Braid Variants and their Applications

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Contents

List of Figures

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Declaration

I declare that, to the best of my knowledge, the material contained herein is original, except where explicitly stated otherwise. This thesis has not been submitted for a degree at another university.

Summary

In Part I we develop the theory of arcbraids and arclinks, which are generalisations of the usual notions of braids and links; an alternative name for arclinks is irrational tangles. A cubical set without degeneracies is called a □-set. Just as braids induce rack automorphisms, arcbraids induce rack homomorphisms. We show that the formulae of the homomorphism induced by certain arcbraids is identical to that of the face maps of □-sets. Thus we can model the face maps of a □-set by arcbraids. However, there are many other arcbraids that do not model the usual face maps. We give a method for constructing new □-sets, with unusual face maps, from arcbraids. Using this method, we construct three □-sets. An alternating sum of the face maps of a □-set is the boundary operator of the chain complex associated to the classifying space of the □-set. So, in theory, new formulae for face maps could give rise to new homology theories. We show that quasi □-maps, a generalisation of □-maps, induce homeomorphisms of the corresponding classifying spaces. Furthermore, we show that we can form quasi □-maps between the three □-sets constructed. Unfortunately, this confounds the hope for new homology theories, but only in this case!

In Part II we define the Welded Jones polynomial, which is a nontrivial, welded isotopy invariant of welded links. In Chapter 5, signed Gauss codes are related to the fundamental rack; we give algorithms to compute the effects of operations such as reversing, mirroring, crossing changing and smoothing on these objects. We recall that a signed Gauss code corresponds to a virtual link. In Chapter 6 we show that permuting consecutive o's in the code is equivalent to the extra isotopy move required for welded links. This allows us to define the Welded Bracket polynomial, which is actually a quotient of the Bracket polynomial of virtual links, and the Welded Jones polynomial can be obtained from this. We give nontrivial examples of computations which distinguish welded links. A theorem of Jones for classical knots, which does not hold for virtual or welded knots, implies that the Welded Jones polynomial is trivial for classical knots. A slight modification leads to the Welded W-polynomial, which is a nontrivial, welded isotopy invariant of classical knots. We end on the entertaining note that whereas the Jones polynomial of the connected sum of classical knots is the product of the individual polynomials, for the Welded W-polynomial it is the sum of the individual polynomials.

Part I

Arcbraids and Boundary Operators

Chapter 1

Arcbraids and Arclinks

1.1 Introduction

Material covering the usual notions of braids and links can be found in any elementary textbook on knot theory; see [4], [6], [24], [32], [37] and [40], for example. We generalise these notions to arcbraids and arclinks, which are, as suggested by the names, unions of arcs and braids or arcs and links. Simply removing the arcs reduces this setting to the usual theory of braids and links. Note that the phrases arcbraids and arclinks may refer either to the equivalence class or to a specific representative. It will be clear which is meant from the context.

We will give an intuitive description here in terms of the smooth category. However, throughout the thesis, unless stated otherwise, we will work in the combinatorial category. In low dimensions these categories are equivalent. Think of a link as a disjoint union of circles properly embedded in a space X, where X is $S^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$ unless otherwise stated. An arclink is a disjoint union of disjoint circles and disjoint arcs properly embedded in X, where X is half space, $\mathbf{H} = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$, unless otherwise stated. Isotopy of links is the equivalence relation which encapsulates the idea of moving one link to another. We introduce this notion for arclinks, allowing points on ∂X, the boundary of X, to move freely around on ∂X, provided that we move the attached arc appropriately.

The main way to study links is via their diagrams, that is projections of them onto a plane.

Reidemeister's theorem gives precisely the combinatorial moves required on diagrams to encapsulate isotopy of the link. We give the analogous theorem for arclinks.

The set of braids with n strands forms a group, B_n, under concatenation, and so we can think of the set of all braids as a groupoid with objects $n \in \mathbb{N}$ and morphisms $n \to n$ given by the n-braids. We generalise this to give the arcbraid category, $\mathcal{A}$, which has extra morphisms joining different objects, corresponding to starting or terminating braid strands.

Markov's theorem for links says precisely when two braids close to give the same link. A strong version of this, using only one type of move, called an L-move, was given in [39]. This easily generalises to arclinks as well.

We generalise the ambient space of arclinks from $\mathbb{H}$ to a general manifold with boundary. Arclinks in $B^3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ are shown to be in one to one correspondence with arclinks in $\mathbb{H}$, and also with freely singular links with one singularity in S^3. Finally it is shown that tangles differ from arclinks precisely by boundary moves. Thus, an alternative name for arclinks is "irrational tangles".

1.2 Arclinks

A (polygonal) **arclink** is a disjoint union of a finite number of disjoint closed non self-intersecting polygonal lines in $int(\mathbb{H})$, the interior of $\mathbb{H}$, and a finite number of disjoint proper non self-intersecting polygonal lines in $\mathbb{H}$. Here **proper** means that the two boundary points of the lines are precisely those points which lie in $\partial\mathbb{H}$.

Let L be a polygonal arclink. We define **elementary moves** of L which are also referred to as **Δ-moves** throughout. Any elementary move is reversible, even if we do not explicitly state this in the following.

Let $[ABC]$ be a triangle in $int(\mathbb{H})$ which meets L in precisely the edge $[AB]$. Replacing $[AB]$ by $[AC] \cup [CB]$ is an **elementary interior move**. See Figure 1.1.

Now suppose that $[ABC]$ is a triangle in $\mathbb{H}$ which meets L in precisely the edge $[AB]$ and intersects $\partial\mathbb{H}$ in precisely the side $[BC]$. Replacing the edge $[AB]$ with the edge $[AC]$ is an **elementary boundary move**. See Figure 1.2.

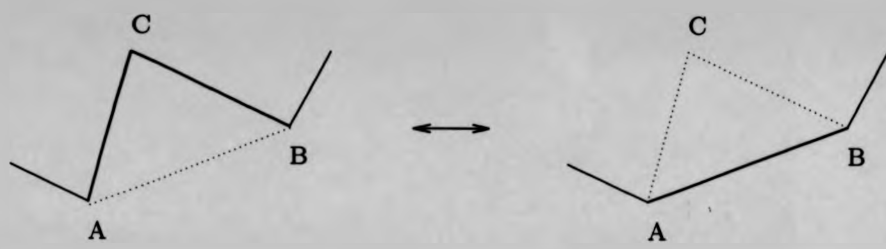

Figure 1.1: An elementary interior move

Figure 1.2: An elementary boundary move

Two polygonal arclinks L and L' are called **equivalent** or **isotopic** (in space) if there is a sequence of polygonal arclinks $L = L_0, L_1, \ldots, L_n = L'$ where adjacent members differ by an elementary move.

Arclinks may be studied via their diagrams. A **diagram** of an arclink, α, is a projection of α onto a plane such that the following conditions hold:

1. The plane of projection, P, intersects $\partial\mathbb{H}$ in a line, Q.
2. The projection onto P is parallel to $\partial\mathbb{H}$. So the image of $\partial\mathbb{H}$ under the projection is precisely Q.
3. The projections of strands of L are not tangent to each other.
4. No point is the projection of three or more points from different local strands of L.
5. No point is the projection of two or more points in $\partial\mathbb{H} \cap L$.

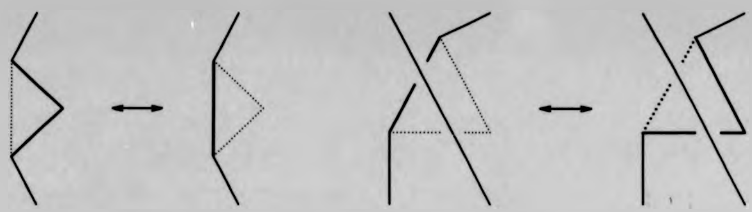

Figure 1.3: Planar isotopy moves

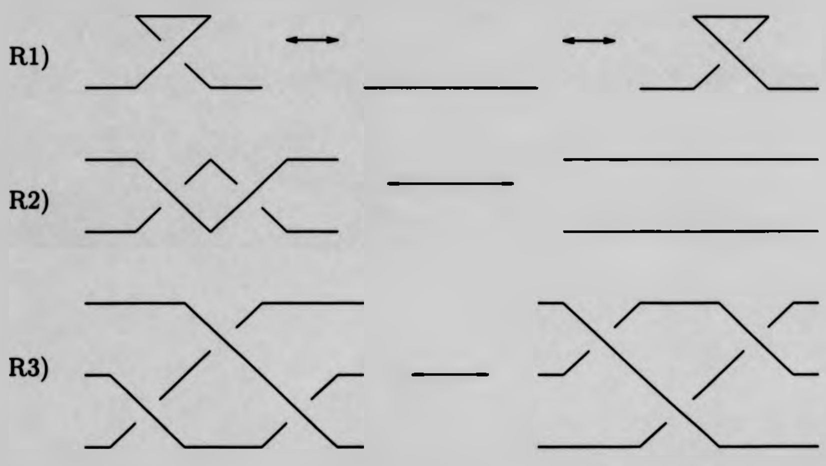

Figure 1.4: The Reidemeister moves; R2) gives rise to the relation $\sigma_i\overline{\sigma_i} = 1 = \overline{\sigma_i}\sigma_i$ and R3) gives the relation $\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}$ in $\mathcal{A}$.

1.2.1 A Reidemeister Theorem for Arclinks

We wish to describe a Reidemeister type theorem for arclinks. Recall the notion of **planar isotopy moves**; essentially, these are moves of the diagram which do not affect the crossing information in the diagram. Examples of these may be seen in Figures 1.3 and 1.2; note that subdividing points are allowed inside the projection of the triangle in Figure 1.3 and that points in $\partial\mathbb{H}$ can be involved, as in Figure 1.2. **Reidemeister's moves** for a link diagram are shown in Figure 1.4; these are now used for arclink diagrams. Figure 1.5 gives the geometric definition of a **P)-move**. The proof of the following theorem extends that in [36, pp 11-12] for links.

