1, \ldots, x_n, y) = 0)$ is the least number k (if such exists) such that, if $0 \leq i \leq k$, $G(x_1, \ldots, x_n, i)$ exists and is not 0, and $G(x_1, \ldots, x_n, k) = 0$.

    We note that, classically, F may not be defined for certain n-tuples; in particular, for those n-tuples $(x_1, \ldots, x_n)$ for which there is no y such that $G(x_1, \ldots, x_n, y) = 0$ (cf. [Me64], p.120-121).

14. **Tarski's inductive definitions:** We shall assume that truth values of 'satisfaction', 'truth', and 'falsity' are assignable inductively to the compound formulas of a first-order theory S under the interpretation $\mathcal{I}_{S(\mathbb{D})}$ in terms of *only* the satisfiability of the atomic formulas of S over $\mathbb{D}$ as usual (see [Me64], pp.50-51; [Mu91]):

    (a) If A is an atomic well-formed formula $A_j^n(t_1, \ldots, t_n)$ of $\mathbb{S}$, and B_j^n is the corresponding relation of the interpretation (of $\mathbb{S}$) with domain $\mathbb{D}$, then the sequence s satisfies A if and only if $B_j^n(s^*t_1, \ldots, s^*t_n)$, i.e., if the n-tuple $(s^*t_1, \ldots, s^*t_n)$ is in the relation B_j^n (where s^* is a function, determined by the sequence s, from the set of terms of $\mathbb{S}$ into $\mathbb{D}$).

    (b) A denumerable sequence s of $\mathbb{D}$ satisfies $[\neg A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, s does not satisfy $[A]$;

    (c) A denumerable sequence s of $\mathbb{D}$ satisfies $[A \to B]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, either it is not the case that s satisfies $[A]$, or s satisfies $[B]$;

(d) A denumerable sequence s of $\mathbb{D}$ satisfies $[(\forall x_i)A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, specified any denumerable sequence t of $\mathbb{D}$ which differs from s in at most the i'th component, t satisfies $[A]$;

(e) A well-formed formula $[A]$ of $\mathbb{D}$ is true under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, specified any denumerable sequence t of $\mathbb{D}$, t satisfies $[A]$;

(f) A well-formed formula $[A]$ of $\mathbb{D}$ is false under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, it is not the case that $[A]$ is true under $\mathcal{I}_{S(\mathbb{D})}$.

The implicit assumption of Aristotle's particularisation is evidenced in Mendelson's assertion—following his formulation of Tarski's definitions (essentially as above)—that:

> "Verification of the following consequences of the definitions above is left to the reader. (Most of the results are also obvious if one wishes to use only the ordinary intuitive understanding of the notions of truth and satisfaction). . . .
>
> (V) (i) A sequence s satisfies $A \wedge B$ if and only if s satisfies A or s satisfies B. A sequence s satisfies $A \equiv B$ if and only if s satisfies both A and B or s satisfies neither A nor B.
>
> (ii) A sequence s satisfies $(Ex_i)A$ if and only if there is a sequence s' which differs from s in at most the ith place such that s' satisfies A."
>
> . . . Mendelson: [Me64], p.53.

15. **Total**: We define a number-theoretic function, or relation, as total if, and only if, it is effectively computable, or effectively decidable, respectively, for any specified set of natural number values assigned to its free variables. We define a number-theoretic function, or relation, as partial otherwise. We define a partial number theoretic function, or relation, as effectively computable, or decidable, respectively, if, and only if, it is effectively computable, or decidable, respectively, for any specified set of values assigned to its free variables for which it is defined (cf. [Me64], p.214).

16. **_Weak_ standard interpretation of PA** (cf. [Me64], p.107): The _weak_ standard interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA over the domain $\mathbb{N}$ of the natural numbers is the one in which the logical constants have their 'usual' interpretations in the first-order predicate logic FOL, and:

(a) The set of non-negative integers is the domain;

(b) The symbol $[0]$ interprets as the integer 0;

(c) The symbol $[']$ interprets as the successor operation (addition of 1);

(d) The symbols $[+]$ and $[\star]$ interpret as ordinary addition and multiplication;

(e) The symbol $[=]$ interprets as the identity relation.

Comment: In this investigation, unless explicitly specified otherwise, we do not assume that Aristotle's particularisation holds under the the standard interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA or under any interpretation of FOL.

Reason: Contrary to what is _implicitly_ suggested in standard literature and texts—Aristotle's particularisation does not form any part of Tarski's _inductive_ definitions of the satisfaction, and truth, of the formulas of PA under the standard interpretation $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ of PA, but is an extraneous, generally _implicit_, assumption in the underlying first-order logic FOL.

Moreover, its inclusion not only makes $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ non-finitary (as argued by Brouwer in [Br08]), but the assumption of Aristotle's particularisation does not hold in any model of PA and, ipso facto, of FOL(see [**?**], Corollary 4.7)!

17. **Weak** **standard model of PA**: The *weak* standard model of PA is the one defined by the classical standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA over the domain $\mathbb{N}$ of the natural numbers.

23.A.   Appendix A1: Turing Algorithm

(*Excerpted from Elliott Mendelson [Me64], pp.229-231.*)

"Attempting to give a precise definition of effective computability, Turing [1936] proposed that a certain class of abstract machines could perform any "mechanical" computing procedure. Such machines are now called Turing machines in honour of their inventor, and can be described in the following way.

There is a two-way potentially infinite tape divided up into squares,

...			S_2	S_1	S_1	S_0		S_3				...

The tape is said to be potentially infinite in the sense that, although at any moment it is finite in length, additional squares always can be added to the right- and left-hand ends of the tape. There is a finite set of *tape symbols* $S_0, S_1, \ldots, S_n$ called the *alphabet* of the machine; at every moment, each square of the tape is occupied by at most one symbol. The machine has a finite set of *internal states* $\{q_0, q_1, \ldots, q_m\}$. At any given moment, the machine is in exactly one of these states. Finally, there is a reading head which, at any given time, stands over some square of the tape. The machine does not act continuously, but only at discrete moments of time. If, at any moment t, the reading head is scanning (i.e., standing over) a square containing a symbol S_i and the machine is in the internal state q_j, then the action of the machine is determined, and it will do one of four things: (1) it may erase the symbol S_i and print a new symbol S_k; (2) it may move left one square; (3) it may move right one square; (4) it may stop. In cases (1)-(3), the machine goes into a new internal state q_r, and is ready to act again at time t+1. We shall assume that the symbol S_0 represents a blank, so that the reading head may always be assumed to be scanning a symbol. The first three actions of the machine just described can be represented by quadruplets: either (1) $q_j S_i S_k q_r$, or (2) $q_j S_i L q_r$, or (3) $q_j S_i R q_r$. The first two symbols stand for the present internal state and scanned symbol, the third symbol represents the action of the machine (print S_k, or move left, or move right one square), and the fourth symbol gives the internal state of the machine after the action has been performed.

If a tape is put into a Turing machine and the reading head is placed on a certain square, and if the machine is started off in one of its internal states, then the machine begins to operate on the tape: printing and erasing symbols and moving from one square to an adjacent one. If the machine ever stops, the resulting tape is said to be the output of the machine applied to the given tape. Now we can associate with any Turing machine T the following algorithm $\mathcal{B}$ in the alphabet A of T. Take any word P in the alphabet A and print it from left to write in the squares of an empty tape. Place this tape in the machine with the reading head scanning the left-most square. Start the machine in the internal state q_0. If the machine ever stops, the word of A appearing on the tape is the value of the algorithm $\mathcal{B}$. $\mathcal{B}$ is called a *Turing algorithm*. (The word appearing on the tape is defined to be the sequence of symbols beginning with the left-most symbol and moving right to the right-most symbol. Remember that a blank symbol encountered in this motion is assumed to have the symbol S_0 printed in it). We have not specified yet the mechanism by which a machine knows when to stop; this will be done below.

Any Turing machine can be determined precisely by a finite set of quadruples of the three kinds: (1) $q_j S_i S_k q_r$; (2) $q_j S_i L q_r$; (3) $q_j S_i R q_r$, such that no two quadruples have the same first two symbols. In fact, we shall now define a *Turing machine* to be such a finite set of quadruples. The *alphabet* of any Turing machine is the set of tape symbols S_m appearing in any of the quadruples. The *internal states* of the machine are the symbols q_r appearing in the quadruples. We assume that q_0 is an internal state of every Turing machine.

An *instantaneous tape description* of a Turing machine T is a word such that (i) all symbols in the word but one are tape symbols S_m; (ii) the only symbol which is not a tape symbol is an internal state q_s; (iii) q_s is not the last symbol of the word.†

> † An instantaneous tape description describes the condition of the machine and the tape at any given moment. When read from left to right, the tape symbols in the description represent the symbols on the tape at the moment. The internal state q_s in the description is the internal state of the machine at the moment, and the tape symbol occurring immediately to the right of q_s in the tape description represents the symbol being scanned by the machine at the moment.

We say that T *moves* one instantaneous tape description α into another one β (abbreviated $\alpha \underset{T}{\to} \beta$) if and only if either (a) α is of the form $Pq_j S_i Q$, β is of the form $Pq_r S_k Q$, and $q_j S_i S_k q_r$ is one of the quadruples of T; or (b) α is of the form $PS_s q_j S_i Q$, β is $Pq_r S_s S_i Q$, and $q_j S_i L q_r$ is one of the quadruples of T; or (c) α is of the form $q_j S_i Q$, β is $q_r S_0 S_i Q$, and $q_j S_i L q_r$ is one of the quadruples of T; or (d) α is of the form $Pq_j S_i S_k Q$, β is $PS_i q_r S_k Q$, and $q_j S_i R q_r$ is one of the quadruples of T; or (e) α is of the form $Pq_j S_i$, β is $PS_i q_r S_0$, and $q_j S_i R q_r$ is one of the quadruples of T.‡

> ‡ Observe that, according to our intuitive picture, "T moves α into β" implies that if the condition at time t of the Turing machine and tape is described by α, then the condition at time t+1 is described by β. Notice that, according to clause (c), whenever the machine reaches the left-hand end of the tape and is ordered to move left, a blank square is attached to the tape on the left; similarly, by clause (e), a blank square is attached on the right when the machine reaches the right-hand end of the tape and has to move right.

