Learning from Experience

Manifestations of young children's learning from pedagogic representations

by

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Declaration

I declare that the material in this thesis has not been presented in any other thesis.

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Summary

This study investigates how influences of teachers' presentations of mathematical ideas, on children's mental constructions for number and number operations, are manifested by the language they use to describe their mental calculation procedures, and by the mental visual imagery they report.

The methodology is described as postpositivist with a phenomenographic orientation. The research method involved the observation of mathematics lessons of two classes of children aged 7- to 9-years over a two year period and structured interviews with a sample of 26 pupils from these classes at the end of each school term.

The language and mental visual imagery described by children in the interviews seemed related to the way the mathematics was presented by their teachers. The use of the 'metaphoric' language related to a previous experience may be seen as a manifestation of the influence of that experience on the child's 'mental construction'. The interdependence of speech style and classroom activity gave evidence of linguistic relativism.

The *commonalities* in the interview responses suggested that children had acquired the cultural tools of the mathematics classroom. The *differences* in response suggested they had constructed their own knowledge. The differences were most apparent in the calculation methods employed and the mode of generality of their expression. Linguistic indicators (use of tense, pronouns and causal connectives) also distinguished groups of children at different achievement levels. In non-calculation contexts these differences were not apparent, suggesting that the responses reveal mental constructions not just linguistic traits.

The conclusion is drawn:

When describing the way they perform mental calculation young children's mental visual imagery and language use is indicative both of the experiences that have influenced their mental constructions and the qualitatively different mental constructions that have been formed.

These manifestations may aid teachers in helping children develop their mental constructions to support further mathematical progression.

CHAPTER ONE

INTRODUCTION

1.1 Background to the study

I came to PhD research with a simple ambition - to find out more about how children think. As a secondary school mathematics teacher I was interested in knowing what was 'going on in their heads' when I asked children questions in my classroom. What was behind the responses they gave? What had influenced the way they thought about procedures I asked them to perform? How much of what they said and did could be traced back to previous experiences in earlier classrooms?

I felt that if I knew how they thought about things it might aid me in helping them learn more effectively.

My view of learning had been influenced by an introduction to Piaget's theories of cognitive development given by tutors on my teacher education course. This suggested that our knowledge is constructed from our experiences, that we learn through doing, and what we learn may be constrained by what we already know. The theory suggests that the way people think is determined by their 'mental constructions'. Teachers need to provide activities that encourage learning by allowing children to develop their own way of thinking. In order to do this the activities need to allow them to build upon their existing mental constructions. Herein lies the teachers' dilemma because we do not know what the pupils' mental constructions are and we often make the assumption that those they do possess will provide appropriate foundations for subsequent learning.

It would not seem difficult to find out what children can already do and what they already know. A teacher can discover if children can do a calculation, for instance, by asking them to perform one. The child who answers "48 + 23" correctly, however, has only indicated that they could calculate that particular addition in that particular instance. Their numerical answer gives no indication of the way they have thought about it. It gives no indication of how that way of thinking might be developed in order to encompass, for instance, addition of rationals. If children have a way of thinking that is peculiar to previous experiences then the teacher needs to build upon that previous experience.

It would not seem impossible to find out what these previous experiences have been by asking colleagues, who have previously taught the children, what classroom activities have been used. However children may not all learn the same thing from the same activities. Teachers need to know what it is that pupils have taken from the experience. The task of the teacher is thus to glean information from children. This requires some way of knowing what their words indicate about their previous experiences. It requires a way of knowing what pupils' words might indicate about their mental constructions.

The purpose of this study was to investigate a way of distinguishing influences on, and differences in, children's learning. It is hoped to find a means of discovering what has been learned from experience.

1.2 Some initial definitions

Theories behind the use of the phrase 'mental construction' will be explored in Chapters Two and Three. Piaget's theories on the way we learn from experience are an important feature of the literature reviewed (for instance Piaget, 1928; Piaget, 1959; Piaget and Inhelder 1971; von Glasersfeld, 1995; Ashcraft, 1994; Wood, 1998; Meadows, 1993). For the purposes of this introduction I will define a mental construction simply as 'that which comes to mind when we think about something'. My mental construction for a concept such as "six" includes recalled experiences which have involved six, my knowledge about six, relationships between six and anything else I might know, processes that involve six, mental images that are evoked when thinking of six, etc..

My mental construction for a process such as addition similarly incorporates previous experiences involving addition. Because I have had many more experiences of addition and have reflected on those experiences my mental construction may be qualitatively different than the mental construction for addition of a young child. The young child may only have experienced counting of objects and formed a mental construction of addition based on these experiences. I have experiences of addition in many more exotic vector spaces which have led to a more sophisticated mental construction.

This attempt to give a first meaning for 'mental construction' suggests that it *might* incorporate 'mental imagery'. This will be explored more fully in the following chapters. The literature (for instance Finke, 1989; Russell, 1956; Richardson 1969; Richardson, 1999, Kosslyn, 1980; Kosslyn, 1996) will lead to a formal definition. Here I will describe a 'mental image' as something in my mind that is 'sensation-like'. When I think of addition I might: 'hear' a previous teacher's words telling me how to add; 'smell' a musty textbook that I have previously used; 'feel uncomfortable' as I did when first learning about metric spaces; 'see' a written calculation like ones I have previously done. None of the objects are present but the sensation is re-created from my experience.

In particular a 'mental *visual* image' can initially be thought of as something 'vision-like' that I might say I can 'see in my head' yet it is not in front of my eyes. If I have a mental visual image for six , for instance, it might not necessarily be as if I were looking at the numeral six. Saying I have a mental visual image would involve my thinking about the spatial characteristics of the numeral, a pattern involving six objects or any visual aspects of previous experience involving "six".

1.3 The research problem

Mental imagery and other aspect of a 'mental construction' have been previously explored at the University of Warwick. This study serves to extend the research which has demonstrated qualitative differences in young children's mental constructions. Gray and Tall (1993) distinguished between 'flexible' thinkers, for whom a symbol is a mathematical object that can be manipulated in the mind, and instrumental, 'procedural', thinkers for whom the symbol signifies a procedure to be carried out. Pitta (1998) explored the association between pupils' achievement levels and their mental imagery.

This study will examine how the mental visual imagery and language used by children can give indications of their type of thinking and the *influences* upon their thinking. It seeks also to discover how these indications relate to the children's achievement levels.

My view that children learn from early experience led to the decision that my study should be based at primary school level. There I might both observe the classroom activities and question children to find indications of how these early experiences influence the way they think. The focus of the study also seemed inevitable since arithmetic plays a predominant role in the primary curriculum. Furthermore the National Numeracy Pilot Project was encouraging an increasing emphasis on mental calculation. Thus ‘number’, and particularly mental calculation, became the focus of the investigation.

In my early lesson observations it was immediately apparent that structure-oriented materials (Resnick and Ford 1981) such as Dienes blocks, hundred squares and number lines were an important feature of the classroom that I visited. It seemed possible that in such a classroom children’s thinking might be influenced by their experiences of the ‘pedagogic representations’ teachers used to present ideas. This presentation involving materials and language, and the children’s own subsequent activities with these materials, could thus be influences on the their mental constructions.

It was also noticeable in early observations that there was distinctive language associated with the manipulation of these materials. This language was used in the classroom when handling the material and was *also* apparent in children’s descriptions of their *mental* calculation. This use of language from one context to talk about a different context is referred to as ‘metaphor’ (Black, 1979; Reddy, 1979; Lakoff and Johnson, 1980). Thus the metaphoric language, associated with a classroom activity but used in the different context of mental calculations in interviews, seemed to suggest that those previous experiences had influenced the children’s thinking.

The external ‘representations’ of mathematical ideas such as the materials mentioned above, the language used, the drawings and symbols used, provide a means of communication for mathematical ideas (Lesh, Post and Behr, 1987; Kang and Kilpatrick, 1992). They are representations in the sense that they ‘stand-in’ for ‘the mathematics’. They provide activities for teaching and learning and thus they will be referred to in this study as ‘pedagogic representations’ of mathematics. Their nature and use will be described more fully in Chapters Two and Three.

Whilst pedagogic representations may be easily observed and described, mental constructions can not be observed and are thus not so easily described. If I am to know about a child’s mental construction I can only get indications about it by asking them questions that allow them to demonstrate their mental construction in action. If what the child says is in language associated with pedagogic representations, or if they describe a mental visual image of the material, it suggests that they are recalling experiences with those representations. This language and imagery could be indications of influences on their mental construction.

The theories of mental imagery (for instance Richardson 1980; Richardson 1999) suggest that mental images are formed from our previous perceptual experiences. Any mental image that a child chooses to describe could be a very graphic indication of previous experiences. The language that children use, on the other hand, may or may not be related to the thinking that is taking place in their mind. This study assumes that the language used *is* indicative of the thinking that has taken place. The literature reviewed is thus presented to provide a theoretical framework on which this assumption is based (for instance Pavio, 1979; Johnson 1987). It is important to this study to provide an argument that language is an outward manifestation of the mental construction.

I stated above that I hoped to find a means for discovering what has been learned from experience. I am suggesting that mental visual imagery and language may provide manifestations of the influence of previous experiences.

The research question for this study is thus:

> *Are influences of pedagogic representations, on children's mental constructions for number and number operations, manifested by the language they use to describe their mental calculation procedures, and by the mental visual imagery they report?*

1.4 The study

The methodology chapter (Chapter Four) will describe in detail the rationale for the operational research questions and the method employed. The methodology chosen for the study draws on methods that are both qualitative and quantitative. The chief means of data collection have been lesson observations and interviews with children.

Mental visual imagery and language have been suggested as manifestations of children's learning from experience. Lesson observations and pupil interviews provided the means of gaining information about the activities and their influences. The majority of interview questions required children to perform and talk about mental calculations. However, there was a possibility that the way children express themselves might simply indicate a preference for a style of language. Qualitative differences in language used after calculations might not then tell us anything about their mental constructions for calculation. Questions were included in the interviews that were non-mathematical in order to help identify similarities, or differences, in language used in the different contexts.

Preliminary studies were conducted in 1997-8 with a group of Year 2 children (aged 6 to 7 years) at Bright Cross Primary School. The main longitudinal study, during 1998-2000, was carried out with the children as they moved first into Year 3 then Year 4. This study involved weekly lesson observations and interviews with a sample of children at the end of each term. In order to compare the language and imagery of children in a different school, where different classroom activities were used, a small sample of children were interviewed at Peacehaven School in July 2000.

The analysis of results focused on four 'indicators of learning' which will be described in detail in Chapter Three. They are: 'image', 'metaphor', 'generality' and 'method'.

- 'Image' relates to the mental visual imagery reported (Pitta, 1998).
- 'Metaphor' relates to the style of metaphor children use (Lakoff and Nunes, 1997).
- 'Generality' relates to the mode of generalisation expressed (Mason and Pimm, 1984; Harel and Tall, 1991)
- 'Method' relates to the method of calculation (Carpenter, Hiebert and Moser, 1981; Beishuizen, 1993; Thompson, 1997a)

Each of the 'indicators' has three categories which relate to abstraction from experience. The choice of three categories was influenced by Bruner (1966) and Mason and Pimm (1984).

In addition to these indicators the study investigated several linguistic devices which were used in qualitatively different ways by pupils in their responses. The tense used (past, present and conditional), pronouns ("I" and "you"), causal connectives ("because", "so", "if", "then") and the word "like" will be referred to as 'linguistic indicators'. They too can be categorised using three categories compatible with the same global scheme of categorisation. Like the 'indicators of learning' they provide clues about qualitative differences in pupils' mental constructions.

1.5 The thesis

The theoretical frameworks presented, the data collected as a result of the research questions, and its analysis using the indicators described above, lead to the following thesis:

> *When describing the way they perform mental calculation young children's mental visual imagery and language use are indicative both of the experiences that have influenced their mental constructions and the qualitatively different mental constructions that have been formed.*

The purpose of the following chapters is thus to investigate how children learn from their classroom experiences and the ways in which this learning is manifested. Two themes are developed:

- Language and mental visual imagery are indicative of previous experiences and of the mental constructions formed.
- Language and mental visual imagery are indicative of the qualitatively different mental constructions of individuals.

1.6 A preview of the literature review

In order to build the argument to support the thesis the literature review is necessarily wide ranging.

In Chapter Two there is an overview of the mechanisms for learning. A selection of literature from fields of psychology is presented to demonstrate the way in which the human brain 'learns' from experience. Consideration is given to models and modes of internal representation including mental imagery. This leads to a closer look at the cognitive structures that are involved in the 'mental construction' and the role of abstraction in the development of cognitive structures. The theory of 'embodied cognition' suggests that language and thought are interdependent.

A review of constructivist and socio-cultural theories of learning suggests how classroom activities may become 'internalised'. There is then a discussion of the role of imagery and language in thought. The argument for linguistic relativism is developed to suggest that language is indicative of the cognitive structure of which it is a part. Metaphor has a key role in this argument.

Chapter Three provides a review of the literature on pedagogic representations and particularly the teaching of algorithms. There are five sections which provide the theory behind each of the 'indicators' and how use of particular words may act as indicators of thought.

1.7 A preview of the argument based on data

In order to illustrate the theme that children learn from their experiences the presentation of results starts with a case study of one child, Elspeth. Her language and mental visual imagery, described in response to interview questions, suggested that she had been influenced by her experiences. She was an articulate child who reported the most mental visual imagery though her language was not atypical. This analysis of her responses indicates the influences of the classroom experiences on her thinking about mental calculation. Chapter Five is thus an in-depth look at the manifestations of one child's mental construction which also serves to introduce the tools that will be used in the analysis for the rest of the sample.

In Chapter Six there is a closer look at the link between classroom activities and mental constructions. The way in which pedagogic representations influence the pupils is demonstrated by extracts from lesson observation and selections of interview responses. These show the metaphoric nature of the language. Children's responses to non-mathematical questions in the interviews are then compared with their responses to mathematical questions to identify characteristics of individuals' language and mental visual imagery.

The responses given in interviews by the teachers of the children are also presented. They were asked some of the same mental calculation questions as the children had been asked. Their answers serve to illustrate the common language and the common procedures that were used by teachers and pupils at Bright Cross. There is an indication in this that the activities determine the language.

This indication of the influence of classroom activities on both language and procedures is further elaborated when a comparison is drawn with another school. The experiences of the children at Peacehaven School were very different from the pupils in the main study. The language and methods used by these other pupils indicated the different influences on their thinking.

The theme of individual differences is taken up in Chapter Seven. Tests of statistical significance are used in arguments to support the thesis. It will be shown that the 'indicators' provide a means of identifying qualitative differences in the language use of children which is related to different levels of achievement. Since these differences in language use by groups of children of different achievement levels are not apparent in non-mathematical questions it is argued that there is a characteristic use of language associated with their learning in mathematics.

Implications are drawn in Chapter Eight. It is suggested here that teachers have much more to learn from children when they ask "How did you work that out?" than simply the strategy that was employed.

1.8 A note on presentation

Mason (1994) suggests that researchers need to present reports in a form that allows others to reconstruct the awarenesses and sensitivities developed during the research. Research should be presented in forms which promote personal construal. I have taken heed of this advice but accept that the presentation I give, on which readers will base their personal construal, is not value-free.

I present the literature to demonstrate the influences on my own thinking and to provide the theoretical framework for the study. It may thus appear to be simply a synthesis of the ideas of others. I have attempted to present their ideas so that the reader might draw their own conclusions but inevitably it is my re-construction of their views and it represents the awarenesses that I have developed over the period of the study.

These awarenesses thus appear to have determined the way that I interpret the data. In reality the literature search and data collection were concurrent. The growing awarenesses in each aspect of the research were interdependent. In keeping with my thesis I believe that I have a mental construction of 'children's thinking', developed from my previous mental construction, as a result of my experiences during the course of this study. It will be communicated here in language associated with that experience. The indications about the experiences and about the mental construction are in the language that I choose to describe it. Also in keeping with my thesis I ask the reader to accept that my thinking, and the attempt at communication of that thinking, are closely related. The language I use is indicative of the experiences that have formed the mental construction.

The distinction between subjectivism and objectivism is a recurring theme in this report. I take the subjectivist view that we each construct our own reality. My hope is that the mental constructions formed by readers in reading this report will at least bear some resemblance to my own.

CHAPTER TWO

THOUGHT, LEARNING, IMAGERY AND LANGUAGE

This chapter provides the theoretical background to the study. There is an examination of the mechanisms by which children might learn from their interactions with their environment, consideration of some theories of the way in which learning occurs and consideration of the roles of mental imagery and language in thinking and learning. Here the arguments are developed to support the claim that children learn from their experiences and that mental imagery and language are indicative of the mental constructions they have developed.

The first part (§2.1) starts with a neuropsychological view of thinking and learning as brain activity. This provides the context for consideration of 'mental representation' and consideration of theoretical models for cognitive structures. Mental imagery is considered here as one aspect of internal representation. The definitions of concepts, schemes and larger scale structures provide a means of describing what constitutes thinking and learning. This section concludes with a description of 'embodied cognition', a theory which suggests that our bodily experiences have a role to play in our understanding of our environment.

The next section (§2.2.1) introduces the Piagetian theory of development and the Vygotskian theory of learning. The views of learning as 'individual construction of knowledge' (Piaget) and as 'acquisition of cultural tools' (Vygotsky) are compared. This allows the role of language in thought and learning to be considered from both perspectives. The 'constructivist' view is that language merely provides labels for communicating thoughts whilst the 'socio-cultural' view suggests that language provides us with the *means* of thinking about our experiences.

The interplay between language and thought is explored in §2.2.2. The theory of 'cognitive linguistics' suggests that language and thought are interdependent and that the use of language indicates the roots of our conceptualisations. The linguistic terms 'metaphor', 'analogy' and 'metonymy' are then examined to consider whether they may be indicative of constraints upon our thinking. The theory of 'linguistic determinism' suggests that the language of our culture *determines* the way we think. This discredited theory would mean that we can only think about things we have words for. The more moderate claim of 'linguistic relativism', which suggests that language use can give indications of the way we are thinking, is supported by studies.

The theories and concepts introduced here provide the foundations for the discussions in Chapter Three which looks more closely at the experiences in the mathematics classroom and the manifestations of the learning that has taken place. The interdependence of mental imagery, language and thought is considered there more particularly in the context of learning mathematics.

NB: Terms may be used initially without a formal definition in order to progress toward a definition. When a term is defined for the purposes of the study it will appear in **bold script** and this definition will be entered in the glossary.

2.1 Theories of Cognition

> I take it as given that all cognition, whatever its nature, relies upon representation, how we lay down knowledge in a way to represent our experience of the world.
>
> (Bruner, 1996, p95)

Bruner was not suggesting that we copy the world into our minds. He is pointing out that it is our experience that is 'represented'. We construct the world through our *subjective* experience of it. Thinking of a 'mental representation' as a simple re-presentation of an external reality *or* thinking of the mental representation as part of a conceptual structure built upon our experiences, is an example of the objective/subjective tension mentioned at the end of Chapter One. It is at the heart of theories of learning which will be discussed in §2.2.

In this section the psychological foundation for those theories is set out. Brain activity is considered and the theoretical notions of 'mental representation' are also examined. Some models of cognitive structures are explored, the key idea of 'mental construction' is introduced and the place of mental imagery in that construction is considered. Finally the notion of 'embodied cognition' is examined. The key idea developed here is that our mental constructions are internal representations based on our experiences and this includes our bodily interactions with our environment.

2.1.1 Internal representations

A '**representation**' can be taken to be something which 'stands-in' for something else. The representation thus takes the place of the represented. A '**mental representation**' can be interpreted as that which takes the place of perceptual experience in our brains and provides the means for thinking about the experience. This section considers first how brain activity might provide the structure of a 'mental representation' and the structure for 'learning'. This leads to a review of models of mental representation and modes of mental representation.

2.1.1a The brain

The complexity of the structure of the brain and limited, though expanding, knowledge of the relationships between brain activity and 'mental' states inevitably means that models of the mind draw only loosely on the actuality of neurological processes. This brief description of the way the brain operates is thus intended to provide a neurological basis for considering the validity of those models.

There are 180 billion cells in the brain, 80 billion of which may be involved in information processing. Each cell receives up to 15 000 connections from other cells (Kolb and Whishaw, 1996). The activity of these 'neurons' has an electro-chemical basis i.e. the cells are separate but communicate through chemical interaction which produces small changes in their electro-static charge. This electrical activity can be measured using brain scanning techniques and provides a picture of the involvement of brain areas related to specific behaviours. Neuropsychologists take the view that 'mentalist terms' such as ideas, beliefs, memories etc. are associated with physical activity and structures of the brain.

In addition to brain scans of humans and studies of brain damaged patients, studies of invertebrates with few, but large, neurons suggest that learning and memory are processes of chemical transmission which alter the strengths of connections. It was once thought that the brain had innately wired neuronal pathways which were modified only slightly by experience, probably through habitual use of nerve cell connections (Meadows, 1993). It is now clearer that, at least in the early years, neurons which integrate nerve impulses develop patterns of connections and activities and transmit substances which help further nerve cell growth. Multiple connections between areas of the cortex are thus further developed and elaborated by 'experience', i.e. familiar and novel nerve impulses.

There is evidence also from laboratory experiments that animals living in complex environments have more neuronal activity and have more neuron connections than those living in sterile environments. This suggests that 'learning' from experience, i.e. adapting to new experiences, is an organic process and that brain structures develop as a result of interactions with the environment.

The neural activity associated with overt behaviour such as perception and movement can thus be studied in some detail but the way information is stored and used at a latter time is not so easily described. Kolb and Whishaw (1996) suggest that information that controls action may arise directly from present experiences or

> it may be 'representational' in the sense that it is an internal representation of what was (and perhaps still is) in the real world. When behaviour is guided by representation it is guided by a neural record that is independent of concurrent sensory input.
>
> (Kolb and Whishaw, 1996, p151)

In the absence of representational information behaviour can only depend on direct information. From this viewpoint the 'neural record' must provide the basis for subsequent reasoning about previously experienced activities (both physical and mental) which are not currently occurring. Whilst perceptual information is available at the time of the stimulus any further processing relies on some representation of it. This representation forms the 'memory' of that previous experience. A **memory** can be taken to be a neural record of an experience.

The complexity of the patterns of neuronal interconnectivity precludes simplistic models of specific storage areas for these representations. Some areas appear to be specialised in processing different *aspects* of sensory experience and some in different psychological processes such as memory and coordination of movement. There are numerous regions of the brain, however, implicated in each of the modalities of sensory perception: around 30 for vision, 15 for auditory and tactile sensations. Some areas also have multi-modal functions so that, for instance, we can visually identify objects we have only touched.

Neural records of past activity provide us with mental representations that allow thought even if they are not simply stored in one identifiable area of the brain. This suggests that models of mental representation in terms of 'memory stores' may simply be theoretical conveniences. Models do, however, allow us to characterise thinking and learning.

2.1.1b Models of mental representation

Halford (1993) distinguished two *levels* of cognitive representation. The first is '**implicit**' . This relates to unconscious thought. Performance is efficient and relatively effortless but without cognitive access to the rules or knowledge employed. We do not have to 'think' about walking, for instance, it happens 'automatically' without conscious effort and we need not be aware of the mental activity which controls it. The other level of representation is '**explicit**'. This is a representation we may be consciously aware of. It relates to a less efficient mode of thought where access to the rules and knowledge represented is greater and it is often communicable to others. Thus implicit knowledge of procedures for performing a task can exist without the conscious access to them that would allow explicit links to be made with other knowledge. To this extent implicit knowledge thus allows performance without 'understanding' i.e. without awareness of why it works or how it relates to other knowledge. Explicit knowledge may be seen as a higher level representation of implicit knowledge that allows it to be expressed and allows for understanding.

This theory is independent of any specific mode or organisation of the cognitive representations. 'Information processing' is seen to provide a meta-theory for cognitive psychology (Ashcraft, 1994). The information processing model treats cognition as the coordinated operation of active mental processes within a multi-component memory system. The 'sensory register' is a brief duration *sensory memory* system which receives and holds perceptual information some of which may be transferred to the 'short-term store'. This is the information we are consciously aware of. If it is 'attended to', then 'rehearsal' or 'comprehension' may lead to transfer to the 'long-term store'.

It is important to note that the internal representation is a consequence of what is 'attended' to and extracted from the initial experience (Kosslyn, 1980). '**Attention**' describes the process of using selected information, either sensory or from internal representations. '**Rehearsal**' is a conscious act of retaining information in memory by repetition (verbal or physical activity) and '**comprehension**' is a conscious act of understanding through relating to previously stored information. Experimental studies to discover the neuronal activity associated with attention suggests an 'executive attention system' i.e. a part of the brain that directs attention, so that 'consciousness' might consist of the information being operated on by this executive system. The evidence from these studies is, however, not overwhelming (Kolb and Whishaw, 1996).

Tulving (1972) distinguished two separate information processing systems which he refers to as 'semantic' and 'episodic' memory. He suggests that **episodic memory** is a record of a person's experiences which receives and stores information about episodes, events and relations between them. Perceptual information is stored in terms of its 'autobiographical reference' to the existing contents of the episodic memory. **Semantic memory** is the memory necessary for the use of language. It is the store of knowledge about words and other verbal symbols, the relations between them and the rules for manipulating both symbols and concepts. It is a store of objects detached from autobiographical reference.

These models are not incompatible with mental representation in terms of brain activity and neural records but it is difficult to give meaning to a mental state such as 'understanding' in terms of electro-chemical activity in the cortex. Theories of cognition have thus developed based on modes of mental representation that are more easily described. In asking what mental representations are 'like' we are forced to use analogies. The 'modality' of the internal representation, i.e. the form in which it exists in the mind, is thus often described in terms of language or in terms of sensations such as sight.

2.1.1c Modes of mental representation

A common theoretical notion for the content of memory stores is the 'proposition'. Ashcraft (1994) simply defines a **proposition** as a representation of meaning that can be stored and retrieved from memory. It is the smallest unit of knowledge that can stand as a separate assertion, a simple relationship between two concepts that can be expressed by a simple declarative sentence (e.g. "six is a number"). The traditional view of a proposition is a statement whose truth value may be ascertained but Kosslyn (1996) makes the point that a propositional representation should be thought of as a 'mental sentence' not a statement in a natural language. Richardson (1998) further suggests that propositional representations are strings of symbols, not necessarily words. Propositional representation could be characterised as neural connections between neural records.

Much of our brain activity may be at an implicit level and the mental representations may be in terms of these mental sentences. In our explicit mental activity, however, we may be consciously aware of thinking in words by forming an internal dialogue with ourselves. A thought expressed in words, to oneself or others, has an underlying mental representation that may not be stored in memory as words. In addition to words forming in our minds we may also be consciously aware of perception-like sensations. In simply thinking about a meal, for instance, we might have an apparent sensation of 'taste', 'smell' or 'sight' of the food or the emotions and physical movements involved in preparing and eating the meal. These sensation-like mental experiences are usually referred to as 'images'.

A functional definition of **mental imagery** is given by Finke (1989):

> the mental invention or recreation of an experience that in at least some respects resembles the experience of actually perceiving an object or an event, either in conjunction with, or in the absence of, direct sensory stimulation.
>
> Finke (1989, p2)

Mental images are seen by some, notably Kosslyn (1980, 1996) with reference to mental *visual* images, as another mode of mental representation distinct from propositional representation. Others, notably Pylyshyn (for instance 1993), have argued for exclusively propositional representation i.e. a distinct internal, amodal, vocabulary of symbols. He insists that thoughts are not thought in a natural language nor in pictures but in terms of propositions. Kempson (1988) suggests that Fodor's Representational Theory of Mind also assumes that cognitive mechanisms extract information from the outside world and process, store, and retrieve such information via an internalised system of representations called 'the language of thought'.

Whatever the underlying internal representation 'mental imagery' usually refers to all quasi-sensory experiences that are sensation-like re-presentations of sensory, perceptual, affective or other experiential states e.g. hunger or fatigue (Richardson, 1969). Evidence that images are based, in part, on previous perceptual experiences is presented in the next section.

2.1.1d Mental visual imagery

Russell (1956) gave a definition of an image that emphasises that previous perceptual experiences are reproduced by the image. An image is:

> a centrally aroused experience which reproduces in part some previous perceptual experience in the absence of the original sensations.
>
> Russell (1956, p68)

Neurophysiology findings indicate that images have similar characteristic patterns of brain activity to the actual experiences, i.e. when an image is evoked there could be similar neuronal activity to the original perception. Kolb and Whishaw (1996) point out, for instance, that experiments to discover features of 'movement images' suggest that they parallel real movements. For example when subjects are asked to imagine rotating a die a similar time elapses to when the physical act is performed. When asked to imagine finger movements the same area of the cortex is involved as for the physical movement though with out the lower level neuronal activity required specifically for the movement of fingers.

Kosslyn (1996) also noted the findings from neuroanatomy and neurophysiology that 'visual areas' of the brain have neurons organised to preserve the topographical structure of the retina and that parts of the brain used in visual perception are also used during visual mental imagery. He thus proposed that mental visual images are 'seen' when information stored as a result of visual perception is projected back to those visual 'buffers'.

Much of the literature on mental imagery concerns the connection between mental visual imagery and perception. There are a variety of opinions on how closely connected they are. Intons-Peterson and McDaniel (1991) note that the 'functional' view of mental imagery is that it simply parallels perceptual processes. They point out that Shepard's idea of 'second order isomorphism', for instance, requires only that an image has something in common with what went on in the brain when the perception occurred, rather than a one-to-one correspondence between components of the image and components of the perceived object. The 'structural' view of mental imagery assumes that there are one-to-one correspondences between the structure of the image and the structure of the perception. The 'interactive' view of imagery assumes that some of the same brain mechanisms are used for each. The research which demonstrates a structural connection between perception and imagery, and in some cases evidence of an interaction between perceptual and imaginal brain activity, is of psychological interest, but for the purposes of this study it is sufficient to assume simply that mental images are in some way based on previous experiences.

2.1.1e Imagery and language in thought

Kosslyn (1980) has suggested that mental imagery will necessarily be used in response to a question about a concrete object where the information has not been stored as part of a propositional representation or can not be deduced from propositional representations. He also proposed a Representational-Development Hypothesis:

1) The type of internal representation that is predominantly used changes with age.

2) The later types are more powerful than earlier ones.

3) Later types supplement and overshadow older ones.

Kosslyn suggested that as propositional knowledge increases and deduction becomes easier then imagery is used less. With increasing use of known facts the mental representation is more likely to take on a propositional form. Kosslyn has also suggested that if a child has few representations in other formats he has little choice but to use the ones he has. For some children mental imagery generation may be quicker and easier than propositional reasoning which requires more or different processing resources.

Paivio (1971, p18) insists that it is "simply asserting a truism" to say that modes of representation evolve within the individual from the more concrete to the more abstract. In Paivio's view images and words constitute two interacting symbolic systems of mental representation so that meanings of words may be stored in both a verbal and imaginal form. Marschark and Hunt (1989) note that laboratory experiments on remembering tasks have mostly been consistent with Paivio's 'dual coding' theory.

The view taken by Piaget and Inhelder (1971) was that the use of mental visual images may be evidence of the individual's preference for the representation of concrete instances and thus abstract concepts are coupled with mental images to give them 'exemplarity'. They regarded it as *self-evident* that individuals will concretise the words they use with personal images and that language alone can not conserve perceptual experiences in memory.

Language and mental imagery are thus thought to interact in short-term memory. Logie (1991) points out that this is the assumption of the Baddely and Hitch 'working-memory' model. This model assumes a 'central executive' which is responsible for reasoning. This central executive coordinates the activities of the 'phonological loop', specialised for holding verbal information, and the 'visuo-spatial scratch pad' which is used for temporary retention of visual and spatial material. In Logie's view the evidence from research literature is consistent with a temporary visual storage system of some sort and that an image can only be retained and used for short periods in thinking whilst a conscious effort is made to keep it in mind. This is consistent with Kosslyn's (1996) visual buffer model which assumes that information held in 'visual' form in the buffer will fade unless continually refreshed.

The theoretical view that thought may involve words and images has some neuropsychological foundation. Kolb and Whishaw (1996) suggest that thoughts are often 'verbal images' which use the same area of the brain as that used for utterance production. They conclude that each kind of image is formed by the same brain areas that produce action so that these 'brain systems' are responsible for action, memory and imagining. This is an important point which provides a basis for claims that cognitive structures develop as a result of experiences and that words, images and memories are all part of those structures. If thinking in words is seen as using 'word images' then the utterances that accompany those thoughts may be a close approximation to the thoughts themselves.

2.1.1f Relevance to present study

The literature reviewed provides a theoretical basis for the view that mental representations result from experiences. The neurophysiological evidence suggests that brain activity provides the mechanisms for representation of experience. The internal representations of previous physical and mental activity could involve similar brain activity to the original experience. The theories of mental representation suggest that the explicit mental representation may be linguistic or imagistic in nature but the underlying implicit representation may rely on some 'language of thought'. This 'language of thought' could relate, however, to patterns of interconnectivity of neurons. This implies that the language of thought relates to neuronal records of previous physical or mental experience.

In much of the psychology literature the mental imagery debate centres on the mode of internal representation. Having acknowledged the different theories it is sufficient for the purposes of this study to concentrate on the mental imagery that is manifested to the conscious mind. In this study it is assumed that mental visual images, which relate to previous experiences, can provide some evidence that the previous experiences have become a part of our memory. This is not, however, an assumption that the images are a 'copy' of classroom materials. It is important to note that what children attend to in classroom activities will determine the image that is formed. Individual differnces in quantity and quality of mental visual imagery will be explored more fully in §3.2.3.

The theories of internal representation suggest that all individuals may have the mechanisms for learning but that 'attention' could be the factor which dictates individual differences in what is learned from the common experiences. The individual's mental representations are a result of the individual's experiences. Similarly the cognitive structures that develop in the mind could be the result of an individual's mental activity. Like the internal representations on which they are based these cognitive structures are usually described in terms of models and analogies. The next section looks at these models.

2.1.2 Models of cognitive structure

The neuropsychological view of cognition presented in the previous section provides a background for this consideration of cognitive structure. Whilst the neurophysiological perspective allows these structures to be set in the context of brain activity, the models described in this section are largely theoretical constructs. This is an examination of the theoretical framework for the theories of learning which are discussed in §2.2.

Concepts and schemes are introduced as base levels of cognitive structure. The theories suggest that cognitive structures develop as a result of abstraction. Abstraction from our experiences allows us to build upon our existing structures and this process is seen as the key to learning. In this section larger scale structures, such as 'scripts' are briefly reviewed and a definition of the term 'mental construction' is then given. Finally the role that bodily experience plays in developing cognitive structures is considered in the context of the theory of 'embodied cognition'.

The aim in this section is to consider what *structures* are involved in 'learning from experience'.

2.1.2a Concepts, schemes and abstraction

The New Shorter Oxford English Dictionary quotes Bertrand Russell as suggesting that

> Awareness of universals is *conceiving*, and a universal of which we are aware is called a *concept*.
> (Brown 1993, p467)

Love and Tahta (1991) also distinguish between the process of conception, which implies conceiving and growing, and the product, concept. If a child has a conception it is their own, a concept cannot simply be received from the outside.

Thus to form a **concept** individuals need to be actively aware of universals, in other words to recognise similarities in their experiences. From the multiplicity of sensory information available in any experience those commonalities with previous experiences need to be attended to. The Piagetian view on the process of concept formation is that new experiences are 'assimilated' to existing conceptual structures. Structures may also be adapted in order to 'accommodate' new experiences which have characteristics that do not fit exactly with the existing structure (von Glasersfeld 1995).

Similarly a '**scheme**' is an active organisation of past experiences and reactions to them (Ashcraft, 1994). In particular 'Sensori-motor schemes' are the learned coordinations between action and their sensory consequences (Wood, 1998). The child uses the experience of the consequences of his previous actions to form a cognitive structure with which to think about and predict the consequence of future actions. Von Glasersfeld (1995) describes these 'action schemes' as having three parts: recognition of perceived situation, activity associated with that situation, expectation of previously experienced result. When the result is unexpected the perceptual recognition pattern may be altered to accommodate this new experience.

From this perspective the biological process of adaptation of an organism to its environment is at the heart of how a child makes sense of his experiences. The infant 'learns' by recognising similarities in sensory information gained from perception and refining schemes of action to accommodate new situations. As children develop, however, this simple organic model does not adequately predict how more advanced skills and abstract concepts are learned. Piaget (1928) recognised that since assimilation is a process of fitting a new experience into an existing conceptual structure each new experience may simply be seen as an instance of something already known. He referred to this comprehensive act of perception which ignores detail as 'syncretism'. Differences between the new experiences and previous similar experiences may be disregarded when the learner only attends to that which fits with existing structures.

Piaget thus distinguished 'empirical abstractions', when sensory patterns that recur in a number of experiences are retained and coordinated to form concepts, from 'reflective abstraction', which results from mental operations (von Glasersfeld, 1995). Active reflective abstraction is required to form new generalisations when new experiences do not fit exactly with existing cognitive structures.

These theories do not suggest a mechanism for the processes of abstraction nor how concepts might be represented in the brain. It can be argued that if each previous experience of a concept has a neuronal record then concepts may be represented by common patterns in these records. If propositions are thought of as the building blocks for mental representation then, Halford (1993) suggests, 'propositional networks' link individual propositions to represent concepts. These more complex structures of meaning representation are thus distributed over a large number of neural units and a concept is represented in the brain by a pattern of activation of these units.

2.1.2b Process, procedure and procept

These theories help to explain how we might comprehend 'concrete' concepts related to everyday experience such as classifications of frequently encountered objects. The theories may also explain the learning of skills required for survival in the world. In mathematics learning, however, concepts need not be related to everyday activity and skill acquisition can not be reduced to simple stimulus-response models.

Gray and Tall (1994) distinguish between 'process', the cognitive representation of a mathematical operation, and 'procedure', the specific algorithm for implementing a process. They note also the dichotomy between things to do (procedures) and things to know (concepts) so that procedural and conceptual learning are both requisites of mathematical learning. Gray and Tall also pointed out the importance of forming a static conceptual entity, a 'mental object', from a dynamic process. This has been variously called 'entification', 'reification' and 'encapsulation' (see Gray and Tall (1994) for review). An example is the process of counting being encapsulated as the concept of number.

Gray and Tall suggested that in learning mathematics the formation of a mental object which can become the subject of thought, and itself be used in more abstract processes, is fundamental to mathematical thinking. They also defined a new component in mathematical cognition. 'Procepts' are an amalgam of processes, concepts formed from them and symbols which represent both. Characteristics of 'proceptual thinking' will be explored in §3.2.2b.

The distinction between 'instrumental' and 'relational' understanding, inherent in Gray and Tall's theories, was previously made by Skemp (1976). Relational understanding requires knowing both what to do and why, whilst instrumental understanding involves simply knowing how to do something. The suggestion here is that steps for procedures may be memorised and performed without an understanding of the process. The process may thus not become a mental object that can be used in thought. If processes are not objects of thought then they can not be the subject of the reflection required to make generalisations or for relationships with other processes to be appreciated.

2.1.2c Large scale cognitive structures

Halford (1993) noted Johnson-Laird's suggestion that there are three, mutually exclusive and exhaustive kinds of mental representation: propositional representations, which are strings of symbols that correspond to natural language; mental models which are structural analogues of the world, and images which are the 'perceptual correlates' of models. Halford argued, however, that they are not mutually exclusive and that mental models could involve both propositions and images.

However they are represented **mental models** are defined as cognitive structures which reflect the knowledge, experience and goals of the learner and are not objective 'copies' of reality (Halford 1993). Children learning arithmetic, for instance, have models of calculation built on their own concepts and autobiographical memories of procedures. Lawler (1996) used the term 'thinkable models' which he described as representations of things and processes simple enough that people can use them in 'thought experiments'. He saw cognitive structures as a network of connected thinkable models.

There are other theories of larger scale knowledge structures that accumulate in memory and guide our interpretation and comprehension of daily experiences (Ashcraft, 1994). One such was suggested by Schank and Abelson (1977) who coined the term 'script' for a cognitive structure that represents an appropriate sequence of events in a particular context. They described a script as made up of 'slots' to be filled with information from the current situations and knowledge of the requirements for what can fill them based on previous experiences. A script is an interconnected whole such that what is in one slot affects what can be in another. Scripts are thus the mental models used in everyday situations, they are the stereotypical sequences of actions that define well known activities.

Minsky (1977) similarly used the word 'frame' for a remembered cognitive framework that can be adapted to fit each situation by changing details as necessary. He described it as a data-structure for representing a stereotyped situation. Attached to each frame are several kinds of information: how to use the frame, what to expect to happen next, what to do if something unexpected happens. The frame can be thought of as a network of relations that are always inherent in the situation and 'terminals' that accept data specific to a new situation.

These theories of large scale cognitive structures provide a terminology for describing how we make sense of our everyday experiences by relating the present situation with similar past experiences. Our experiences in the world provide frameworks for coping with future experiences. In mathematics another large scale structure, the 'concept image' has been defined by Tall and Vinner (1982).

Tall and Vinner suggested that a collection of *particular experiences* represents our initial knowledge of mathematics. In their view particular instances of most 'advanced' mathematical concepts have been encountered before they are formally defined. A complex cognitive structure may exist in the mind which yields personally meaningful mental images when the concept is evoked. They used the term '**concept image**' for this cognitive structure which includes 'mental pictures', properties and processes. The individual's concept image develops with new experiences.

Tall and Vinner suggested, however, that sensory input excites certain neuronal pathways and inhibits others so that different stimuli can activate different parts of the concept image. Furthermore, they suggested, concept images may develop in a way which need not make a coherent whole and different parts of the concept image can be evoked at different times. The suggestion here is that our concept image can develop as a collection of unrelated particular instantiations rather than as a connected conceptual structure.

In this study the term '**mental construction**' will be used as an umbrella phrase for the various cognitive structures that have been described. A mental construction for an object or process incorporates all previous experiences related to the object or process. It is an amalgam of concepts, schemes, mental models, and scripts. It involves both language and images related to those experiences. It is similar to the 'concept image' described by Tall and Vinner (1982). The different terminology was chosen to overcome the restricted view of this vast mental structure that 'concept' and 'image' might in themselves imply.

For the purposes of this study '**learning**' will be used to imply that a change in an individual's mental construction has occurred.

2.1.2d Embodied cognition

The cognitive structures described above seem to be constructed from interactions between the brain and the environment. The body merely provides the means by which the brain is provided with information. Much greater prominence is given to the role of the body in the interaction by Lakoff and Johnson (Lakoff and Johnson, 1980; Lakoff 1987; Johnson 1987, 1992) who have developed a theory which has come to be termed 'embodied cognition'. A key aspect of this theory is the 'image schema'. Johnson (1987) defines this as:

> a recurring, dynamic pattern of our perceptual interactions and motor programs that gives coherence and structure to our experience.
>
> Johnson (1987, pxiv)

Image schemas are formed from patterns of recurring bodily experiences that emerge throughout sensori-motor activity and from our perceptual understanding of actions and events in the real world. Johnson assumes that much of our knowledge is thus 'grounded in', and structured by, these patterns of our perceptual interactions, bodily actions and manipulation of objects.

Much of the elaboration of this theory depends upon subsequent use of metaphoric language (described in more detail in §2.2.2) that has its roots in these bodily experiences. Embodied cognitivists also believe that these cognitive structures exist pre-linguistically and that they structure our thinking not just our language. Johnson (1987) gives 'balance' as an example of an image schema of patterns common to all situations requiring bodily equilibrium. We all experience bodily responses to imbalance in our internal bodily functions as well as the more obvious feeling of being upright. The 'balance schema', he suggests, gives rise to meaning for equivalence relations (symmetry, transitivity and reflexivity) and gives rise to the concept of equality of magnitudes. Thus Johnson (1992) suggests:

> The nature of our embodied experience constrains and motivates how things are meaningful to us.
>
> Johnson (1992, p346)

Lakoff (1987) characterised himself as an 'experiential realist'. He suggests that 'objectivists' believe that thought is the mechanical manipulation of abstract symbols (words and mental representations) and that these symbols 'represent' an external objective reality. 'Experientalists', however, believe that our conceptual systems grow out of perception and bodily experience along with social interactions. Thus meaningful thought concerns the nature and experience of the person doing the thinking and the nature and experience of the species and of communities. 'Experience' is here taken to be the totality of human experience which involves the nature of our bodies, our genetically inherited capacities and our social organisation. So meaning is not independent of the knower but meaning is characterised by 'embodiment' that is in terms of the collective biological, physical and social experiences of functioning in our environment.

From this viewpoint our learning of mathematics is in terms not only of our own bodily experiences in and out of the classroom but also in terms of the collective experiences of previous mathematicians.

2.1.2e Relevance to present study

A developing mental construction may evolve as a result of a variety of experiences related to a concept, but not necessarily related to one another. Reflective abstraction is required for connections to be made. This suggests that different classroom activities could evoke memories of different previous experiences without connections being made between the previous experiences. This may suggest why there are differences in the learning of individuals. In the same way that mental representations differ because of differences in attention, the differences in cognitive structures may be due to differences in reflective abstraction. This is an important point for this study and the differences in what and how children abstract from their experiences will be explored further in §3.2.2b.

The theory of embodied cognition assumes that common physiology give individuals common image schemas and hence common understandings. When teachers and pupils use equipment such as Dienes blocks, for instance, they experience similar bodily movements and use language associated more generally with movement and manipulation. Their common understandings are thus rooted in their common image schemas. These schemas may have developed as a result of common neuronal activity in response to individuals' earliest interactions with the environment.

Both these theories are relevant to the present study because they help explain differences and commonalities in the learning of pupils who partake in the same classroom activities.

2.1.3 Summary of theories of cognition

Whilst propositions, concepts and schemes are central to theories of cognitive development, models of large scale cognitive structures, such as scripts or frames, provide a way of describing how learners cope with everyday classroom experiences. The combination of semantic and episodic memories provides mental models of our previous experiences which guide our actions in similar new situations. These structures may be implicitly or explicitly available to learners, but their presence may be suggested by observed behaviour.

The theme that connects the ideas presented in this section is simply that humans learn from their experiences. The brief review of neuropsychology literature was intended to set out the physiological basis for this belief. It appears that cognition can be described in terms of organic activity in the brain. Similarly the theoretical models of cognitive structure suggest the ways in which we adapt to our environment by assimilation and accommodation and by abstraction from previous experiences. The picture of learning from experience that is common to all these models is further enhanced by the notion of embodied cognition which suggests our common bodily experiences could explain why we may learn some of the same things.

2.2 Theories of learning and language use

The theories of Piaget concerning the formation of cognitive structures have already been introduced in §2.1. In §2.2.1 the Piagetian view of development will be contrasted with that of Vygotsky. Particular attention is given to their views on the role of language in *thought*. Both saw experience gained in the world as the key to development and learning yet they assigned different roles to teachers. They had different views on the place of language in *learning*. This first section thus considers the role of activity in learning, the role of society in communication of ideas, and the part played by representations, including language, in the development of cognition.

The role of mental imagery and language in thought is explored further in §2.2.2. 'Symbolist' theories assume that both images and language can provide the medium of thought whilst 'conceptualist' theories assume they are the product of thought. Imagery is seen as a potential medium for thought. The discussion of the role of language begins with consideration of the literary devices 'metaphor', 'analogy' and 'metonymy'. It will be suggested that these are more than mere 'figures of speech' and that they are a manifestation of underlying cognitive structures. It is concluded that our imagery and language do not determine our thinking but that the images and language we use can give indications of our modes of thought.

2.2.1 The learner as individual and community member

The neuropsychological perspective presented in §2.1 makes it clear that the brain is capable of learning from experiences. This section reviews theories of how learning can occur in the classroom. The Piagetian view of an organism's adaptation to its environment is contrasted with the Vygotskian view of the learner supported in learning by others in the community. These perspectives taken together add to our understanding of the differences and commonalities in learning by children who have participated in the same classroom activities.

2.2.1a Construction of reality and internalised action

Von Glasersfeld (1995) suggests that Piaget was profoundly influenced by Kant's insight that:

> ... whatever we call knowledge is necessarily determined to a large extent, if not altogether, by the knower's way of perceiving and conceiving.
>
> (von Glasersfeld 1995, p55)

Thus Piaget, as noted in §2.1.2a, considered that learners construct their own knowledge, their own picture of reality, from their experiences. He saw cognition as a process of adaptation to the environment and assumed that humans have a natural tendency to organise knowledge into coherent cognitive systems. This suggests that the 'way of perceiving and conceiving' can vary with individuals and that the individual's existing conceptual structures can determine what is conceived.

Building upon Piaget's 'constructivist' theory von Glasersfeld proposed 'Radical Constructivism' as "a way of knowing and learning" (von Glasersfeld, 1995). He gave two basic principles:

> 1) Knowledge is not passively received either through the senses or by way of communication; knowledge is actively built up by the cognising subject.
>
> 2) The function of cognition is adaptive in the biological sense of the term, tending toward fit or viability; cognition serves the subject's organisation of the experiential world, not the discovery of ontological reality.
>
> (von Glasersfeld 1995, p51)

From this perspective knowledge does not constitute a picture of what 'is' but rather the learner's conceptualisation of how they fit into the world they have experienced. Humans construct both an internal and external world, the internal being the 're-presentations' of the experiences that have given rise to their constructions of an external world. Both are based on *subjective* experiences. In terms of the elementary mathematics classroom this suggests that it is the pupil's experience of mathematical activities that is the basis for learning. Since those experiences vary and the conceptual structures that individuals bring to those activities vary the conceptualisations individuals form will also vary.

Piaget's theory placed action and self-directed problem solving at the heart of learning and development (Wood, 1998). The theory suggested that 'mental operations' develop from internalised action and abstract thinking is abstracted from physical actions. Learning how to act on the world and discovering the consequences of actions thus forms the basis for thinking. From this perspective it may be conjectured that the learner's interaction with the physical world, and particularly experiences encountered in the classroom, might provide the main constraints and contributions to the development of their thinking.

Vygotsky also argued that the foundations of mental processes lie in activity, suggesting that all forms of thought are activities. Mental activity is not a simple re-enactment of external activities, but activity in the social world becomes 'internalised' to form the intellectual process. Thought is thus a substitute for action (Wood, 1998).

Piaget suggested how this internalisation might occur (Meadows, 1993). 'Mental actions' are learned coordinations between action and their sensory consequences. 'Mental operations' are mental constructions which are classes of mental actions; they are systems of actions not just single action-consequence relations. Thus one stage in intellectual development, 'concrete operational' thinking, uses these mental operations. Piaget's developmental theory assumes a natural progress towards a balanced cognitive system developed from an infant's earliest experiences where what is perceived is dependent on previous actions, where an object "is" what has previously been done to it. It may thus be conjectured that pupils need to perform actions on physical materials as a basis for learning. This point will be discussed further in the next chapter.

2.2.1b The mind in society

In a comparison of the views of Piaget and Vygotsky, Meadows (1993) notes that Piaget's theory assumes cognitive structures (concepts, schemes, mental operations) which are essentially individual and independent of social practices. His psychology is the study of the mind's inner working. By contrast the notion that cognitive processes are formed through interaction with the social environment is at the centre of Vygotskian theory. Vygotsky (quoted in Meadows, 1993) was of the opinion that

> any higher mental function was once external because it was social at some point before becoming an internal, truly mental functioning ... first it appears between people as an inter psychological category, and then within the child as an intra psychological category.
>
> (quoted in Meadows, 1993, p 237)

in his view, for instance

logical argumentation first appears among children and only later is united within the individual and internalised.

(quoted in Meadows, 1993, p 237)

Richardson (1998) suggests that Vygotsky sought to explain the development of higher level thought processes within a Marxist framework, viewing humans as social beings influenced by their social environment, whilst Piaget set out to provide a description of cognitive development in terms of adaptation between individual organism and environment. Piaget recognised, however, that society was part of that environment and stressed that the most frequent cause of accommodations is linguistic interaction with others (von Glasersfeld, 1995).

Von Glasersfeld was also of the opinion that social interaction can provide corroboration for the constructions of individuals. Thus, whilst the constructivist view is that all knowledge is subjective knowledge, knowledge is taken as objective when it is corroborated by others. We infer that others have the same knowledge as we have when they behave in a way that produces the same result as that produced by our use of our own knowledge. This 'intersubjective' knowledge is then regarded as real or true. Whilst empirical 'facts' may be constructs based on regularities in a learner's experiences they become 'common knowledge' when corroborated by others.

Vygotsky saw instruction as the key to human development. He defined 'intelligence' as the capacity to learn through instruction. His primary concern lay with the nature, evolution and transmission of human culture and in his view the young are assisted in their learning by more expert peers and adults, including teachers, who already possess knowledge of the culture (Wood, 1998). Thus children unable to perform tasks when left to their own devices can be helped by someone who already can perform them and this co-operatively achieved success is the key to learning. From this perspective the child is trained to behave in ways which the culture has found cognitively useful (Meadows, 1993). Development is thus from the outside to the inside and as a result, it is suggested, cognition is extended beyond the limits inherent in the individual (Richardson, 1998).

2.2.1c Cultural tools and the role of the teacher

Vygotsky emphasised the role played by the culture's system of symbols (languages, sciences, books, diagrams, pictures and other materials) in the learner's development. In his view these systems not only structure learning but are also part of the structure and activity of thinking (Wood, 1998).

Luria (1982) noted the Vygotskian distinction between 'everyday concepts' learned at home and 'scientific concepts' learned in school. Whilst children understand those things that are common features of their familiar social experience they need activities to provide experience of the unfamiliar 'scientific concepts' encountered in more formal school settings. At school the child may initially use words or performs actions without knowing why but the full range of meanings develops through further social interaction. Thus children can use words and perform actions initially without having the same understandings as the teacher.

Piaget's view that "Thought is internalised action" also leads to the conclusion that children need to be active and constructive to develop their understanding of the world. From the constructivist perspective, however, teaching of abstract procedures, without establishing the connections with activities involved in the solution of practical concrete problems, is bound to fail. In other words the learner's thinking develops as a result of their activities not simply as a result of being told what to do. Piaget also believed (Wood, 1998), however, that social interactions may facilitate development. The teacher may encourage the child to review their own constructions, but only when the child is in an appropriate state of readiness for change. A teacher thus can provide appropriate materials and contexts for development but the construction of understanding occurs through the child's activity not through direct efforts of the teacher .

Lesh and Kelly (1997) gave a summary of the constructivist view which indicates that materials and contexts, in other words cultural tools, are an important aspect of the child's environment for learning mathematics (my italicised emphasis):

> ...we view learners as active, purposive, self-regulating and continually adapting agents who structure their environments as well as their own experiences in these environments. Humans interpret their experiences using internal conceptual structures, which cannot merely be received from others, but which must be developed, actively, by each individual. Further we assume that the *meanings* of these constructions tend to be partly *embedded in a variety of external systems of representation*, which might involve written symbols, spoken language, experience-based metaphors, or graphic images.
>
> (Lesh and Kelly 1997, p398)

2.2.1d Social constructivism and situated cognition

The Piagetian and Vygotskian views have been presented as two theories for classroom activity and children's learning. Many mathematics educators have seen the benefits of using both perspectives and embrace the theories of both individual and social construction. Fuson, Smith and Lo Cicero (1997), for instance, in a study involving a variety of representations for place value, used a constructivist view of learning and a Vygotskian view of teaching. Thus the teacher assists the child to build conceptual structures, that are useful within a cultural domain, by adapting their assistance to the child's existing conceptual structures.

This, and constructivist inspired problem-solving-based learning (Cobb, 1995; Fuson, et al., 1997) might be seen as 'social constructivism'. Proponents recognise that the social context of the classroom is an essential part of the environment that the learner experiences. The learner is still seen as an individual constructor of a private reality even though it is corroborated by the community.

Those who see the learner as one who 'appropriates' knowledge and meaning in response to experience in social contexts (Ernest, 1994) may be regarded as taking a more purely socio-cultural perspective. This assumes a shared construction of meaning in a 'community of practice'. The socio-cultural perspective (Goos, Galbraith and Renshaw, 1999) assumes the inherently social and cultural nature of cognition. It regards knowledge acquisition as progress towards fuller participation in communities of practitioners, and not primarily as the acquisition of mental structures. From this perspective the mathematics classroom:

> enables the practices, values, conventions and beliefs characteristic of the wider communities of mathematicians to be progressively enacted and gradually appropriated by students.
>
> (Goos, Galbraith and Renshaw, 1999, p 36)

This is not, however, the 'traditional' classroom where mathematics is seen as mastering a predetermined body of knowledge, where pupils' activity is limited to imitating the techniques prescribed by the teacher, where the teacher is the external 'validating' authority for knowledge. In such circumstances, it is recognised, children can give the appearance of mathematical competence by simply reproducing the correct ways to manipulate symbols.

Socio-culturalists believe that human action is mediated by cultural tools and that cognition is located both in the individual and in the functional systems of actions and interaction that this tool use allows (Richardson, 1998). Learning is thus seen as a process of appropriating cultural tools. The teacher *and* learners need to weave together the 'everyday' and previously acquired concepts of the learner with new 'scientific' knowledge encountered in the classroom so that learning is not seen as merely 'transmission' of skills but a re-constructing of the culture by members of the community (Goos, Galbraith and Renshaw, 1999).

Lerman (1999) noted the body of research which suggests that knowledge is culturally and socially 'situated' and expressed the concern that school and everyday experience might be divorced. He pointed out that if teaching and learning of mathematics is seen simply as apprenticeship into the practices of school mathematics, and induction into the language of mathematics, then skills learned in the classroom may not transfer to other contexts. Furthermore he quoted Gergen:

> different vocabularies construct the world differently, and as they do so they have different implications for action.
>
> (quoted in Lerman 1999, p 95)

There are two implications here. First, that 'situated-knowledge' carries the risk of being devoid of meaning in different contexts. Second, that the 'vocabulary' of the cultural tools, appropriated in specific contexts, could constrain the way that aspects of the world are collectively constructed by members of the community. The concern that language may constrain thought is considered more fully in §2.2.2. The concern that classroom activities may constrain thought will be explored in Chapter 3.

2.2.1e Relevance to present study

Whether the learner is seen as an individual making sense of his environment or a member of a society being introduced to the culture common to that society, it is clear that the child's efforts and existing conceptual structures may be the key to successful learning. Lampscher (1984) suggests that:

> The quality of learning does not directly depend on the teacher's activities ... On the contrary the critical point is the quality of the pupils' own learning activity - how they themselves work on the learning material in intellectual and/or practical ways, how they perceive and process what the teacher, the text or other sources offer.
>
> (Lampscher 1984, p21)

Piaget and Vygotsky each suggest that the learner's activity is the key to learning and to their intellectual development. The cultural tools are not, per se, determining factors in this intellectual development. The classroom activities do, however, provide the context for that intellectual development.

The perspectives of constructivism and socio-culturalism are relevant to this study in providing frameworks for the analysis of the children's cognitive constructions revealed by their responses to interview questions. The two perspectives also provide a framework for a description of the classroom context in which they were constructed. The community of practice in the classrooms observed for this study involved the use of cultural tools. The influence of these tools on children's thinking will be described more fully in Chapter 3.

The way the system of *symbols*, developed by the community, influences the thinking of the children will be explored in the rest of this chapter. One perspective suggests that mental imagery and language may simply provide a means of communication of ideas that have been formed as a result of the activity. The other suggests that internalising the symbols (including language and mental imagery) that accompanies those activities can provide the structure for thought.

2.2.2 Mental imagery and language in learning and thinking

This section considers first the role of mental imagery and language in learning and in thinking. There is further exploration of the ways in which thought and language use interact and a closer examination of the way embodied cognition may be manifested.

The literary terms metaphor, metonymy and analogy are defined in order to provide the context for a key notion in this section that use of language associated with one concept to describe another is more than just a 'figure of speech'. The 'conceptual metaphor' is introduced as a constraint on problem solving. This is seen as an instance of interaction between thought and language rather than language determining thought. The section concludes with details of a study which indicates that language and thought may be interdependent and that speech style can indicate modes of thought.

The aim of this part of the literature review is to develop the argument that the mental imagery and language people use is indicative of their mental constructions.

2.2.2a Role of mental imagery in thinking

A distinction is made between 'symbolist' and 'conceptualist' theories of the role of mental imagery in thought by Denis (1991). 'Symbolist' theories of thinking assume that thought is linked to mental representations (symbols) so that these symbols, which include images and verbal representations, are the medium of thought. 'Conceptualism' assumes thinking involves mental entities of a conceptual and abstract nature so that symbols are a product of thinking. From this view symbols are the means of expressing thought but not necessary for it. Vygotsky was thus a symbolist whilst the Piagetian view is conceptualist. Denis (1991) argues that mental imagery is a set of processes that can be brought into play at various levels of cognitive activity and is thus a potential medium for thought.

There is certainly anecdotal evidence that images have a role in thinking for some people. Wood (1998), for instance, quotes Einstein:

> The words or the language, as they are written down do not play any part in my mechanism of thought ... in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined ... the elements are in my case visual and some of muscular type ... conventional signs and words have to be sought for laboriously only in the second stage.
>
> (quoted in Wood, 1998, p226)

The work on mental visual imagery conducted by Galton, published in 1880, suggests however that at that time many people were sceptical that imagery even existed let alone played a part in thinking. Richardson (1999) notes that Galton, who devised a questionnaire to investigate people's mental visual imagery, found that many friends in the scientific community dismissed the idea of mental imagery as 'fanciful'. From this he inferred that

> an over-readiness to perceive clear mental pictures is antagonistic to the acquirement of habits of highly generalised and abstract thought.
>
> (quoted in Richardson, 1999, p10)

There is some confirmation of this view in the studies of Krutetski (1976) and Presmeg (1986a, 1986b). Galton subsequently discovered that a wider cross section of society showed considerable diversity in quantity and quality of mental visual imagery related to everyday events and that mental visual imagery was not uncommon.

Modern research into mental imagery does not question that mental imagery is an empirical phenomenon which is open to scientific investigation (Richardson, 1980). Much of the laboratory experimentation centres on connections between perception and mental image, however, and does not have immediate implications for the elementary mathematics classroom. Those studies which reveal the value of imagery as a mnemonic device may suggest that similar activities might have a use in learning number facts but no such experiments were discovered in the literature search.

Two other findings, however, are worth noting (Richardson, 1999):

1) The reported vividness of mental visual imagery seems unrelated to general academic performance.

2) The reported vividness of mental visual imagery seems unrelated to spatial ability.

There is also some evidence to suggest that mental visual imagery is used most often in response to questions where a pictorial image is helpful, for instance people report an image when asked "How many windows in your house". This suggests that images are generated in situations where thinking is supported by visual images i.e. in the absence of a physical picture a mental one is generated when needed.

Other studies have given evidence that when some words are heard, or thought about, they more often evoke mental visual imagery than others and furthermore that the mental visual imagery is more vivid for some words than others (see Richardson, 1999 for review). These 'high-imagery' words tend to be names of common objects. Abstract nouns such as 'thing' are 'low-imagery'. High-imagery words are found to be more easily recalled in laboratory experiments involving remembering lists of words. Instruction on use of mental visual imagery for the purpose of recall is found to be beneficial for high-imagery words but not for low. This would seem to suggest that mental visual imagery is more easily related to 'everyday' concepts than 'scientific' concepts.

From the studies reviewed it is not clear what part mental visual imagery plays in thinking. Kosslyn (1996) noted that some people believe images to be 'epiphenomenal' in that they play no part in cognitive processing. Images merely accompany the thinking. This may be true, but mental visual imagery appears to have a role in *assisting* thinking in those situations where information about spatial characteristics are required. The studies give indications of individual differences in quantity and quality of mental visual imagery. This will be explored more thoroughly in the context of mathematics learning in §3.2.3.

2.2.2b The role of language

Wood (1998) contrasts the Piagetian and Vygotskian view of the role of language in development. For Piaget language was a system of symbols for representing the world, as distinct from actions and operations which form the process of reasoning. The processes of thought are thus derived from action not talk. Whilst language does not create the structure of thinking, however, it can facilitate the emergence of thinking when the child is ready.

The Vygotskian view is that a learned language does not simply provide labels for significant features of the environment but a way of construing and constructing the world. For the infant, speech is regulative, used by adults to regulate the child's actions and then by the child for self-regulation. It thus initially serves for communication but later becomes a tool for thought when speech is internalised. Internalised speech that serves for verbal thinking is a form of mental activity and thus speech comes to form a higher mental process (Wood, 1998).

From the Vygotskian perspective signs of all kinds (gestures, symbols, images, words) can be used for communication between people or for communication with oneself in thinking. They are embedded in activity and constructed through one';s interactions with the social and physical world. The interweaving of thought and language in social interactions with more skilled partners allows the learner to move from fragmentary use of signs, perhaps without understanding, to a more flexible use of cognitive skills (Meadows, 1993). The very processes of reasoning are developed as a result of the ways in which explanations are given by others so that not only the 'local knowledge' about that activity is gained but the instructional process itself is internalised. The nature and quality of classroom discourse thus plays a role in developing children's ability to learn and reason analytically (Wood, 1998).