Figure 1.5: P)-moves give rise to the relation $s\,\sigma_1\,t = t\,s = s\,\overline{\sigma_1}\,t$ in $\mathcal{A}$.

Theorem 1.2.1 (Reidemeister's Theorem for Arclinks) *Two arclinks are isotopic if and only if the diagrams representing them may be obtained from each other by planar isotopy and a finite sequence of Reidemeister moves and P)-moves.*

Proof: The moves for planar isotopy, R)-moves and P)-moves can clearly be obtained from elementary moves, and so generate a space isotopy.

Conversely, assume that the polygonal arclinks are isotopic in space. It suffices to consider the case when two links are obtained from one another by one elementary move. The first type of elementary move takes $[AB] \mapsto [AC] \cup [CB]$, the second type takes $[AB] \mapsto [AC]$. Let L_0 denote the projection of the arclink L onto the plane containing the triangle $[ABC]$. A small isotopy ensures that the plane containing $[ABC]$ is not horizontal (so that its projection satisfies condition 1 in the definition of a diagram).

First of all, we may assume the edges of L_0 which meet any vertex of the triangle $[ABC]$ do not intersect the interior, $int[ABC]$. In such a case a R1)-move suffices to produce a new triangle $[A'BC]$ say, where A' is a point near A, with the above property. See Figure 1.6 for example.

The case when $[ABC]$ meets $\partial\mathbb{H}$ and L_0 meets all the defining vertices of $[ABC]$ can be excluded by simply choosing another defining point C' near C and using the new triangle $[ABC']$. See Figure 1.7.

We subdivide the triangle $[ABC]$ into smaller triangles, whose sides contain no vertices in $L_0 \cap int[ABC]$, of five types. The first three types involve triangles in $int(\mathbb{H})$, the fourth may involve any triangles in $\mathbb{H}$ (note that this includes the case of a triangle having precisely one point in $\partial\mathbb{H}$), and the last type involves triangles in $\mathbb{H}$ with one side, called the **boundary side**, in $\partial\mathbb{H}$. Figure 1.8 shows examples of the following five types:

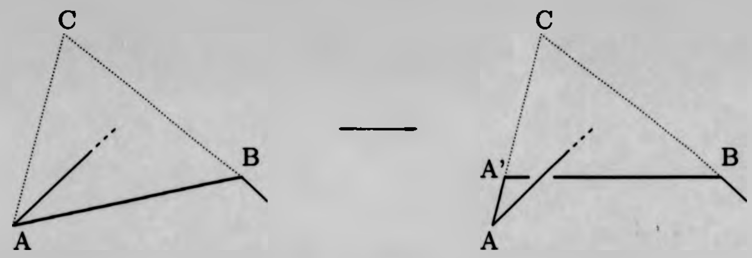

Figure 1.6: Eliminating an entering edge

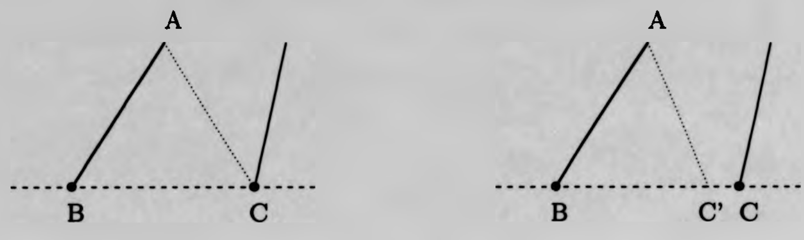

Figure 1.7: Eliminating an extra vertex

1. Triangles, Δ, whose intersection with L_0 is the neighbourhood of a crossing point with branches intersecting only two sides of Δ.

2. Triangles containing only one vertex of L_0 and parts of two edges issuing from it.

3. Triangles containing only one part of one edge and none of its vertices.

4. Triangles whose interiors have empty intersection with L_0.

5. Triangles whose interiors meet L_0 in only one part of one edge which has a vertex on the 'boundary side' of the triangle.

To construct such a triangulation of $L_0 \cap [ABC]$ we simply construct pairwise non-intersecting triangles of Type 1 for all crossing points, Type 2 for all vertices in $[ABC]$ coming from $int(\mathbb{H})$, and Type 5 for all vertices in $[ABC]$ coming from $\partial\mathbb{H}$. Then triangulate the rest into triangles of Types 3 and 4.

Now replace the original elementary move through $[ABC]$ by a sequence of moves through these smaller triangles starting from $[AB]$ and working up to either $[AC] \cup [CB]$ or just $[AC]$ depending on the original elementary move.

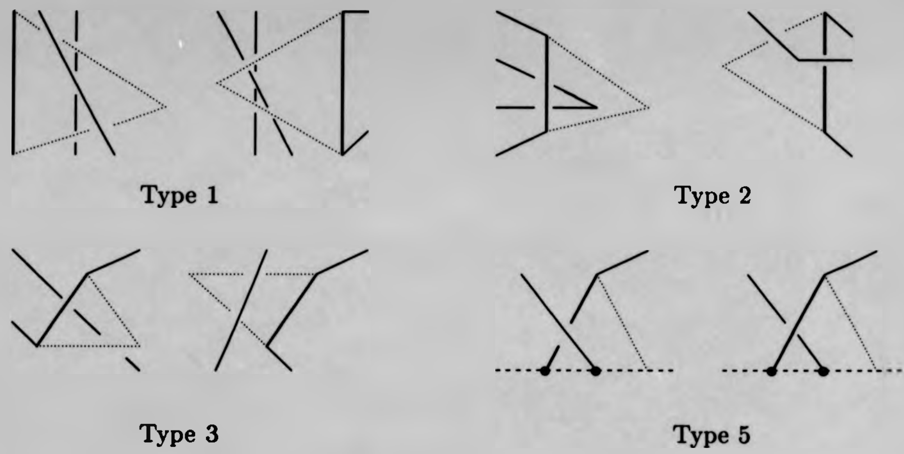

Figure 1.8: Examples of small triangles of types 1, 2, 3 and 5.

Now we have the following correspondences:

- Type 1 move $\longleftrightarrow$ R3)-move.
- Type 2 move $\longleftrightarrow$ R2)-move or planar isotopy.
- Type 3 move $\longleftrightarrow$ R2)-move or planar isotopy.
- Type 4 move $\longleftrightarrow$ planar isotopy.
- Type 5 move $\longleftrightarrow$ P)-move.

$\square$

1.3 Arcbraids

In order to enhance understanding of the following the reader is referred to Figure 1.10 which shows an example of an arcbraid diagram. Let the set $\{(x, y, 0) : x \in (0, 1)\} \subset \mathbb{H}$ be called the **floor**. Note that we may also refer to the projection of the floor in a diagram as the floor. The plane $\{x = t\}$ is called **level** t. A polygonal line is called **increasing** if it is monotonically increasing in the x-direction; that is, each level cuts an increasing line in

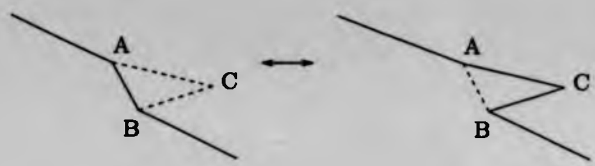

Figure 1.9: Not an elementary move for an arcbraid.

exactly one point. Let the **beginning points** be denoted by $A_i = (0, 0, i)$ and the **ending points** by $B_i = (1, 0, i)$ for $i \in \mathbb{N} - \{0\}$.

An **arcbraid**, α, is a set of pairwise non-intersecting increasing polygonal lines in $\mathbb{H}$ with the following properties:

1. The lines may start at either a beginning point or at a point in the floor.
2. The lines may end either at an ending point or at a point on the floor.
3. If $A_k \cap \alpha \neq \emptyset$ then $A_j \cap \alpha \neq \emptyset$ for all $j \leq k$.
   Similarly, if $B_k \cap \alpha \neq \emptyset$ then $B_j \cap \alpha \neq \emptyset$ for all $j \leq k$.

The **starting** and **terminating points** are points in the floor where the lines start or terminate; they may, collectively, be referred to as **pegs**. The beginning and ending points of α are the sets $\{A_i \cap \alpha\}$ and $\{B_i \cap \alpha\}$ respectively.

Two arcbraids, α and α', are said to be **isotopic**, written $\alpha \sim \alpha'$, if there exists a sequence of arcbraids $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \alpha'$ where adjacent members differ by an elementary move or its inverse (see Figures 1.1 and 1.2). Note that the requirement that each α_i is an arcbraid restricts the elementary moves to those which keep the strands increasing. Figure 1.9 shows an example of an elementary move which is not allowed here because it does not keep the strands increasing. Let $\alpha \equiv \alpha'$ mean the equivalence of arcbraids where the arcbraids α and α' differ by any sequence of elementary moves.

Lemma 1.3.1 *The two equivalence relations $\sim$ and $\equiv$ are equivalent.*

Proof: Suppose that $\alpha \equiv \alpha'$. Isotope all of the starting pegs of α into a neighbourhood of the plane $\{x = 0\}$, called P_0, and all of the terminating pegs of α into a neighbourhood of the plane $\{x = 1\}$, called P_1. Let this arcbraid be denoted by γ, and let $\eta(P_0 \cup P_1)$ denote a regular neighbourhood of $(P_0 \cup P_1)$ which is small enough so that $\eta(P_0 \cup P_1) \cap \gamma$ contains no crossings. Now $\gamma - \eta(P_0 \cup P_1) \cap \gamma = \beta$ is a braid on n strands, and $\eta(P_0 \cup P_1) \cap \gamma = \epsilon$ is the disjoint union of two 'ends' of the arcbraid. The same process is applied to α', giving rise to β' and ϵ'.