We say that T *stops* at an instantaneous tape description α if and only if there is no instantaneous tape description β such that $\alpha \underset{T}{\to} \beta$. (This happens when $q_j S_i$ occurs in α but $q_j S_i$ are not the first two symbols of a quadruple of T.)

A *computation* of a Turing machine T is a finite sequence of instantaneous tape descriptions $\alpha_0, \ldots, \alpha_m$ (m $\geq$ 0) such that the internal state occurring in α_0 is q_0; for $0 \leq i < m$, $\alpha_i \underset{T}{\to} \alpha_{i+1}$; and T stops at α_m. This computation is said to begin with α_0 and end with α_m. The algorithm $\mathcal{B}_{T,C}$ in any alphabet C containing the alphabet A of T is defined as follows: for any words P, Q in C, $\mathcal{B}_{T,C}(P) = Q$ if and only if there is a computation of T which begins with the instantaneous tape description $q_0 P$ and ends with an instantaneous tape description of the form $R_1 q_j R_2$, where $Q = R_1 R_2$. An algorithm $\mathcal{U}$ in an alphabet D is called Turing-computable if and only if there is a Turing machine T with alphabet A and an alphabet C containing A $\cup$ D such that $\mathcal{B}_{T,C}$ and $\mathcal{U}$ are fully equivalent relative to D.

We let 1 stand for S_1. Remember that $\overline{m}$ stands for 1^{m+1}, for any natural number m. Also, let $\star$ be an abbreviation of S_2. Given a partial number-theoretic function f($x_1, \ldots, x_n$), we say that a Turing machine T (whose alphabet A includes $\{1, \star\}$) *computes* f if and only if, for any natural numbers $k_1, \ldots, k_n$, and any word Q, $\mathcal{B}_{T,A}(\overline{k_1} \star \overline{k_2} \star \ldots \star \overline{k_n}) = Q$ if and only if Q is $R_1 \overline{f(k_1, \ldots, k_n)} R_2$, where both R_1 and R_2 are certain (possibly empty) words consisting only of S_0's. (The form $R_1 \overline{f(k_1, \ldots, k_n)} R_2$ is allowed for the result since S_0 is interpreted as a blank.) The function f is called *Turing-computable* if and only if there is a Turing machine T which computes f."

. . . Mendelson: [Me64], pp.229-231.

23.B. Appendix A2: Gödel-numbering

(*Excerpted from [Go31], pp.13-14.*)

"We now set up a one-to-one correspondence of natural numbers to the primitive symbols of the system P in the following manner:

"0"	... 1	" $\vee$ "	... 7	"("	... 11
"f"	... 3	"Π"	... 9	")"	... 13
" $\sim$ "	... 5				

and furthermore, to the variables of n-th type we assign the numbers of the form p^n (where p is a prime number > 13). Thus, to every finite sequence of primitive symbols (hence also to every formula), there corresponds in a one-to-one fashion a finite sequence of positive integers. We map (again in a one-to-one fashion) the finite sequences of positive integers into the natural numbers by letting the number $2^{n_1}.3^{n_2}\ldots p_k^{n_k}$ correspond to the sequence $n_1, n_2, \ldots, n_k$, where p_k denotes the k-th prime number (according to magnitude). Hence a natural number is correlated in one-to-one fashion not only to every primitive symbol but also to every finite sequence of such symbols."

. . . Gödel: [Go31], pp.13-14.

24.   Appendix B: Definitions, Queries, Theses, Lemmas, Theorems and Corollaries

24.A.   Definitions

§1. **Definition 1. (Well-defined logic)** *A finite set λ of rules is a well-defined logic of a formal mathematical language L if, and only if, λ assigns unique, evidence-based, values:*

   (a) *Of provability/unprovability to the well-formed formulas of L; and*

   (b) *Of truth/falsity to the sentences of the Theory $T(U)$ which is defined semantically by the λ-interpretation of L over a given mathematical structure U that may, or may not, be well-defined; such that*

   (c) *The provable formulas interpret as true in $T(U)$.*

§2. **Definition 2. (Algorithmic verifiability)** *A number-theoretical relation $F(x)$ is algorithmically verifiable if, and only if, for any given natural number n, there is an algorithm $AL_{(F,\ n)}$ which can provide objective evidence for deciding the truth/falsity of each proposition in the finite sequence $\{F(1), F(2), \ldots, F(n)\}$.*

§2. **Definition 3. (Algorithmic computability)** *A number theoretical relation $F(x)$ is algorithmically computable if, and only if, there is an algorithm AL_F that can provide objective evidence for deciding the truth/falsity of each proposition in the denumerable sequence $\{F(1), F(2), \ldots\}$.*

§2.A. **Definition 4.** *If $[A]$ is an atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of a formal language S, then the denumerable sequence $(a_1, a_2, \ldots)$ in the domain $\mathbb{D}$ of an interpretation $\mathcal{I}_{S(\mathbb{D})}$ of S satisfies $[A]$ if, and only if:*

   (i) *$[A(x_1, x_2, \ldots, x_n)]$ interprets under $\mathcal{I}_{S(\mathbb{D})}$ as a unique relation $A^*(x_1, x_2, \ldots, x_n)$ in $\mathbb{D}$ for any witness $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$;*

   (ii) *there is a Satisfaction Method that provides objective evidence by which any witness $\mathcal{W}_{\mathbb{D}}$ of $\mathbb{D}$ can objectively **define** for any atomic formula $[A(x_1, x_2, \ldots, x_n)]$ of S, and any given denumerable sequence $(b_1, b_2, \ldots)$ of $\mathbb{D}$, whether the proposition $A^*(b_1, b_2, \ldots, b_n)$ holds or not in $\mathbb{D}$;*

   (iii) *$A^*(a_1, a_2, \ldots, a_n)$ holds in $\mathbb{D}$ for any $\mathcal{W}_{\mathbb{D}}$.*

§2.A.a. **Definition 5.** *A denumerable sequence s of $\mathbb{D}$ satisfies $[\neg A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, s does not satisfy $[A]$;*

§2.A.a. **Definition 6.** *A denumerable sequence s of $\mathbb{D}$ satisfies $[A \rightarrow B]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, either it is not the case that s satisfies $[A]$, or s satisfies $[B]$;*

§2.A.a. **Definition 7.** *A denumerable sequence s of $\mathbb{D}$ satisfies $[(\forall x_i)A]$ under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, given any denumerable sequence t of $\mathbb{D}$ which differs from s in at most the i'th component, t satisfies $[A]$;*

§2.A.a. **Definition 8.** *A well-formed formula $[A]$ of $\mathbb{D}$ is true under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, given any denumerable sequence t of $\mathbb{D}$, t satisfies $[A]$;*

§2.A.a. **Definition 9.** *A well-formed formula [A] of $\mathbb{D}$ is false under $\mathcal{I}_{S(\mathbb{D})}$ if, and only if, it is not the case that [A] is true under $\mathcal{I}_{S(\mathbb{D})}$.*

§2.B. **Definition 10.** *An atomic formula [A] of PA is satisfiable under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ if, and only if, [A] is algorithmically verifiable under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

§2.C. **Definition 11.** *An atomic formula [A] of PA is satisfiable under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$ if, and only if, [A] is algorithmically computable under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

§4. **Definition 12. (Aristotle's particularisation)** *If the formula $[\neg(\forall x)\neg F(x)]$ of a formal first order language L is defined as 'true' under an interpretation, then we may always conclude unrestrictedly that there must be some unspecified object s in the domain D of the interpretation such that, if the formula $[F(x)]$ interprets as the relation $F^*(x)$ in D, then the proposition $F^*(s)$ is 'true' under the interpretation.*

§4.C. **Definition 13. (Weak quantification)** *A PA formula $[(\forall x)F(x)]$ is algorithmically verifiable as true under an interpretation if, and only if, $F^*(x)$ is algorithmically verifiable as always true.*

§4.C. **Definition 14. (Deterministic algorithm)** *A deterministic algorithm is a well-defined mechanical method, such as a Turing machine, that computes a mathematical function which has a unique value for any input in its domain, and the algorithm is a process that produces this particular value as output.*

§4.C. **Definition 15. (Weak negation)** *The PA formula $[\neg(\forall x)F(x)]$ is algorithmically verifiable as true under an interpretation if, and only if, there is no algorithm which will evidence that $[(\forall x)F(x)]$ is algorithmically verifiable as always true under the interpretation.*

§4.C. **Definition 16. (Strong quantification)** *A PA formula $[(\forall x)F(x)]$ is algorithmically computable as true under an interpretation if, and only if, $F^*(x)$ is algorithmically computable as always true.*

§4.C. **Definition 17. (Strong negation)** *The PA formula $[\neg(\forall x)F(x)]$ is algorithmically computable as true under an interpretation if, and only if, there is no algorithm which will evidence that $[(\forall x)F(x)]$ is algorithmically computable as always true under the interpretation.*

§4.F. **Definition 18. (Well-defined sequence)** *A Boolean number-theoretical sequence $\{F^*(1), F^*(2), \ldots\}$ is well-defined if, and only if, the number-theoretical relation $F^*(x)$ is algorithmically verifiable.*

§4.F. **Definition 19. (Well-definedness)** *A mathematical concept is well-defined if, and only if, it can be defined in terms of algorithmic verifiability.*

§4.G. **Definition 20. (Relative randomness)** *A number-theoretic sequence is relatively random if, and only if, it is algorithmically verifiable, but not algorithmically computable.*

§4.G.a. **Definition 21. (Effective computability)** *A number-theoretical sequence $\{F^*(n)\}$ is effectively computable if, and only if, it is well-defined.*

§7.C. **Definition 22. (Integer specifiability)** *A natural number n in $\mathbb{N}$ is specifiable if, and only if, it can be explicitly denoted as a PA-numeral by a PA-formula that interprets as an algorithmically computable constant.*

§8.C. **Definition 23. (Well-defined interpretation)** *An interpretation $\mathcal{I}$ of a formal language L, over a domain D of a structure $\mathcal{S}$, is constructively well-defined relative to an assignment of truth values $T_{\mathcal{I}}$ to the formulas of L if, and only if, the provable formulas of L interpret as true over D under $\mathcal{I}$ relative to the assignment of truth values $T_{\mathcal{I}}$.*

§9. **Definition 24. (Proof-theoretic logic)** *The proof-theoretic logic of a first-order theory S is a set of rules consisting of:*

- *a selected set of well-formed formulas of S labelled as 'axioms/axiom schemas' that are assigned the value 'provable'; and*

- *a finitary set of rules of inference in S;*

that assign evidence-based values of 'provable' or 'unprovable' to the well-formed formulas of S by means of the axioms and rules of inference of S.