When members of a community use a common language it could be an indication that learners are merely copying the language use of their teachers. However it was suggested in §2.1.2c that 'embodied cognition' could imply that common language use could emanate from common bodily experiences. In order to explore this further some linguistic terms need to be defined.

2.2.2c Metaphor, analogy and metonymy

As a working definition **'metaphor'** can be taken to be the use of the language of one domain to communicate thoughts about another. Paivio (1979) suggests that:

> For the student of language and thought, metaphor is a solar eclipse.
>
> (Paivio, 1979, p 150)

Paivio makes the point that during a solar eclipse aspects of the sun's appearance may be viewed that are usually not apparent. In the context of the study of thinking the suggestion is that metaphor gives us a view of the thought process that would otherwise not be available to us.

The metaphor 'Metaphor is a solar eclipse' is couched in the conventional 'target-domain' *is* 'source-domain' style. This convention is used to denote a mapping between domains of knowledge, a set of conceptual correspondences; it is not read as a statement of fact (Paivio and Walsh, 1993). This 'structure-mapping' view (Gentner and Jeziorski, 1993) assumes that the metaphor evokes a mapping of knowledge from one domain of knowledge (variously called the source, base, vehicle, literal or primary) into another domain (the target, topic, tenor, metaphoric or secondary).

Metaphors can allow large chunks of information to be transferred or converted from the source to the target provided the 'ground' is established. The **'ground'** refers to the similarities that exist between the two domains. In the eclipse example the ground is 'what may be seen'. Thus our knowledge about what is revealed in a solar eclipse is mapped to what may be revealed by metaphor. Even when the ground is established, however, metaphors are only as useful as our knowledge of the source allows. If we know, for instance, only that in an eclipse the sun is obscured then we might infer that metaphors block our view of thought.

Language associated with the source may also be used when talking about the target. For instance inherent in the theories of learning described in §2.2.1 are two distinct metaphors: 'learning is construction' and 'learning is acquisition'. Thus we may use language related to construction in talking about learning e.g. 'building' knowledge, 'laying foundations' for theories, 'fitting' ideas together etc. Alternatively we may use the language of acquisition e.g. we may talk of 'getting' trained, 'delivering' lessons, 'buying' ideas etc.

Black (1979) suggests that metaphors can help generate new knowledge when the source and target subjects interact in the mind of the hearer so that each is enriched by the other. Metaphors can only achieve this however if they are 'active', if both producer and receiver can identify the ground and the source. 'Extinct' metaphors are merely figures of speech because the source can not be identified and a 'dormant' metaphor implies that the source is not generally recognised. Metaphors that support high levels of implicative elaboration are referred to as 'resonant'.

Each of the metaphors for learning is resonant in providing rich sources of language and related ideas. They are both active in that the ground is easily established and language of construction and acquisition is commonly understood. The distinction between extinct and dormant metaphors is subtle and for most purposes it is sufficient to refer to inactive metaphors. The metaphor 'a triangle is a pair of legs' is an inactive metaphor. Most people do not know that 'scalene' derives from skalinos (Greek word for limping) and is thus distinguished from 'isosceles' (equal legs) (Lopez-Real, 1989). Phrasing metaphors in this way can feel forced when it would be more natural to see this as an analogy or simile i.e. 'the sides of a triangle are like a person's legs'.

At its simplest 'analogy' may be viewed as the basis for both metaphor and simile. In simile the comparison between source and target is expressed more explicitly than metaphor with similarities clearly referred to. For example the simile 'he ate like a horse' is a conventional analogy made even more explicit by the use of the word 'like' and the ground, large appetite, is commonly understood. On the other hand 'Adding decimals is like adding whole numbers' is a common classroom analogy but the similarities, and dissimilarities need to be explored.

Gentner and Jeziorski (1993) as noted above, have provided a technical view of analogy. Gentner's 'structure-mapping theory' assumes that an **analogy** is a mapping of knowledge from one domain (the base) into another (the target) such that the system of relations that holds among the base objects also hold among the target objects. Thus 'analogical transfer' involves mapping a system of relations from the base to the target whilst surface features of the base are ignored (Gholson, Smithers, Buhrman, Duncan and Pierce, 1997).

Another linguistic term which helps with understanding the role of language in thought is 'metonymy'. **Metonymy** has traditionally been seen as the use of a word related to a cognitive domain to refer to the whole or part of the domain. Thus metaphor involves mappings across cognitive models and metonymy maps within the model. Ungerher and Schmid (1996) list the types of relations as: part for whole, whole for part, container for content, material for object, producer for product, place for institution, place for event, controlled for controller, cause for effect. 'I was livid' is a metonymy which uses one aspect of anger to stand for the emotion. 'I'll walk the dog' uses walk for the whole activity but also 'the dog' rather than my particular pet uses the category name rather than the particular member.

The value of these linguistic terms in discussion of cognition can now be examined.

2.2.2d Cognitive metaphors and analogies

Lakoff and Nunez (1997) insist that:

> Metaphor does not reside in words; it is a matter of thought. Metaphorical linguistic expressions are surface manifestations of metaphorical thought.
>
> (Lakoff and Nunez, 1997, p32)

The use of the language of one domain to communicate thoughts about another can indicate a 'cognitive metaphor'. This implies that our conceptualisation of the target domain has the same structure as the source domain. Reddy (1979) gave examples of the 'conduit metaphor' of communication to illustrate that everyday behaviour reflects our metaphorical understanding of experience. In his view the English language suggests a preferred framework for conceptualising 'communication' that can bias the thought process. For example use of "give me an idea" or "put it into words" appears to assume that language transfers thoughts. Words are seen as containers of thought and language functions like a conduit to transfer thought between people. Reddy argued that the language of containers and conduits is so all-pervasive that it requires great conscious effort to communicate about communication in any other way. Its pernicious influence is that teachers and learners can feel cheated when the supposed transfer of knowledge from one to the other is not achieved.

Schon (1979) used the term 'generative metaphor' to imply that the *perspective* of one domain of experience is applied to another. He suggested that metaphor can account for the way in which we think about things, make sense of reality, solve problems and subsequently frame questions about reality. In particular when problem posing derives from the generative metaphor the range of solutions is constrained. This is because attention is focused by the perspective determined by the generative metaphor. His examples were taken from the field of social policy where if the problem is seen as 'fragmentation' then the obvious solution is 'amalgamation'. Framing the problem by use of the word "fragmented" in relation to social services generates the solution of joining them up whilst if they are called "autonomous" then those services may be left alone. Pylyshyn (1993) noted that the use of visual terms for mental images are metaphoric. They do not explain the phenomenon but give us a way of describing it which may inhibit further thinking because there is no need to explain processes in the source.

When learning is viewed as an active construction process based on recognising similarities between new and existing ideas then analogical reasoning may be seen to provide a mechanism for linking new with existing concepts (Gholson, Smithers, Buhrman, Duncan and Pierce, 1997). Teaching through the use of analogy and metaphor, however, presents us with the previously mentioned 'learning paradox' that the structure needs to be understood both in the source and target concepts (Bereiter, 1985). If the metaphor is active for the teacher but at the best dormant, or at the worst extinct, for the learner, then the desired communication of ideas is unlikely to occur.

Despite these reservations the use of analogy is widespread in teaching. The pedagogic use of metaphors can be divided between 'functional metaphor', relating novel to familiar concepts, and 'technical metaphors' which describe abstract concepts in terms of more concrete ones (Petrie and Oshlag, 1993). Boyd (1979) suggests that the use of metaphor in science tends to be at the pre-theoretic level but metaphors can also be used to introduce theoretical terminology. These metaphors can thus become 'theory-constitutive' in that the theory is only understood in terms of the concrete concepts of the source domain. The information processing metaphor for cognition, for instance, is 'theory-constitutive' and literal paraphrases are not easily given.

2.2.2e Cognitive linguistics

As outlined in the embodied cognition section (§2.1.2d) the common basis for language and thought can be explained in terms of common human perceptual systems. This view is the foundation of a theory of language use referred to as 'cognitive linguistics' (Ungerher and Schmid, 1996). Lakoff and Johnson (1980) argue that communication is based on the same conceptual system that we use in thinking and acting so that the language a person uses is a source of evidence for what their conceptual system is like. They also suggest that conceptual systems are fundamentally metaphoric in that we think of concepts in terms of others. Those concepts that are understood *directly* are those related to bodily experience, for example 'object' and 'container', and words such as 'up' are understood in terms of experience of the environment. They believe, moreover, that thought is 'imaginative' in that concepts which are not grounded in bodily experience are conceived through metaphor, metonymy and mental imagery.

Lakoff and Johnson go further to suggest that metaphors partially structure our everyday concepts and that this structure is reflected in our literal language *and* in our behaviour. They give the example of the metaphor 'argument is war' where we not only use the language of battle we also adopt an adversarial approach to it. We may, however, use a variety of different metaphors which indicates that we have different ways of thinking about things. We talk of 'ideas', for instance, as: plants (ideas come to fruition), products (ideas need refining), commodities (ideas are worthless), resources (we use an idea) etc. The language we use indicates how we are conceptualising 'ideas' at the time of the utterance. Furthermore successful functioning in daily life can require shifting of metaphors when we need to use a variety of metaphors to understand some concepts.

Some metaphors organise systems of concepts with respect to one another, for example the 'up-down' spatial orientation provides related metaphors: 'happy is up', 'health is up', 'wealth is up', 'control is up', 'virtue is up'. These provide coherence for both thought and communication: positive aspects are up, negative are down so 'more is up' is associated with 'good is up'. This may lead, for instance, to the view that bigger is better, rather than smaller is better, and lead to the view that there is virtue in accumulating more of anything.

Since metaphor is a way of conceiving of one thing in terms of another its primary function may be to aid understanding. The role of metonymy is primarily referential. It is the use of one entity to refer to another related to it, including the use of a part to stand for a whole. Metonymy can, however, serve also to aid understanding by allowing us to focus more specifically on certain aspects of what is being referred to. Again experience with physical objects provides the basis for metonymy because the 'part-whole' image schema is grounded in experience of our bodily parts. Lakoff and Johnson (1980) also argue that homonymy, the use of the same word for different concepts, never occurs since there is always an underlying metaphor that links the two concepts and which explains why the same word is used.

2.2.2f Linguistic relativism

The notion that language and thought are in some way related is not problematic but the degree of *influence* of language on the way we think is contested. The term 'linguistic relativism' is used when language and thought are considered to be interdependent, i.e. thinking and language are related to the same mental construction. In its strongest form this interdependence is termed 'linguistic determinism', suggesting that our thinking is determined by our language. This view is frequently referred to as the Sapir-Whorf Hypothesis. Whorf (1956a) quoted Sapir as stating:

> The fact of the matter is that the 'real world' is to a large extent unconsciously built up on the language habits of the group ... We see and hear and otherwise experience very largely as we do because the language habits of our community predispose certain choices of interpretation.
>
> (Whorf, 1956a, p134)

Whorf (1956b, p214) stated his own 'principle of linguistic relativity' as:

> All observers are not led by the same physical evidence to the same picture of the universe unless their linguistic backgrounds are similar, or can in some way be calibrated.
>
> (Whorf, 1956b, p214)

Whorf drew on his knowledge of the language of Hopi Indians to suggest that the way we think is *determined* by our language. He also gave examples taken from reports of fire damage which illustrate how habitual use of words can influence behaviour. Thus 'empty' petrol cans were deemed safe because, he suggested, empty usually carries a connotation of inert, so precautions to prevent fire were not taken.

Even radical constructivists see language as having an influence on our mental constructions. Von Glasersfeld (1995) recalled his own experience of learning languages through everyday use (expressing everyday experiences in the mother tongue) and through school study (translating from mother tongue). He suggests that initially the way the world is viewed and described is determined by the mother tongue and that there are things which can be expressed and believed to be true in one language which can not be translated into another.

The suggestion that the language we use determines our view of the world and that different language communities thus think differently has largely been dismissed, (see for instance Pinker, 1994), but the view that our culture can constrain both experiences and the language associated with those experiences is not so contentious.

Vygotsky (1962) noted the distinction between theorists who consider thought simply to be speech without the sound, and those who see speech as simply the means for communicating thought and felt that both ignore the relations between language and thought. In his view a word refers to a group or class of objects and is thus a generalisation. It is generalisation that distinguishes thought from sensation. An individual's experiences can only be communicated, however, when included in a category which is acceptable to others. Thus thought is necessarily expressed in words common in the culture.

In an elegant study Pederson (1995) attempted to test the relations that might exist between linguistic and non-linguistic thought. He questioned whether linguistic parameters determine the non-linguistic cognitive operations, whether the reverse might be true or whether there is a general cognitive structure for both. He recognised that even when a difference in cognitive performance is demonstrated by different linguistic populations there might be other cultural or environmental factors which determine the difference. His comparative study thus used two populations sharing the same cultural features but differing linguistically i.e. two Tamil sub-communities in Southern India. He found that performance in experiments *did* correlate with language use.

The experiments involved spatial relationships because spatial reasoning is believed to be based on a common perceptual system and on universal elements of human environments, such as gravity and permanence of objects. Thus effects due to the different language ought to be apparent if they exist. Subjects were asked to look at objects on one table then go to another table and stand at 180° to their original position. They were tested on: memory of position of objects, simplified return journey for a complicated outward journey, relative positions of three objects.

Pederson found that those who habitually used relative position terms (left, right, front, back) gave answers which were the opposite orientation to those who habitually used absolute position terms (north, south, east, west). He thus claimed this as evidence of linguistic relativity but not of linguistic determinism. His study suggests that whether the internal mental coding was linguistic or not, language and thought seemed to be based on the same conception.

2.2.2g Speech communities

Each of the Tamil communities had a different mode of expression that was common within the community. A common term used to describe collective use of language is the '**speech community**'. Hudson (1980) gives an historical perspective on definitions of the phrase and suggests it is frequently taken to be any group characterised by regular and frequent interaction by means of a shared body of verbal signs. He pointed out that each of us is a member of a number of different speech communities and we that have little difficulty in adapting our speech to a variety of social contexts. Individuals use different varieties of 'dialect' and 'register' at different times and under different circumstances.

Halliday (1979) refers to '**register**' as the language of a particular linguistic situation determined by what is taking place, who is taking part and the part the language is playing. Thus the context determines the register. In contrast **dialects** are habitual, merely different ways of saying the same thing; the differences may be in sound or grammar but not in meaning. In the primary school classroom the 'mathematics register' consists of those words used by the teacher and pupils. The register is partly dictated by the National Curriculum which specifies content and thus some of the language to be used. Children in different classrooms may, however, habitually use different words, "plus" or "add" for instance. In one room "plus" evokes the same procedure that "add" does in another room so they signify the same process yet are from different dialects.

Whilst speech communities are characterised by the words they use, the communities may also have quite different practices. Hymes (1974) suggests that 'speech styles' should be interpreted as 'ways of speaking' because it is analogous to 'ways of life' and thus emphasises more than just use of words. In the classrooms the ways of speaking of teachers and children are influenced by the activities they are engaged in. Thus the **speech style** incorporates not only the register but the life style of the community which gives rise to the style of speech.

2.2.2h Relevance to present study

Whilst pupils may construct their own meanings they do so in the context of the activities in the learning environment. Their learning may thus be constrained by the way ideas are presented by teachers. Vygotsky characterised a culture's system of symbols as the means by which the culture is communicated.

It has been suggested that images and language may provide symbols with which to think and with which to communicate. We may view images and words merely as symbols attached to thoughts or as having played a part in forming that thought. In either case the theories presented suggest that words can give indications about thought. More importantly there is a strong suggestion that thought and language are interdependent. This comes from the theories of cognitive metaphor and cognitive linguistics. The study which showed that speech communities had distinctive modes of thought which matched the language they used gives strong confirmation that speech style is indicative of thought style.

When we want to know about another person's thoughts we have to accept their description of an image or the words they use to describe what they were thinking about. These descriptions may or may not be an accurate representation of what was in the mind but they are the only indications we have. The theory of cognitive linguistics suggests that our thoughts and our language are interdependent. Use of particular language peculiar to a speech community may only indicate that the practices of the community have influenced the mode of communication of its members. The language might, however, indicate the influences on their thinking.

For the purposes of this study it will be assumed that the words people use *are* related to their cognitive structures. Speech style is taken to be a manifestation of experiences that have given rise to an individual's mental construction.

2.2.3 Summary of theories of learning, imagery and language

The theories of Piaget and Vygotsky have permeated this section and they provide frameworks for the discussion not only of cognitive development and learning but also of teaching. Teachers may use both perspectives to inform their classroom practice. They would provide the cultural tools and the support of the community whilst acknowledging that the learners construct their own knowledge. They would expect children to use the common symbols but also acknowledge that the use of the symbols does not necessarily imply understanding.

Some theorists take the view that classroom activities provide the basis for children's own construction of knowledge; others see the cultural symbols used in these activities as providing the very structure of thought. It is clear that use of these symbols by learners indicates that learning has occurred. The activities of the mathematics classroom may be for many children a first introduction to some mathematical ideas. These activities may simply provide the vocabulary for communication of ideas but they may also provide the vocabulary for the language of thought.

It has been suggested that communities with common practices and common language may have common conceptual structures. There is some evidence to indicate that language use is linked to conceptualisation. It can be conjectured that distinctive use of language is thus indicative of distinctive mental constructions. The use of metaphor is seen as an indication of understanding of one situation in terms of previous experience. The notion that conceptual metaphors provide a basis for both language and behaviour might imply that language and behaviour are both indicators of the underlying conceptualisation.

The strong claims of Lakoff and Johnson that all conceptual understanding relates to bodily experience are noted and taken to imply that language related to physical experiences indicates that the physical experience has influenced both our conceptual understanding and how we choose to express it.

2.4 Summary of chapter and implications for study

This chapter sought to introduce the key concepts which underlie the discussion of how children learn from experience and to consider the roles of mental visual imagery and language in thinking and learning.

The children in the present study have had common experiences yet have learned in different ways. A naive empiricist view of learning would suggest that all perceptions are mentally represented and that these representations are synthesised into a coherent concept. Presented in this chapter is the constructivist view that each individual builds their own reality. Their mental construction is based on their subjective experience of the environment and is constrained by their existing cognitive structures. The workings of the brain, and the rest of the body, influence the ways in which we think.

The theories presented give no reasons for individual differences in mental constructions. Differences in activities could be a reason for different constructions being formed by individuals but common activities seem also to be experienced in different ways. Differences in attention can explain why activities are experienced differently. Differences in reflective abstraction can explain why mental constructions develop differently. The models of cognitive growth allow us to describe and explain but not easily to give reasons for differences. We may simply conclude that our knowledge of the world is *our own* and thus, as different individuals, what and how we learn will be different.

This study seeks to explore these differences. We may expect different activities to lead to different mental constructions but when common activities seem to be experienced differently we need to consider the manifestations of those differences. We may also then consider the relation between these manifestations and performance.

Children bring their own existing mental constructions to the start of lessons and learn from their experiences of the activities in the classroom. The context in which learning takes place, the experiences and language shared by teachers and pupils, could clearly influence the language used by individuals to communicate their learning. It is suggested by the theories of learning that the activities and accompanying language can influence both learning and thinking. It may be conjectured that the language and mental visual imagery related to classroom activities could thus be manifestations of the influences on learning. The following chapter explores further the association between classroom activities and these manifestations of learning.

CHAPTER THREE

PEDAGOGIC REPRESENTATIONS AND MANIFESTATIONS OF LEARNING

This chapter builds on the theories presented previously which suggest that mental constructions develop from children's experiences and that mental visual imagery and language have a role in thinking and learning. There is an examination of the classroom environment in which children's conceptualisations are formed and an exploration of the manifestations which might indicate that children have learned from their classroom experiences

In §3.1 we look more closely at the activities which give rise to the experiences and particularly at the 'pedagogic representations' which are the means by which teachers and pupils communicate.

In §3.2 we consider in more detail how expression of thought might indicate the nature of the influences on that thinking.

The manifestations of learning considered in this study will be referred to as 'indicators of learning'. They are:

- Expressions of generality
- Calculation methods
- Mental visual imagery
- Metaphor

Use of particular words are also considered as indicators of conceptual structures and are referred to as 'Linguistic indicators'.

This chapter explores the cultural tools used in the classroom and the ways in which the influence of the culture on thinking and learning may be apparent. The pedagogic representations are intended to provide *common* experiences for pupils' development of mental constructions but studies presented here show that they may learn differently. The indications of both the common aspects of learning and difference in learning may be given by the 'indicators of learning' and by 'linguistic indicators'.

3.1 The pedagogic representations

Teachers present mathematical ideas to children in a variety of ways, using language, drawings and materials in an attempt to engage pupils in mathematical activity. These are the cultural tools. Materials, such as counters, and verbal presentations are used to communicate, to model procedures and to stimulate mathematical learning. In this section the emphasis is on these 'external' representations. In §3.1.1 we consider how teachers attempt to communicate mathematical ideas. In §3.1.2 we look more particularly at how procedures for calculation are presented.

3.1.1 Means of communication

In this section there is clarification of the term 'pedagogic representations'. Some theoretical justification for the use of these representations are considered and some of their limitations examined. Previous studies, involving the use of pedagogic representations, will be briefly reviewed and the points of relevance to the present study will be discussed and summarised.

It is important to note that the representations that teachers use are not 'the mathematics' but a transformation of the mathematics into a communicable form (Kang and Kilpatrick, 1992). The argument developed in this section is that the representations can not 'carry' meaning. From the constructivist perspective the pedagogic representations provide activities but pupils construct their own knowledge from *their experience* of those activities. It will be noted however that children may be learning how to use the representation rather than learning the mathematics that it is supposed to represent.

3.1.1a Varieties of pedagogic representation

Lesh, Post and Behr (1987) identify five distinct types of representation systems:

1) experience-based 'scripts' - in which knowledge is organised around 'real world' events that serve as general contexts for interpreting and solving other kinds of problem situations.

2) manipulatable models - such as Cuisenaire rods, arithmetic blocks, fraction bars, number lines etc., in which the elements in the system have little meaning per se, but the 'built in' relationships and operations fit many everyday situations.

3) pictures or diagrams.

4) spoken languages - including specialised sub-languages related to domains like logic etc.

5) written symbols - which, like spoken languages, can involve specialised sentences and phrases.

An external representation of a mathematical concept or process is thus taken to be any linguistic or physical device (written or spoken words or symbols, pictures or concrete objects) which stands in for, illustrates or exemplifies that concept or process. Thus pedagogic representations are the external means by which the teachers attempt to communicate mathematical ideas.

The 'manipulatable' models, listed above, are commonly referred to as 'manipulatives' or 'concrete models' and include classroom resources such as Dienes blocks and hundred squares. Models such as these are intended to be 'structure oriented' (Resnick and Ford, 1981) because they are constructed to exemplify place value and to have the relationships inherent in the decimal numeration system 'built in'. Resnick and Ford were of the opinion that such materials could 'embody' qualitative characteristics of mathematics. This view is an example of the 'conduit metaphor' discussed in §2.2.2d. It suggests that it will be possible for the material to transfer ideas from the teacher to the learner. This view is strongly criticised and these criticisms will be explored in the following sections.

Pedagogic representations are the external representations that teachers use in their attempt to communicate mathematical ideas but they could also be used by pupils to communicate their understanding. Thus external representations could allow two-way communication about mathematics. We shall also see in the following sections, however, that there are concerns that this communication may be about the representations rather than about the mathematics that they are intended to represent.

3.1.1b Reasons for use of representations

Hall (1991) notes that whilst numerous publications, used for teacher education and in schools, promote the use of concrete materials for teaching, none of them explains the relationship between their use and learning. He suggests that the belief in their effectiveness is based on ideological preference, common sense and classroom experience.

The current belief in the efficacy of use of physical materials could have its roots in the developmental psychology of Piaget and Bruner. Piaget's notion of stage development appears to suggest that children need to pass through a concrete-operational stage, when thought represents coordinated mental actions based on experience of physical action, in order to attain more formal, abstract, levels of thinking (Meadows 1993).

In Bruner's view (Bruner, 1966) there are three modes of representing our experiences of the world:

- enactive .....related to action
- iconic ....related to physical or mental image, independent of action
- symbolic.....related to language and other symbol systems

Anghileri (1995) suggests that teachers and children use these three modes of representation to focus instruction and communicate thinking. The three modes could be interpreted to mean that there is a natural *progression* in learning and that physical materials and iconic representations are a necessary pre-cursor for abstract thinking. Bruner (1966) however was of the opinion that there may be changes of emphasis on one or other mode during learning but that the interplay of the three persists as a feature of adult intellectual life.

Dufour-Janvier, Bednarz and Belanger (1987) questioned the motives for using external representations in mathematics teaching and suggested some reasons for their use:

(a) Some representations are so closely associated with a concept that it is hard to see how the concept can be conceived without them.

(b) In presenting "multiple concretizations" the teacher hopes the learner will be able to grasp the common properties of these diverse representations and ultimately "extract" the intended structure.

(c) A child is given a task and several representations in the hope he will be able to find one which will help.

The first, (a), suggests that in a sense the representation *is* the mathematics. The example given by Dufour-Janvier et al is of the concept 'function' and 'cartesian equation' as a representation. It could be argued that mapping diagrams offer a viable alternative representation. More importantly it can be argued that the concept of function need not be conceived when working with equations.

Inherent in (b) and (c) is the "multiple embodiment approach" which follows Dienes' principle that an ideal method of learning mathematics would be to use several representations of the same mathematical object (Janvier, 1987). Underlying this view could be the belief that connections and relationships are at the heart of mathematics and that making connections between a variety of representations and abstracting relationships between them is a key aspect of mathematical thinking.

The words "concretize", "embodiment" and "extract" could be indicative, however, of a quite different view of teaching and learning than this 'relational' one. These words may indicate an approach which treats mathematics as a body of knowledge that may be transmitted by the teacher and received by the pupils. This is an example of Reddy's (1969) 'conduit' metaphor. It has also been characterised as using the 'conveyance metaphor' of teaching and learning (Mason, 1987) i.e. that meaning is carried by representations and the learner must take out the meaning when the representation is received. It is unease with this view that lies at the heart of the reservations about the use of representations.

3.1.1c Some problems of representation

In response to his concerns over the value of using representations, noted above, Hall (1998) developed 'Procedural Analogy Theory' which assumes that the function of an 'embodiment' is to provide a procedure which leads to the development of an algorithm by analogy. The usefulness of an embodiment thus lies in the degree of similarity between the embodiment procedure and the procedure for the initial symbolic representation.

This appears to be an example of the first of two views on "transparency" of external representations noted by Meira (1998). Meira suggests that the effectiveness of a representation for communication of a mathematical idea may be judged by one of two criteria:

1) the quality of the formal correspondence between the material and the target knowledge domain which experts consider to be inherent in the material.
2) the quality of the meaning constructed by the learner through participation in activities in the cultural setting of the classroom.

This is an example of an objectivist/subjectivist duality. Hall takes the objectivist view that there is an external reality that can be represented. Using his 'isomorphism index' Hall can give an expert's assessment of the formal correspondence between the material and the idea it is intended to communicate, i.e. how transparent the material is in allowing the 'reality' to show through. In Meira's subjectivist view it is the quality of the learning as constructed by the learner that is the only true measure of the transparency.

Boulton-Lewis, Cooper, Atweh, Pillay, Wilss and Mutch (1997) also note the conventional wisdom that understanding is assisted by use of concrete representations that can be physically manipulated:

> The belief is that students can construct a mental representation from the concrete representations as long as the manipulative material is sufficiently isomorphic to the mathematical concept or action.
>
> (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss and Mutch, 1997, p379)

They claim that a growing body of evidence shows that the material often fails to produce the expected positive outcome. Edwards (1998) went so far as to suggest that the search for a transparent container of meaning is *futile* because representation systems need to provide contexts in which meanings can be individually constructed and socially negotiated. The materials themselves cannot contain the meaning.

The fact that children do not learn what they are expected to learn from their experiences with material representations is seen by some to be due to the 'Learning Paradox' (Bereiter 1985):

> If one tries to account for learning by means of mental actions carried out by the learner, then it is necessary to attribute to the learner a prior cognitive structure that is as advanced or complex as the one to be acquired .
>
> (Bereiter 1985, p202)

In the context of concrete materials this may be interpreted to mean that pupils are expected to discern the mathematical meaning represented by the material before they have constructed the meaning (Chronaki, 1992). Pimm (1995) suggests that Dienes materials provide physical symbols of the numeral system and are manipulated to mimic the manipulation of the written system. Thus they symbolise the way the notational system works. The real experience is then twice removed because the rods are used to model the manipulation of symbols which are themselves substitutes for numbers. Pimm suggests that structural material will fail to serve its purpose if, when children manipulate the physical objects, they have the experience but miss the meaning.

Underlying all these concerns is the view that the learner must construct their own meanings and can not be given them by a teacher. Another major concern is that even when children take the meaning that was intended their future learning may be inhibited if the pedagogic representations leads to a restricted conceptualisation. This is explored next.

3.1.1d Alternative representations of number

Wing (1996) was concerned that the common representations of number, as objects to be counted, and operations on numbers, as groups of objects to be combined in various ways , in themselves lead to problems. He argues that counting as a means of developing arithmetic operations may be 'counter'-productive and that children remain 'counters' to the detriment of their mental mathematics ability. Wing was influenced by Stern. Stern (1949) insisted that relations between numbers, and particularly between visual patterns for numbers, need to be stressed in the classroom, rather than 'real life' counting situations. She argued that 'number' needs to be detached from 'numbers of objects' in order to develop arithmetical concepts such as prime or odd and even. In her view arithmetic needs to be conducted at a level of awareness beyond the actual counting activity. Instead of disorganised collections of objects to be counted she recommended structural collections that allow quick recognition. The 'pattern plates' she advocated (see Appendix 1.1) allow exploration of relations by recognition of pattern rather than counting e.g. 2 x 4 is visually the same as 8, 3 + 4 is visually the same as 7.

Problems with the ways in which number is represented, particularly the representation of place value, have been highlighted by Wigley (1994). He insisted that structural material such as Dienes blocks is not required for the teaching, or understanding, of place value and noted Gattegno's argument that place value should be treated as a part of language learning. In Gattegno's approach number notation is worked on directly using a number chart that emphasises the regularity of the system and the patterns in the verbalisation of number words are stressed. He valued chanting as an induction into the language of numeration.

These alternative means of representing number could lead to quite different conceptualisations for number and number operations. Stern's ideas have influenced the mathematics teaching at Peacehaven School and this was visited as part of the study. The approach adopted there will be described more fully in later sections.

3.1.1e Studies involving pedagogic representations

Some studies have indicated that pedagogic representations can cause confusion for learners when the use of the representation becomes an end in itself. Herbst (1997) for instance analysed the use of number lines as illustrations in a high school textbook and concluded that the text sets out to prove the validity of the representation not to clarify the properties of the number system. Ernest (1985) gave a fuller critique of number line use. He referred to Liebeck, who was of the opinion that a number line is not a useful visual aid for helping younger children to add and subtract whole numbers. Ernest also referred to Hart, who concluded from her studies that the number line model should be abandoned for integer subtraction.

The number line, and the number track, may be criticised on a number of levels but the fact that they are seen to be of questionable value illustrates the point previously made about transparency. These are pedagogic representations that, from the viewpoint of some experts, *represent* the number system. Other experts may disagree. The important point is that if children do not learn from their experiences with these representations then the representations do not serve their purpose.

The number line in conjunction with other materials was seen as valuable by the teachers at Peacehaven School (Wing and Tacon, 1999). They rejected the view that arithmetic develops, without being taught, in 'ineluctable stages' (against which it is useless to strive), from children's early counting experiences toward derived-fact strategies. They used planned teaching interventions and the use of structured images and apparatus (Cuisenaire rods, number lines and Stern pattern plates) to develop children's awareness of number relationships. Children in the infant school (Reception, Y1 and Y2) were taught using this approach. At the end of Key Stage One (KS1) the Standard Assessment Tests (SATs) results improved dramatically but, more importantly, it was found that the majority of children by the end of KS1 were using strategies based on composition and decomposition of numbers rather than counting. Wing and Tacon concluded that the effectiveness of the combination of visual representation for number and an emphasis on visual and numerical patterns merited further investigation.

Some other studies have explored the value of representations as part of innovative teaching approaches. Beishuizen (1993; 1997a; 1997b), for instance, is a strong advocate of the empty number line as a means of communication for both teachers and pupils. It is used as part of the Realistic Mathematics Education (RME) movement in Dutch schools as an aid to demonstrating and developing mental calculation strategies and its use is part of a philosophy of building formal algorithms on children's own informal strategies.

Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter and Fennema (1997) also gave an enthusiastic account of a variety of representations used in four projects based on a problem-solving approach. They noted however, that what a child sees when looking at the representations depends on his existing conceptual structures. This point was also made by Cobb (1995). In an investigation of the role of the hundred square in supporting the conceptual development of four Grade 2 pupils over a 10 week period he distinguished between 'image-supported' and 'image-independent' solutions to problems i.e. when the pupils used or did not use the hundred square to communicate their solution to two-digit addition questions. He found that use of the hundred square did not support the development of their conceptions of place value. These pupils used image-independent strategies before they had formed the concept of place value and use of image-dependent strategies was made possible by their developing conceptions. Thus the relationships that pupils saw in the hundred square were "constrained by their current interpretative possibilities". The pupils could use the pedagogic representation as a means of communication after they had constructed their own knowledge.

These studies suggest that whilst some pedagogic representations may appear to fail as teaching aids, i.e. in themselves eliciting understanding, they have value as part of a two-way process of communication between teacher and learner. Those educators (above) who are most enthusiastic about materials are those who use representations as part of a new approach to teaching mathematics not simply as a tool for teaching a concept or procedure.

3.1.1f Relevance to present study

There is clear concern expressed above that:

- the common concrete material representations are not necessarily of value to pupils in helping them develop their understanding.
- children may end up learning to use the materials and not the mathematics the materials are intended to represent.
- use of the materials might confuse learners and at might even inhibit their understanding.

This could also suggest that any overt behaviour which demonstrates that pupils make use of their experiences with the representations will not necessarily indicate an understanding of the mathematics. There is an indication, however that children may only make use of a pedagogic representation when they have understood the concept. This suggests that those who make use of pedagogic representations to explain their ideas have developed an understanding of the concept supposedly inherent in the representation.

This theoretical perspective on pedagogic representations provides a framework for the description of the classroom materials and activities used in the present study.

3.1.2 Written algorithms and mental strategies

The teachers of the Y2 pupils in Bright Cross used a variety of pedagogic representations such as Dienes blocks to demonstrate procedures. With older children the pedagogic representation for numbers was almost exclusively written numerals. The pedagogic representation of operations on numbers was almost exclusively the written algorithms. Written algorithms for calculations are intended to be efficient and performed in such a way that only a limited number of known facts are required. By emphasising manipulation of symbols, however, the meaning of the quantities involved and even the meaning of the operations can be neglected. There is a danger when teaching written algorithms that pupils may be trained to use this representation without understanding what it represents. This may be no less true if children are taught mental calculation strategies without an understanding of the underlying process.

This section gives a brief review of the extensive literature on children's misunderstanding of algorithms. The strategies for mental calculations advocated by the National Numeracy Strategy for English schools (DfEE, 1999; Qualifications and Curriculum Authority, 1999) are also introduced.

3.1.2a Procedures and algorithms

Hughes' (1986) extensive study of young children's understanding of number suggests that the mathematical symbols that they meet in the classroom have very little meaning for them. He argues that children only learn about operation signs in the artificial context of the classroom and with the very specific classroom activity of 'doing sums'. Thus there is a "serious and disturbing split" (Hughes 1986 p 78) between children's use of symbols in the classroom and their ability to apply them to problems encountered elsewhere. Hughes suggests that the ability to translate between written and concrete representations is fundamental to understanding of arithmetic yet when conventional symbols are introduced children often fail to establish connections between the symbolism and their concrete understanding. In his opinion written procedures and manipulation of materials are seen as fundamentally unrelated activities by children.

This view was endorsed by Harries (1997) who described one pupil's use of the written algorithm for two-digit addition where she failed to 'carry' (her answer for 49 + 22 was 611). The pupil's consideration of the correctness of the answer was based on perceived accuracy of the procedure not on the basis of the objects she was working with. In the conference presentation of his paper Harries also suggested that low attaining pupils are not concerned when two different representations for the same calculation give different answers because they see no connection between the representations. He concluded that the focus on procedure of low attaining pupils hinders their mathematical development.

Several other studies demonstrate the potentially harmful effects of the teaching of procedures for operations on numerals. For instance:

McNeal (1995) in a paper aptly titled "Learning Not to Think in a Textbook-Based Mathematics Class" gave a case study of one child's development as he moved from a

Grade 2 inquiry-based maths course to a textbook-based Grade 3 course. He had a variety of procedures initially but he interpreted worksheets as requiring standard algorithms. He saw algorithms as steps to be remembered and he did not use his previous strategies to check answers.

The study presented by Markovits and Sowder (1994) focused on number magnitude, mental computation, and computational estimation. They suggested that abilities related to magnitude are not developed when the focus of instruction is symbol manipulation. They also demonstrated that other research evidence gives abundant examples of the dissociation between symbol and referent when symbol manipulation is emphasised.

Boulton-Lewis (1998) conducted a study of Y1-Y3 Australian pupils learning place value, multi-digit addition and subtraction. She found that children were most successful when *not* using either using Dienes blocks or a written algorithm. She noted, however that they often talked about the numbers as if they were in columns and thus dealt with separate single-digit additions or subtractions. Furthermore these pupils made the common error, associated with the written algorithm, of routinely subtracting smaller from larger digits. Boulton-Lewis suggests that the children had practised so many calculations using the written algorithm with no need for decomposition that they failed to use it mentally when it was required.

Kamii and Dominick (1997) concluded from their study of American Grade 2-4 children, some taught standard algorithms, some encouraged to invent their own strategies, that the teaching of algorithms can hinder children's development of number sense.

The suggestion in each of these studies is that the pedagogic representation of operations on numbers, as algorithms for manipulation of numerals, can lead to faulty conceptualisations of the processes. This is not surprising in the light of the Tall and Gray distinction between process and procedure discussed in §2.1.2b. Algorithms are necessarily procedural if they are to be efficient i.e. a mechanical means of calculation. What these studies illustrate is that 'process' may not be learned as a result of activities which are designed to teach procedure. They also illustrate that instrumental knowledge of procedures can be divorced from relational knowledge of process. The manifestations of this qualitatively different learning, the difference between 'procedural' and 'proceptual' thinking, will be explored further in §3.2.2b.

3.1.2b Mental calculation strategies

In September 1999 the National Numeracy Strategy (NNS) 'Framework for Teaching Mathematics from Reception to Year 6' (DfEE, 1999) was introduced into schools in England. This gives a high priority to the teaching of mental calculation strategies as distinct from procedures related to written calculations.

"Teaching Mental Calculation Strategies: Guidance for Teachers at Key Stages 1 and 2" was produced by the Qualifications and Curriculum Authority (1999) as a supplement to the 'Framework'. The authors suggest that an ability to calculate mentally lies at the heart of numeracy, that mental methods should be emphasised and that written algorithms should develop out of mental methods. They distinguish between 'mental arithmetic', which requires rapid recall of number facts, and 'mental calculation' which uses strategies and known facts. It is made clear that:

> children should learn number facts 'by heart' and be taught to develop a range of mental strategies for quickly finding from known facts a range of related facts that they cannot recall rapidly... For much work at key stage 1 and 2, a mental approach to calculation is often the most efficient and needs to be taught explicitly.
>
> (Qualifications and Curriculum Authority 1999, p3)

It is stressed that mental calculation is not the same as mentally picturing a written algorithm and mental calculations are different in nature from written algorithms in that they treat numbers as quantities rather than digits. Some of the suggested strategies are listed in Appendix 2.1. Discussion of methods with children is regarded as essential in order for them to see why some strategies are more appropriate and efficient than others.

This initiative to improve pupils' numeracy skills was a reaction to the poor performance of pupils in international comparative studies of mathematics. This was in part thought to be attributable to over emphasis on written calculation. Bierhoff (1996) noted the predominance of vertical calculation in English textbooks, that children are introduced to numbers beyond 100 through calculations in columns and learn to count the number of units, tens and hundreds separately rather than partition numbers. By contrast, she pointed out, mental calculation is regarded as a priority in other parts of Europe, to the exclusion of formal paper and pencil calculations, until the age of 9. Bierhoff stressed that teachers should see it as their responsibility to teach more efficient methods if pupils have developed their own inefficient methods.

3.1.2 c Relevance to present study

The children observed and interviewed for the present study were taught written algorithms in Y2. The emphasis on procedures related to place value and manipulating individual digits of multi-digit numbers was common in the classroom throughout the period of the study. In the final year of the longitudinal study implementation of the NNS 'Framework' meant some increase in both teaching of, and discussion of, different strategies.

The findings of previous studies indicate that written algorithms may not be understood by children and one study has noted that the same mistakes are made when similar strategies are employed mentally. There is a suggestion here that the influence of pedagogic representation, i.e. the written algorithm, is apparent in the mental construction that children use for subsequent calculations even when they are not written.

3.1.3 Summary of pedagogic representations

This account of features of the classroom environment provides the background for consideration of how children might learn from the activities described. Pedagogic representations have been described as a means of communication of ideas. We have seen, however, that children need to construct meaning and that meaning is not inherent in pedagogic representations. The picture that has emerged is that children do not necessarily learn mathematics as a result of their interactions with the representations. Particular representations can be used to aid learning but the reports of successful use of physical devices, such as empty number lines, are part of innovative teaching approaches. There is nothing to suggest that the material in itself has beneficial effects on learning.

What is clear from the literature on algorithms is that children can be influenced in the way they approach calculation by their previous experience of manipulation of written numerals. The studies suggest that they may lose sight of the numbers when the focus of their attention is the individual digit. When they focus on the procedure they may not be concerned about the meaning of the outcome of that procedure.

This description of pedagogic representation as an aspect of classroom activities gives the context in which to explore what children learn from them. The commonality of the activities for the pupils does not imply commonality of experience for individuals and hence what and how children learn will vary. The final section now turns to the indicators of learning.

3.2 Manifestations of learning

The literature presented in the previous chapter indicated how children learn from their experiences in the classroom and in §3.1 the classroom activities were examined. This section builds on these ideas to investigate the ways in which children's learning from these experiences may be manifested.

It was noted in Chapter One that the list of 'indicators of learning' evolved as a result of both observation and reading the literature. This section presents the indicators and supporting literature which builds on the theories elaborated in Chapter Two. The purpose is to show how each indicator may give indications about children's learning.

'Generality' (§3.2.1) draws on the theories of abstraction from experience and looks more closely at the distinctions between the 'particular' and the 'general' and examines the role of 'generic' examples.

'Method' (§3.2.2) is concerned with calculation strategies and distinguishes 'holistic' methods, which conserve a sense of the number, from 'algorithmic' methods which involve manipulating single digits.

'Image' (§3.2.3) is concerned with modes of mental visual imagery and draws on previous studies which have indicated individual differences in imagery.

'Metaphor' (§3.2.4) has been introduced as the language of one context used in another. This is elaborated to distinguish between metaphors that indicate differences in the way calculation is conceptualised.

'Linguistic indicators' (§3.2.5) are the linguistic devices (individual words and tense) which may also indicate qualitative differences in mental constructions.

At the end of each section the categories for each indicator are given. These categories will be used in the analysis of interview responses in Chapters Five, Six and Seven. Each of the indicators will be seen to have three categories which will be described as 'concrete', 'representative' and 'abstract'. The rationale for this is described in §3.2.1c.

3.2.1 Generality

> To generalise is to derive or induce from particulars, to identify commonalities, to expand domains of validity.
>
> (Dreyfus, 1991, p 35)

In order to 'generalise' we need to be aware of commonalities and differences in our experiences. In §2.1.2a assimilation and accommodation were seen as process by which mental constructions develop as a result of experiences. Reflective abstraction was distinguished from empirical abstraction and was seen as a requirement for generalisation. In this section further distinctions are made and the terms 'generic' and 'prototypical' are introduced. Each may be seen as 'representative'.

3.2.1a Types of generalisation

Harel and Tall (1991) make distinctions in types of generalisation in 'Advanced Mathematical Thinking'. I suggest that these types are also applicable to elementary mathematics and give examples related to arithmetic:

- 'expansive generalisation' occurs when the learner expands the applicability range of an existing schema without reconstructing it

The schema 'to multiply by ten add a nought', learned in the context of natural numbers is expanded to include decimals.

- 'reconstructive generalisation' occurs when the existing schema is reconstructed in order to widen its applicability range

The 'add a nought' rule is reconstructed to become 'move each digit to the left' in order to allow application to decimals.

- 'disjunctive generalisation' occurs when, on moving from a familiar context to a new one, the learner constructs a new, disjoint, schema to deal with the new context and adds it to the array of schemas available

'Move the decimal point one place right' is added to the 'add a nought' schema without necessarily making connections between them.

- 'generic abstraction' occurs when particular examples are seen as typical of a wider range of examples embodying an abstract concept

The known fact '2 x 10 = 20' may be seen as typical of all multiplications by ten. From this we may infer that they all end in a nought.

The last category of generalisation, 'generic abstraction' is of particular interest in the context of the mathematics classroom because new concepts and procedures are frequently introduced through particular examples. Children are encouraged to see this as a representative example of the concept or procedure. Some children may not, however, 'see the general' in the particular example.

3.2.1b Generic examples and prototypes

Mason and Pimm (1984) in their paper "Generic Examples: Seeing the General in the Particular" use the term **'generic example'** when a particular number is used to stand in for a class and doesn't rely on any specific property of that number. More generally, a generic example has been described by Balacheff as:

> an object that is not there in its own right, but as a characteristic representative of the class.
>
> (Balacheff, 1988, p219)

The use of the word 'generic' to imply a representative example is common. Johnson (1987) for instance, gave what he regarded as a 'mainstream' definition of 'schema': a cluster of knowledge representing a particular generic procedure, object, percept, event, sequence of events, or social situation. He suggests that this cluster provides a 'skeleton structure' for a concept that can be instantiated with the detailed properties of the particular instance being represented.

The notion that a particular instance is recognised as typical of a class, and is thus used to represent that class, is also the key to Rosch's theory of categorisation. She insists that:

> One of the most basic functions of all organisms is the cutting up of the environment into classifications by which non identical stimuli can be treated as equivalent.
>
> (Rosch 1977, p212)

In her view categories are not coded in the mind as lists of individual members of the category nor as lists of category inclusion criteria but are coded as 'prototypes' of the most characteristic members of the categories. Prototypes have the most attributes and motor movements in common with the most members of the category and are typical of the category in overall look.

These theories suggest that learners construct concepts by comparing new experiences with prototypical or generic examples which represent their current knowledge. Von Glasersfeld (1991) describes 're-presentation' as a regeneration of a prior experience for the purposes of classification and reflection. He suggests that what we re-present has the character of a particular experience. This does not mean we can not have general ideas, only that we may not re-present them. Thus our representation of the general is in terms of the particular.

For von Glasersfeld that re-presentation was frequently in the form of a mental visual image. Rosch (1977) also suggested that:

> the most cognitively economical code for a category is, in fact, a concrete image of an average category member.
>
> (Rosch 1977, p214)

The suggestion is that mental visual imagery may also be representative rather than simply depicting a specific object. This point will be pursued in §3.2.2.

3.2.1c Individual differences in generalisation

Krutetski (1976), in his study of mental abilities that may be specific to high achieving mathematics students, hypothesised that pupils with different abilities in mathematics

> are characterised by differences in degree of development of both the ability to generalise mathematical materials and the ability to remember generalisations
>
> (Krutetski 1976, p 84)

Working with older children than the present study he collected data on the ease with which children could generalise a rule, such as multiplication of powers of x by addition of indices, from particular examples. Some where able to give a general rule 'on the spot' (his highest level of generalisation) others could not do so even with considerable help from the teacher (his lowest level of generalisation). Children were also given non-mathematical material to generalise.

Krutetski's colleague, Dubrovina, conducted a similar study with primary aged children (Grade 3 and 4). In each of the studies children varied in their ability to generalise. The pupils who were 'very capable' in mathematics were able to generalise mathematical material at level 5 (on a scale from 1 to 5) whilst not necessarily being able to generalise non mathematical materials at above level 3. The pupils identified as 'incapable of mathematics' could only generalise mathematical material at level 1 yet could generalise non-mathematical material at levels 3 and 4.

Krutetski concluded that

> mental abilities that are general by nature (such as the ability to generalise) in a number of cases can appear as specific abilities (the ability to generalise mathematical objects).
>
> (Krutetski 1976, p360)

3.2.1d Relevance to present study

There is no indication here of why individuals may vary in what they learn from their experiences but there is a framework in which we might describe those variations. All pupils may start with similar schemes but some may fail to generalise from new experiences. Some pupils may make inappropriate expansive generalisation or make disjunctive generalisations where connections are not made with previous schemes . Some pupils may fail to see particular experiences as being generic examples typical of a wider range. Their mental constructions may thus remain rooted in the particular.

Krutetski's (1976) study has suggested that it is possible to use levels of generalisation to distinguish between pupils of differing mathematical achievement but that the differences might not be apparent in non-mathematics contexts.

In the analysis of children's responses to interview questions three categories of 'generality' will be used

- 'particular' when they talk only of particular instances.
- 'generic' when they talk of a particular instance as if it is representative of other similar instances.
- 'general' when particular instances are not mentioned and they give a rule applicable in a wide range of circumstances.

These categories will be used in mathematical and non-mathematical contexts. These different contexts and the categories of 'generality' will be elaborated with examples in Chapter Four.

The categories that will be described for the other indicators will conform to the same global pattern which may be termed 'concrete', 'representative' and 'abstract'. The rationale for this global categorisation is drawn from the constructivist theory of Piaget.

This suggests (Meadows, 1993) that cognitive structures evolve from a 'representational operations stage' when concrete internal images lead to production of other internal or external representations then to logical and formal operation through a process of abstraction.

The Bruner (1966) categorisation of representations (enactive, iconic, and symbolic) also provides a theoretical framework for three categories which fits broadly with this scheme. Iconic representations may be interpreted as representing the concrete and thus may provide a bridge between action with concrete objects and processes with symbols. The view taken in this study is that these may be recognisably different but that they do not necessarily indicate stages of development.

Different names are chosen for the categories of each of the indicators to avoid confusion between them. Each of these indicators can indicate differences in learning by individuals.

3.2.2 'Imagery'

Mental imagery was introduced in §2.1.1d as a mental representation of previous experience and its role in thinking was considered in §2.2.2a. In this section the use of mental imagery in mathematics will be reviewed before consideration is given to individual differences in cognitive styles. The research into children's visual and verbal thinking characteristics has a long pedigree but the recent work at Warwick has highlighted the qualitatively different mental imagery of procedural and proceptual thinkers.

In mathematics 'visual devices' such as diagrams, graphs, number lines etc. have a key role for illustration and exemplification of concepts whether spatial or numeric. It is possible that mental visual images based on experiences with these pedagogic representations might have a role in mathematical thinking. Mental visual imagery for numbers has been of interest to previous researchers and this literature is also briefly reviewed.

3.2.2b Imagery in mathematics

Pictures, diagrams and other external visual representation clearly have a place in mathematics to illustrate and exemplify concepts that have spatial characteristics. In some instances, however, the image may be more than merely an illustration. Dreyfus (1991), for example, argues that visualisation should be regarded as a tool for mathematical reasoning rather than merely a helpful learning aid. Fischbein (1993) suggested that concept and mental image may usually be distinct but geometric figures are '**figural concepts**' because they are general representations that have intrinsically conceptual properties. He distinguished three categories of mental entity: the definition, the image (based on perception) and the figural concept which is devoid of sensorial detail. Thus a geometric figure is a mental image controlled by a definition as opposed to a drawing which is a concrete embodiment.

From this point of view an image of a number line may be a figural concept if its underlying conceptual basis is understood. If the image is of a physical number line seen previously there need be no understanding of the number system to generate it. This is an important idea because it suggests that being able to generate a mental visual image of a pedagogic representation may indicate that the underlying mathematics is understood.

The National Numeracy Strategy promotes the use of visual materials and particularly the empty number line. The script for a training session for teachers, for instance, insists that when developing new strategies teachers will need to demonstrate, model, explain and describe these strategies. The empty number line is given as an example of the model for demonstration. One of the foci for the 'oral and mental starters' to lessons is "visualise and use mental strategies". Part of the audit carried out in schools before the Numeracy Strategy was implemented asked if lessons "allow practice and use of mental imagery" (DfEE, 2000). In all of this there appears to be an assumption that mental images might be formed as a result of the teachers' demonstrations, and as a result of the use by children themselves of drawings and other materials to communicate calculation strategies.

This assumption is common amongst authors who have written books to complement the NNS. Frobisher, Monaghan, Orton, Orton, Roper, and Threlfall (1999), for instance, suggest that children should be to required to draw pictorial representations of the methods used. Children should also make pictures 'in their heads' of what they did and describe them to others. They note, however, that none of this comes easily to children and that they will need continual encouragement. Harries and Spooner (2000) go further and suggest that 'visual competence' can be 'accelerated' by providing structured and regular exercise. They give examples: stories that could be visualised to answer questions;

visualised words for letter matching; visualised multilink for multiplication and division activities; 3D visualisation exercises; number tracks and number squares.

Elphinstone (1998) also gives many mathematical activities which aim to develop and consolidate children's mental images. Pupils are instructed to close their eyes and are encouraged to do mathematical thinking in their heads. She suggests that the activities employ a range of senses - sight, sound, touch, physical activity and muscular coordination so that children have a chance to evolve different kinds of mental imagery for mathematical ideas. For example with eyes closed a number is said and children hold up that number of fingers, a drum is beaten and children hold up fingers to show the number of beats, a number is 'written' with a finger on the child's hand and they say the number. In another activity children first look at a number grid then with eyes closed have to say what number is underneath another. No research evidence is cited to endorse the suggestion that these activities will lead to increased use of mental imagery by children.

3.2.2c Number imagery

Personal mental images for number, as opposed to mental visual images of pedagogic representation, have been investigated extensively. Seron, Pesenti, Noel, Deloche and Cornet (1993) referred to the mental images for number described by Galton in 1880. Galton's subjects had stable patterns that came to mind when they thought of individual numbers or the number sequence. He noted that these mental visual images of number sequences, referred to as 'number forms', had developed in infancy and were automatically activated when any number was heard, seen or thought of. Galton estimated that it could be present in 1 in 30 adult males and 1 in 15 females. There was, however, no correlation between possessing a number form and arithmetic proficiency.

Seron et al's own study revealed that 49 out of 194 psychology students had some 'number form' (NF). They noted that several studies suggest that many people can visualise numerals or even written calculation when required to do so but they investigated whether these NF would be used spontaneously in mental calculation either to perform the calculation or simply as a memory aid. All of those having a NF claimed to visualise the result or elements of the operation or both when performing a mental calculation. Nearly half of them reported using the NF for addition though very few for subtraction, multiplication or division. The researchers noted however that the explanations were ambiguous and that the subjects performed the calculation using known facts simultaneously with evoking their images. They thus suggested that these activations of a NF may only represent the output in working memory of a calculation performed in verbal or abstract code.

Some studies have been conducted into young children's 'number forms'. Thomas, Mulligan and Goldin (1996) for instance, asked Australian children to imagine the numbers from 1 to 100 then draw what was in their minds. They conjectured that it is possible to infer aspects of the internal representation from these pictures.

Irwin (1995) explored the images of 36 10-12 year old children in New Zealand. She noted the research evidence which shows that children deal with decimals as though they are whole numbers and concentrate on the symbolic features of the notation rather than relating to the quantity expressed. Samples were chosen from two schools in a multi-cultural, lower economic, area where pupils had been shown to have less understanding of decimals than in more affluent areas. With eyes closed pupils were asked to think about what came between zero and one and report the image they saw (verbal and drawn). Similarly their mental visualisation of zero point one and zero point zero one was investigated. Irwin found that pupils from one school predominantly described number lines whilst the others predominantly described symbols. In a task requiring dividing a rectangle into 10, 100, and 1000 pupils appeared not to make use of their experience of Dienes blocks.

3.2.2d Individual differences in imagery

Love and Tahta (1991) note the remark in the Mathematical Association's primary report of 1955:

> A teacher's life would be a great deal simpler ... if all children formed mental images of the same kind in the same way and at the same speed
>
> (Love and Tahta, 1991, p258)

In reality they do not. This section looks at the research on individual differences in terms of quality and quantity of imagery use, particularly in mathematical thinking.

One of the first psychologists to relate memory performance to people's learning strategies was Bartlett (Richardson, 1999). In 1932 he found he could classify his subjects as 'vocalisers' or 'visualisers' who relied, respectively, on language cues or visual imagery for remembering . Whilst Bartlett's methodology was questionable the distinction between the two has come to be accepted as characteristic of an individual's 'cognitive style'.

An individual difference questionnaire (IDQ) was devised by Paivio (1971) to assess the degree to which people habitually used mental imagery or verbal processes for thinking. Research has been conducted into the implications of the differences in cognitive style for spatial and verbal abilities and for use of 'remembering strategies'. The students with number sequence images in Seron et al's study, for instance, were given Paivio's IDQ and were found not to be significantly high visualisers but were significantly low verbalisers.

Specht and Martin (1998), however, noted that an easily administered measure of imaginal and verbal thinking habits of children did not exist. They thus adapted the IDQ and administered it to 214 11-12 year old children. The frequency distribution for verbal items showed a normal distribution but the distribution for imaginal items was negatively skewed. They classified 'pure' types as those whose score was 1 standard deviation above the mean on one set of items and 1 sd below the mean on the other. They found that only three of the 214 were pure verbalisers and fifteen were pure visualisers.

Krutetski (1976) attempted to identify 'verbal-logical' and 'visual-pictorial' components of very high achieving mathematics pupils' mental activity. He distinguished three types of mathematical giftedness:

- 'analytic' pupils who favour verbal-logical thinking even when the problem suggests visual concepts
- 'geometric' pupils who prefer to interpret abstract mathematical relationships visually and have difficulty reasoning without visual supports
- 'harmonic' pupils who show an equilibrium of verbal-logical and visual-pictorial thinking

Of 34 gifted pupils he described 6 as analytic, 5 as geometric, 13 as 'abstract-harmonic' (those who can use visual-pictorial thinking but feel no need to) and 10 pupils were 'pictorial-harmonic' (those who use visual images to simplify a solution though can manage without). He noted, however, that boundaries between types are not entirely clear-cut.

Low levels of visualisation amongst high achieving South African high school students has also been reported by Presmeg (1986a; 1986b). Presmeg defined 'visual methods' of solution as involving visual imagery, constructions, drawings, diagrams, charts, tables and graphs, whether on paper *or* in the mind. Her measure of 'mathematical visuality' was then defined as the extent to which a preference is shown for visual methods. In her sample of 277 final year senior high school students the 7 identified as having outstanding abilities were almost always non-visualisers. Of the 27 who were 'very good' only five were visualisers. Presmeg's international study of Grade 11 students (Presmeg, 1995) also showed that visuality scores were normally distributed.

3.2.2e Previous research at the University of Warwick

Children's use of mental imagery for mental calculation does not appear to have been extensively investigated. Recent research at Warwick has provided some evidence of qualitatively different mental imagery for number reported by low and high achieving pupils.

Pitta (1998) set out to investigate what kinds of mental representations children report and how these might be associated with their level of achievement. Her pilot study had suggested that children may have a disposition toward different kinds of 'mental representations' which transcends arithmetical and non-arithmetical boundaries. A mental representation was defined as "the product of imaging in any modality".

She characterised the images reported by low achievers as having descriptive emphasis. High achievers reported images which were descriptive but also had relational characteristics. Pitta also suggested that there is qualitatively different thinking in number processing associated with a disposition toward qualitatively different kinds of mental representation. She hypothesised that if an individual's mental representation does not embrace an encapsulated object but is a mental analogue of a procedure, in visual or verbal form, then it will have a significant effect on calculation.

Pitta interviewed children aged 7 to 12 years. They were asked to say what first came into their heads when nouns such as "ball", "five" and "fraction" were spoken and then to say what else came to mind. They were also asked to perform number combinations up to 20, some presented verbally, some visually. Calculation strategies were classified as: count all, take away, count on, count up, count back, derived fact, transformation, accumulation, algorithmic, known fact. She found that higher order strategies were used more frequently with visually presented additions. The styles of response to numerical combination items were categorised as: automatic, abstract representation, perceptual representation, counting, figural representation. She found that low achievers were more likely to have images of objects to be counted and high achievers most likely to have images of symbols. She suggested that procedural thinkers retrieve the general number sequence and re-enact the counting episode because they have not encapsulated the concept of sum. In her view the higher achiever is more disposed toward searching long term memory whilst low achievers emphasise short term memory for procedures.

3.2.2f Relevance to present study

There is a common assumption that visual representations in mathematics, used for communication, might 'provide' mental images as a medium for thinking about mental calculation strategies but there is no evidence to suggest that this is so. Theories noted in §2.1.1e suggest that images have to be consciously refreshed in the mind and this might infer that maintaining a mental image could interfere with other thinking. Whilst mental visual imagery is an accepted and well researched phenomenon its value as a tool for aiding calculation is not proven.

Irwin's study suggests that pedagogic representations have an influence on mental visual imagery in that children from one school had images involving number lines whilst the others claimed only mental visual images of symbols. The experiences of Dienes blocks, however, were not recalled as a source for mental imagery for either group and this may suggest that this pedagogic representation had not formed a part of the pupils' mental constructions.

Individual difference in cognitive style is well documented and it appears that high levels of mental visual imagery are not common in mathematical thinking. The research that indicates that some children are more likely to mentally visualise objects to be counted for simple number bonds might suggest that they may also use mental visual images of pedagogic representations for more difficult mental calculations. The possible connection between styles of mental visual imagery and styles of number processing suggested is consistent with the theories of growth of cognitive structures. Those who have not generalised from particular activities may have images related to specific situations.

The 'representative' character of mental visual images suggested by von Glasersfeld and Rosch was noted in 3.2.1b. Together with Pitta's findings this suggests three main categories for mental visual images. These relate to particular objects, representative objects and symbols:

- 'specific' when specific objects and autobiographic incidents are visualised.
- 'iconic' when representative, prototypical objects, non-autobiographic incidents are visualised.
- 'symbolic' when words and other symbols are visualised.

An associated classification may be used to categorise mental visual images reported for mental calculation:

- 'specific' when objects to be counted are visualised.
- 'iconic' when pedagogic representations such as Dienes blocks and number lines are visualised.
- 'symbolic' when numerals and other mathematical symbols are visualised.

The choice of the names 'iconic' and 'symbolic' have been inspired by the Bruner (1966) classifications of representations.

These categories of 'image' will be elaborated with examples in Chapter Four.

3.2.3 'Method'

Mental calculation provided the context for many of the questions used in interviews. In this section there is a brief review of previous classifications of mental calculation strategies in order to provide the frame work for the categorisation that will be used in this study. There is also further elaboration of differences in children's thinking that might be described as 'procedural' or 'proceptual'.

3.2.3a Pupils' strategies

The variety of children's own calculation strategies, whether invented by themselves or constructed as a result of classroom experiences, is well documented (See for instance Carpenter, Hiebert and Moser (1981)). A number of studies have provided a hierarchy of mental calculation strategies, commonly used by children, which range on a scale of sophistication from counting to a variety of ways to derive number facts using known facts. In these studies the assumption is often made that students construct their own mathematical knowledge irrespective of how they are taught (Newstead and Murray, 1998).

It is important to note that whilst these studies show difference between pupils, and may suggest a hierarchy, this does not necessarily indicate a natural progression that all pupils will follow. Dehaene (1993) reviewed research which demonstrates that use of calculation strategies does not follow a strict developmental sequence. The studies suggest that individual children switch strategies from trial to trial and that the strategy selected by an individual depends on the reliability and speed of the strategies available to them. Memory retrieval becomes a dominant strategy when it is most efficient.

In a similar vein Siegler (1996) pointed out the ‘staircase metaphor’ of cognitive development. This assumes that children’s thinking is at a given level for a prolonged time then undergoes a sudden upward movement to a new higher level where it remains for a period of time. Siegler noted that this does not help with understanding the change. He also suggested that there is variability between *and within* individuals and he gave data to demonstrate variability of strategy use for addition and subtraction.