Now $\beta \equiv \beta'$ (as braids) up to a twisting of the ends of the braid, corresponding to twisting the starting or terminating pegs. The equivalence of the two equivalence relations for braids was established in [2]. Thus if we twist the start and end pegs of γ the appropriate number of times as done in the in the original equivalence and take δ to be the braid remaining on removing ζ, a neighbourhood of the planes which contains precisely the crossings coming from the twisting, then $\delta \sim \beta'$ (as braids). So

$$\begin{aligned} \alpha \sim \gamma &= \beta \cup \epsilon \\ &\sim \delta \cup \zeta \\ &\sim \beta' \cup \epsilon' \\ &= \gamma' = \alpha'. \end{aligned}$$

□

1.3.1 The Arcbraid Category

We now describe a natural categorical structure obtained from the set of equivalence classes of arcbraids. The objects in the category are simply the elements $n \in \mathbf{N}$ (including 0). An arcbraid with l beginning points and m ending points describes a morphism from the object l to the object m. Composition of such morphisms is inherited from the operation of concatenation and reduction; that is, given two arcbraids, α and β, where the number of end points of α is the same as the number of beginning points of β, we identify the B_i of α with the A_i of β and then shrink by half its horizontal height (i.e. its x-coordinate) in order to obtain the arcbraid $\alpha\beta$. The set of arcbraids under this operation gives rise to the **arcbraid category** $\mathcal{A}$. We will give a presentation for $\mathcal{A}$.

First note that in the equivalence class of any arcbraid there is a representative possessing a

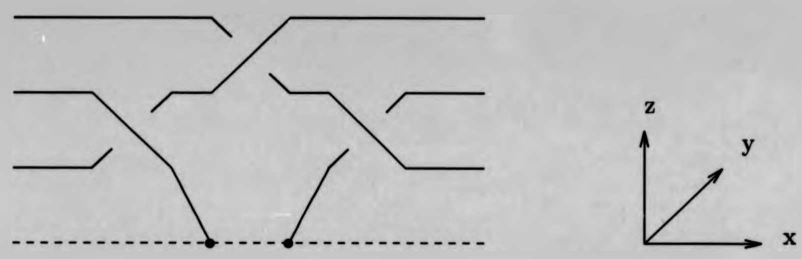

Figure 1.10: An example of the arcbraid $\sigma_1\, t\, \overline{\sigma_2}\, s\, \sigma_1$.

projection onto the (x, z)-plane with the following properties:

1. the projections of the strands are not tangent to each other.
2. No point of the (x, z)-plane is the projection of three or more points from different strands.
3. No point on the line $\{z = 0\}$ is the projection of two or more points.
4. All the crossings, starting points and terminating points are at different levels.

Figure 1.10 shows an example of such a representative. Any arcbraid may be represented as a product of the σ_i's, the $\overline{\sigma_i}$'s, the s's, and the t's, which are shown in Figures 1.11 and 1.12. Thus these are generating morphisms for the category $\mathcal{A}$.

Consider the starting object n in our category; that is, we suppose that we have an arcbraid, α, with n beginning points. The defining relations for the category may be obtained by considering the transformations of arcbraids for which the properties 1-4 above break down.

Property 1 breaks down for transformations of the type $\sigma_i\overline{\sigma_i} = 1 = \overline{\sigma_i}\sigma_i$. See Figure 1.4 (and its rotation by π). Figure 1.4 also depicts Property 2 breaking down for transformations of the type $\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}$. Transformations of the type $ts = s\sigma_1 t = s\overline{\sigma_1}t$ are yielded by the breakdown of Property 3. See Figure 1.5. This property also breaks down for the transformations $\sigma_1 t^2 = t^2$ and $s^2\sigma_1 = s^2$. See Figure 1.13. The **far commutativity** relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$ is obtained when Property 4 breaks down for pairs of crossing points; similarly we obtain the **far commutativity** relations $\sigma_i t = t\sigma_{i-1}$ for $i > 1$ and $s\sigma_i = \sigma_{i-1}s$ for $i > 1$ when Property 4 breaks down for a crossing point and a terminating

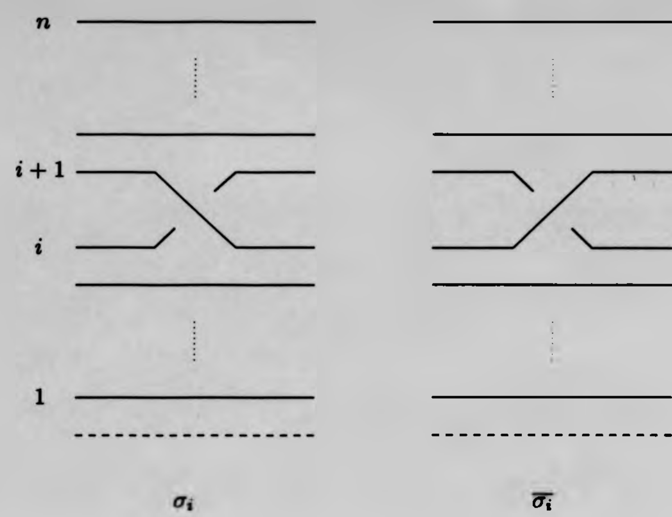

Figure 1.11: The generating morphisms σ_i and $\overline{\sigma_i}$ of the arcbraid category.

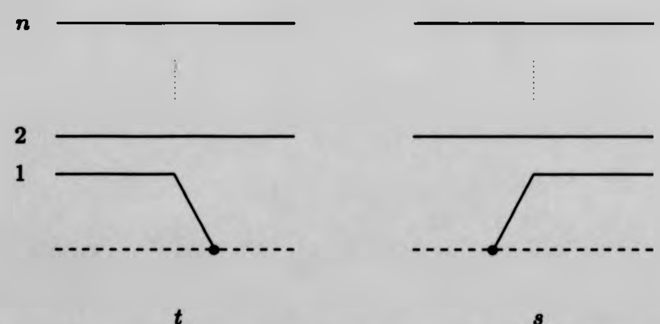

Figure 1.12: The generating morphisms t and s of the arcbraid category.

Figure 1.13: The relations $\sigma_1 t^2 = t^2$ and $s^2\sigma_1 = s^2$ in $\mathcal{A}$.

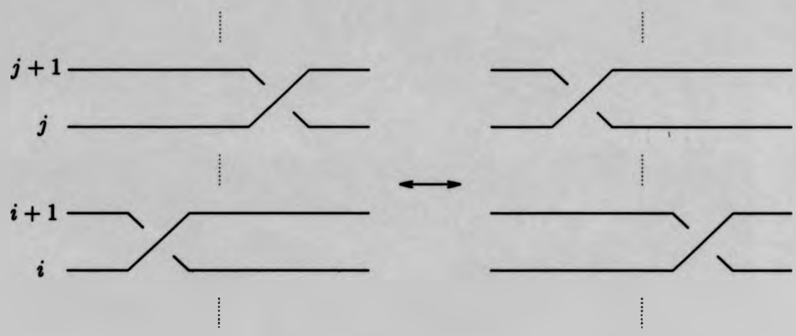

Figure 1.14: The far commutativity relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$.

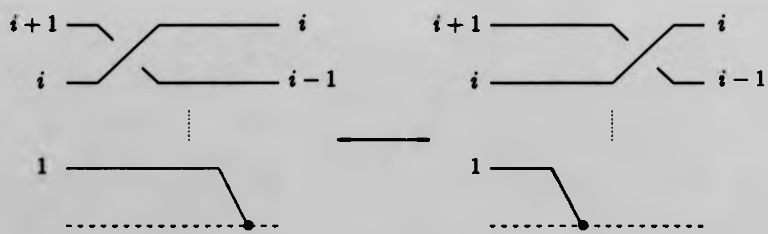

Figure 1.15: The far commutativity relation $\sigma_i t = t\sigma_{i-1}$ for $i > 1$.

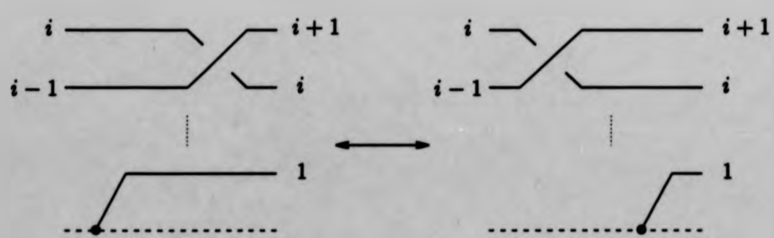

Figure 1.16: The far commutativity relation $s\sigma_i = \sigma_{i-1} s$ for $i > 1$.

or a starting point. See Figures 1.14, 1.15 and 1.16. Thus these relations are necessary. We show that they are also sufficient.

Theorem 1.3.2 *The arcbraid category, $\mathcal{A}$, is the category with objects $n \in \{0, 1, 2, \ldots\}$ and generating morphisms given by:*

$$\begin{array}{ll} \sigma_i, \overline{\sigma_i} : n \to n & \text{for } 1 \leq i \leq n-1 \text{ and } n \geq 2, \\ t : n \to n-1 & \text{for } n \geq 1 \text{ and} \\ s : n \to n+1 & \text{for } n \geq 0. \end{array}$$

The relations are given by:

$$\begin{array}{rcllr} \sigma_i\sigma_j\sigma_i & = & \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 & \text{(i)} \\ \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{for } |i-j| > 1 & \text{(ii)} \\ \sigma_i\overline{\sigma_i} & = & 1 \;=\; \overline{\sigma_i}\sigma_i & & \text{(iii)} \\ \sigma_1 t^2 & = & t^2 & & \text{(iv)} \\ \sigma_i t & = & t\sigma_{i-1} & \text{for } i > 1 & \text{(v)} \\ s^2\sigma_1 & = & s^2 & & \text{(vi)} \\ s\sigma_i & = & \sigma_{i-1}s & \text{for } i > 1. & \text{(vii)} \\ ts & = & s\sigma_1 t \;=\; s\overline{\sigma_1}t & & \text{(viii)} \end{array}$$

Proof: Consider the case of a morphism from the object $n \in \mathbb{N}$; this corresponds to an arcbraid with n beginning points. We show that any relation between arcbraids arises as a consequence of the given relations.