§9. **Definition 25.** (**Model-theoretic logic**) *The model-theoretic logic of a first-order theory S with a proof-theoretic logic is a set of rules that assign evidence-based truth values of 'satisfaction', 'truth', and 'falsity' to the well-formed formulas of S under an interpretation $\mathcal{I}$ such that the axioms of S interpret as 'true' under $\mathcal{I}$, and the rules of inference of S preserve such 'truth' under $\mathcal{I}$.*

§9. **Definition 26.** (**Well-defined model**) *The Theory $T(\mathcal{U})$ defined semantically by the λ-interpretation of a formal mathematical language $\mathcal{L}$ over the structure $\mathcal{U}$ is a constructively well-defined model of $\mathcal{L}$ if, and only if, λ is a constructively well-defined Logic of $\mathcal{L}$.* □

§9. **Definition 27.** (**Constructive mathematics**) *Constructive mathematics is the study of formal mathematical languages that have a constructively well-defined logic.* □

§9.A.d. **Definition 28.** (**Wittgensteinian interpretation**) *An atomic formula $[A(x)]$ of PA is satisfied under $\mathcal{I}_{PA(PA, \ W)}$ if, and only if, for any substitution of a specified PA-numeral $[n]$ for the variable $[x]$, there is a deterministic algorithm which will evidence that the formula $[A(n)]$ is provable in PA.*

§9.A.d. **Definition 29.** *The formula $[(\forall x)A(x)]$ of PA is true under $\mathcal{I}_{PA(PA, \ W)}$ if, and only if, $[A(x)]$ is satisfied under $\mathcal{I}_{PA(\mathbb{N}, \ W)}$.*

§10.D.b. **Definition 30.** (**Meaning of a formula under interpretation**) *If a number-theoretic relation $R(x_1, \ldots, x_n)$ is expressible by a PA-formula $[A(x_1, \ldots, x_n)]$ then, under any well-defined interpretation of PA, $A(x_1, \ldots, x_n)$ means $R(x_1, \ldots, x_n)$ if, and only if, $R(x_1, \ldots, x_n)$ is the well-defined interpretation, under $\mathcal{I}$ in $\mathbb{N}$, of some PA-formula that expresses $R(x_1, \ldots, x_n)$ in PA.*

§12.A. **Definition 31.** *A number-theoretic function $f(x_1, \ldots, x_n)$ is said to be representable in the first order Peano Arithmetic PA if, and only if, there is a PA formula $[F(x_1, \ldots, x_{n+1})]$ with the free variables $[x_1, \ldots, x_{n+1}]$, such that, for any specified natural numbers $k_1, \ldots, k_{n+1}$:*

(i) *if $f(k_1, \ldots, k_n) = k_{n+1}$ then PA proves: $[F(k_1, \ldots, k_n, k_{n+1})]$;*

(ii) *PA proves: $[(\exists_1 x_{n+1})F(k_1, \ldots, k_n, x_{n+1})]$.*

The function $f(x_1, \ldots, x_n)$ is said to be strongly representable in PA if we further have that:

(iii) *PA proves: $[(\exists_1 x_{n+1})F(x_1, \ldots, x_n, x_{n+1})]$.* □

§17.F. **Definition 32.** (**Non-terminating loop**) *A non-terminating loop is any repetition of the instantaneous tape description of a Turing machine during a computation.*

§17.J. **Definition 33.** (**Analog process**) *A physical process is analog if, and only if, it's states can be represented mathematically by a number-theoretic function that is algorithmically verifiable.*

§17.J. **Definition 34.** (**Digital process**) *A physical process is digital if, and only if, it's states can be represented mathematically by a number-theoretic function that is algorithmically computable.*

§18. **Definition 35.** (**Signature of a natural number**) *The signature of a specified integer n is the sequence of residues $< a_{n,i} >$ where $n + a_{n,i} \equiv 0 \ mod \ (p_i)$ for all primes p_i such that $1 \le i \le \pi(\sqrt{n})$.*

§18. **Definition 36.** (**Value of a natural number**) *The value of a specified integer n is any well-defined interpretation—over the domain of the natural numbers—of the (unique) numeral $[n]$ that represents n in the first-order Peano Arithmetic PA.*

§18.A. **Definition 37.** *Two events are independent if the occurrence of one event does not influence (and is not influenced by) the occurrence of the other.*

§18.A.c. **Definition 38.** (**Bazeries Cylinder**) *A modified Bazeries Cylinder is a set of polygonal wheels—not necessarily identical (such as B_i and B_j in Fig. 1)—rotating independently on a common spindle, whose faces are coded with, say, integer values 0 to $i - 1$ and 0 to $j - 1$ respectively, where the event $B_i(u)$ (Fig 2) is the value $0 \le u \le i - 1$ yielded by a spin of a single i-faced Bazeries wheel B_i, and the event $B_{ij}(u,v)$ (Fig, 3) is the value (u,v)—where $0 \le u \le i - 1$ and $0 \le v \le j - 1$—yielded by simultaneous, but independent, spins of an i-faced Bazeries wheel B_i and a j-faced Bazeries wheel B_j.*

§18.A.e. **Definition 39.** *A deterministic algorithm computes a number-theoretical function $f(n)$ in polynomial-time if there exists k such that, for all inputs n, the algorithm computes $f(n)$ in $\le (log_e \ n)^k + k$ steps.*

§*18.B.* **Definition 40.** $n + r_i(n) \equiv 0 \ (mod \ i)$ *where* $i > r_i(n) \geq 0$.

§*18.B.a.* **Definition 41.** *Two events* e_i *and* e_j *are mutually independent for* $i \neq j$ *if, and only if,* $\mathbb{P}(e_i \cap e_j) = \mathbb{P}(e_i).\mathbb{P}(e_j)$.

§*18.C.b.* **Definition 42.** $\pi_D(n) = \sum_{l=1}^{n} (\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^{k} (1 - \frac{1}{q_i})^{-1} \cdot \prod_{j=1}^{\pi(\sqrt{l})} (1 - \frac{1}{p_j}))$.

§*18.D.* **Definition 43.** *An integer* n *is a* $\mathbb{TW}(k)$ *integer if, and only if,* $r_{p_i}(n) \neq 0$ *and* $r_{p_i}(n) \neq 2$ *for all* $1 \leq i \leq k$, *where* $0 \leq r_i(n) < i$ *is defined for all* $i > 1$ *by:*

$$n + r_i(n) \equiv 0 \ (mod \ i).$$

§*21.A.c.* **Definition 44. (Quantum verifier)** *A physical device is a quantum verifier if it verifies only the first* m *values of an algorithmically verifiable, but not algorithmically computable, number-theoretic function* f *whose unique Gödel beta-function for* m *is* g_m, *where* $g_i = f_i$ *for* $0 \leq i \leq m$, *and the value* $f(i)$ *is determined by a probability function governed by a function such as Schrödinger's* ψ-*function.*

§*21.A.c.* **Definition 45. (Knowledge verifier)** *A physical device is a knowledge verifier if, for any given* m, *it can verify the first* m *values of an algorithmically computable, number-theoretic, function* f *whose unique Gödel* β-*function for* m *is* g_m, *where* $g_i = f_i$ *for* $0 \leq i \leq m$.

§*21.A.c.* **Definition 46. (Quantum computer)** *A physical device is a quantum computer if, and only if, it is a quantum verifier.*

24.B.   Theses, Propositions, Queries

§1. **Thesis 1 (Complementarity Thesis).** *Mathematical 'provability' and mathematical 'truth' need to be interdependent and complementary, 'evidence-based', assignments-by-convention towards achieving:*

(1) *The goal of proof theory, post Peano, Dedekind and Hilbert, which is:*

— *to uniquely characterise each informally defined mathematical structure S (e.g., the Peano Postulates and their associated, classical, predicate logic),*

– *by a corresponding, formal, first-order language L, and a set P of finitary axioms/axiom schemas and rules of inference (e.g., the first-order Peano Arithmetic PA and its associated first-order logic FOL),*

- *which assign unique provability values (provable/unprovable) to each well-formed proposition of the language L without contradiction;*

(2) *The goal of constructive mathematics, post Brouwer and Tarski, which must be:*

— *to assign unique, evidence-based, truth values (true/false) to each well-formed proposition of the language L,*

– *under an, unarguably constructive, well-defined interpretation $\mathcal{I}$ over the domain D of the structure S,*

- *such that the provable formulas of L are true under the interpretation.*

§2.F. **Thesis 2.** *There can be no mechanist model of human reasoning if the assignment $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions, and the assignment $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.*

§5. **Proposition 5.1. (Algorithmic (weak) ω-Rule of infinite induction)** *If it is proved that the PA formula $[F(x)]$ interprets as an arithmetical relation $F^*(x)$ that is algorithmically computable as true for any specified natural number n, then the PA formula $[(\forall x)F(x)]$ can be admitted as an initial formula (axiom) in PA.*

§5. **Proposition 5.2. (Hilbert's (*strong*) ω-Rule of infinite induction)** *If it is proved that the PA formula $[F(x)]$ interprets as an arithmetical relation $F^*(x)$ that is algorithmically verifiable as true for any specified natural number n, then the PA formula $[(\forall x)F(x)]$ can be admitted as an initial formula (axiom) in PA.*

§5.B. **Proposition 5.4. (Gentzen ω-rule of infinite induction)** *If the S_∞-formula $[\mathcal{A}(\overline{n})]$ interprets as true for any specified natural number n, then we may conclude that $[(\forall x)\mathcal{A}(x)]$ is provable in S_∞.*

§5.B. **Thesis 3.** *Hilbert's ω-Rule of infinite induction is equivalent to Gentzen's ω-Rule of infinite induction.* □

§6.A. **Query 1.** *Can the above argument in §6.A. be extended to ordinals below ϵ_0 by defining higher order ordinals similarly in terms of the ordered n-tuples $(W, W_1, W_2, \ldots, W_n)$, where $W_i = \omega_i^n.x_{i,n} + \ldots + \omega_i^4.x_{i,4} + \omega_i^3.x_{i,3} + \omega_i^2.x_{i,2} + \omega_i.x_{i,1}$, and so on recursively?*