Beishuizen (1993) distinguished two main strategies for two-digit addition and subtraction. He referred to decomposition into tens and units and working with each separately as a ‘1010’ strategy. Counting on in tens then units from one number is ‘N10’. Thompson (1997a) made a similar distinction. Thus for 56+38:

‘partial sums’ (1010)	50 + 30 = 80, 6 + 8 = 14, 80 + 14 = 94
‘cumulative sums’ (N10)	56 + 30 =8 6, 86 + 8 = 94
‘cumulo-partial sums’	50 + 30 = 80, 80 + 6 = 86, 86 + 8 = 94

Thompson suggests that use of ‘partial sums’ is most common with pupils who invent their own strategies for written methods and this is the most common mental strategy. Beishuizen suggests that ‘1010’ is preferred by weaker pupils because ‘N10’ is more difficult to learn.

These strategies have two things in common:

- they are distinct from simple counting

- they retain a sense of the size of the numbers

This category of methods will be referred to as 'holistic'.

Neither author gives a name to 'separating the digits and adding them as separate digits'. This may be referred to as 'separate-digits'. Thus for 56 + 38:

'separate-digits' 5 + 3 = 8, 6 + 8 = 14, 8 + 1 = 9, answer 94

This is a distinctive method since the size of the number is lost. This category of method will be referred to as 'algorithmic'.

3.2.3b Procedural and proceptual thinking

There is evidence from a number of studies that low attaining pupils use different strategies than more successful pupils. Barrington, Hamilton and Harries (1997), in a careful analysis of strategies for mental mathematics used by a large sample of Y5 and Y6 pupils in England, found that low attaining pupils predominantly used counting strategies for mental calculation.

Gray and Pitta (1996) looked at strategies for number combinations to 10 and to 20 used by children aged 8 - 12. They suggested that some pupils were able to hold symbols in mind as objects of thought whilst low achievers reconstructed the counting process because they had not 'compressed' the counting procedures into numerical concepts. In an earlier study of addition and subtraction of numbers between 10 and 20 Gray (1991) found that at age 7 years approximately 25% of pupils, of average and above average attainment, used non-counting derived fact strategies or known facts. There was a subsequent divergence, however, so that at age 8 years approximately 80% of pupils of above average attainment were using derived fact strategies whilst the proportion of average attainment pupils using derived facts remained approximately the same as those aged 7 years.

Gray and Tall (1993) argued that those who fail at mathematics have failed to progress satisfactorily from the procedures of counting to the processes of arithmetic and that they similarly fail to generalise from other learned procedures in other areas of mathematics. Gray and Tall distinguished between 'flexible thinkers', for whom a symbol is a mathematical object that can be manipulated in the mind, and instrumental thinkers for whom the symbol signifies a procedure to be carried out. They argued that those who are unsuccessful in mathematics are doing a more difficult mental task in trying to use a mental analogue of the procedure. Number lines and their mental analogues, for instance, provide a physical representation which encourages counting on and counting back rather than use of known facts. Unsuccessful children may use a mental number line, rather than numbers, as mental objects. Gray and Tall suggested that when children develop idiosyncratic methods based on counting that are not generalisable then they may have short term success but long term failure. In their view, however, trying to impose flexible methods on those who are tied to thinking about procedures will only increase the cognitive burden.

As noted in §2.1.2b Gray and Tall (1994) defined a 'procept' as a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either or both. They argued that learners who see only the process not the concept in a symbol will eventually be unsuccessful in mathematics, but they did not exclude short-term success. Thus the term 'proceptual thinking' implies that conceptual and procedural thinking is taking place. Since a symbol, verbal or written, can represent both the process and the product of the process, making use of the ambiguity inherent in the symbol is characteristic of successful mathematical thinking. Those who think proceptually can use known facts whilst 'procedural thinkers' have to carry out the procedure.

3.2.3c Relevance to present study

The literature described above provides a background for the analysis of the calculation strategies used by pupils and a theory for individual differences in pupils' performances. Many studies have documented stages in children's strategy use for mental calculation without reference to the classroom activities that may have contributed to their development. In the present study both physical materials, such as Dienes blocks, and written algorithms have been used by teachers and pupils. These may have influenced the way in which pupils perform mental arithmetic.

Three categories of 'method' may be distinguished:

- 'Counting'. Counting individual objects is often children's first introduction to the processes of arithmetic. Counting as a method for calculation may be seen as the most 'concrete' of the methods in that it relates to these earliest physical experiences. In

Bruner's (1966) terms it is a representation of number operations that relates to enactive experiences.

- 'Holistic'. These methods retain the sense of the number. Both 1010 and N10 methods for addition and subtraction involve re-grouping based on place value but they refer to 20 as 'twenty' not '2 tens'. Similarly other derived fact methods make use of the size of the number (see NNS strategies, Appendix 2.1). The verbal symbols appear to be representative of numbers not just alternative symbols for written numerals.
- 'Algorithmic'. These methods relate to written algorithms. They are, in Bruner's (1996) terms, 'symbolic' representations. When children use a 'single-digit' method they need not have a sense of the size of the numbers. This is the most abstract of the 'method' categories in that the manipulation of symbols is the furthest removed from counting activities.

These categories of ''method'' will be elaborated with examples in Chapter Four.

3.2.4 'Metaphor'

It was suggested in §2.2.2 that the use of metaphor and metonymy may give indications of the influence of previous experiences on both our mode of expression and the related conceptualisation. In this section we look more particularly at the use of signs in mathematics and at how the use of metaphor can give indications of the signified.

3.2.4a Signified and signifiers in mathematics

Humans use signs to communicate. We use gestures, words, symbols and other external physical representations but these 'referents' or 'signifiers' can not constitute a 'sign' without the signifier being unified with the signified in the minds of the communicators. Saussure is credited with recognising the importance of the fact that the relationship between signified and signifier is arbitrary in the sense that it is conventional rather than necessary (Walkerdine, 1988). The act of 'signification' when signifier and signified are linked is, however, necessarily individual even if the signifiers are used universally. Von Glasersfeld (1995) suggested that:

> We can not share our experience with others, we can only tell them about it, but in doing so, we use the words *we* have associated with it. What others *understand* when we speak or write is necessarily in terms of the meanings their experience has led them to associate with the sound images of the particular words - and their experience is never identical with ours.
>
> (von Glasersfeld, 1995, p48)

This suggests that what is signified by a signifier is our own mental construction. Words, mathematical symbols and pedagogic representation do not have intrinsic meaning but interpretative action is needed to unify signifier and signified to create a sign (Presmeg, 1998).

The terms 'metaphor' and 'metonymy' were used by Walkerdine (1988) in her analysis of young children's conceptualisation of signs. She noted that Saussure's use of fraction notation for a sign, i.e. $\frac{\text{signified}}{\text{signifier}}$, links with Jakobson and Halle's description of metaphor as the vertical axis of language whilst metonymy provides a horizontal axis. The fraction was inverted by Lacan to emphasise the primacy of signifier over signified and that the signified underlies the signifier. Metaphor thus allows the 'descent to the signified'. Tahta (1991) tells us that Lacan claimed that all discourse uses metonymic 'chains of signifiers' in which reference to reality can become increasingly problematic when signifiers become less obviously linked to the signified.

Walkerdine (1988) pointed out that a statement such as $2 + 3 = 5$ is purely metonymic whilst when it is articulated as 'two plus three equals five' or 'two add three makes five' it can have two distinct metaphoric meanings. "Equals" may use the metaphor of balance whilst "makes" points to the metaphor of construction. Tahta (1991) suggested that metaphor is used in the early development of mathematics both in society and in individuals. He argued, however, that as a mathematical skill is developed the metaphor that gave rise to the activity can become inactive. Furthermore the use of algorithms is necessarily metonymic because they involve manipulating signifiers (numerals) without involvement with the signified (numbers). Place value activities involving experiences with different materials can involve metaphoric thinking but the ultimate aim may be to achieve metonymic fluency.

Metaphor and metonymy are thus essential components of pedagogic representations. In mathematics 'let x be any integer' where the variable name stands for a set of numbers or 'let ABC be any triangle' where a particular stands for the general, are both instances of metonymy (Presmeg, 1997). Tahta (1991) pointed out that 'metonymy' means a change of names so all mathematical equivalence is metonymic because equivalence can be interpreted as 'is another name for'.

Barwell (2000) has suggested that mathematical expressions may be 'decoded to meaning' when they are read , thus involving metaphor, or simply articulated into components, a purely metonymic process. Whilst the signifier '1/2' may be decoded to the signified 'half', '3y + 4' may be simply decoded to sounds i.e. other signifiers . He also argued that metonymy applies to all mathematical procedures where symbols are manipulated without thought to their individual meanings and that this is what makes algorithms efficient. Metaphor, on the other hand highlights connections, giving meaning by connecting with the known, so 6 + 3 = 9 can provide a metaphor for all expressions involving the sum of 6 and 3 e.g. 56 + 3, 600 + 300, 6x + 3x. Barwell argued that metaphor is the process by which learners develop meaningful known facts. He thus suggests that procedural thinkers may only use metonymy whilst proceptual thinkers can use metaphor and metonymy.

3.2.4b Metaphor in mathematics

The external representations used by the mathematics teacher (words, drawings, physical materials, real life contexts) can become theory-constitutive when used with the intention to communicate a mathematical idea. Concrete-material representations such as Dienes blocks, hundred squares and number tracks, used for place value, which are intended to be 'structure-oriented', are sometimes referred to as 'physical metaphors' (Resnick and Ford, 1981). As noted in §3.1.1 the representations that teachers use are not 'the mathematics' but a transformation of the mathematics into a communicable form. It can thus be argued that these representations are *intended* to provide the source for metaphors.

Sfard (1994) suggested that in mathematics the meaning of abstract concepts is often created through the construction of an appropriate metaphor and that metaphors are projections from the tangible world onto the universe of ideas. In her view 'reification' (when mental objects replace processes) is the birth of a metaphor. Thus, in mathematics, metaphor can bring the target concept into being rather than simply make comparisons between existing concepts (Sfard, 1997).

Whilst literary metaphor may work at the microscopic word or sentence level, a macroscopic view (Ortony, 1979) is also needed for the systems of metaphoric models used in teaching and learning mathematics. Pimm (1995), for instance, has drawn attention to 'manipulation' as the core metaphor for 'doing' mathematics.

The manipulation of concrete referents of numbers, for instance adding more counters or taking some away, provide the physical and linguistic metaphors for mathematical operations. Addition, putting together or counting more, then becomes synonymous with increasing. Subtraction becomes synonymous with taking away, thus decreasing. When the metaphor 'Subtraction is Take Away' is the theory-constitutive model for a child then subtraction of negative numbers becomes problematic. Similarly the metaphors 'Multiplication is Lots Of' and 'Division is Sharing' leave children ill equipped for calculations with anything other than natural numbers. In the same way manipulation of symbols can provide restrictive metaphors, for example 'Multiplication By Ten is Adding a Nought'.

It has been suggested (Lakoff and Nunez, 1997) that there are three basic 'grounding' metaphors for arithmetic:

'Arithmetic is Object Collection' - numbers are collections of objects and operations are acts of forming collections.

'Arithmetic is Object Construction' - numbers are physical or mental objects and operations are acts of object construction.

'Arithmetic is Motion' - numbers are locations on a path and operations are acts of moving along the path.

The first two may be seen as instances of the more general 'Arithmetic is Object Manipulation'.

3.2.4c Relevance to present study

Many of the metaphors used in mathematics are extinct. We may be perfectly adequate geometers without knowing that line comes from linea (Latin - linen thread), parallel from para (Greek - beside) and allos (Greek - one another), that 'perpendicular' comes from perpendiculum (Latin - plumb line) and 'rhombus' was Greek for spinning top (Lopez-

Real, 1989). We may have mental constructions for the signified using what appear to be arbitrary signifiers without realising their metaphoric origins.

There are, however, generative cognitive metaphors apparent in the primary school classroom which may be 'restrictive'. For instance children, in using the language of counting and grouping of objects, may be prevented from developing deeper understandings of arithmetic. The metaphoric language that children use may signify an understanding of the link between signifier and signified but it may also indicate that their thinking is rooted in one particular pedagogic representation.

The categories of metaphor suggested by Lakoff and Nunez (1997) have been adopted for this study and they can be seen to conform with the global categories 'concrete', 'representative' and 'abstract' described in §3.2.3f, i.e.

- 'collection' uses the language of manipulation of concrete objects and counting.
- 'motion' is 'representative' in that numbers are represented as distances and positions.
- 'creation' is the most abstract in using a language of manipulation of symbols.

Note 'creation' has been used instead of 'construction' to avoid confusion with 'mental construction'. These categories of 'metaphor' will be elaborated with examples in Chapter Four.

3.2.5 Linguistic indicators

This is the final section detailing literature pertinent to indications of mental constructions. Previous studies are reviewed which have suggested that individual words may be indicative of conceptualisations. Use of pronouns and causal connectives have each been shown to indicate children's ability to reason. They are thus potentially useful indicators of differences in the mental constructions that individuals have formed.

3.2.5a Words as indicators

Deictic terms, "this", "it", "you", (from Greek 'deixis' meaning to point) serve to identify objects, people, times and places without reference to particular things. Their use could thus be indicative of a generalisation. This was the view taken by Rowland (1995) who stated as a 'deictic principle' that language is a code to express and 'point to' concepts, meanings and attitudes. So not only deictic terms per se but use of other words could be indicators of cognitive structures.

Rowland (1999) noted that the use of the pronoun "you", to refer to generalities, i.e. what usually happens, is common in non-mathematical situations where "you" is used in place of the more formal "one". This is particularly true of children in their description of rules of games. The use of "we" may imply acceptance of a common code of practice for the classroom and a wider community of mathematicians (Pimm, 1987). Teachers may seek to identify themselves with the community of mathematicians in such phrases as "We only need one counter example to disprove a conjecture". They also indicate procedures that all in the classroom are expected to follow "We add the units first".

Rowland also suggested that the use of "you" is an effective 'pointer' to a quality of thinking. For the pupil in Rowland's study the shift from "I" to "you" in a problem solving situation signified her move from working with particular numbers to expressing a generalisation. In Rowland's view the shift from "I" to "you" indicates a shift to a mathematical generalisation. In an earlier study (Rowland, 1992) he also illustrated that the use of "it" is a linguistic pointer to a child's attempts at generalisation.

Piaget (1959) suggested that the use of "why?" before the age of 7 years does not show a desire for logical justification but a desire for an explanation of physical phenomena. He hypothesised (Piaget, 1928) that the use of "because" as a logical justification would be equally rare even though the word is used by children from the age of 3 years. He believed that young children have a tendency to "connect everything with everything else" as a result of a comprehensive act of perception which ignores detail. Thus, because of this insufficient discrimination, anything can be related to anything and young children suggest inappropriate analogies and inappropriate causal relations. He categorised the majority of young children's spontaneous use of "because" as a 'motive for action' rather than as logical relation between cause and effect. He suggested that children's logic is typified by 'transduction' - proceeding from particular to particular rather than induction (from particular to general) or deduction (from general to particular). He argued that young children can only reason about particular cases but admitted that his studies might simply show aptitude for use of language.

Tough (1977) also noted that frequently, in the speech of 3 year-old children, it was clear that they had not yet discovered the agreed meaning of some words that they used. In particular "because" did not necessarily indicate a causal relationship. By the age of 5, however, some children gave justifications and alternative consequences when interpreting pictures but children from disadvantaged home backgrounds tended to merely describe objects. Tough suggested that these differences were not necessarily due to a lack of cognitive resources but because of a lack of motivation to use language in this way at home, and thus lack of experience. She found that these children could use the appropriate words when encouraged to do so.

In a study conducted by Vygotsky (1962) 80% of children in his sample, at both 7 and 9 years, were able to correctly complete sentence fragments ending in "because" when related to scientific concepts. Only 60% of the 7 year-olds could do so with sentences related to 'spontaneous' everyday concepts. He attributed this to the fact that scientific concepts had been learned in collaboration with teachers. He assumed that the ability to use "because" appropriately in everyday concepts is improved by being able to do so in scientific concepts.

Mercer, Wegerif and Dawes (1999) noted that, although one of the aims of education is the induction of children into ways of using language for seeking, sharing and constructing knowledge, teachers rarely give explicit guidance on language use. In their study children were taught to use particular language in reasoning and, subsequently, the key linguistic features of the children's talk in small groups, which led to correct answers, were use of "because", "I think", "agree" and long turns at talk.

Donaldson (1986) questioned whether children's inability to give an explanation is due to poor understanding of the concept to be explained or to lack of ability to explain per se. She argued that adequate linguistic competence for explanation is demonstrated by appropriate use of the causal connectives "because" and "so". She also argued that the development of explanation may be investigated through the use of causal connectives. She followed Piaget in distinguishing between explanations in terms of physical (empirical), psychological or logical causal relations. Modes of explanation were categorised as empirical, intentional, deductive and procedural. In procedural explanations temporal connectives are more often used ("now", "first", "then"), though rule based explanations usually involve "if", "when", and the present tense.

She noted that "because" and "so" are converses, i.e. 'X because Y' is equivalent to 'Y so X', and that their primary role is to convey information about the direction of the causal direction. They can also be used, however, for temporal order i.e. 'X followed Y'. Her studies show that young children do use "because" and "so" appropriately and by the age of 8 years can even use them in the deductive mode. She argued that this shows that they have the cognitive ability to distinguish between cause and effect, action and intention, conclusion and evidence and that in using the words they demonstrate they possess the cognitive abilities. Thus when they use these words they also possess the cognitive abilities. She concluded that the linguistic ability of understanding causal connectives and the cognitive ability of understanding causality are interdependent.

3.2.5b Relevance to present study

The final point that Donaldson made about the 'interdependency of language and cognition' is the theme of this section and provides an important theoretical perspective for the present study. The case has been made that certain words can be indicators of modes of thinking. The switch from "I" to "you" and the use of logical connectives are indicative of an ability to generalise and an ability to give an explanation based on an understanding of causality. The use of "you" and "if" along with expression in the present tense are common characteristics of rule based explanations.

It is to be expected that a common speech style will characterise the language use of children from the same speech community. This could imply that when different language from that usually used by the community is used by an individual it may be attributable to different conceptual structures than those common in the community. The linguistic indicators can thus indicate both commonalities and differences in the learning that has occurred.

Following Rowland's suggestion that the use of pronouns may provide information about children's ability to generalise, the three categories used for 'generality' are appropriate in an analysis of how pronouns are used. The use of "I" or "you" may be seen to fall into the following categories:

'concrete' when used in relation to a particular instance

'representative' when used in relation to a representative example

'abstract' when used in relation to a general rule.

The causal connectives similarly may be used to argue causality based on a particular example, a representative example or based on a general rule. These categories are, however, complicated because the words may also be used as temporal connectives and may not imply causality. The categories of connectives use will be elaborated with examples in Chapter Four.

The final linguistic indicator, tense, may also be categorised. As noted by Donaldson the use of present tense is associated with descriptions of procedures and may be categorised as 'representative'. Past tense, being associated with events that have occurred, may be seen as 'concrete' whilst future and conditional tense may be seen to be more 'abstract'. This classification of tense is not based on any previous studies and may be peculiar to the present study.

3.2.5 Summary of manifestations of learning

The distinction made between 'particular', 'generic' and 'general', as different categories of 'generality', is at the heart of the distinctions made for each of the other indicators. Thinking related to particular instances is distinguished from thinking related to general rules. Thinking about 'representative' examples, i.e. generic examples and prototypical members of categories, is seen as a bridge between the 'concrete' and the 'abstract'. Each of the indicators has been categorised in ways to complement this format.

Each of the indicators of learning suggested in this section could provide a means of distinguishing qualitative differences in the mental constructions of individuals. The use of metaphor is seen to be an indication of influences on conceptual structures and thus differences in metaphor use may indicate differences in mental constructions. Previous studies of language use have also indicated that qualitative differences in use of words may also indicate qualitative differences in conceptualisations. Rowland found that a switch from use of "I" to "you" indicated a move toward generalisation. Piaget, Vygotsky and Donaldson suggested that use of causal connectives may be related to understanding of causality.

Pitta's study of the reported mental imagery of children conforms with the studies of Krutetski (1976) and Presmeg (1986a, 1986b) which indicated qualitative differences in mental images reported by pupils of different achievement levels. It may be conjectured that qualitative differences suggested by each of the other indicators of learning may also be associated with differences in achievement. Krutetski (1976) has demonstrated that ability to generalise may be a specific mental ability associated with high achievement in *mathematics* and that differences in ability to generalise might not be apparent in non-mathematics contexts.

It is important to note however that literature on mental calculation strategies indicates variability within as well as between individuals. Strategies used by an individual may vary from trial to trial and need not indicate progression or regression. This variability may thus indicate the variety of influences on their mental construction. It may be conjectured that each of the other indicators will demonstrate the same variability within, as well as between, individuals.

3.3 Summary of chapter and implications for study

The two parts of this chapter have first presented those aspects of the mathematics classroom that provide the experiences for learning and then suggested how learning might be manifested. It will be seen in subsequent chapters that the children in this study have had common classroom activities using pedagogic representations yet their learning is manifested in different ways.

The variety of pedagogic representations for mathematical concepts and procedures are not seen as 'carriers' of ideas but a means by which learners are encouraged to construct their own ideas. The language and activities of the mathematics classroom does, however, provide a basis on which learners build upon their existing cognitive structures to develop mental constructions that are meaningful to them. It may be expected that classroom experiences will influence the mental constructions of pupils.

Calculation strategies used by pupils could be interpreted as an example of this influence. Previous studies indicate that counting is a predominant strategy of calculation for all pupils initially, and subsequently remains so for the least able. The fact that children use counting as a procedure for addition and subtraction may be because this is a predominant pedagogic representation rather than because it is a natural human behaviour. Large scale studies have indicated that children may display progression through levels of sophistication in strategy use for mental calculation but this may be attributed as much to common classroom experiences as to common neurological structures. Some studies have indicated that experience of different pedagogic representations can lead to different strategies.

It may be conjectured that reported mental visual imagery and language use may also indicate that pedagogic representations have influenced the way children think about mathematics. It would then be expected that distinctive environments, such as those classrooms where unusual pedagogic representations are used, will influence mental construction differently. This will be manifested by distinctive language and distinctive mental visual imagery.

Research on mental visual imagery has indicated that it is not common in children's reports of their thinking in mathematics. Where it does occur there is some indication that it may be epiphenomenal (Seron et al, 1993) but some indication also that types of images may be related to achievement (Presmeg 1986a, 1986b, Pitta, 1998). Pitta found that images of concrete objects were common for the least successful pupils and images of symbols for the most successful.

The research on individual differences in ability to generalise (Krutetski, 1976) suggests that generalisation of mathematics material may be a characteristic of pupils who are very capable in mathematics. Pupils may not vary so markedly in their ability to generalise in non-mathematics contexts.

It may be conjectured that language use which is related to particular instances in mathematics might be characteristic of the speech of low achieving pupils whilst higher achieving pupils may use more 'abstract' forms of expression. Furthermore, in the same way that children of different achievement levels begin to diverge in their strategy use, they also may begin to diverge in their language use.

If children are influenced by their experiences, and if those influences are manifested by language and mental visual imagery, it is to be expected that language and imagery related to pedagogic representations will be evident when children perform calculations. It may be conjectured that profound changes such as those proposed by the NNS will be reflected by a change in language and mental visual imagery. The move away from written algorithms and toward holistic strategies for mental calculation, supported by empty number line use, could be manifested not only by changes in the methods used but also the associated language and mental visual imagery.

3.4 Overview of literature

From the literature presented in Chapters Two and Three the following points emerge:

- Neurophysiology research indicates that the brain 'learns from experience'. Neuronal records and interconnections between neurons provide the physiology for representation of experience.
- Mental imagery may be one mode of representation of experience.
- Theoretical models of cognitive structures allow descriptions of how new experiences lead to development of mental constructions by processes of abstraction.
- Bodily experiences may have a role to play in the development of mental constructions.
- Interactions with others in our community and particularly experience of cultural tools, including language, may influence what is learned.
- Language use may indicate how experiences have been conceptualised.
- Pedagogic representations may not lead to the learning that teachers intend.
- Differences in language use, calculation strategies and mental visual imagery may all indicate different conceptualisation.

Common physiology may lead to common understandings of bodily experiences and thus these 'grounding' metaphors may be universal. Different language groups may merely associate a different signifier with the same signified. However where environments differ, and this includes schooling, people from different cultures with different experiences of the environment may have different conceptual systems and use different metaphors for understanding. The mental visual imagery and language used by groups and by individuals may thus be important indicators of the experiences that have shaped their conceptual structures.

Learning activities in the community of the classroom are significant factors in the development of the pupil's procedural and conceptual knowledge. Children's active construction of mathematical meaning is based on these classroom experiences and they may provide generative or restrictive metaphors both for present communication and for future thought. Categorisation of the language used, and mental visual imagery reported, by children in response to interview questions will provide a tool for the analysis of the influences on their thinking and their developing conceptual structures.

CHAPTER FOUR

METHODOLOGY AND METHOD

This chapter is in two parts. In §4.1 there is a description of the methodology. It begins with the research problem and the purpose of the study. The conjectures and operational research questions are then listed. The methodology chosen is rooted in a philosophical framework and draws on the methodology used in a previous study at the University of Warwick. The influences of this philosophical perspective and the methodology of Pitta's (1998) study are described. The methodology employed for the present study may be described as postpositivist with a phenomenographic orientation. These terms are defined and the implications for method are considered.

The method has been determined by the methodology and is an adaptation of methods used in a number of other studies. The method is detailed in §4.2. Classroom observations and structured interviews have been used for data collection. The rationale for observations, sample selection and interview items is given. There is detailed description and examples of the categories used for coding and a description of the data handling procedure adopted. Consideration is given to the reliability and validity of the methodology and limitations on subsequent interpretations. Finally there is an overview of the results chapters that follow.

4.1 Methodology

This section looks at the factors which have determined the methodology for this study.

4.1.1 The study

The study began with a research problem and its subsequent direction was largely determined by conjectures. These are described here.

4.1.1a The research problem

The problem addressed by this study is:

Are influences of pedagogic representations, on children's mental constructions for number and number operations, manifested by the language they use to describe their mental calculation procedures, and by the mental visual imagery they report?

The purpose of the study was to investigate whether the effect of previous experiences on a pupil's conceptual and procedural knowledge may be identified through what they choose to communicate. If it is possible to tell something about the way a pupil is thinking from what they say then the teacher has an opportunity to provide activities which could lead to progression.

4.1.1b Conjectures

The choices of operational research questions and method were influenced by conjectures. The conjectures are listed, with comments on consequences for method, before the research questions are given.

The first conjecture was a result of my early observations:

- Types of calculation strategy, mental visual imagery and language use may be associated with different pedagogic representations.

This would require comparisons between classroom activities and the calculation strategy, mental visual imagery and language used in interviews. It would also require comparison between environments.

Two other conjectures, which have been noted in the literature review, were influential. The first was made by Pitta (1998). She described a 'mental representation' as "the product of imaging in any modality" and conjectured that children may have a disposition toward different kinds of mental representations which transcends arithmetical and non-arithmetical boundaries.

This led to my conjecture that:

- Types of calculation strategy, mental visual imagery and language use may be characteristic of an individual rather than of context.

In order to test this the characteristics of imagery and language use in different contexts would need to be identified. Comparisons between individuals could then be made.

Pitta's findings also suggested that children of different achievement levels might have different mental representations. This led to my third conjecture that:

- Types of calculation strategy, mental visual imagery and language use may be associated with achievement levels.

In order to test this, the characteristics of imagery and language used in responses associated with correct calculations are needed. The imagistic and linguistic characteristics of groups of pupils of different achievement levels would also need to be investigated.

The second influential conjecture was Kosslyn's (1980) 'representational-development hypothesis':

1) The type of internal representation that is predominantly used changes with age.

2) The later types are more powerful than earlier ones.

3) Later types supplement and overshadow older ones.

Kosslyn suggested that the internal representation would develop from mental visual imagery to propositional representation. If Pitta's conjecture and findings were true then the child's disposition and achievement level could affect, and be affected by, their mental representations. The development that Kosslyn conjectured could be constrained by the individual's characteristics.

This led to my final conjecture that:

- Types of calculation strategy, mental visual imagery and language use may vary over time.

In order to test this it would need to be a longitudinal study and questions would need to be set which could identify variations over time.

4.1.1c Operational research questions

The operational research questions based on these conjectures were as follows:

Variation with pedagogic representation:

1 What are the indications in the mental visual imagery, language and calculation strategy used by pupils that pedagogic representations have influenced their thinking?

Variation with context:

2 What are the characteristics of mental visual imagery and language used in mathematical and non-mathematical contexts?

3 Is there evidence that individuals or groups have a style of language or mental visual imagery which is common to mathematical and non-mathematical contexts?

Variation with achievement level:

4 What are the characteristics of mental visual imagery, language and calculation strategy associated with successful and unsuccessful calculations?

5 How do mental visual imagery, language and calculation strategy relate to achievement?

Variation with time:

6 How does the mental visual imagery, language and calculation strategy used for mental calculation, by individuals and groups, change over time?

The study set out to explore the linguistic and imagistic manifestations of the pupils' mental constructions over an extended time period, and in contexts other than mental calculation, in order to test the conjectures.

4.1.2 Aspects of methodology

The methodology adopted for this study is postpositivist with a phenomenograhic orientation. The theories behind these terms are described here.

4.1.2a Philosophical and practical considerations

> Qualitative research is a form of inquiry that explores phenomena in their natural settings and uses multi-methods to interpret, understand, explain and bring meaning to them. ... diverse methods of collecting, analysing and interpreting data blend together to reveal a deep and rich form of research.
>
> (Anderson and Arsenault 1998, p119)

In a purely qualitative 'ethnographic' study the emphasis is on exploring phenomena within their natural setting and working with data which is not pre-coded. This often involves an investigation of a small number of cases and the analysis emphasises description rather than quantification and statistical analysis.

The research problem for this study required a modified qualitative approach because in preliminary observations it was apparent that talk about pupils' mental visual imagery was not part of the usual classroom discourse. Information concerning mental visual imagery could not have been gained simply through observation. Children did talk about the way they performed their mental calculations occasionally but during the course of a lesson the quantity and quality of these interactions between teachers and pupils was limited. The lessons were very busy and to have had prolonged conversations between researcher and individuals or between researcher and groups would have been impractical without changing the nature of the lessons. In order to gain information about pupils' mental visual imagery and ways of thinking about calculation an interview seemed the only practical solution.

This is an example of the tension that exists between practical considerations which partly determine method and philosophical dispositions which might determine methodology. Cohen, Manion and Morrison (2000) note that research approaches start with the researcher's view of reality. These ontological assumptions give rise to epistemological assumptions which in turn give rise to methodological considerations and these determine method.

A purely 'subjectivist' view of reality assumes that the world is construed differently by individuals . Knowledge is constructed and does not pre-exist. Research is then a search for meaningful relationships between the sets of meanings which people use to make sense of their world. The researcher attempts to give a representation of this reality for purposes of comparison. In contrast, an 'objectivist' view assumes that the world is knowable as it is. Theories may be built to explain behaviour of individuals responding to the existing reality. Research is then a validation of theory and the researcher abstracts reality through quantitative analysis.

This study does not fit neatly into either of these paradigms but draws its method from both. Pring (2000) argues persuasively that this is a necessary course of action. He refers to the 'false dualism' of educational research, that of the quantitative and qualitative paradigms, and suggests that 'the world of real life' cannot be captured by either one or other of them. There needs to be an integration of the two. Qualitative research can clear the ground for a quantitative investigation and quantitative approaches can be suggestive of differences to be explored qualitatively.

Pring characterises quantitative research as following the 'scientific paradigm' i.e. assuming that a there is an objective reality which may be discovered and causal explanations given and tested. Qualitative research, he suggests, adopts the 'constructivist paradigm' which assumes that reality is subjective and no laws of causality exist. Pring argues that it is possible to reject the 'positivism' of the scientific paradigm, which suggests that only quantifiable behaviour is eligible for study, without abandoning the realism of the physical sciences i.e. that there is a reality that is not simply a construction of the researcher.

4.1.2b The postpositivist approach

From Pring's perspective how we see the world does depend upon the ideas we have inherited, and the distinctions we choose to make are influenced also by the society we live in. The distinctions are possible, however, because of the features which exist independently of us. This is seen as a 'postpositivist' approach (Mertens, 1998) which assumes that a reality does exist, but it is known through the researcher's interpretation of events. The researcher's own knowledge and theories will influence what is observed so that 'facts' that result from the research are 'theory laden'.

The present study gives an example of this influence of the researcher's theories on what is observed. My preliminary observations and conversations with pupils indicated that pupils used language in interviews that had similarities with the language used in the classroom. It could be argued, however, that those similarities were a part of the researcher's construction of the world. In noticing the procedural nature of the explanations given by the teachers it was these aspects of the pupils' expressions that were attended to when the pupils were interviewed. A different researcher could have focused on different aspects of the observations and interviews. The postpositivist approach suggests, however, that these differences would not have been apparent to the researcher if they did not exist.

The categorisation that has been introduced in §3.2 is 'theory laden' in that various aspects of the expressions used and the images described have been coded according to an apparently pre-determined mode of classification. This classification is based on the theory that individuals learn from their interactions with the environment. They develop their concepts and schemas by abstraction from their experiences. As we have seen in the previous chapters the theories of Piaget and Bruner suggest that the individual's mental model of an aspect of the world may involve representation of concrete features of their experience, or more formal iconic or symbolic representation. Thus a coarse grained analysis involving three 'categories of abstraction' was developed which used labels from these theories. These were not however pre-determined classifications used to pre-code data arrived at without reference to the context. The use of the 'theory laden' labels anchors this study into a theoretical framework, but the observed difference in responses could have been similarly coded with different labels.

It was noted in Chapter One that my awarenesses of theory developed concurrently with awarenesses gained in preliminary empirical studies. Classroom observations and reading commenced at the same time; there was not an extensive literature search prior to school visits. My initial observation of classroom and preliminary interviews provoked my initial choice of reading. Theories of learning informed what I observed and what I observed informed my understanding of theory.

4.1.2c. The phenomenographic orientation

In addition to the philosophical considerations given above, the methodology for this study has been influenced by Pitta (1998). The methodology she adopted has been adapted for the present study. An important aspect of the methodology is its phenomenographic orientation. Pitta saw her study as having a phenomenographic orientation because this provides an approach which supports the classification of data obtained in semi-structured interviews.

Phenomenography is a research approach originally developed by researchers in the Department of Education of the University of Gothenburg (Marton, 1988). A phenomenographic study is an investigation of people's understanding of phenomena which seeks to categorise and explain the qualitatively different ways in which people think about the phenomena. The categorisation of descriptions is based on structurally distinctive characteristics of the responses in semi-structured interviews. The categories developed in one context are, however, potentially part of a larger structure of categories applicable in other contexts. The initial discovery of previously unspecified categories of thinking may be peculiar to the researcher and context but the test of their validity is in their applicability for other researchers and as a source of explanation of differences in learning outcomes.

In content-related phenomenograhic studies pupils' responses in interviews are categorised. These studies focus on the relation between the pupils' conceptions and the conditions and processes from which they originate. The present study is thus described as 'phenomenographic' in that both the original phenomena (the classroom activities) and the conceptualisations based on these phenomena are analysed. This allows the relation between conceptions and 'originating' conditions and processes to be explored.

4.1.3 Influences on methodology

The methodology used by Pitta (1998), and influences upon it, are described here.

4.1.3a Influences on Pitta's methodology

The phenomenographic orientation provided a framework for Pitta's study but several other influences directed her methodology. She recognised that how we account for differences in children's arithmetical behaviour may be seen to have direct links with actions interiorised as concepts (Piaget), to different forms of mathematical understanding (Skemp) and to different forms of mathematical thinking (Gray and Tall).

Her descriptions of the strategies in elementary arithmetic drew on the approaches used by Carpenter, Hiebert and Moser (1981), Siegler and Jenkins (1989) and Gray (1991). Her insight into different kinds of mental representation drew upon psychological influences (for instance De Beni and Pazzaglia, 1995). In drawing on psychology research methods for her item bank structure and categorisation she recognised that the qualitative methodology was in contrast to the controlled laboratory-based experiments favoured by cognitive psychologists. She also noted that such experimental studies tend to focus on common cognitive processes with little regard for individual differences. Her study sought to *explain the differences* that were apparent in the achievement of pupils.

The present study recognises the same influences on methodology, though categorisation of mental calculation strategies draw on the studies cited in §3.2.2a. The categorisation of children's descriptions of mental visual imagery used in this study is a modified version of Pitta's finer grained categorisation which she based on De Beni. In addition I have drawn on the methodology adopted in the linguistic studies of Donaldson (1986) and Vygotsky (1962) described in §3.3.3a.

4.1.3b Aspects of metacognition

Pitta initially used the question "What comes to mind when you hear the word ...?" in an attempt to explore mental imagery that might be evoked by words without specifically directing children to talk about something seen in the mind. She also used the question "What was in your head as you were saying this?" in her attempt to establish the form of the mental representation. After children had performed a calculation she asked "Tell me how you did that". These questions may be seen as an invitation for children to think about their own thinking and Pitta (p71) cited Morris (1984) in her recognition that the metacognitive knowledge of young children may be extremely limited and frequently erroneous.

Pitta also recognised the distinction between the world of mental operation and the external manifestation of that world. She noted (p96) Kaput's caution that we can only hypothesise about mental representation since it is not observable.

In the present study "What is the first thing that comes into your head when I say ..." and "Can you tell me more about ..." have been used to elicit indications of children's mental constructions. Any mental visual images described and any other descriptions of what came to mind are taken to be indications of the mental construction in use at that particular moment. It is recognised, however, that the only indication of what the mental construction might be is the words that pupils choose to use.

Similarly a mental construction for a calculation procedure is taken to be indicated by the language used, and mental visual imagery described, in response to "What was in your head when you were thinking of that?" after they had performed a mental calculation. If, for instance, their response is to give a general rule then this is an indication that the mental representation is at this level of abstraction. If a description of the calculation is given simply in terms of what was done with those particular numbers then pupils may or may not have a mental representation that includes a general rule.

It may seem unsatisfactory to focus on pupils' own descriptions, which may or may not correspond to what was in their minds, but it is a characteristic of classroom discourse that teachers can only be informed of children's thinking by what pupils choose to tell them. What pupils choose to say is thus important. For instance those who give a general description of their procedure have not only abstracted the procedure but also have a sense of the efficiency of expressing it in this compressed form. This study is concerned with those manifestations of a child's mental construction that could also be observed by a teacher.

4.2 Method

The methodology described above has determined the method employed in this study. This section details the method used.

4.2.1 Data collection

Lesson observations and interviews were used for data collection. Details of the school setting, the observations and the sample are given here.

4.2.1a The school

The school selected for the main study was chosen for its convenient geographical location close to the researcher's home and to the University of Warwick. The school is a large primary school (645 pupils in 2000) serving a large middle-income village near Birmingham. It has pupils in the age range 3 to 11 years. An Office for Standards in Education (OFSTED) report in 2000 indicated that:

Attainment on entry is above average. Approximately one in ten pupils is on the special educational needs register and two pupils have statements of special educational need. This is well below the national average. The pupils come from a range of social backgrounds, and very supportive homes. Fewer pupils than usual take free school meals. The school is very popular and oversubscribed.

In 1997 96% of children in Y2 achieved level 2 or above in the KS1 SATs in comparison with the national average of 83%.

4.2.1b Rationale for classroom observations

Because the focus of this study is the influence of classroom activities on children's mental representation it was important to be present in the classrooms. The intention was to document the words and actions of the teacher to provide a context for the subsequent enquiry into the influence these might have had on the children.

The decision was taken that this should be a naturalistic study in that there would be no input from the researcher to change the content or style of presentation of the lessons. As far as possible the researcher thus needs to adopt a 'complete observer' role (Scott and Usher, 1999) and, though not entirely ignored by pupils and teachers, attempts to have minimal impact on the lesson. A preliminary study in which an audio-tape recorder was used to record the lesson resulted in a recording with poor sound quality. More importantly the teacher was uncomfortable with having her voice recorded, was nervous and did not perform as she did in non-recorded lessons. The descision was made to take notes rather than use recording equipment. The focus for the observations was the language and pedagogic representations used by the teacher and thus the notes attempted to be a record of these aspects of the lesson. The pupils' responses to the teacher's questions during the lessons were also noted (For a sample lesson observation see Appendix 2.2).

4.2.1c Observation schedule

Preliminary observations commenced with a Y2 class in September 1997. There were 80 children in this year group and they had been placed in three sets determined by their achievements during Y1. The set observed had the 33 highest achieving pupils. At the end of the year a few pupils changed sets to take in to account their performances in the Key Stage 1 Standard Assessment Tests.

In order to observe a variety of teachers' presentations it was decided that more than one class would be observed when these children moved into Y3. The lowest achieving pupils were in Set 3 and were to be taught in Y3 by a newly qualified teacher. In consultation with the Y3 teachers it was decided that only Set 1 (the heighest achievers) and Set 2 (average achievers) would be observed.

The original intention was to observe one lesson each week with each of the two groups of pupils over the following two years. This was largely the case during the first year. When the pupils moved from Y3 to Y4, however, the children were moved to different sets and observations became more difficult to arrange.

Set 1 was left largely unchanged but the pupils from sets 2 and 3 were placed in two new parallel sets which each had a wider range of ability. This meant that the previously observed pupils from Set 2 were now in two different classes. An arrangement was made to observe each of the two parallel sets on alternate weeks and Set 1 each week.

Early in the autumn term the teacher of one of the parallel sets resigned due to ill health and her class was covered initially by supply teachers and then by two job-share teachers. It became impractical to observe lessons with this class. The teacher of Set 1 was a Deputy Head who frequently had to leave her class to be covered by other teachers and this also disrupted the observations. Observations ceased with this set in the Spring term. The teacher of the other parallel set was very accommodating and observations in her class occurred weekly throughout the year.

At the start of the second year of observations (September 1999) the 'Mathematics Framework Reception to Y6' (DfEE, 1999) was implemented in the school and medium term planning was collaborative between the teachers of Y4. The observations of one set thus gave an impression of the content and activities followed by all the classes.

4.2.1d The sample

Three preliminary interviews took place with pupils from Y2 Set 1 during the course of 1997-8 to refine the interview questions and to identify pupils who might form part of the sample for the longitudinal study in Y3 and Y4. A sample of 14 pupils were subsequently chosen. Criteria for the sample selection were:

1) The distribution of KS1 SAT scores for the sample to be similar to that of the set as a whole.
2) Half of the sample to have given evidence of having mental visual imagery during the course of the interviews. The other half of the sample had made no mention of mental visual imagery.

This last criterion was to ensure that there would be some possibility of reporting of mental visual imagery in the longitudinal study. It is a high proportion in comparison with studies (§3.2.3) which indicate lower proportions of 'visualisers' in the population. This was deliberate but may bias the quantities of mental visual imagery reported.

In order to investigate the influence on pupils of the different representations that might be used by the two teachers, a sample of 14 pupils were chosen from Set 2. In order to select the sample an interview was conducted with all pupils from Y3 Set 2 in October 1998. This also allowed the format of interviews for the longitudinal study to be piloted. The same criteria for selection were applied.

Overall 73% of the Bright Cross pupils attained National Curriculum level 2 at the end of Y2 and the rest were level 3. The proportion of the sample at level 2 was 71%. At the end of Y4 the written non-statutory 'optional' SATs scores for the sample were higher than the school as a whole. This is explained by the absences of the lowest achieving pupils (in Set 3) from the sample. (Appendix 2.3 gives the SAT results for the sample and the year group)

During the course of the two years, two girls from the Set 2 sample left the school so that the final analysis involves the six interviews completed by 26 pupils.

4.2.1e The comparison school

Time constraints limited the main study to one school but for comparison another school which used quite different pedagogic representations was visited for a period of two days. Teachers at this school use Stern's materials and methods described in §3.1.1d, 3.1.1e.

It is referred to as one school but the children of Peacehaven Infant school transfer to Hoddern Junior school at the age of 7 years. The teachers of these two schools selected 6 pupils from each of Y2, Y3 and Y4 for interview. The children were representative of the ability range.

Peacehaven is a coastal town in the south of Sussex. The 1996 OFSTED report indicated that:

Some of the children come from private homes, some from council housing or housing association developments. There is high entitlement to free school meals and a large number of children with special needs. Whilst most children come from stable homes there is a measure of privation.

In 1996 83% of children in Y2 achieved level 2 or above in the KS1 SATs in comparison with the national average of 82%.

The interviews followed the same format and had questions in common with those used for the main study. The responses will be used to make comparisons with the main study pupils' responses.

4.2.2 The interviews

This section gives the rationale for the interview format and details the interview items.

4.2.2a Rationale for format of interviews

The item bank of questions and format of interviews used in the longitudinal study was developed from the model set by Pitta's study but modified as a result of preliminary interviews conducted with Y2 and Y3 pupils in 1997-8. Pitta had used calculation questions that she could expect children to be able to perform in order to explore the mental imagery that might be associated with these calculations. In the present study the influence of classroom activities were a key focus so it was important to choose questions that were related to the classroom activities.

Pitta used some visually presented calculations. In a preliminary study that I conducted with some Y3 pupils a visually presented two-digit addition was answered by all pupils as if it were a written calculation. Using the visually presented calculation they all described the written algorithm and none claimed any mental visual imagery. In comparison, when I asked a similar question presented orally there was a greater range of strategies and some claimed mental visual imagery. It was thus decided to confine interview questions to oral presentation.

The preliminary studies also revealed three common ways in which pupils described the calculations they had just performed:

- descriptions of what they did with the numbers.
- descriptions of what to do with these sorts of numbers. This included illustrating how to do this type of calculation with numbers of their own choosing and not the ones in the calculation they had performed.
- giving a general rule for what they had just done with little mention of the numbers.

In order to discover whether pupils would do either of these when particular numbers were not given, 'procedure' questions were introduced into preliminary study interviews. These were phrased "tell me how to ...". These questions were important in identifying characteristics of language use in situations where a procedure is described.

One final lesson learned from preliminary studies was that children could be led into using linguistic devices by the example set by the interviewer. An example is the use of tense. When I asked "How would you ...?" children tended to respond with "I would ...". When I asked "How do you ...?" there was a tendency to reply in the present tense and to describe the common classroom procedure. It was thus important to achieve consistency in the way the questions were phrased and to avoid too many follow-up questions where children's language might be inadvertently influenced by the mode adopted by the interviewer. It was decided that "What was in your head when you were thinking of that?" would avoid directing the children into necessarily describing either an image or a procedure.

In order to collect data to answer the research questions it was decided that comparisons would most easily be made if the interviews followed a common format and each pupil was asked the same questions. Hence, though the interviews allowed some follow up questioning, the interviews could best be described as 'structured'. The interview questions fell into three sections:

Mental calculations	A calculation followed by "What was in your head when you were thinking of that?"

Procedures	"Tell me how to ..."
Concept questions	"What is the first thing that comes into your head when I say ...?" and "Can you tell me more about ...?"

The rationale for this framework was to compare the strategies, language and images over time and in different contexts in order to answer the research questions The concept questions had some similarities with Pitta's questions.

The reason for having the procedure questions was to identify a style of language that might be associated with the giving of instructions. This could indicate whether the children's descriptions of their mental calculations were phrased in this way. This is important because when a child simply describes what they have done it may indicate that they have not generalised the procedure. If they use language characteristic of that used when giving instructions for procedures it could indicate that they have generalised. These questions could also indicate whether some pupils might be incapable of describing procedures. Finally these questions would indicate characteristics of styles of language and mental visual imagery in mathematics and non-mathematics contexts to help identify linguistic and imagistic pre-dispositions by individuals.

The 'concept' questions i.e. those which asked about words, were again designed to identify differences in language and mental visual imagery in the different contexts. Comparisons could then be made between them. The purpose of the comparison was to find whether there was evidence of a disposition toward a mode of expression which might mask differences in mental constructions. If for instance a child consistently uses 'particular' expressions across all contexts it could point to a preference for that mode of expression rather than giving information about the mental constructions used for mental calculation. Similarly the use of 'general' expressions in calculation contexts but 'particular' elsewhere might indicate that the rule had been abstracted and that it was not just a linguistic trait of the child.

The longitudinal nature of the study allowed comparison over time in order to find evidence of any development in mental constructions. Thus over the period:

Some calculations were kept constant.

Some required the same calculation process but with numbers increased in difficulty.

New questions were related to recently taught calculation procedures.

Interviews started with the following statement:

> "After each question I will ask 'What was in your head when you were thinking about that?'. It could be pictures, words, written numbers, memories, anything."

After giving the answer to a mental calculation question most pupils interpreted "What was in your head ..." to mean that a description of how they had performed the calculation was required. Few pupils spontaneously described mental visual images. After pupils had given the description of their method the follow-up question "Was there anything to see when you were thinking of that?" was thus frequently used. The biasing effect of this question will be discussed in §4.2.4b.

4.2.2b The interview items

A summary of the items is given in the following tables with a brief description of why they were chosen (For a sample interview script see Appendix 2.4). Over the six interviews 78 questions were used. They were classified into 10 calculation types and 6 non-calculation types. Each was presented verbally and followed by "What was in your head when you were thinking of that?"

The first six types were all about familiar procedures that had been learned in Y2 or early in Y3. The questions progressed in difficulty as children performed more demanding questions in class. Two questions 17+9 and 48+23 were kept constant to allow comparison over time. Analysis of responses allow the operational research questions (§4.1.1c) 1, 4, 5, 6 to be addressed:

Type	Description	Examples of questions
1	1-digit addend	17 + 9 (repeated in each interview)
2	Missing addend	13 + * = 18, 30 + * = 80, 27+*=65
3	2-digit addition	48 + 23 (repeated in each interview)
4	Add multiple of 10	97 + 10, 597 + 10, 1097 + 10, 1197 + 10, 36 + 20
5	Counting	What comes before 380, 2380, 12100; after 12386
6	Rounding	Round 2462 (nearest ten), 239 (nearest hundred)

Table 4.1 Familiar procedure items

The next four types allowed the influence of recent activities related to newly learned topics to be explored. They were introduced into interviews when similar questions had been introduced in the classroom. These allow the same operational research questions to be addressed:

7	Recent topic	What is the difference between 27 and 65, 0.6+0.7
8	Recent topic	65 subtract 29, Read time (11:40), 0.1 times by 10
9	Division and fractions	quarter of 40, third of 48, 140 divided by 3
10	Multiplication	48 multiplied by 3, 47 multiplied by 5

Table 4.2 Recent activity items

The Numerical procedure questions and mathematics concept questions were again a mixture of recently taught and familiar ideas. This allows operational research questions 2 and 3 to be addressed:

11	Numerical procedure	Tell me how to add 23, find a third, times by ten,
12	Maths concept, first	First thing in head ... centimetre, three, million
13	Maths concept, more	What else ... about centimetre, three, million

Table 4.3 Numerical procedure and mathematics concept items

The non-mathematical items related both to familiar everyday procedures and concepts and to procedures and concepts encountered in the classroom. This allows operational research questions 2 and 3 to be addressed:

14	Non-numeric procedure	Tell me how to cross road, tell the time
15	Non-maths concept first	First thing in head ... shadow, ball, adjective
16	Non-maths concept more	What else ... about shadow, ball, adjective

Table 4.4 Non-mathematics procedure and concept items

With 28 pupils involved the interviews were of approximately 10 minutes duration to minimise disruption to lessons. Preliminary studies had suggested that this was an optimum time for pupils' concentration spans.

An interview was conducted at the end of each term i.e. in December 1998, April 1999, July 1999, December 1999, April 2000, July 2000. The interviews will be referrred to subsequently as Y3/1, Y3/2, Y3/3, Y4/1, Y4/2 and Y4/3.

4.2.3 Data analysis

This section expands upon the categorisation described in §3.2. The global classification system is described before exemplars are given for each of the indicators. Finally there is a description of the way the data were analysed.

4.2.3a Overview of categories of response

As noted in §3.2.3f the rationale for categorisation was drawn from the constructivist theory of Piaget which suggests that cognitive structures evolve from representations of concrete experiences to more abstract mental representations through processes of abstraction.

The preliminary observation gave evidence of qualitatively different responses which have been referred to as 'particular', 'generic' and 'general' in §3.2.1c. This may be seen to conform with Piaget's theory. It was also noted in §3.2.1c that Bruner's (1966) classification of representation was an influence on the choice of the three global descriptions of categories as 'concrete', 'representative' and 'abstract'. The global overview of the categorisation used for all of the modes of response is:

'Concrete'	Specific objects and particular numbers.
'Representative'	Representative objects and numbers.
'Abstract'	Symbols and general rules.

These are simply labels but they give an indication of the influences that have determined their choice. The categories for individual indicators have been described in §3.2. They are now summarised as:

	Categories		
Global	**Concrete**	**Representative**	**Abstract**
Generality	Particular	Generic	General
Method	Counting	Holistic	Algorithmic
Image	Specific	Iconic	Symbolic
Metaphor	Collection	Motion	Creation

Table 4.5 Categories for 'indicators of learning'

In the following sections each is elaborated with examples.

4.2.3b Categories of 'generality'

The description of each category is given below followed by characteristics of the categories which aided coding. Examples of responses to a calculation question (17 + 9) and a 'procedure' question (Tell me how to tell the time) are given for each:

'particular' in response to calculation and procedure items.

Children talked about the particular numbers given in the question or gave an explicit example in a procedure question.

Responses were frequently in the past tense, "I" was often used and reference was only to the particular numbers of the question given. This category included responses where the answer to a calculation was guessed or known. When children counted on their fingers without any explanation this was also included in this category.

17+9	I thought well 17 add 9 so I got 5 and then I added the 5 and then I just, and then I got and then I just added 4 on.
time	When the number's on the 6 it's half past.

'generic' in response to calculation and procedure items.

Children talked about the numbers given in the question as if they could be any other numbers or they chose examples to illustrate a procedure.

The numbers given in the question were used but the response suggested that this is a procedure to follow. The responses were often in the present tense and used "you". They made use also of causal connectives.

17 + 9	9 is nearly 10, so you add that onto the 17 ... then you remember you had 9 instead of 10, so you take one off.
time	If it was like at 8 past I'd round it to the nearest 5, which would be 10 and if the little hand was at 6 I'd say like 10 past 6.

'general' in response to calculation and procedure items.

Children did not mention the numbers given in the question when describing what they did or they gave general rules in a procedure question.

Responses often referred to 'the' tens and units and other place value terms without reference to particular numbers.

17 + 9	Make the 9 to 10, add a ten and take one away.
time	The big hand tells time in minutes and the little hand tells how many hours.

Pupils' expressions of 'generality' in their responses to concept questions were categorised similarly. The mode of language use in terms of tense, pronouns and causal connectives were similar. The examples given here are responses to "Can you tell me anything more about centimetre?".

'particular' in response to 'concept' items.

Children spoke of particular objects.

centimetre	They're that long (showed with fingers).

'generic' in response to 'concept' items.

Children gave a prototypical property.

centimetre	You see centimetres on a ruler.

'general' in response to 'concept' items.

Children attempted a definition.

centimetre	If you want to measure something you measure with a ruler.

4.2.3c Categories of 'method'

The categories of method, with an example of responses to 97 + 10, were as follows

'counting'

Children counted for whole number calculations. Fractions were treated as parts of a whole.

97 + 10	Added 5 on then added another 5 on and see what I got.

'holistic'

The sense of the size of the number was maintained even when procedures were followed. Responses to fractions and division items involved sharing.

97 + 10	You keep the 7, 90 add 10 is a hundred, add the 7.

'algorithmic'

Separate digit methods similar to written algorithms were used.

97 + 10	You just use 9 and then go to ten then just a zero but put ten's zero to the 7.

4.2.3d Categories of 'image'

The mental visual images that pupils described after performing a calculation, explained a procedure or that were evoked by the 'concept' questions can also be assigned to one of three categories. The category descriptors, with an example of responses to "What comes before 380?" and "What's the first thing that comes into your head when I say Christmas" were as follows:

'specific'

Children reported images of objects to be counted or episodic images related to 'real life' experiences.

before 380	I got fingers in my head.
Christmas	There's my Christmas tree with lots of presents underneath.

'iconic'

Children reported mental visual images of physical pedagogic representations and representative images related to the concept words.

before 380	It was like a number line to 380
Christmas	Not like the presents that we get, but like - big ones I normally draw with like - cardboard square box ones with a ribbon on.

The majority of responses for non-calculation mathematical questions were 'iconic' in that a prototypical image is common. If a child has an image of 'a Christmas tree' rather than a particular tree then it is regarded as 'iconic'. Attempts to find out from pupils if it was a particular object they had in mind proved largely inconclusive. Thus unless the image was spontaneously described as a particular object it has been classified as 'iconic'.

'symbolic'

Children reported pictures of numerals, words and other symbols

before 380	Like a written down sum. Take away one and an equals.
Christmas	The word Christmas.

This response to "... before 380" and all responses which suggest images of written numerals are not regarded as 'concrete' if they are not related to an episodic memory. This is not a sum that the child has previously seen written and the child is not recalling a particular incident. The imagined sum may be in terms of numerals imagined to have been written and thus have a 'concrete' appearance but it is none-the-less 'symbolic'. These are images generated at this instant not a specific object previously seen.

4.2.3e Categories of 'metaphor'

Metaphors were categorised in accord with Lakoff and Nunez (1997) 'grounding metaphors' which have been seen to conform to the global scheme above (§3.2.4c). An example of responses to "30 add something is 80. What is the something?" is given to illustrate each:

'collection'

Children used the language of manipulation of concrete objects and counting. This included counting on fingers, including counting in tens. Use of the words "add", "take", "more", "gives", "with".

30 add ... Just got my hand like and just added 30, 40, 50, 60, 70, 80.

'motion'

Children used language related to position and directed movement such as the words "go", "up", "down", "back".

30 add ... 40 add 40 is 80 but it goes one ten down so I have to put a ten up.

'creation'

Children used the language of manipulation of symbols. This included place value language, derived facts and known facts. Use of the words "is", "equals", "make", "sums", "the", "it".

30 add ... I was thinking of 3 add something equals 8.

4.2.3f Categories of linguistic indicators

The categories of pronoun use, causal connective use and use of "like" were chosen to complement the categories used above in that they are in a sense 'concrete', 'representative' or 'abstract'. They are summarised as:

	Category 1	Category 2	Category 3
I	did for this one	do for these	do always
you	do for this one	do for these	do always
if	then in this instance	then for this type	then always
then	next	it follows that/from	in general
so	next	it follows that/from	in general
because	it is	it follows from	it always is /must be
like	something	analogy, example	a general rule

Table 4.6 Category descriptors for linguistic indicators

Examples of responses to calculation questions are given for each:

	Category 1	Category 2	Category 3
I	*I did for this one* Estimate length of room. I did it in fives	*I do for these* 30 + something = 80 **I** know that 3 add 5 is 8 so just turn it into tens	*I do always* Read this time (11:40) Because **I** count the thingy-bobs and then **I** look at the other number
You	*You did for this one* 200 more than 4360 **you** know, um, 2, um, 3 add 2 is 5	*You do for these* Quarter of 40? 40 has 4 tens in it so **you** know it's ten	*You do always* 48 + 23? the ten of there and **you** put it back and **you** put it together
If	*If ...then in this instance* What comes before 380? **if** you get 80 and then	*tIf ... then for this type* Read number (26,365) well **if** there's a ten thousand there must be a 20 thousand	*If ... then always* 97 + 10? **if** you're adding a ten, you just, like, take away the units
Then	*Then next* 597 +10? six hundred **then** add the 7	*Then it follows that/from* 200 more than 4360? but **then** I knew that you have to add 2 noughts on the end	*Then in general* Round 246 (nearest 10) and **then** you just do it without it
So	*So next* 48 + 23? **So** you take the 8 off and you take the 3 off	*So it follows that/from* 48 + 23? 8 and the 3 made 11, **so** I carried a figure	*So in general* Round 2462 (nearest 10) if it's below 5 you have to go down **so** I knew
Because	*Because it is* 17+8? um **because** it was	*Because it follows from* 17+9? I added the 6 on **because** you've got 9 units so I added 3 to the 10	*Because it always is /must be* 13 + something = 18? start with 14 **because** you always start with the number next to it
Like	*Like something specific* Round 246 we had to **like** write the numbers then round them to the nearest ten	*Like example, analogy* 30 + * = 80 it's **like** units, you've got 3 and then , then you count up in your head to, 8	*Like a general rule* 97 + 10 when you've got a number **like** 97 and you add ten it has to be the same number but the front number has to change.

Table 4.7 Category exemplars for linguistic indicators

4.2.3g The database and quantitative analysis

A database has been used in order to facilitate coding and analysis of the data. The package was FileMaker Pro 2.1. The unit of analysis chosen was the individual question response. Full details on fields and how each record was developed are in Appendix 2.5.

The data for each pupil was exported to Excel spreadsheets to allow correlation analysis on frequencies of responses of each category for each pupil.

The data for each question were also exported to Excel spread sheets to allow analysis of trends in pupil responses.

The counts of frequencies of each category of response for different groups of children were used to form contingency tables.

Tests for statistical significance were applied to the correlation coefficients and chi-square tests were applied to the contingency tables.

4.2.4 Interpretation

This section relates to the limitations of the study. Consideration is given to validity and reliability and the factors which limit generalisations from the results.

4.2.4a Validity and reliability

Internal consistencies in the 'indicators of learning' will be demonstrated in Chapter Seven but their validity as 'measures' can not be proven. It is important to note that it is suggested that these may give indications of mental constructions and, whilst it is easy to slip into describing these as 'levels', no claim is made that they can provide a measure of progression.

In order to check the reliability of the coding used, exercises in inter-rater comparability were conducted with Fourth year BA(QTS) students at the University of Warwick in February 2001 after the data analysis had been completed. This provides some indication of how well other teachers might understand the system of categorisation employed in the study.

These students were briefly shown the criteria and examples of coding for 'generality', 'image' and 'method'. They then coded a sample of 20 responses for each of the indicators. The exercise gave an average 81% inter-rater reliability coefficient across the indicators. The agreement with my coding varied from 70% to 95%. Borg and Gall (1983) suggest that in situations where raters are required to make inferences or evaluation of responses 70% to 80% agreement is considered satisfactory.

4.2.4b Limitations of the methodology

The limitations of the methodology have been alluded to in the preceding sections but they are listed here.

1) The verbal descriptions given by children can give indications about their mental constructions but no precise claims can be made about the mental constructions themselves. This issue is partly addressed by making explicit that it is the outward manifestations that are the subject of the analysis not the mental constructions they are thought to indicate. Those who might make use of the results of this study may choose simply to note the association between the previous classroom activities and the mental visual imagery and language of the responses, or the association between the language and mental visual imagery and the achievement levels of pupils, without seeing these as manifestations of a mental construction.

2) The verbal expressions of pupils may not be true to their original thoughts. They may choose to say something that they think is more socially acceptable in the interview situation than that which they actually thought. A counter to this criticism is that the children used very similar expressions to those used in the classrooms. This however only suggests that they have seen the interview as a similar situation to the classroom. This is a check on the validity of the interview as a measure of the classroom language characteristics but the interview, like the classroom interactions between teacher and pupil, may not elucidate the pupil's actual thoughts.

3) The interview format of calculation followed by "What was in your head ...?" was almost universally interpreted as a request for a description of the method employed. This may indicate that pupils have seen this as a 'test' and that they have felt driven to say what they think is expected of them. This is true of both the classroom and interview questioning. What children choose to say under these circumstances is no less an indication of the influences of previous experiences even if it is only the 'acceptable face' of that thinking. The question "Was there anything to see in your head" will almost certainly have provoked descriptions of images that would not have been evoked had the question not been asked. This is accepted but it is argued that what pupils chose to describe

may also be indicative of the influence of previous experiences. The question might also be used in the classroom and elicit similar responses.

4) The results are peculiar to the situation in which they were collected and the validity of any generalisations must ultimately depend upon the degree to which that situation is seen as typical. The researcher may claim that the defining characteristics of the situation are to be found in other educational settings but there may be other characteristics of that situation which are not to be found elsewhere. It may be these undisclosed characteristics which have determined the outcome of the study.

5) The categorisation across contexts may not be consistent. An image of a pedagogic representation in calculation questions may not demonstrate the same mode of thinking as a 'representative' picture of a non-mathematics word. Similarly a 'generic' expression in calculation may not be indicative of the same mode of thinking as when a property of an object is given in response to concept questions. This is accepted and generalisations would have to be viewed with caution. The justification for their assumed compatibility is that the 'representative' category in each instance may be seen as a bridge between the 'concrete' and 'abstract'.

6) The theoretical constructs that overlay the categorisation of the distinguishing features of children's responses, and the 'theory laden' interpretation of results, both mark this as a study that might not have been conducted in this way by any other researcher. There is no counter to this criticism. It may, however, be judged as a valid piece of research if it makes sense to others in the education community.

4.2.5 Overview

This section summarises the methodology and gives a preview of the results chapters.

4.2.5a Summary of methodology

My methodology is described as postpositivist with a phenomenographic orientation. It follows in the tradition of research into children's strategies for mental calculation, research into mental visual imagery and research into children's use of language. The principle influence has been the methodology employed in recent research at the University of Warwick. The interviews may be described as structured in having a bank of items conforming to a rationale and the same questions were given to all pupils in a consistent mode of delivery.