It suffices to show that any elementary arcbraid equivalence can be performed by using these relations. Thus we have to consider the cases where $[AB] \mapsto [AC] \cup [CB]$ and $[AB] \mapsto [AC]$. Subdivide the triangle $[ABC]$ into triangles of Types 1-5 as in Theorem 1.2.1 and replace the elementary move by a sequence of elementary moves on small triangles starting with the side $[AB]$ and moving either to $[AC] \cup [CB]$ or to the side $[AC]$ depending on the type of the original elementary move.

This is done as in Theorem 1.2.1, with the following notes. Entering edges (Figure 1.6) cannot occur in a braid since the strands are increasing. Projections of the forbidden types that

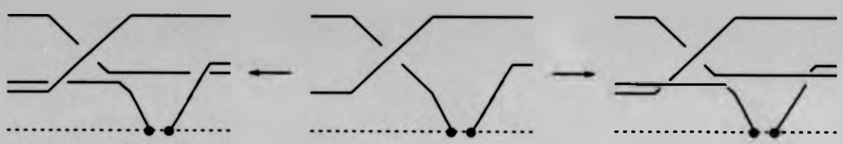

Figure 1.17: L-moves.

occur obviously correspond to the given relations. The relations $\sigma_1 t^2 = t^2$ and $s^2 \sigma_1 = s^2$ are necessary here, but were not needed in 1.2.1, since the strands must remain increasing for arcbraids, but not for arclinks.

□

1.3.2 Alexander and Markov Theorems for Arclinks

An arcbraid, α, with n beginning points and n ending points is called an **n-arcbraid**. The **closure** of α, denoted by $\overline{\alpha}$, is given by joining the ending points, B_i, to the beginning points, A_i, by n polygonal lines which are unlinked with respect to each other and α. Thus a diagram of α gives rise to an arclink diagram of $\overline{\alpha}$ by joining its beginning and ending points with arcs that introduce no new crossings.

A **under (over) L-move** on an arcbraid, shown in Figure 1.17, consists of cutting one arc of the arcbraid open and splicing two new strands into the broken strand, both under (over) the rest of the arcbraid. An equivalence relation on arcbraids, called **L-equivalence**, is generated by L-moves and isotopy.

Note that replacing arcbraid by braid in the definitions above gives rise to the setting of [39] where it was shown that "L equivalence classes of braids are in bijective correspondence with isotopy classes of oriented links in S^3, where the bijection is induced by closing the braid to form a link".

We extend Alexander and Markov theorems to arcbraids and arclinks (c.f. Alexander/Markov theorems for generalised braids in [12]). This is achieved by reducing Theorem 1.3.3 to the situation for links as in [39].

Theorem 1.3.3 *Closure, C, induces a bijection between the set of L-equivalence classes of k-arcbraids (where k may vary) and the set of isotopy types of (oriented) arclink diagrams.*

Proof: We define a braiding process, B, which is the inverse to the closure operation, C. Let us assume that we are in the same (combinatorial) setting as in [39], except for switching the horizontal and vertical directions. Any point in the floor is considered to not be a subdividing point. Note that, in this setting, we also consider **subdivision** of diagrams, that is, introduction or deletion of a vertex in a diagram.

We first give a brief outline of the proof for braids and links given in [39]. Assume all arcs are oriented throughout. An arc in a diagram which slopes "leftwards" with respect to its orientation is called an **opposite arc**. In order to obtain a braid from a link diagram we:

- Keep the arcs that go rightwards.
- Eliminate the opposite arcs and replace with braid strands.

We subdivide every opposite arc into smaller arcs, called **left arcs**, each containing crossings of only one type. These smaller arcs may cross over, under or not cross other strands; call them **over, under** or **free** left arcs correspondingly, and label them by "o" for over, "u" for under and either "o" or "u" for free arcs. Eliminate the opposite arcs by eliminating their left arcs one by one, creating braid strands in the process.

Associated to each left arc is a **sliding triangle**, which is a right angled triangle whose hypotenuse is the left arc and whose right angle lies to the right of the left arc. The sliding triangle is of type **over** or **under** depending on the label associated to the left arc.

The braiding process is given by performing an over L-move (under L-move) at the head of a left arc labelled "o" ("u") followed by a Δ-move across the associated sliding triangle. See Figure 1.18 where a smooth arc has been used purely for aesthetic value. This replaces the left arc by arcs, none of which are left arcs. Repeating the process for each left arc in turn gives a braid.

Sliding triangles are said to be **adjacent** if the corresponding left arcs have a common vertex (and so the sliding triangles have a common corner). The **triangle condition** says that:

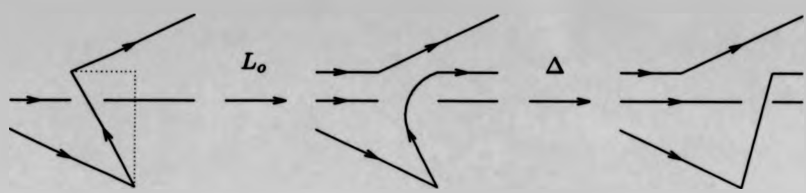

Figure 1.18: The braiding process at an overarc.

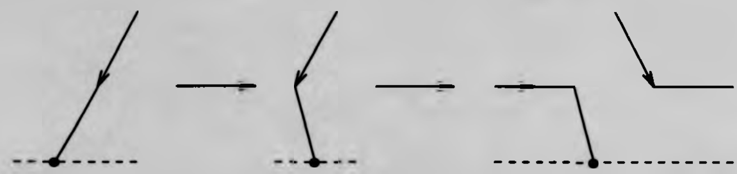

Figure 1.19: An example of the braiding process applied to an opposite pegged arc.

Nonadjacent sliding triangles are allowed to meet if they are of opposite types (i.e. one over and one under).

A **generic arclink diagram** is an arclink diagram put in general position with respect to the horizontal height function (i.e. the level) such that the following conditions hold:

1. there are no vertical arcs,
2. no two disjoint subdividing points are in horizontal alignment, where disjoint means that the subdividing points do not share a common edge,
3. any two nonadjacent sliding triangles satisfy the triangle condition and if they intersect then they do so along a common interior (and not in a single point).

Recall that a point in the floor of a diagram is called a **peg**. Let a **pegged line** be a line joining a peg to a subdividing point, and let a **pegged arc** be any arc with a peg as one endpoint.

Given a generic arclink diagram, D, we proceed as follows:

Choose a regular neighbourhood, N, of the floor which contains no crossing or subdividing points. We may obtain $N \cap D$ by simply taking one pegged arc which contains no crossing points and no subdividing points for each peg, and then taking their union.

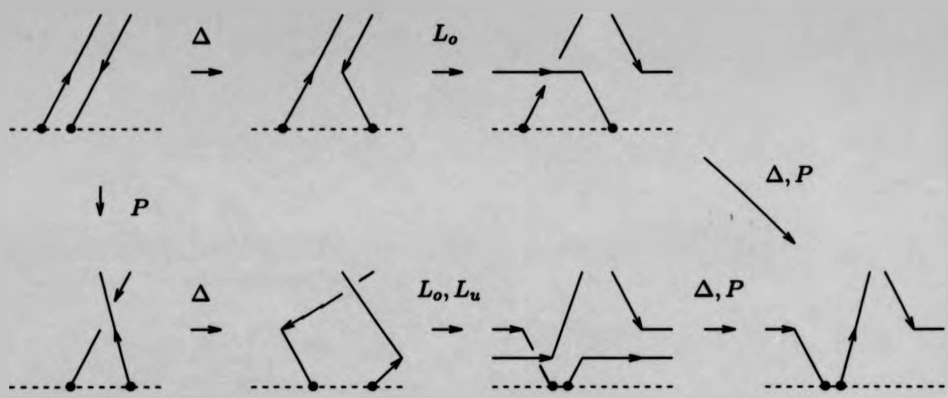

Figure 1.20: An example of the independence of P)-moves with respect to our braiding process; the top right and the bottom right diagrams are the results of braiding before and after a P)-move, respectively.

Subdivide any left-arc in N, making sure that no two subdividing points are in horizontal alignment. Change all pegged left arcs to pegged right arcs by Δ-moves (actually, just planar isotopy), as shown in the first part of Figure 1.19. Now choose a regular neighbourhood $N' \subset N$ which does not contain any subdividing points.

Locally, $D - \{D \cap N'\}$ is a link diagram. Also, all arcs entering or leaving N' are right arcs by construction, so they are unaffectd by the braiding process given above. We obtain an arcbraid diagram by applying this process to $D - \{D \cap N'\}$.

In order to show that B is well defined it suffices to check independence of P)-moves. This is because independence of all other choices (such as subdivision and labelling of free left arcs) and Δ-moves before braiding is identical to that shown in [39].

Independence of P)-moves is easy to see; the obvious isotopy takes the strings involved in any extra L-move that is introduced to a point where undoing the L-move yields the required result. See Figure 1.20 for an example of such a process. The other cases are similar. □

1.4 Generalising the Ambient Space of Arclinks

Let M be a combinatorial manifold with boundary ∂M. The **interior** of M, denoted $int(M)$, is $M - \partial M$. An **arclink** in M is a disjoint union of a finite number of disjoint closed non self-intersecting polygonal lines in $int(M)$ and a finite number of disjoint proper non self-intersecting polygonal lines in M.