§7.F. **Query 2.** *Are both the interpretations $\mathcal{I}_{PA(\mathbb{N}, \ SV)}$ and $\mathcal{I}_{PA(\mathbb{N}, \ SC)}$ of PA over the structure $\mathbb{N}$ well-defined, in the sense that the PA axioms interpret as true, and the rules of inference preserve truth, relative to each of the assignments of truth values T_V and T_C respectively?*

§10.D. **Query 3.** *When can a formal assertion claim, under a well-defined interpretation, to 'mean' what it represents?*

§10.D. **Query 4.** *When is an arbitrary number-theoretic function or relation representable in PA?*

§10.D.b. **Query 5.** *If a number-theoretic relation $R(x_1, \ldots, x_n)$ is expressible by a PA-formula $[A(x_1, \ldots, x_n)]$, when may we assert that, under a well-defined interpretation $\mathcal{I}$ of PA, $A(x_1, \ldots, x_n)$ 'means' $R(x_1, \ldots, x_n)$?*

§10.D.b. **Query 6.** *When is a number-theoretic relation a well-defined interpretation of some PA-formula that expresses it in PA?*

§10.G.a. **Query 7.** *If an assertion of L is decidable as true/false under an interpretation $\mathcal{I}_{L(D)}$ in the domain D of L', then does such decidability also ensure an effective method of deciding its corresponding provability/unprovability in L?*

§10.G.a. **Thesis 4.** *If there is no effective method for the unambiguous decidability of the assertions of a mathematical language L under any interpretation $\mathcal{I}_{L(D)}$ of L in the domain D of a language L', then L can only be considered a mathematical language of subjective expression, but not a mathematical language of effective, and unambiguous, communication under interpretation in L'.*

§10.G.b. **Query 8.** *How can we assert that a PA formula (whether PA-provable or not) is true under the standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ of PA, so long as such truth remains effectively unverifiable under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$?*

§11.B. **Query 9.** *Can Gödel be held responsible for not distinguishing—in his seminal 1931 paper on formally undecidable propositions—between the implicit circularity in (masked by the non-constructive nature of) his set-theoretical proof of arithmetic undecidability in PM; and the lack of any circularity in his finitary proof of arithmetic undecidability in his Peano Arithmetic P?*

§11.F. **Query 10. Is there any natural number n for which $mBSUBn$ is true?**

§11.F.b. **Thesis 5.** *There can be no interpretation of Gödel's definition of his formally undecidable arithmetical proposition $[(\forall x)R(x,p)]$ over the domain $\mathbb{N}$ of the natural numbers—whether expressed mathematically or in any language of common discourse—that could lead to a contradiction.*

§15. **Query 11.** *Are we free to postulate the existence of such a natural number z, and conclude that some member of $G(m)$ must evaluate to 0 in $\mathbb{N}$?*

§16.C.c. **Query 12.** *In what evidence-based mathematical sense of a metric 'space' can Fig.4 be described mathematically as defining a space-filling curve?*

§16.D. **Query 13.** *Since the raison d'être of a mathematical language is—or ideally should be—to express our abstractions of natural phenomena precisely, and communicate them unequivocally, in what sense can we sensibly admit an interpretation of a mathematical language that constrains all the above cases by 'limiting' configurations in a putative, set-theoretical, 'completion' of Euclidean Space?*

§16.D.b. **Thesis 6.** *There are no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena.*

§16.D.b. **Thesis 7.** *If:*

(a) *a physical process is representable by a Cauchy sequence (as in the above cases §16.C.a., §16.C.b.);*

and:

(b) *we accept that there can be no infinite processes, i.e., nothing corresponding to infinite sequences, in natural phenomena;*

then:

(c) *in the absence of an extraneous, evidence-based, proof of 'closure' which determines the behaviour of the physical process in the limit as corresponding to a 'Cauchy' limit, the physical process must tend to a discontinuity (singularity) which has not been reflected in the Cauchy sequence that seeks to describe the behaviour of the physical process.*

§16.D.b. **Thesis 8.** *The perceived barriers that inhibit mathematical modelling of a cyclic universe—one which admits broken symmetries, dark energy, and an ever-expanding multiverse—in a mathematical language that admits unambiguous communication are illusory; they arise out of an attempt to ask of the language selected for such representation more than the language is designed to deliver.*

§16.D.d. **Thesis 9.** *The anti-matter in $\mathcal{U}$ produces a repulsive, anti-gravitational, field:*

(a) *that is consistent with both general relativity and Newtonian gravity;*

(b) *whose state at any instant is either exploding, steady, or imploding;*

(c) whose 'energy anti-confinement' strength at any instant is determined by an anti-gravitational dimensionless 'cosmological constant' asp that can assume any of three values $asp = 1$ (exploding at the instant of the Big Bang), $asp = \frac{1}{3}$ (steady between an explosion and an implosion) or $asp = \frac{1}{3} + \frac{2}{3}(1 - \frac{1}{n+1})$ (imploding at the instant of the extinguishing of the nth 'critical black hole' for all $n \geq 1$);

(d) which constantly opposes the 'energy confinement' strength of the Newtonian gravitational field whose state is determined at any instant by only one dimensionless gravitational constant $gsp = \frac{1}{2}$.

§17.   **Query 14.** *Does recognition of the 'truth' of Gödel's formally undecidable arithmetical proposition under the classical standard interpretation of a Peano Arithmetic imply that such recognition cannot be duplicated in any artificially constructed mechanism or organism whose design is based on classical logic?*

§17.   **Thesis 10. (Gödelian Thesis)** *Gödel's construction of an arithmetical proposition that is not provable in Peano Arithmetic, but true under interpretation if the Arithmetic is consistent, entails that there can be no mechanistic model of human reasoning.*

§17.A.   **Thesis 11. (Lucas' Gödelian Thesis)** *There can be no mechanistic model of human reasoning.*

§17.B.   **Query 15.** *Are the concepts 'non-algorithmic' and 'non-constructive' necessarily synonymous in classical logic and mathematics?*

§17.B.   **Thesis 12. (Penrose's Gödelian Thesis)** *Any formal system of logic cannot completely assign unique Tarskian truth values of satisfiability and truth to the propositions of any language that seeks to adequately represent, and unequivocally communicate, an individual's perception of a Platonic mathematical reality.*

§17.C.   **Query 16.** *Are the concepts 'algorithmically verifiable' and 'non-constructive' necessarily synonymous in classical logic and mathematics?*

§17.E.   **Query 17. (Turing Test)** *Can you prove that, for any well-defined numeral $[n]$, Gödel's arithmetic formula $[R(n)]$ is a theorem in the first-order Peano Arithmetic PA, where $[R(x)]$ is defined by its Gödel number r in eqn.12, and $[(\forall x)R(x)]$ is defined by its Gödel number $17Gen\ r$ in eqn.13, on p.25 of [Go31]? Answer only either 'Yes' or 'No'.*

§17.E.a.   **Query 18.** *Does Query 17 yield a definitive Turing Test?*

§17.E.a.   **Query 19.** *Can a Turing machine be forced to reply in a finite time whether or not a partial recursive function $F(x_1, \ldots, x_n)$ is effectively computable as total?*

§17.F.   **Query 20. (Turing's Halting Problem)** *Is it always decidable by a Turing machine whether or not a specified partial recursive function $F(x_1, \ldots, x_n)$ is total?*

§17.F.a.   **Query 21. (Halting-decidability problem for T)** *Given a Turing machine T, can one effectively decide, given any instantaneous description alpha, whether or not there is a computation of T beginning with alpha?*

§17.F.a.   **Query 22. (Halting-decidability problem for T)** *Is it always possible to effectively determine whether a Turing machine will halt or not when computing any partial recursive function $F(x_1, \ldots, x_n)$?*

§17.J.   **Hypothesis 1.** *Whilst an organic brain can evidence that an arithmetical proposition is algorithmically computable as true under an interpretation, only the sensory organs (such as those of sight, smell, hearing, taste and touch) can evidence that an arithmetical proposition is algorithmically verifiable as true under an interpretation.*

§17.J.a.   **Hypothesis 2. (Awareness)** *Awareness is the primary conceptual metaphor that corresponds to the ability of an organic intelligence to—not necessarily symbolically—express sensory perceptions in the first person as 'I sense'.*

§17.J.a.   **Hypothesis 3. (Self-awareness)** *Self-awareness is the secondary conceptual metaphor that corresponds to the ability of an organic intelligence to symbolically postulate the existence of an id that can be subjectively identified as aware, and which is implicitly expressible as 'I sense, therefore I am'.*

§18.A.   **Query 23.** *Are the prime divisors of an integer n mutually independent?*

§18.A.   **Proposition 18.2.** *Whether or not a prime p divides an integer n is independent of whether or not a prime $q \neq p$ divides the integer n.*

§18.A.c.   **Hypothesis 4.** *The event yielded by the simultaneous spins of a set of Bazeries wheels is random.*

§19.D.   **Thesis 13. (Dimensionless constants)** *Some of the dimensionless physical constants are only representable in a mathematical language as 'unmeasurable' real numbers that are defined by functions which are algorithmically verifiable, but not algorithmically computable.*

§19.D.a.   **Thesis 14. (Unmeasurable constants)** *Whilst a symbol for an 'unmeasurable' physical constant may be introduced into a physical theory as a primitive term without inviting inconsistency in the theory, the sequence of digits in the decimal representation of the 'measure' of an 'unmeasurable' physical constant cannot be treated in the mathematical language of the theory as a 'completed' infinite sequence whose 'measure' is the Cauchy limit of the sequence.*

§19.D.a.   **Thesis 15. (Measurable constants)** *The sequence corresponding to the decimal representation of the 'measure' of a 'measurable' physical constant, when introduced as a primitive term into a physical theory, can be treated as a 'completed' infinite sequence, whose 'measure' is the Cauchy limit of the sequence in the mathematical language of the theory, without inviting inconsistency.*

§19.D.c.   **Thesis 16. (Classical laws)** *Classical laws of nature determine the nature and behaviour of all those properties of the physical world which are mathematically describable completely at any moment of time $t(n)$ by algorithmically computable functions from a given initial state at time $t(0)$.*

§19.D.d.   **Thesis 17. (Neo-classical laws)** *Neo-classical laws of nature determine the nature and behaviour of those properties of the physical world which are describable completely at any moment of time $t(n)$ by algorithmically verifiable functions; however such properties are not completely describable by algorithmically computable functions from any given initial state at time $t(0)$.*