The analysis is qualitative in seeking to categorise responses that show qualitative differences but the labels were determined by theories of learning. The qualitative analysis uses an individual case study and explores the common features of the sample in comparison to a sample from a different environment. The quantitative analysis uses statistical techniques to identify differences between individuals and groups.

4.2.5b The results chapters

The data collected will be presented in three parts.

Chapter Five is a case study of one child, Elspeth. The analysis of her responses to interview questions is used to develop the theme that children's language and mental visual imagery are indicative of their previous experiences.

Chapter Six further develops this theme by showing the similarities in responses given by others in the sample. There are examples in this chapter of lessons which may have influenced the language and mental visual imagery of pupils in response to interview questions. The commonality of speech style is highlighted by an analysis of the teachers' responses when they were asked some of the questions the pupils had been asked. The influence of the classroom activities on the children in the sample, as manifested by their language and mental visual imagery, is contrasted with the influences on pupils in the other school.

Chapter Seven presents the results of the quantitative analysis of the data collected in the main study. Each of the research questions is addressed and the second theme, that pupils are influenced in different ways by the same classroom activities, is developed.

The multi-level analysis that has been adopted means that some research questions are addressed in several different sections of these chapters. In order to collect together the evidence that is gathered from the different levels of analysis the answers to the research questions will be presented in Chapter Eight.

CHAPTER FIVE

ELSPETH : A CASE STUDY

The purpose of this chapter is to set the scene for the data analysis which follows in Chapters Six and Seven. The indicators of learning and the linguistic indicators will be illustrated through the analysis of Elspeth's responses.

Elspeth has been chosen for this case study because her response in the first interview in the first exploratory study gave such a strong impression of the influence of classroom activities, and pedagogic representations, on her thinking. Her response in that interview, just two weeks after observations began, can be said to have influenced the course of the whole study.

Elspeth's answers to subsequent interview questions serve to exemplify the variety of responses given by many of the others. I will present evidence from seven interviews with Elspeth to illustrate the way in which children 're-present' experiences. They do this both to assist them with calculation and to give explanations for why they perform the calculations in the way they do. Elspeth was an articulate pupil who reported mental visual imagery more than any other child in the sample, yet her language was not atypical. The mental visual images she described and the language she used when talking about her calculations seemed to be formed from classroom activities that I had observed.

Though unusual in many respects, Elspeth has provided a cipher for de-coding the comments of other pupils. Others may not have re-presented experiences in the form of mental visual imagery yet their language suggested that they had been influenced in their thinking by the same experiences that Elspeth described.

In §5.1 Elspeth is placed in context before examples of aspects of re-presentation of her experiences are given in §5.2. In §5.3 Elspeth's responses are categorised for each of the indicators.

5.1 Elspeth in context

At the end of Y2 Elspeth was one of the highest achievers in the KS1 SAT. Her score of 30 (out of 36) placed her 4th in the year overall and 3rd in the sample (mean for year 21.5, for sample 22.2). At the end of Y3 Elspeth did less well. Her score of 38 (out of 60) placed her 28th in the year and 14th in the sample (mean for year 32.7, sample 36.0). At the end of Y4 she was 15th in the year and 7th in the sample (she scored 45 out of 60, the year mean was 33.5 and the sample mean was 37.6). Thus in Y2 Elspeth's score, in the 95th percentile of the year, placed her as one of 11 pupils who achieved at greater than one standard deviation above the mean. In Y3 and Y4 she was not one of these highest achievers being placed in the 55th and 80th percentiles in these two years.

Elspeth (d.o.b. 7/12/90) was close to the average age for the year. She was aged 6 years 10 months at the time of the first interview in October 1997 and 9 years 7 months at the time of the last interview in June 2000. An assessment of her reading age in May 2000 placed her in the 50th percentile for the year with a reading age of 11; she was ranked 38th in the year and 15th in the sample. She was similarly close to the average in spelling being 26th in the year and 10th in the sample.

5.2 Elspeth's re-presentation of experience

Von Glasersfeld (1995) suggested that thought may involve 're-presenting' our previous experiences. Manifestations that children have learned from experiences may thus involve re-presenting those previous experiences. These are not copies of past events but are representations of our subjective experiences of those events. In this section examples are presented of the way previous experiences were evident in what Elspeth talked about.

5.2.1 The first manifestation of learning

The very first indication of the influence of a pedagogic representation on a child's mental construction are presented here. The lesson and then the subsequent interview response are presented.

Note: Some words such as "um" and repetitions have been removed from transcripts to aid clarity. This will be indicated by '-'

5.2.1a The lesson

This lesson, observed in October 97, was the first in which the Y2 pupils were to work with Dienes 'tens' and 'ones'. It is described in detail because it illustrates the representations used by the teacher. Mr. K. was explicit in his use of analogies, using the word "like" frequently to make connections with previous representations. The important feature of this lesson is the handling of 'tens' and 'ones' as separate objects, particularly placing 'tens' in one hand and 'ones' in the other.

Mr. K. reminded the pupils that in the previous day's lesson they were adding-on tens using a number square.

Mr. K. It is like a number line cut up so we can use it when our number line is too short.

He also reminded them that in Year 1 they used unifix cubes for counting and pointed out that they are like the Dienes 'ones' and 'tens' he showed them yesterday. He then balanced ten Dienes 'ones' on his hand, one on top of another, to show that they were the same height as a 'ten'. When the tower collapsed the 'ones' were laid along the 'ten' to show again that ten 'ones' made a 'ten'.

Two pupils were selected to help. Mr. K. gave Mandy two 'tens'.

Mr. K. Another way of putting it?

Mandy Twenty.

Mandy was then given four 'ones'.

Mr. K. How many altogether?

Mandy Twenty-four.

Mr. K. gave Nina one 'ten' and two 'ones'

Mr. K. How many altogether?

Nina Twelve.

Mr. K. Now put them together in my hands.

Mandy and Nina put their 'tens' in one of his hands and the 'ones' in the other hand. He then held out both hands for the pupils to see.

Mr. K. How many altogether ?

Pupils Thirty-six.

Mr. K. Look how easy it is to add them instead of all individual cubes.

Mr. K. also told the class "we can say 'ones' or 'units'".

Another pair of pupils were chosen, the first was given four 'tens' and two 'ones'

Mr. K. How many altogether?

Pupil Sixty, no, no, 42.

Mr. K. showed a one pound coin and a one penny piece from his pocket and pointed out that it would be two coins but not two pounds.

The example was completed and another acted out in which the same procedure was used, with individual pupils placing 'tens' and 'ones' in separate hands.

Abacus Number 2 Workbook 1 was distributed and many pupils started to use a number square for the two-digit additions even though half the questions showed numbers represented by pictures of Dienes 'tens' and 'units'. This was partly due to lack of sufficient Dienes materials. Mr. K. quickly re-distributed the equipment and insisted that the whole class did one example together. He wrote numerals and drew pictorial representations of the Dienes materials on the white-board and the pupils used the equipment on their desks. Pupils subsequently worked quickly but many were making errors.

During the lesson I asked some pupils a two-digit question and many of them counted in ones with their fingers to get the answer. This lesson illustrates the way the pedagogic representation was used and it also illustrates that it was not immediately accepted by pupils.

5.1.1b The manifestations of influence

Interviews with a sample of eleven pupils were conducted one week after this lesson in order to investigate whether pupils might call to mind either the use of Dienes material or the hundred square when asked to perform a two-digit addition. The pupils were shown 24 + 53 (printed in 100pt in the centre of an A5 sheet) and asked to work it out. Only two pupils calculated correctly, one of whom used fingers saying the numbers semi-audibly, using just one hand, counting on five at a time. She showed no sign of having been influenced by the activities with materials. Another pupil, however, may have been thinking of 'tens' and 'units' as objects to be counted when he gave the following response:

Neal	You get 24 in your head then add on the 5 and the 3 .. 32.

The most interesting response, however, came from the only other pupil who gave a correct answer, Elspeth:

I	How would you do it?
Elspeth	**Put those together**. (Points at numbers.)
I	Why would you put those together?
Elspeth	Well you add the tens together then you add the units because . **it's like in one hand you have the tens and in the other you have the units.**

In saying "put those together" Elspeth used words for physical action on physical objects: "put", "together" and "those". It appears that the previous classroom activities had provided a metaphor for the addition of numerals in that she used words associated with that activity for the mental calculation. Elspeth's use of the word "like" appears to be a more explicit recognition that the manipulation of the individual digits is analogous to the handling of the Dienes 'tens' and 'units'. The phrase "in one hand" seems particularly to emphasise recall of the experiences in the classroom. Furthermore, her use of "you" might indicate that this was a generally accepted routine and her use of "**the** tens" and "**the** units" rather than the particular numbers in the example could be further evidence that she recognised this as a general procedure.

After this exchange the interview continued:

I	That's very good, it's like using those rods and blocks.
Elspeth	Is it?

This is important because in using the words "rods and blocks", which were not the usual classroom words, an opportunity was lost to explore whether Elspeth was explicitly thinking about Dienes material. This indicates that if the words common to a speech community are not used then communication may fail. This point was noted for subsequent interviews.

5.2.2 Further evidence of influence

The following examples come from the three interviews in Y3 (Y3/1, Y3/2, Y3/3). They illustrate the way Elspeth seemed to re-present her previous experiences and thus, perhaps, indicate influence of these experiences on her mental construction. The manifestations of the influences are: references to pedagogic representations, metaphoric language, mental visual imagery and expressions of 'generality'. In non-calculation questions Elspeth also demonstrated the influence of previous experiences.

5.2.2a Counting

The questions concerning counting were included in interviews to explore the influence of pedagogic representations on children's use of the numeration system.

I (Y3/1)	When you are counting what comes before 380?
Elspeth	379
I	What was in your head when you were thinking of that?
Elspeth	Well - when you're counting on, like you go 1, 2, 3, 4, 5, 6, 7, 8, 9 and then it comes to a ten and that's in your tens and you think 'Oh that must be 80' it's 80, it was, it's 79 because - it's - only one from 80.

This appeared to be a re-presentation of the experience of counting and the vocalising of the word sequence, emphasised again by use of "like", but further conversation indicated an alternative source:

Elspeth	I don't really have anything in my head, I only have, numbers.
I	Was that sort of saying the numbers or seeing them?
Elspeth	It was a ruler.

This 'ruler' may have been prompted by the questioning and may not to have been in her mind originally. It suggests, however, that Elspeth's mental construction of the natural numbers was forming from more than one experience. The response to a similar question six months later was unequivocal. This time she used a word associated with a more recent experience and reported a mental visual image :

I (Y3/3)	When you are counting what comes before 2380?
Elspeth	2379
I	OK so what was in your head that time?
Elspeth	Well the scale of the numbers.
I	So how do you mean the scale?
Elspeth	Like there was 2000, 2001, 2002, like it didn't go 1, 2, 3, but it went 10, then when you got to the 70 the scale went like 71, 72,
I	So was that something you could see?
Elspeth	Yes.

Again Elspeth re-presented the idea of a scale but not merely a scale she had seen. She needed to imagine the scale appropriate to this question. It would appear that the number line was part of her mental construction as a figural concept.

5.2.2b Calculation

Elspeth also demonstrated that she could re-present vertical written calculations as well as the more concrete classroom representations for number, such as Dienes blocks and number lines. After giving the correct answer to "597 add 10" (Y3/2), she explained:

Elspeth	- if there's ninety you - like you only have to add ten to make a hundred, and so I - went to hundred, yeh and then like the last number from the other one was seven, was it seven? Yeh? I just had to add 7 on the end.
I	OK so what was in your head there?
Elspeth	- tens and units and then there was the, and a hundreds and then - I set it out downwards and put the add sign and the numbers and the number.
I	So you are pointing to it as if it was written?

Elspeth	(nods)

Though Elspeth reported a mental visual image of a vertical 'sum' her language was not related to the algorithm. She added the 7 on at the end rather than adding the units first. The other important point is that in her explanation she forgot the numbers. This suggests that her mental visual image of the calculation, if it was part of her original calculation, had faded. This also illustrates that it was the procedure for "the last number" that was important rather than the number itself.

5.2.2c The language of manipulation

Whilst Elspeth often re-presented recent experiences, in Y3 she demonstrated that her Y2 experiences in Mr K's class were still important to her and the metaphoric language of manipulation was part of the re-presentation. The following seems to be a re-presentation of the movement of Dienes blocks on pupils' desks in lessons such as the one above (§5.2.1a):

I (Y3/1)	What is 246 rounded to the nearest ten?
Elspeth	250.
I	What was in your head when you were thinking of that?
Elspeth	Well 6 is nearer to 10 than, nought and if you just add them on you can just do it without the, **like push the hundreds** - away and **the tens away** and then you just do it without it then you **put them back together** again.
I	So when you were saying "push away" you were showing me by moving your hand. Were you thinking of moving something away then? What were you thinking of moving away?
Elspeth	Moving cubes cause there were loads of cubes.
I	So is that something you did in class on your desk?
Elspeth	No, but I can remember it from Mr K's class.

In the first instance "6 is nearer to 10 than nought" may indicate a re-presented number line approach to rounding number which her Y3 teacher had used. When she felt she had to justify to me why she was concentrating on one digit, however, she reverted to previous experiences. Dienes materials had not been used by her Y3 teacher.

In each interview Elspeth was asked to calculate "48 add 23" mentally and in each her response to "What was in your head when you were thinking of that?" used the metaphoric language associated with the teachers' demonstrations and manipulation of materials or digits. For instance (Y3/1):

Elspeth	It was **take the tens** out and then add the other ones up and it comes to 11 and then you just take 1, the ten of there and you **put it back** and you **put it together** and then there's one unit **there.**

5.2.2d Mental visual imagery for mental calculation

Elspeth often reported some form of mental visual image. When asked to calculate 48 add 23 in March and July of Y3 her image *and* language appeared first to re-present the written algorithm then in the second to involve a physical representation.

(Y3/2):

Elspeth	It was the tens and units and then I had to **carry a figure** thing - 'cause it was, 8 and the 3 made 11, so I carried a figure and **put it under** there then I add a 6.
I	So that's just how you would do it on paper? So did you see the numbers in your head like that?
Elspeth	Yeh.

In the next interview again language and image are complementary and indicate the influence of Dienes. She made an error which may have been due to concentrating on the separated 'tens' and 'ones'

(Y3/3):

Elspeth	So you **get the units** and that's 8 add 3 and you know that that's 11, so you'd add the **one from the tens** unit onto the one of the 48, 4, and then you'd add the 20 onto it to make 50, - so you take the units and **put it next to** the tens.
I	So when you are saying that is there anything you see there?
Elspeth	There was um, cubes, there was cubes with um like, they're all stacked up with tens and the units are on their own.

There were other occasions when Elspeth made an error through calculating with separated digits and yet reported some mental visual imagery. This suggests that her image was not such a vivid picture that she could inspect it and use it to remind her of the separate digits. The image did seem to be quite 'real' for her. On several occasions when I probed to find out more about the image she reported written numerals appearing:

I	Right so you were moving your finger like writing on the desk, did you have anything written in your head or was that just words?
Elspeth	Well there was some, some like tens and units.
I	That you could see in your head?
Elspeth	Yeh. I normally have that.

I	So is that like writing as it would be on the board or in your book or in a textbook or something else?
Elspeth	It's just like in mid air.

5.2.2e Language of procedures

In addition to specific calculations Elspeth was also asked in December and July of Y3 to say how she would "add 23 on to any number". In each instance she chose to give an example involving her knowledge of bonds to ten. She re-presented a pattern of bonds to ten, not a single known fact. For example in July:

Elspeth	**If it was like** 23 add like 17 you would know, - 1 add 9 then it would be 8 add 2, 7 add 3 and then it would be 7 add 3 because 7 add 3 equals ten. They are all the numbers that equal ten, - you'd like, add that nought and then you'd add the one of the tens onto the 20 in 23.

Whilst 17 was the chosen number her use of "if it was like" indicates that she was using this as an example of what to do in general. It is thus considered to be a generic example in that she would have described just the same procedure if she had chosen any other two-digit number.

In March, in response to two questions of the "Tell me how to ..." type, her Y3 teacher (Mrs. I.) was part of the re-presentation. In the first she also used the phrase "if it was like" but then gave the general rule "add the nought on the end" without reference to the number:

I	Tell me how to times any number by ten.
Elspeth	Well say if it was like 55 times ten **all you do is add the nought on the end.**
I	What was in your head when you were telling me that?
Elspeth	A number, and Mrs I is saying - "what's something times ten?"
I	Were you seeing Mrs I or just remembering what she'd said?
Elspeth	Remembering what she'd said.

Similarly in the second question she spoke first of the particular instance of folding a circle but then attempted to give a general rule without reference to particular numbers. This time it was not so clear in her mind:

I	Can you tell me how to find a third of something?
Elspeth	Well if you have like a circle - you can fold it in half, and then fold it half in again, but that would make, -
I	Make a third?

Elspeth — Well Mrs I said like you can go in the one times table, . and then like, like in the two, like, like you would have a whole then it would be a half - then a third, no, is that right? No it would be a fourth wouldn't it, then you would just do a different thing, like different table thing.

Here Elspeth recalled two experiences, one involving folding of paper circles to form a quarter, the other of using times tables for division, and their appeared to be some cognitive conflict for her.

5.2.2f The other three rules

The two questions used in each interview were both addition. The other three 'rules of arithmetic', subtraction, multiplication and division were introduced into interviews as they became part of the children's classroom activity. These calculation questions were sometimes given in interview before the procedures were as firmly established classroom routines as those for addition, counting and rounding already were at the start of the study.

Elspeth initially struggled with fractions but by Y4/2 had developed her procedure sufficiently to calculate a third of 48. When asked in the same interview "Tell me how to divide by three" she showed an understanding of the underlying principal:

Elspeth — Well you need to know your time table, like 3.

The standard written algorithm for division was introduced to the children soon after this and Elspeth subsequently attempted to perform a calculation mentally by dealing with individual digits:

I (Y4/3) — What is 140 divided by three?

Elspeth — Well 100 divided by 3 is 3 and one left over. That's 10. And then 40 divided by 3 is 1 with ten left over so then, that is 4 with 20 left over. I hope. Is it? 4 Oh I know what I've done. Divided by 30. Ha! Um, that's 40 with -

Asked if there was anything to see she replied:

Elspeth — Well, there's kind of - like a dividing, division sign with the sum. It's - 140 divided, set out like that with a divide sign there (points at written vertical multiplication). You can't actually do that. It's like this (writes horizontally as $3\overline{)140}$).

It seemed that Elspeth had a mental visual image of the the numerals and a division sign written vertically rather than the conventional written format for division which she had recently been taught.

Multiplication showed a similar trend to division. "48 times 3" was first calculated by repeated addition with no reported visual image (Y4/1):

Elspeth	Well I was trying to add 48 to 48 which equalled - 96. And then I add 96 to 48.

Then, in the next interview, after the written algorithm had become the common classroom practice, she gave the wrong answer, 153, and explained (Y4/2):

Elspeth	Well, there was, 'cause we're doing it - we're doing multiplying like big numbers by one digit numbers. It was set out in, like the sums where you times. - And you just, and I put like, you put a carrier there and stuff like that. It had like the number and then it had times sign and it had, well, the number you're timesing by, yeh? and it had (indicates lines underneath)

The previous responses to addition and subtraction questions and these responses for division and multiplication indicate that Elspeth has a tendency to attempt mental calculations using written algorithm techniques when taught them. They are accompanied by mental visual images of written calculation. Even though she might have been able to do calculations using 'holistic' methods she made mistakes after adopting the 'algorithmic' methods.

There was an indication earlier in Elspeth's final response for "48 add 23" that she was willing to adopt mental strategies when they had been taught and this was also apparent with 'multiplication by five'.

I (Y4/3)	If I gave you a two digit number how would you multiply it by five?
Elspeth	Well, I could - change it to 10
I	Mmm.
Elspeth	Multiply it by 10 and then I half it. So say it was - 5 times 35, multiply that by 10 that's 350. And then that's 155.

This mental strategy had recently been taught and we see that Elspeth had adopted it but as in so many questions she followed the procedure and gave the wrong answer due to a slip. She seemed to focus on the procedure and did not have a feel for whether it produced the right answers.

5.2.2g Use of pronouns

It has already been noted that Elspeth's use of "you" in the expression "in one hand **you** have the tens and in the other **you** have the units" might indicate that she was reporting a procedure that is common in her experience. The use of "you" may indicate that this is what everyone does.

She used "I" exclusively in circumstances where she was talking specifically about herself and particularly when using known facts to derive others:

I	13 add something is 18, what is the something?
Elspeth	5 add 3 is 8 so **I** knew that so **I** could just do it with the tens on as well.

She also used "I" exclusively when she was less confident (indicated here by her faltering response):

I	1097 add 10
Elspeth	Well we've been doing - adding tens and stuff. Well 'cause 97 yeh is at the end - like only 3 more till add the next hundred, - Well - what **I** did is **I** rounded it up to the nearest - hundred - and then **I** added ten to the hundred, that's 110 and then **I** just like took off that 9, **I** mean **I** put it back to 97, took off the 9 and then ended up with 1017.

"I" was also used exclusively when not following taught procedures:

I (Y4/1)	What is 48 times 3?
Elspeth	Well **I** was trying to add 48 to 48 which equalled, um, 96. And then **I** add 96 to 48.

By contrast use of "you" exclusively accompanied well used procedures :

I (Y4/1)	17 add 9?
Elspeth	Well if **you** round up 9 to the nearest 10 it would be 10, and then **you** would add 10 onto 17 and it would, 27 and **you** have to just take off 1, 26

Elspeth also frequently used a mixture of pronouns in the same response. Here again the "I" was associated with known facts, personal or little used procedures and lack of confidence. "You" was used when describing well practised procedures and algorithms :

I (Y4/1)	What is two thirds of 24?
Elspeth	**You** halve it which is 12 and then **you** halve it again which is - 6 and then, is it? Is that right? - Is it 12? It's just, um, well that's a quarter isn't it. Two quarters is - Is it, is it 18? Well, um, **I** was just like, thinking, um, 6, yeh, because there was like, **I** don't really know.

The pronouns "I" and "you" were used almost exclusively by teachers and pupils in this study. The pronoun "we" which might be common in other speech communities (Pimm 1987) was very seldom used. Elspeth gave one of only 10 responses (out of nearly 2000 for the sample as a whole) which used "we" to indicate a generalisation:

I (Y3/1)	Tell me how to add 23 onto any number.

Elspeth	If it was a 7 and you knew it was going to be 30 because you know 3 add 7 is 30. I mean 3 add 7 is 10, and you just, all you put them back, tens back and then we have it.

Many other pupils used "we" to refer to the class and Elspeth did so when referring explicitly to previous experiences. She also frequently re-called the collective activity which had given rise to her individual mental visual image. For instance she had a mental visual image when given the question 0.6 + 0.7 in Y4/3 and said:

Elspeth	Well, we've been doing about decimals and it was set out kind of like that - vertical.

5.2.2h Non-calculation questions

Elspeth may again have re-presented Year 2 experiences in March when asked "What is the first thing that comes in to your head when I say three?"

Elspeth	Well it's an odd number and, and it's,
I	That's the first thing that comes in your head? OK. Did any pictures come into your head or writing or anything?
Elspeth	Yeh, well there was these cubes with um well there was two cubes there and one on top of it.
I	Then you said it's an odd number. Were you thinking of when you did that with cubes in class?
Elspeth	Yes.

This may well have related to the demonstration given by Mr.K. that even numbers are formed by cubes in pairs whilst odd numbers have an extra cube on top.

When asked "Can you tell me anything more about three?" her response seemed spontaneous but it also turned out to have its roots in previous experience:

Elspeth	It's odd, it's in the 3 times table, it's in the 1 times table, - 2 add 1 is 3, 10 take away 7 is three - 5 take away 2 is three - 7 take away 4 is three - 50 take away, 47 is 3.
I	So when you were telling me those things was there anything written in your head then? Any pictures?
Elspeth	Well we did do something like, Mrs I gave us some minutes to do, to talk about like two. So I just - I remembered her words.

In the other questions where Elspeth was asked "What is the first thing that comes into your head ..." previous experience was also in evidence. For instance:

I (Y3/2)	What is the first thing that comes into your head when I say ball?
Elspeth	It's bouncy

I	Did you have anything to look at when you said that?
Elspeth	Yeh
I	What?
Elspeth	Well my little sister has got this video called "Spot" and it was playing with a ball.

In the non-mathematical procedure questions Elspeth always seemed to re-present the experience in order to describe the process. For example after describing how to cross the road:

I (Y3/1)	Right so that was a very full list of instructions. What was in your head when you were thinking of that?
Elspeth	Well I was crossing the road.

5.2.2i Summary of influences

There is evidence in all these responses that Elspeth used previous experience in her thinking both in mathematical and 'everyday' contexts. Her mental visual imagery and language relate to previous classroom activities in which teachers and pedagogic representations are either explicitly recalled or metaphoric language gives indications of influences. 'Episodic' memories of classroom experiences were mentioned in 17 of her responses (88th percentile, mean 9)

Elspeth demonstrated in Y3, as she had in Y2, that she was capable of abstracting from her experiences. She could re-call previous activities to explain how and why she performed her calculations. Her mental construction for whole number involved mental visual images and language related to concrete materials such as rulers, scales and Dienes blocks. Her procedures related to manipulation of concrete referents for numbers and symbol manipulation as well as counting. Her language and her mental visual imagery give evidence of these various influences on her thinking. She emphasised the use of a general procedure, i.e. one which is commonly used by herself and others, by her use of "you". She often used generic examples to illustrate these procedures indicated by "if it's like".

5.2.3 Summary of Elspeth's re-presentation of experience

There is an emerging picture of the way influences on learning may be manifested. In particular there is an emerging picture of the influences of previous experiences on

Elspeth's thinking. This qualitative snap-shot suggests that the influences are manifest in the language and mental visual imagery.

- By talking about and reporting images of pedagogic representations Elspeth gave explicit signs that these were part of her learning.
- Her use of pronouns, generic examples and metaphoric language gave more subtle information about her thinking.

The next section seeks to quantify these manifestations.

5.3 Elspeth's abstraction from experience

In this section Elspeth's responses over the two years of the longitudinal study are analysed more carefully following the themes identified above. We have seen that her methods of calculation, her use of metaphoric language, her reported mental visual imagery and her expressions of generality all indicate influences on her thinking. The categories of these 'indicators', introduced in Chapter Three and exemplified in Chapter Four, help quantify the qualitative trends. Elspeth's use of each of the indicators of learning and her use of linguistic indicators is analysed to provide information about the characteristics of her thinking.

5.3.1 Indicators of learning

Each of the indicators of learning provides a means of categorising responses to interview questions. This section develops a profile of Elspeth's responses which gives an indication of her thinking.

5.3.1a 'Metaphor' in Elspeth's responses

Elspeth frequently indicated the influences on her thinking of previous activities through the use of words associated with those activities. The metaphoric language of manipulation was described above as an indication of re-presented experience. The metaphors in use have been categorised in line with the three 'grounding metaphors' for arithmetic (Lakoff and Nunes, 1997). Table 5.1 illustrates Elspeth's use of the categories of 'metaphor' in comparison with the sample. The table shows Elspeth's frequency for each category, the sample mean and Elspeth's percentile rank in the sample. It shows that Elspeth was a relatively high user of 'object creation' and 'motion' metaphors and a relatively low user of 'object collection':

	Elspeth	Mean	Percentile
collection	9	14	20th
motion	7	6	76th
creation	33	21	100th

Table 5.1 Categories of 'metaphor' for Elspeth's responses compared with sample mean

The fact that she used a variety of metaphors indicates, again, the variety of experiences which have led to her conceptual and procedural knowledge. Examples are:

object 'collection':

I (Y4/2) What comes before 12 100 when you are counting?

Elspeth Obviously a hundred **take away 1** is 99.

'motion':

I (Y3/1) What comes before 380 when you are counting?

Elspeth You go 1, 2, 3, 4, 5, 6, 7, 8, 9 and then it **comes to** a ten - it's 79 - 'cause it's - **only one from** 80.

object 'creation':

I (4/2) What is a third of 48?

Elspeth It's **8 on the end** and - **8 divided by** three.

It might be argued that children's language could be influenced by the context of the question, that questions involving "comes before" or "round to the nearest" are most likely to evoke 'motion' metaphors. In the first two examples of 'metaphor' Elspeth demonstrated that she could think in terms of object 'collection' as well as 'motion' and many other pupils used object 'creation' metaphors for these questions. Similarly "a third of 48" gave rise to language of symbol manipulation for Elspeth but many others used object 'collection' metaphors.

When restricted to those questions requiring a calculation to be performed Elspeth's distribution of metaphors indicated that no category was particularly associated with accuracy. Elspeth's responses to 42 of the 45 calculation questions fell into one of the categories of metaphor. Table 5.2 indicates the way Elspeths use of metaphor was associated with correct and incorrect answers:

	Wrong	Right	Totals
collection	3	5	8
motion	2	5	7
creation	10	17	27
Totals	15	27	42

Table 5.2 Categories of 'metaphor' for Elspeth's correct and incorrect responses

5.3.1b 'Method' in Elspeth's responses

Metaphors of counting, number lines and place value are closely connected with mental calculation strategies. The calculation methods described in the various calculation and mathematics procedure questions were categorised as 'counting', 'holistic' or 'algorithmic'. Elspeth's responses to 50 of the 53 computation questions fell into one of these categories. Table 5.3 gives the comparison between Elspeth's categories and the sample means. She was more likely to use 'holistic' and 'algorithmic' methods and less likely to use 'counting' methods than the majority of the sample:

	Elspeth	Mean	Percentile
counting	7	8	40th
holistic	22	17	76th
algorithmic	21	14	88th

Table 5.3 Categories of 'method' for Elspeth's responses compared with sample mean

There were 45 questions where a calculation was required to be performed and Elspeth gave descriptions of her method in 42 of them. As with her use of grounding metaphors, Table 5.4 suggests that there does not appear to be any advantage for accuracy in using any particular 'method':

	Wrong	Right	Totals
counting	2	3	5
holistic	6	11	17
algorithmic	7	13	20
Totals	15	26	42

Table 5.4 Categories of 'metaphor' for Elspeth's correct and incorrect responses

Like 'metaphor', 'method' indicates that Elspeth shows variety but that she has preference for the more abstract methods. The large proportion of 'algorithmic' methods gives an indication that manipulation of single digits has been a significant influence on her thinking. It is apparent in what she does and what she says.

5.3.1c 'Generality' in Elspeth's responses

The descriptions given when asked "What was in your head when you were thinking of that?" have also been categorised by the level of 'generality' they exhibit. When Elspeth had just completed a mental calculation and was then asked this question her responses fitted into one of the categories of 'generality' ('particular', 'generic' and 'general').

Table 5.5 illustrates the distribution of use of these different levels of 'generality' in all calculation questions. It can be seen that she was not typical of the sample. She was a much higher user of 'generic' and 'general' descriptions of 'generality' than the majority and much less likely to describe simply what she did with the particular numbers.

	Elspeth	Mean	Percentile
particular	5	12	4th
generic	26	17	96th
general	10	5	88th

Table 5.5 Categories of 'generality' for Elspeth's responses compared with sample mean

When restricted to those questions requiring a calculation to be performed Elspeth's tendency to get right those which she subsequently described in the most general terms was quite marked. A much higher proportion of 'general' responses were associated with correct answers than the other categories:

	Wrong	Right	Totals
particular	3	3	6
generic	11	14	25
general	1	9	10
Totals	15	26	41

Table 5.6 Categories of 'generality' for Elspeth's correct and incorrect responses

The same categories of 'generality' have been used for the responses to non-calculation questions. For non-arithmetic procedures children may again talk about the particular context, use an example to describe a rule or state a general rule. When they were asked "What is the first thing that comes into your head when I say ..." the three categories involved particular objects, properties of the named item or an attempt at a definition.

Table 5.7 illustrates that Elspeth showed a greater tendency to talk about properties and use examples (i.e. 'representative') than the rest of the sample:

	Elspeth	Mean	Percentile
particular	5	9	8th
generic	21	13	100th
general	7	7	44th

Table 5.7 Categories of 'generality' for Elspeth's non-calculation responses compared with sample mean

This suggests that Elspeth was as capable of generalising from her experiences in non-calculation contexts as she was in arithmetic. She was also most likely to express herself in 'generic' or prototypical terms. In calculation questions she gave a relatively high proportion of 'general' responses. Taken together this might suggest that, rather than merely a predisposition toward a mode of expression, her use of general rules in calculations indicates a genuine abstraction. It is important to note that in all contexts Elspeth was much less likely than the majority to talk simply of the particular calculation or bring to mind particular objects.

5.3.1d 'Image' in Elspeth's responses

It has previously been noted that Elspeth described more mental visual images than any other pupil and some examples have already been given. All her images were again categorised using the three categories 'specific' (objects and episodic images), 'iconic' (pedagogic representations, representative pictures of real life) 'symbolic' (symbols). Table 5.8 illustrates that the same trend was apparent in her images as has been noted for the other 'indicators' i.e. Elspeth was much more likely than the average to have 'representative' or 'abstract' images:

	Elspeth	Mean	Percentile
specific	5	7	68th
iconic	15	4	92nd
symbolic	26	11	96th

Table 5.8 Categories of 'image' for Elspeth's responses compared with sample mean

Table 5.9 illustrates that when mental visual images associated with her mental calculations are considered separately her categories showed a similar distribution:

	Elspeth	Mean	Percentile
specific	1	1	56th
iconic	7	3	92nd
symbolic	21	8	96th

Table 5.9 Categories of 'image' for Elspeth's calculation responses compared with sample mean

These tables (5.8 and 5.9) suggest that Elspeth has a tendency to have mental visual images which are not simply evoking specific objects in both mathematical and non-mathematical contexts.

Whilst mental visual images were commonly reported by Elspeth this was not necessarily associated with accuracy in her calculations. Table 5.10 shows that her accuracy (two thirds of her answers were correct) was not much different whether she reported an image or not.

	Wrong	Right	Totals
No image	6	10	16
image	9	20	29
Totals	15	30	45

Table 5.10 Reported imagery for Elspeth's correct and incorrect responses

Elspeth's categories of 'image' are given in Table 5.11 for correct and incorrect answers. There is an indication that mental visual images of Dienes blocks and number lines most frequently accompanied correct answers whilst more symbolic mental visual images were less of an advantage:

	Wrong	Right	Totals
specific	0	1	1
iconic	1	6	7
symbolic	8	13	21
Totals	9	20	29

Table 5.11 Categories of 'image' for Elspeth's correct and incorrect responses

The frequencies in Table 5.11 are too small to draw many conclusions but it could indicate that mental visual images most frequently accompany responses to questions where Elspeth was confident. In fact the questions were some of the easier ones. (Differences in responses to 'easy' and 'hard' questions will be analysed more carefully in Chapter Seven).

5.3.1e Summary of 'indicators of learning'

The categories of response for each of the indicators have given a clearer picture of how Elspeth distinguished herself from other pupils. 'Metaphor', 'method', 'generality' and 'image' each give suggestions that Elspeth had abstracted from her experiences. In each of the indicators she was much more likely than the majority of the sample to give responses which indicated 'abstract' or 'representative' thinking. She was much less likely to respond in 'concrete' terms.

The analysis of 'linguistic indicators' in the next section serves to reinforce this picture.

5.3.2 Linguistic indicators

Metaphoric language of manipulation provides information about influences on children's thinking. Their use of individual words may also be used to classify different modes of expression. More particularly the different categories of use of pronouns, causal connectives and "like" may give indications of abstraction from experience.

Elspeth, in addition to being an articulate pupil, was also nearly the most verbose. In the six interviews she used a total of 5481 words in comparison with a mean of 3437 words. She was in the 96th percentile of word use and her total was exceeded by only one other pupil who used 6038. In consequence she used more of the linguistic indicators than most other pupils. Her responses serve to illustrate their use.

5.3.2a Pronouns

Table 5.12 indicates that Elspeth was a relatively high user of "you" (92nd percentile) and a relatively low user of "I" (44th percentile) in calculation questions. She was also much more likely than the majority to use "you" exclusively or in combination with "I" and much less likely to use "I" exclusively.

	Elspeth	Mean	Percentile
I exclusively	16	26	8th
I and you	16	10	84th
you exclusively	21	11	92nd

Table 5.12 Elspeth's use of pronouns in calculation responses

This is important in the light of the indication that exclusive use of "I" is associated with lack of confidence (her faltering responses noted in §5.2.2g) and with own methods (§5.2.2g) , "you" with following known procedures (5.2.2g). This table thus suggests that Elspeth was more likely than most to describe well rehearsed procedures in which she had confidence.

It is worth noting that the connection between her exclusive use of "I" and lack of confidence may be supported by the fact that it was more frequently associated with wrong answers. Table 5.13 also indicates that exclusive use of "you" was associated with right answers:

	Wrong	Right	Total
I exclusively	7	4	11
I and you	5	7	12
you exclusively	3	8	11
	15	19	34

Table 5.13 Elspeth's use of pronouns in correct and incorrect responses

This seems to indicate that Elspeth's use of known procedures, expressible in general terms using "you", was associated with successful calculation.

Elspeth's categories of use of "you" (descriptors and exemplars for cartegories given in §4.2.3f) was also distinctive and commensurate with her tendency toward abstraction noted in the 'indicators of learning' (§5.3.1e) . She was much more likely than the majority to use "you" in the sense of "What you do in general".

	Category 1	Category 2	Category 3
Sample mean	7%	60%	33%
Elspeth	0%	22%	78%

Table 5.14 Categories of pronouns in Elspeth's calculation responses

Elspeth's use of "I" was similar to the sample averages. Elspeth seemed to signal her knowledge of procedures, and her acceptance of them, by use of the common classroom practice of describing procedures using "you". She reserved "I" for own methods. This may be an indication that she frequently followed taught procedures.

5.3.2b Causal connectives

Table 5.15 shows that, as with pronouns, Elspeth made more frequent use of causal connectives than most others in the sample:

	If	Then	Because	So
Sample mean	11	24	11	15
Elspeth	17	39	17	29
Percentile	88	96	84	92

Table 5.15 Use of causal connectives in Elspeth's responses compared with sample mean

More important than mere quantity, however, is the style in which she used these words. Their use has again been categorised using the three categories described in Chapter 4. Table 5.16 indicates that Elspeth's distribution for "if" and "then" showed less use in category 1 (reasoning about particulars) than the sample average and higher use of category 2 (reasoning about representatives). She had a similar distribution of "because" and "so" to the sample averages. The table shows Elspeth's percentage of word use in each category compared with the sample means (in brackets). It indicates more non-trivial use of causal connectives than the sample norm.

	Category 1	Category 2	Category 3
if	... then in this instance 29% (40%)	... then for this type 53% (31%)	... then always 18% (29%)
then	next 74% (88%)	... it follows, that, from 21% (10%)	... in general 5% (2%)

Table 5.16 Categories of causal connectives for Elspeth's responses compared with sample mean

This fits with a picture of Elspeth as someone who is comfortable with abstraction.

5.3.2c "Like" as an indicator of analogy

This chapter started with Elspeth's first interview in which she made explicit her re-presentation of previous experiences by use of the word "like". She continued to do so frequently in subsequent interviews. She used "like" in 48 of her 77 responses (96th percentile, mean 25).

The categories of "like" were given in Chapter Four as:

Category 1	comparison with something particular
Category 2	comparison with another example or an analogy drawn with a similar situation.
Category 3	comparison with a general rule

"Like" was frequently used by Elspeth when describing her mental visual images, comparing them with things previously experienced, i.e. category 1:

Elspeth	It was just **like** - tens and units and then - I set it out downwards and put the add sign and the numbers and the number (pointing as if sum written in air).

In 25 responses Elspeth used "like" to indicate an analogy more than any other child (mean 10). For this analysis an analogy is taken to be an explicit comparison with another situation which has aspects in common with the present one. For instance:

I (Y3/2)	30 add something is 80, what is the something?
Elspeth	Well I know that 5 , um 5 add 3 is, 8, so I just **like** added a 0 to that.
I (Y3/2)	597 add 10
Elspeth	If there's ninety you, just **like**, **like** you only have to add ten to make a hundred.
I (Y4/3)	Now add 20 to 36.
Elspeth	It's just **like**, if you take off the 6 and then you add the 30, 20, 50 and then you put the 6 back on.

In each instance Elspeth does not simply do this particular calculation but compares it with another. She did 3 add 5 instead of 30 add 50 and said it was like adding a nought (as one would on paper). For 597 add 10 she compared with 90 add 10. She compared her partitioning strategy to the removal of the 6 when adding 20 to 36.

Whilst the word "like" may appear to be overused in everyday conversation, particularly by children, the above examples suggest that they may be searching to explain themselves by suggesting similarities with familiar events. It may seem superfluous to use "like" in "**like** you only have to add ten to make a hundred" but Elspeth may be indicating, consciously or unconsciously, recognition that digit manipulation is like manipulation of objects. A similar indication is given by: "97 yeh is at the end of, of, **like** only 3 more till add the next 100", "so I just **like**, um, just put the next 100", "I had to **like** take another ten off", "it was going **like** 2, 3, um, 2, 3, 5, 7, and **like** that". All these expressions show a search to give meaning by relating to the known. "Like" may not simply be a 'filler' in discourse, used to give time to think, but may be an indicator of a mind seeking connections.

Elspeth's categories of use of "like" were not so markedly different from the rest of the sample as some of the other indicators (Table 5.17). She did however show a tendency toward more analogy and less simple comparison than the sample norm:

	Category 1	Category 2	Category 3
Sample mean	32%	63%	5%
Elspeth	23%	70%	7%

Table 5.17 Categories of "like" for Elspeth's responses compared with sample mean

5.3.2d Summary of linguistic indicators

Elspeth's use of pronouns, causal connectives and "like" has emphasised that she was more inclined toward indications of abstraction than the majority of the sample. She used "you", "if" and "then" frequently in non-particular terms. Whilst this may be an indication of abstract thinking it may also be a sign of procedural thinking. The use of "you", causal connectives and "like" are all common features of the speech community. Elspeth's use of these indicators may be no more than an indication that she has acquired the cultural tools. What we do know is that Elspeth gave more evidence of using the tools than the majority of the sample.

What has also become apparent is that there appear to be consistencies in Elspeth's categories in the indicators which suggests at least a mode of expression if not a mode of thought. These consistencies will now be quantified.

5.3.3 Signs of Elspeth's consistency

The analysis in the previous two sections indicates that Elspeth could use a variety of categories of linguistic indicators and indicators of learning but seemed to have a preference towards expressions which were at least 'representative'. In this section this consistency is explored and an attempt is made to identify trends in the way she responded to a few questions which occurred in each interview. A comparison is also made between arithmetic questions and non-arithmetic questions in order to explore the possibility that Elspeth has a pre-disposition toward a mode of expression which is common across contexts.

5.3.3a Consistencies between 'indicators of learning'

In the analysis above there is evidence of a style of thinking that is characterised by methods other than 'counting', metaphors of manipulation and images of symbols. A picture emerges of a pupil who used procedures largely based on individual digit manipulation which she expressed in a language of object 'creation' and these were often accompanied by mental visual images of written calculation. This becomes clearer when corresponding categories of 'image', 'generality', 'metaphor' and 'method' are considered.

Mental images of calculations ('symbolic')were most frequently associated with descriptions of procedures that gave indications of a general process ('generic' and 'general'). The descriptions were couched in terms of instructions for what to do with these types of questions. Table 5.18 illustrates this by giving the number of each response in each category of 'image' which was accompanied by each category of 'generality'. The predominant cell (bold typed 13) accounts for nearly half of her responses.

	specific (image)	iconic (image)	symbolic (image)	Totals
particular (generality)	0	0	4	4
generic (generality)	0	4	**13**	17
general (generality)	1	3	4	8
Totals	1	7	21	29

Table 5.18 Associations between Elspeth's categories of 'generality' and 'image'

'Metaphor' and 'image' show a predictable correspondence (Table 5.19). Metaphors of object 'creation' are those related to manipulation of individual digits, treating numbers as things to be manipulated not treating numbers as collections of objects. Elspeth used this language most frequently and it was often combined with a mental visual image of written calculation or numerals.

	specific (image)	iconic (image)	symbolic (image)	Totals
collection (metaphor)	0	2	1	3
motion (metaphor)	1	2	2	5
creation (metaphor)	0	3	**18**	21
Totals	1	7	21	29

Table 5.19 Associations between Elspeth's categories of 'metaphor' and 'image'

Table 5.20 shows that Elspeth used both 'holistic' and 'algorithmic' strategies and in each case it was most frequently accompanied by mental visual images of symbols:

	specific (image)	iconic (image)	symbolic (image)	Totals
counting (method)	0	1	1	2
holistic (method)	0	3	**9**	12
algorithmic (metaphor)	1	3	**11**	15
Totals	1	7	21	29

Table 5.20 Associations between Elspeth's categories of 'method' and 'image'

When describing her calculations Elspeth was most likely to do so in non-'particular' terms and this was most frequently accompanied by 'creation' metaphors (Table 5.21):

	particular	generic	general	Totals
collection (metaphor)	2	3	1	6
motion (metaphor)	1	4	2	7
creation (metaphor)	3	**17**	7	27
Totals	6	24	10	40

Table 5.21 Associations between Elspeth's categories of 'metaphor' and 'generality'

When Elspeth spoke of the particular numbers of the question this was most frequently related to 'holistic' methods (Table 5.22). General expressions were most frequently related to 'algorithmic' methods. This could indicate that the 'algorithmic' methods had been internalised as general rules. Her most frequent mode of expression was 'generic' for both 'holistic' and 'algorithmic' methods:

	particular	generic	general	Totals
counting (method)	1	1	2	4
holistic (method)	**4**	12	0	16
algorithmic (method)	1	11	**8**	20
Totals	6	24	10	40

Table 5.22 Associations between Elspeth's categories of 'method' and 'generality'

There is an indication here that taught algorithms had been learned as general procedures whilst 'holistic' methods had not been learned in this way.

Most of the calculations using separate-digit methods were, predictably, described in language of object 'creation':

	collection	motion	creation	Totals
counting (method)	2	1	1	4
holistic (method)	4	5	7	16
algorithmic (method)	1	1	**18**	20
Totals	7	7	26	40

Table 5.23 Associations between Elspeth's categories of 'method' and 'metaphor'

This analysis demonstrates inner consistencies in Elspeth's cognitive structures as revealed by language and mental visual imagery. Elspeth most frequently seemed to express herself in non-'concrete' ways.

5.3.3b Consistency over time

Responses to the two questions which were common to all of the first five interviews show some variations but a considerable consistency in expression. Table 5.24 summarises Elspeth's responses for the addition of 17 and 9. With the exception of the occasion on which she counted on her fingers, Elspeth consistently treated the numbers as objects to be manipulated. Her mental visual images were all of symbols. Her responses were categorised as:

	metaphor	method	generality	image
Y3/1	creation	holistic	generic	symbolic
Y3/2	creation	algorithmic	generic	symbolic
Y3/3	collection	counting	-	-
Y4/1	creation	algorithmic	generic	symbolic
Y4/2	creation	algorithmic	general	-

Table 5.24 Elspeth's consistency over time the single-digit addend question

The other question to appear in each interview was a two-digit addition (48 add 23) which Elspeth tackled with mixed success but again consistencies in 'metaphor', 'method', 'generality' and 'image'. Table 5.25 illustrates the consistencies:

	metaphor	method	generality	image
Y3/1	creation	algorithmic	general	-
Y3/2	creation	algorithmic	generic	symbolic
Y3/3	creation	algorithmic	generic	iconic
Y4/1	creation	algorithmic	generic	symbolic
Y4/2	creation	holistic	generic	symbolic

Table 5.25 Elspeth's consistency over time in the two-digit addition question

Here Elspeth consistently used the method of the written algorithm, starting with the units and carrying the ten, until the last interview when she used a mental calculation strategy which had been taught as part of the Numeracy Strategy adopted by the school. This suggests that the emphasis on written calculation in Y3 had had the biggest influence on Elspeth's mode of mental calculation but when taught a mental strategy she made use of it. It should also be noted that much of this consistency may be due to the fact that these were both easy questions. They involved familiar routines which could be performed, and expressed, in stereotypical ways. Elspeth may be said to have a 'script' for these.

5.3.3c Consistency across arithmetic and non-arithmetic procedures

Elspeth again showed consistency in her response to those questions which asked her to describe how to do something. They may be summarised as:

	metaphor	method	generality	image
add 23 (Y3/1)	creation	holistic	generic	-
find a third (Y3/2)	creation	counting	generic	-
times by ten (Y3/2)	creation	algorithmic	generic	symbolic
add 23 (Y3/3)	creation	holistic	generic	-
find two thirds (Y4/1)	collection	counting	generic	-
divide by 3 (Y4/2)	creation	holistic	general	symbolic
multiply by 5 (Y4/3)	creation	holistic	general	symbolic

Table 5.26 Elspeth's consistency in arithmetic procedure questions

It seems that in those questions where she was confident (indicated by fluency in response and correct answers) she used the metaphor of object 'creation' and articulated the procedure in terms of a generic example or in more general terms. The two questions on thirds where she was confused between finding quarters and using times tables (as noted above §5.2.2e) caused her to struggle and the pattern in the indicators is different.

Elspeth also showed consistency in those questions where she was asked "tell me how to ..." in non-calculation situations. In each instance she gave a generalisation that was either 'generic' or 'general' and this was accompanied by an episodic mental visual image. Whilst some of Elspeth's mental visual images in other contexts may have been provoked by my questioning there was a sense here that the mental visual image was an accompaniment to her instructions. As previously noted (§5.2.2h) she gave a very full set of instructions for crossing the road and said she could see herself crossing the road.

5.3.3d Images for concepts

"What is the first thing that comes into your head when I say ...?" evoked similar responses over the two years of interviewing. Table 5.27 shows the consistency in her expression of generality in all the mathematical items items over the period:

	Response	generality	image
centimetres	They're like inches.	representative	-
three	It's an odd number.	representative	iconic
million	A hundred thousands make a million.	representative	symbolic
fraction	Parts and sections.	representative	iconic
polygon	Shapes.	representative	symbolic

Table 5.27 Elspeth's consistency in mathematics concept questions

In each instance Elspeth responded, as the first thing in her head, with a property of the named object or a characteristic feature. This was also true of her responses to non-mathematical items (Table 5.28):

	Response	generality	image
shadow	Well if you're standing and the light, the light, the sun comes down that way (gestures with hands) it can't get past you, so it shines straight down and it forms a shadow underneath you.	generic	specific
adjective	say if there was like, um, there was a dog or there was a scruffy dog, that would be the scruffy that was the adjectives.	generic	-
ball	it's bouncy	generic	specific
Christmas	Presents?	particular	iconic
animal	Loads of animals like hopping around.	particular	iconic

Table 5.28 Elspeth's consistency in non-mathematics concept questions

There is some confirmation here that everyday and scientific concepts evoke different responses. The two words learned in school, shadow and adjective, both gave rise to generic examples in an attempt to define them. Shadows had been the subject of a Science lesson one week before the interviews and distinctions between nouns, verbs and adjectives was a feature of literacy lessons that term. Christmas and ball both evoked particular objects and 'iconic' images. The low-imagery word, adjective, evoked no mental visual image.

5.3.3e Summary of consistencies

Elspeth's tendency toward modes of expression that were not simply 'concrete' has been confirmed by this analysis. Her images, methods, metaphors and expressions of 'generality' show inter-indicator consistencies. Her mental construction is manifested in consistent ways.

Elspeth demonstrated her use of routines for two addition questions over the period of time which suggests a 'script' for these familiar situations. Her expression of 'algorithmic' methods in 'general' terms also indicates the use of a procedure that was in a sense automatic, used almost at a subconscious level, and expressed as a general rule without reference to the numbers involved.

There also appear to be some consistencies in Elspeth's modes of expression across contexts to the extent that her use of non-'concrete' categories was evident in non-calculation contexts. Her mental visual images in non-mathematics situations were, however, more likely than in mathematical situations, to be of specific objects.

5.3.4 Summary of Elspeth's abstraction from experience

The explicit and implicit manifestations of re-presented experience noted in §5.2.3 have been quantified in this section. Here we have been concerned with 'abstraction from experience'. The categories of the indicators are qualitatively different in terms of their relation to 'concrete' experience, even though they may not 'measure' abstraction. Elspeth's profile indicates a mental construction that is manifested by abstraction from concrete experiences. In support of this claim the following points have been noted:

- In mental calculation questions Elspeth most frequently gave responses which: used non-'counting' methods, used 'creation' metaphors and were expressed in 'generic' or 'general' terms. Her mental visual images were most frequently of symbols.
- Her use of pronouns, causal connectives and analogy indicate that she has abstracted procedures which can be termed 'scripts'. She has routines for what "you" do which she explains with appropriate use of "if", "then", "because", "so" and "like".
- Her responses in non-calculation questions similarly demonstrate her ability to explain general rules, not simply describe the particular.

5.4 Characteristics of Elspeth's thinking

Elspeth was a relatively high achieving pupil who was both articulate and reported a relatively high number of mental visual images. Throughout this chapter the transcripts suggest that Elspeth made use of her experiences to build routines for mental calculation and the evidence is provided by language, mental visual images and episodic references to classroom activity.

The role of mental visual imagery in her thinking is difficult to ascertain. There is the possibility that she did not have mental visual images and simply described a picture to please me. If this is the case then what she chose to describe is as useful in distinguishing influences on her conceptualisation as 'real' mental visual images would be. If she chose to pretend that she 'saw' a number line, for instance, then there is an indication that it is part of her mental construction. There is also the possibility that she had quite different images from the ones she described. This seems unlikely in calculation contexts given the degree of consistency between how she described her calculation and the mental visual images she reported. In non-calculation contexts it seems unlikely that she would have a mental visual image of one thing and describe another. However there is no check on this.

Elspeth emerges from this analysis as a procedural thinker. She had internalised the routines learned in the context of written calculation and re-enacted them usually with a description of what "you" do. When taught a different mental calculation strategy such as 'round to the nearest multiple of ten then adjust' or 'to multiply by 5 first multiply by ten then halve' she adopted it with alacrity. Her focus on procedures was, however, often at the expense of correct answers. She forgot the number in her manipulation of its individual digits and did not check whether the answer was reasonable.

Yet she also had a rich mental construction of number and seemed to be able to relate written calculation to iconic representations. She showed variety in her responses and seemed to have learned from a variety of her experiences. She thus demonstrated the potential for flexible, proceptual thinking and this came through when she was asked for ideas about three.

In the next two chapters the responses of the whole sample will be analysed and Elspeth will be seen in the context of her peers.

CHAPTER SIX

INFLUENCES ON PUPILS

Elspeth is a 'particular' example. In this chapter the other pupils' reactions to classroom experience are examined in order to discover whether Elspeth might be considered a 'generic' example. Further evidence is presented to demonstrate the way in which experience of classroom activities becomes manifested in pupils' responses to interview questions. It will be suggested that there *is* an influence of classroom experiences on the way children perform and describe mental calculations.

The classroom experience involves the teacher and the pedagogic representations that are used. Thus the materials used and language aspects of the teachers' presentations are the focus of attention in lesson descriptions given in this chapter. In § 6.1.1 extracts from lesson observations and children's interview responses are given which demonstrate that the classroom experiences with pedagogic representations are re-presented by pupils. This influence of classroom activities is manifested by the pupils' language and mental visual imagery. The data point to qualitative similarities and differences between the language and mental visual imagery pupils use. The commonality in metaphoric language indicates influences on their learning but individual differences suggest individual mental constructions.

Beside classroom experiences children's own disposition toward a mode of expression has been noted as a possible determining factor of the way children might respond in interview questions. It has been conjectured that a disposition toward a mode of expression of generality and type of visual image might be independent of context. It has also been conjectured that mode of expression of generality and other language use may vary over time. In §6.1.2 evidence of both consistency and variation over time is presented. The data collected in the non-mathematical questions in the interviews are compared with the mathematical items to explore the possibility that pupils' responses to calculation questions might be affected by individual preferences in mode of expression.

The data analysis suggests that there is not an identifiable tendency toward a mode of expression that is independent of context.

In order to identify peculiarities in the language and mental visual imagery of the pupils in the sample the language and mental visual imagery of their teachers and of children in a different school are considered in §6.2.

The lesson observation extracts give an indication of how teachers presented the ideas to the pupils but at the end of the study teachers were also asked some of the same questions that had been given to the children. In §6.2.1 the teachers' responses demonstrate that they chose to describe their calculations with similar language and used similar methods to the pupils' when they performed the same calculations. This suggests that these classrooms constitute a community of practice where teachers and pupils perform and describe their calculations in culturally determined ways. It also suggests that the language is related to the activities.

In order to emphasise that these pupils' mental constructions are distinctive, and may be in part attributable to the influences of the classroom, a contrast is drawn with pupils from another school who have had a different experience from the pupils at Bright Cross. In §6.2.2 the influence of the different pedagogic representations in use at Peacehaven School are explored. This serves to illustrate the ways in which the influences of different experiences are manifested. The metaphoric language and the mental visual imagery of a sample of pupils from Peacehaven School was markedly different from those of the pupils at Bright Cross.

This first chapter of results for the whole sample is intended to provide a broad picture of the manifestations of influences on young children's mental calculations. The second part of the data analysis, 'Comparisons between quantities and qualities', forms Chapter 7. The data collected during the longitudinal study is presented there in a quantitative analysis which seeks to identify characteristics of the different ways in which the influences are manifested by the children.

6.1 Manifestations of Influences

Chapter Five gave some examples of lessons and the way experiences of the activities in those lessons were manifested by one child. In §6.1.1 examples are given to illustrate the commonalities and differences in individuals' manifestations of learning from experiences. In §6.1.2 consistencies over time and context give some indication that the context of questions can determine the response. There is evidence that mental visual imagery may be a characteristic of the individual but differences in expressions of generality seem to be an effective indicator of differnces in learning.

6.1.1 The influence of pedagogic representations

A presumption of this study is that the materials and presentations of the teachers are likely to be a major influence on the way pupils think about and perform calculations. The effect of pedagogic representations at Bright Cross is illustrated by providing examples of lessons which may have influenced the way in which pupils later performed and described calculations. The responses to interview questions are presented to demonstrate the influences which may have prompted those responses.

The teachers in Bright Cross used several representations for the numeral system and to demonstrate two-digit addition and subtraction to Year 2 classes during the course of the year. These included: a number track (from 1 to 105), a hundred square (from 1 to 100), Dienes base ten blocks, numeral cards printed with single digits, coins, chanting number sequences and the written algorithm. The children practised a 'representation-specific' procedure with each of the materials. They added tens, for instance, by: taking ten steps on a number track, going down a column on a number square, having an extra ten-block, replacing the tens digit card with one higher, having an extra 10p coin, saying the next word in the word sequence and adding one to the tens column.

In Y3 and Y4 the representations were almost entirely symbolic. Written algorithms were the focus of many lessons. In Y4, with the introduction of the National Numeracy Strategy 'Framework for Mathematics' (DfEE, 1999), mental calculation strategies were also taught. This teaching of strategies was, like written algorithms, largely in terms of how to manipulate numerals. The emphasis changed from written to oral.

This section gives a qualitative analysis of the influence of pedagogic representations on children's mental constructions.

6.1.1a Separating digits

In the previous chapter a lesson was described in which Dienes 'tens' and 'ones' blocks were manipulated separately. Another lesson from Y2 is detailed here since the language and procedures which this lesson typifies were frequently apparent when pupils were interviewed. On 20/3/98 a supply teacher, Mrs. J., gave this lesson about three-digit numbers. Some language that was subsequently used by pupils is in bold for emphasis:

Mrs. J — You have been throwing dice and all sorts of things. You also looked at rolls of raffle tickets like this:

Draws on white board

	186	

Mrs. J. — What comes next?

Pupil — 187.

Mrs. J — What comes before?

Pupil — 185.

Mrs J — Why wasn't **the 1 or the 8 changed**?

Pupil — Because you are not adding tens or hundreds.

The teacher then replaced 186 on the white board with 199

Mrs. J. — What comes before?

Pupil — 198.

Mrs. J. — What comes after?

Pupil — 200.

Mrs J — But that means I'm **altering the tens and hundreds**. That's because I can't have more than 9 in any column.

To illustrate "**going to**" the next number she held up three numeral cards and then **changed** the units digit card for a different one. She indicated that only the **units digit changed** except when **the 9 is changed for a 0** and then the ten digit is changed as well.

Whilst some pupils played a game (making three-digit numbers using digits shown on dice) the teacher talked to those pupils who had had difficulties in the previous lesson. She showed three-digit numbers by holding up three numeral cards and suggested keeping two **the same** and only changing the end digit to add-on one. She told them to use these cards to help and then gave a new raffle ticket sheet.

When their work was checked half of them had made mistakes by altering the wrong digit on some strips.

The other teachers also made use of these individual digit cards to illustrate changing digits to increase a number by one, ten or a hundred. This language of separating the digits was common, however, even when digit cards were not in use. For instance in a mental warm-up to a lesson pupils were asked how they would add 20p on to an amount of money:

Pupil 1	See what **the first number** is and **add 2 on to it**.
Pupil 2	If you had 59, **take the 9 off** and **add the tens on** then **add the 9 back**.

Here the "first number" is the tens-digit so "it" is the only part of the number that is being altered. "Take the 9 off" does not imply subtraction but rather physically removing the unit digit in order to alter the tens digit then replace the 9.

In interviews conducted with pupils at the start of Y3 to select the sample from Set 2, the language of handling the tens or units and of changing digits suggested that classroom activities such as these had influenced the pupils' mental constructions.

Their responses to the question "What comes next after 379", fell broadly into two categories:

1) counting i.e. 'object collection', for instance:

Joe	Well I thought um I knew that, that, I just **went one over.**
Shaun	**One more added on** to it.
Paddy	I thought, so if I count from 79, I **just have to count one.**

2) digit manipulation i.e. 'object construction', for instance:

Rose	Because take the three hundred off and - just **turn the seven into an 8**.
Myles	Well you know after nine **it comes on to a different number**.

Digit manipulators, however, most frequently gave the wrong answer:

Nikki	Three hundred and *eighty* nine. Well if you go 379, **in the middle it's a 7**, so when you go 6, 7 then if it's 8 so you go "right that's 8 then".
Suzy	400. 'Cause, after 3 comes 4?
Geoff	Four hundred and, seventyyyy, . ten?
Dave	310? Well I thought it went 9, 10 , so it's 10.

Some children who gave the correct answer separated the three hundred then counted:

Pam	I knew that after 79, eighty came, and **300 would come first.**
Nora	I thought when you go 1, 2, 3, 4, like that I thought that and then I went to 79 and like 80 and I just **added a 3 in front.**

The children whose mental construction for 'increasing a number by one' involves separating the digits seem to have been most influenced by the pedagogic representations involving physical materials. For Robert experiences related to 'rounding' on a number track (like the raffle tickets) were more obvious. His language was of position.

Robert	You get 370, in your head, and then you like think is it **lower**, **is it low** or **is it high**, and I thought, it's 370. So you had to go three hundred and eighty. **Like a square** and it's (379), **it's in the middle**.

6.1.1b The variety in the language of manipulation

The metaphoric language of manipulation, movement and position suggests that many of the children have mental constructions formed from their interactions with the pedagogic representation. "Take the three hundred off", "added a 3 in front", "300 would come first" seen above are sufficiently different to suggest that these are the children's own constructions. They are also sufficiently similar to suggest that they have a common source in the pedagogic representations. Elspeth's' "..in one hand you have the tens ..." was the most blatant re-presentation of an experience but others signalled the influence:

Terry	I **took the tens** then I added them together and then **I took the units** and added them together.

The use of the word 'took' indicates that the handling of the digits separately, even if not necessarily handling the Dienes blocks, had become a metaphor.

Many other words are also indicative of an 'arithmetic is object manipulation' metaphor:

17 + 8	**I make** it 8 and 7 separate, and **make** the 8, 7, so 7 add 7 equals 14 and add 1 equals 15, add the ten equals 25.
48 + 23	You **take the 8 off** and you take the 3 off so you add them back - and you think of tens and **put them on** and you add them all up.
48 + 23	Well I **got the tens** and then added them up then that make, made, 60 and then 8 and the 3.
97 + 10	**I move the 7** and then I knew which one, I know I got to add a ten on, I can't add a ten so I **put the one in front** of it and then a **nought in the middle.**
97 + 10	You **keep the** - seven, the units, and you just, if you knew that hundred was after ninety, you just add, ten and seven takes it, so we have **nought where the nine is.**
97 + 10	I just **knock the 7 out** and just put ten onto 90 which makes a hundred then I just added on 7.
Round 246	6 is nearer to 10 than nought and if you just add **them** on you can just do it without the - like **push** the hundreds - away, and the tens away and then you just **do it** without it then you **put them back together** again.
Round 2462	**Look at the** 5 and which is the higher number on the units, add on to the ten
30 + * = 80	I know that 3 add 5 is 8 so just **turn it into** tens.
30 + * = 80	I know 5 add 3 is 8 so you just add, **change** the 30, the 3 **into** the 30, and add 5, that should be 50.
200 more	I was trying to **get away** all the units um and tens and hundreds, and, add another one on to the thousands.