Remark 1.4.1 *Arcs may join different boundary components. A restricted class of arclinks could be obtained by requiring that each arc has its endpoints in a single boundary component.*

Isotopy of arclinks in M is via a sequence of elementary moves, as usual. See Section 1.2 for the definition, noting that we replace $\mathbb{H}$ by M, $\partial\mathbb{H}$ by ∂M, and $int(\mathbb{H})$ by $int(M)$.

Proposition 1.4.2 *Isotopy classes of arclinks in $\mathbb{H}$ are in one to one correspondence with isotopy classes of arclinks in B^3.*

Proof: We show that stereographic projection, s, from any non-pegged point in ∂B^3 is a well defined map from the set of isotopy classes of arclinks in B^3 to the set of isotopy classes of arclinks in $\mathbb{H}$. It is clear that s is an inverse to the one point compactification of $\mathbb{H}$.

Let s_p denote stereographic projection from a non-pegged point p in ∂B^3. Suppose that β and β' are isotopic arclinks in B^3 and that p and q are non-pegged points in ∂B^3. We show independence of choice of representative of the isotopy class of arclink in B^3 and independence of choice of point of projection, that is,

1. $s_p(\beta) = s_p(\beta')$ and
2. $s_p(\beta) = s_q(\beta)$.

1. The only difficulty occurs when a peg passes through p, via a Δ move, prior to projection. We simply replace this Δ-move by an arbitrarily close sequence of Δ-moves which do not pass through p. In Figure 1.21, $[CA] \to [CB]$ is replaced by $[CA] \to [CD] \to [CB]$, for example.

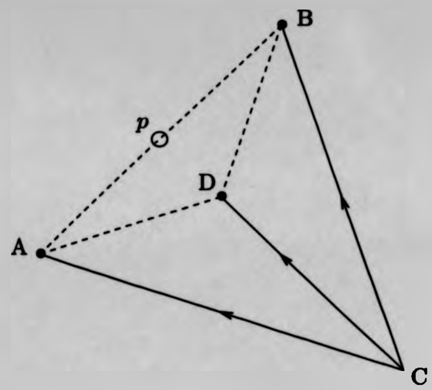

Figure 1.21: A path through $p \in \partial B^3$ and an alternative isotopy path.

2. There are only a finite number of pegs so there is a polygonal line in ∂B^3 joining p and q, which does not meet any other peg. It suffices to consider the case when p and q are joined by a single straight line, since we may apply the following to each straight line in turn. Note that if we were in a setting with ambient isotopy then a rotation of B^3 taking p to q would do, but we have a fixed ambient space and so we just move the arclink in B^3 by a sequence of Δ-moves to its position after such a rotation. Clearly we can do this without the need for any other peg to pass through the line pq . So if this new arclink is called β' then we have that $s_q(\beta') = s_p(\beta)$. The above case gives us that $s_q(\beta') = s_q(\beta)$. □

1.5 Related Geometric Objects

1.5.1 Freely Singular Links

A **freely singular link** in S^3 is the union of a finite number of closed non self-intersecting polygonal lines in S^3 which are disjoint except for at a finite number of **multiple (singular) points**. Let γ be a freely singular link in S^3. A **non-singular arc** is an arc in γ which contains no multiple points.

Let $[ABC]$ be a triangle in S^3 which intersects γ in precisely the non-singular arc $[AB]$. A Δ-move involves replacing the arc $[AB]$ by $[AC] \cup [CB]$, and is called an elementary interior move, as usual.

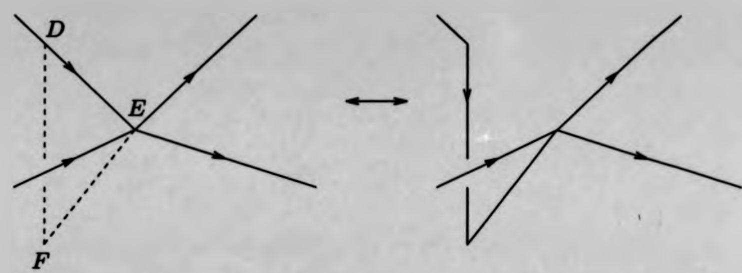

Figure 1.22: An elementary singular move of type 1

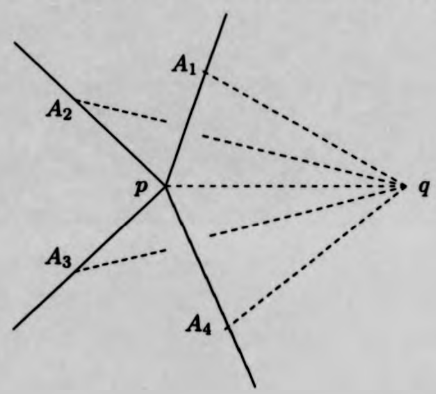

Figure 1.23: An elementary singular move of type 2

We extend the notion of Δ-moves to those involving multiple points. Let $[DEF]$ be a triangle in S^3 which intersects γ in precisely the arc $[DE]$, which is non-singular except for the multiple point E. Replacing the arc $[DE]$ by $[DF] \cup [FE]$ is a Δ-move, called an **elementary singular move of type 1**. See Figure 1.22.

Notice that this fixes the position of the multiple point in S^3. We wish to allow movement of the multiple points as well. So we let p be a singular point of multiplicity n. Suppose that $A_1, \ldots, A_{2n}$ are non-singular points of α, one on each arc meeting p. Further suppose that there exists a q in $S^3 - \alpha$ and $2n$ triangles $[A_i\, p\, q] = [A_i\, p] \cup [p\, q] \cup [q\, A_i]$ which intersect α precisely in $A_i\, p$. Then an **elementary singular move of type 2** is the $2n$ simultaneous Δ-moves involving the triangles $[A_i\, p]$. See Figure 1.23.

Two freely singular links are **isotopic** if they differ by a sequence of elementary moves. Let us elaborate on isotopy of freely singular links. Consider a small ball, B_p, around p, which intersects γ in $2n$ straight lines, each with one endpoint p and the other on ∂B_p, which are non-singular except at p. These lines are the tangential directions at p. Isotopy allows us to move the points on ∂B_p freely (provided, of course, that we move the attached arcs and these remain non-singular).

Proposition 1.5.1 *Arclinks in B^3 are in $1 \leftrightarrow 1$ correspondence with freely singular links with one singularity in S^3.*

Proof: It suffices to note that $S^3 = B_p \cup_\partial B^3$. □

1.5.2 Tangles

Let M be a combinatorial manifold with boundary ∂M. A **tangle** in M is a disjoint union of a finite number of disjoint closed non self-intersecting polygonal lines in $int(M)$ and a finite number of disjoint proper non self-intersecting polygonal lines in M.

Note that this is the same definition as for arclinks in M. However, two tangles are **isotopic** if they differ by an elementary interior move. So ∂M is fixed for tangles, unlike for arclinks.

Lemma 1.5.2 *Isotopy classes of arclinks in M are in one to one correspondence with isotopy classes of tangles in M modulo the equivalence relation generated by boundary elementary moves.* □

Remark 1.5.3 *If one thinks of "Rational Tangles" as tangles that differ from the identity tangle by rational twists, i.e. applying elementary boundary moves to the tangle, then this suggests an alternative name for arclinks is* ***irrational tangles****.*

Remark 1.5.4 *In the smooth category, framed tangles in B^3 modulo diffeomorphisms of S^2 were used in [3] as a step towards a model of Quantum Gravity.*

Chapter 2

Racks

2.1 Introduction

A large part of the following is a review of material that may be found in more detail in the literature. The notion of a rack [17] has been widely studied, under a variety of names, such as Right Automorphic Set, Crystal, or Distributative Groupoid. Given a link, L, [17] shows how to obtain its fundamental rack, $\Gamma(L)$, which is a complete invariant of framed links in S^3, up to the reverse mirror operation.

A $\square$-set [19] is a cubical set without degeneracies. Given a rack X we review how to obtain its trunk, $\mathcal{T}(X)$, which is analagous to a category, but is based on squares rather than triangles. The nerve of this trunk, $\mathcal{N}\mathcal{T}(X)$, is a $\square$-set and its geometric realisation is called the rack space, BX.

Given a rack X, an n-braid induces an automorphism of X^n, which is equivalent to a map of the n-cubes in BX; this was shown in [17], but the automorphism was read in the opposite direction there. This is extended to show when an arcbraid with n beginning points and r ending points induces a homomorphism from X^n to X^r. In the special case when we have a terminating braid, that is, an arcbraid with no starting arcs, this homomorphism is always defined. The terminations correspond to face maps in the $\square$-set, that is, projections onto the faces in the classifying space. Any such face map can be represented by a terminating braid, with projection onto the i-th front or back face being represented by the terminating braid whose i-th strand passes, respectively, behind or in front of the strands $i-1, i-2, \ldots, 1$ before terminating. However, terminating braids can induce homomorphisms which are not achievable as products of face maps, for instance, take the terminating braid which terminates the i-th strand after first passing alternately over and under the strands $i-1, i-2, \ldots, 1$.

We introduce notions of Δ-sets, Δ-spaces, and their geometric realisations, which are called their classifying spaces; see [33] for instance. Sometimes Δ-sets are referred to as pre-simplicial sets; these are simplicial sets without degeneracies. We give some notation for the complex of a Δ-set which gives the homology groups of its classifying space. The same is done for a $\square$-set; in fact, we show that any linear combination of the front and back face maps, with coefficients in some commutative ring with a 1, defines a differential.