§19.D.f.   **Thesis 18. (Conjugate properties)** *The nature and behaviour of two conjugate properties F_1 and F_2 of a particle P that are determined by neo-classical laws are described mathematically at any time $t(n)$ by two algorithmically verifiable, but not algorithmically computable, functions f_1 and f_2.*

§19.D.g.   **Thesis 19. (Entangled particles)** *The nature and behaviour of an entangled property of two particles P and Q are determined by neo-classical laws, and are describable mathematically at any time $t(n)$ by two algorithmically verifiable—but not algorithmically computable—functions f_1 and g_1.*

§19.D.h.   **Thesis 20. (Schrödinger's cat)** *I may either assume the cat in Schrödinger's gedunken to be alive until a given time t (in the future), or assume the cat to be dead until the time t, without arriving at any logical contradiction in my existing Quantum description of nature.*

§21.A.a.   **Thesis 21. (Computational equivalence)** *Any polynomial-time computation by a quantum device can be executed in polynomial time by a classical Turing machine.*

§21.A.g.   **Thesis 22. (Quantum algorithm)** *The quantum algorithm appealed to in Deutsch's quantum computer in [DJ92] is a provable formula of PA.*

§21.A.g.   **Thesis 23. (Quantum oracle)** *The quantum oracle appealed to in Deutsch's quantum computer in [DJ92] is the first-order Peano Arithmetic PA.*

§22.   **Query 24.** *Is there a rational danger to humankind in actively seeking an extra-terrestrial intelligence?*

§22.A.   **Query 25.** *Is fear of actively seeking an ETI merely paranoia, or does it have a rational component?*

§22.C.   **Query 26.** *Can we responsibly seek communication with an extra-terrestrial intelligence actively (as in the 1974 Aricebo message) or is there a logically sound possibility that we may be initiating a process which could imperil humankind at a future date?*

§22.C.   **Thesis 24.** *Any communication with an extra-terrestrial intelligence will involve periods of upto thousands of years between the sending of a message and receipt of a response.*

§22.C.   **Thesis 25.** *We can only communicate with an essentially different form of extra-terrestrial intelligence in a platform-independent language of a mechanistically reasoning artificial intelligence.*

§22.C.   **Thesis 26.** *Nature is not malicious and so, for an ETI to be malevolent towards us, they must perceive us as an essentially different form of intelligence that threatens their survival merely on the basis of our communications.*

§22.D.   **Thesis 27.** *The language of algorithmically computable functions and relations is platform-independent.*

§22.D. **Thesis 28.** *All natural phenomena which are observable by human intelligence, and which can be modelled by algorithms, are interpretable isomorphically by an extra-terrestrial intelligence.*

§22.D. **Thesis 29.** *There are innumerable, distinctly different, observable natural phenomena.*

§22.D. **Thesis 30.** *A language of algorithms with a denumerable alphabet does not admit effective communication with an ETI.*

§22.F. **Thesis 31.** *Any advanced ETI can communicate with us by recognising that the signals we are beaming:*

> *(a) are not random;*
> *(b) contain a primer that defines PA;*
> *(c) contain theorems that describe common natural phenomena as observed by us.*

§22.H. **Thesis 32.** *A malevolent ETI could perceive us as an essentially different form of intelligence that threatens their survival if they have an interpretation of PA that is essentially different to our standard interpretation $\mathcal{I}_{PA(Standard/Tarski)}$ of PA.*

§22.H. **Thesis 33.** *An ETI would perceive their interpretation $\mathcal{I}_{PA(Non-Standard/ET)}$ of PA as essentially different to our standard interpretation $\mathcal{I}_{PA(Standard/Tarski)}$ of PA if, and only if, there is a PA formula which either interprets as false under $\mathcal{I}_{PA(Non-Standard/ET)}$ and true under $\mathcal{I}_{PA(Standard/Tarski)}$, or as as true under $\mathcal{I}_{PA(Non-Standard/ET)}$ and false under $\mathcal{I}_{PA(Standard/Tarski)}$.*

§22.J. **Thesis 34.** *Any extra-terrestrial intelligence which is capable of learning the language PA would interpret the satisfaction, and truth, of the formulas of PA—under a well-defined interpretation of PA—precisely as would a human intelligence.*

§22.J. **Thesis 35.** *Since thousands, if not millions, of years would elapse before a putative first, spatially separated, contact with any extra-terrestrial intelligence, and any putative subsequent, proximal, contact between the two species:*

(a) any species that survives over eons must evolve in harmony with its environment through intellectual evolution and respect for natural law;

(b) categorical communications between the two species in the interregnum would engender, and nurture, respect for life forms with:

> *— seemingly comparable respect for natural law; and*
> *— the capacity for survival through intellectual evolution.*

§22.J. **Thesis 36.** *Any extra-terrestrial intelligence would not rationally perceive us as being an essentially different form of intelligence that would necessarily be inimical to their interests and/or survival.*

24.C. Lemmas, Theorems, Corollaries

(*The symbol □ indicates an immediate consequence without a formal proof.*)

§2.A.b. **Lemma 2.1.** *Under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$, an atomic formula $A^*(x_1, x_2, \ldots, x_n)$ is both algorithmically verifiable and algorithmically computable in $\mathbb{N}$ by $\mathcal{W}_{(\mathbb{N},\ S)}$.*

§2.B. **Theorem 2.2.** *The atomic formulas of PA are algorithmically verifiable as true or false under the algorithmically verifiable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

§2.B.a. **Lemma 2.3.** *The PA axioms PA_1 to PA_8 are algorithmically verifiable as true over $\mathbb{N}$ under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

§2.B.a. **Lemma 2.4.** *For any given PA formula $[F(x)]$, the Induction axiom schema $[F(0) \to (((\forall x)(F(x) \to F(x'))) \to (\forall x)F(x))]$ interprets as an algorithmically verifiable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

§2.B.a. **Lemma 2.5.** *Generalisation preserves algorithmically verifiable truth under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.*

§2.B.a. **Lemma 2.6.** *Modus Ponens preserves algorithmically verifiable truth under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.* □

§2.B.a. **Theorem 2.7.** *The axioms of PA are always algorithmically verifiable as true under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, and the rules of inference of PA preserve the properties of algorithmically verifiable satisfaction/truth under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.* □

§2.B.a. **Theorem 2.8.** *If the PA formulas are algorithmically verifiable as true or false under $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, then PA is consistent.* □

§2.C. **Theorem 2.9.** *The atomic formulas of PA are algorithmically computable as true or as false under the algorithmically computable interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

§2.C. **Lemma 2.10.** *The formulas of PA are algorithmically computable finitarily as true or as false under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

§2.C.a. **Lemma 2.11.** *The PA axioms PA_1 to PA_8 are algorithmically computable as true under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

§2.C.a. **Lemma 2.12.** *For any given PA formula $[F(x)]$, the Induction axiom schema $[F(0) \to (((\forall x)(F(x) \to F(x'))) \to (\forall x)F(x))]$ interprets as an algorithmically computable true formula under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

§2.C.a. **Lemma 2.13.** *Generalisation preserves algorithmically computable truth under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.*

§2.C.a. **Lemma 2.14.** *Modus Ponens preserves algorithmically computable truth under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.* □

§2.C.a. **Theorem 2.15.** *The axioms of PA are always algorithmically computable as true under the interpretation $\mathcal{I}_{PA(\mathbb{N},\ SC)}$, and the rules of inference of PA preserve the properties of algorithmically computable satisfaction/truth under $\mathcal{I}_{PA(\mathbb{N},\ SC)}$.* □

§2.C.a. **Theorem 2.16.** *PA is strongly consistent.* □

§2.E. **Theorem 2.17. (Provability Theorem for PA)** *A PA formula $[F(x)]$ is PA-provable if, and only if, $[F(x)]$ is algorithmically computable as always true in $\mathbb{N}$.*

§2.E. **Corollary 2.18.** *PA is categorical.* □

§2.F. **Lemma 2.19.** *If $\mathcal{I}_{PA(\mathbb{N},\ M)}$ defines a model of PA over $\mathbb{N}$, then there is a PA formula $[F]$ which is algorithmically verifiable as always true over $\mathbb{N}$ under $\mathcal{I}_{PA(\mathbb{N},\ M)}$ even though $[F]$ is not PA-provable.*

§2.F. **Corollary 2.20.** *The PA formula $[\neg(\forall x)R(x)]$ defined in Lemma 2.19 is PA-provable.* □

§2.F. **Corollary 2.21.** *In any well-defined model of PA, Gödel's arithmetical formula $[R(x)]$ interprets as an algorithmically verifiable, but not algorithmically computable, tautology over $\mathbb{N}$.*

§2.F. **Corollary 2.22.** *PA is not ω-consistent.*

§2.F. **Corollary 2.23.** *The classical standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ S)}$ of PA does not well-define a model of PA.*

§4. **Theorem 4.1.** *Hilbert's 'principle of excluded middle' is 'stronger than Aristotle's particularisation.*

§*4.G.* **Theorem 4.2.** *There are well-defined number theoretic functions that are algorithmically verifiable but not algorithmically computable.*

§*4.G.a.* **Theorem 4.3. (CT falsifiable)** The classical Church-Turing Thesis is false in any interpretation of the first-order Peano Arithmetic PA that admits *evidence-based* reasoning.

§4.H. **Theorem 4.4. (Specifiability Theorem for Reals)** *Every real number is specifiable in PA.*

§*4.H.a.* **Theorem 4.5. (Invalid Cauchy Limit Theorem)** *The values of n for which Gödel's arithmetic formula* $[R(x)]$*, with Gödel number r as defined in [Go31], p.25, eqn.12, interprets as a true arithmetic proposition $R^{*}(n)$ in $\mathbb{N}$ do not define the real number $c = \sum_{n=1}^{\infty}(1/10^n) = 0.c_1 c_2 \ldots c_n \ldots$; which is classically treated as defining the real number* $1/9$.