It would seem unlikely that such expressions were chosen deliberately from a range of options simply to conform with the pupil's perception of what was the required response for the interview. These seemed spontaneous responses and show an influence of classroom activities on the language.

6.1.1c Written algorithms

Written algorithms were developed in the classroom based upon activities with Dienes blocks. For instance in a lesson to Y2 set 1 (6/2/98):

> Mr. K. told the class they would be playing the 'exchange game'. He held up 3 'tens' and asked what was in his hand, the pupils replied "thirty". He explained that pupils had to get rid of the thirty by throwing dice and putting back the number of cubes shown on the dice. He pointed out that this would need a 'ten' to be exchanged for ten 'little ones'. He repeated his performance with a pile of 'ones' to show that a 'ten' is ten 'ones'.
>
> Pupils played in groups, taking turns to replace into the box the number on the die. There was some confusion so Mr. K. demonstrated by drawing three 'tens' on the board then rubbing one out and replacing it with ten 'ones' (Fig 6.1). He talked of 'swapping' so that "you still have the same number. It's like unwrapping a packet of Rolos, you still have the same even though one lot is unwrapped."

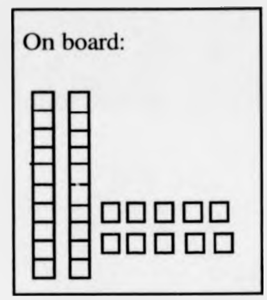

Figure 6.1 Teacher's diagram for thirty

> The pupils returned to the game.
>
> Later in the lesson A3 sheets of paper (Fig 6.2) were distributed and pupils, in pairs, put 3 'tens' blocks and 4 'ones' in the top row. Mr. K. explained that he wanted them to take away 16. He told them that "on paper we always take away units first. So we will pick up a 'ten' and swap it for 10 'ones'. Now take away 6 and then a ten."

Hundreds	tens	units

Figure 6.2 Template for addition of Dienes blocks in columns

In a lesson one month later this experience was part of the background to Mr K's explanation of how to do 48 take 14 simply as "1 ten off 4 tens is 3 tens then 4 off of 8 is 4." Here he still dealt with tens and units separately but tens first.

The addition algorithm had been introduced similarly. Unit cubes and tens blocks were placed in the appropriate places, units counted and ten exchanged for a 'ten' which was then placed with the other 'tens' and added up. After further practice pupils dispensed with the template and wrote the numerals in the conventional vertical format in their exercise books. Dienes blocks were initially available to help with the calculations.

In a lesson with Y3 set 2 (4/11/98) Miss P quickly re-capped the written algorithm for addition. Figure 6.3 illistrates the written algorithm format she drew on the white board:

Teacher	I don't know if, when you were in the infants, you did sums like this. Which side do you always add up first?
Pupil	The right.
Teacher	Which is?
Pupil	The units.
Teacher	This adds up to 15 we **put the 5** here and **carry the one** - it's not really one is it?
Pupil	Tens.
Teacher	And **add up the tens**. Be careful to write these sums properly.

On board:

T U

2 9

+ 1 2

2 4

Figure 6.3
Addition algorithm.

The expressions "the units", "put the five", "carry the 1", "the tens" are all appropriate in the context of the written calculation. However, these expressions also occur in the language of children who are explaining what was *in their heads*, after they have performed a mental calculation. This is taken as evidence of a mental construction based on these classroom experiences. Such expressions are thus classified as metaphoric. They use the language of one context (written algorithm) in describing another (mental calculation). This was common in the responses to "48 add 23", for instance (Y4/1):

Dennis	I was adding the 8 and the 3 Which makes 11 and then I was adding the 2 and 4 and that equals 6. But then **it needs to be carried by one**.
Mandy	Well it was like tens and units and all that, and then I just **added the units then added the tens**.

Irene	Well **I added 8 and 3**, which makes 11, then I added **the 4 and 2** which makes 6 but the 11 is over 10 so I added another one, **another ten to the 6** to make 70.

The use of the language of the algorithm indicates that the pupils have been influenced by their practice of both the techniques and the associated language in their thinking about mental calculation.

A distinction *was* made by teachers between the written algorithm and mental calculation i.e. 'In written calculation start with the units. Start with the tens when calculating mentally'. The distinction was only in which digit to start with. For example in a lesson with 3(1) (7/1/99) Mrs I recapped first how to add two-digit numbers when they are written horizontally. She first wrote 99 + 35 on the board, took the answer then asked how pupils had done it. The class were evenly divided between those who had added tens first and those who had added the units and then the tens. No other mental strategy was mentioned by Mrs I or the pupils. The lesson continued. Notice that the examples used (Fig. 6.4) might have provided a vehicle for discussion of a mental strategy rather than the algorithmic approach:

Mrs I	What did we do in the Abacus books when we were adding horizontally?
Pupil	Tens first.
Mrs I	I have a series of numbers.

Writes examples on board (see fig 6.4) Pupils are asked which are right and which wrong.

Mrs I	How did you add them up, what did you add first?
Pupil	Tens.
Mrs I	Good girl.

On Board
35 + 75 = 100
35 + 65 = 100
25 + 70 = 100
65 + 45 = 100
30 + 70 = 100
5 + 85 = 100
45 + 55 = 100
85 + 15 = 100

Figure 6.4 Classroom addition examples

Later in the lesson the written algorithm was practised:

Mrs I	If you've got something bigger than 10 in your unit column what do you do? What's the word I am looking for?
Pupil	Carry
Mrs I	Some of you forget even if you put it down.

A little later in the lesson she reminded them:

Mrs I	We only add tens (first) when we are doing it horizontally or in our heads.

The teachers, before the implementation of the Numeracy Strategy, saw this order as the chief distinction between written and mental calculation. Separating tens and units was emphasised for each.

6.1.1d Training in procedures

In common with Elspeth many other pupils used "you" to indicate that they were using a general rule. For instance in Y3/1 "97 + 10" elicited many such responses, e.g.:

Jeremy	I just like added the ten on because when **you've** got a number like 97 and **you** add ten **it has to be the same number** but the front number has to change.

This may have been as a result of reflective abstraction of the rule from Jeremy's experiences of classroom 'counting in tens' activities or digit card manipulation. It may, however, have been influenced by experiences such as:

Miss P	50 from 82?
Pupil	17.
Miss P	What has it got to end in? You know that. (Writes on board, see Fig 6.5) 8 tens take the five tens off then just bring the 2 down. That's a lot quicker. 97 take off 7 tens? That way **the units number will always be the same**.

On board:

T U
8 2
5 -
3 2

Figure 6.5 Subtraction algorithm

Most of the teachers used "you" to indicate what to do and encouraged pupils to do the same. For example Miss P had introduced the idea of "splitting a number in two" on the day previous to an observed lesson and had shown a method for halving an odd number. In this lesson to Y3 set 2 (7/10/98) she checked if pupils could remember what to do. As the child talked the teacher drew a diagram on the white board (Fig 6.6):

Miss P	How to half an odd number equally? Do you remember what to do?
Pupil	11 is made out of 10 and 1 and **you put** ...
Miss P	Carry on "**you put** ..."
P	**You put** half of 10 is 5 and half of 1 is half so it's 5 and a half.

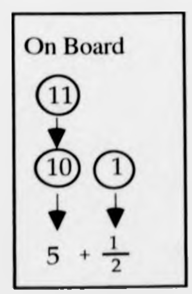

Figure 6.6 Diagram for halving eleven

In an interview two weeks later half the 20 children interviewed could not answer the question "What is 9 split in two?". Two said "4 and 5". The rest gave responses which quite accurately re-presented this experience when asked "What was in your head when you were thinking of that?". For example:

Dave	It's like when she splits in half and writes the number in the halves. Well **you** put a kind of circle, a number in a circle and then another number that it can be split into half, then 1 , then **you** put the number in the half.
Jeremy	**You** get um 9 like **you** put 1 - and 9 and then **you** put a little arrow down and the one's like a half and - **you** add it up, and - the number nine **you** put nine and a half?
Nora	Well - **you** get 9 is made out of 8 add 1 and half of 8, is, 4, and half of 1 is half, - so **you** add 8 and a half together and **you** get - 4 and a half, 8 and a half.
Bobby	**You** say half of them - it's made up of 8 and one - then **you** - say what half of 8 is, and then - **you** can - say - what half the 9 is.

This example serves to illustrate the way in which recall of procedure may be characterised by the use of the word "you".

6.1.1e Counting as a procedure

Many pupils used counting when asked to perform calculations. For instance:

17+8	**Counting-on** to - 8 and it ended on 25, it wasn't actually 1, 2, 3, 4, 5, 6, 7, 8, I just counted the numbers I was saying
17+9	Well I just got, the number and just counted-on. Just started at 17 and then **counted-on** 9.

'Counting-on' is a strategy that many children are thought to develop for themselves but at Bright Cross it has been specifically taught. For example Miss P (11/11/98) showed counting on as a method of addition both on the number line and using cubes:

Miss P	Remember yesterday I mentioned something about 'counting-on'? Well there is a quick way for doing 16 add 5. Count-on 5.

Miss P drew a 0 to 30 number line on the white-board. She put a cross under the 16, counted 1, 2, 3, 4, 5, and put a cross under 21.

Miss P	If you have 87 add 5 there is no way you can have a number line but you can count on from 87. 88, 89, 90, 91, 92 (shows on fingers)

Multi-link cubes were distributed.

Miss P	Now you have cubes. I am going to give you a sum by counting on. Get 8 cubes joined together and 7 separate cubes on your desk (not enough for everyone) OK change it to 6 add 5. Pick up each of the 5 and join them on saying 7, 8, 9, 10, 11.

Pupils copy.

Miss P — I'm going to do bigger numbers. 27 add on 6 - you will go like this: you've got 27 in your mind and you will count like this (holds up 6 cubes joined together and counts them) 28, 29, 30, 31, 32, 33. Try 32 add 8. (pupils count cubes).You should have done this : 32 in my mind, count on 33, 34, 35, 36, 37, 38, 39, 40 (points at cubes).

In my mind 77 add 11 look 78, 79, ... ,88 write it down 77 add 11 equals 88 (writes on board). Listen to the numbers , do it in your mind 43 add 7. (pupils count cubes). Show me how you did it Pat.

Pat — I got the 7 and then I added them on.

Miss P — How did you add the seven.

Pat — I went 44, 45, ...

Miss P — Good I want to see you all do it this time.

Much of the language of 'counting-on' used in this lesson was apparent in pupils' responses. For instance for "17 add 9" there were *explicit* re-presentations:

Teddy — I was just, **like counting cubes in my head** - just adding on one at a time, nine.

The experiences were also *implicit* in the language used in many of the other responses:

Hester — Well **I got 17** and then 18, 19, 20 and I got, kept **counting and got 9 on my fingers.**

Many pupils frequently used counting for addition of single digits.

6.1.1f "Because" in explanations

The use of "you" for descriptions of procedures is common in everyday life and it is not surprising that it should be so in the mathematics classroom. Its use, as we have seen above, is also reinforced in the classroom discourse. There were other examples which illustrate the similarities in language by teachers and pupils and which suggest that pupils are being encouraged to adopt the speech styles common in mathematics. For instance explanations involving "because" were used both by teachers and pupils. Again this is not uncommon in everyday life but pupils frequently followed the teacher's lead in the use of the word. For example in a lesson with Y4 Set 2 (29/9/99) the class checked which questions they should do in an exercise in the textbook about measurement:

Mrs F — Now if you look at the book you can't do part (g) or (h) **because it's in the hall.**

Pupil — We can't do (h) either **because it's the breadth.**

In a lesson on 'adding and subtracting near-multiples of ten', using a hundred square, Mrs F first talked about 36 add 19 to Y4 set 2 (11/1/00):

Mrs F	What is 19 nearly?
Pupil	20.
Mrs F	If you wanted to add 20 to 36 how would you do it?
Pupil	Go down.
Mrs F	So 36 go down 2 is 56 but we have to take 1 **because** we are only adding 19 not 20.

More examples of addition were worked through as a class then subtraction was similarly described. Mrs F wrote 55 -19 on the board.

Mrs F	See if you can do it in your head but using that strategy.
Pupil	36.
Mrs F	Tell us how you did it.
Pupil	**'Cause** 19 is near 20 so I just took 2 tens from 55 and added one.

Pupils also showed that they could remember and use an explanation from one year to the next. Miss P gave a lesson on 'telling the time' to Y3 Set 2 (14/10/98):

Miss P	Why is it quarter past? **Because** it has only gone quarter way round.

Mrs F asked the same children, Y4 Set 2, the following year (25/1/00):

Mrs F	We don't say 15 past 8 we say quarter past 8. Why?
Pupil	**Because** it is a quarter of the clock.

Notice that this is not a copy of the teacher's words ("quarter of the clock" instead of "quarter way round") but the use of "because" in explanation is common to both situations.

Pupils also demonstrated that they could use "because" appropriately in their own reasoning. In a lesson about odd and even numbers Mrs F put a cross on the even numbers of a number square as pupils in Y4 set 2 said each one (29/2/00)

Mrs F	All the even numbers have a cross. Is 81 going to have a cross on it?
Pupil	No.
Mrs F	Why?
Pupil 1	**Because** it is an odd number.
Mrs F	Will 67?
Pupil 2	No **because** none of the ones with a 7 has got a cross.
Pupil 3	You could do it like **because** 7 isn't in the 2 times table. No number in the 2 times table has a 7.

These examples suggest that pupils may seem on occasions to simply copy the teacher's speech. They also demonstrate their own mental constructions when using the words independently and appropriately on other occasions.

6.1.1g Mental visual images of pedagogic representations

Some pupils, like Elspeth, described pedagogic representations when asked if there had been anything to see in their heads after they had performed the mental calculation questions:

Mandy	A number line and I went to 17 and counted on 10
Clara	Lots of things lying around - lots of cubes in a big long line.
Myles	Well it's just like Miss P's rulers.
Malcolm	Yeh there was the numbers, like with cards and I took um, like, one of them away, and then it changed to a 9.

Only in 5% (66 out of 1158) of all responses to interview calculation questions in the longitudinal study were teachers' physical representations evoked as images. In three times as many responses (218) pupils claimed to have seen 'numbers' and in a third of these (77) a mental visual image of written 'sum' in either horizontal or vertical format, was described :

Terry	Little picture of a sum. Like 17 there, 9 there, add there, equals there.
Peter	The numbers were underneath each other for a sum and I took away the numbers and I put it in and added tens like we'd write down.

Very few pupils (21 responses, i.e. less than 2%) mentioned mental visual images of other objects or episodic images after they had performed a calculation. Only one pupil saw objects for counting, and he did so on three occasions for "17 add 9"

Simon	Teddies. Just took them away as I counted 9 of them.

Only four pupils mentioned mental visual images of fingers and others saw teachers or themselves engaged in classroom activities

Spontaneous description of mental visual images occurred in under 10% (102) of responses to calculation questions but there were double that when children were asked explicitly if they had seen anything after they had performed the calculation. In the final interview pupils were also asked to try to 'mentally visualise'. In response to "Try to picture something in your head to help you work out 65 take 29" 15 of the 26 pupils reported to have a mental visual image. One reported a mental visual image of a teacher, two reported a number line or ruler and the other 12 all saw numerals. Half of these saw a vertical written-algorithm layout.

When asked to try to get a picture of a number line in their head, in the final interview, all but two said they were able to do so. When asked to picture 36 on the line then add 20 only 6 pupils claimed to use their image of a line. The others often did not even use a 'holistic' 1010 method related to the number line. For instance:

Mandy I added 2 onto the 3 and that made 5 then I added the 6 and that makes 56.

Whilst there may have been a mental visual image the mode of calculation was not necessarily in accord with it.

6.1.1h Summary of manifestations of learning

The examples given above illustrate how the pedagogic representations may influence the way in which the children respond to mental calculation questions. Physical materials have been used by the teachers to demonstrate procedures. The language related to the manipulation of the materials, or to manipulation of numerals, is apparent in the children's answers. 'Counting-on' and written algorithms have both been taught as procedures, with and without materials, and the classroom language of these procedures was used by the children, often phrased in terms of what "you" do. Similarly the classroom culture of explanation and seeking explanation was apparent when causal connectives such as "because" were used by teachers and pupils.

An explicit mode in which children might re-present their experience, i.e. by describing mental visual images, was uncommon. Even though part of the agenda for the interviews was mental visual imagery and children became used to "Was there anything to see there?" only 10% of responses involved a mental visual image being described without the explicit question asked. In total 30% of responses involved some mental visual image but some of these may simply have been described to satisfy my persistent questioning.

Whether 'real' or made up to satisfy me the pupils had choice over what to describe and very few chose to describe images of physical materials. What they chose to describe indicates what they feel to be important and what they can recall from previous experiences. There is thus an indication of the influences on their thinking in what mental visual images they report, even if they were manufactured because of the unnatural interview situation. When asked specifically to imagine something, pupils appeared to make little use of their image for the calculation. The mode of calculation was not in accord with their mental visual imagery of the number line or of the mental visual image of the vertical calculation.

Pupils demonstrated the classroom influences on their modes of expression by their use of language, counting on fingers, or use of mental visual imagery. Much of this language and associated procedures may be common to pupils in other schools in the UK. My suggestion is that it is likely to have been influenced by experiences similar to those described at Bright Cross.

6.1.2 Consistencies over time and across contexts

It has been suggested in the literature reviewed and in the data analysis in the first part of this chapter that the pedagogic representations, both the materials and the language used in the classrooms, will influence and constrain the way pupils perform their calculations. The representations will influence the way pupils describe calculations and the mental visual imagery which accompanies the calculation. Yet the examples given above show *variations* between pupils' responses. These could be due to individual differences in cognitive structures but could simply be due to a preferred mode of expression.

If pupils had a preferred mode of description or mode of mental visual imagery this could explain the differences in the ways in which pupils describe calculations and the images that are reported. This disposition could result in individual pupils using similar modes of expression over time or across contexts. The following analysis will lay the foundation for the analysis of consistencies over time and over context which follows in Chapter Seven.

- Comparison of responses over time is achieved through analysis of similar questions in each interview. These could indicate changes which may occur over the period.
- Comparison of responses across contexts is achieved through analysis of mathematical and non-mathematical items. These could give an indication of any disposition by pupils toward a mode of expression, or a type of mental visual imagery, common across contexts .

The indications from the qualitative analysis presented here is that some children show consistency over time in their responses to calculations of a similar type. Other children show variations. Evidence is also presented in this section to demonstrate that the pupils did not all show consistency of language, or mental visual imagery, in either the procedural or 'concept' interview items.

One characteristic of pupils' explanations of their calculations is their use of causal connectives. The data on use of "because', "so" "if" and "then" that will be analysed in Chapter 7 arose spontaneously as pupils described their calculations, procedures or images. However a pair of questions was also included in Y4/2 to explore further the use of causal connectives. Pupils' use of this language of logic seemed to be determined by the context rather than some disposition of their own.

The picture that emerges is one of variation within and between contexts in terms of the language used and there is no clear evidence of a predisposition toward a mode of expression by individual pupils which is apparent across contexts.

6.1.2a Consistency in language over time

There was an initial indication in preliminary interviews conducted in Y2 that pupils might have some consistencies in their language over time:

	March 81 add on 10	**July 38 add 10**
Christine	Well I - sort of **ignore** the units for a minute and just added like a ten on.	I would **leave** - the units and just add on ten.
Hannah	Because you just add a ten on - so you **keep** the - the last number then you add a ten onto - the tens.	Well you **leave** the eight as it is and add one to the three.
Jack	If you just go eight, nine then you just make it into a ten and you **put the one on**, ninety one.	I know my ten times table and then I just **put the eight on.**
Ann	**Add a ten would take you to a next column** and the unit would just be the same. You know what it is just saying, instead of saying eight you say nine. **Leave** the one how it was.	Well you just **add another ten onto the tens column**. So three so you count one, two, three and the next one is four. It's the columns.

The longitudinal study allowed comparisons over a longer period and most pupils showed greater variety than these examples suggested, for instance Irene in response to the "What was in your head ...? after the 'rounding' questions. (Note: 246 rounded to the nearest 10 is abbreviated as 246(10) etc):

246 (10)	I had the 200 - and I just - **forgot about the 2** - then knew that 6 was near to, well 10, and not nearer to whatever like 40.
2462 (10)	**Looking at** the last two numbers and seeing what's the nearest ten.
239 (100)	'Cause I know that, **if it's over 4** it goes to the next hundred up.
2462 (100)	Well one of those lines where, like, you use **like a ruler** but longer. And then just, like, 2000 or whatever.

Irene used a different metaphor to explain her reasoning on each occasion and this provides evidence of a mental construction which incorporates a variety of experiences. Jeremy also showed variety in his language in the questions about counting. In each, however, there is the experience of manipulating numerals. There is evidence that 'adding-on' is seen as equivalent to counting:

Before 380	I just like **thought of** three hundred and **took 1 away** from the 80.
Before 2380	I had like a number in my head, and then **took away 1** and **saw like the number change.**
After 12386	I just tried to **remember the number** and then just **added on** 1.

These few extracts indicate some aspects of consistency in language for some pupils in some circumstances. The global picture that is revealed by quantitative analysis does not, however, confirm that pupils, in general, have a fixed mode of expression common across calculation contexts, nor even, when the calculations are similar, over a longer period of time. This is analysed further in §7.3.2b.

6.1.2b Consistency of method over time

There were some signs of consistency in the method used by some individuals for the question "48 add 23" over the period. Jack for instance gave the following responses:

Y3/1	I added the tens first, and then I added the others, the units, then put them both together.
Y3/2	Just adding numbers.
Y3/3	I just added the tens and added the units and then added them both together.
Y4/1	Just - adding the tens then adding the units.
Y4/2	Just added the tens and the units. I did the tens, then I did the units 68 add 3, 71.

Jack's language is of separating tens and units but he added the tens first. This may have been his own idea but may have been a result of explicit instruction such as the lesson by Mrs I in §6.1.1c. Ellain, like Jack, may also have been influenced by lessons such as this and was similarly consistent in her descriptions but showed more of the influence of the written algorithm. Like Jack, however, she started with tens rather than units when calculating 48 add 23:

Y3/1	So you take the 8 off and you take the 3 off so you add them back, and you think of tens and put them on and you add them all up and it makes 60 and you add the 3 onto the 8, and it makes 11 and you know that it is 60, . 71
Y3/2	I had a little picture in my head, because I had - tens and units, I had that in my head and I saw the like the big numbers first and then I saw the little numbers and then I added the big numbers up on that side, and then I added the little numbers on to the, whatever it makes.
Y3/3	It was the tens again, and I added the 2 onto the 40 and it came to, 60, and then I added the 3 onto the 8.
Y4/1	I had the 40 and the 20 and then I had the 8 and the 3 and that I had in my head and then I added the tens together and I added the units together and that came to an answer.
Y4/2	I added the 20 to the 40, so that makes, 6, 60 and then I added the 8 and the 3 together and then that made the number and then I added the units on to the other tens.

This consistency in algorithmic language was not common to all. Others used more holistic language in that 'tens' and 'units' were not mentioned. John, for instance:

Y3/1	48 add 20 comes 68 and then you add 3 on.
Y3/2	Add the 20 is 60, then got 8 and then add the 3 on.
Y3/3	40, that's 60, yeh then 8 add 2 equals 10.
Y4/1	That equals 60 and then 8 add 3 is 11.
Y4/2	I knew 40 add 20, 60 then 8 add 2 is 70 and then you just add the one.

Jack, Ellain and John do show consistencies in their language and method. This is a relatively easy question and their mental construction seems to involve following a routine. Most other pupils showed a variety of influences. Hester for instance gave the following responses:

Y3/1	I pictured it to work it out. I had some cubes and - I was thinking of cubes but I didn't have a picture
Y3/2	I, added, 20 on and the 8.
Y3/3	I added up, 2, 20, then I put the 3 on it
Y4/1	Oh I just lost the number (counting on fingers).
Y4/2	I had just like - a hundred square and I added up - so I went 50, 60 68 add 3, 71

Hester did not appear to have a mental construction that had evolved to include a preferred method. She seemed to re-present a variety of experiences and there is no sense of generalisation.

As with use of metaphoric language of manipulation there is some evidence that some individuals showed consistency in the methods they employed over the period. Others showed a greater variety.

6.1.2c Consistency in 'generality' across 'procedure' items

The interviews of Y3 Set 2, used for selecting a sample for the main study, included a pair of questions "Tell me how to add 20 to any number" and "Tell me how to write the date". This gave an indication of a similarity in the style of response that some individual pupils gave for each.

Two of the twenty children interviewed gave a procedure in 'general' terms for both questions:

	how to add 20	**how to write the date**
Shaun	**You** just add a ten and then again.	Just like get what the day it is and then put the month and then the years.
Bobby	**You** count how many, ones, um units, to make um the sum and then **you** add the rest on.	**You** put the - what day it is, the month it is and the year it is.

Six of the twenty gave an illustrative example for each, indicated by the use of 'if' and 'like'. For example:

Jonathan	**If** there was **like** 47 or something **like** that then **you** knew it would be sixty something so **you** would just go 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, on **your** fingers and do it again.	**If** it was **like** September the 6th I'd write, and - 98, **You**'d write 6 dot nine, nineteen ninety eight .
Geoff	**You**'ll **like** have 20, then **you** have 5 there and then **you** just like 20 and then **you** add on like 5 **if you** were ...	**You** write - well **if** it's the 20th of October then **you** write 20 and then t, h and then - c, t, o, o, b, e,r.

However the remaining 12 pupils were not consistent in their style of response. Even the consistencies of responses in these examples may have been due to the fact that the pair of questions were together in the interview. Pupils may simply have followed their own example when replying to the second. For the main study interviews the questions were separated.

In the main study there were similarly a few pupils who gave 'general' expressions in both mathematical and non-mathematical contexts. In questions requiring descriptions of familiar procedures 'general rules' were quite commonly given. Thus comparing answers to "Tell me how to add 23" with "Tell me how to cross the road" (both in Y3/1) showed that 4 pupils gave a 'general' expression for each. For example:

	how to add 23	**how to cross the road**
Myles	You like write down 2 and then 3 and put an add sign there and put something there and then two lines and you add them all together and you get the answer.	You need to look where the traffic's coming and you need to look and listen and then if there's any cars coming you stop, and if there's no cars coming you just walk across.
Clara	Use a number line, or a ruler, jump to your next number.	Look left and right see if there's any cars coming. If there isn't, walk across.

One pupil gave a generic example for each and in each instance it was a special case:

Irene	If you had 23 and add them on to like add 7 say, 1, 2, ..	If there's a traffic light you just press this button and wait till there's (no cars).

The majority of pupils, however did *not* show any consistency in their responses to mathematics and non-mathematics procedure items. A quarter of them gave a generic example for the numerical procedure and a general rule for the familiar everyday procedure of crossing the road.

Individual pupils were not consistent in the type of 'generality' they used when the "Tell me how to add 23" question was repeated seven months later in Y3/3. Only seven of the 26 pupils gave the same type of 'generality' and five of these gave a generic example for each. One of them chose the same number in each instance and gave a similar description:

Christine	I'll do 23 add on 30 it's easy what I normally do is I just forget the 3 altogether and just count in my head, and that's 50.	I'll pick 30, well I normally start with the units first but there's nothing to add on to the units so you'd leave that as it is.

The important point here is that these were exceptions. The other pupils gave a variety of combinations of 'particular', 'generic' and 'general' responses in each instance.

Tables of categories of 'generality' and 'image' have been compiled for all pupils across contexts. To maintain the flow of the general discussion these are not included in the body of the text but are available as Appendices.

The categories of 'generality' allow comparisons between individuals and between responses for each individual . When all questions of this type are considered (Appendix 3.1 has full summary) there is no evidence of consistency for individuals in either the mathematical or non-mathematical contexts nor across the contexts. No pupil gave the same category of response in each question. Thus pupils can not be labelled as consistently using 'particular' 'generic' or 'general' expressions in these procedure questions.

What is clear is that in these procedure questions 'particular' responses were uncommon. In other words few children gave a response using a particular example. Where an example was given by the pupil it was usually 'generic' in that it was used to explain the procedure as, for instance, Christine did above. All pupils in the sample gave a 'generic' or 'general response in at least 8 of the 11 questions. All pupils used a mixture of categories in each context.

There was an indication in the total numbers of each category for the whole sample (Appendix 3.1) that pupils were more likely to express themselves in 'generic' or 'general' terms in procedures with which they were most familiar. The very high level of 'general' expressions (21 out of 26) for " ... cross the road" suggests that everyday situations are most easily expressed in 'general' terms. This could also have been a result of the pupils having been taught the rule. When asked how they knew the rule, however, few said they had been taught it. The highest number (11 out of 26) of 'general' responses to mathematics procedure questions was for " ... multiply a two-digit number by 5". It is possible that this was because the procedure had recently been taught.

The two "... add 23" questions (given in Y3/1 and Y3/3) showed a trend away from 'no response' toward more 'generic' and 'general' responses. This suggests that the pupils as a whole were more able to describe the two digit addition procedure in more general terms as it became more familiar. Difficult mathematical procedures " ... find a third" and "...

find two thirds" were frequently not answered. When answers were given the responses were not often in the most general terms (4 and 5 'general' responses respectively).

When pupils' accuracy in the calculation part of the interviews is taken into account there is not a strong distinction between high and low accuracy pupils in terms of the type of 'generality' they used in these procedure questions. (Appendix 3.1 orders pupils in ascending accuracy in calculation items). Low- and high-accuracy pupils were all capable of describing a procedure in general terms in both mathematical and non mathematical contexts.

Some of the least accurate pupils used 'general' expressions even for difficult mathematical procedure questions and some of the most accurate did not necessarily do so. Two of the most accurate, Kath and Jack, had a high proportion of 'generic' and 'general' in both the mathematical and non-mathematical procedures, but the same could also be said of one of the least accurate, Myles. The three pupils with the highest number of 'general' responses were, however, three of the four most accurate.

This does not suggest that a particular mode of expression for procedures is a characteristic quality of pupils of different accuracy levels. There is evidence, however, that higher accuracy pupils may have a tendency to give 'general' responses. This point will be examined more closely in §7.3.1.

The overall impression is that context may influence the mode of response. It appears that easy, familiar procedures are most frequently explained in 'general' terms. Pupils' differences in their responses can not be attributed to a disposition toward a mode of response. Differences in pupils' responses in these procedure questions seem not to be easily associated with their ability to answer calculation questions correctly.

6.1.2d Consistency in 'generality' across 'concept' items

In the interviews of Y3 Set 2, used for selecting a sample for the main study, the children were asked "What is the first thing that comes into your head when I say 'hundreds, tens and units'?" and "What is the first thing that comes into your head when I say 'sentence'?" All but three of them were consistent in their type of mental construction. The majority responded with a 'particular' example of each. For instance:

	"hundreds tens and units"	**"sentence"**
Pam	Two hundred, and sixty, seven.	I jumped on a cat.

Three talked in more 'general' terms, for instance:

Joe	Well just put, the hundreds, but it has to be over a hundred - then you have to do tens, and then you just put the um ones next.	You just write down - lots of words, until like you get to a place where you can put a full stop, to make a sentence.

Some gave responses which used an illustrative example. For instance:

Robert	Well I get hundred, I get **like** one, oh, oh, hundred.	It's **like** doing sentences - like "One day a boy walked the street" It's **like** that.

In the main study the mathematical and non-mathematical concept items were separated in the interview to avoid children simply following their own example for mode of response. Even so there were again high levels of consistency between questions in the same interview when the words used were similar in terms of their familiarity to pupils. Thus "centimetre" and "shadow" had both been the subject of recent classroom activities when they were used in the "What is the first ..." questions in Y3/1. Nearly half (11 of the 26) of the pupils gave 'particular' response to each. e.g.

	"centimetre"	**"shadow"**
Malcolm	I just saw lines, with numbers.	A black picture of someone.

whilst five gave a 'generic' or 'prototypical' response for each. For instance:

Hester	You measure in cms.	The sun's out and you're standing there and you've got a picture of your self, and um if you didn't have any light and you weren't standing there you wouldn't have any shadow.

The other pupils gave mixed responses. The pair "three" and "ball" which are both very familiar concepts for these children again evoked similar types of responses to each from 14 of the 26 pupils. When the pair involved difficult words in each context ("million" and "adjective") only a quarter of responses were of the same type for each. When the maths item was not so familiar and the non-mathematics word was a high-imagery everyday concept ("fraction" paired with "Christmas" and "polygon" with "animal") then the number of pupils giving similar responses was less than half.

Responses to all the 'concept' questions were again compared (Table of categories of response for each pupil are in Appendix 3.2). The following points were apparent:

- In comparison to procedure questions the concept questions evoked more 'particular' responses and more pupils showed consistencies.
- Mathematics concept questions evoked a higher proportion of 'generic' responses than non-mathematics questions.
- Seven of the pupils (including Elspeth) used either all 'particular' or all 'generic' when they responded to mathematics concept questions. Fifteen pupils had all but one of their responses of the same category.
- Five pupils were consistent in their response to non-mathematics questions giving either all 'particular' or all 'generic' when they responded. Fourteen pupils had all but one of their responses of the same category.
- Only one pupil, Suzy, who answered only 4 out of the 10 questions, gave 'generic' responses to each across the two contexts. Six pupils had all but one of their responses of the same category in both contexts.
- Though there are signs of consistency for these pupils the others showed little consistency either within or between the contexts.

As with procedure questions a comparison between pupils of different accuracy levels does not reveal a strong association between category of expression in these questions and accuracy in calculation questions.

The analysis in this and the previous section suggest that pupils, in general, do not necessarily have a disposition toward a type of 'generality' in response to questions about procedures or about concepts in mathematical and non-mathematical contexts. There is not sufficient evidence of consistency to suggest a disposition by individuals' toward a type of expression of generality which is consistent across contexts in either procedural or conceptual items.

There was some indication that context may determine the mode of response in that a higher proportion of mathematics concept questions, than the non-mathematics context, evoked 'generic' responses. 39% of all mathematics responses were 'generic' compared with 22% of non-mathematics responses. There appeared to be a higher proportion of 'general' responses in non-mathematics contexts but this is almost entirely due to a high number of 'general' responses for "... adjective". Here nearly half the pupils gave a definition ("it's a describing word") and this may be attributed to the fact that this had been taught in school.

6.1.2e Consistency in 'image' across contexts

These same questions "Tell me how ..." and "What is the first ..." also allow comparison between the mental visual images evoked over the period of the interviews and in the two different contexts. In the responses to maths procedure questions 13% involved a reported image (28 of the 207 responses) whilst 49% of the non-maths procedure involved a reported image (38 out of 78 responses). In the concept questions 34% (44 out of 130) of responses to maths words involved a reported image and 44% (57 out of 130) of non-maths words evoked a reported image. It is important to note that there was a higher proportion of mental visual imagery related to non-mathematical procedures and to non-mathematical words. This suggests that context may be a factor in determining whether a mental visual image is evoked.

"Tell me how to cross the road" evoked a mental visual image for 13 pupils and 14 pupils had an image for telling the time. These were, predictably, mental visual images of roads and clocks and frequently episodic in nature. For example:

Myles	I just saw a picture of some people walking across the road.
Kath	I was picturing a road, I was picturing the school road actually.
Paddy	Just a big clock.
Malcolm	a clock on this, um like a cardboard and it says like past.

When pupils were asked how to draw a 'times-table-picture', which had been drawn in class three weeks before the interview, 11 of them reported an image of the one they had drawn. This suggests that both 'everyday' procedures and procedures related to spatial activities may evoke more images.

The very few images related to mathematical procedures were mostly 'symbolic'. "Tell me how to multiply a two-digit number by 5" evoked images of symbols from 6 pupils. For instance:

Dennis	I just saw the number 54 times 5. In a multiply sum.
Bobby	The numbers in the sum like 26 and then take off the 6.

This question also evoked two episodic images

Paddy	I've got a picture of this old man telling me about it.
Hester	I just saw a picture of when I do like counting my fives.

These were two of only seven instances (out of a total 207 responses) of episodic mental visual imagery in the mathematics procedure questions and no pupil reported a mental visual image of objects in response to these questions. Only four pupils made any mention of a mental visual image of a pedagogic representation:

find a third	Saw picture like Mrs I has drawn.
times by 10	The ten times table line.
two thirds	We have like a number line.
divide by 3	Just like the numbers on cards.

The mathematical words (centimetre, three, millions, fraction, polygon) were similarly most likely to evoke images of symbols though the non-numeric words, "centimetre" and "polygon", evoked images of objects. The non-mathematical words (shadow, ball, Christmas, animal) produced the most reports of mental visual images of particular objects and episodic images. The word which evoked the fewest images was "adjective". Not surprising since it is a low-imagery word.

A comparison between the images evoked by the mathematical and non- mathematical procedure questions (Appendix 3.3) revealed no obvious disposition of individual pupils toward a particular category of image. Most pupils demonstrated a greater likelihood of images of symbols in numerical contexts and of episodic images in non-mathematical.

Mental visual imagery for procedures thus gives no conclusive evidence of a preference for a type of image by individuals. This was also the case in the 'concept' questions and again there appears to be little difference between pupils of different accuracy levels (See Appendix 3.4 for table of categories). A more thorough quantitative analysis will be applied in Chapter 7.

6.1.2f Consistency in use of causal connectives

Pupils spontaneously made use of causal connectives, "because", "so", "if" and "then" when describing their calculations and when responding to the procedure and concept questions. Some of the use of "because", "so" and "then" was at a mundane level which did not explicitly imply causality. For instance:

Jack	**Because** a tenth is one tenth.
Paddy	60 take away 2 is that and it's 59, 58 **so** then you've got the answer.
Peter	She did pronouns, nouns, and adjectives, **then** verbs.

The majority of pupils, however, used these words appropriately to indicate that one statement was a consequence of another. This was mostly in the sense that 'it follows from' or 'it is like':

Naomi	I remember they're quite small **because** there's a hundred in 1 metre.
Irene	7 add 7 is 14 and 8 is one over 7 **so** I just added a one to 24.
Hannah	You would see the 3 times table like 3, 6, 9, 12 and **then** it would be 4, 4 threes are 12.

Some pupils also used "because" and "then" when applying a general rule to indicate that one statement was a consequence of what must always be true:

Myles	**Because** when you've got a number like 97 and you add ten it has to be the same number but the front number has to change.
Teddy	If they've got those traffic lights that says "wait" **then** the sign tells you when you can go across.

"So" was never used in this way.

Similarly with "if" three categories of use were in evidence. At the most concrete level "if" may be used to connect statements about particular calculations or properties:

Clara	**If** you add the 3 on to the 17 its 20.
Bobby	**If** you've got a basket ball, you could bounce it.

"If" was also used frequently to indicate a generic example so that there is a sense that if one statement is true then for this type of situation there is a consequence:

Terry	**If** you had 9 and you've got to add something you have to take off one unit and then make it - a 'teen' like seven take away 1 is 6 'teen'
Jeremy	**If** you're like writing a story it would make it more interesting

At the most general level "if" may be used to indicate a general consequence, something that always follows:

Ann	I knew **if** it was over 5 it would go up.
Jack	**If** you flatten a ball then you have a circle.

In order to check whether pupils could use "if ... then ..." appropriately two sentence fragments were included in one interview. Pupils showed themselves more adept at completing "If it is raining then ..." than the mathematical sentence "If a polygon has four sides ..." which were given as consecutive questions in Y4/2.

All but four pupils gave an appropriate consequence of "raining". This is predictable in that the consequence of rain is well known to pupils from everyday life. Many gave a 'prototypical' response from their own experience phrased in general terms. e.g. "if it is raining then ..." "you go inside", "you can't go out to play". It is interesting to note that those pupils who did not give a deduction were not the lowest achievers:

Terry	... a minute later the sun comes out.
Max	It was sunny, it was raining but it's, but then it started - started to get windy

The two others, Jeremy and Teddy could not give a response. All four are far from being the lowest achievers in either SATs or accuracy in the calculation questions. Teddy did give a satisfactory response to "If a polygon has four sides then..."

Teddy	It's the same as a quadrilateral

Jeremy and Terry were both unable to respond but Max gave a response which was common to half the class

Max	It is like a square

This is again a prototypical response in that the pupils are most familiar with squares.

The final interview included the sentence fragment completion task "If 13 times 3 equals 39 then ... because ...". Again some of the highest achievers, including Teddy, could not give a response. Elspeth's inappropriate completion of the sentence was noted in Chapter Five and other high achievers were similarly unable to use the connectives appropriately:

Clara	Then - 33 add 6 makes 39 because ...

Ten others gave the rules that they had been taught, either inverse operation or commutativity :

Hannah	Then 39 divided by 3 is 13 because that's the inverse operation.
John	Then 39 divided by 3 is 13 because (points at question on script).
Jack	Then 3 times 13 would be 39 because if you switch numbers around when it's times then it's still the same answer.
Hester	Then 3 times 13 would be 93 because you just switch the numbers around?

There is evidence here of the influence of their classroom activities, though for Hester it had become confused. Hester appears to use "because" appropriately and had remembered that there is a rule but she had no sense of the size of the numbers. "You switch the numbers around" was common classroom language and is typical of the algorithmic, digit-manipulation, approach.

An implication that may be drawn is that use of causal connectives in these artificial settings shows no association with pupils' accuracy. Use of "because" may be independent of understanding of the mathematics. This suggests that simply having the linguistic ability to use causal connectives is not a factor related to pupils' ability to perform procedures.

6.1.2g Summary of time and context consistencies

The purpose of this section was to lay the foundations for the quantitative analysis of differences between pupils and between contexts that follows in Chapter Seven. Children's responses to interview questions have been presented to demonstrate that the context of the question is a factor in their response and that there is little evidence of a disposition toward a mode of response which is independent of the context. This may be interpreted to imply that children manifest their mental constructions, rather than just a mode of expression, when replying to questions.

The comparison of pupils' responses in mathematical and non-mathematical contexts suggests that pupils can not easily be identified as having a particular style in terms of the category of 'generality' of expression they use nor in the type of mental visual imagery that is evoked. This is true both in those items that require description of a procedure in mathematical and in non-mathematical contexts and in those 'conceptual' items that require them to describe what words evoke for them.

The context can have an important role in determining the response. The mathematical items, not surprisingly, are more likely to evoke mental visual images of symbols. Routine procedures, both mathematical and non-mathematical, are more often described in 'general' terms. Difficult, and not frequently practised, procedures did not often evoke 'general' responses. This can be interpreted to imply that children have abstracted from their experiences in familiar situations. Where children have explicitly been taught a definition ("an adjective is a describing word") they were able to use it and repeat it.

Appropriate use of causal connectives is also most apparent in an everyday situation. There is evidence here too that experience is an influence on pupil's responses in that prototypical or taught consequences were most frequently given in the sentence completion tasks.

This first analysis suggests that the mode of response may be determined by the context and does not indicate a disposition by the individual toward a mode of expression which is consistent across contexts. This suggests that qualitative differences in response to mathematics questions may indicate qualitative differences in learning. This will be pursued further in Chapter Seven

6.1.3 Summary of manifestations of learning

The conclusions that may be drawn from the evidence presented are:

- These children manifested their learning from experience of pedagogic representations through their language, methods and reported mental visual imagery.
- The differences in learning are manifested by qualitatively different use of language and method.
- Differences in language may be indicative of qualitative differences in learning not merely dispositions by individuals toward a mode of expression.
- Differences in mental visual imagery may be dependent on context. 'Symbolic' images are most common in response to calculation questions and 'specific' images most common in everyday situations.

6.2 Language and imagery of other groups

In order to highlight characteristics of language, 'method' and mental visual imagery of this sample of children we turn to two other groups. If the speech community and its practices are influences on its members we would expect the teachers to give similar responses to the children. A different environment ought to produce different influences and thus different responses.

6.2.1 Teachers' interview responses

In this very brief section the responses that the teachers gave to some of the same items that had been used in the pupils' interviews are presented to demonstrate the common language and methods that this community shares.

6.2.1a Teachers' mental calculation.

In response to "When you are counting what comes before 12 100?" teachers used a language of manipulation of digits similar to the pupils. Some of them reported mental visual images of numerals:

Mrs I	12 000 Oh, it's not is it? it's 12 099. I looked at the number in my head. I'd **got the** 12 000 and then I'd **got a one** floating around there, that's why I said 12 000 because I'd forgotten, oh, **there were two zeros**
Miss P	12 099. I saw one, two, one, nought, nought. Then I thought what's 100, 99 **put 12 000 on.**
Mrs F	12 000. aghhh (laughter), how funny. I visualised 12 000 and then I heard **the hundred**.
Mr K	12 000, 12 099 I did have a panic because you always suspect a trick question when **you're around the hundreds.**

In response to "1197 add 10" they talked of place value partitioning and manipulation. Note "changing", "make", "chop"; these are metaphoric references to the classroom activities:

Mrs I	I **got the** 1197 in my head I just saw the 1000, but not the 197 because I knew **that would be changing**
Miss P	I had to think about that hard. I see 1, 1, 9, 7. Then I **add ten on to the** 97 - **to make the hundreds a** 7.
Mrs F	What I do is I actually **add on to get to** 1217 and **take ten away.**
Mr K	I knew I had to **chop the hundreds off**, so I **dealt with that first**, and I think I guessed at **the 7 at the end**

In response to "48 add 23" single-digit procedures similar to written algorithms were in evidence though one teacher talked of a number line and used a 1010 strategy:

Mr K	**I added the tens first, then the units**. 4 and 2, 8 and 3 is 11 and **added the 1 on**.
Mrs I	It's 40 plus 20 is 60, **8 and 3 is 11** and 60, 71
Mrs F	48 add 20 and **add on the** 3

Miss P	I just see **48 add 2 tens** and 3. 48 and then my mind would go between 58, 68, 1, 2, 3 I'm looking at a number line I suppose, I am.

These responses serve to emphasise that the classrooms observed for this study make up a speech community and a community of practice in which pupils and teachers share language and ways of performing calculations. The teachers described procedures that almost always involved place value partitioning. Elspeth's language had similar characteristics. The other pupils' responses also demonstrate how close pupils were to the teachers in their description of calculation. There is a suggestion here of linguistic relativism in that the language related to the activities becomes part of the mode of communication of the thinking.

6.2.1b Teachers' mental visual images

There are allusions to mental visual images in the responses above and the three female teachers seemed to be strong mental visualisers in that they declared vivid mental visual images that were present whilst they were calculating. Miss P gave evidence that recent changes in the representations she used with children had influenced her own mental visual imagery:

48+23	I'm looking at a number line, I certainly don't think vertically not now anyway because I think that's the way it's going, the (numeracy) strategy, we don't think vertically hardly at all.

Mrs I was convinced that a mental visual image was of use to her as a memory aid but did not carry out the written algorithm on her image:

48+23	Just saw the numbers, the two numbers you gave me, clear printed underneath one another like a sum, I did it in the mental way instead of the written way. I've got quite a good visual memory.

For a more difficult question she again reported a mental visual image of a vertical calculation though she used a mental strategy. It may have triggered thinking but it was 'lost' and did not appear to have served a memory-aid purpose:

65-27	65 subtract 27 so you could say 65 minus 20, 45, minus, and then you could say minus 5, where have I got to, oh I've forgotten now I did (see something) to begin with and then I lost it, like a sum again.

Mrs F also frequently saw numerals. For "48 add 23" she began

Mrs F	This 20 separates itself off from the rest of the number, goes onto the 48 and add on the 3. I think as you go through each step the images appear. I think there's the visualising the 40, 48, then across comes the 20 and then there's no more picture.

but later explained:

Mrs F	What I tend to do is see the written method. Now what I see is 8 and 3 is 11, That's how we were taught. So I sort of put them, set them out, two lines. What I can see is the carrying one on.

She had not formed a mental visual image of this initially because she said "immediately there's 'Oh I've got to do it quickly' ". There is an indication here that the images described do not necessarily relate to the method that was used initially but seem to form in the mind when the explanation is given subsequently.

Each of these teachers had a mental visual image of a numeral when asked "What is the first thing that comes into your head when I say three?"

Mr K claimed not to have mental visual images in response to any of the calculation questions and even when asked "What ... three?" replied:

Mr K	Not a lot! I mean, three is totally abstract to me. You know, I don't picture anything.

He could, however, mentally visualise a number line when asked. He could imagine jumping on the number line after performing 36 add 20. The other teachers responded similarly.

6.2.1c Summary of teachers' responses

This section serves to illustrate that the teachers showed similarities with the children in both their language and mental visual imagery. Three of the four teachers reported vivid mental images of 'sums'. There was some spontaneous use of number line mental visual imagery but no other physical representation was mentioned. This is similar to the children. They also were more inclined to report images of symbols and occasionally number lines, rather than other pedagogic representations.

Like the children, clues to the teachers' modes of thought came in their language and the methods they employed. They did not use any of the 'compensation' mental strategies that they had recently been teaching but for both "48 add 23" and "65 subtract 27" dealt with tens and units separately just as the pupils did. They also referred to "adding on" and "taking away" just as the children did. This suggests that place value representations and object 'collection' have been influences on their own thinking.

There are indications in these similarities that the community of practice is an influence on all participants and that the language of the common activities is a manifestation of the learning that is taking place.

6.2.2 A different environment

In 1996/7 teachers at Peacehaven Infant School, in collaboration with Tony Wing (see Wing, 1999), began to develop a programme of classroom activities in which Stern plates (see Appendix 1.1), number lines and Cuisenaire rods were predominant classroom materials. By July 2000 their programme was well established and the Junior School had continued with the approach. Y2 and Y3 were thus two cohorts of children that could offer a contrast to Bright Cross in terms of the influence of pedagogic representations.

To provide a comparison I conducted one interview with a small sample of pupils in Peacehaven to find if there might be evidence of the influence of these different materials.

6.2.2a The materials and approach

The teachers at Peacehaven took advantage of a Teacher Training Agency Teacher Research Pilot Project Grant and have reported the results of their approach in 'Developing Mental Arithmetic A 'Stern' Approach in the 1990s' (Tacon, Atkinson and Cooper, 1999). Details of the classroom activity programme are taken from this report.

The teachers were dissatisfied with published mathematics schemes "because these invariably encourage counting procedures in preference to the overt establishment of relational number understanding". They found in early experiments with Stern structured apparatus that children were much more capable of grasping relationships between numbers on a number line than in their previous experience of working with Reception and Year 1 children using other materials and approaches. The teachers wished to develop children's 'arithmetic technique' so that pupils would use numbers as related wholes, rather than understand them as counts. They believed that numbers should be added or subtracted by using the relations between numbers embodied in the plates not by 'counting-on' or 'taking away.'

The teachers in Peacehaven were not happy with the traditional emphasis in England on written arithmetic "with mental arithmetic as an add-on". They were concerned that teaching children written arithmetic does not establish relational understanding of number. In their view the children are then doing writing, not arithmetic. They drew on Gray (1991) to suggest that when children do not have a rich understanding of relations between numbers they fall back on counting procedures which are inefficient and error prone. They questioned whether counting needs to be the foundation for arithmetic. They argued instead that structured images would allow children to relate numbers to each other and this would form the basis of their conceptual knowledge of number.

The programme they have developed starts with:

1. 'Pre-counting'- Learning the images before numbers are attached (Stern plates, Cuisenaire rods)
2. Counting and finding how many (associating counted groups of objects with plates and number line)

It goes on to introducing written numerals. Addition and subtraction are done by fitting plates together or "chopping off" one plate from another. Bonds to ten are seen in terms of plates. Numbers to 100 are introduced through rods placed on number tracks and number lines are also used for addition and subtraction as movements on the line.

This is a different approach to that used at Bright Cross where one-to-one matching and counting are predominant in Reception and Y1.

6.2.2b Interviews with pupils

The interviews took place in July 2000 and it was immediately obvious that the approach had influenced the children. Six children were interviewed in each of Y2, Y3 and Y4, chosen by the teachers to represent the range of pupil achievement in the school. Questions similar to those used at Bright Cross were used.

Year 2 pupils were asked "17 add 9" and if there was "anything to see" in their heads when they had worked it out. One pupil gave a response similar to the pupils at Bright Cross:

Susan 26. Add 10 onto 17 makes 27 and take one away.

But the others had very different language and methods:

Barbara	26. That was where I had the 17 on the **plates**, and then I had the 9 - I **chopped the 9** - took 3, chopped it off and then there was, I added the 3 onto the seven, teen and then I had the 6 left and then that was 26.
Geoffrey	26? I had 10 in my head and then **I put the 9** on and it's 19 and add the 7
Lawrence	26. I like see like 17 cars and 9 people and **I put them in the pattern** then I add them up I took the three off - the 9 and I put it onto the 17, that made 20 then there were 6 left from the 9.
David	26. **I got numbers up to a hundred in my head** and I got 9 and 17 and I added the 3 onto the 17 and added the 6 onto the 20. They looked like, squares with the numbers on.
Clive	26. I got a picture of the 7 and I got a picture of the 9 that looked like, **them joined together and making the number** I just said.

An important point here is that *none* of these pupils counted. The pupils at Bright Cross were asked "17+8" in December of Y3 (five months older than these Peacehaven children) and 16 of the 26 counted either in their heads or on fingers. The other ten used a variety of derived fact methods and there was no consistency of metaphoric language. Only three of them used the complement to 20 approach. At Bright Cross 11 of the 26 pupils still counted for "17 add 9" at the end of Y3 and, at the end of term 2 in Y4, 7 children still counted. This indicates that the influence of classroom counting activities was long lasting on the pupils at Bright Cross whilst at Peacehaven, where counting is not taught as a procedure for addition, the pupils did not use it. The pupils were of similar achievement levels.

The Y3 pupils in Peacehaven were less successful on "17 add 9" since 2 of the 6 gave a wrong answer. There was, however, still evidence of the 'complement to 20' approach and the treating of numbers as objects not as collections of objects. One of these pupils did count but this compares with nearly half of the Bright Cross children who counted and only two of the 26 Bright Cross pupils used the 'complement to 20' strategy. The one pupil who counted did not say 'count-on' as the Bright Cross pupils did:

Jacob	26. I was going (moves fingers) **counting up**, I was counting to 9

Eleanor said she had a mental visual image of plates and gave the wrong answer because she recalled the wrong number fact

Eleanor	24. 'Cause 7 add 9 is - 16 - it's actually 27 isn't it? I got confused with 7 and 5 (mental visual image of plates).

The others all used language and method related to the plates:

Derek	15, no 25 I mean. Well first I took 17 and 5 and I said, I borrowed 3 from it, I took **the three from the 9** to make that, that gave me 5 left so I just added the 5 on to make it 25.
Denise	26. You've got 17, **you add 6**, then add **another 3**, that will be 26.
Miranda	26. A **number plate ten** and **a 7**, and um, **a 3** and **a 6**, I added the three on to the 7, equal 20, I've got 6 left, add the 3 onto the 20 equals 26.
Conrad	26. You **take the 3** off of, **the 9** and you **put it on to the 17**, makes 20 and then - what you're left with is the 6 and then you put it onto the twenty.

The children at Bright Cross seem to use counting as a 'fall-back' method when they feel at risk of getting an answer wrong. At Peacehaven, where counting is discouraged, the pupils do not resort to it.

For "48 add 23" the Y3 Peacehaven pupils sounded almost as algorithmic as the Bright Cross pupils did at the same age. The success rate was the same with half the pupils giving incorrect answers. The big distinction however is that none of the Peacehaven pupils referred to 4 and 2 instead of 40 and 20 whilst a quarter of the Bright Cross pupils did this. Furthermore none of them talked of "carrying" or "adding one to the tens" but a third of the Bright Cross pupils did this. Even when one of them said she saw the sum written down her language was of plates:

Eleanor	71. Added the two tens together and then the units. I had to **fit the** two numbers **together**, the two tens together, and then the two units (mental visual image of sum).
Derek	71. Well I added the 40, and to the 60, and then the 8 and the 3 that gave me 11, added the two together
Denise	71. I thought, you've got to add the tens together, which would make 60 and then you add the units together it would make, 11, so, then you **put them together**, it made, 71.
Miranda	.. 51, add the 40 and 20 equals sixty - the 8 and 3 equals 11
Conrad	91. I'm adding the tens up together so it's 40 add 20 and then it's 8 add 3 and then I'm sort of **taking the numbers away** that are making the numbers what are not in tens and then **putting the left out numbers** what are not in tens to the number
Jacob	. 63 I've got my 20 and I add 20 onto the 40 that made 50 and then **I got the 3** and **put it onto** the 50

The teachers at Peacehaven have made a determined effort to work with materials in order to provide images for numbers. In response to "What is the first thing that comes in to your head when I say three" two Y2 pupils mentioned plates, two said number lines and one said the numeral three. Four of the Y3 pupils said numerals and one said rods (this *is* a Peacehaven word). There is some evidence that there is more mental visual imagery here than at Bright Cross where only 9 of the 26 pupils had an image for this question in Y3/2; two were of 'cubes' and the rest symbols. There is also a suggestion that the greater emphasis on symbol manipulation in Y3 at Bright Cross could have led to a greater number of images of symbols being reported.

The images reported after the calculation questions were qualitatively different at Peacehaven than those evoked by pupils at Bright Cross. The question "17 + 9" evoked 5 images from the 6 Y2 pupils at Peacehaven (4 of pedagogic representations and 1 of symbols) and 4 from the six Y3 pupils (3 of pedagogic representations and 1 of symbols). For "48 add 23' there were 4 images from the Y2 pupils (3 of pedagogic representations and 1 of symbols) but only 1 from the Y3 pupils (a pictured written calculation). These are not dissimilar quantities to Bright Cross where half of those asked in Y3 said they had an image for "17 + 9" and a quarter for "48 + 23". At Bright Cross, however there were 8 images of pedagogic representations and 13 of symbols in Y3 whilst in Y4 there were 4 images of pedagogic representations and 18 of symbols. There is thus a suggestion that pedagogic representations were more common with the younger Peacehaven pupils and that at Bright Cross images of symbols were more common in Y3 and became more so in Y4.

The Y4 pupils at Peacehaven had not used any of the materials or activities of the program and appeared to have had similar classroom experiences to the pupils at Bright Cross. Their responses in interview questions show a similar variety of responses and images as the Bright Cross pupils in Y4. This provides some further justification for the claim that the pedagogic representations have an influence on the way that children describe their calculations. It is further evidence also of linguistic relativity since the Y4 Peacehaven pupils were from a similar cultural background, apart from their mathematics lessons. They spoke and performed differently from the Y3 Peacehaven pupils in mathematics because their mathematics classroom activities had been different.

6.2.2c Summary of 'the Peacehaven effect'

The teachers at Peacehaven have a clear vision of a style of teaching and learning which contrasts with that in evidence at Bright Cross. When the pupils at Peacehaven were interviewed the effect of this difference in pedagogic representation was apparent in both the language and methods employed for mental calculation. The pupils who had not been through the programme of classroom activities developed by these teachers used very similar language and methods to those in use at Bright Cross. There is clear evidence here that the classroom activities are an important influence on pupils' learning and this is manifested by explicit talk about the materials and implicit metaphoric language associated with the materials and their manipulation. Without having counting as a first resort pupils at Peacehaven used strategies of number bonds that were not in use at Bright Cross. They expressed themselves in the language of manipulation of the materials they had been using. Their mental visual imagery was also influenced by these pedagogic representations.

6.2.3 Summary of comparisons with other groups

The responses of teachers at Bright Cross serve to illustrate the influence of the community and its practices on the speech styles of its members. The Children at Peacehaven have a speech style that is different and is commensurate with the different activities.

There is some support here for the claims of linguistic relativists in that the language and practices are complementary. The quite different methods, with accompanying language, of the two communities demonstrate the interdependence of practice and language.

6.3 A summary of the evidence presented

The variety of metaphoric language of manipulation serves to illustrate that, whilst the pupils express themselves in different ways, there is a common source in the manipulation of materials and subsequent manipulation of numerals. The common language of written algorithms, for instance, indicates that the mental calculation may have been performed using the strategies of written calculation.

The language of procedure, what "you" have to do to perform a calculation, is part of the classroom speech style. Both teachers and pupils make use of this form of expression and when pupils do so it indicates that they have been influenced by the classroom experiences. The lesson on counting-on was detailed above (§6.1.1e) to illustrate the way in which procedures and strategies that many pupils are thought to develop for themselves may also be actively taught. They are then expressed in the language of the shared procedure. Similarly the use of causal connectives is common in all aspects of life but there is evidence of pupils adopting the style of explanation that is part of the mathematics classroom culture.

Mental visual images of teachers' physical representations were comparatively rare and mental visual images of objects for counting was mentioned by only one pupil. More common were mental visual images of symbols which may well reflect the much greater prominence given to non-material representations in Y3 and Y4. This was further evidenced when pupils were explicitly asked to mentally visualise something for a subtraction. The majority of pupils reported mental visual images of numerals. Even when they were required to, and had succeeded in, visualising a number line none 'used' their mental visual image to perform an addition. They reverted to mental strategies of separating tens and ones which was their common practice.

The teachers and pupils at Bright Cross, not surprisingly, share both language and strategies when describing mental calculations. Whilst the teachers may have had different personal methods they gave no evidence of this in the interviews and performed the calculations in the same way as the majority of the pupils.

The different environment at Peacehaven and the mental visual imagery and language evoked for the pupils there provide a contrast with Bright Cross which demonstrates the influence of pedagogic representations on pupils. The quite different thinking manifested in the language used is evidence of the way pedagogic representations provide the early structure, and mode of expression, for pupils' calculations. No claim is made here that one approach is in any sense 'better' than the other, simply that the pedagogic representations are influential in the way pupils think about and describe mental calculations. The influence of the classroom activities was emphasised by the different responses of the Y4 pupils at Peacehaven.

The information provided by comparing pupils' responses to the procedural and conceptual questions in mathematical and non-mathematical contexts also suggests that the context rather than pupil disposition is the biggest factor in determining mode of response. There is no hard evidence to suggest that an individual is disposed toward a particular style of 'generality' or of 'image'. There is evidence that familiarity with a procedure is signalled by use of "you" and giving of 'general' rules. When children abstract from their experiences they express this in 'general' ways. They may thus be indicating their mental constructions by their mode of response.

The suggestion that the context rather than the child is the main factor in determining the style of response is given further weight by the analysis of use of causal connectives in the artificial setting of sentence completion tasks. Here pupils of all achievement levels can make appropriate responses in an everyday situation but not necessarily in a mathematical context. In each instance however it is important that again the influence of experience is in evidence in the common prototypical responses given. This point was reinforced when pupils most commonly gave, as consequence of $3 \times 13 = 39$, those consequences they had been taught. They did not use this fact to derive others.

This chapter is intended to provide a link between the case study of one child and the statistical analysis that is to follow. Many more examples have been given of the categories of response that pupils have used and further issues have been raised. This brief summary seems to be pointing to environmental factors, i.e. pedagogic representations, being the prime influence on pupils. There is, however, ample evidence here that given similar experiences pupils will re-present those experiences in different ways. There is evidence of the variations in responses between and by individuals which suggests that children have different 'mental constructions'.

The following chapter attempts to quantify these similarities and differences in order to explore the divergence between those who become more successful in mental calculation and those who do not.

CHAPTER SEVEN

COMPARISONS BETWEEN QUANTITIES AND QUALITIES

This chapter seeks to develop the theme that individuals may form qualitatively different mental constructions from the common classroom experiences. In particular it addresses the research questions:

Variation with context:

RQ2 What are the characteristics of mental visual imagery and language used in mathematical and non-mathematical contexts?

RQ3 Is there evidence that individuals or groups have a style of language or mental visual imagery which is common to mathematical and non-mathematical contexts?

Variation with achievement level:

RQ4 What are the characteristics of mental visual imagery, language and calculation strategy associated with successful and unsuccessful calculations?

RQ5 How do mental visual imagery, language and calculation strategy relate to achievement?

Variation with time:

RQ6 How does the mental visual imagery, language and calculation strategy used for mental calculation, by individuals and groups, change over time?

Every response has been categorised for use of linguistic indicators, 'image', and 'generality'. For computation questions the two other 'indicators of learning', 'metaphor' and 'method', have also been categorised. In this chapter the first level of analysis involves the comparison of the frequency of use of each category by each pupil to explore the correlation between achievement and these indicators. Correlations between the indicators themselves are also investigated as a check on validity.