Finally, as an interesting diversion, we define a bicomplex, as in [33], and we show that a $\square$-module defines a bicomplex. This can artificially be made into a first quadrant bicomplex from which we can obtain its total homology. We show that restricting to the diagonal of the total complex is equivalent to taking the usual homology of the classifying space of the original $\square$-set.

2.2 Rack Homomorphisms

2.2.1 Racks

A **rack** is a non-empty set X with a binary operation (written exponentially) satisfying the following axioms:

1. Given $a, b \in X$ there is a unique $c \in X$ such that $a = c^b$.
2. Given $a, b, c \in X$ the formula $a^{bc} = a^{cb^c}$ holds.

This second axiom is called the **Rack Identity**.

The two axioms can easily be seen to be the algebraic distillation of two of the Reidemeister moves R2 and R3; that is, these moves, shown in Figure 1.4, induce the same automorphism of any rack, as explained in Section 2.7.

Let $a, b \in X$. Write $a^{\bar{b}}$ for the unique element c appearing in axiom 1 above. Define an equivalence relation, $\equiv$, on $F(X)$, the free group with generating set X, called **operator equivalence** via:

$$w \equiv z \Leftrightarrow a^w = a^z \text{ for all } a \in X \text{ and } w, z \in F(X).$$

The following, equivalent forms of the Rack Identity can be found in [17, pp 348-349]:

- Given $a, b, c \in X$ the formula $a^{b^c} = a^{\bar{c}bc}$ holds.
- Given $a, b \in X$ we have $a^b \equiv \bar{b}ab$.

A **quandle** is a rack, X, such that $a^a = a$, for all $a \in X$.

2.2.2 The Fundamental Rack of a Link

This subsection is taken almost exactly from [17, pp 358-359] and is set in the smooth category. A **link** is defined to be a codimension two embedding $L : M \hookrightarrow Q$ of one manifold into another. We assume that each normal disc to M in Q has an orientation which is locally and globally coherent. A **framing** is just a choice of trivialisation of the normal disc bundle; that is, a given cross section $\lambda : M \to \partial N(M)$. Call $M^+ := \lambda(M)$ the **parallel manifold** to M. For a link, L, in S^3 this is just a link parallel to L, on the boundary of a union of regular (solid torus) neighbourhoods each of whose core is a component of the link L. The linking number of a component of this parallel manifold with its corresponding core is equivalent to the framing of the component.

Consider homotopy classes Γ of paths in $Q_0 = \overline{Q - N(M)}$ from any point in M^+ to a base point, $*$, relative to $*$ (noting that the endpoint in M^+ is not fixed). The fundamental group of Q_0 acts on Γ by post-composition; that is, acting $\gamma \in \pi_1(Q_0)$ on $\alpha \in \Gamma$ gives $\alpha \circ \gamma \in \Gamma$ (go along the path α and then around the loop γ). We use this action to define a rack structure on Γ. There is a unique meridian circle of the normal circle bundle containing $p \in M^+$. Let m_p be the loop based at p which follows the meridian in the positive direction (i.e. the oriented meridian which has linking number 1 with L in a small neighbourhood of p). Let $a, b \in \Gamma$ be represented by the paths α, β respectively. Let $\partial(b)$ be the element

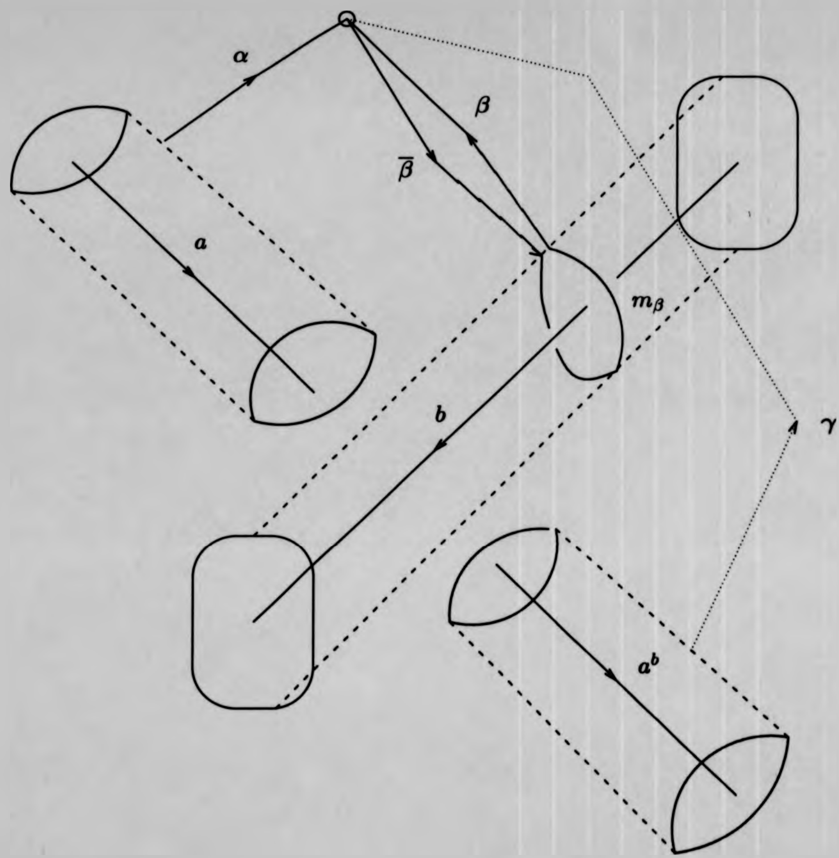

Figure 2.1: The fundamental rack operation

of the fundamental group determined by the loop $\overline{\beta} \circ m_\beta \circ \beta$ (meaning follow β backwards, then go around the meridian at the initial point of the path β and finally follow β). The **fundamental rack** of the framed link L is defined to be the set $\Gamma = \Gamma(L)$ of homotopy classes of paths together with the operation $a^b := a.\partial(b) = [\alpha \circ \overline{\beta} \circ m_\beta \circ \beta]$.

Figure 2.1 is the visual aid; here γ is the path representing a^b, which can be seen to be homotopic to the path $\alpha \circ \overline{\beta} \circ m_\beta \circ \beta$ shown. Note that this figure also gives the diagram labelling.

2.3 An Introduction to □-sets.

As in [19, page 334] we define a **□-set**, $X = \{X_n\}$, to be a collection of sets X_n, one for each natural number $n \geq 0$, and functions $d_i^\epsilon : X_n \to X_{n-1}$, with $1 \leq i < j \leq n$ and $\epsilon \in \{f, b\}$, satisfying the following **face relations**:

$$d_i^\epsilon d_j^\eta = d_{j-1}^\eta d_i^\epsilon \qquad \text{for } 1 \leq i < j \leq n \text{ and } \epsilon, \eta \in \{f, b\}.$$

The functions d_i^f and d_i^b are called the **front** and **back face maps** respectively. Note that in [19] the numbers 0 and 1 are used instead of the letters f and b respectively. We will use the notation (X, d^f, d^b) for the □-set.

The geometric realisation of the □-set (X, d^f, d^b) is given by:

$$|(X, d^f, d^b)| = \bigcup_{n \geq 0} (X_n \times I^n) / \approx$$

where the equivalence $\approx$ is generated by the face maps.

2.4 Trunks

A **trunk** consists of a directed graph, Γ, together with a collection of oriented squares in Γ, called **preferred squares**. A **corner trunk** is a trunk which also satisfies the following axioms:

<u>Axiom 1</u>: Given edges $a : A \to B$ and $b : A \to C$ there are unique edges $c : C \to D$ and $d : B \to D$ such that the following square with base a is preferred. It is given an anticlockwise orientation.

$$\begin{array}{ccc} C & \xrightarrow{c} & D \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle d} \\ A & \xrightarrow{a} & B \end{array}$$

We write $c = a^b$ and $d = b_a$ for the two partially defined binary operations which determine the other edges.

<u>Axiom 2</u>: Three co-original edges determine the entire cubical diagram of preferred squares. See Figure 2.2; the outside square has the anticlockwise orientation.

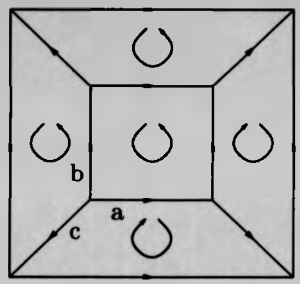

Figure 2.2: Axiom 2 for a corner trunk

Notice that in a corner trunk the corner of a cube has a unique completion.

A **Trunk map** between trunks is a map which takes vertices to vertices, directed edges to directed edges, and preferred squares to preferred squares. If S and S' are trunks then the set of trunk maps from S to S' is written $\mathrm{Hom}(S, S')$.

2.5 The Nerve of a Trunk

Recall [19, pp 334-335] that the n-cube, $I^n \subset \mathbb{R}^n$, can be regarded as a trunk by taking its vertices and edges as the vertices and edges of the trunk and 2-faces as preferred squares. Let S be a trunk. The **nerve** of S, denoted NS, is the following $\Box$-set. The set, NS_n, of n-cubes of NS is defined to be $\mathrm{Hom}(I^n, S)$, that is, the set of trunk maps from the cube trunk to S. The face maps are given by $\lambda^*(f) = f \circ \lambda$ where $\lambda : I^p \to I^n$ is a face map and $f : I^n \to S$ is an n-cube.

Theorem 2.5.1 ([19], page 340) *Let S be a corner trunk. Then the nerve NS has the following description: There is one n-cube for each co-original n-tuple $(x_1, \dots, x_n)$ of edges in S. The face maps are given by:*

$$\begin{aligned} d_i^0(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) & \text{(i)} \\ d_i^1(x_1, \dots, x_n) &= ((x_1)^{x_i}, \dots, (x_{i-1})^{x_i}, (x_{i+1})_{x_i}, \dots, (x_n)_{x_i}) & \text{(ii)} \end{aligned}$$

for $1 \leq i \leq n$.