§*5.* **Lemma 5.3.** *If we meta-assume Hilbert's ω-rule of infinite induction for PA, then a consistent PA is ω-consistent.*

§*5.C.* **Lemma 5.5.** *Assuming that PA admits Hilbert's ω-Rule of infinite induction is stronger than assuming that PA is ω-consistent.* □

§*5.D.* **Lemma 5.6.** *If PA is consistent but not ω-consistent, then there is some PA formula $[F(x)]$ such that, under any interpretation—say $\mathcal{I}_{PA(N)}$—of PA over $\mathbb{N}$:*

  (i) *the PA formula $[\neg(\forall x)F(x)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$;*

  (ii) *for any specified numeral $[n]$, the PA formula $[F(n)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$.*

§*5.D.* **Lemma 5.7.** If PA is consistent and the interpretation $\mathcal{I}_{PA(N)}$ admits Aristotle's particularisation over $\mathbb{N}$, then:

  (i) if the PA formula $[\neg(\forall x)F(x)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$,

  (ii) then there is some *unspecified* natural number m such that the interpreted arithmetical proposition $F^{*}(m)$ is algorithmically verifiable as false in $\mathbb{N}$.

§*5.D.* **Corollary 5.8.** *If PA is consistent and Aristotle's particularisation holds over $\mathbb{N}$, then there can be no PA formula $[F(x)]$ such that, under any interpretation $\mathcal{I}_{PA(N)}$ of PA over $\mathbb{N}$:*

  (i) *the PA formula $[\neg(\forall x)F(x)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$;*

  (ii) *for any specified numeral $[n]$, the PA formula $[F(n)]$ interprets as an algorithmically verifiable true arithmetical proposition under $\mathcal{I}_{PA(N)}$.* □

§*5.D.* **Corollary 5.9.** *If PA is consistent and Aristotle's particularisation holds over $\mathbb{N}$, then PA is ω-consistent.* □

§*5.D.* **Corollary 5.10.** *If Aristotle's particularisation holds over $\mathbb{N}$, then PA is consistent if, and only if, it is ω-consistent.*

§*5.D.* **Corollary 5.11.** *If PA is consistent but not ω-consistent, then Aristotle's particularisation does not hold in any interpretation of PA over $\mathbb{N}$.* □

§*5.D.* **Theorem 5.12.** *Aristotle's particularisation does not hold in any interpretation of PA under which the PA-axioms interpret as true, and the PA rules of inference preserve such truth.* □

§*5.D.* **Theorem 5.13.** *The first-order logic FOL is finitarily consistent.*

§*5.D.* **Corollary 5.14.** *The Law of the Excluded Middle $[P \lor \neg P]$ is a theorem of the first-order logic FOL.* □

§*5.D.* **Corollary 5.15.** *The Law of the Excluded Middle does not entail Aristotle's particularisation.* □

§*5.E.* **Corollary 5.16.** *Markov's principle: $\neg(\forall x)(f(x) = 0) \rightarrow (\exists n)(f(n) = 1)$, where $f(n)$ is a Boolean number-theoretic function such that $f(n) = 0$ or $f(n) = 1$, does not hold in PA.*

§5.F. **Lemma 5.17.** *The Axiom of Choice is true in any well-defined interpretation of the Zermelo-Fraenkel set theory ZF_ε that admits Aristotle's particularisation.* □

§5.F. **Lemma 5.18.** *The postulation of an 'unspecified' object in Aristotlean particularisation is a stronger postulation than the Axiom of Choice.* □

§5.G. **Lemma 5.19.** *If:*

(i) *from the P-provability of $[(\exists x)F(x)]$ we can always conclude the existence within a proof sequence of an unspecified P-term $[a]$ such that $[F(a)]$ is provable;*

then:

(ii) *we cannot have that a P-formula $[(\exists x)F(x)]$ is P-provable and also that $[\neg F(a)]$ is P-provable for any specified, constructively well-defined, term $[a]$ of P;*

§5.G. **Lemma 5.20.** *Rosser's Rule C entails Aristotle's particularisation.*

§5.G. **Corollary 5.21.** *Rosser's Rule C is stronger than Gödel's ω-consistency.*

§7.A. **Lemma 7.1.** *Aristotle's particularisation holds under every well-defined interpretation of Hilbert's ε-calculus L_ε.* □

§7.A. **Lemma 7.2.** *The Law of the Excluded Middle (LEM) holds under every well-defined interpretation of Hilbert's ε-calculus L_ε.*

§7.C. **Theorem 7.3.** *The Church-Turing Thesis entails Aristotle's particularisation.* □

§9.A.d. **Theorem 9.1.** *The atomic formulas of PA are algorithmically verifiable as true or false under the algorithmically verifiable, weak 'Wittgensteinian', interpretation $\mathcal{I}_{PA(\mathbb{N},\ W)}$.*

§9.A.e. **Theorem 9.2.** *The interpretations $\mathcal{I}_{PA(\mathbb{N},\ SV)}$ and $\mathcal{I}_{PA(PA,\ W)}$ of PA are isomorphic.*

§9.A.e. **Corollary 9.3.** *A formula of PA is true (by §9.A.d., Definition 28) under the weak 'Wittgensteinian' interpretation $\mathcal{I}_{PA(PA,\ W)}$ if, and only if, it is true (by §2.B., Definition 10) under the weak standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$.* □

§9.A.e. **Corollary 9.4.** *The weak standard interpretation $\mathcal{I}_{PA(\mathbb{N},\ SV)}$, and the weak 'Wittgensteinian' interpretation $\mathcal{I}_{PA(PA,\ W)}$, are both weak models of PA.*

§9.A.f. **Theorem 9.5.** *PA is not ω-consistent.*

§10.E.c. **Lemma 10.1.** *The Axiom of Choice is false in N.*

§10.E.d. **Theorem 10.2.** *Any model of ZF, in which the quantifiers are interpreted according to Aristotle's particularisation, is a model of ZF_ε if the expression $[\varepsilon_x B(x)]$ is interpreted to yield Cohen's symbol '$\overline{x}_B$' whenever $[B(\varepsilon_x(B(x)))]$ interprets as true in the model.*

§10.E.d. **Corollary 10.3.** *ZF_ε has no model that admits Aristotle's particularisation.*

§10.E.d. **Corollary 10.4.** *ZF has no model that admits Aristotle's particularisation.*

§11.B. **Lemma 11.1.** *The structure of the finite ordinals under any putative interpretation of ZF is not isomorphic to the structure $\mathbb{N}$ of the natural numbers.* □

§12.A. **Lemma 12.1.** *For any non-terminating sequence of values $f(0), f(1), \ldots$, we can construct natural numbers b_k, c_k such that:*

(i) $j_k = max(k, f(0), f(1), \ldots, f(k))$;

(ii) $c_k = j_k!$;

(iii) $\beta(b_k, c_k, i) = f(i)$ for $0 \le i \le k$.

§12.A. **Lemma 12.2.** *$\beta(x_1, x_2, x_3)$ is strongly represented in PA by $[Bt(x_1, x_2, x_3, x_4)]$, which is defined as follows:*

$$[(\exists w)(x_1 = ((1 + (x_3 + 1) \star x_2) \star w + x_4) \wedge (x_4 < 1 + (x_3 + 1) \star x_2))].$$

§12.B. **Theorem 12.3.** *The cardinality $2^{\aleph_0}$ of the real numbers cannot exceed the cardinality $\aleph_0$ of the integers.*

§12.B. **Corollary 12.4.** $\aleph_0 \longleftrightarrow 2^{\aleph_0}$ □

§13. **Lemma 13.1.** *If P is ω-consistent, then neither $[(\forall x)R(x)]$ nor $[\neg(\forall x)R(x)]$ are P-provable.* □

§14.A.a. **Theorem 14.1.** *No model of PA can admit a transfinite ordinal under the standard interpretation of the first-order logic FOL.*

§14.A.b. **Corollary 14.2.** *The language of PA admits of no constant that interprets in any well-defined model of PA as the set N of all natural numbers.* □

§14.D.a. **Lemma 14.3.** *If the collection $Th(\mathbb{N})$ of all true $\mathcal{L}_A$-sentences is the $\mathcal{L}_A$-theory of the standard model of Arithmetic, then we may consistently add to it the following as an additional—not necessarily independent— axiom:*

$$(\exists y)(y > x).$$ □

§14.D.a. **Theorem 14.4.** *The Compactness Theorem does not entail that:*

(*) $\cup_{k \in \mathbb{N}}\{Th(\mathbb{N}) \cup \{c > \underline{n} \mid n < k\}\}$

is consistent and has a model M_c which contains an 'infinite' integer.

§15. **Theorem 15.1.** *Goodstein's sequence $G_o(m_o)$ over the finite ordinals in any putative model $\mathbb{M}$ of ACA_0 terminates with respect to the ordinal inequality '$>_o$' even if Goodstein's sequence $G(m)$ over the natural numbers does not terminate with respect to the natural number inequality '$>$' in $\mathbb{M}$.*

§15. **Corollary 15.2.** *The relationship of terminating finitely with respect to the transfinitely defined ordinal relation '$>_o$' over the set of finite ordinals does not entail the relationship of terminating finitely with respect to the finitarily defined natural number relation '$>$' over the set of natural numbers.* □

§15.A. **Corollary 15.3.** *The subsystem ACA_0 of second-order arithmetic is not a conservative extension of PA.*

§15.A. **Theorem 15.4.** *The subsystem ACA_0 of second-order arithmetic is not a conservative extension of PA.*

§17.D. **Theorem 17.1.** *There can be no mechanist model of human reasoning if the standard interpretation $\mathcal{I}_{PA(N,SV)}$ of the first-order Peano Arithmetic PA can be treated as circumscribing the ambit of human reasoning about 'true' arithmetical propositions, and the finitary interpretation $\mathcal{I}_{PA(N,SC)}$ of PA can be treated as circumscribing the ambit of mechanistic reasoning about 'true' arithmetical propositions.*

§17.D. **Corollary 17.2.** *Although no mechanical intelligence can evidence that Gödel's formula $[(\forall x)R(x)]$ is algorithmically verifiable, a human intelligence can evidence that $[(\forall x)R(x)]$ is algorithmically verifiable.*

§17.F.b. **Theorem 17.3.** **(Forced Halting Theorem)** *It is always possible to effectively determine whether a Turing machine will halt or not when computing any partial recursive function $F(x_1, \ldots, x_n)$.*

§17.F.b. **Corollary 17.4.** **(Halting-decidability Theorem for T)** *Given a Turing machine T, one can effectively decide, given any instantaneous description alpha, whether or not there is a computation of T beginning with alpha.* □

§17.F.b. **Corollary 17.5.** **(Turing's Halting Theorem)** *Whether or not a specified partial recursive function $F(x_1, \ldots, x_n)$ is total is not always decidable by a Turing machine.* □

§17.F.c. **Theorem 17.6.** **(Essential Uncomputability Theorem)** *Some functions that are algorithmically verifiable, but not algorithmically computable, are essentially uncomputable.*

§17.I. **Corollary 17.7.** *The classical Church-Turing thesis is false in any interpretation of the first-order Peano Arithmetic PA that admits evidence-based reasoning.*

§18.A. **Theorem 18.1.** **(Fundamental Theorem of Arithmetic)** *Every positive integer $n > 1$ can be represented in exactly one way as a product of prime powers:*

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} = \prod_{i=1}^{k} p_i^{n_i}$$

where $p_1 < p_2 < \ldots < p_k$ are primes and the n_i are positive integers (including 0).