The second style of analysis explores the associations between the indicators as they relate to each response. The variety of pairs of indicators for each response are tabulated and chi-square tests of significance computed for these contingency tables. Finally pupils are divided into 'upper' and 'lower' groups for achievement, accuracy and quantity of mental visual imagery. In each case the two halves of the sample are then compared. A finer level of analysis is also employed in which pupils are grouped into 'high' (scores greater than 1 sd above mean), 'middle' (scores within 1 sd of mean) and 'low' (scores less than 1 sd below mean) for the number of questions correct, and the number of mental visual images they reported. The performance of the three groups are then compared.

The final level of analysis compares the response of individual pupils over the six interviews to explore signs of consistency or progression over the two year period. With the question as the unit of analysis the data suggest that the determining factor in accuracy is not simply the difficulty of the question but may be due to characteristics of the child.

These differing levels of analysis allow the research questions to be addressed in a variety of ways and thus provide a check on the reliability of the analysis.

In §7.1 the analysis provides indications of how variation with context and with achievement may be manifested. It appears that

- Use of 'representative' and 'abstract' categories of 'generality', 'metaphor' and 'method' is associated with high achievement.
- Children show a disposition toward a mode of mental visual imagery.

In §7.2 the characteristics of the contexts are explored. Firstly the variables associated with accurate answers and secondly characteristics of mathematical and non-mathematical contexts are addressed. The key findings are:

- Accurate answers are associated with non-'concrete' 'image' and 'generality'. Accuracy is independent of 'method' and 'metaphor'.
- Accurate answers are associated with distinctive use of pronouns, tense and causal connectives.
- Responses to mental calculation questions are associated with language which is different from responses to procedure and concept questions.

In §7.3 the characteristics of children are examined. Firstly different groups are compared and then comparisons are made over time. A comparison is also made between children who were successful in difficult questions and those who were not. The key findings are:

- Successful calculators have distinctive language and demonstrate their flexibility by use of a variety of methods.
- Different achievement level groups show distinctive development over time.

Note on presentation of results

Percentages are used to aid interpretation when tables are included in the body of the text. The raw data have been included as tables in the Appendix. Tests of statistical significance on the raw data contingency tables provide p values which are also recorded in the Appendix. In the body of the text 'statistically significant' will imply that $p<0.05$. 'Highly statistically significant' will imply that $p<0.005$.

7.1 Indications of learning

In §7.1.1 the analysis suggests that there are associations between pupils' achievement levels and frequency of use of some categories of the 'indicators of learning'. Analysis in §7.1.2 suggests that there are consistencies in the categories of use of the various indicators.

7.1.1 Correlations between quantities

In this section the analysis is at the level of correlations between quantities for each pupil. The individual categories of the indicators are considered for each response to explore both how they might be related individually to accuracy and how they might be associated with one another. This gives a partial answer to the question "How do mental visual imagery, language and calculation strategy relate to achievement?".

7.1.1a Measures of achievement

Analysis of associations between different measures of achievement is presented here in order to indicate relevant measures for the groupings of pupils by achievement level that follows later in the chapter. The analysis also indicates the reliability of the interview questions as a measure of achievement.

The end-of-year assessment tests in mathematics provide a measure of the pupils' mathematical achievement over the period of the study. For convenience in this analysis they are all referred to as SATs though it is only mandatory that pupils take the end of KS1 test in Y2. The three written test scores have been totalled to provide a 'combined SAT score'. Scores in end-of-year mental arithmetic tests, spelling tests and NFER reading tests have also been recorded. These provide standardised measures of mathematical and linguistic achievement. The number of calculations that pupils answered correctly in the interview is referred to as 'accuracy'. (The table of distributions of achievements for each pupil are in Appendix 4.1).

There were some associations between these measures of achievement. The Pearson-r correlation coefficient was calculated for pairs of measures. Statistically significant correlation coefficients are summarised in Table 7.1:

	NFER Reading Age	Comb SAT score	Comb. Mental Arith	accuracy
Comb Spelling	0.74	0.63	0.65	0.53
NFER Reading Age		0.66	0.62	0.58
Comb SAT score			0.89	0.76
Comb Mental Arith				0.77

Table 7.1 Statistically significant correlations between measures of achievement

There is a highly statistically significant correlation between accuracy in interviews and both combined written SAT tests and combined mental arithmetic scores which may be interpreted to suggest that the interview questions test similar arithmetic skills to the tests. The strong correlation between combined SATs and combined mental arithmetic suggest they too are measuring similar skills.

The lower correlation between both spelling and reading age with accuracy and the two combined mathematics measures of achievement indicate that the levels of linguistic achievements of pupils may not be so closely related to their mathematical achievements.

The use of language is an important aspect of this study. Comparisons between language use and arithmetic achievement will provide information about characteristics of arithmetic achievement. It is possible, however that childrens distinctive use of language could also be related to their linguistic achievement. This first analysis suggests that factors associated with mathematical achievement may also be associated with linguistic achievement.

7.1.1b Mental visual imagery

The quantity and quality of the mental visual imagery reported by pupils in interviews varied considerably. In responses to 78 questions the number of times an image was reported ranged from 3 to 48 (mean = 22, sd = 13). In the 45 questions where a calculation was required the range was 1 to 29 mental visual images (mean = 12, sd = 8). The number of times that individual pupils were specifically asked if they saw anything in their heads varied between 37 and 60 (mean = 50, sd = 6).

Frequencies of reported imagery and categories of 'image' were tabulated and ordered by total number of images reported (Table in Appendix 4.2). Correlation analysis indicates that there was no relation (r = 0.04) between quantity of images reported and the number of times pupils were specifically asked if they had anything to see. This suggests that the question has not biased the outcome. Other conclusions may be drawn:

- There is a tendency to be a high or low reporter of mental visual images in both calculation and non-calculation contexts.

The number of calculation images and the number of non-calculation images are positively correlated (r = 0.61). This is highly statistically significant.

- There is a tendency to be a high or low reporter of non-'specific' mental visual images in both calculation and non-calculation contexts.

When the frequency of 'iconic' and 'symbolic' are combined in each context then the correlation between frequencies of these non-'specific' images is high (r = 0.75). This is highly statistically significant.

Taken together these indicate that:

There is a disposition by pupils toward both quantity and quality of mental visual imagery that is consistent across contexts.

7.1.1c Other indicators

The other 'indicators of learning' and linguistic indicators were analysed in the same way as mental visual imagery. Correlation coefficients were calculated based on tables of frequencies for each category of each indicator. (Table in Appendix 4.3). The statistically significant correlations are summarised in Table 7.2:

	creation'	counting'	holistic'	algorithmic'
generic'	0.77		0.72	
general'	0.51			0.6
collection'		0.63		
creation'		-0.62		0.79
counting'				-0.52

Table 7.2 Statistically significant correlation coefficients between 'indicators of learning'

There is a picture here of consistencies between modes of expression and methods.

- Pupils who can express themselves in 'non-particular' terms tend to do so using metaphors of 'object creation' and to use 'non-counting' methods.
- Metaphors of object 'collection' are associated with 'counting' methods.
- High levels of use of 'creation' metaphors are associated with high levels of 'algorithmic' methods and low use of 'counting'
- High levels of 'counting' use are associated with low use of 'algorithmic' methods.

When frequencies of the separate categories of 'image' were correlated with frequencies in the separate categories of the other indicators there were also consistencies. The statistically significant associations were:

- The frequency of reported mental visual images of pedagogic representations is associated with the frequency of use of 'motion' metaphors ($r = 0.71$).
- The frequency of images of symbols in calculations is related to the frequency of 'creation' metaphors used ($r = 0.55$).

The suggestion here is that pupils' use of 'metaphor' are consitent with the image they report. We now turn to the linguistic indicators.

The frequency of use of "I", "you", "like", "if", "then", "because", and "so" were compared in a similar way (Table in Appendix 4.4). The strongest relationship existed between frequency of use of "you" and the total number of causal connectives used ($r = 0.84$) which suggests that children who describe their calculations in terms of what "you" do also explain themselves using causal connectives.

This initial analysis has given indications about the characteristics of the language and imagery used by this sample of pupils and the way these indicators are related. This will be pursued in the later sections when language and imagery used in individual questions becomes the focus.

7.1.1d Relations between the indicators and achievement

When pupils' achievements and the frequency of their use of the different categories of the indicators are correlated (Appendices 4.1 - 4.4) we may begin to address the question "How do mental visual imagery, language and calculation strategy relate to achievement?".

The following points emerge:

- There is no statistically significant relation between any of the measures of achievement (SAT scores, accuracy in interviews, or reading age) and the quantity of images pupils reported in calculation or non-calculation questions.
- There is no statistically significant relation between any of the measures of achievement and the frequency of images in any of the separate categories.

This tells us that frequency of reported mental visual imagery is not associated with achievement. However the statistically significant correlations (summarised in Table 7.3) indicate that:

- Achievement is associated with frequency of use of 'general' expressions of 'generality'.
- Achievement is associated with frequency of use of 'creation' metaphors.
- Achievement is associated with frequency of use of 'algorithmic' methods.

	'general' expression	'creation' metaphor	'algorithmic' method
Comb Spelling	0.54		0.50
Y4 Reading	0.60		0.63
Comb SAT scores	0.58	0.72	0.77
Comb Mental Arith	0.54	0.61	0.64
Accuracy in interviews		0.72	0.50

Table 7.3 Statistically significant correlation coefficients between measures of achievement and 'indicators of learning'

When frequency of use of the indicators in non-calculation contexts were correlated with levels of achievement there were no statistically significant correlations. It is thus of note that this level of analysis gives no evidence that high achievers or low achievers have a disposition toward any particular category of 'generality' or of 'image' in non-calculation contexts.

7.1.1e Summary of connections between achievement and indicators

The number of images a pupil described in these interviews was independent of their achievement levels and largely independent of other indicators. There is, however, some evidence that pupils have a disposition toward non-'specific' images, i.e. 'iconic' or 'symbolic', that is consistent in calculation and non-calculation contexts. This provides some support for Pitta's findings and is thus a check on the reliability of this indicator.

'Metaphor' and 'method' relationships give evidence of consistency in that the quantity of 'counting' methods is associated with quantity of 'collection' metaphors. Quantity of 'algorithmic' methods is associated with quantity of 'creation' metaphors. There are also consistencies between the category of 'generality' and 'metaphor'. Pupils who frequently used 'holistic' methods also frequently used 'generic' generality whilst 'general' expressions and 'algorithmic' methods also correlate. Predictably pupils who use more 'counting' methods tend to use fewer 'algorithmic' methods.

There is evidence that pupils who more frequently use the most 'abstract' forms of 'generality', 'metaphor' and 'method' are also those who are high achievers in SATs, have a high reading age and were accurate in interview calculations. Whilst correlations between the frequencies of use of the indicators is a crude measure of association this analysis has given indications of the characteristics of language and mental visual imagery worth exploring further. This section also provides a check on the reliability and validity of the indicators.

7.1.2 Consistencies between indicators

In this section the individual response is the unit of analysis. Contingency tables were compiled for each combination of categories for each pair of indicators. Chi-square values were computed for these contingency tables. This level of analysis provides a further check on consistencies between indicators. This analysis confirms the findings of the previous section i.e. compatible categories are associated with one another. Responses are often in similar categories of 'image' 'metaphor', 'method' and 'generality'.

7.1.2a Associations between pairs of indicators

The expected association was apparent when categories of 'metaphor' were tabulated with 'image' categories. Table 7.4 shows the percentage of each category of 'image' for each 'metaphor' (raw data in Appendix 4.5(i)). The differences in the distributions are statistically significant. Bold type highlights the strongest associations. The table shows the percentages of each row total. For instance 65% of all the 72 responses involving 'collection' metaphors were accompanied by a 'symbolic' images.

	specific	iconic	symbolic	Total
collection	11%	24%	**65%**	72 (100%)
motion	8%	**62%**	30%	60 (100%)
creation	7%	9%	**84%**	174 (100%)

Table 7.4 Percentaged contingency tables for 'metaphor' against 'image'

This table indicates that:

- A 'symbolic' image most frequently accompanied 'collection' and 'creation' metaphors. An image of a number line was accompanied most frequently by language of 'motion'.

This analysis of different distributions in the contingency tables for other combinations of indicators suggest the following:

- When pupils described an image of numerals or written calculation they were most likely to talk in 'generic' terms. When their image was of a pedagogic representations they were most likely to talk simply of the numbers in the calculation and not generalise (Appendix 4.5(ii)).
- There was a tendency for the 'image' category to be consistent with the method described for mental calculation. 'Symbolic' images were associated with both

'holistic' and 'algorithmic' methods. 'Iconic' images were most frequently accompanied by 'holistic' methods. Only 6% of images reported were of 'specific' objects and could accompany any method (Appendix 4.5(iii)).

The three categories of 'metaphor' and the three categories of 'method' provide a contingency table which indicates a very strong association between them (Appendix 4.5(iv)).

This is in part explained because when children simply counted, with or without any description of their method, this was categorised as both a 'collection' metaphor and a 'counting' method. Similarly descriptions involving digit manipulation were inevitably categorised as 'creation' metaphor and 'algorithmic' method. The 'motion' metaphor could accompany any method. The table was thus quite predictable and showed:

- 'Counting' methods most frequently accompany 'collection' metaphors, 'algorithmic' methods accompany 'creation' metaphors, 'holistic' methods are accompanied most frequently by either 'collection' or 'creation' metaphors.

The high proportion of compatible categorisations ('concrete' in both or 'abstract' in both) demonstrates not only the consistency in the categorisation but also a consistency between pupils' language use and method employed.

The consistency here could be explained away in terms of similarities between category descriptors. However the associations between categories of 'generality' and the categories of each of 'metaphor' and 'method' suggests that the abstraction represented by the categories is consistent across these indicators. The differences in distributions were statistically significant and suggest that:

- 'Counting' methods are most often associated with 'particular' expressions of generality, 'holistic' and 'algorithmic' methods are most often associated with 'generic' expressions of generality. 'General' expressions are less frequently used but the highest proportion of them are associated with 'algorithmic' methods (Appendix 4.5(v)).
- 'Collection' metaphors are most often associated with 'particular' expressions of generality, 'motion' and 'creation' methods are most often associated with 'generic' expressions of generality. The highest proportion of 'general' expressions are associated with 'creation' metaphors (Appendix 4.5(vi)).

7.1.2b Summary of indicator consistency

Since the frequencies of mental visual images involved are small some caution must be attached to any generalisations about these but there is evidence here that the language that pupils used was at least consistent with the images that they reported.

The data presented also suggest that the children demonstrated consistencies in the category of 'generality' they expressed, the metaphors they used and the methods they employed. This provides confirmation that the category descriptors are complementary.

7.1.3 Summary of indications of learning

There is evidence, to add to that presented in the previous two chapters, to suggest that competence and confidence in mental calculation is characterised not just by correct answers but by a mode of expression. This mode of expression shows consistency across the indicators used in this study. These consistencies suggest that the individual's mental construction may be manifested in a variety of different but associated ways.

7.2 Characteristics of contexts

In this section the characteristics of the distributions of the categories of the indicators are explored for each context, i.e. calculation, procedures and concepts. The purpose is to examine whether differences in outward manifestations tell us about learning or simply about a disposition of the individual. There are further indications that a mode of expression, or mental visual imagery, may be characteristic of the context rather than the child. The distinction is important for the following reason.

If comparisons of pupils show differences in calculation contexts that are not apparent in non-calculation contexts it could indicate differences in learning from experiences of calculation. If profiles of categories are consistent across contexts then it could merely indicates a consistency in mode of expression.

Furthermore, if pupils begin to use modes of expression in calculation that are characteristic of 'everyday' familiar contexts then this could indicate a growing confidence in a mathematical routine. It is important to identify differences and similarities in the category profiles of different contexts in order to identify the indications of learning.

7.2.1 Variables associated with accuracy

In this section the analysis continues at the level of the individual response. The question addressed here is "What are the differences in language and mental visual imagery associated with successful and unsuccessful calculations?".

It is important to identify the characteristics of language and mental visual imagery associated with correct and incorrect calculation in order to identify characteristics of successful calculators. When successful calculators do something that is not simply characteristic of successful calculation we may be able to identify what makes them different. When unsuccessful calculators do something which is characteristic of successful calculation we may have evidence that they are becoming successful.

The analysis involves a comparison of percentage of correct answers associated with each category. Some categories of the indicators accompany higher proportions of correct answers than others. This allows us to summarise the 'characteristics of accurate calculations'. 'Generality' and some linguistic indicators are the most important. Accuracy is independent of 'image', 'metaphor' and 'method'.

7.2.1a Image, metaphor, generality and method

The discussion in this section relates to contingency tables in Appendix 4.6.

Table 7.5 indicates that in the mental calculation questions pupils who reported a mental visual image were more likely to also have given a correct answer. There is a statistically significant difference in the proportions (Appendix 4.6(i)):

	image	no image	Totals
Percentage correct	69%	60%	63%
Number of responses	305	853	1158

Table 7.5 Percentage of correct calculations when mental visual image reported

There appears to be an advantage in having a mental visual image that accompanies a mental calculation.

Of the 86 responses where a mental visual image of a 'sum' was reported 62% were correct whilst 59% of the other responses were correct. This difference was not statistically significant (Appendix 4.6(ii)). Thus a mental visual image of a written calculation did not appear to be an advantage in terms of association with a correct answer.

The majority of images were of symbols (71%) and 'specific' images were uncommon (7%). The distribution of categories of 'image' were similar for correct and incorrect answers (Appendix 4.6(iii)). This suggests that the category of 'image' is not a factor in accuracy. There was, however, a slight (but not statistically significant) tendency for 'symbolic' images to be associated with correct answers and for images of objects to be associated with wrong answers.

In common with mental visual imagery the style of metaphor and the method described did not appear to be factors in the accuracy of response (Appendix 4.6(iv)). 'Creation' metaphors were the most common (51%) and 'motion' the least common (15%). 'Counting' methods were used in 21% of responses, 'holistic' in 41% and 'algorithmic' in 38%. Again there was a slight (but not statistically significant) tendency for correct answers to be associated with more 'abstract' categories and 'concrete' categories with incorrect answers (Appendix 4.6(v)).

The difference in distributions of 'generality' categories for correct and incorrect answers indicated in Table 7.6 was highly statistically significant (Appendix 4.6(vi)). It appears that those who gave a 'generic' or 'general' response were most likely to be correct. Those who expressed themselves in particular terms were less likely to have given a correct answer:

	particular	generic	general	Totals
Percentage correct	56%	75%	78%	63%
Number of expressions	331	434	137	902

Table 7.6 Percentage of correct calculations for each category of 'generality'

These results confirm the impression given by the previous correlation figures that 'generic' and 'general' responses are associated with accuracy and that inaccuracy is associated with expressions in 'particular' terms.

7.2.1b Pronouns and Tense

Tables of raw data for this section appear in Appendix 4.7.

When distributions of pronoun use in correct and incorrect responses are compared the difference is statistically significant (Appendix 4.7(i)). Table 7.7 suggests that pupils whose response was described exclusively using "you" were more likely to have been correct than those who used "I" exclusively or a mixture of the two :

	"I" only	"I" and "you"	"you" only	Totals
Percentage correct	65%	66%	79%	67%
Number of expressions	520	166	97	783

Table 7.7 Percentage of correct calculations for combinations of pronouns

The difference in distribution of category of "I" use for correct and incorrect answers was highly statistically significant (Appendix 4.7(ii)). It appears that when "I" was used to describe specifically what the individual had done in that particular question the answer was less likely to have been correct than if "I" indicated what the individual did in general:

	"I" categ 1	"I" categ 2	"I" categ 3	Totals
Percentage correct	63%	68%	79%	66%
Number of expressions	489	47	133	669

Table 7.8 Percentage of correct calculations for categories of "I"

By contrast "you" was used much less often than "I" and the difference in distribution for correct and for incorrect answers was not statistically significantly (Appendix 4.7 (iii)). The association of low category with wrong answers and high category with correct is, however, also apparent.

The difference between the distribution of tenses in which children chose to phrase their answers when they had given a correct answer and after an incorrect answer was also statistically significant (Appendix 4.7(iv)). It appears from Table 7.9 that exclusive use of past tense was accompanied by a lower proportion of correct answers than present and mixed tense expressions:

	past	present	mixed	Totals
Percentage correct	62%	71%	71%	67%
Number of expressions	391	244	259	894

Table 7.9 Percentage of correct calculations for different tenses

These two linguistic indicators, use of pronouns and tense, provide characteristics of descriptions given after correct answers. The exclusive use of "you", the use of "I" in a 'general' sense and expressions in non-past tense terms are characteristic of successful attempts at questions. Unsuccessful attempts are more often couched in particular terms and in the past tense.

7.2.1c Causal connectives

Tables of children's use of causal connectives are in Appendix 4.8.

Children's use of "if", "then" and "so" again followed the trends common to the pronouns. The lowest category for each was more often associated with a wrong answer than higher categories (Appendices 4.8(i) - 4.8(iv)). "Because" was an exception in that low category use was associated more with correct than incorrect answers. "Because" is peculiar in that children who simply said the word without following up with a justification (category 1) often gave the correct answer whilst more appropriate use of the word could accompany an incorrect answer.

Low frequencies in the contingency tables, however, meant that a test of statistical significance was not possible (Appendices 4.8(i) - 4.8(iv)) . Combining categories to conduct a significance test is not possible for "because" but for each of "if", "then" and "so" categories 2 and 3 were combined to give a two by two table. The test then showed statistically significant differences in the use of the causal connectives in correct and incorrect responses. The tables (Appendices 4.8(v) - 4.8(vii)) suggest that higher category use of "if" and "then" is associated with more correct answers and category 1 with incorrect. The distribution of categories of "so" was not significantly different for correct and incorrect answers.

Table 7.10 indicates that when any of the causal connectives were used, however, it was more likely that the correct answer had been given than those responses where no causal connective had been used. The differences in the proportions correct were all highly statistically significant (Appendices 4.8(viii) - 4.8(xi)):

	% correct when used	% correct when not used
"because"	79% (of 189)	59% (of 969)
"so"	72% (of 284)	59% (of 874)
"if"	82% (of 135)	60% (of 1023)
"then"	68% (of 448)	59% (of 710)

Table 7.10 Percentage of calculations correct when connectives used and not used

Here there are again indications of linguistic characteristics of descriptions of successful calculations. Explanations given after a correct answer are more likely to involve the use of a causal connective than those given after incorrect answers. The higher categories of use of "if" and "then" are also associated with these correct answers.

7.2.1d Analogy

Unlike causal connectives the category of use of "like" appears to be independent of the accuracy of answers. A test of statistical significance on the contingency table (Appendix 4.9(i)) revealed no significant difference in the distribution of category of use of "like" in correct and incorrect responses. Grouping of categories (Appendix 4.9(ii)) and comparing use of "like" and non-use of "like" (Appendix 4.9(iii))both confirm that how, or whether, pupils used the word was not associated with accuracy .

The use of "like" may give indications about a pupil's ability to make appropriate connections and thus give evidence of relational understanding but from this analysis it does not discriminate between accurate and inaccurate responses.

7.2.1e Summary of associations with accuracy

Children were asked to perform a calculation then asked what was in their head when they were thinking about it. They were then often asked if they saw anything in their heads. The mental visual image described may or may not have been present when the calculation was performed and the pupils may not even have had an image when they said they had. What comes out of the analysis of the mental visual image that they chose to report is that those who describe symbols are more likely to have given a correct answer. Incorrect answers are more often accompanied by descriptions of specific objects. Whether it is a real or fictitious image its character is associated with accuracy so if children are choosing to invent an image they are more likely to invent a 'symbolic' one if they are correct.

Whilst there may be reservations concerning the presence, or not, of an image, the language children use is likely to be more spontaneous and not so dependent on a wish to please the interviewer. This said, children may feel they ought to give a general expression of a rule and it is clear that when they do so they are more likely to have given the correct answer. Those who do not talk of anything other than the particular numbers of the question are more likely to be wrong. This may be because they did not feel a need to give a general rule or were incapable of doing so; in either case the lack of 'generic' or 'general' terminology is characteristic of inaccuracy.

The method employed and the metaphoric language used does not show the same strong associations with accuracy. The children are almost as successful when using 'counting' as they are when using methods which make use of place value and mental calculation strategies. Similarly language of 'collection', 'motion' or 'creation' are almost equally likely to accompany success or failure in a question. These indicators are thus not useful discriminators of accuracy.

Use of pronouns and the tense of an answer are linguistic indicators of competence in calculation. Correct answers are more likely to be accompanied by exclusive use of "you" and by tenses other than past. Descriptions associated with incorrect answers are more often in the past tense and use "I" exclusively. The category of pronoun use is also important. Accurate answers are more likely accompanied by use of "I" and "you" in the sense of what is done with this type of number or what to do in general. Inaccuracy is associated with descriptions of what the individual did with those particular numbers.

The use of "like" is not associated with accuracy but causal connectives tend to be used by children who have given the correct answer. This suggests that accuracy and sophisticated use of language are associated.

This section provides evidence that there are some associations between accuracy and the 'indicators of learning' and categories of linguistic indicators. No claim is made that the one determines the other, linguistic competence may or may not affect arithmetic competence, but it appears that certain aspects of language use are associated with accuracy in mental calculation. It could be argued that listening to the way children describe their calculations is a better indicator of their progress than merely finding out the method they employ.

7.2.1f The characteristics of accurate answers

The characteristics of accurate answers are:

- Mental visual images were more frequently associated with correct answers
- 'Generic' or 'general' responses were more likely to be correct and 'particular' responses were more likely to be wrong.
- Exclusive use of "you" was more often associated with correct answers than use of "I" exclusively or a mixture of the two.
- If "I" indicated what the individual did in general the answer was more likely to be correct. If "I" indicated only what the individual had done in that particular question the answer was more likely to be wrong.
- Expressions in non-past tense terms were most likely to be correct. Past tense was most frequently associated with wrong answers.

- The use of "if" and "then" to imply 'generic' or 'general' causality were most frequently associated with correct answers.
- The use of causal connectives was most frequently associated with correct answers.

7.2.2 Mathematical and non-mathematical consistencies.

In addition to the 45 calculation questions there were 33 questions not requiring calculation. Of these non-calculation questions 8 required a description of a mathematical procedure making a total of 53 computation questions. Of the 25 non-computation questions 12 required a description of what came to mind when a mathematics word was given. There were thus 13 non-mathematical procedure or 'concept' questions. In this section these different groups of questions will be compared to determine whether the quality of mental visual imagery and language are different in these contexts. The analysis is for all responses and involves tabulating the number of responses in each category for each different context to give a view of whole-sample differences.

The research question addressed here is:

What are the characteristics of mental visual imagery and language used in mathematical and non-mathematical contexts?

The usefulness of this analysis is twofold

- When characteristics of 'everyday' situations are evident in mathematical situations it can indicate confidence and familiarity.
- When individuals behave differently in different contexts it indicates different learning has occurred as a result of experience of that context.

7.2.2a 'Image'

The mental visual images evoked by questions in the different contexts were qualitatively different. The differences in distributions of the categories was highly statistically significant. Table 7.11 indicates that 'specific' and 'representative' images are more common in non-mathematics contexts than in both calculations and non-calculation questions related to mathematics. Note that with these tables the percentages are for row totals so for instance 71% of the 305 calculation responses involving mental visual images were 'symbolic':

	specific	iconic	symbolic	responses
Calculation	7%	22%	**71%**	305
Other mathematics	22%	28%	**50%**	120
Non mathematics	**46%**	**54%**	7%	118
Total images	93	164	286	543

Table 7.11 Percentaged distributions of 'image' for each context

This difference may be due to the influence of the pupils' previous experiences in mathematics where symbolic representation is predominant. The high-imagery non-mathematics words are also more likely to give rise to 'specific' and 'iconic' images.

From this it appears that the style of mental visual imagery may be partly determined by the context.

7.2.2b 'Metaphor', 'method' and 'generality'

Metaphor and 'method' may only be compared across the mathematics contexts. There was a suggestion here that metaphoric language of 'motion' is less common in describing procedures than in calculation (Appendix 4.10(i)). Also that 'holistic' methods are more common than digit manipulation (Appendix 4.10(ii)). Both of these may be explained by a relatively high proportion of procedures relating to fractions and division. Neither fractions nor division had been taught using number lines and the division algorithm only taught in time for the last interview. This gives some corroboration for the suggestion that pupils' metaphors and methods are largely determined by what have been common classroom activities.

Categories of 'generality' may be compared across all aspects of the interviews. The differences in distributions of the categories in the different contexts were highly statistically significant (Appendix 4.10(iii)). Table 7.12 indicates that pupils demonstrated in non-mathematical contexts that they could give responses at all levels of 'generality'. In calculation, mathematics procedure and mathematics concept items they were more likely to express themselves in 'particular' or 'generic' terms. :

	particular	generic	general	responses
Calculation	37%	**48%**	15%	902
Other mathematics	27%	**53%**	20%	386
Non mathematics	**31%**	**36%**	**33%**	281
Total responses	521	741	307	1569

Table 7.12 Percentaged distributions of 'generality' for each context

This suggests that mode of expression is linked to context and this is also apparent when math concept responses are compared with non-mathematics concept responses (Appendix 4.10(iv)). Table 7.13 shows that pupils were more inclined to use 'generic' expressions for mathematics words than non-mathematics words:

	particular	generic	general	responses
Mathematics word	33%	**54%**	13%	227
Non-maths word	38%	37%	26%	213
Total responses	156	200	84	440

Table 7.13 Percentaged distributions of 'generality' for each non-calculation mathematics contexts

When the mathematics procedure questions are compared with the non-mathematics procedure in Table 7.14 (Appendix 4.10(v)) there is a pronounced tendency for non-mathematics procedures to be expressed in general terms:

	particular	generic	general	responses
Mathematics procedure	17%	**53%**	30%	159
Non-maths procedure	10%	34%	**56%**	68
Total responses	34	107	86	227

Table 7.14 Percentaged distributions of 'generality' for each procedure context

This may be interpreted to demonstrate that pupils have abstracted, or have learned, general rules in these familiar everyday contexts.

7.2.2c Pronouns and tense

Linguistic indicators are easier to compare across question contexts and most show marked differences in their distributions. This is important because it makes clear the distinction between the styles of language which have been shown to accompany accuracy (§7.2.1f) from the style of 'everyday' use of language.

Use of tense is distinctive in mathematics contexts with past tense predictably used frequently when children described how they had performed a calculation. In Table 7.15 the two more comparable sets of questions, 'non-calculation mathematics' and 'non-mathematics' questions, illustrate that mathematical talk more often uses mixed tenses. The differences in the distributions is highly statistically significant (Appendix 4.11(i)).

	past tense	present	mixed	responses
Calculation	**44%**	27%	29%	894
Other mathematics	7%	**65%**	28%	328
Non mathematics	6%	**81%**	13%	232
Total responses	426	646	382	1454

Table 7.15 Percentaged distributions of tenses for each context

Tables of pronoun use also indicate differences in distributions. Comparison of the mixture of "I" and "you" use (Appendix 4.11(ii)) demonstrates that exclusive use of "you" was common in both non-calculation mathematics questions and non-mathematics procedure questions. It was relatively uncommon in descriptions of calculation. This emphasises that the use of "you" is characteristic of descriptions of procedures without particular numbers given. Exclusive use of "you" has been shown to accompany accuracy in calculations so it could be an indicator of both a familiar and successful procedure.

Use of "I" in the three categories has similar distribution in mathematics and non-mathematics (Appendix 4.11(iii)). When "I" is used to indicate what the individual does in general (category 3) it is uncommon, and thus since it has been shown to accompany accuracy it indicates use of a successful strategy. This point is emphasised when questions on mathematical procedure are compared to those requiring calculations (Appendix 4.11(iv)). Here higher proportions of 'generic' and 'general' use of "I" is seen to be characteristic of procedures.

The distribution of the categories of use of "you" also demonstrates the same pattern (Appendix 4.11(v)). There is thus a confirmation that use of pronouns in the sense of what "I" or "you" do in non-particular terms is characteristic of talk of successful procedures. When the frequency of use of pronouns is compared in the non-calculation contexts (Appendices 4.11(iii) and 4.11(iv)) extensive use of "you" is seen clearly to be characteristic of mathematics. This adds weight to the suggestion that children are adopting the mathematics classroom speech style when using "you".

7.2.2d Causal connectives and "like"

"Like", "because" and "so" each have very similar distributions across the contexts (Appendix 4.11(viii) gives "like" as an example). "If" (Appendix 4.11(ix)) and "then" (Appendix 4.11(x)) are, however used quite differently in responses to mathematics and non-mathematics questions. There is evidence here that children are capable of using each at a level of describing general consequences in non-mathematics contexts but are less likely to do so in mathematics. When the word is used in this way in mathematics it has been seen above to be associated with accuracy. This confirms use of "if" and "then" as important indicators when used in mathematics because they are associated with accuracy.

The *total frequency* of use of each of the connectives was compared in the different contexts. The tables indicate that use of "then" and "so" (of any category) is more common in mathematics but "if" and "because" are used in non-mathematics in similar proportions to mathematics contexts (Appendices 4.11(xi) - 4.11(xiv)).

7.2.2e Summary of characteristics of contexts

- Images related to previous experiences and images of specific objects are more common in non-mathematics contexts. A high proportion of mathematics images are of symbols.

There is support here for the suggestion that children are re-presenting the common classroom experiences.

- Expressions of generality are equally likely in each category in non mathematics contexts. In mathematics 'generic' expressions are most common.

It could thus be argued that the teachers' practice of describing procedures in terms of a generic example is reflected in the pupil's modes of expression.

- Familiar procedures are more likely to be expressed in 'general' terms.

When children use 'general' expressions in calculation contexts it may indicate that this is a familiar routine.

- Use of present and mixed tenses is characteristic of non-calculation contexts.

Past tense is to be expected when pupils are asked how they performed calculations. When they use present tense in these circumstances it could indicate confidence in following a common routine.

- Exclusive use of "you" is characteristic of procedures.

This indicates that not only is "you" common to the speech community of this mathematics classroom but more generally to our culture. When it is used in calculation contexts it is indicative of a familiar routine.

- Use of "if" and "then" in non-'concrete' senses is less common in calculation than other contexts.

When these two connectives are used in this way in calculations it suggests that there is confidence in the explanations.

7.2.3 Summary of characteristics of contexts

The comparisons between responses to mathematical and non-mathematical items in the interviews reveals characteristics of the mathematical talk of this sample. The comparison between calculations and mathematical procedures emphasises distinctive use of language associated with procedures. The statistically significant differences between distributions in the different contexts are summarised in Table 7.16:

Indicator	Comparisons			
	mathematics/ non-maths	non-calc/ non-Maths	calculation/ procedure	correct/ incorrect
Image	different	different		
generality	different	different	different	different
"I" categories	different	different		different
"you" categories	different		different	
"I" and "you"	different	different		different
tense	different	different	different	different
"like" categories			different	Similar
"if" categories	different	different	different	different
"then" categories	different	different		different
"so" categories				
"because" categories				

Table 7.16 Summary of context comparisons

This analysis of differences and the characteristic of accurate calculations leads to the following conclusions:

- In mathematics mental visual images of symbols are more usually reported than in other contexts but accuracy is independent of the category of 'image'. Reporting an image is associated with accuracy.
- 'General' expressions are characteristic of familiar procedures and also of accurate answers. When children start to use them they may be indicating a firmly established and effective mental construction for the process.
- There is a higher proportion of the use of "I" in the sense of "What I do 'in general'" in procedures than in other contexts. This suggests that its use in this sense is a

characteristic of procedural language. When it occurs it is also associated with accuracy. This suggests it is indicative of a successful and familiar routine.

- "You" in the sense of "What one does in general" is also characteristic of procedures (mathematical and non-mathematical) but is used more frequently in mathematics. Its use, however, is not necessarily associated with accuracy. Exclusive use of "you" is characteristic of mathematical and non-mathematical procedures and is associated with accuracy.
- Use of past tense is characteristic of descriptions of calculations. Procedures and non-mathematical items are most usually described in present or mixed tenses and since this use of tense is usually associated with accuracy it is an indicator of a familiar and successful procedure.
- The frequencies involved in the analysis of the causal connectives are in some instances small and thus generalisations are suspect. There is some evidence however that "if" and "then" are used to infer a consequence in a higher proportion of non-mathematical contexts than mathematical. When "if" is used as part of a pupil's description of their calculation it is associated with a correct answer so it is again a useful indicator in mathematical contexts.

7.3 Characteristics of children

The characteristics of successful calculations and of different contexts allow us to make comparisons between children which take these global distinctions into account. Discrepancies between category profiles of groups, and of individuals, with these characteristic profiles have been noted as important in the introduction to §7.2. When groups of children are distinguishable in calculation contexts and not in other contexts it suggests that there is not a global disposition toward a mode of expression.

In this section we look first at profiles of groups (§7.3.1) then at individuals (§7.3.2). The differences that become apparent in terms of the 'indicators' are indications of the differences in learning. This allows us to look at variation with achievement levels and variation with time.

Many of the characteristics associated with accuracy listed in §7.2.1f are associated with higher achievers but there are indications also of the extra flexibility demonstrated by pupils who are most successful in difficult questions.

The picture that emerges gives no evidence of a pre-disposition, by individuals or groups, toward a mode of expression which is consistent across contexts. This suggests that language use is indicating different mental constructions in mathematics and not merely a linguistic trait.

7.3.1 Differences between groups of children

In the previous analysis associations between variables have been tabulated without reference to which pupils have given the response. In this section characteristics of different types of children are explored. This was achieved by tabulating the responses given by different groups of children in each category of the indicators of learning and linguistic indicators. The differences in the distributions thus provides information about characteristics of children at different achievement levels. The sample was divided by: levels of achievement in written tests, mathematics set, reading ages, levels of accuracy in the interviews, quantities of mental visual imagery reported. Analysis at this level provides more data in order to answer the questions:

- Is there evidence that individuals or groups have a style of language or mental visual imagery which is common to mathematical and non-mathematical contexts?
- How do calculation strategy, mental visual imagery and language relate to achievement?

7.3.1a Comparison of upper- and lower-achievers

The children were grouped into the 13 pupils who achieved above the mean in the combined SATs and the 13 pupils who were below the mean.

Tables of frequencies of categories of each indicator for the two groups were compiled first for the calculation questions then for the other contexts. (Appendix 4.12)

The indicators which are associated with accuracy (see § 7.2.1f) all showed highly statistically significant differences in distributions of categories for the two groups in calculations. This is not surprising since the upper group achieved 71% correct answers whilst the lower group achieved 54% (Appendix 4.12(i)).

The differences in distributions for the two groups were as follows:

Table 7.17 summarises the data in Appendix 4.12(ii). It illustrates that the lower achievers were most likely to talk in terms simply of the calculation they had just performed and the upper achievers to talk in 'generic' and 'general' terms:

	particular	generic	general
Upper Combined SATs	28%	**52%**	**20%**
Lower Combined SATs	**47%**	**44%**	9%

Table 7.17 Percentaged distribution of 'generality' for upper- and lower-combined-SATs score groups

Table 7.18 summarises the data in Appendix 4.12(iii). It illustrates that the lower achievers were more likely than the upper achievers to use 'collection' metaphors. The upper achievers most frequently used 'creation' metaphors:

	collection	motion	creation
Upper Combined SATs	28%	13%	**59%**
Lower Combined SATs	**42%**	16%	42%

Table 7.18 Percentaged distribution of 'metaphor' for upper- and lower-combined-SATs score groups

Table 7.19 summarises the data in Appendix 4.12(iv). It illustrates that the lower achievers were more likely to use 'counting' methods and the upper achievers to use 'algorithmic' methods:

	counting	holistic	algorithmic
Upper Combined SATs	13%	41%	**46%**
Lower Combined SATs	**29%**	42%	29%

Table 7.19 Percentaged distribution of 'method' for upper- and lower-combined-SATs score groups

Table 7.20 summarises the data in Appendix 4.12(v). It illustrates that the responses of the lower achievers were most frequently in the past tense whilst the upper achievers showed more variation:

	past	present	mixed
Upper Combined SATs	**37%**	**31%**	**32%**
Lower Combined SATs	**52%**	23%	25%

Table 7.20 Percentaged distribution of tense for upper- and lower-combined-SATs score groups

Table 7.21 summarises the data in Appendix 4.12(vi). It illustrates that the lower achievers predominantly used "I" in the sense of "What I did". This was true also of the upper achievers but they had a higher proportion of "What I do":

	"I" categ 1	"I" categ 2	"I" categ 3
Upper Combined SATs	67%	8%	**25%**
Lower Combined SATs	**86%**	5%	10%

Table 7.21 Percentaged distribution of "I" categories for upper- and lower-combined-SATs score groups

There were no other statistically significant differences except in mental visual imagery. Table 7.22 summarises the data in Appendix 4.12(vii). It illustrates that the higher achievers have a tendency toward more symbolic images whilst lower achievers have a higher proportion of images of pedagogic representations:

	specific	iconic	symbolic
Upper Combined SATs	6%	15%	**79%**
Lower Combined SATs	8%	**30%**	62%

Table 7.22 Percentaged distribution of 'image' for upper- and lower-combined-SATs score groups

The differences between the two groups may be predictable since each of these indicators, except 'image', is associated with accuracy. It is important to note that there are *no* significant differences in the distributions of these indicators in non-mathematics contexts. This suggests that high achievers do not have a global disposition toward non-'particular' expressions of generality, symbolic mental visual image etc. but that they do so in the context of mental calculation.

The implication here is that these indicators give indications about the mental constructions related to mental calculations. They do not simply point to a preferred mode of response by this group of children.

7.3.1b Comparison of the teaching groups

In a comparison between the two teaching groups, Set 1 and Set 2, the same indicators, except 'image', showed the same differences in distributions as those listed for the two achievement groups above. In the calculation items Set 1 pupils were more accurate in interviews than Set 2 pupils , expressed themselves in more general terms, used more object 'creation' language and more single-digit methods. They used mixed tenses more often and more "I" of categories 2 and 3. The difference in the various distributions were all highly statistically significant (Tables in Appendix 4.13)

There was no significant difference between the two sets in respect of any of the other indicators. The distribution of categories of mental visual image, "you", "like" and causal connectives were very similar for each set.

It is not surprising that the differences occurred for those indicators which are associated with accuracy since the pupils in Set 1 were more accurate. It is thus likely that differences between the sets are explained more by their higher mathematical performance than by differences in experiences. This view is endorsed when non-calculation items are compared. There is then no significant difference between the two sets of children. This is important because it indicates that there were no differences between the two sets which could have indicated differences in experiences. The children in both sets had similar classroom activities and we would expect there to be only the differences related to accuracy since the pupils had been assigned to the sets by achievement in mathematics.

7.3.1c Comparison of upper- and lower-reading-age pupils

All but one of the 13 upper-reading-age pupils were also upper-combined-SATs pupils so not surprisingly the contingency tables divided by reading age showed much the same differences in distributions as when divided by SAT scores. It was also noted in §7.1.1a that there is a strong correlation between mathematics achievement and reading age so the similarities are to be expected.

The issue concerning the correlation between achievement in mathematics and linguistic achievement, as measured both by reading age and the indicators used in this study, will be raised in the concluding chapter. There are implications for further research.

7.3.1d Comparison of high-, middle- and low-accuracy pupils

It has been suggested in the analysis of children grouped by achievement in SATs that the main factor in the differences in distributions of the indicators is the greater accuracy of the upper-SAT-score pupils. In order to examine differences between pupils grouped by their accuracy in the calculation questions in the interviews, they have been grouped into high- (scores greater than 1 sd above mean), middle- (scores within 1 sd of mean) and low- (scores less than 1 sd below mean) accuracy types. There were 5 high-accuracy and 5 low-accuracy pupils. The differences then become more marked between the most and least accurate. Most of this difference can, however, again be explained in terms of the distributions typical of correct and incorrect answers seen above.

Predictably the higher-accuracy pupils reported higher proportions of 'symbolic' images than low-accuracy pupils in the calculation questions. This also confirms Pitta's findings. Table 7.23 summarises the data in Appendix 4.14(i). It illustrates that low-accuracy pupils were distinctive in reporting a high proportion of images of pedagogic representations whilst the high-accuracy pupils reported a high proportion of 'symbolic' images. The middle-accuracy pupils were similar in their distributions to the high-accuracy group. It is also important to note that the middle-accuracy pupils' distribution lies numerically between the other two. Since they have been grouped by accuracy there is some verification here for her hypothesis that 'specific' 'image' is associated with inaccuracy and 'symbolic' is associated with an increasing accuracy:

	specific	iconic	symbolic	responses
High accuracy	5%	16%	**79%**	82
Middle accuracy	6%	19%	**74%**	175
Low accuracy	13%	**40%**	48%	48

Table 7.23 Percentaged distribution of 'image' in calculations for three accuracy groups

Table 7.24 summarises the data in Appendix 4.14(ii). It illustrates that when their categories of 'image' in non-calculation questions are compared the high-accuracy pupils distinguish themselves by reporting a lower proportion of 'specific' images and consequently higher proportions of 'representative' and 'general' images. There is some confirmation here of Pitta's findings that the highest and lowest achievers have a dispositions toward a mode of mental visual imagery that is consistent in mathematics and non-mathematics contexts.

	specific	iconic	symbolic	responses
High accuracy	**19%**	**50%**	**31%**	54
Middle accuracy	30%	42%	28%	141
Low accuracy	**33%**	**42%**	**26%**	43

Table 7.24 Percentaged distribution of 'image' in non-calculation questions for three accuracy groups

Table 7.25 summarises the data in Appendix 4.14(iii). It illustrates that low-accuracy pupils mostly expressed themselves in particular terms in calculation questions. High-accuracy pupils used a much higher proportion of 'general' expressions and again the positioning of the middle-accuracy group emphasises the apparent association of 'particular' with inaccuracy and 'general' with accuracy:

	particular	generic	general	responses
High accuracy	32%	48%	**20%**	184
Middle accuracy	33%	52%	15%	583
Low accuracy	**60%**	30%	10%	135

Table 7.25 Percentaged distribution of 'generality' in calculations for three accuracy groups

It is also important to note that differences in the distributions of categories of 'generality' in non-mathematics contexts for the three groups were *not* statistically significant (Appendix 4.14(iv). There is thus no evidence of a disposition by these groups of children to a mode of expression which is consistent across contexts. The differences in distributions for calculation questions seem attributable to differences in learning not to linguistic characteristics of the children.

The distinction between high-accuracy and low-accuracy pupils in terms of their expressions of generality is emphasised when their responses for correct and incorrect answers are tabulated . For high-achievers expressions of a 'generic' or 'general' type are more emphatically associated with correct answers. 89% of 'generic' expressions were associated with correct answer and 92% of 'general expressions' in comparison with 66% of 'particular' responses (Appendix 4.14(v)). It seems that when these pupils have given a correct answer they are much more likely to express themselves in a non-'particular' way. Thus their use of higher level expressions of generality is associated with successful use of the procedure described. For low-accuracy pupils none of the expressions was necessarily associated with ability to use the procedure accurately. Only 46% of 'particular', 48% of 'generic' and 43% of 'general' expressions accompanied correct answers (Appendix 4.14(vi)).

Table 7.26 summarises the data in Appendix 4.14(vii). It illustrates that metaphors also show the trends associated with accuracy in that low-accuracy pupils also used predominately 'collection' language, whilst high-accuracy pupils used predominantly 'creation' metaphors. The high-accuracy pupils also, however, were more inclined than the middle accuracy group to use 'object collection' rather than 'motion' metaphors. Metaphor is thus not as clearly associated with achievement.

	collection	motion	creation	responses
High accuracy	36%	10%	**55%**	185
Middle accuracy	30%	16%	54%	595
Low accuracy	**49%**	16%	35%	145

Table 7.26 Percentaged distribution of 'metaphor' in calculations for three accuracy groups

Table 7.27 summarises the data in Appendix 4.14(viii). It illustrates that in choice of method low-accuracy pupils predictably used the most 'counting' methods and higher accuracy pupils used both 'holistic' and 'algorithmic' methods. This may indicate their flexibility in choice of methods. It is also an indication that the progression is not from 'counting' through 'holistic' to 'algorithmic' but rather that 'holistic' methods are most associated with accuracy. The middle-accuracy group's distribution, which is again between the other two groups, adds to this impression:

	counting	holistic	algorithmic	responses
High accuracy	11%	**47%**	42%	177
Middle accuracy	19%	42%	40%	577
Low accuracy	**43%**	30%	27%	125

Table 7.27 Percentaged distribution of 'method' in calculations for three accuracy groups

These tables for 'generality', 'metaphor' and 'method' taken together give an indication that the low achievers, irrespective of method used, are much more likely to simply describe what they had done. The higher achievers are more likely to use the numbers to explain a procedure or to simply state general rules. When low-accuracy pupils used more generalised expressions they were as likely to have been wrong as when they simply described what they did in terms of the particular numbers. This suggests that they had acquired the language of the general procedure without the ability to carry it out.

This analysis has also given an impression of which of the indicators show the strongest correspondence between category of abstraction and level of accuracy. 'Generality' and 'image' show the clearest associations in that the 'concrete' category is associated with least accuracy and 'abstract' is associated with greatest accuracy. 'Holistic' methods though not as abstract as the 'algorithmic' methods are slightly more associated with accuracy. Metaphor gives mixed messages.

The linguistic indicators also show differences between the three groups of pupils. There were however distinctions between the highest accuracy pupils and the middle-accuracy pupils that are not simply explained by the trends associated with correct answers.

Table 7.28 summarises the data in Appendix 4.14(ix). It illustrates that high-accuracy pupils used more past tense than might have been expected from comparison with the accurate answers distribution. This could indicate a confidence in the method they have just employed:

	past	present	mixed	responses
High accuracy	**49%**	31%	21%	183
Middle accuracy	39%	27%	34%	571
Low accuracy	**58%**	23%	19%	140

Table 7.28 Percentaged distribution of tense in calculations for three accuracy groups

High-accuracy pupils also used "I" exclusively (Appendix 4.14(x)) more frequently than is predictable from the accurate answers distribution and this fits with their predominant use of past tense.

These anomalies can largely be explained by the fact that the high-accuracy pupils answered difficult questions with their own methods when a taught procedure was not available. This point will be elaborated in the section on difficult questions below (§7.3.2c).

Table 7.29 summarises the data in Appendix 4.14(xi). It illustrates that the higher-accuracy pupils' category of use of "I" was exactly as would be expected of accurate answers and low-accuracy pupils showed a pronounced preference for 'specific' use of "I".

	"I" category 1	"I" category 2	"I" category 3	responses
High accuracy	**71%**	7%	**22%**	126
Middle accuracy	73%	3%	20%	499
Low accuracy	**93%**	4%	**3%**	136

Table 7.29 Percentaged distribution of "I" categories in calculations for three accuracy groups

In non-mathematics items there were also differences in use of pronouns. High-accuracy pupils demonstrated that they were also able to use "I" and "you" to talk about what they did in general in contexts where they had not been taught procedures and also gave general rather than particular properties for non-mathematical concepts. The frequencies involved were, however, too small for tests of significance (Appendices 4.14(xii) and (xiii).

Use of causal connectives in responses to calculation questions also distinguished the groups but there was no difference in their use of these words in non-calculation contexts. Tables of frequencies of use of each of the connectives show statistically significant differences in the proportions used by high- and low-accuracy pupils in calculation questions and no significant difference in non-mathematics questions (Appendices 4.14(xiv) - (xxi)). In calculation questions high accuracy pupils used a higher proportion of each of the connectives. An interpretation of this is that use of causal connectives is a characteristic of language associated with higher accuracy pupils in calculation but it is not a characteristic of their language per se. This could indicate that the use of causal connectives is an indication of learning in mathematics not simply an indication of linguistic ability.

7.3.1e Summary of differences between groups of children

The differences between Set 1 and Set 2 children may be explained by the high proportion of upper achievers and upper-accuracy pupils in Set 1 and the corresponding high number of lower achievers and lower-accuracy pupils in Set 2. When the achievement groups are compared their differences may be explained by the patterns of distributions in correct and incorrect answers. The indicators, apart from 'metaphor', showed a correspondence between categories. 'Concrete' categories were associated with low accuracy, 'abstract' with greater accuracy and 'representative' placed between them. The finer distinction between higher-, middle- and lower-accuracy pupils is almost entirely predictable from these patterns. Here we saw that 'generality' and image were better indicators of learning whilst 'method' showed that 'holistic' is more associated with accuracy than 'algorithmic'.

The important point to note is that the distinctions between the groups is peculiar to responses associated with calculations. There is no evidence that the differences are due to a trait in their language use in other contexts. The tendency of high-accuracy pupils to use "I" and past tense more than would be predicted from accuracy alone may be attributable to their responses to difficult questions and this point is addressed in the next section. Since the differences in use of pronouns in non-mathematics contexts do follow the pattern of accuracy it would seem that this talk in particular terms relates to a confidence in what they have just done. The greater use of connectives by the high-accuracy group in calculations may indicate their greater understanding of causality in this context.

The differences in groups may be summarised. High-accuracy pupils more frequently than low-accuracy pupils:

- report 'symbolic' images
- use 'generic' and 'general' expressions of generality
- use 'creation' metaphors
- use 'holistic' and 'algorithmic' methods
- use "I" in a 'general' sense

low-accuracy pupils more frequently than high-accuracy pupils:

- report 'specific' images
- use 'particular' expressions of generality
- use 'collection' metaphors
- use 'counting' methods
- use "I" in a 'particular' sense

7.3.2 Comparisons of questions and individual pupils

In this section the question is first taken as the unit of analysis to compare questions of different facility levels. The pupils then become the focus of attention to make comparisons between performance in different types of questions over the period. With the information gained about category profiles, for contexts and for groups of children, we are now in a position to identify the indications of differences between individuals and groups of children over time. We can also identify how the most successful children distinguish themselves in difficult questions.

7.3.2a Comparison of questions

The preceding analysis which shows differences in responses for correct and incorrect answers could be construed to suggest that the facility level of the question (the number of correct answers given) might determine the category of response in calculation questions. From this view point the questions which children have answered most accurately would lead to the most 'abstract' responses for each indicator. There is no evidence of this. When the number of responses in each category for 'generality', 'image', 'metaphor' and 'method' are tabulated for each of the calculation questions there are no statistically significant correlations with the facility level (Appendix 4.15). This suggests that the difficulty of the question does not determine the style of the response.

It is worth considering questions at each end of the facility scale. High facility questions that do not evoke high levels of 'abstract' generality are those which can be done most easily by 'counting' or the answer is known. In these circumstances there is little need for expressions of generality. "13 add something is 18. What is the something?" and "What comes after 12386" are like this. This is an important point because it gives a clue to why high accuracy pupils do not necessarily express their descriptions of their calculations in general ways. For these pupils an answer may be known or mentally calculated without apparent thought. Pupils who need to think more consciously about the question are likely to explain in more detail what they did, what procedure they used and perhaps express it as a general rule.

Low facility questions which evoke 'generic' and 'general' expressions associated with wrong answers are also of importance. Response to both "65 subtract 29" and "1097 add 10" indicate, by the high number of pupils using 'generic' expressions, that the procedures for adding ten and for subtracting were well known. The relatively low facility of these questions perhaps indicates that pupils knew what to *do* with simpler examples yet were unable to do it with large numbers or where decomposition was involved. Children who know, and can express, the rules yet get the answers incorrect may be procedural rather than proceptual thinkers. This mis-match between 'generality' and accuracy will be explored further in the next section on difficult questions.

7.3.2b Difficult questions

High facility questions such as "17 add 9" (26 correct answers in y4/1) and "48 add 23" (23 correct answers in y4/2) do not bring into focus what might be characteristics of the most successful pupils since all pupils show variations in categories of the indicators over the period of the study. This variation with time was suggested in §6.12 and will be explored further in § 7.3.2c.

When the low facility questions are tabulated separately the characteristics of the most successful pupils become more apparent. A facility range from 0 to 10 has been chosen to distinguish 'difficult' questions. There were 11 questions in this range (Appendix 4.16).

Pupils were ranked by the number correct in these 11 questions (mean = 2.2, sd = 2.1). The pupils were then grouped into 'high', 'middle' and 'low' groups. There were 6 high-scorers (score 4 to 7), and 7 low-scorers (score 0).

The percentages in each category of each indicator of learning were tabulated for the difficult questions and for the other questions for each of the groups. In Table 4.30 comparisons are made between pairs of rows. First the 'other' row gives the percentages of each category of each indicator for all the other calculation questions for the sample overall. The next row gives the percentages for the whole sample in the 'difficult' questions. It will be seen that there is an overall tendency: for 'general' category to decrease; for use of 'creation' metaphors to increase; for a higher proportion of responses not to give a method and the number of 'holistic' methods to increase; for there to be fewer reported mental visual images.

	generality				metaphor				method				image			
	No response	particular	generic	general	no response	collection	motion	creation	no reponse	counting	holistic	algorithmic	no response	specific	iconic	symbolic
Overall others	23	26	38	13	21	27	14	38	23	18	29	30	74	2	7	18
Overall difficult	24	34	35	8	20	28	6	47	31	8	37	23	75	2	2	21
High others	17	23	45	16	17	30	10	43	17	13	37	33	75	0	2	23
High difficult	9	38	39	14	11	32	8	50	18	5	55	23	76	2	0	23
Middle others	21	25	41	13	19	24	15	42	19	17	30	33	70	2	9	19
Middle difficult	19	34	41	7	15	26	4	55	25	9	38	27	71	3	2	23
Low others	32	33	26	10	30	30	13	27	34	23	19	24	79	3	7	11
Low difficult	45	31	19	4	35	27	6	31	53	10	19	17	82	0	4	14

Table 7.30 Comparisons of percentages of indicator categories between difficult and other questions

With these as comparison the table next gives two rows related to the 'high' group. This group usually uses more 'representative' and 'abstract' categories than the overall percentages. In the difficult questions the trends noted for the sample as a whole are apparent for this group. They were however more likely to give an answer that could be classified in these difficult questions. Their percentage of 'general' was less diminished than for the sample overall. They switched, however, more to 'particular' explanations and their increase in 'holistic' methods was more pronounced. This may indicate that they were able to switch to personal methods in these difficult questions.

By contrast the mid and low groups were much less likely than usual to use 'general' expressions. The change of methods was most distinctively different for the low group. They answered far fewer questions and were unable to switch to 'holistic' methods.

A higher proportion of 'creation' metaphors were used in these difficult questions by all pupils. This may be because the 'dificult' questions include a higher number of multiplication and division questions which had not been related to a number line in lessons. 'Image' showed little difference in the distributions for any of the groups.

Whilst trends are apparent, the differences in the distributions between difficult and other questions were *not* statistically significant for any of the groups, except in 'method'. This analysis suggests that the difficulty of the question is not a determining factor in 'generality', 'image' or 'metaphor' use. Success in these difficult questions demanded more 'holistic' methods.

This suggests that when familiar routines are not available (indicated by fewer 'general' expressions and fewer 'algorithmic' methods) then the successful pupils can switch to 'holistic' methods and explain them in terms of the particular numbers. The successful pupils do not need to rely on a routine.

7.3.2c Comparison over time

The two questions kept consistent in each interview allow comparisons to be made between the categories of the 'indicators' used over the period of the study .

17 add 9 was practised in Y3 as "add 10 and take 1" and the profile of categories in all 'indicators' remained similar over the year (Appendix 4.17). In Y4 children again showed similar category profiles except in 'method'. Here initially more 'holistic' methods were used but by the second term the 'algorithmic' method became the most common in response to this question. This may be because the children were taught the mental strategy of 'rounding to the nearest ten, adding tens separately then adjust'. However they added the 'tens' as single digits.

The other question used in each interview was 48 add 23. In Y3 term 3 the peak in use of single-digit methods coincided with the highest levels of 'generic' and 'general' expressions of 'generality', and highest 'creation' metaphor use, but lowest accuracy for this question. This may be attributable to the heavy emphasis on written algorithms in the classroom so that the procedure was familiar to pupils but carrying it out in their heads proved difficult. By the second term of Y4 'holistic' methods became more common and the facility level rose.

Both questions revealed little change in pupils' mental visual images. Images of symbols were common throughout. This, as we have seen (§7.2.2a), was true for other questions.

Individuals' responses to these questions were also tabulated (Appendix 4.17) and do not reveal any particular pattern. Pupils used a variety of methods, metaphors, and expressions of generality but showed some consistency in mental visual imagery. There was a greater consistency in pupils' methods and metaphors amongst the pupils who had had the most success but the same could be said of some of the less successful, such as Elspeth.

These two questions represented very familiar types of question for these children and some of the wide variations are perhaps due to this familiarity. A more interesting and informative picture emerged when difficult questions were examined.

7.3.2d Comparisons between years

Trends of change in categories of the indicators are not easy to describe for individuals. The variation that has been tabulated for different contexts (§6.1.2c, d) and for the addition questions (§7.3.2d) show wide variations within and between pupils. When children are grouped, however, then totals may be compared.

The upper half of the sample, ranked by accuracy in interviews, have been compared with the lower half in each of the two years. The percentages of total responses in each category of each indicator of abstraction for each year has been tabulated. Table 7.31 gives percentages of total responses in each category for each year. It follows a similar format to the 'difficult questions' analysis above so the rows are compared in pairs. Overall in Y4 there were more questions to which pupils failed to give a response that could be categorised for 'generality', metaphor or 'method'. There was little other change in the expressions of generality used but there was an increase in 'creation' metaphors at the expense of 'collection' and 'motion'. There was also a trend away from 'counting' methods and there was an increase in the numbers of 'symbolic' mental visual images with fewer images of pedagogic representations. The difference in the overall distributions for 'metaphor' is statistically significant. The difference in the overall distributions for both 'method' and 'image' were highly statistically significant:

	generality					metaphor					method					image			
	No response	particular	generic	general		no response	collection	motion	creation		no reponse	counting	holistic	algorithmic		no response	specific	iconic	symbolic
Overall Y3	21	30	38	12		18	30	14	38		22	22	27	29		75	2	8	14
Overall Y4	25	27	36	12		23	25	10	42		27	10	34	29		73	2	3	22
Upper Y3	16	26	43	14		15	27	12	46		18	16	30	36		76	1	7	16
Upper Y4	17	24	45	14		14	26	9	51		18	8	44	29		67	2	2	29
Lower Y3	25	33	33	10		21	32	16	30		27	27	25	22		75	3	9	13
Lower Y4	33	30	28	9		32	23	11	34		36	13	24	28		79	2	5	15

Table 7.31 Comparisons of indicator categories between years

There was little difference in the changes made by each separate group in their use of 'generality' except for the fewer answers given by the lower accuracy pupils. The trend in 'metaphor' use was common to both groups of children but the failure by many more lower-accuracy pupils to give an answer made their change in distribution statistically significant. The change in 'method' is statistically significant for both groups. For the upper-accuracy pupils this is because of a swing toward 'holistic' methods whilst the lower-accuracy pupils did not make this change and moved toward more 'algorithmic' methods. The higher accuracy pupils also showed a tendency toward more images of symbols.

The failure to answer questions can mask the changes in proportions in the other categories. A clearer picture of change is given with these removed. It is then apparent in Table 7.32 that the lower-accuracy pupils had changed from 'counting' methods to single-digit methods whilst the upper-accuracy pupils had changed toward 'holistic' methods.

	generality				metaphor				method				image		
	particular	generic	general		collection	motion	creation		counting	holistic	algorithmic		specific	iconic	symbolic
Overall Y3	37	48	15		36	17	47		28	35	37		7	34	59
Overall Y4	36	49	15		32	13	55		14	47	39		6	12	81
	1	2	3		1	2	3		1	2	3		1	2	3
Upper Y3	31	52	17		32	14	54		20	36	44		3	31	66
Upper Y4	29	54	17		30	10	60		10	54	36		6	6	88
	1	2	3		1	2	3		1	2	3		1	2	3
Lower Y3	44	43	13		41	20	39		37	34	30		12	36	52
Lower Y4	45	42	13		34	16	50		20	37	43		7	22	70

Table 7.32 Comparisons of indicator categories between years without 'no response'

This confirms the point made earlier that 'holistic' methods are more associated with accuracy. It could also indicate that the lower-accuracy pupils are the more procedural in using a mental analogue of the written algorithm. There is some confirmation here for the widening proceptual divide (Gray and Tall, 1994). These lower achievers are using a mental analogue of the written algorithm whilst the higher achievers move to methods which demonstrate their proceptual flexibility.

An important point to note is that there is not evidence of global change in expressions of generality, thus any change in expressions of generality can not be accounted for simply in terms of maturation or greater practice in procedures. It is possible, however, that the decrease in 'general' expressions noted for the difficult questions have balanced any increase in 'general' expressions in easier questions (due to developing confidence in routines).

7.3.2e Summary of comparisons

The number of pupils' responses in each category of the indicators for each question was contrasted with the facility level. There were no significant correlations. In other words the difficulty of the question is not a determining factor in the overall styles of response. Some difficult questions have similar proportions of 'general' expressions and 'algorithmic' methods to the proportions for easy questions, which suggests that pupils may simply state the procedure without being able to perform it accurately. By contrast easy questions may be answered without any category of expression of generality when the answer is simply known.