2.6 The Rack Space

Let X be a rack. Then the trunk $\mathcal{T}(X)$ has one vertex, and an n-tuple of co-original edges is prescisely an n-tuple of elements of X. So we have the following description:

$$N\mathcal{T}(X) = X^n.$$

Face maps, for $1 \leq i \leq n$, are given by:

$$d_i^0(x_1,\ldots,x_n) = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \tag{iii}$$

$$d_i^1(x_1,\ldots,x_n) = (x_1^{x_i},\ldots,x_{i-1}^{x_i},x_{i+1},\ldots,x_n). \tag{iv}$$

The geometric realisation of this $\square$-set is called the **Rack Space**, and is denoted by BX.

2.6.1 An Informal Description

Informally, BX is the topological space obtained by taking the disjoint union of all possible n-cubes, whose edges are labelled by elements of X, for all $n \in \mathbb{N}$, and glueing them together in the obvious manner. Note that the face maps d_i^0 and d_i^1 are projections onto the front and back i-th faces respectively.

2.7 Automorphisms Induced by Braids

Let X be a rack, and let β be an n-braid diagram. We obtain an automorphism of X^n as follows. Label the n points of β in level 0 by elements $x_1, x_2, \ldots, x_n \in X$, in order. This labels the leftmost arcs of the braid diagram by elements of the rack. The labelling on the rest of the arcs is determined by Figure 2.4.

By the definition of a rack, this labelling respects the relations in the braid group.

Recall that a **critical level** of a diagram is a level which contains a crossing point. Any level which is not critical is called **regular**. Now every regular level of the diagram is labelled by n elements of X. In particular, level 1 is labelled by the elements $x_{\pi(1)}^{w_1}, \ldots, x_{\pi(n)}^{w_n}$, where π is the permutation induced by the braid and w_i is a word in the free group with generating set

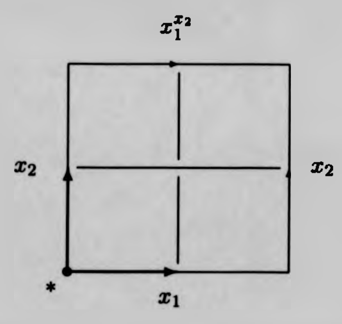

Figure 2.3: The preferred square (x_1, x_2) in the trunk $\mathcal{T}(X)$. The faces of the square are given by:

$$\begin{aligned} d_1^0(x_1,x_2) &= x_2 \\ d_1^1(x_1,x_2) &= x_2 \\ d_2^0(x_1,x_2) &= x_1 \\ d_2^1(x_1,x_2) &= x_1^{x_2}. \end{aligned}$$

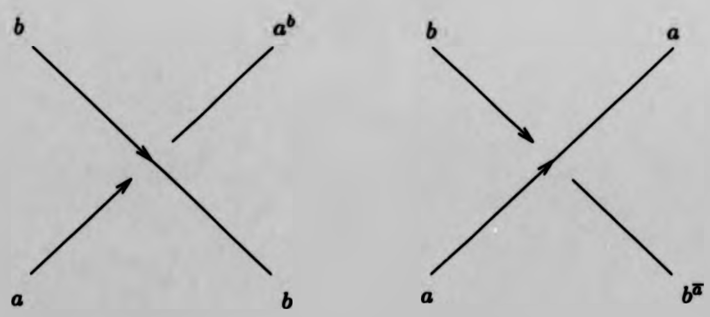

Figure 2.4: Labelling the braid strings

X. Thus the automorphism $X^n \to X^n$ given by $(x_1, \ldots, x_n) \to (x_{\pi(1)}^{w_1}, \ldots, x_{\pi(n)}^{w_n})$ is induced by the braid.

Recall that σ_i and $\overline{\sigma_i}$, for $1 \leq i \leq n$, denote the generators of the braid group B_n. The action of B_n on X^n is given by:

$$(x_1, \ldots, x_n)^{\overline{\sigma_i}} = (x_1, \ldots, x_{i-1}, x_{i+1}^{\overline{x_i}}, x_i, \ldots, x_n) \quad \text{(v)}$$

$$(x_1, \ldots, x_n)^{\sigma_i} = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i^{x_{i+1}}, x_{i+2}, \ldots, x_n). \quad \text{(vi)}$$

Note that the action of any element $a \in X$ on X^n is given by:

$$(x_1, \ldots, x_n)^a = (x_1^a, \ldots, x_n^a).$$

This commutes with d_i^ε, for $\varepsilon \in \{0, 1\}$.

Let an n-**cube** of BX be an n-cube, I^n, together with a labelling of the n co-original edges of I^n by elements of X. This determines the labelling on the rest of the edges of the n-cube, as in Figure 2.3. Thus every regular level in β corresponds to an n-cube of BX, and the automorphism defined gives rise to a map of n-cubes of BX.

2.8 Homomorphisms Induced by Arcbraids

Let $\mathcal{A}_n$ denote the set of arcbraids with n beginning and n ending points. We wish to extend the action of B_n on the n-cubes of BX, to an action of $\mathcal{A}_n$ on the n-cubes of BX. In order to achieve this we need to define the action of t and s on an n-cube of BX and show that the relations of $\mathcal{A}_n$ are respected.

An (n,r)-**arcbraid** is an arcbraid with n beginning points and r ending points. Let $\mathcal{A}_n^r$ denote the set of such arcbraids (under the equivalence relation defined earlier). An arcbraid with no starting arcs, that is, a sequence of σ_i's, $\overline{\sigma_i}$'s and t's, is called a **terminating braid**. Let $\mathcal{T}_n^r$ denote the set of all (n,r)-terminating braids (also under the equivalence relation defined earlier).

$$\text{Let } t \text{ act as :} \qquad (x_1, \ldots, x_n)^t = (x_2, \ldots, x_n).$$

This is termination of the first string, which corresponds to the projection which removes the first coordinate. The other face maps can be realised as the appropriate sequence of crossings followed by termination of the first string; the sequence of crossings comprises of all over or all under crossings. Explicitly, the following relations hold:

$$d_i^0(x_1,\dots,x_n) = (x_1,\dots,x_n)^{\overline{\sigma_{i-1}\dots\sigma_1 t}} \tag{vii}$$

$$d_i^1(x_1,\dots,x_n) = (x_1,\dots,x_n)^{\sigma_{i-1}\dots\sigma_1 t} \tag{viii}$$

Now $\mathcal{T}_n^r$ operates on the n-cubes of BX since the relations in $\mathcal{T}_n^r$ involving t are respected:

$$\begin{aligned}
(x_1,\dots,x_n)^{\sigma_1 t^2} &= (x_2, x_1^{x_2}, x_3,\dots,x_n)^{t^2} \\
&= (x_3,\dots,x_n) \\
&= (x_1,\dots,x_n)^{t^2}, \qquad \text{and} \\
(x_1,\dots,x_n)^{\sigma_i t} &= (x_2,\dots,x_{i+1},x_i^{x_{i+1}},\dots,x_n) \\
&= (x_1,\dots,x_n)^{t\sigma_{i-1}} \qquad \text{for } i > 1.
\end{aligned}$$

We wish to extend this to an operation of $\mathcal{A}_n^r$ on the n-cubes of BX. Let $x \in X$ be a fixed element satisfying the identity $x^a = x$ for all $a \in X$. Then we may define

$$(x_2,\dots,x_n)^s = (x, x_2,\dots,x_n).$$

As seen below, the required property of x is necessary if we wish the relations of $\mathcal{A}_n^r$ to hold. Let $x, y \in X$ be labels of two adjacent starting arcs. Then the relations enforce the conditions:

$$\begin{aligned}
(x_3,\dots,x_n)^{s^2} &= (x, y, x_3,\dots,x_n) \\
= (x_3,\dots,x_n)^{s^2\sigma_1} &= (y, x^y, x_3,\dots,x_n), \\
(x_2,\dots,x_n)^{s\sigma_i} &= (x, x_2,\dots,x_{i-1},x_{i+1},x_i^{x_{i+1}},x_{i+2},\dots,x_n) \\
&= (x_2,\dots,x_n)^{\sigma_{i-1}s} \qquad \text{for } i > 1, \\
\text{and } (x_1,\dots,x_n)^{ts} &= (x_2,\dots,x_n)^s \\
&= (x, x_2,\dots,x_n) \\
= (x_1,\dots,x_n)^{s\sigma_1 t} &= (x^{x_1}, x_2,\dots,x_n) \\
= (x_1,\dots,x_n)^{s\overline{\sigma_1}t} &= (x, x_2,\dots,x_n).
\end{aligned}$$

Thus we must have $x = y = x^y = x^{x_1}$ for all $x_1 \in X$ as indicated.

Remark 2.8.1 *Note that an identity of the rack satisfies this condition, but an identity also satisfies the condition $y^x = y$ for all $y \in X$. If there is no such x, then we could always point the rack, as in [19], and use this identity.*

So a given element of $\mathcal{A}_n^r$ or $\mathcal{T}_n^r$ determines a map $X^n \to X^r$. In particular an element of $\mathcal{A}_n$ determines an element of $End(X^n)$. We may extend linearly from X^n to $\mathbb{Z}X^n$.