§*18.A.* **Theorem 18.3.** *There is no deterministic algorithm that, for any specified n, and any specified prime $p \geq 2$, will evidence that the probability $\mathbb{P}(p \mid n)$ that p divides n is $\frac{1}{p}$, and the probability $\mathbb{P}(p \nmid n)$ that p does not divide n is $1 - \frac{1}{p}$.*

§*18.A.* **Theorem 18.4.** *For any specified n, there is a deterministic algorithm that, for any specified prime $p \geq 2$, will evidence that the probability $\mathbb{P}(p \mid n)$ that p divides n is $\frac{1}{p}$, and the probability $\mathbb{P}(p \nmid n)$ that p does not divide n is $1 - \frac{1}{p}$.*

§*18.A.c.* **Lemma 18.5.** $P_3(B_i(u)) = \frac{1}{i}$.

§*18.A.c.* **Corollary 18.6.** $P_3(i|n) = P_3(B_i(0)) = \frac{1}{i}$.

§*18.A.c.* **Corollary 18.7.** $P_3(i \nmid n) = 1 - \frac{1}{i}$.

§*18.A.c.* **Lemma 18.8.** $P_3(B_{ij}(u,v)) = P_3(B_i(u)).P_3(B_j(v)) = \frac{1}{ij}$.

§*18.A.c.* **Lemma 18.9.** $P_3(i|n \ \& \ j|n) = P_3(i|n).P_3(j|n)$ *if, and only if, $n > i, j > 1$ and i, j are co-prime.*

§*18.A.c.* **Corollary 18.10.** *If p and q are two unequal primes, $P_3(p|n \ \& \ q|n) = P_3(p|n).P_3(q|n)$.* □

§*18.A.c.* **Theorem 18.11.** *The prime divisors of an integer are mutually independent.* □

§*18.A.d.* **Theorem 18.12.** *The probability $P_3(n \ is \ co-prime \ to \ p \leq \sqrt{n})$ of determining that n is not divisible by any prime $p \leq \sqrt{n}$ is $\prod_{i=1}^{\pi(\sqrt{n})} (1 - \frac{1}{p_i})$.* □

§*18.A.e.* **Theorem 18.13.** *The minimum number of events needed for determining that the signature yielded by simultaneous spins of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig.4)—is that of a number not divisible by any prime $p \leq \sqrt{n}$ is of order $O(log_e n)$.* □

§*18.A.e.* **Corollary 18.14.** *Determining whether the signature yielded by simultaneous spins of the $\pi(\sqrt{n})$ Bazeries wheels—where p_i is the i'th prime and B_i has p_i faces (Fig.4)—is that of a prime, or not, can be simulated by a deterministic algorithm in polynomial time $O(log_e n)$.* □

§*18.A.f.* **Theorem 18.15.** (*DPTvNDPT* **Thesis**) *There are integers n such that no deterministic algorithm can compute a prime factor of n in polynomial-time.*

§*18.B.* **Lemma 18.16.** $r_i(n) = 0$ *if, and only if, i is a divisor of n.* □

§*18.B.a.* **Lemma 18.17.** *For any $n \geq 2$, $i \geq 2$ and any specified integer $i > u \geq 0$:*

- *the probability $\mathbb{P}(r_i(n) = u)$ that $r_i(n) = u$ is $\frac{1}{i}$;*
- $\sum_{u=0}^{u=i-1} \mathbb{P}(r_i(n) = u) = 1$;
- *and the probability $\mathbb{P}(r_i(n) \neq u)$ that $r_i(n) \neq u$ is $1 - \frac{1}{i}$.* □

§*18.B.a.* **Theorem 18.18.** *For any $i \geq 2$, $\mathbb{M}_i = \{(0, 1, 2, \ldots, i-1), \ r_i(n), \frac{1}{i}\}$ yields a probability model for each of the values of $r_i(n)$.* □

§*18.B.a.* **Corollary 18.19.** *For any specified n, i and u such that $r_i(n) = u$, the probability that the roll of an i-sided cylindrical die will yield the value u is $\frac{1}{i}$ by the probability model defined in Theorem 18.18 over the probability space $(0, 1, 2, \ldots, i-1)$.* □

§*18.B.a.* **Corollary 18.20.** *For any $n \geq 2$ and any prime $p \geq 2$, the probability $\mathbb{P}(r_p(n) = 0)$ that $r_p(n) = 0$, and that p divides n, is $\frac{1}{p}$; and the probability $\mathbb{P}(r_p(n) \neq 0)$ that $r_p(n) \neq 0$, and that p does not divide n, is $1 - \frac{1}{p}$.* □

§*18.B.b.* **Lemma 18.21.** *If $n \geq 2$ and $n > i, j > 1$, where $i \neq j$, then:*

$$\mathbb{P}((r_i(n) = u) \cap (r_j(n) = v)) = \mathbb{P}(r_i(n) = u).\mathbb{P}(r_j(n) = v)$$

where $i > u \geq 0$ and $j > v \geq 0$.

§*18.B.b.* **Corollary 18.22.** $\mathbb{P}((r_i(n) = 0) \cap (r_j(n) = 0)) = \mathbb{P}(r_i(n) = 0).\mathbb{P}(r_j(n) = 0)$. □

§*18.B.b.* **Theorem 18.23.** *If i and j are co-prime and $i \neq j$, then whether, or not, i divides any specified natural number n is independent of whether, or not, j divides n.*

§18.B.b. **Corollary 18.24.** *The prime divisors of any integer n are mutually independent.*   □

§18.B.c. **Lemma 18.25.** *The asymptotic density of the set of all integers that are not divisible by any of a specified set of primes $Q = \{q_1, q_2, \ldots, q_k\}$ is:*

$$\prod_{q \in Q}(1 - 1/q).$$ □

§18.B.c. **Lemma 18.26.** *The expected number of integers in any interval (a,b) that are not divisible by any of a specified set of primes $Q = \{q_1, q_2, \ldots, q_k\}$ is:*

$$(b - a)\prod_{q \in Q}(1 - 1/q).$$ □

§18.C. **Lemma 18.27.** *If p_i and p_j are two primes where $i \neq j$ then, for any $n \geq 2$, $\alpha, \beta \geq 1$, we have:*

$$\mathbb{P}((r_{p_i^\alpha}(n) = u) \cap (r_{p_j^\beta}(n) = v)) = \mathbb{P}(r_{p_i^\alpha}(n) = u).\mathbb{P}(r_{p_j^\beta}(n) = v)$$

where $p_i^\alpha > u \geq 0$ and $p_j^\beta > v \geq 0$.   □

§18.C. **Corollary 18.28.** $\mathbb{P}((r_{p_i^\alpha}(n) = 0) \cap (r_{p_j^\beta}(n) = 0)) = \mathbb{P}(r_{p_i^\alpha}(n) = 0).\mathbb{P}(r_{p_j\beta}(n) = 0).$ □

§18.C. **Theorem 18.29.** *For any two primes $p \neq q$ and natural numbers $n, \alpha, \beta \geq 1$, whether or not p^α divides n is independent of whether or not q^β divides n.*   □

§18.C.a. **Lemma 18.30.** *For any co-prime natural numbers $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_k}$ where:*

$$q_1 < q_2 < \ldots < q_k \text{ are primes and } \alpha_1, \alpha_2 \ldots \alpha_k \geq 1 \text{ are natural numbers;}$$

the natural number n is of the form $a + m.d$ for some natural number $m \geq 1$ if, and only if:

$$a + r_{q_i^{\alpha_i}}(n) \equiv 0 \ (mod \ q_i^{\alpha_i}) \text{ for all } 1 \leq i \leq k$$

where $0 \leq r_i(n) < i$ is defined for all $i > 1$ by:

$$n + r_i(n) \equiv 0 \ (mod \ i) \ .$$

§18.C.a. **Corollary 18.31.** *The probability that $a + r_{q_i^{\alpha_i}}(n) \equiv 0 \ (mod \ q_i^{\alpha_i})$ for any $1 \leq i \leq k$ is $\frac{1}{q_i^{\alpha_i}}$.*   □

§18.C.a. **Corollary 18.32.** *The joint probability that $a + r_{q_i^{\alpha_i}}(n) \equiv 0 \ (mod \ q_i^{\alpha_i})$ for all $1 \leq i \leq k$ is $\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}}$.* □

§18.C.a. **Corollary 18.33.** *The asymptotic density of Dirichlet integers, defined as numbers of the form $a + m.d$ for some natural number $m \geq 1$ which are not divisible by any specified set of primes $\mathbb{R} = \{r_1, r_2, \ldots, r_l\}$, where $1 \leq a < d = q_1^{\alpha_1}.q_2^{\alpha_2}\ldots q_k^{\alpha_l}$ is:*

$$\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{r \in \mathbb{R} \ \& \ r \neq q_i}(1 - \frac{1}{r}).$$

§18.C.a. **Corollary 18.34.** *The expected number of Dirichlet integers in any interval (a, b) is:*

$$(b - a)\prod_{i=1}^{k} \frac{1}{q_i^{\alpha_i}} \cdot \prod_{i=1}^{k}(1 - \frac{1}{q_i})^{-1} \cdot \prod_{r \in \mathbb{R}}(1 - \frac{1}{r}).$$ □

§18.C.b. **Lemma 18.35.** $\pi_{(a,d)}(n) \approx \pi_D(n) \to \infty$ *as $n \to \infty$.*

§18.C.b. **Theorem 18.36. (Dirichlet's Theorem)** *There are an infinity of primes in any arithmetic progression $a + m.d$ where $(a, d) = 1$.*   □

§18.D. **Lemma 18.37.** *If n is a $\mathbb{TW}(k)$ integer, then both n and $n + 2$ are not divisible by any of the first k primes $\{p_1, p_2, \ldots, p_k\}$.*