The comparison of the two 'addition' questions kept in each interview did not reveal a trend of changes over time though in the very last interview "48 add 23" was answered more accurately and using more 'holistic' methods. This suggests a greater sense of the numbers involved with less emphasis on separating digits. This may have been due to the greater emphasis in the classroom on mental strategies. When the difficult questions were analysed it was discovered that this move toward 'holistic' methods was particularly pronounced amongst the pupils who were most successful in the hardest questions. The least successful showed no greater proportion of 'holistic' methods in these difficult questions than in the others in the study. This is a marker of the flexibility demonstrated by successful pupils.

The contrast was also drawn between upper- and lower-accuracy pupils in terms of the different performances measured by the indicators over the two years. In Y4 the upper-accuracy pupils gave a similar proportion of classifiable responses as they had in Y3 but the lower-accuracy pupils failed to respond to a higher proportion of questions. The gap was similarly widening between the two groups in terms of the methods they employed. The upper accuracy group increased their proportion of non-'counting' methods largely due to an increase in 'holistic' methods. The increase in non-'counting' methods was even more marked for the lower-accuracy pupils but it was 'algorithmic' methods which increased for them. These pupils did not show the capacity for mental strategies which require connections to be made and mental objects to be used. They remain procedure focused.

7.3.3 Summary of characteristics of children

In §7.3.1 high-accuracy pupils were distinguished from low-accuracy pupils by their different profiles of categories of the indicators. A fuller picture of differences in category profiles has emerged in §7.3.2. through the analysis of difficult questions and of variation over time. This may be summarised:

Comparison between difficult and other questions:

'image'	Differences between groups remained. High accuracy was still associated more with mental visual images of symbols than 'specific' images.
'metaphor'	Differences between groups remained. High accuracy was still associated more with 'creation' metaphors than 'collection' metaphors. 'Creation' was more common.
'generality'	Differences between groups remained. High accuracy was still associated more with 'generic' and 'general' than 'particular' expressions of generality. Fewer 'general' expressions were used by all. The most successful used more 'particular' expressions than in other questions.
'method'	Differences become more pronounced between groups in difficult questions. High accuracy was associated in difficult questions more with 'holistic' methods than either 'counting' or 'algorithmic'. Low-accuracy pupils failed to follow the trend to 'holistic' methods.

Variation with time:

'image'	Differences between groups remained. High accuracy is associated more with mental visual images of symbols than 'specific' images. More mental visual images of symbols were reported by all.
'metaphor'	Differences between groups remained. High accuracy is still associated more with 'creation' metaphors than 'collection'. More 'creation' metaphors used by all.
'generality'	Differences between groups remained. High accuracy is still associated more with 'generic' and 'general' than 'particular' expressions of generality. Little change overall.
'method'	Differences become more pronounced between groups in Y4. High accuracy is associated more with 'holistic' methods than either

'counting' or 'algorithmic'. The least accurate pupils move to 'algorithmic' methods.

7.4 Summary of quantitative analysis

Chapter 6 presented a picture of the influence of the classroom activities on pupils' subsequent modes of expressions. The pedagogic representations were re-presented by pupils in their use of metaphoric language as much as in the procedures they employed. This appeared to suggest that pupils with similar experiences will perform similarly in mental calculation. Yet there were also the signs of variations in their language, mental visual images and methods employed. All the pupils at Bright Cross have had similar classroom activities yet they behaved differently. This chapter has attempted to identify indications of learning, the characteristics of descriptions of calculation, and of other contexts, and to identify characteristics of children.

These characteristics have been summarised at the end of each section. The results will be used to answer the research questions in Chapter Eight and will not be simply repeated here. Instead it is important to summarise some of the arguments that have been associated with the findings from this chapter before moving to the final conclusions in Chapter Eight.

- **'Image' categories reported by highest and lowest achievement groups were consistent across contexts.** (§7.3.1d)

This supports Pitta's findings that highest achievers report more 'abstract' and fewer 'concrete' images than low achievers. This provides a check on validity and reliability of this aspect of the study.

- **'Generality' use is context dependent.** (§7.2.2b, §7.3.1d)

Non-mathematical contexts have different category profiles from mathematical contexts. Pupils of all achievement levels used expressions of generality in procedure questions in similar ways. The implication is drawn that when individuals and different groups use different modes of expression in calculations it indicates that they have different mental constructions for mental calculations. Expression of generality is not simply a linguistic trait that is independent of the learning in the mathematics classroom.

Non-mathematical procedures are most frequently expressed in 'general' terms. When 'general' is used it could thus be indicative of a familiar procedure. Correct calculations have a different profile of categories of 'generality' from incorrect calculations. This implies that when individuals and groups use more 'generic' and 'general' expressions they may be indicating confidence with a procedure. When these modes of expression are adopted without correct answers there is an indication of having simply acquired the cultural tool. Pupils who use the procedures but get wrong answers seem capable of metonymic reasoning about signifiers without neccessarily having an understanding of the signified.

A higher proportion of 'algorithmic' methods are accompanied by 'general' expressions than 'holistic' methods. This is evidence of the emphasis given to 'algorithmic' methods. They have been taught as general rules and have been most frequently practised.

- **There are inter-indicator consistencies.** (§7.1.2a)

Consistency between reported mental visual image and the language used adds some check on the 'existence' of the image.

Consistency between 'metaphor' and 'method' suggests a mental construction that is based on classroom experiences. Language of 'creation' with 'algorithmic' methods is indicative of the influence of the written algorithm. This was often accompanied by a mental visual image of symbols which emphasises the re-presented nature of the response. There is support here for linguistic relativism when language and practice are consistent.

Consistencies between compatible categories suggest that the abstraction represented by the categories is consistent across these indicators.

- **Linguistic indicator use varies with context.** (§7.2.1e)

Exclusive use of "you", "I" in a non-'particular' sense and use of present tense are all typical of procedures. When they are used after a calculation they may indicate that a routine, familiar procedure has been used. The use of "you" and use of causal connectives are associated with correct answers. They signal that a familiar and successful routine is in use. Use of "I" in a 'general' sense is indicative of a successful and perhaps personal procedure. Changing use of pronouns may signal changing mental constructions.

These two chapters of results have demonstrated that whilst all pupils are influenced by pedagogic representations the way in which this influence is manifested and the way pupils abstract from those experiences varies. The characteristics of descriptions associated with successful calculation, and characteristics of successful calculators, are potentially useful indicators of learning. The implications will be discussed in the following chapter.

CHAPTER EIGHT

CONCLUSIONS

The aim of this chapter is to show how the results fit with, and add to, the literature and support the thesis. There have been frequent summaries throughout the first seven chapters of this report which have sought to identify the key points made in each section. In this chapter there is first an overview summary of the argument that has been developed from the literature, then the answers to the research questions are given. Conclusions are drawn and some implications suggested.

8.1 Leading up to conclusions

8.1.1 The literature and the results

8.1.1a Summary of the literature

The argument developed, with reference to literature, was that children's language and mental visual imagery give indications of the experiences which led to their individual mental constructions.

In order to provide a context in which theories of learning might be discussed consideration was given to cognitive structures. The brief description of brain activity indicated that learning from experience is a natural biological reaction and that mental representation is in terms of a neural record of the brain activities associated with that experience (Kolb and Whishaw, 1996). The models of mental representation, particularly those related to information processing, provide a way of characterising learning as stored memories of experiences (Halford, 1993; Ashcraft, 1994; Richardson, 1998; Tulving, 1972).

The notion of implicit and explicit representation suggests that those procedures we perform 'automatically' without conscious effort may be difficult to explain in words

(Halford, 1993). Thus differences in response to questions about procedures may be in part due to the different levels at which it was represented and performed by individuals (Gray and Tall, 1994). A clue to the differences in learning by individuals from the same common experiences is provided by the notion of 'attention' and 'comprehension' (Kosslyn, 1980; Ashcraft, 1994; Kolb and Whishaw, 1996). Different individuals attend to different aspects of the activities and relate their new experiences to different stored information (Piaget, 1928; von Glasersfeld, 1995; Wood, 1998).

The theories of mental structures and modes of mental representation provide a language for describing what might be in children's minds as a result of their experiences. Concepts, schemes, mental models, concept images, frames and scripts are all cognitive structures which allow us to talk about how we make sense of our experiences (Halford, 1993; Schank and Ableson, 1977; Minsky, 1977). In this study the term 'mental construction' has been used as a phrase that incorporates these various terms. Children's mental constructions, developed as a result of their experiences, are thus the means by which children represent experiences and they provide the mechanisms for future learning (Buner, 1966, 1996; von Glasersfeld 1991, 1995; Meadows, 1993). Mental constructions develop from experiences in the classroom as a result of active abstraction on the part of the learner (Piaget, 1928, 1959; Piaget and Inhelder, 1971; von Glasersfeld, 1995).

The theories of learning associated with the names of Piaget and Vygotsky were compared (Wood 1998, Meadows 1993). Piagetians view the learner as an individual who constructs meaning from his experiences (Lesh and Kelly, 1997). Vygotskians emphasise the role of the individual's interaction with others in this construction of meaning (Vygotsky, 1962; Luria, 1982). Socio-cultural theories see learners as members of a community of practice who appropriate the cultural tools of that community (Goos, Galbraith and Renshaw, 1999; Lerman, 1999). From the Vygotskian perspective the culture's system of signs provides the structure for thought (Vygotsky, 1962; Luria, 1982). This 'symbolist' view is in contrast to the 'conceptualist' view of Piaget which assumes that words and images are the product of thought (Denis, 1991). The role of mental visual imagery and language was thus examined.

Imagery research has a long tradition but much of the laboratory based experiments offer little information about the value of an image in assisting thinking (Richardson, 1999).

There is some evidence that images may aid memory and much 'common sense' advice that, since physical visual images are an important aspect of mathematical communication, children should be encouraged to form mental visual images (DfEE, 2000; Frobisher, Monaghan, Orton, Orton, Roper, and Threlfall, 1999; Harries and Spooner, 2000). There is some suggestion, however, that using memory space in forming images may be a disadvantage when performing calculations (Richardson, 1999; Seron, Pesenti, Noel, Deloche and Cornet, 1993). An important point to emerge from this consideration of imagery was its connection with experience. Firstly 'everyday' concepts and situations where a picture is helpful are more likely to evoke images (Richardson, 1999; Kosslyn 1980, 1996). Secondly children in one study seemed to have different imagery which could have been influenced by their different environments (Irwin 1995). The other important point to emerge is that mental visual imagery appears not to be a common feature in children's thinking (Presmeg, 1986a, 1986b, 1995; Krutetski 1976; Pitta 1998).

Pitta's (1998) research indicated that pupils may have similar styles of imagery in mathematics contexts and non-mathematics contexts. Krutetski's (1976) studies suggest that the ability to generalise in mathematics may be a characteristic of high achievers even though pupils of all levels of mathematical achievement may generalise in non-mathematics contexts.

'Pedagogic' representations were described as a medium for teaching and learning (Lesh, Post and Behr, 1987; Kang and Kilpatrick, 1992; Dufour-Janvier, Bednarz and Belanger 1987). It was not suggested that materials or words could 'carry' mathematical ideas but that they could provide the experiences upon which a learner could construct their knowledge (Mason, 1987; Hall, 1998; Meira, 1998). It was noted however that 'the medium' might become 'the message' and that children might simply learn the procedures for manipulating materials or for manipulating symbols without constructing knowledge of the underlying mathematical ideas (Edwards, 1998; Boulton-Lewis, Cooper, Atweh, Pillay, Wilss and Mutch, 1997; Pimm, 1995). Algorithms, as an instance of pedagogic representation, are specifically designed to be performed with efficiency without necessarily engaging with the numbers represented by the numerals (Tahta 1991).

Mental calculation is seen as an increasingly important aspect of mathematics teaching and learning (DfEE, 1999, 2000). It has often been assumed that children develop their own calculation strategies of varying degrees of sophistication independently of teaching (Carpenter, Hiebert and Moser, 1981; Beishuizen, 1993; Thompson, 1997a). The least successful mental calculators use 'counting' and mental analogues of pedagogic representations whilst the more successful children use flexible holistic methods (Gray, 1991). New initiatives in England (National Numeracy Strategy) suggest that all children can be taught the sophisticated mental procedures (Qualifications and Curriculum Authority, 1999). There is a danger however that the mental strategies, like written algorithms, may be taught and learned as procedures without the required underlying sense of the meaning of the numbers which are being manipulated in this way (Gray and Tall 1993, 1994). Some studies have shown that a teaching approach that encourages the individual's construction of strategies before algorithms are taught is more successful than when algorithms have been taught first (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter and Fennema, 1997; Cobb, 1995; Kamii and Dominick, 1997). Other studies suggest that different pedagogic representations could lead to different mental constructions and to different mental calculation strategies (Boulton-Lewis, 1998; Markovits and Sowder, 1994).

Language used by children is the chief indication of what they may have been thinking about and the reviewed literature suggested how closely related language and thought might be. The initial discussion of metaphor and metonymy suggested that our language which links concepts, or which links different parts of a conceptual structure, is indicative of the relational thinking which underlies these links (Black, 1979; Lakoff and Johnson, 1980, Lakoff, 1987; Johnson, 1987, 1992). It was also suggested that the language used may be indicative of a conceptual metaphor which constrains our thoughts (Reddy, 1979). Whilst linguistic determinism is not widely accepted there is growing evidence for linguistic relativism (Ungerher and Schmid, 1996; Rosch, 1977). This is an important point for this study. Language and thought were shown to be related in a study in which communities with different languages also demonstrated different modes of thought commensurate with the different language (Pederson, 1995). The theory of cognitive linguistics suggests that metaphoric language is indicative of the experiences that have formed the conceptualisations and that bodily experiences have a primary role in developing that language (Lakoff, 1987; Johnson, 1987, 1992).

The literature on 'linguistic indicators' gave some evidence that use of words and the understanding that underpins it are interdependent. Two studies suggested that pronouns and causal connectives could be pointers to generalisation and to understanding of causality (Rowland, 1992, 1995; Donaldson, 1986). The notions of a 'speech community', 'speech style' and 'register' all arise from the commonality of both the language and the activities of groups of people such as a teacher and pupils (Hudson, 1980; Hymes, 1974; Halliday 1979). This again suggests that speech, thought and activity are linked (Vygotsky, 1962; Luria, 1982). It is to be expected that children in the same classroom will share a common language but it is also possible that their individually different use of language might indicate the differences in the ways they have conceptualised from their common activities.

As a result of this review I adopted the position in the presentation of my results that children's language and reported mental visual imagery in interview responses were indicative of their mental constructions.

8.1.1b Answering the research questions

The twin themes of commonalities and differences in children's language and mental visual imagery are reflected in the research questions. The multi-level approach to the development of the argument has meant that the questions have been addressed over the three chapters. The results are listed here.

1 **What are the indications in the mental visual imagery, language and calculation strategy used by pupils that pedagogic representations have influenced their thinking?**

Elspeth's response "it's like in one hand you have the tens ..." was an explicit suggestion of the influences on her thinking. Her mental visual imagery also suggested that she was re-presenting her experiences. The data presented in Chapter Six suggest that metaphoric language related to previous classroom activities is a manifestation of the influences of those activities on children's thinking. The distinctive use of pronouns and causal connectives in explanations by teachers seemed to be common also to the pupils.

The language and mental visual imagery that I have reported in the main study could have been a universal language for primary classrooms that did not relate to the pedagogic representations. The comparison with the other school emphasised how speech communities have a style of speech that is relevant to the activities in which they engage. The mental visual images of pedagogic representations were also peculiar to the environments. The comparison with the teachers served to emphasise the commonality of speech and practice at Bright Cross.

The indicators also provide evidence of the influence of pedagogic representations.

'image'	The high incidence of mental visual imagery of symbols may be indicative of the emphasis given to symbols in the classroom.
'generality'	The high proportion of 'generic' expressions could be a reflection of the common speech style where procedures are described in terms of what "you" do with these numbers.
'metaphor'	The high proportion of 'creation' and 'collection' metaphors and low proportion of 'motion' could also be indicative of the prominence given to Dienes blocks and written algorithms. The number line was not frequently used.
'method'	The high proportion of 'algorithmic' methods is an indication of the influence of the written algorithms. When mental strategies were introduced in Y4 a higher proportion of 'holistic' methods were used by pupils.
'linguistic indicators'	Use of pronouns and causal connectives may have been influenced by the speech style of the community.

2 What are the characteristics of mental visual imagery and language used in mathematical and non-mathematical contexts?

Elspeth and the other pupils reported some mental visual imagery related to a physical pedagogic representation, but the most common images reported after calculations had been performed were of numerals. Some children reported images of 'sums' in either horizontal or vertical format but the majority simply 'saw numbers'. Mental visual images in non-calculation contexts were more frequently of specific objects and often involved episodic memories. There was evidence of the use of 'generic' examples indicated by "if it's like" in procedures. Descriptions of calculations just performed also used "you" when a familiar procedure was used and "I" for more personal methods.

The indicators provide the following characteristics of different contexts:

'image'	More likely to be 'symbolic' in mathematics and 'specific' or 'iconic' in non-mathematics contexts. Procedures were not accompanied by reported images that were characteristically different from images in other contexts.
'generality'	'Generic' expressions most common in mathematics and 'general' was more common in non-mathematics procedures. 'Particular' is characteristic of calculations.
'metaphor'	'Collection' and 'creation' most frequent and common across mathematics contexts.
'method'	'Holistic' and 'algorithmic' methods most frequent and common across mathematics contexts.
'linguistic indicators'	Past tense most common in calculation. Present tense use most common in non-mathematics contexts. Exclusive use of "you" and non-'particular' use of "I" is characteristic of procedures. Causal connectives were used more frequently in mathematics but in non-mathematics a higher proportion of use of "if" and "then" indicated 'general' consequences.

3 Is there evidence that individuals or groups have a style of language or mental visual imagery which is common to mathematical and non-mathematical contexts?

Elspeth and some of the other pupils gave indications of consistencies in their mental visual imagery and language use across contexts. Some pupils who reported non-'specific' imagery tended to do so in mathematics and in non-mathematics questions. Some others did not. Some pupils' expressions of generality were of complementary categories in mathematics and non-mathematics contexts. The majority of responses for more than half the pupils were of consistent levels in all the non-calculation questions. Other pupils showed greater variation. There was no conclusive evidence of a disposition by individuals toward a mode of expression in Chapter Six.

The analysis of categories of indicators used by individuals in Chapter 7 gave indication of consistencies between quantities of mental visual images reported. Pupils who reported mental visual image in mathematics tended to do so in non-mathematics contexts. There was also a tendency to report non-'specific' images across the contexts. When children were grouped by achievement the lowest achievers had a tendency to report 'specific' in both contexts and higher achievers to report non-'specific' in both contexts.

This was the only indication of a disposition toward a mode of response that was consistent across contexts. The categories of the other indicators were compared between groups in contexts other than calculation. No significant differences were found. There was thus no other evidence of a disposition toward a mode of expression that was consistent across contexts.

4 What are the characteristics of mental visual imagery, language and calculation strategy associated with successful and unsuccessful calculations?

Responses to "What was in your head ..." after calculations were performed correctly were qualitatively different from the responses after the calculation had resulted in an incorrect answer. The characteristics were:

'image'	Mental visual images were more frequently associated with correct answers. No significant difference between categories.
'generality'	'Generic' or 'general' responses were more likely to be associated with correct calculations and 'particular' responses were more likely to be associated with incorrect calculations.
'metaphor'	No metaphor particularly associated with accuracy.
'method'	No method particularly associated with accuracy.
'linguistic indicators'	Exclusive use of "you" was more often associated with correct answers than use of "I" exclusively or a mixture of the two. The answer was more likely to be correct when "I" was used in a 'general' sense . If "I" indicated only what the individual had done in that particular question the answer was more likely to be wrong. Expressions in non-past tense terms were most likely to be correct. Past tense was more frequently associated with wrong answers. The use of "if" and "then" to imply 'generic' or 'general' causality were most frequently associated with correct answers. The use of causal connectives was most frequently associated with correct answers.

5 How do mental visual imagery, language and calculation strategy relate to achievement?

In common with studies of children's mental arithmetic strategies the data have shown that the least successful pupils used more 'counting' methods than the most successful pupils. This study has shown that their language and mental visual imagery were also different in the calculation questions.

Children were grouped by their achievement in SATs and also by their accuracy in interview calculations. A different grouping was formed by consideration of performance in difficult questions. Characteristics of the different groups were as follows.

'image'	The least accurate pupils reported a higher proportion of non-'symbolic' mental visual images. The higher achievers predominantly reported mental visual images of symbols.
'generality'	High achievers used more 'general' expressions than other pupils and the majority of responses were 'particular' for the least able.
'metaphor'	'Creation' metaphors were most common for the higher achievers and 'collection' for the lowest achievers.
'method'	The majority of calculations were performed using 'holistic' or 'algorithmic' methods by the high achievers. The lowest achievers used a high proportion of 'counting' methods. In difficult questions the most successful pupils used 'holistic' methods more than they did in other questions.
'linguistic indicators'	The lower achievers used "I" in a 'particular' sense and used past tense most frequently. Higher achievers used higher proportions of "I" in a 'general' sense and non-'past tense'. Higher achievers also used more causal connectives.

It should be noted, again, that the groups did not respond differently in non-calculation questions except in 'image'.

6 How does the mental visual imagery, language and calculation strategy used for mental calculation, by individuals and groups, change over time?

The picture here was of variation within and between individuals. There was no clear evidence of change in either mental visual imagery or language use over the period of the study for individuals, but trends were apparent in the sample as a whole. The comparison between Y4 and Y3 revealed:

'image'	More 'symbolic' images were reported.
'generality'	There were no overall changes in expressions of generality.
'metaphor'	More 'creation' metaphors were in use.
'method'	Overall more 'holistic' methods were used but low accuracy pupils switched from 'counting' to 'algorithmic' methods. Higher achievers switched from 'algorithmic' to 'holistic'.

There could be an indication in this that the lower achievers have been more influenced by written algorithms whilst the higher achievers have been able to make more use of the 'holistic' mental calculation strategies.

8.1.2 Conclusions from the study

8.1.2a Reservations about, and justifications for, drawing conclusions

The inter-rater exercise demonstrated that it is not easy to train other teachers in a style of categorisation but this does not detract from the value of a categorisation process. Instead of training in using a particular set of category descriptors there could be value in groups of teachers developing their own systems of categories. The ones used in this study showed inner consistencies and showed associations with achievement and accuracy. They have been used for comparisons so that across-the-board re-categorisation would lead to some changes in categories for all groups of children. The same differences would still be apparent.

In interview situations it is well known that some people will choose to say things which they think will be most acceptable to the hearer. There is thus a possibility that children's responses are not indicative of their mental constructions. My argument is that if children's mental constructions are not at least as sophisticated as the language they use then they would not be able to say those things. If these children have chosen to give a general rule, for instance, because they think this is what is required in the interview it demonstrates that they have the general rule as part of their mental construction. These indicators may provide indications of development in mental constructions when children start to use language related to abstraction when they had not done so before. It is also true, however, that even if a rule is known and well understood it need not be expressed.

Implicit mental representations of procedures such as walking are difficult to explain in words. We can give an overview (one foot in front of the other) but the details of muscle movements may not even be known to us. Experts in any field may reach a stage of expertise in a procedure so that they may perform it without consciously thinking about it. When asked about that procedure they may simply give an overview without giving details which they are not consciously aware of having used. Novices are more likely to be conscious of the steps they have followed and will explain in more detail. In the interview responses in this study children who gave an almost immediate response frequently gave the overview in terms of a 'general' rule. This may not have been the way that they performed the calculation but it again indicates that knowledge of the general rule is part of their mental construction.

The language and mental visual imagery children use may have nothing in common with the way they were thinking. If this is so then what they say and the image they describe can still indicate influences on their mental construction even if it does not indicate the way they think. Context has been seen to be a determining factor in the style of language used. This is taken to suggest that when pupils use a peculiar speech style in describing their mental calculations it indicates the influences of their experiences on their mental constructions. When asked "What was in your head ... ?" many children needed to be reminded of the numbers and then they talked through what to do with those numbers. What they choose to say could thus be what they think is the best procedure even if it was not what they did. This gives an impression of some of the influences on their mental construction even if there were other thoughts in their mind which they have not revealed. Children sometimes give an impression that they are talking through their procedure in real-time, using the present tense and "you". This suggests that it is a well rehearsed routine they are following.

Quantitative 'measures' of abstraction are difficult to achieve. I do not claim that the categories that I have used represent stages in development of abstraction. Other teachers and researchers could use different 'level descriptors', but equally well do the task of distinguishing modes of expression which allow them to discriminate amongst pupils (and thus aid them in building on pupils' existing mental constructions). Though the category descriptors may be queried, their application has demonstrated that differences in mode of expression are not simply attributable to a mode of expression peculiar to an individual. There is evidence that different modes of expression are peculiar to groups of children of different achievement levels when they perform calculations. In the case of 'generality' there is a correspondence between the categories and levels of accuracy. These differences in modes of expression are not apparent in other contexts. This suggests that it is not just a disposition toward a mode of expression. The mode of expression could be associated with the different mental constructions formed by the different pupils as a result of their experiences.

8.1.2b Conclusions

Children have been shown to give manifestations of their learning in terms of the language and mental visual imagery that they have employed in response to interview questions. The cultural tools that they have acquired in classroom activities are apparent when common metaphoric language and common modes of expressions indicate the speech style that is common to the community.

Not all pupils learn the same things from their experiences. Indeed not all children have the same experience from common activities. If individual children's existing mental constructions are different then new activities will be construed in different ways. They will attend to different aspects of the activity and abstract different experiences. The external manifestations of these different mental constructions are the different modes of verbal expression and the different descriptions of reported mental visual imagery.

The conclusions have been signalled in the statement of the thesis in Chapter One. The conclusion that is drawn from this study is summarised as:

When describing the way they perform mental calculation young children's mental visual imagery and language use are indicative both of the experiences that have influenced their mental constructions and the qualitatively different mental constructions that have been formed.

8.1.2c Thoughts that led to conclusions

The language used by these children may be described as 'image-like'. The metaphoric language, like the mental visual imagery reported by some children, seems to be rooted in the previous activities. This re-presentation of the child's experience indicates the influences on their thinking. A Vygotskian view of learning helps explain how these children have common metaphors and common speech styles whilst the Piagetian perspective suggests how the individual differences occur.

Counting-on as a process for addition is thought to develop through a natural progression of stages. In this study we have seen how this procedure was taught in a classroom activity and was adopted by pupils. In an environment where counting as a procedure for addition and subtraction was not encouraged it did not flourish. This is one illustration of the influence of pedagogic representations on pupils' mental constructions. Some pupils continued to use this method and the associated language for longer than others. This is an illustration of the qualitative differences in pupils' mental constructions.

The initial mental constructions of the children in the main study were based on classroom activities involving counting, manipulation of place-value materials and manipulation of digits in written algorithms. When also encouraged to learn mental calculation strategies involving holistic methods the least successful pupils seemed not to have incorporated the new ideas into their mental constructions to the same extent that the more successful pupils had done. Their language showed the qualitative differences in their mental constructions. The least successful used mental analogues of the counting procedures and of the written algorithms.

It could be argued that these children would have learned in the ways that they have irrespective of the teaching approach. What this study has attempted to show is how the common language and conceptualisations may have developed as a result of the classroom activities. The comparison with the other school has demonstrated how big an impact the classroom activities can have on pupils' language and methods. What may be argued is that the mental constructions that these two groups of pupils have developed are qualitatively different. The two communities have different speech styles not simply different dialects.

The socio-culturalist view of learning gives no account of individual differences in conceptualisations. The children in this study have appropriated different cultural tools and this could be due to their different mental constructions. Those who predominately use counting methods have mental constructions for number which have not developed beyond the experiences of early classroom activities. They also demonstrate that their thinking is in 'concrete' terms by predominantly describing the calculation they have just performed rather than explaining their procedure. This hints that there are differences in mental constructions for mental calculations and not merely differences in verbal expressions.

Are mental visual images and words necessary for thought or a product of thought? This study gives evidence of some interdependence. Mental visual images seemed to be an advantage in that these more often accompanied correct answers, but the image may not have been the only factor in this. The style of language used in terms of the level of abstraction was also associated with accuracy. Language and image were usually related to one another. When the children were asked to mentally visualise a number line, however, their responses suggested that they can form mental visual images of a pedagogic representation but that the method of calculation need not be associated with that representation. Questioning of pupils about whether the image they reported was in their head before or after they had done the calculation indicated that it was concurrent but the answers were in many cases equivocal.

Linguistic expressions, whilst only being what an individual chooses to say, may still give indications of the cultural influences on the individual's mode of thought. In writing this report, for instance, I have tried to use the convention of using past tense to describe what was done and found. I use the present not only to describe what I am doing but also to say what I think may be true in general. Children may not have been taught this convention yet seem to have adopted this aspect of the culture's use of signs. Similarly use of "the tens" has been characterised as 'metaphoric' language in that it is associated either with Dienes blocks or with the algorithm. It is also, however, an instance of metonymy and as such demonstrates a generalisation. Talk of 'the tens' when talking through a procedure suggests that the child knows what to do in general and applies that procedure for the particular. This is just as the teacher does.

The comparison between two different environments could be simply claimed by behaviourists as evidence that different training leads to different behaviours. Differences in the 'behaviours' of individuals in the main study sample, however, suggest that children have responded differently to the common 'training'. Children learn from their experiences and the different language and mental visual imagery they employ indicates the differences in the learning that has taken place. This study adds weight to the theory of linguistic relativism in that the qualitative differences in language have been shown to be related to qualitative differences in performance of calculations. Furthermore the language of the two communities was different and the methods they employed was commensurate with that language.

8.2 Leading from conclusions

8.2.1 Implications

This section aims to explore the implications of the present research in terms of the links with existing theory, consideration of improvements of the method and implications for future research. Some implications of the findings for classroom teachers are also suggested.

8.2.1a Links with, and development of, existing theory

- **This study has indicated that children may have a style of mental imagery that is consistent across contexts. This is also related to their mathematical achievement levels.**

This confirms the findings of Presmeg (1986a, 1986b) and Pitta (1998). The present study has shown that higher achieving pupils are more likely to have non-specific mental visual imagery and lower achieving pupils are more likely to have mental visual images of specific objects and episodic events (§7.3.1a, §7.3.1d). These distinctions are apparent in both mathematical and non-mathematical contexts.

This study adds to the existing studies in this area by demonstrating that children may mentally visualise a pedagogic representation, such as a number line, yet not use a mental calculation strategy consistent with that representation.

- **This study has demonstrated the interdependence of language and thought in two different communities.**

Theories related to metaphor (Black, 1979; Lakoff and Johnson, 1980; Lakoff, 1987; Johnson, 1987, 1992; Reddy, 1979) suggested that language might be indicative of a mode of thought. Pederson (1995) demonstrated linguistic relativism in two communities.

The present study (Chapter 6) has shown how children's language and method in mental calculation are related. In each school the metaphoric language related to the pedagogic representations was marked.

- **This study suggests that commonalities in language may be due to the speech style of the classroom but differences in language may be related to children's achievement levels.**

The common metaphors and common use of both causal connectives and pronouns appear to be related to the classroom activities. This conforms with the theory of acquisition of cultural tools (Vygotsky 1962, Luria 1982). The present study also provides evidence that children of different achievement levels may use language differently. Although all have been members of the same speech community they choose different language in their descriptions of mental calculations. Use of past tense, the pronoun "I", and reference to specific numbers were characteristics of descriptions given by the lowest achievers. Use of present tense, the pronoun "you" and statements of general rules were more characteristic of higher achievers (§7.3.1a, §7.3.1d).

- **This study shows that accuracy is independent of metaphor but that pupils of different achievement levels have tendencies towards metaphor related to different methods.**

Analysis of the data in terms of the three categories of metaphor, suggested by Lakoff and Nunez (1997), has indicated that no category is particularly associated with accuracy (§7.2.1a). When children are grouped by their achievement in mathematics however the highest achievers are least likely to use language and methods related to counting. There is agreement here with previous studies of strategy use (Carpenter, Hiebert and Moser, 1981; Beishuizen, 1993; Thompson, 1997a; Gray, 1991; Gray and Tall, 1994).

The main study at Bright Cross provides an indication of the levels of use of each metaphor. This may indicate that early emphasis on counting methods and subsequent emphasis on written algorithms have influenced the pupils in their choice of method and associated language.

- **This study demonstrates that generalisation is a characteristic of the mental constructions of higher achievers that is specific to mathematics.**

This is confirmation of Krutetski's (1976) findings. Krutetski gave tasks for pupils to demonstrate their ability to generalise. The present study suggests that this ability is also demonstrated when pupils describe their mental calculations. Children's *spontaneous* expressions of generality in descriptions of their mental calculation are associated with accuracy (§7.2.1a) and achievement levels (7.3.1a, 7.3.1d). This variation in ability to generalise was, like Krutetski's study, not apparent in non-mathematics contexts.

Expression of generality in descriptions of mental calculation is thus a characteristic of children of high mathematical achievement. This may be seen as evidence of the different mental constructions formed from common activities.

8.2.1b Refinements to the method

The limitations of the method have been detailed in §4.2.4b and have been alluded to in §8.1.2a. In future research it would be useful to include data collected from group discussions by pupils where the influence of the researcher might be less obtrusive. In situations where pupils discuss methods with their peers there might be less of a tendency to use language that is thought to be acceptable to the teacher or researcher.

The limitations of the lesson observation might be addressed by video-recording or tape-recording lessons more frequently. The optimum situation would be continuous recording in order to detail all of the classroom activities that could influence children's mental constructions. Continuous recordings could also provide the data for an analysis of children's language and imagery in a more 'natural' setting than the interviews conducted for the present study. In classrooms where mental visual imagery is actively encouraged spontaneous descriptions of imagery might be a feature of classroom discussion and discussions between individuals.

The influences of activities in the home might also be investigated by interviewing parents and other carers. This could give an indication of language and stimuli for imagery that is peculiar to individual children and not common to the speech community and community of practice in the school classroom.

Each of these refinements were beyond the resources of the present study.

8.2.1c Implications for future research

It almost seems obvious that people are influenced in their thinking by their previous experiences, yet many studies of children's development in calculation strategies ignore the classroom activities that may have given rise to those strategies. This study has indicated the effectiveness of research which investigates the different conceptualisations which arise from common experiences. Knowing the expected similarities helps distinguish the distinctive mental constructions formed by individuals from classroom activities.

Future research needs to explore the longer term effects of mental constructions, developed as a result of a particular pedagogic representation, on children's learning. Children with exceptional backgrounds such as the pupils of Peacehaven would provide a sample for comparison with children from mainstream schools such as Bright Cross. When these groups of children encounter extensions to the number system how will their learning differ? Will their previous activities have provided them with restrictive metaphors?

No claim has been made that movement through the categories marks a progression in 'understanding'. Future explorations will be needed to investigate how the different modes of expression may relate to any deeper understandings of the number concepts and procedures.

The modes of language that were employed by high achievers in mental calculation did not appear to represent a peculiar mode of expression across contexts. The less successful pupils could use similar expressions to those of the most successful in non-calculation contexts. Linguistic 'ability' would thus not appear to be a factor in determining mental constructions. Yet, as previously noted, linguistic achievement as measured by reading-age does seem to accompany mathematical achievement. It seems that the more successful pupils' higher reading age is commensurate with their greater success in expressing generality in mathematical procedures. Pupils with lower reading ages can express generality in 'everyday' situations yet do not choose to do so following calculations. Further research could elaborate on this distinction.

8.2.1d Implications for the classroom

The comparison between the two schools in this study seems to suggest that the two environments have given rise to something more than two dialects. These children not only talk differently they also perform calculations differently. It could be that their subsequent learning in mathematics will be affected by the different mental constructions that these different behaviours and related language seem to indicate.

The 'first language' of mathematics learned in the early mathematics classroom may be an indication of the mental constructions that have been formed. It is possible that later concepts learned in the 'second language' of more formal mathematics will need to be translated into the 'mother language' of the primary classroom in order for them to be understood. The secondary school mathematics teacher could thus face the same problems as a second-language teacher who has no knowledge of the learners' first language.

Metaphoric language may go largely unnoticed in the classroom where it is a commonplace, where it merely expresses the common experiences, but it may be incomprehensible to an outsider, and this includes subsequent teachers. This language *can* be noticed by an outside observer of both experience and mode of expression. Few teachers, however, have the opportunity to both observe and talk with children prior to teaching them. This suggests that teachers need to attune themselves to notice the different speech styles of the children and to try to discover through further discussion what might underlie the differences in language. A teacher also needs to be aware that if they start using language of a different metaphor than that used by pupils it is likely to be incomprehensible to some pupils.

Identifying differences does nothing to improve the learning of pupils who are less successful in mathematics unless the information is used by teachers to move those pupils to greater understanding. Identifying 'where they are' through the use of the indicators can help with moving them on. Identifying influences on their thinking through the indicators may help with identifying new experiences which might add to and extend the mental construction. This sample, for instance, used very little language of 'motion' and position. This could well change and is likely to be different in classrooms where the number line is a more frequently used pedagogic representation. Teachers who can recognise the differences in language of those who have been influenced by number lines may thus be aware of the different experiences of the children, and build upon the diversity of experiences, to make links with different pedagogic representations.

The inter-rater exercise has indicated that teachers might find difficulty in categorising some of the expressions that children use but in the classroom they would have more than one response from which to draw conclusions. Not all previous experiences that have given rise to language used will be identifiable. However it will be a start in a teacher's attempt to build on the present mental constructions of pupils if teachers can distinguish qualitatively different language.

Characteristics of descriptions given after correct answers may be seen as providing clues about pupils' state of development toward mastery of calculation strategies. It is not, however, as simple as "'general' is associated with success" since it is clear that children can express themselves in 'general' terms when they can not answer the question correctly. This is also worth noting by the teacher because if children have acquired the cultural tool without the skill to use it then action is required. It could be beneficial to add new experiences which could allow children to descend the chain of signifiers and thus to develop their mental constructions to include meaning for the general rules they already posses.

The teacher needs to be aware also that the highest-accuracy pupils do not necessarily use 'general' expressions in the present tense using "you" when they are not simply following taught procedures. Descriptions of use of non-standard procedures may well be couched in 'particular' terms, using "I" in the past tense, when it is their own procedure.

This study presents results that are perhaps peculiar to the time at which it was conducted. If the best intentions of the National Numeracy Strategy are realised then children will be encouraged to construct their own mental calculation strategies and informal written procedures before written algorithms are introduced. At Bright Cross children learned written algorithms at an earlier age than will be expected if the NNS 'Framework for Mathematics' is adhered to. However there are many children, who started in the same way as those in this study, who will be continuing their mathematics learning with mental constructions based on the sorts of activities described here.

Teachers' language and constructions have also been formed from their experiences. It was noticeable that teachers in interview at Bright Cross did not use the newer strategies they had been recently teaching and, moreover, their language and mental visual imagery related to previously taught algorithms. There are two issues here. First, teachers may continue to use the language of their previous pedagogic representations even though it may be incompatible with new approaches. Second, teachers may not be able to easily communicate with children who have had different experiences from themselves. This second issue is important to secondary school teachers who will teach children from many different primary school backgrounds. Even new primary school teachers will face the same difficulties of mismatch in experiences and language use between themselves and their pupils.

8.2.2 Final remarks

The study shows that children have learned from a variety of activities. The variety in their language, image, and method demonstrates the various influences on their mental constructions. Each individual has also given evidence of a variety of modes of expressions and of the 'metaphors they calculate by'. This is taken to be a manifestation that the pupils have rich mental constructions which teachers can build upon in a variety of ways.

Medium and message have been an implicit theme throughout the preceding chapters:

- The medium of models becomes the message of the way the mind works.
- The medium of pedagogic representation becomes the message of the way mathematics works.
- The medium of the procedure becomes the message of the way the way the process works.
- The medium of the signifier becomes the message of the way the signified works.

There is a danger that the medium of the indicator becomes the message of the indicated. We can not assume we know about the child's mental construction simply because we know something about their manifestations of learning. These manifestations can, however, be a start on the descent to the signified.

The final remark returns to awarenesses. As noted in the introduction researchers need to present reports in a form that allows others to reconstruct the awarenesses and sensitivities developed during the research. Children may not be so considerate to their teachers. Their presentation of the influences that have formed their mental constructions will be in terms of their metaphoric language. Indicators of learning suggested in this study may provide clues to the state of their mental constructions. These clues need to be pursued.

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Glossary

analogy is a mapping of knowledge from one domain (the base) into another (the target) such that the system of relations that holds among the base objects also hold among the target objects.

attention describes the process of using selected information, either sensory or from internal representations.

comprehension is a conscious act of understanding through relating to previously stored information.

concept image is a cognitive structure which includes mental pictures, properties and processes. It yields personally meaningful mental images when the concept is evoked and develops with new experiences.

concepts are cognitive structures formed as a result of an individual's active awareness of universals and recognition of similarities in their experiences.

dialects are habitual, merely different ways of saying the same thing; the differences may be in sound or grammar but not in meaning.

episodic memory is a record of a person's experiences which receives and stores information about episodes, events and relations between them. Perceptual information is stored in terms of its autobiographical reference to the existing contents of the episodic memory.

explicit representations of procedures are those we may be conscious of. It is a less efficient means of representation than implicit representation but access to the rules and knowledge represented is greater and it is often communicable to others.

external representation of a mathematical concept or process is taken to be any linguistic or physical devise (written or spoken words or symbols, pictures or concrete objects) which stands in for, illustrates or exemplifies that concept or process.

figural concepts are general representations that have intrinsically conceptual properties; a mental image controlled by a definition.

generic example is an object that is not there in its own right but as a characteristic representative of the class.

ground (of a metaphor or analogy) refers to the similarities that exist between the two domains.

image schemas are formed from patterns of recurring bodily experiences that emerge through sensori-motor activity and from our perceptual understanding of actions and events in the real world

implicit mental representation is unconscious in that performance of a represented procedure is efficient and relatively effortless but without cognitive access to the rules or knowledge employed.

learning implies that a change in an individual's mental construction has occurred.

mental construction is the term used to describe the mental structure that develops as a result of experiences. It incorporates concepts, schemes, mental models and scripts. It is also referred to as a concept image.

mental imagery involves the mental invention or recreation of an experience that, in at least some respects, resembles the experience of actually perceiving an object or an event, either in conjunction with, or in the absence of, direct sensory stimulation.

mental models are mental structural analogues of the world. They are cognitive structures which reflect the knowledge, experience and goals of the learner and are not objective 'copies' of reality

a **mental representation** is that which takes the place of perceptual experience in the brain and provides the means for thinking about the experience.

a **memory** is a neural record of a previous experience.

metaphor is the use of the language of one domain to communicate thoughts about another.

metonymy is the use of a word related to a cognitive domain to refer to the whole or part of the domain.

pedagogic representations are the means by which teachers attempt to communicate mathematical ideas using 'external representations'.

a **proposition** is a representation of meaning that can be stored and retrieved from memory. It is the smallest unit of knowledge that can stand as a separate assertion, a simple relationship between two concepts that can be expressed by a simple declarative sentence (e.g. "six is a number").

register is the language of a particular linguistic situation determined by what is taking place, who is taking part and the part the language is playing.

rehearsal is a conscious act of retaining information in memory by repetition (verbal or physical activity).

a **'representation'** can be taken to be something which 'stands-in' for something else. The representation thus takes the place of the represented.

semantic memory is the store of knowledge about words and other verbal symbols, the relations between them and the rules for manipulating both symbols and concepts.

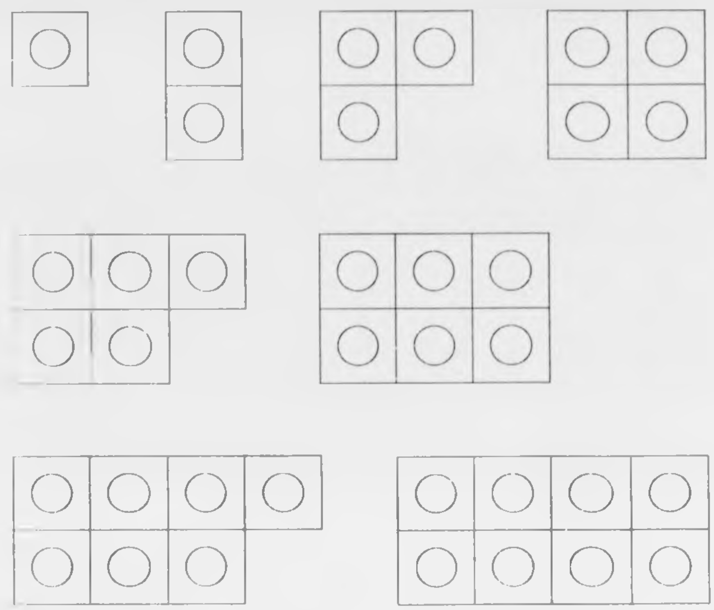
scheme is an active organisation of past experiences and reactions.

speech community is taken to be any group characterised by regular and frequent interaction by means of a shared body of verbal signs.

speech style incorporates not only the register but the life style of the speech community.

Appendices

Appendix 1.1 Stern Plates



Appendix 2.1 Mental strategies for addition and subtraction

"Teaching Mental Calculation Strategies: Guidance for Teachers at Key Stages 1 and 2." (Qualifications and Curriculum Authority, 1999) lists the following strategies for addition and subtraction:

'Counting forward and backwards' in ones, twos, tens and hundreds

'Re-ordering' when several numbers are written down to make use of complements to a multiple of ten e.g. 3 + 6 + 7 = 3 + 7 + 6

There are also five different 'partitioning strategies'

(i) 'Using multiples of 10 and 100' e.g. 68 - 32 = 60 - 30 + 8 - 2

(ii) 'Bridging through multiples of 10' e.g. 57 + 15 = 57 + 3 +12

(iii) 'Compensating' i.e. rounding numbers to the nearest multiple of ten and adding or subtracting to compensate e.g. 25 + 32 = 25 + 30 + 2 and 67- 29 = 67 - 30 + 1

(iv) 'Using near doubles' e.g. 40 + 39 is double 40 subtract 1

(v) 'Bridging through numbers other than ten' e.g. it is 8.35, how many minutes to 10.00

Appendix 2.2 Sample lesson observation

Y3 Set 2 15/5/98 Teacher MP.

Each pupil has 20 cubes in a line on desk.

MP Share these 20 between four people

Pupils move cubes into four columns.

MP How would we write a division sum?

Pupil 20 divided by 4 equals 5.

MP writes 20 ÷ 4 = 5 on board.

MP Now share these 20 between five people

Pupils move cubes into five columns.

MP How many would each person get?'

Pupil 20 divided by 5 equals 4.

MP writes 20 ÷ 5 = 4 on board.

MP Now 3 people, how many does each get?

Pupil 6 remainder 2

MP How do we write that down as a dividing sum?

Pupil 20 divided by 3 equals 6 remainder 2.

MP writes 20 ÷ 3 = 6 r 2 on board.

MP Now try 12 cubes. How many different ways can they be shared <u>equally</u> with no remainders? Try different ways each time.

Takes answers from pupils. Then records systematically on board:

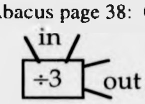
12 ÷ 2 = 6
12 ÷ 3 = 4
12 ÷ 4= 3
12 ÷ 5 = 2 r 2
12 ÷ 6 = 2
12 ÷ 7 = 1 r 5
12 ÷ 8 = 1 r 4

MP Without doing any more look at the pattern'

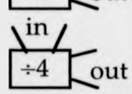
12 ÷ 9 = 1 r 3
12 ÷ 10= 1 r 2
12 ÷ 11 = 1 r 1
12 ÷ 12 = 1

MP Now Abacus page 38 - a bit like a function machine'

Abacus page 38: Copy and complete the table:



in	12	18	30	9	21	6	27	15
out								



in	8	28	12		24			
out				8		10	9	4

Explore: use counters, share each of these numbers equally among 3. How many left over. Guess first. Try some larger numbers.

Two boys are conviced that they need their 4 times table for second table but can not say why.

MP If you don't need to use the cubes, don't use them.

Many pupils are using the cubes

MP If 8 has come out and you are dividing by 4 what went in?'

Writes ☐ ÷ 4 = 8 on board.

Pupil Well I think what's in the 4 times table so it is 2'

MP How many people will have 8 each?'

Pupil The answer is 32 because its greater than the other two. I always go backwards on the sum I do 8 x 4 =32'

MP Good girl. These can all be done in the same way.

Pupil Dividing is just putting the sum the other way round

MP then does a few oral questions

£1 to share between 10 people?	10p
24 apples to share between 6 children	4
1m of string cut up into 2 equal parts	50 cm
£2 shared between 4 people	50p
30 stickers between 5 children	6

MP You are all working these out so quickly. How do you do that?

Pupil We are doing our times tables in our heads.

MP Well done

Appendix 2.3 SAT level percentages for sample and overall

	Percentage of pupils			
SAT level	Overall Y2	Sample Y2	Overall Y4	Sample Y4
2	73%	71%	12%	4%
3	27%	29%	42%	35%
4			44%	61%

Appendix 2.4 Sample interview script

Interview Script 1/12/98

After each question I will ask "What was in your head when you were thinking about that?" It could be - pictures, words, written numbers, memories, anything

What is the first thing that comes into your head when I say centimetres?

Can you tell me anything more about it?

When you are counting what comes before 380?

What is 97 add 10?

What is 17 add 8?

13 add something is 18. What is the something?

What is 246 rounded to the nearest ten?

Tell me how to add 23 on to a number.

What is 48 add 23?

Tell me how to cross the road safely.

Estimate the length of this room.

Last question - not maths - What is the first thing that comes in to your head when I say shadow

Can you tell me anything more about it.

Appendix 2.5 Details of database construction

Each record contained initially the following fields:

Question

Year, Term, question type

Pupil name

Answer to calculation

Description of how answer was obtained

Visual imagery described.

The last three fields contained the edited responses from the transcripts. For instance:

I	What is 17 add 8?
Clara	. 25
I	What was in your head when you were thinking of that?
Clara	three, and five
I	3 and 5? Why was it 3 and 5?
Clara	'cause 3 and 5 makes 8 and then if you add the 3 on to the 17 its 20, add the 5 it was 25
I	OK so what was in your head there?
Clara	.lots of things lying around
I	So did you, um, see the three and the 5 making 8? How was it lying around was it numbers written down or was it things
Clara	um, lots of cubes in a big long line
I	So what the 17 all in a line and the 8 in a line? OK?

This became:

Answer	25
Method	'cause 3 and 5 makes 8 and then if you add the 3 on to the 17 its 20, add the 5 it was 25
Image	.lots of things lying around ... um, lots of cubes in a big long line

After the initial importing of this raw data the records were sorted and data about individual pupils was imported. The new fields were:

Y3 Set, Y4 Set, Y2 SAT, Y3 SAT, Y4 SAT, Y3 Reading Score, Y4 Reading Score, Y3 Mental Arithmetic Score, Y4 Mental Arithmetic Score. A combined SAT score field was computed.

Each record was then coded. Fields for the indicators of learning and linguistic indicators are used to record the category of response for: Generality, Metaphor, Method, Image, "I", "you", "like", "because", "so", "if", "then", tense.

Binary fields record:

Accuracy	1 if right, 0 wrong
Attempt	1 if question given, 0 not given
Asked	1 if asked if image seen, 0 not asked
Sum	1 if visualised sum described
lang	1 if metaphoric language used

"I only", "I and you" and "you only" record if the response uses these combinations of pronouns in the method section of calculation questions.

Sorting records on each of these fields allowed the number of responses for each pupil in each category to be counted. This data concerning 'pupil performance' was recorded in fields. For instance "pacc" records how many questions the pupil answered correctly, "pcvis3" records how many times a pupil described an image of category 3 (symbolic) in a calculation question.

The records were also sorted by pupil achievements and by pupil performance and fields established to code for different groups of pupils:

Upacc	(1 if pupil in top 50% by accuracy in interview questions)
Upcomb	(1 if pupil in top 50% by combined SAT score)
Upvis	(1 if pupil in top 50% by number of imagess claimed)
Upcvis	(1 if pupil in top 50% by number of images in calculation questions)
Upread	(1 if pupil in top 50% by reading age)

A finer grouping for pupil 'types' was also recorded:

H(igh)	indicates that pupils performed higher than 1 standard deviation above the mean
M(iddle)	scores between 1 sd below and 1 sd above the mean.
L(ow)	scores less than 1 sd below the mean.

The following fields used this coding for pupil types:

acc t	records type of pupil by accuracy score
vis t	records type of pupil by number of images.
cvis t	records type of pupil by number of images in calculations

These fields allowed counts to be made of categories of responses given by different groupings of pupils. Counts could also be made to distinguish the different categories of each indicator associated with correct and incorrect answers.

Appendix 3.1 Categories of Generality in response to 'procedure' questions

0	No response	2	generic
1	particular	3	general

Table ordered by accuracy in interview calculation questions. Note the totals in the last four rows give the overall frequencies in each question. The totals in the last four columns give the frequencies in each category for each pupil.

		Mathematical Procedure									Non-Maths				Overall			
accuracy		How to add 23	How to find a third	How to times by ten	How to add 23	How to do this subtraction	Two thirds of a number?	Divide a number by 3	Multiply a 2-digit number by 5		How to cross the road	How to draw picture	How to tell the time		no response	particular	generic	general
		Y3/1	Y3/2	Y3/2	Y3/3	Y3/3	Y4/1	Y4/2	Y4/3		Y3/1	Y3/2	Y4/1					
12	Hester	2	0	0	3	0	0	0	3		3	2	3		5	0	2	4
13	Myles	3	3	2	2	2	2	2	0		3	2	1		1	1	6	3
16	Simon	3	0	2	1	2	3	2	3		3	2	1		1	2	4	4
19	Suzy	0	0	2	0	2	0	0	0		0	0	1		8	1	2	0
20	Sean	3	1	0	0	2	0	1	2		2	2	1		3	3	4	1
22	Mandy	3	2	3	2	2	0	0	2		3	2	3		2	0	5	4
25	Peter	1	2	2	1	2	0	0	2		3	0	3		3	2	4	2
27	Ann	0	0	2	2	2	0	3	3		3	2	3		3	0	4	4
28	Christine	2	1	3	2	2	2	2	2		3	2	3		0	1	7	3
28	Ellain	2	1	2	3	2	2	2	0		3	3	3		1	1	5	4
28	Jeremy	2	3	3	2	2	3	0	0		3	0	0		4	0	3	4
28	Naomi	0	0	0	3	2	1	3	0		3	0	3		5	1	1	4
30	Irene	2	0	2	2	2	0	2	2		2	2	3		2	0	8	1
31	Elspeth	2	1	2	2	2	2	3	3		3	2	2		0	1	7	3
31	Hannah	1	2	2	3	2	2	2	1		3	2	3		0	2	6	3
32	John	0	0	2	1	1	0	1	2		3	2	2		3	3	4	1
32	Malcolm	1	0	3	2	2	1	3	0		3	2	3		2	2	3	4
32	Terry	0	1	3	0	2	0	1	3		0	0	0		6	2	1	2
33	Max	2	3	1	1	2	1	2	2		3	2	3		0	3	5	3
34	Bobby	0	0	2	2	2	3	3	2		3	2	3		2	0	5	4
34	Dennis	0	0	2	3	2	1	3	3		3	2	1		2	2	3	4
35	Paddy	1	0	2	2	2	0	2	3		3	0	2		3	1	5	2
36	Clara	3	0	3	3	2	1	2	3		3	1	2		1	2	3	5
37	Kath	2	3	3	3	1	3	3	3		2	3	3		0	1	2	8
38	Jack	2	2	3	2	2	3	0	3		3	3	3		1	0	4	6
38	Teddy	1	0	1	2	2	2	3	3		3	1	2		1	3	4	3
	no response	7	13	3	3	1	10	6	6		2	6	2					
	particular	5	5	2	4	2	5	3	1		0	2	5					
	generic	9	4	13	12	23	6	9	8		3	15	5					
	general	5	4	8	7	0	5	8	11		21	3	14					

Appendix 3.2 Categories of generality in response to 'concept' questions

0	No response	2	generic
1	particular	3	general

Table ordered by accuracy in interview calculation questions. Note the totals in the last four rows give the overall frequencies in each question. The totals in the last four columns give the frequencies in each category for each pupil.

		Mathematical word						Non-maths word						Overall			
accuracy		centimetre	three	millions	fraction	polygon		shadow	ball	adjective	christmas	animal		No response	particular	generic	general
		Y3/1	Y3/2	Y3/3	Y4/1	Y4/2		Y3/1	Y3/2	Y3/3	Y4/1	Y4/2					
12	Hester	2	2	2	0	0		2	3	0	2	2		3	0	6	1
13	Myles	1	2	1	0	1		1	1	2	1	1		1	7	2	0
16	Simon	1	1	1	2	3		3	2	1	3	2		0	4	3	3
19	Suzy	0	2	2	0	0		0	0	0	2	2		6	0	4	0
20	Sean	1	2	1	1	1		1	2	3	1	1		0	7	2	1
22	Mandy	1	1	1	1	1		2	1	3	1	0		1	7	1	1
25	Peter	2	2	2	2	2		2	2	3	1	1		0	2	7	1
27	Ann	1	2	1	3	1		1	2	1	1	2		0	6	3	1
28	Christine	1	1	1	2	0		1	1	3	2	1		1	6	2	1
28	Ellain	2	3	3	2	2		3	2	0	3	3		1	0	4	5
28	Jeremy	0	3	2	3	2		0	0	3	2	3		3	0	3	4
28	Naomi	0	1	0	1	2		1	0	1	1	2		3	5	2	0
30	Irene	1	3	1	1	1		1	1	1	1	0		1	8	0	1
31	Elspeth	2	2	2	2	2		2	2	2	1	1		0	2	8	0
31	Hannah	1	2	2	1	0		1	2	3	1	3		1	4	3	2
32	John	2	2	2	1	2		1	2	3	3	3		0	2	5	3
32	Malcolm	1	1	1	1	1		1	1	3	1	1		0	9	0	1
32	Terry	0	1	2	2	1		0	1	3	1	1		2	5	2	1
33	Max	1	2	1	2	2		1	1	1	1	1		0	7	3	0
34	Bobby	2	3	2	2	1		2	2	3	1	1		0	3	5	2
34	Dennis	1	1	1	1	1		2	1	1	1	1		0	9	1	0
35	Paddy	1	1	2	2	1		1	1	1	1	1		0	8	2	0
36	Clara	1	2	1	1	2		1	1	1	1	1		0	8	2	0
37	Kath	1	2	2	3	1		1	1	0	1	2		1	5	3	1
38	Jack	1	1	1	1	2		3	1	3	1	1		0	7	1	2
38	Teddy	2	2	2	1	2		2	3	3	1	2		0	2	6	2
	No response	4	0	1	3	4		3	3	4	0	2					
	particular	15	9	12	11	11		13	12	8	19	13					
	generic	7	13	12	9	10		7	9	2	4	7					
	general	0	4	1	3	1		3	2	12	3	4					

Appendix 3.3 Categories of image in response to 'procedure' questions

0	No response	2	iconic
1	specific	3	symbolic

Table ordered by accuracy in interview calculation questions. Note the totals in the last four rows give the overall frequencies in each question. The totals in the last four columns give the frequencies in each category for each pupil.

		Mathematical procedure								Non-Maths			Overall			
accuracy		How to add 23	How to find a third	How to times by ten	How to add 23	How to do this subtraction	Two thirds of a number?	Divide a number by 3	Multiply a 2-digit number by 5	How to cross the road	How to draw picture	How to tell the time	No response	specific	iconic	symbolic
		Y3/1	Y3/2	Y3/2	Y3/3	Y3/3	Y4/1	Y4/2	Y4/3	Y3/1	Y3/2	Y4/1				
12	Hester	0	0	0	0	0	0	0	1	0	0	2	9	1	1	0
13	Myles	0	0	0	0	0	0	3	0	2	0	2	8	0	2	1
16	Simon	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0
19	Suzy	0	0	0	0	0	0	0	0	2	0	0	10	0	1	0
20	Sean	0	0	0	0	0	0	0	0	3	1	0	9	1	0	1
22	Mandy	0	1	2	0	0	0	0	0	3	1	0	7	2	1	1
25	Peter	0	0	0	0	0	3	0	0	0	1	1	8	2	0	1
27	Ann	0	0	0	0	0	0	3	0	0	1	0	9	1	0	1
28	Christine	0	0	0	0	0	0	3	3	1	1	0	7	2	0	2
28	Ellain	0	0	3	0	0	0	0	0	1	1	0	8	2	0	1
28	Jeremy	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0
28	Naomi	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0
30	Irene	0	0	0	0	0	0	0	3	1	1	2	7	2	1	1
31	Elspeth	0	0	3	0	0	0	3	3	1	1	2	5	2	1	3
31	Hannah	0	0	0	0	0	0	0	0	0	0	0	11	0	0	0
32	John	0	0	0	3	0	0	2	0	0	0	2	8	0	2	1
32	Malcolm	0	0	0	0	0	1	1	0	0	1	2	7	3	1	0
32	Terry	0	2	0	0	0	0	0	0	0	0	0	10	0	1	0
33	Max	0	0	0	0	0	2	0	0	1	1	0	8	2	1	0
34	Bobby	0	0	0	3	0	3	0	3	0	0	2	7	0	1	3
34	Dennis	0	0	3	0	0	1	0	3	2	1	2	5	2	2	2
35	Paddy	0	0	0	0	0	0	0	1	0	0	2	9	1	1	0
36	Clara	0	0	0	0	0	0	1	3	2	0	2	7	1	2	1
37	Kath	0	0	0	0	0	0	0	0	1	0	1	9	2	0	0
38	Jack	0	0	0	0	0	0	0	0	2	0	2	9	0	2	0
38	Teddy	0	0	0	0	0	0	0	0	0	0	3	10	0	0	1
	No response	26	24	22	24	26	21	19	18	13	15	12				
	specific	0	1	0	0	0	2	2	2	6	11	2				
	iconic	0	1	1	0	0	1	1	0	5	0	11				
	symbolic	0	0	3	2	0	2	4	6	2	0	1				

Appendix 3.4 Categories of image in response to 'concept' questions

0	No response	2	iconic
1	specific	3	symbolic

Table ordered by accuracy in interview calculation questions. Note the totals in the last four rows give the overall frequencies in each question. The totals in the last four columns give the frequencies in each category for each pupil.

		Mathematics word						Non-maths word						Overall			
accuracy		centimetre	three	millions	fraction	polygon		shadow	ball	adjective	christmas	animal		No response	specific	iconic	symbolic
		Y3/1	Y3/2	Y3/3	Y4/1	Y4/2		Y3/1	Y3/2	Y3/3	Y4/1	Y4/2					
12	Hester	1	3	0	0	0		0	1	0	2	2		5	2	2	1
13	Myles	3	0	3	0	1		2	3	0	2	1		3	2	2	3
16	Simon	0	0	0	0	0		0	0	0	2	0		9	0	1	0
19	Suzy	0	0	0	0	0		0	0	0	2	2		8	0	2	0
20	Sean	3	0	3	3	1		0	2	0	1	1		3	3	1	3
22	Mandy	0	0	0	2	0		0	0	0	0	0		9	0	1	0
25	Peter	0	0	0	0	0		0	1	0	1	0		8	2	0	0
27	Ann	0	0	0	0	0		0	0	0	1	0		9	1	0	0
28	Christine	1	2	1	0	0		1	1	3	2	1		2	5	2	1
28	Ellain	0	3	3	3	0		2	0	0	0	2		5	0	2	3
28	Jeremy	0	0	0	0	0		0	0	0	0	0		10	0	0	0
28	Naomi	0	0	0	2	2		0	0	3	0	2		6	0	3	1
30	Irene	0	0	3	0	0		1	0	0	2	0		7	1	1	1
31	Elspeth	0	2	3	2	2		2	1	0	2	2		2	1	6	1
31	Hannah	0	2	0	0	0		0	1	0	0	0		8	1	1	0
32	John	0	0	0	0	0		0	0	0	0	2		9	0	1	0
32	Malcolm	0	3	2	1	3		2	1	0	1	2		2	3	3	2
32	Terry	0	0	0	0	1		0	3	0	0	2		7	1	1	1
33	Max	3	0	0	3	0		2	0	0	0	0		7	0	1	2
34	Bobby	0	0	0	0	2		0	0	0	0	2		8	0	2	0
34	Dennis	0	3	3	3	0		2	2	0	2	2		3	0	4	3
35	Paddy	0	0	0	3	0		1	0	0	1	0		7	2	0	1
36	Clara	2	0	2	3	2		2	2	0	2	2		2	0	7	1
37	Kath	1	3	0	0	0		2	2	0	2	0		5	1	3	1
38	Jack	0	3	0	3	0		0	0	0	0	0		8	0	0	2
38	Teddy	0	0	0	0	0		1	0	0	1	2		7	2	1	0
	No response	19	17	17	15	18		14	14	24	10	11					
	specific	3	0	1	1	3		4	6	0	6	3					
	iconic	1	3	2	3	4		8	4	0	10	12					
	symbolic	3	6	6	7	1		0	2	2	0	0					

Appendix 4.1 Measures of achievement for sample

Table ordered from highest to lowest 'combined SAT scores', i.e. the total score achieved in all the written mathematics tests:

	Age	Set	Y2 SAT	Y3 SAT	Y4 SAT	Comb SAT score	Comb Spelling	Y4 Reading	Y3 Mental Arith	Y4 Mental Arith	accuracy
Clara	9.59	1	33	49	51	133	27	11.5	16	17	36
Jack	9.63	1	26	48	50	124	38	13.9	16	19	38
Hannah	9.63	1	28	50	43	121	30	11.8	18	18	31
Kath	9.83	1	29	42	48	119	37	12	12	18	37
Terry	9.19	1	32	42	45	119	18	12.6	13	15	32
Irene	9.48	1	23	49	43	115	32	11.3	17	12	30
Elspeth	9.57	1	30	38	45	113	32	11	12	14	31
Bobby	9.6	1	26	41	45	112	40	13.5	12	14	34
Christine	9.53	1	23	43	45	111	21	12	13	14	28
Max	9.24	1	21	47	43	111	20	9.8	12	16	33
Peter	8.86	1	24	41	43	108	30	11.5	11	14	25
Teddy	9.79	2	26	42	40	108	27	11.5	17	13	38
Ann	9.32	1	23	42	42	107	22	11.8	13	10	27
Dennis	9.6	1	23	38	42	103	18	11.8	14	11	34
John	8.98	2	21	34	34	89	23	10.5	15	12	32
Naomi	9.2	2	19	32	38	89	26	10.5	9	9	28
Mandy	9.75	1	21	40	27	88	17	8.8	9	9	22
Suzy	8.85	2	23	33	32	88	21	9.6	9	13	19
Paddy	9.53	2	25	31	31	87	10	8.7	9	13	35
Malcolm	9.71	2	18	27	41	86	15	10.2	10	5	32
Jeremy	9.11	2	18	23	37	78	35	12.2	11	14	28
Ellain	9.71	2	17	27	28	72	19	9.1	10	8	28
Hester	8.97	2	18	22	23	63	15	8.5	6	1	12
Simon	9.56	2	14	25	24	63	11	7	9	9	16
Sean	9.17	2	17	24	20	61	19	11.5	6	6	20
Myles	9.75	2	15	26	18	59	10	9.9	4	8	13
Median	9.545		23	39	41.5	105	21.5	11.4	12	13	30.5
Mean	9.429		22.8	36.8	37.6	97.19	23.6	10.87	11.7	12	28.4
St dev	0.302		5.07	8.94	9.42	21.63	8.7	1.612	3.59	4.31	7.41

Appendix 4.2 Frequencies of reported imagery and categories of 'image'

The table has been ordered by the total number of images reported by each child in the interviews:

	asked	images	calculation images	Visualised sum	specific' images	iconic' images	symbolic' image	calculation 'specific'	calculation 'iconic'	calculation 'symbolic'	non-calculation 'specific'	non-calculation 'iconic'	non-calculation 'symbolic'
Elspeth	54	48	29	15	12	10	26	1	7	21	11	3	5
Clara	55	45	27	4	19	9	17	4	8	15	15	1	2
Dennis	52	45	26	12	12	0	33	0	0	26	12	0	7
Christine	49	35	15	8	16	3	16	2	2	11	14	1	5
Myles	41	34	19	2	9	9	16	0	6	13	9	3	3
Malcolm	50	33	14	0	14	6	13	1	4	9	13	2	4
Mandy	37	33	22	2	8	18	7	2	16	4	6	2	3
Irene	56	29	19	12	8	3	18	2	1	16	6	2	2
Kath	45	29	18	2	6	3	20	0	1	17	6	2	3
Bobby	59	28	18	3	5	2	21	0	1	17	5	1	4
Hester	54	25	10	1	14	7	4	2	6	2	12	1	2
Ellain	48	24	10	3	7	2	15	1	1	8	6	1	7
Sean	38	23	11	0	7	3	13	1	3	7	6	0	6
Peter	50	21	11	7	7	2	12	0	1	10	7	1	2
Max	60	20	10	3	5	2	13	1	0	9	4	2	4
Teddy	54	19	10	1	4	5	10	0	3	7	4	2	3
Paddy	54	15	8	3	6	2	7	1	1	6	5	1	1
Ann	47	13	5	4	5	1	7	0	0	5	5	1	2
Naomi	53	12	4	0	5	2	5	0	0	4	5	2	1
Terry	48	10	5	1	2	2	6	0	0	5	2	2	1
John	51	9	2	2	3	1	5	0	0	2	3	1	3
Simon	48	9	6	0	5	4	0	3	3	0	2	1	0
Jack	46	8	1	0	3	2	3	0	1	0	3	1	3
Suzy	38	7	2	0	4	2	1	0	1	1	4	1	0
Hannah	53	5	1	1	1	2	2	0	0	1	1	2	1
Jeremy	50	3	2	0	1	0	2	0	0	2	1	0	0
Median	50	22	10	2	6	2	11	0	1	7	5.5	1	3
Mean	49.6	22.4	11.7	3.31	7.23	3.92	11.2	0.81	2.54	8.38	6.42	1.38	2.85
St dev	6.09	13	8.4	4.15	4.68	3.95	8.23	1.1	3.62	6.91	4.06	0.8	1.99

Appendix 4.3 Frequencies of categories of 'generality', 'metaphor' and 'method'

Ordered by overall frequency of 'general' responses.

	Overall generality			Calculation generality			Non-calculation gnerality			calculation metaphor			calculation method		
	particular	generic	general	Calculation particular	Calculation generic	Calculation general	Non-calc particular	Non-calc generic	Non-calc general	collection	motion	creation	counting	holistic	algorithmic
Jack	19	28	24	10	16	14	9	12	10	17	2	3	8	13	22
Christine	16	30	22	9	14	12	7	16	10	18	4	22	13	13	19
Kath	20	27	21	12	16	10	8	11	11	7	8	29	2	28	15
Bobby	14	34	18	8	22	9	6	12	9	15	3	27	5	24	16
Elspeth	11	46	17	6	25	10	5	21	7	9	7	33	7	22	21
Ellain	14	40	15	11	23	4	3	17	11	9	6	27	5	25	11
Hannah	23	33	15	16	17	5	7	16	10	12	6	31	9	14	22
Teddy	18	38	15	10	24	6	8	14	9	20	4	24	10	27	10
Ann	16	29	14	9	16	5	7	13	9	7	3	25	5	9	20
Clara	23	27	14	11	16	4	12	11	10	10	4	26	2	17	18
Jeremy	24	23	14	22	15	3	2	8	11	29	2	12	16	19	11
Terry	22	29	12	14	18	8	8	11	4	7	5	31	4	16	18
Peter	19	36	11	12	20	8	7	16	3	16	10	21	5	18	19
Simon	26	22	11	18	9	3	8	13	8	24	5	10	17	13	4
John	22	32	10	15	18	4	7	14	6	21	4	19	7	27	9
Malcolm	29	28	10	14	19	2	15	9	8	21	4	23	16	19	10
Mandy	26	26	10	17	13	3	9	13	7	8	20	13	13	17	10
Hester	18	25	9	3	8	3	15	17	6	20	5	13	10	12	9
Max	23	43	9	9	26	5	14	17	4	9	8	27	6	23	16
Myles	34	18	9	21	5	4	13	13	5	14	9	16	15	6	12
Naomi	21	21	9	13	12	1	8	9	8	10	6	15	8	12	10
Paddy	16	38	9	6	24	4	10	14	5	9	5	27	7	21	13
Irene	20	40	8	10	23	4	10	17	4	7	3	32	2	13	25
Dennis	29	29	7	16	17	2	13	12	5	18	2	24	3	19	17
Sean	26	27	7	13	14	3	13	13	4	14	7	14	8	15	8
Suzy	19	13	1	13	4	1	6	9	0	11	1	3	9	5	4
Median	21	29	11	12	17	4	8	13	7.5	13	5	24	7.5	17	14
Mean	21	30	12	12	17	5.3	8.8	13	7.1	14	5.5	21	8.2	17	14
St dev	5.3	7.8	5.2	4.5	5.9	3.4	3.5	3.1	2.9	6.1	3.7	8.6	4.5	6.3	5.7

Appendix 4.4 Frequency of use of linguistic indicators

Ordered by frequency of use of "I".

	I	You	I exclusively	you exclusively	I and you	Like	Analogy	Calc analogy	If	Then	Because	So
Hester	57	16	31	9	9	23	13	10	12	16	16	11
Ellain	55	37	24	23	18	49	24	20	27	37	29	26
Irene	55	24	38	7	15	32	10	10	16	31	15	20
Teddy	54	16	45	8	8	18	9	5	8	21	12	23
Kath	53	19	37	9	13	22	12	12	13	23	8	10
Malcolm	53	17	30	7	12	26	8	7	12	34	9	17
Paddy	53	32	25	9	24	34	17	13	14	24	26	32
Peter	52	20	35	9	13	19	8	7	10	18	11	19
Bobby	51	30	31	11	19	30	9	8	18	28	17	31
Terry	49	18	35	6	14	14	4	4	15	22	13	15
Sean	48	21	25	12	9	28	5	4	14	16	8	15
Christine	46	31	23	19	13	43	18	16	18	20	13	16
Mandy	46	16	33	10	6	25	6	5	7	30	7	12
Jeremy	44	7	35	5	2	20	13	11	3	21	2	6
Elspeth	41	37	16	21	16	48	25	21	17	39	17	29
Ann	39	15	24	5	12	36	12	11	11	16	7	9
Myles	36	21	28	12	9	40	11	8	8	18	4	3
Naomi	36	7	2	1	1	6	1	1	10	18	4	7
Hannah	33	24	21	15	12	25	10	10	5	30	10	18
Max	32	46	10	26	20	28	11	8	16	40	18	25
Simon	31	14	21	12	2	22	11	10	3	23	2	6
John	30	23	21	16	8	18	6	5	11	27	11	10
Clara	24	19	16	16	3	24	5	4	11	12	6	7
Jack	23	17	21	15	2	6	3	2	8	24	6	7
Suzy	23	6	19	3	4	7	6	5	2	8	0	4
Dennis	21	11	17	7	5	10	4	4	7	38	6	8
Median	45	19	25	9.5	10.5	24.5	9.5	8	11	23	9.5	13.5
Mean	41.7	20.9	26	11.3	10.3	25.1	10	8.5	11.4	24.4	10.7	14.8
St dev	11.6	9.77	9.5	6.17	6.21	11.9	5.91	5.03	5.59	8.55	7.01	8.62

Appendix 4.5 Contingency tables for combinations of indicators

(i) $p<0.005$

	Number of responses with percentages of row totals in bold							
	specific		iconic		symbolic		Totals	
collection	8	**11**	17	**24**	47	**65**	72	**100**
motion	5	**8**	37	**62**	18	**30**	60	**100**
creation	13	**7**	15	**9**	146	**84**	174	**100**
Totals	26	**8**	69	**23**	211	**69**	306	**100**

(ii) $p<0.1$

	Number of responses with percentages of row totals in bold							
	specific		iconic		symbolic		Totals	
particular	6	**7**	28	**32**	53	**61**	87	**100**
generic	7	**6**	19	**15**	98	**79**	124	**100**
general	4	**10**	10	**24**	28	**67**	42	**100**
Totals	17	**7**	57	**23**	179	**71**	253	**100**

(iii) $p<0.05$

	Number of responses with percentages of row totals in bold							
	specific		iconic		symbolic		Totals	
counting	3	**9**	16	**47**	15	**44**	34	**100**
holistic	5	**5**	28	**29**	65	**66**	98	**100**
algorithmic	7	**6**	13	**12**	92	**82**	112	**100**
Totals	15	**6**	57	**23**	172	**70**	244	**100**

(iv) $p<0.005$

	Number of responses with percentages of row totals in bold							
	collection		motion		creation		Totals	
counting	142	**81**	22	**13**	12	**7**	176	**100**
holistic	135	**38**	65	**18**	155	**44**	355	**100**
algorithmic	28	**9**	36	**11**	265	**81**	329	**100**
Totals	305	**35**	123	**14**	432	**50**	860	**100**

(v) $p<0.005$

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
counting	110	**72**	21	**14**	21	**14**	152	**100**
holistic	114	**32**	215	**60**	28	**8**	357	**100**
algorithmic	62	**19**	188	**57**	81	**24**	331	**100**
Totals	286	**34**	424	**50**	130	**15**	840	**100**

(vi) $p<0.005$

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
collection	165	**58**	87	**31**	32	**11**	284	**100**
motion	36	**29**	63	**50**	27	**21**	126	**100**
creation	96	**21**	278	**62**	76	**17**	450	**100**
Totals	297	**35**	428	**50**	135	**16**	860	**100**

Appendix 4.6 Contingency tables for correct and incorrect answers with each 'indicator of learning'

(i) p<0.01

	Number of responses with percentages of row totals in bold					
	Visual Image		No Visual Image		Totals	
Right	211	**29**	513	**71**	724	**100**
Wrong	94	**22**	340	**78**	434	**100**
Totals	305	**26**	853	**74**	1158	**100**

(ii) not significant

	Number of responses with percentages of row totals in bold					
	Visualised Sum		No Visualised Sum		Totals	
Right	53	**7**	671	**93**	724	**100**
Wrong	33	**8**	401	**92**	434	**100**
Totals	86	**7**	1072	**93**	1158	**100**

(iii) not significant

	Number of responses with percentages of row totals in bold							
	specific		iconic		symbolic		Totals	
Right	11	**5**	46	**22**	154	**73**	211	**100**
Wrong	10	**11**	20	**21**	64	**68**	94	**100**
Totals	21	**7**	66	**22**	218	**71**	305	**100**

(iv) not significant

	Number of responses with percentages of row totals in bold							
	collection		motion		creation		Totals	
Right	203	**33**	95	**15**	324	**52**	622	**100**
Wrong	113	**37**	41	**14**	149	**49**	303	**100**
Totals	316	**34**	136	**15**	473	**51**	925	**100**

(v) not significant

	Number of responses with percentages of row totals in bold							
	counting		holistic		algorithmic		Totals	
Right	119	**19**	247	**40**	248	**40**	614	**100**
Wrong	63	**24**	114	**43**	88	**33**	265	**100**
Totals	182	**21**	361	**41**	336	**38**	879	**100**

(vi) p<0.005

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
Right	184	**30**	324	**53**	107	**17**	615	**100**
Wrong	147	**51**	110	**38**	30	**10**	287	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

Appendix 4.7 Contingency tables for correct and incorrect answers with pronouns and tense

(i) $p<0.025$

	Number of responses with percentages of row totals in bold							
	"I" only		"I" and "you"		"you" only		Totals	
Right	338	**64**	111	**21**	77	**15**	526	**100**
Wrong	182	**71**	55	**21**	20	**8**	257	**100**
Totals	520	**66**	166	**21**	97	**12**	783	**100**

(ii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	"I" Category 1		"I" Category 2		"I" Category 3		Totals	
Right	306	**69**	32	**7**	105	**24**	443	**100**
Wrong	183	**81**	15	**7**	28	**12**	226	**100**
Totals	489	**73**	47	**7**	133	**20**	669	**100**

(iii) not significant

	Number of responses with percentages of row totals in bold							
	"you" Category 1		"you" Category 2		"you" Category 3		Totals	
Right	23	**13**	119	**65**	41	**22**	183	**100**
Wrong	11	**17**	42	**65**	12	**18**	65	**100**
Totals	34	**14**	161	**65**	53	**21**	248	**100**

(iv) $p<0.025$

	Number of responses with percentages of row totals in bold							
	Past tense		Present tense		Mixed tense		Totals	
Right	243	**40**	174	**29**	184	**31**	601	**100**
Wrong	148	**51**	70	**24**	75	**26**	293	**100**
Totals	391	**44**	244	**27**	259	**29**	894	**100**

Appendix 4.8 Contingency tables for correct and incorrect answers with causal connectives

(i) test not applicable

	Number of responses with percentages of row totals in bold							
	"if" Category 1		"if" Category 2		"if" Category 3		Totals	
Right	52	**47**	26	**23**	33	**30**	111	**100**
Wrong	18	**75**	3	**13**	3	**13**	24	**100**
Totals	70	**52**	29	**21**	36	**27**	135	**100**

(ii) test not applicable

	Number of responses with percentages of row totals in bold							
	"then" Category 1		"then" Category 2		"then" Category 3		Totals	
Right	268	**88**	31	**10**	6	**2**	305	**100**
Wrong	136	**95**	7	**5**	0	**0**	143	**100**
Totals	404	**90**	38	**8**	6	**1**	448	**100**

(iii) test not applicable

	Number of responses with percentages of row totals in bold							
	"so" Category 1		"so" Category 2		"so" Category 3		Totals	
Right	12	**6**	192	**94**	0	**0**	204	**100**
Wrong	7	**9**	73	**91**	0	**0**	80	**100**
Totals	19	**7**	265	**93**	0	**0**	284	**100**

(iv) test not applicable

	Number of responses with percentages of row totals in bold							
	"Because" Cat 1		"Because" Cat 2		"Because" Cat 3		Totals	
Right	12	**8**	120	**81**	17	**11**	149	**100**
Wrong	1	**3**	37	**93**	2	**5**	40	**100**
Totals	13	**7**	157	**83**	19	**10**	189	**100**

(v) not significant

percentages of row totals in bold						
	"so" category 1		"so" categories 2/3		Totals	
Right	12	**11**	93	**89**	105	**100**
Wrong	7	**9**	73	**91**	80	**100**
Totals	19	**10**	166	**90**	185	**100**

(vi) $p<0.025$

percentages of row totals in bold						
	"if" category 1		"if" categories 2/3		Totals	
Right	52	**47**	59	**53**	111	**100**
Wrong	18	**75**	6	**25**	24	**100**
Totals	70	**52**	65	**48**	135	**100**

(vii) $p<0.05$

percentages of row totals in bold						
	"then" category 1		"then" categories 2/3		Totals	
Right	268	**88**	37	**12**	305	**100**
Wrong	136	**95**	7	**5**	143	**100**
Totals	404	**90**	44	**10**	448	**100**

(viii) $p<0.005$

percentages of row totals in bold						
	because		not because		Totals	
Right	149	**21**	575	**79**	724	**100**
Wrong	40	**9**	394	**91**	434	**100**
Totals	189	**16**	969	**84**	1158	**100**

(ix) $p<0.005$

percentages of row totals in bold						
	"so"		not "so"		Totals	
Right	204	**28**	520	**72**	724	**100**
Wrong	80	**18**	354	**82**	434	**100**
Totals	284	**25**	874	**75**	1158	**100**

(x) $p<0.005$

percentages of row totals in bold						
	"if"		not "if"		Totals	
Right	111	**15**	613	**85**	724	**100**
Wrong	24	**6**	410	**94**	434	**100**
Totals	135	**12**	1023	**88**	1158	**100**

(xi) $p<0.005$

percentages of row totals in bold						
	"then"		not "then"		Totals	
Right	305	**42**	419	**58**	724	**100**
Wrong	143	**33**	291	**67**	434	**100**
Totals	448	**39**	710	**61**	1158	**100**

Appendix 4.9 Contingency tables for correct and incorrect answers with "like"

(i) not significant

	Number of responses with percentages of row totals in bold							
	"like" Category 1		"like" Category 2		"like" Category 3		Totals	
Right	73	**30**	157	**65**	10	**4**	240	**100**
Wrong	40	**32**	79	**63**	7	**6**	126	**100**
Totals	113	**31**	236	**64**	17	**5**	366	**100**

(ii) not significant

	percentages of row totals in bold					
	"like"cat1		"like" 2/3		Totals	
Right	73	**30**	167	**70**	240	**100**
Wrong	40	**32**	86	**68**	126	**100**
Totals	113	**31**	253	**69**	366	**100**

(iii) not significant

	percentages of row totals in bold					
	"like"		not "like"		Totals	
Right	240	**33**	484	**67**	724	**100**
Wrong	126	**29**	308	**71**	434	**100**
Totals	366	**32**	792	**68**	1158	**100**

Appendix 4.10 Contingency tables for indicators of learning across contexts

(i) $p<0.005$

	Number of responses with percentages of row totals in bold							
	collection		motion		creation		Totals	
Maths Calculation	316	**34**	136	**15**	473	**51**	925	**100**
Maths procedure	41	**32**	7	**5**	81	**63**	129	**100**
Totals	357	**34**	143	**14**	554	**53**	1054	**100**

(ii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	counting		holistic		algorithmic		Totals	
Maths Calculation	182	**21**	361	**41**	336	**38**	879	**100**
Maths procedure	29	**23**	66	**52**	33	**26**	128	**100**
Totals	211	**21**	427	**42**	369	**37**	1007	**100**

(iii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
Maths Calculation	331	**37**	434	**48**	137	**15**	902	**100**
Maths Non-calc	103	**27**	206	**53**	77	**20**	386	**100**
Non-Mathematics	87	**31**	101	**36**	93	**33**	281	**100**
Totals	521	**33**	741	**47**	307	**20**	1569	**100**

(iv) $p<0.005$

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
Math concept	76	**33**	122	**54**	29	**13**	227	**100**
Non-math concept	80	**38**	78	**37**	55	**26**	213	**100**
Totals	156	**35**	200	**45**	84	**19**	440	**100**

(v) $p<0.005$

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
Maths Calculation	331	**37**	434	**48**	137	**15**	902	**100**
Maths Procedure	27	**17**	84	**53**	48	**30**	159	**100**
Non-Math procedure	7	**10**	23	**34**	38	**56**	68	**100**
Totals	365	**32**	541	**48**	223	**20**	1129	**100**

Appendix 4.11 Contingency tables for linguistic indicators across contexts

(i) p<0.005

	Number of responses with percentages of row totals in bold							
	Past tense		Present tense		Mixed tenses		Totals	
Maths Calculation	391	**44**	244	**27**	259	**29**	894	**100**
Maths Non-calc	22	**7**	213	**65**	93	**28**	328	**100**
Non-Mathematics	13	**6**	189	**81**	30	**13**	232	**100**
Totals	426	**29**	646	**44**	382	**26**	1454	**100**

(ii) p<0.005

	Number of responses with percentages of row totals in bold							
	"I" only		"I" and "you"		"you" only		Totals	
Maths Calculation	520	**66**	166	**21**	97	**12**	783	**100**
Maths Non-calc	93	**36**	56	**22**	109	**42**	258	**100**
Non-Mathematics	52	**33**	33	**21**	72	**46**	157	**100**
Totals	665	**56**	255	**21**	278	**23**	1198	**100**

(iii) not significant

	Number of responses with percentages of row totals in bold							
	"I" category 1		"I" category 2		"I" category 3		Totals	
Mathematics	688	**74**	71	**8**	168	**18**	927	**100**
Non-mathematics	82	**82**	4	**4**	14	**14**	100	**100**
Totals	770	**75**	75	**7**	182	**18**	1027	**100**

(iv) p<0.005

	Number of responses with percentages of row totals in bold							
	"I" category 1		"I" category 2		"I" category 3		Totals	
Maths Calculation	578	**76**	49	**6**	134	**18**	761	**100**
Maths procedure	31	**48**	18	**28**	15	**23**	64	**100**
Totals	609	**74**	67	**8**	149	**18**	825	**100**

(v) p<0.005

	Number of responses with percentages of row totals in bold							
	"you" categ. 1		"you" categ. 2		"you" categ. 3		Totals	
Maths Calculation	34	**14**	161	**65**	53	**21**	248	**100**
Maths procedure	1	**1**	76	**69**	33	**30**	110	**100**
Totals	35	**10**	237	**66**	86	**24**	358	**100**

(vi) p<0.005

Number of responses with percentages of row totals in bold						
	"I"		Not "I"		Totals	
Non-calc. Maths	166	**33**	342	**67**	508	**100**
Non-Mathematics	100	**32**	217	**68**	317	**100**
Totals	266	**32**	559	**68**	825	**100**

(vii) p<0.005

Number of responses with percentages of row totals in bold						
	"you"		Not "you"		Totals	
Non-calc. Maths	407	**80**	101	**20**	508	**100**
Non-Mathematics	104	**33**	213	**67**	317	**100**
Totals	511	**62**	314	**38**	825	**100**

(viii) Not significant

	Number of responses with percentages of row totals in bold							
	"Like" categ. 1		"Like" categ. 2		"Like" categ. 3		Totals	
Maths Calculation	113	**31**	236	**64**	17	**5**	366	**100**
Maths Non-calc	50	**31**	101	**63**	10	**6**	161	**100**
Non-Mathematics	34	**37**	54	**59**	3	**3**	91	**100**
Totals	197	**32**	391	**63**	30	**5**	618	**100**

(ix) $p<0.05$

	Number of responses with percentages of row totals in bold							
	"if" category 1		"if" category 2		"if" category 3		Totals	
Mathematics	104	**45**	73	**32**	52	**23**	229	**100**
Non-mathematics	12	**22**	14	**25**	29	**53**	55	**100**
Totals	116	**41**	87	**31**	81	**29**	284	**100**

(x) test not applicable

	Number of responses with percentages of row totals in bold							
	"then" categ. 1		"then" categ. 2		"then" categ. 3		Totals	
Mathematics	510	**90**	50	**9**	8	**1**	568	**100**
Non-mathematics	32	**84**	1	**3**	5	**13**	38	**100**
Totals	542	**89**	51	**8**	13	**2**	606	**100**

(xi) $p<0.005$

Number of responses with percentages of row totals in bold						
	"so"		Not "so"		Totals	
Mathematics	357	**21**	1309	**79**	1666	**100**
Non-Mathematics	23	**7**	294	**93**	317	**100**
Totals	380	**19**	1603	**81**	1983	**100**

(xii) $p<0.005$

Number of responses with percentages of row totals in bold						
	"then"		Not "then"		Totals	
Mathematics	568	**34**	1098	**66**	1666	**100**
Non-Mathematics	38	**12**	279	**88**	317	**100**
Totals	606	**31**	1377	**69**	1983	**100**

(xiii) not significant

Number of responses with percentages of row totals in bold						
	"because"		Not "because"		Totals	
Mathematics	225	**14**	1441	**86**	1666	**100**
Non-Mathematics	33	**10**	284	**90**	317	**100**
Totals	258	**13**	1725	**87**	1983	**100**

(xiv) not significant

Number of responses with percentages of row totals in bold						
	"if"		Not "if"		Totals	
Mathematics	229	**14**	1437	**86**	1666	**100**
Non-Mathematics	55	**17**	262	**83**	317	**100**
Totals	284	**14**	1699	**86**	1983	**100**

Appendix 4.12 Contingency tables for comparison of upper- and lower-SAT-score pupils

(i) $p<0.005$

Number of responses with percentages of row totals in bold						
	Incorrect		Correct		Totals	
Upper Comb. SAT	167	**29**	410	**71**	577	**100**
Lower Comb. SAT	267	**46**	314	**54**	581	**100**
Totals	434	**37**	724	**63**	1158	**100**

(ii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
Upper Comb. SAT	136	**28**	253	**52**	100	**20**	489	**100**
Lower Comb. SAT	195	**47**	181	**44**	37	**9**	413	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

(iii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	collection		motion		creation		Totals	
Upper Comb. SAT	135	**28**	65	**13**	290	**59**	490	**100**
Lower Comb. SAT	181	**42**	71	**16**	183	**42**	435	**100**
Totals	316	**34**	136	**15**	473	**51**	925	**100**

(iv) $p<0.005$

	Number of responses with percentages of row totals in bold							
	counting		holistic		algorithmic		Totals	
Upper Comb. SAT	62	**13**	192	**41**	218	**46**	472	**100**
Lower Comb. SAT	120	**29**	169	**42**	118	**29**	407	**100**
Totals	182	**21**	361	**41**	336	**38**	879	**100**

(v) $p<0.005$

	Number of responses with percentages of row totals in bold							
	Past tense		Present tense		Mixed tenses		Totals	
Upper Comb. SAT	179	**37**	150	**31**	155	**32**	484	**100**
Lower Comb. SAT	212	**52**	94	**23**	104	**25**	410	**100**
Totals	391	**44**	244	**27**	259	**29**	894	**100**

(vi) $p<0.005$

	Number of responses with percentages of row totals in bold							
	"I" category 1		"I" category 2		"I" category 3		Totals	
Upper Comb. SAT	263	**67**	31	**8**	99	**25**	393	**100**
Lower Comb. SAT	315	**86**	18	**5**	35	**10**	368	**100**
Totals	578	**76**	49	**6**	134	**18**	761	**100**

(vii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	specific		iconic		symbolic		Totals	
Upper Comb. SAT	10	**6**	25	**15**	134	**79**	169	**100**
Lower Comb. SAT	11	**8**	41	**30**	84	**62**	136	**100**
Totals	21	**7**	66	**22**	218	**71**	305	**100**

Appendix 4.13 Contingency tables for comparisons between the two sets

(i) $p<0.005$

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
Set 1	159	**31**	259	**50**	99	**19**	517	**100**
Set 2	172	**45**	175	**45**	38	**10**	385	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

(ii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	collection		motion		creation		Totals	
Set 1	141	**27**	82	**16**	298	**57**	521	**100**
Set 2	175	**43**	54	**13**	175	**43**	404	**100**
Totals	316	**34**	136	**15**	473	**51**	925	**100**

(iii) $p<0.005$

	Number of responses with percentages of row totals in bold							
	counting		holistic		algorithmic		Totals	
Set 1	141	**27**	82	**16**	298	**57**	521	**100**
Set 2	175	**43**	54	**13**	175	**43**	404	**100**
Totals	316	**34**	136	**15**	473	**51**	925	**100**

(iv) $p<0.005$

	Number of responses with percentages of row totals in bold							
	Past tense		Present tense		Mixed tenses		Totals	
Set 1	182	**36**	162	**32**	155	**31**	499	**100**
Set 2	209	**53**	82	**21**	104	**26**	395	**100**
Totals	391	**44**	244	**27**	259	**29**	894	**100**

(v) $p<0.005$

	Number of responses with percentages of row totals in bold							
	"I" category 1		"I" category 2		"I" category 3		Totals	
Set 1	284	**71**	29	**7**	89	**22**	402	**100**
Set 2	294	**82**	20	**6**	45	**13**	359	**100**
Totals	578	**76**	49	**6**	134	**18**	761	**100**

(vi) $p<0.005$

Number of responses with percentages of row totals in bold						
	Incorrect		Correct		Totals	
Set 1	195	**31**	428	**69**	623	**100**
Set 2	239	**45**	296	**55**	535	**100**
Totals	434	**37**	724	**63**	1158	**100**

Appendix 4.14 Contingency tables for comparison of high-, middle-, and low-accuracy pupils

(i) $p<0.005$ Calculation

	Number of responses with percentages of row totals in bold							
	specific		iconic		symbolic		Totals	
High-accuracy	4	**5**	13	**16**	65	**79**	82	**100**
Middle-accuracy	11	**6**	34	**19**	130	**74**	175	**100**
Low-accuracy	6	**13**	19	**40**	23	**48**	48	**100**
Totals	21	**7**	66	**22**	218	**71**	305	**100**

(ii) $p<0.005$ Non calculation

	Number of responses with percentages of row totals in bold							
	specific		iconic		symbolic		Totals	
High-accuracy	10	**19**	27	**50**	17	**31**	54	**100**
Middle-accuracy	42	**30**	59	**42**	40	**28**	141	**100**
Low-accuracy	14	**33**	18	**42**	11	**26**	43	**100**
Totals	66	**28**	104	**44**	68	**29**	238	**100**

(iii) $p<0.005$ Calculation

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
High-accuracy	59	**32**	89	**48**	36	**20**	184	**100**
Middle-accuracy	191	**33**	305	**52**	87	**15**	583	**100**
Low-accuracy	81	**60**	40	**30**	14	**10**	135	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

(iv) $p<0.005$ Non calculation

	Number of responses with percentages of row totals in bold							
	particular		generic		general		Totals	
High-accuracy	22	**39**	16	**28**	19	**33**	57	**100**
Middle-accuracy	50	**29**	63	**36**	61	**35**	174	**100**
Low-accuracy	15	**30**	22	**44**	13	**26**	50	**100**
Totals	87	**31**	101	**36**	93	**33**	281	**100**

(v) $p<0.005$ High-accuracy pupils

	Number of responses with percentages of column totals in bold							
	particular		generic		general		Totals	
Right	39	**66**	79	**89**	33	**92**	151	**82**
Wrong	20	**34**	10	**11**	3	**8**	33	**18**
Totals	59	**100**	89	**100**	36	**100**	184	**100**

(vi) $p<0.005$ low-accuracy pupils

	Number of responses with percentages of column totals in bold							
	particular		generic		general		Totals	
Right	37	**46**	19	**48**	6	**43**	62	**46**
Wrong	44	**54**	21	**53**	8	**57**	73	**54**
Totals	81	**100**	40	**100**	14	**100**	135	**100**

(vii) $p<0.005$ Calculation

	Number of responses with percentages of row totals in bold							
	collection		motion		creation		Totals	
High-accuracy	66	**36**	18	**10**	101	**55**	185	**100**
Middle-accuracy	179	**30**	95	**16**	321	**54**	595	**100**
Low-accuracy	71	**49**	23	**16**	51	**35**	145	**100**
Totals	316	**34**	136	**15**	473	**51**	925	**100**

(viii) $p<0.005$ Calculation

	Number of responses with percentages of row totals in bold							
	counting		holistic		algorithmic		Totals	
High-accuracy	19	**11**	84	**47**	74	**42**	177	**100**
Middle-accuracy	109	**19**	240	**42**	228	**40**	577	**100**
Low-accuracy	54	**43**	37	**30**	34	**27**	125	**100**
Totals	182	**21**	361	**41**	336	**38**	879	**100**

(ix) $p<0.005$ Calculation

	Number of responses with percentages of row totals in bold							
	Past tense		Present tense		Mixed tenses		Totals	
High-accuracy	89	**49**	56	**31**	38	**21**	183	**100**
Middle-accuracy	221	**39**	156	**27**	194	**34**	571	**100**
Low-accuracy	81	**58**	32	**23**	27	**19**	140	**100**
Totals	391	**44**	244	**27**	259	**29**	894	**100**

(x) $p<0.005$ Calculation

	Number of responses with percentages of row totals in bold							
	"I" only		"I" and "You"		"You" only		Totals	
High-accuracy	101	**76**	16	**12**	16	**12**	133	**100**
Middle-accuracy	328	**62**	130	**25**	68	**13**	526	**100**
Low-accuracy	91	**73**	20	**16**	13	**10**	124	**100**
Totals	520	**66**	166	**21**	97	**12**	783	**100**

(xi) $p<0.005$ Calculation

	Number of responses with percentages of row totals in bold							
	"I" category 1		"I" category 2		"I" category 3		Totals	
High-accuracy	89	**71**	9	**7**	28	**22**	126	**100**
Middle-accuracy	363	**73**	34	**7**	102	**20**	499	**100**
Low-accuracy	126	**93**	6	**4**	4	**3**	136	**100**
Totals	578	**76**	49	**6**	134	**18**	761	**100**

(xii) test N/A Non calculation

	Number of responses with percentages of row totals in bold							
	"I" category 1		"I" category 2		"I" category 3		Totals	
High-accuracy	9	**53**	1	**6**	7	**41**	17	**100**
Middle-accuracy	55	**90**	2	**3**	4	**7**	61	**100**
Low-accuracy	18	**82**	1	**5**	3	**14**	22	**100**
Totals	82	**82**	4	**4**	14	**14**	100	**100**

(xiii) test N/A Non calculation

	Number of responses with percentages of row totals in bold							
	"you" categ. 1		"you" categ. 2		"you" categ. 3		Totals	
High-accuracy	0	**0**	6	**30**	14	**70**	20	**100**
Middle-accuracy	0	**0**	29	**47**	33	**53**	62	**100**
Low-accuracy	2	**9**	14	**64**	6	**27**	22	**100**
Totals	2	**2**	49	**47**	53	**51**	104	**100**

(xiv) $p<0.005$ Calculation

percentages of row totals in bold						
	"because"		Not "because"		Totals	
High	29	**13**	195	**87**	224	**100**
Low	19	**9**	204	**91**	223	**100**
Tot	48	**11**	399	**89**	447	**100**

(xv) $p<0.005$ Non-Calculation

percentages of row totals in bold						
	"because"		Not "because"		Totals	
High	7	**3**	217	**97**	224	**100**
Low	8	**4**	215	**96**	223	**100**
Tot	15	**3**	432	**97**	447	**100**

(xvi) $p<0.005$ Calculation

percentages of row totals in bold						
	"so"		Not "so"		Totals	
High	43	**19**	181	**81**	224	**100**
Low	26	**12**	197	**88**	223	**100**
Tot	69	**15**	378	**85**	447	**100**

(xvii) $p<0.005$ Non-Calculation

percentages of row totals in bold						
	"so"		Not "so"		Totals	
High	11	**5**	213	**95**	224	**100**
Low	11	**5**	212	**95**	223	**100**
Tot	22	**5**	425	**95**	447	**100**

(xviii) $p<0.005$ Calculation

percentages of row totals in bold						
	"if"		Not "if"		Totals	
High	20	**9**	204	**91**	224	**100**
Low	14	**6**	209	**94**	223	**100**
Tot	34	**8**	413	**92**	447	**100**

(xix) $p<0.005$ Non-Calculation

percentages of row totals in bold						
	"if"		Not "if"		Totals	
High	25	**11**	199	**89**	224	**100**
Low	25	**11**	198	**89**	223	**100**
Tot	50	**11**	397	**89**	447	**100**

(xx) $p<0.005$ Calculation

percentages of row totals in bold						
	"then"		Not "then"		Totals	
High	87	**39**	137	**61**	224	**100**
Low	57	**26**	166	**74**	223	**100**
Tot	144	**32**	303	**68**	447	**100**

(xxi) $p<0.005$ Non-Calculation

percentages of row totals in bold						
	"then"		Not "then"		Totals	
High	28	**13**	196	**88**	224	**100**
Low	22	**10**	201	**90**	223	**100**
Tot	50	**11**	397	**89**	447	**100**

Appendix 4.15 Categories of the indicators of learning for each question

(i) image and generality

					image						generality				
Year	Term		Facility		No response	specific	iconic	symbolic	non-specific		no response	particular	generic	general	non-particular
4	1	17 + 9 (Y4/1)	26		20	1	1	4	5		4	7	12	3	15
4	3	Now do 36 add 20	26		20	0	5	1	6		1	6	19	0	19
3	3	17 + 9 (Y3/3)	24		14	1	3	8	11		3	7	12	4	16
4	2	17 + 9	24		19	1	1	5	6		1	8	13	4	17
3	2	30 + * = 80	24		22	1	1	2	3		1	12	13	0	13
3	1	97 + 10	24		23	0	1	2	3		3	4	15	4	19
4	2	48 + 23	23		19	0	0	7	7		2	8	11	5	16
3	3	97 + 10	23		18	0	1	7	8		2	10	12	2	14
4	2	Round 2462 (100)	23		19	0	3	4	7		4	7	8	7	15
3	1	Round 246 (10)	22		17	1	3	5	8		3	7	10	6	16
3	1	13 + * = 18	21		23	0	0	3	3		5	15	5	1	6
3	1	Before 380	21		16	1	5	4	9		2	9	12	3	15
3	3	Before 2380	21		17	0	5	4	9		3	13	4	6	10
4	1	After 12,386	21		21	0	0	5	5		16	3	5	2	7
3	2	17 + 9 (Y3/2)	20		21	0	2	3	5		4	7	12	3	15
4	1	Round 239 (100)	20		19	0	5	2	7		7	2	12	5	17
4	3	0.6 add 0.7	20		17	1	0	8	8		3	2	20	1	21
4	3	Tenths make a whole one	20		25	0	0	1	1		3	10	11	2	13
4	3	70 multiplied by 5	20		18	0	0	8	8		6	8	10	2	12
4	2	1197 + 10	19		18	0	1	7	8		11	2	10	3	13
3	3	Quarter of 40	19		19	1	1	5	6		5	7	9	5	14
4	2	Before 12,100	18		18	0	0	8	8		13	7	3	3	6
3	3	Round 2462 (10)	18		20	1	1	4	5		7	4	9	6	15
4	1	48 + 23	17		16	2	1	7	8		3	7	7	9	16
3	2	597 + 10	17		19	0	3	4	7		2	6	13	5	18
3	1	17 + 8 (Y3/1)	16		21	0	3	2	5		6	9	9	2	11
3	2	48 + 23	16		18	1	0	7	7		5	4	14	3	17
4	2	How to write that	16		13	0	0	13	13		26	0	0	0	0
3	2	Read time (11:40)	15		25	1	0	0	0		20	3	0	3	3
4	3	0.1 times by 10	15		21	0	1	4	5		9	7	4	6	10
3	3	48 + 23	14		16	1	2	7	9		3	3	15	5	20
3	3	200 more than 4360	14		21	1	0	4	4		9	4	11	2	13
3	1	48 + 23	13		20	0	2	4	6		6	3	14	3	17
3	1	Estimate the length	13		17	0	9	0	9		2	23	1	0	1
4	2	48 x 3	10		20	0	0	6	6		5	5	14	2	16
4	2	65 subtract 29	9		13	1	1	11	12		2	7	13	4	17
4	1	1097 + 10	8		19	0	0	7	7		3	6	14	3	17
3	3	Read number (26,365)	8		24	0	0	2	2		16	1	7	2	9
4	3	47 multiplied by 5	7		15	1	0	10	10		5	13	7	1	8
4	1	27 + * = 65	5		16	2	0	8	8		4	11	11	0	11
4	2	A third of 48	5		21	2	0	3	3		10	10	4	2	6
3	2	Diff 27 and 65	2		21	0	3	2	5		5	11	10	0	10
4	1	two thirds of 24	2		20	0	1	5	6		8	11	5	2	7
4	1	48 x 3	2		24	0	1	1	2		7	10	4	5	9
4	3	140 divided by 3	0		22	0	0	4	4		3	12	10	1	11

(ii) metaphor and method

					metaphor						method				
Year	Term		Facility		no response	collection	motion	creation	non-collection		no response	counting	holistic	algorithmic	non-counting
4	1	17 + 9 (Y4/1)	26		2	14	1	9	10		2	10	12	2	14
4	3	Now do 36 add 20	26		3	5	1	17	18		1	0	15	10	25
3	3	17 + 9 (Y3/3)	24		0	13	3	10	13		1	10	9	6	15
4	2	17 + 9	24		0	12	2	12	14		0	7	10	9	19
3	2	30 + * = 80	24		1	11	1	13	14		1	8	5	12	17
3	1	97 + 10	24		2	5	3	16	19		3	3	6	14	20
4	2	48 + 23	23		2	5	4	15	19		2	1	11	12	23
3	3	97 + 10	23		0	6	4	16	20		1	2	8	15	23
4	2	Round 2462 (100)	23		5	0	16	5	21		9	2	11	4	15
3	1	Round 246 (10)	22		6	5	14	1	15		6	4	8	8	16
3	1	13 + * = 18	21		2	16	0	8	8		2	15	1	8	9
3	1	Before 380	21		4	6	6	10	16		4	7	9	6	15
3	3	Before 2380	21		3	8	6	9	15		7	12	2	5	7
4	1	After 12,386	21		16	6	0	4	4		16	4	0	6	6
3	2	17 + 9 (Y3/2)	20		0	12	1	13	14		1	12	6	7	13
4	1	Round 239 (100)	20		4	1	19	2	21		6	2	12	6	18
4	3	0.6 add 0.7	20		2	5	0	19	19		2	0	11	13	24
4	3	Tenths make a whole one	20		9	11	0	6	6		7	5	8	6	14
4	3	70 multiplied by 5	20		6	4	0	16	16		6	0	9	11	20
4	2	1197 + 10	19		10	3	6	7	13		11	0	7	8	15
3	3	Quarter of 40	19		7	9	1	9	10		6	15	3	2	5
4	2	Before 12,100	18		14	7	1	4	5		13	8	4	1	5
3	3	Round 2462 (10)	18		7	0	14	5	19		7	1	12	6	18
4	1	48 + 23	17		2	5	1	18	19		2	2	6	16	22
3	2	597 + 10	17		2	5	5	14	19		2	5	9	10	19
3	1	17 + 8 (Y3/1)	16		2	16	2	6	8		2	14	7	3	10
3	2	48 + 23	16		4	11	1	10	11		7	0	9	10	19
4	2	How to write that	16		24	0	0	2	2		26	0	0	0	0
3	2	Read time (11:40)	15		22	2	0	2	2		21	0	5	0	5
4	3	0.1 times by 10	15		10	1	0	15	15		11	0	2	13	15
3	3	48 + 23	14		1	3	3	19	22		1	0	8	17	25
3	3	200 more than 4360	14		10	2	0	14	14		10	0	4	12	16
3	1	48 + 23	13		2	11	3	10	13		2	4	11	9	20
3	1	Estimate the length	13		5	17	2	2	4		4	5	15	2	17
4	2	48 x 3	10		3	8	0	15	15		3	1	15	7	22
4	2	65 subtract 29	9		1	9	1	15	16		2	0	16	8	24
4	1	1097 + 10	8		4	3	4	15	19		8	0	4	14	18
3	3	Read number (26,365)	8		12	0	0	14	14		25	0	1	0	1
4	3	47 multiplied by 5	7		4	2	0	20	20		9	1	6	10	16
4	1	27 + * = 65	5		2	4	2	18	20		5	2	10	9	19
4	2	A third of 48	5		7	10	0	9	9		8	10	7	1	8
3	2	Diff 27 and 65	2		6	5	7	8	15		8	1	11	6	17
4	1	two thirds of 24	2		8	14	1	3	4		9	7	9	1	10
4	1	48 x 3	2		6	15	1	4	5		7	2	15	2	17
4	3	140 divided by 3	0		3	9	0	14	14		5	0	12	9	21

Appendix 4.16 Difficult questions: categories of response in each indicator for each pupil

(i) incorrect/correct answer

0 incorrect 1 correct

			incorrect/correct												
comb SAT	accuracy		27 + * = 65	1097 + 10	Diff 27 and 65	Read number (26,365)	65 subtract 29	two thirds of 24	A third of 48	140 divided by 3	48 x 3	48 x 3	47 multiplied by 5		Total correct
63	12	Hester	0	0	0	0	0	0	0	0	0	0	0		0
59	13	Myles	0	0	0	0	0	0	0	0	0	0	0		0
63	16	Simon	0	0	0	0	0	0	0	0	0	0	0		0
88	19	Suzy	0	0	0	0	0	0	0	0	0	0	0		0
61	20	Sean	0	0	0	0	0	0	0	0	0	0	0		0
108	25	Peter	0	0	0	0	0	0	0	0	0	0	0		0
107	27	Ann	0	0	0	0	0	0	0	0	0	0	0		0
88	22	Mandy	0	0	0	1	0	0	0	0	0	0	0		1
72	28	Ellain	1	0	0	0	0	0	0	0	0	0	0		1
89	28	Naomi	0	0	1	0	0	0	0	0	0	0	0		1
86	32	Malcolm	0	0	0	0	0	1	0	0	0	0	0		1
78	28	Jeremy	0	1	0	0	0	0	0	0	0	0	1		2
115	30	Irene	1	0	0	0	1	0	0	0	0	0	0		2
121	31	Hannah	0	1	0	0	0	0	0	0	0	1	0		2
119	32	Terry	0	0	0	1	0	0	0	0	0	1	0		2
133	36	Clara	0	0	1	0	0	0	1	0	0	0	0		2
111	28	Christine	0	0	0	0	1	0	0	0	0	1	1		3
113	31	Elspeth	0	0	0	1	0	0	1	0	0	0	1		3
112	34	Bobby	1	0	0	0	1	0	0	0	0	1	0		3
87	35	Paddy	0	1	0	0	1	0	0	0	0	0	1		3
89	32	John	0	1	0	0	1	0	0	0	0	1	1		4
111	33	Max	1	0	0	1	1	0	0	0	0	1	1		5
103	34	Dennis	0	1	0	1	0	0	1	0	1	1	0		5
124	38	Jack	0	1	0	1	1	0	0	0	1	1	0		5
119	37	Kath	0	1	0	1	1	0	1	0	0	1	1		6
108	38	Teddy	1	1	0	1	1	1	1	0	0	1	0		7

(ii) generality and metaphor

generality

0	No response	2	iconic
1	specific	3	symbolic

metaphor

0	No response	2	collection
1	motion	3	creation

			generality												metaphor										
comb SAT	accuracy		27 + * = 65	1097 + 10	Diff 27 and 65	Read number (26,365)	65 subtract 29	two thirds of 24	A third of 48	140 divided by 3	48 x 3	48 x 3	47 multiplied by 5		27 + * = 65	1097 + 10	Diff 27 and 65	Read number (26,365)	65 subtract 29	two thirds of 24	A third of 48	140 divided by 3	48 x 3	48 x 3	47 multiplied by 5
63	12	Hester	0	2	0	0	1	0	0	2	3	1	0		1	3	2	3	1	0	0	3	1	1	0
59	13	Myles	1	1	1	0	1	0	1	1	0	0	1		1	3	0	0	1	2	1	1	2	0	3
63	16	Simon	1	1	1	0	0	1	0	3	1	2	1		3	1	0	0	1	1	0	3	1	3	1
88	19	Suzy	0	0	0	0	2	0	0	0	0	0	0		0	0	0	0	2	0	0	0	0	0	0
61	20	Sean	1	0	1	0	2	0	2	1	0	0	2		3	0	1	3	1	0	1	3	0	1	3
108	25	Peter	2	2	2	2	3	1	2	2	0	0	1		3	2	1	3	3	1	1	3	0	1	3
107	27	Ann	1	2	1	0	2	0	0	1	0	1	0		3	3	3	0	3	0	3	3	0	3	0
88	22	Mandy	1	2	1	0	0	2	1	1	0	2	2		2	3	2	3	0	3	1	3	0	1	3
72	28	Ellain	2	1	0	0	2	1	0	1	1	1	1		3	0	0	0	3	1	0	3	3	3	3
89	28	Naomi	2	2	2	2	1	1	0	0	0	0	1		3	3	2	3	3	1	0	0	0	0	3
86	32	Malcolm	2	2	1	0	1	1	1	2	1	2	1		3	3	1	0	1	1	1	3	1	3	3
78	28	Jeremy	2	1	2	0	2	1	0	1	1	1	2		3	1	2	3	1	1	1	1	1	3	1
115	30	Irene	2	2	2	2	1	0	1	1	1	2	2		3	3	3	0	3	0	3	1	1	3	3
121	31	Hannah	0	2	2	3	2	0	1	1	1	2	1		3	3	3	3	3	1	3	3	1	3	3
119	32	Terry	2	1	2	3	1	1	0	0	2	2	1		3	3	3	3	3	3	3	0	1	3	3
133	36	Clara	0	0	1	1	2	2	1	2	3	2	0		0	0	1	3	3	3	3	3	1	1	3
111	28	Christine	1	3	0	2	3	1	0	2	3	1	2		3	3	1	3	3	1	0	3	1	1	3
113	31	Elspeth	2	2	2	2	2	2	1	1	1	3	3		3	2	3	3	3	0	3	3	1	3	3
112	34	Bobby	1	2	0	0	3	2	3	2	2	2	1		1	3	0	0	1	1	1	1	3	3	3
87	35	Paddy	2	2	2	0	2	0	2	2	1	2	2		3	3	2	0	3	0	1	1	1	3	3
89	32	John	1	1	2	0	1	1	1	2	1	2	2		3	1	2	3	1	1	1	3	1	1	3
111	33	Max	2	3	1	0	2	3	2	1	2	2	1		3	2	0	0	1	1	3	1	3	3	3
103	34	Dennis	1	2	1	0	2	1	1	1	2	2	1		3	3	3	0	3	1	3	1	1	3	3
124	38	Jack	1	2	1	2	3	2	0	2	3	2	0		1	3	3	3	3	1	0	1	1	1	0
119	37	Kath	1	3	1	0	2	1	3	2	3	3	1		3	2	2	0	3	0	1	3	3	3	3
108	38	Teddy	2	2	2	2	2	3	1	1	1	2	1		2	3	3	3	3	1	3	1	1	3	3

(iii) method and image

method

0	No response	2	holistic
1	counting	3	algorithmic

image

0	No response	2	iconic
1	specific	3	symbolic

				method												image										
comb SAT	accuracy			27 + * = 65	1097 + 10	Diff 27 and 65	Read number (26,365)	two thirds of 24	A third of 48	65 subtract 29	140 divided by 3	48 x 3	48 x 3	47 multiplied by 5		27 + * = 65	1097 + 10	Diff 27 and 65	Read number (26,365)	two thirds of 24	A third of 48	65 subtract 29	140 divided by 3	48 x 3	48 x 3	47 multiplied by 5
63	12	Hester		0	3	0	0	0	0	2	0	1	2	0		0	0	2	0	0	0	3	0	0	0	0
59	13	Myles		1	0	0	0	0	1	3	2	0	0	3		0	3	3	0	2	0	3	0	2	0	3
63	16	Simon		2	0	0	0	0	0	2	0	3	2	1		0	0	0	0	0	0	0	0	0	0	0
88	19	Suzy		0	0	0	0	0	0	2	0	0	0	0		0	0	0	0	0	0	3	0	0	0	0
61	20	Sean		2	0	0	0	0	1	3	3	0	1	3		0	0	0	0	0	0	3	0	0	0	0
108	25	Peter		2	3	2	0	3	1	2	3	0	2	0		0	3	0	0	0	0	0	0	0	0	3
107	27	Ann		2	3	3	0	0	1	3	2	0	2	0		0	0	0	0	0	0	3	0	0	0	3
88	22	Mandy		2	3	2	0	2	1	3	3	0	2	3		0	0	2	0	3	0	3	0	0	0	3
72	28	Ellain		3	0	0	0	2	0	2	2	2	2	0		0	0	0	0	0	0	0	0	0	0	0
89	28	Naomi		3	3	2	0	1	0	3	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0
86	32	Malcolm		2	3	2	0	1	1	2	3	2	2	3		1	3	0	0	3	0	2	0	0	0	0
78	28	Jeremy		3	2	2	0	1	1	2	2	1	2	2		0	0	0	0	0	0	0	0	0	0	0
115	30	Irene		3	0	3	0	0	3	2	2	2	3	3		1	0	0	0	0	0	3	0	0	3	3
121	31	Hannah		0	3	3	0	1	2	2	3	2	3	0		0	0	0	0	0	0	0	0	0	3	0
119	32	Terry		3	3	2	0	2	0	0	0	2	3	2		0	0	0	0	0	0	0	0	0	0	0
133	36	Clara		0	0	2	0	2	2	2	3	2	2	0		3	3	2	3	0	1	0	3	0	3	1
111	28	Christine		3	2	1	0	1	0	3	3	2	3	3		0	0	0	3	0	0	3	3	0	3	3
113	31	Elspeth		3	2	3	0	1	2	2	3	2	3	3		3	0	3	0	0	3	0	3	0	3	3
112	34	Bobby		3	3	0	0	2	1	2	2	2	2	2		3	3	0	0	3	3	3	3	0	0	0
87	35	Paddy		2	2	2	0	0	1	2	2	2	3	3		0	0	0	0	0	1	3	0	0	0	3
89	32	John		2	3	2	2	2	1	3	2	2	2	3		3	0	0	0	0	0	3	0	0	0	0
111	33	Max		2	0	0	0	1	2	2	2	2	2	2		3	0	0	0	0	0	0	0	0	0	0
103	34	Dennis		0	3	3	0	0	2	3	2	2	2	3		3	3	0	0	3	0	0	0	3	0	3
124	38	Jack		1	3	3	0	2	0	0	2	2	2	0		0	0	0	0	0	0	1	0	0	0	0
119	37	Kath		3	3	2	0	2	2	2	3	3	3	2		3	3	0	0	0	0	0	0	0	0	3
108	38	Teddy		2	3	2	0	2	2	2	2	2	2	2		3	0	0	0	3	3	0	0	0	3	0

Appendix 4.17 Categories of response for two common questions

(i) frequency in each category of learning indicators

			Generality						Image			
Year	Term		Accuracy	no response	particular	generic	general		no response	specific	iconic	symbolic
3	1	17 + 8	16	6	9	9	2		20	0	3	2
3	2	17 + 9	20	4	7	12	3		21	0	2	2
3	3	17 + 9	24	3	7	12	4		13	1	3	8
4	1	17 + 9	26	4	7	12	3		19	1	1	4
4	2	17 + 9	24	1	8	13	4		18	1	1	5

			Method						Metaphor			
Year	Term		Accuracy	no response	counting	holistic	algorithmic		no response	collection	motion	creation
3	1	17 + 8	16	2	14	7	3		2	16	2	5
3	2	17 + 9	20	1	12	6	7		0	12	1	12
3	3	17 + 9	24	1	10	9	6		0	13	3	9
4	1	17 + 9	26	2	10	12	2		2	14	1	8
4	2	17 + 9	24	0	7	10	9		0	12	2	11

(ii) frequency in each category of learning indicators

			Generality						Image			
Year	Term		Accuracy	No response	particular	generic	general		No response	specific	iconic	symbolic
3	1	48 + 23	13	6	3	14	3		19	0	2	4
3	2	48 + 23	16	5	4	14	3		17	1	0	7
3	3	48 + 23	14	3	3	15	5		15	1	2	7
4	1	48 + 23	17	3	7	7	9		15	2	1	7
4	2	48 + 23	23	2	8	11	5		19	0	0	6

			Method						Metaphor			
Year	Term		Accuracy	No response	counting	holistic	algorithmic		No response	collection	motion	creation
3	1	48 + 23	13	2	4	11	9		2	10	3	10
3	2	48 + 23	16	7	0	9	10		4	10	1	10
3	3	48 + 23	14	1	0	8	17		1	3	3	18
4	1	48 + 23	17	2	2	6	16		2	4	1	18
4	2	48 + 23	23	2	1	11	12		2	5	4	14

(iii) Categories for each pupil in common questions

In this table cells are shaded when the question was answered correctly. It is ordered by the number answered correctly from Myles with 3 to Malcolm etc with 10:

generality

0	No response	2	iconic
1	specific	3	symbolic

method

0	No response	2	holistic
1	counting	3	algorithmic

	generality												method										
	17 + 8	17 + 9	17 + 9	17 + 9	17 + 9		48 + 23	48 + 23	48 + 23	48 + 23	48 + 23		17 + 8	17 + 9	17 + 9	17 + 9	17 + 9		48 + 23	48 + 23	48 + 23	48 + 23	48 + 23
Myles	0	3	1	1	2		0	2	2	1	2		0	1	1	1	3		0	3	2	1	3
Jeremy	1	1	1	1	2		2	1	2	3	3		1	1	1	1	2		3	2	3	3	3
Simon	1	2	0	1	1		1	2	2	2	3		1	1	0	1	1		2	2	2	3	3
Christine	3	1	0	2	2		2	2	0	3	0		1	1	1	2	3		2	3	3	2	0
Hester	0	1	1	1	1		0	1	1	0	1		1	1	1	1	1		1	2	3	1	2
Mandy	1	1	3	0	2		0	0	2	3	1		1	1	1	0	2		1	0	3	3	2
Elspeth	2	2	0	2	3		3	2	2	2	2		2	3	1	2	3		3	3	3	3	2
Suzy	0	1	1	1	1		0	0	0	3	1		0	1	1	1	1		0	0	0	3	1
Max	2	2	2	2	2		2	2	3	0	2		3	2	2	2	2		3	3	3	0	2
Sean	2	1	2	2	1		0	3	2	0	1		1	1	3	1	1		2	3	2	0	2
Ann	3	0	2	2	2		2	1	3	3	3		1	1	3	2	3		3	0	3	3	3
Hannah	1	0	2	2	2		2	2	3	1	2		1	1	2	3	3		3	3	3	3	3
John	0	2	1	2	2		2	2	1	1	2		1	2	3	2	2		2	2	2	2	2
Naomi	0	0	2	1	1		1	0	2	1	2		1	1	2	2	1		1	0	3	3	3
Paddy	1	2	1	0	2		2	0	2	2	0		1	3	1	1	2		3	0	3	2	0
Peter	1	2	2	3	2		2	2	3	3	1		3	2	2	2	2		2	3	3	3	3
Terry	2	1	3	2	3		2	2	2	1	1		2	0	3	2	3		2	3	2	2	2
Bobby	2	2	2	2	1		2	0	2	2	1		2	3	3	3	3		2	0	2	2	2
Dennis	1	2	3	3	2		1	1	2	2	2		2	3	2	2	2		2	2	3	3	3
Jack	2	2	3	3	1		3	3	3	3	3		2	3	1	1	1		3	0	3	3	3
Kath	1	3	2	2	3		2	2	2	1	1		1	2	2	2	2		2	2	3	2	2
Teddy	0	2	1	1	3		2	2	2	2	3		1	3	1	1	2		2	2	2	3	3
Clara	2	3	2	0	2		3	2	0	1	2		2	3	2	0	2		3	2	3	3	2
Ellain	2	2	2	2	2		2	3	2	3	2		2	2	2	2	3		3	3	3	3	3
Irene	2	2	2	2	1		2	2	1	2	2		3	2	2	2	3		2	3	3	3	3
Malcolm	1	0	2	0	0		0	2	2	3	2		1	1	3	1	1		1	2	2	3	2

(iv) Categories for each pupil in common questions

In this table cells are shaded when the question was answered correctly. It is ordered by the number answered correctly from Myles with 3 to Malcolm etc with 10:

image

0	No response	2	iconic
1	specific	3	symbolic

metaphor

0	No response	2	motion
1	collection	3	creation

	image												metaphor										
	17 + 8	17 + 9	17 + 9	17 + 9	17 + 9		48 + 23	48 + 23	48 + 23	48 + 23	48 + 23		17 + 8	17 + 9	17 + 9	17 + 9	17 + 9		48 + 23	48 + 23	48 + 23	48 + 23	48 + 23
Myles	0	0	0	2	0		3	0	2	2	0		0	1	1	2	3		0	3	2	2	3
Jeremy	0	0	0	0	0		0	0	0	0	0		1	1	1	1	3		3	1	1	3	3
Simon	2	0	2	1	1		0	0	0	0	0		2	1	2	1	1		1	1	3	3	3
Christine	0	0	0	0	0		0	3	3	1	0		1	1	1	3	3		1	3	1	1	0
Hester	0	0	0	0	0		0	0	0	0	0		1	1	1	1	1		1	1	3	1	2
Mandy	2	2	2	0	2		2	1	1	0	0		2	2	2	0	2		2	2	2	3	2
Elspeth	3	3	0	3	0		0	3	2	3	3		3	3	1	3	3		3	3	3	3	3
Suzy	0	2	0	0	0		0	0	0	0	0		0	1	1	1	1		0	0	0	3	1
Max	0	0	3	0	0		0	3	3	0	0		3	3	3	3	1		2	3	3	0	3
Sean	0	0	3	0	0		3	0	0	3	0		1	1	3	1	1		3	3	3	0	1
Ann	0	0	3	0	0		0	0	0	0	3		1	1	3	1	3		3	1	3	3	3
Hannah	0	0	0	0	0		0	0	0	0	0		1	1	3	3	3		3	3	3	3	2
John	0	0	0	0	3		0	0	0	0	0		1	3	1	3	3		1	1	3	3	3
Naomi	0	0	3	0	0		0	0	3	0	0		1	1	2	1	1		1	0	3	3	2
Paddy	0	0	0	0	3		3	0	0	0	0		1	3	1	1	3		3	0	3	3	0
Peter	0	0	0	0	0		0	0	0	3	3		1	3	1	1	1		1	1	3	3	3
Terry	0	3	0	0	0		0	0	0	0	3		3	3	3	3	3		1	1	3	1	3
Bobby	0	0	3	3	0		0	0	3	3	0		3	3	3	3	3		1	0	3	1	1
Dennis	0	3	3	3	0		0	3	3	3	0		1	3	1	1	1		1	1	3	3	3
Jack	0	0	0	0	0		0	0	0	0	0		1	3	1	1	1		3	1	3	3	3
Kath	0	0	0	0	3		0	3	0	3	3		1	1	3	3	3		2	3	2	1	1
Teddy	0	0	2	0	0		0	0	0	0	3		1	3	1	1	1		3	3	1	3	1
Clara	2	0	3	0	3		2	0	3	1	0		3	3	1	0	1		1	1	3	3	3
Ellain	0	0	0	3	0		0	3	0	3	0		1	3	3	1	2		3	1	3	3	3
Irene	3	0	1	0	3		3	3	0	0	3		3	3	3	3	3		3	3	3	3	3
Malcolm	0	0	3	0	0		0	0	3	0	0		1	1	3	1	1		1	3	3	3	3