§18.D. **Lemma 18.38.** $\mathbb{D}(\mathbb{TW}(k)) = \prod_{i=2}^{k}(1 - \frac{2}{p_i}).$ □

§18.D. **Lemma 18.39.** *If $p_k^2 \leq n \leq p_{k+1}^2$ is a $\mathbb{TW}(k)$ integer, then n is a prime and either $n + 2$ is also a prime, or $n + 2 = p_{k+1}^2$.*

$\S18.D.$ **Lemma 18.40.** $\pi_{\text{TW}(k)}(b) - \pi_{\text{TW}(k)}(a) \approx (b-a)\prod_{i=2}^{k}(1 - \frac{2}{p_i}).$ $\qquad\square$

$\S18.D.$ **Lemma 18.41.** $\pi_{\text{TW}(k)}(p_{k+1}^2) - \pi_{\text{TW}(k)}(p_k^2) + 1 \geq \pi_2(p_{k+1}^2) - \pi_2(p_{k)}^2) \geq \pi_{\text{TW}(k)}(p_{k+1}^2) - \pi_{\text{TW}(k)}(p_k^2)$ $\qquad\square$

$\S18.D.$ **Lemma 18.42.** $\pi_{\text{TW}(k)}(p_{k+1}^2) - \pi_{\text{TW}(k)}(p_k^2) \approx (p_{k+1}^2 - p_k^2)\prod_{i=2}^{k}(1 - \frac{2}{p_i}).$ $\qquad\square$

$\S18.D.$ **Lemma 18.43.** $\sum_{j=1}^{k}(\pi_2(p_{j+1}^2) - \pi_2(p_j^2)) = \pi_2(p_{k+1}^2) \approx \sum_{j=1}^{k}(p_{j+1}^2 - p_j^2)\prod_{i=2}^{j}(1 - \frac{2}{p_i}).$ $\qquad\square$

$\S18.D.$ **Theorem 18.44.** **(Twin Prime Theorem)** $\pi_2(n) \to \infty$ *as* $n \to \infty$.

$\S18.D.a.$ **Theorem 18.45.** **(Generalised Prime Counting Theorem)** $\sum_{j=1}^{n}\prod_{i=a}^{\pi(\sqrt{j})}(1 - \frac{b}{p_i}) \to \infty$ *as* $n \to \infty$ *if* $p_a > b \geq 1$.

$\S21.A.c.$ **Lemma 21.1.** *A classical computer is a knowledge verifier but not a quantum verifier.*

$\S21.A.c.$ **Lemma 21.2.** *A quantum computer is a knowledge verifier and a quantum verifier.*

$\S22.E.$ **Lemma 22.1.** *Every algorithmically computable number-theoretic function can be formally expressed by some formula of a first-order Peano Arithmetic, PA.*

25.   Acknowledgements

I am immensely grateful to Dr. Danko D. Georgiev [1], Professor Anthony Beavers [2], Professor Jean-Yves Beziau [3, 4, 6] and Professor Christian Retoré [5], for their personal interest, indulgence, and encouragement in providing me an opportunity in which to present before critical academic audiences some of the paradigm-challenging—and seemingly-heretical—perspectives offered in this investigation:

1. *Do Gödel's incompleteness theorems set absolute limits on the ability of the brain to express and communicate mental concepts verifiably?* Invited article in the Special Issue on *Minds, Brains and Mathematics*. In *NeuroQuantology*, [S.l.], v.2, n.2, 2004, pp.60-100. ISSN 1303-5150. DOI: 10.14704/nq.2004.2.2.39. [An04]

2. *Evidence-Based Interpretations of PA.* Presented in the *Symposium on Computational Philosophy* at the AISB/IACAP World Congress 2012-Alan Turing 2012, 2-6 July 2012, University of Birmingham, Birmingham, UK. [An12]

3. *A suggested mathematical perspective for the EPR argument.* Presented on 7th April at the workshop on '*Logical Quantum Structures*' at UNILOG'2013, 4th World Congress and School on Universal Logic, 29th March 2013 - 7th April 2013, Rio de Janeiro, Brazil. [An13] [An13p]

4. *Aristotle's particularisation: The Achilles' heel of Hilbertian and Brouwerian perspectives of classical logic.* Presented on 6th April at the Contest on '*Scope of Logic Theorems*' at UNILOG'2013, 4th World Congress and School on Universal Logic, 29th March 2013 - 7th April 2013, Rio de Janeiro, Brazil. [An13a] [An13b]

5. *Why Hilbert's and Brouwer's interpretations of quantification are complementary and not contradictory.* Presented on 10th June at the Epsilon 2015 workshop on '*Hilbert's Epsilon and Tau in Logic, Informatics and Linguistics*', 10th June 2015 - 12th June 2015, University of Montpellier, France. [An15a] [An15b]

6. *Algorithmically Verifiable Logic vis à vis Algorithmically Computable Logic: Could resolving EPR need two complementary Logics?* Presented on 26'th June at the workshop on '*Emergent Computational Logics*' at UNILOG'2015, 5th World Congress and School on Universal Logic, 20th June 2015 - 30th June 2015, Istanbul, Turkey. [An15] [An15p]

I also take this opportunity to record my respect for the professional standards of the *three* Editors—Vasant Hanovar, Rajiv Khosla and Peter Èrdi—of *Cognitive Systems Research*, *each* of whom deliberated for between *two to six* months before accepting—*in toto*—for publication the paradigm-challenging—and seemingly-heretical—conclusions of the relatively short, 10-page, paper:

> *The truth assignments that differentiate human reasoning from mechanistic reasoning: The evidence-based argument for Lucas' Gödelian thesis.* [An16]

If I have seen a little further it is by standing on the shoulders of Giants ... *Isaac Newton: Attributed quote in* http://www.phrases.org.uk/meanings/268025.html.

Prior to Isaac Newton's reported tribute in 1676 to René Descartes and Robert Hooke, in a letter to the latter, it was reportedly the 12th century theologian and author, John of Salisbury, who was recorded as having used an even earlier version of this humbling admission—in a treatise on logic called *Metalogicon*, written in Latin in 1159, the gist of which is translatable as:

> "Bernard of Chartres used to say that we are like dwarfs on the shoulders of giants, so that we can see more than they, and things at a greater distance, not by virtue of any sharpness of sight on our part, or any physical distinction, but because we are carried high and raised up by their giant size.
>
> *Dicebat Bernardus Carnotensis nos esse quasi nanos, gigantium humeris insidentes, ut possimus plura eis et remotiora videre, non utique proprii visus acumine, aut eminentia corporis, sed quia in altum subvenimur et extollimur magnitudine gigantea.*"
>
> ... *Wikipedia:* http://en.wikipedia.org/wiki/Standing_on_the_shoulders_of_giants.

Contrary to a contemporary interpretation of the remark 'standing on the shoulders of' as describing 'building on previous discoveries', it seems to me that what Bernard of Chartres apparently intended was to suggest that it doesn't necessarily take a genius to see farther; only someone both humble and willing to:

- first, clamber onto the shoulders of a giant and have the self-belief to see things at first-hand as they appear from a higher perspective (achieved more by the nature of height—and the curvature of our immediate space as implicit in such an analogy—than the nature of genius); and,

- second, avoid trying to see things first through the eyes of the giant upon whose shoulders one stands (for the giant might indeed be a vision-blinding genius)!

It was this latter lesson that I was incidentally taught by—and one of the few that I learnt (probably far too well for better or for worse) from—one of my Giants, the late Professor Manohar S. Huzurbazaar, in my final year of graduation in 1964.

The occasion: I protested that the axiom of infinity (in the set theory course that he had just begun to teach us) was not self-evident to me, as (he had explained in his introductory lecture) an axiom should seem if a formal theory were to make any kind of coherent sense under interpretation.

Whilst clarifying that his actual instruction to us had not been that an axiom should necessarily 'seem', but only that it should 'be treated', as self-evident, Professor Huzurbazar further agreed that the set-theoretical axiom of infinity was not really as self-evident as an axiom ideally ought to seem in order to be treated as self-evident.

To my natural response asking him if it seemed at all self-evident to him, he replied in the negative; adding, however, that he believed it to be 'true' despite its lack of an unarguable element of 'self-evidence'.

It was his remarkably candid response to my incredulous—and youthfully indiscreet—query as to how an unimpeachably objective person such as he (which was his defining characteristic) could hold such a subjective belief that has shaped my thinking ever since.

He said that he had 'had' to believe the axiom to be 'true', since he could not teach us what he did with 'conviction' if he did not have such faith!

Although I did not grasp it then, over the years I came to the realisation that committing to such a belief was the price he had willingly paid for a responsibility that he had recognised—and accepted—consciously at a very early age in his life (when he was tutoring his school going nephew, the renowned physicist Jayant V. Narlikar):

> Nature had endowed Professor Huzurbazar with the rare gift shared by great teachers—the capacity to reach out to, and inspire, students to learn beyond their instruction!

It was a responsibility that Professor Huzurbazar bore unflinchingly and uncompromisingly, eventually becoming one of the most respected and sought after teachers of Modern Algebra (now Category Theory), Set Theory and Analysis of his times in India at both the graduate and post-graduate levels.

At the time, however, Professor Huzurbazar pointedly stressed that his belief should not influence me into believing the axiom to be true, nor into holding it as self-evident.

His words—spoken softly as was his wont—were:

> Challenge it.

Although I chose not to follow an academic career, he never faltered in encouraging me to question the accepted paradigms of the day when I shared the direction of my reading and thinking (particularly on Logic and the Foundations of Mathematics) with him on the few occasions that I met him over the next twenty years.

Moreover, even if the desired self-evident nature of the most fundamental axioms of mathematics (those of first-order Peano Arithmetic and Computability Theory) might eventually be accepted as formally inconsistent with a belief in the 'self-evident' truth of the axiom of infinity (as argued in §14.), I choose to believe that the shades of Professor Huzurbazaar would rest more liberated than grieved by the demise of a compulsive belief.

Finally, I can only ascribe with gratitude the philosophical motivation underlying these investigations to what was once quoted to me half a century ago by another Giant—my erstwhile classmate, friend, mentor, and prematurely late Ashok Chadha:

> 'Let not posterity view us as having spent the gift of life polishing the pebbles and tarnishing the diamonds'.

Bhupinder Singh Anand

Blog: https://foundationalperspectives.wordpress.com/2009/07/06/hello-world/

Part VI
BIBLIOGRAPHY

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