Remark 2.8.2 *The map $\mathbb{Z}X^n \to \mathbb{Z}X^{n-1}$ determined by an alternating sum of the face maps, where $\mathbb{Z}X^n$ is the free abelian group generated by the elements of X^n, is the coboundary operator in [7].*

2.9 An Introduction to Δ-sets

2.9.1 Δ-sets

A **simplicial object** in a category $\mathcal{C}$ is a covariant functor $X : \Delta^{op} \to \mathcal{C}$. That is, X is, for all $n \geq 0$, a set of objects, X_n in $\mathcal{C}$, and a set of morphisms, $d_i : X_n \to X_{n-1}$ with $0 \leq i \leq n$, called **face maps**, and $s_j : X_n \to X_{n+1}$ with $0 \leq j \leq n$, called **degenaracies**, satisfying the following relations:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{for } i < j, \\ s_i s_j &= s_{j+1} s_i \quad \text{for } i \leq j \qquad \text{and} \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{for} \quad i < j \\ id_{X_n} & \text{for} \quad i = j, \text{ or } i = j+1 \\ s_j d_{i-1} & \text{for} \quad i > j+1. \end{cases} \end{aligned}$$

We refer to a simplicial object as a **Δ^+-object**. Ignoring the degenaracies gives a **presimplicial object**; this may be referred to as a **Δ-object**. When the category involved is the category of sets, for instance, then a presimplicial object is called a **Δ-set**. If we wish to specify the face maps in the notation for a Δ-set then we will write (X, d).

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2.9.2 Δ-spaces

A **pre-simplicial space (Δ-space)** is a covariant functor $\Delta^{op} \to$ *Spaces*.

The **geometric realisation** of the Δ-space (X, d) is given by:

$$|(X,d)| = \bigcup_{n \geq 0} (X_n \times \Delta^n)/ \approx$$

where the equivalence $\approx$ is generated by the face maps. Note that if we have a simplicial space, then the degeneracy maps, together with the face maps, generate the equivalence.

Suppose, for example, that the spaces X_n are discrete. Then the geometric realisation of the Δ-space is obtained by taking one n-cell for every element of X_n and gluing them together via the face maps d_i. If X is a Δ^+-space then we simply take one n-cell for every non-degenerate n-simplex and glue them together via the face maps.

2.9.3 The Complex of a Δ-set.

Let (X, d) be a Δ-set and let k be a ring with a 1. Replacing the sets X_n by $k[X_n]$, the free modules over k with basis the sets X_n, gives rise to the Δ-module $(k[X], d)$.

We contruct the **complex** of (X, d) with coefficients in k. Form the chain complex $C(X, d; k)$ whose n-chains are the sets $k[X_n]$ and whose n-th differential is given by $d = \sum_{i=1}^{n}(-1)^{i+1}d_i$, an alternating sum of the face maps. The fact that this is a differential follows from the face map relations:

Lemma 2.9.1 *With notation as above, we have $d \circ d = 0$.*

Proof:

We have
$$\begin{aligned} d \circ d &= \sum_{i=1}^{n-1}(-1)^{i+1}d_i \left(\sum_{j=1}^{n}(-1)^{j+1}d_j\right) \\ &= \sum_i \left(\sum_{i<j}(-1)^{i+j}d_i d_j + \sum_{i \geq j}(-1)^{i+j}d_i d_j\right) \\ &= \sum_i \left(\sum_{i<j}(-1)^{i+j}d_{j-1} d_i + \sum_{i \geq j}(-1)^{i+j}d_i d_j\right) \\ &= 0, \end{aligned}$$

since each term in the bracket occurs twice with opposite sign. □

Note that this construction of a chain complex can be applied to any Δ-module.

2.9.4 The Complex of a □-set.

Let (X, d^f, d^b) be a □-set and let k be a commutative ring with a 1. Then we define a chain complex, $C(X, d^f, d^b; k)$, whose n-chains are the modules $k[X_n]$ and whose n-th differential is given by $c = \sum_{i=1}^{n}(-1)^{i+1}(d_i^b - d_i^f)$. Call this the **complex** of (X, d^f, d^b), with coefficients in k. We will take coefficients in $\mathbf{Z}$ unless stated otherwise. We show that the differential above really is a differential. In fact we show more:

Proposition 2.9.2 *Let (X, d^f, d^b) be a □-set and let k be a commutative ring with a 1. Take $u, v \in k$ and set $e_i = ud_i^b + vd_i^f$. Then $e = \sum_{i=1}^{n}(-1)^{i+1}e_i$ is a differential from $k[X_n]$ to $k[X_{n-1}]$. In particular, taking $k = \mathbf{Z}, u = 1$, and $v = -1$ shows that c, defined above, is a differential.*

Proof: As usual, it suffices to check the relation $e_ie_j = e_{j-1}e_i$, for $i < j$. The face map relations give the third equality in the following:

$$
\begin{aligned}
e_ie_j &= (ud_i^b + vd_i^f)(ud_j^b + vd_j^f) \\
&= u^2d_i^bd_j^b + uvd_i^bd_j^f + vud_i^fd_j^b + v^2d_i^fd_j^f \\
&= u^2d_{j-1}^bd_i^b + uvd_{j-1}^fd_i^b + vud_{j-1}^bd_i^f + v^2d_{j-1}^fd_i^f \quad \text{by face map relations} \\
&= u^2d_{j-1}^bd_i^b + uvd_{j-1}^bd_i^f + vud_{j-1}^fd_i^b + v^2d_{j-1}^fd_i^f \quad \text{since } uv = vu \\
&= (ud_{j-1}^b + vd_{j-1}^f)(ud_i^b + vd_i^f) \\
&= e_{j-1}e_i.
\end{aligned}
$$

□

2.10 Total Homology

2.10.1 From □-modules to Bicomplexes

A **bicomplex**, X_{**}, is collection of modules, $X_{p,q}$, indexed by two integers, p and q, together with a **horizontal differential**, $d^h : X_{p,q} \to X_{p-1,q}$, and also a **vertical differential**, $d^v : X_{p,q} \to X_{p,q-1}$, satisfying the following identities

$$d^v d^v = d^h d^h = d^v d^h + d^h d^v = 0.$$

Proposition 2.10.1 *Let (X, d^f, d^b) be a □-module. This defines a bicomplex with modules $X_{p,q} = X_{p+q}$, horizontal differential $d^h = d^b = \sum_{i=1}^n (-1)^{i+1} d_i^b$ and vertical differential $d^v = -d^f = \sum_{i=1}^n (-1)^i d_i^f$.*

Proof: The face map relations $d_i^\epsilon d_j^\epsilon = d_{j-1}^\epsilon d_i^\epsilon$, for $\epsilon \in \{f, b\}$, give $d^v d^v = d^h d^h = 0$, as before. We show that $d^v d^h + d^h d^v = 0$:

$$\begin{aligned}
d^v d^h &= \sum_{i=1}^{n-1} (-1)^i d_i^f \left(\sum_{j=1}^{n} (-1)^{j+1} d_j^b \right) \\
&= \sum_i \left(\sum_{i<j} (-1)^{i+j+1} d_i^f d_j^b + \sum_{i \geq j} (-1)^{i+j+1} d_i^f d_j^b \right) \\
&= \sum_i \left(\sum_{i<j} (-1)^{i+j+1} d_{j-1}^b d_i^f + \sum_{i \geq j} (-1)^{i+j+1} d_i^f d_j^b \right) \\
\text{and} \qquad d^h d^v &= \sum_{i=1}^{n-1} (-1)^{i+1} d_i^b \left(\sum_{j=1}^{n} (-1)^{j} d_j^f \right) \\
&= \sum_i \left(\sum_{i<j} (-1)^{i+j+1} d_i^b d_j^f + \sum_{i \geq j} (-1)^{i+j+1} d_i^b d_j^f \right) \\
&= \sum_i \left(\sum_{i<j} (-1)^{i+j+1} d_{j-1}^f d_i^b + \sum_{i \geq j} (-1)^{i+j+1} d_i^b d_j^f \right).
\end{aligned}$$

Summing and rearranging terms gives:

$$
\begin{aligned}
d^v d^h + d^h d^v &= \sum_i \left[\left(\sum_{i<j} (-1)^{i+j+1} d^b_{j-1} d^f_i + \sum_{i \geq j} (-1)^{i+j+1} d^b_i d^f_j \right) \right. \\
&\quad + \left. \left(\sum_{i<j} (-1)^{i+j+1} d^f_{j-1} d^b_i + \sum_{i \geq j} (-1)^{i+j+1} d^f_i d^b_j \right) \right] \\
&= 0 + 0 \\
&= 0 .
\end{aligned}
$$

Each of the two brackets contribute 0 to the sum since every term in them occurs twice with opposite sign (remember that $1 \leq j \leq n$ and $1 \leq i \leq n-1$). □

We may make this bicomplex into a first quadrant bicomplex, also denoted $X_{\bullet\bullet}$, by simply truncating in the appropriate places. This is depicted in the diagram below; we have set $X_{p,q} = 0$ for $p < 0$ or $q < 0$.

$$
\begin{tikzcd}
{} \arrow[d, "d^v"'] & {} \arrow[d, "d^v"'] & {} \arrow[d, "d^v"'] & \\
X_2 \arrow[d, "d^v"'] & X_3 \arrow[l, "d^h"'] \arrow[d, "d^v"'] & X_4 \arrow[l, "d^h"'] \arrow[d, "d^v"'] & {} \arrow[l, "d^h"'] \\
X_1 \arrow[d, "d^v"'] & X_2 \arrow[l, "d^h"'] \arrow[d, "d^v"'] & X_3 \arrow[l, "d^h"'] \arrow[d, "d^v"'] & {} \arrow[l, "d^h"'] \\
X_0 & X_1 \arrow[l, "d^h"'] & X_2 \arrow[l, "d^h"'] & {} \arrow[l, "d^h"']
\end{tikzcd}
$$

The **total complex** of the bicomplex $X_{\bullet\bullet}$, denoted by $\mathrm{Tot}(X)$, is given by the k-modules:

$$(\mathrm{Tot}\, X_{\bullet\bullet})_n := \bigoplus_{p+q=n} X_{p,q} = \bigoplus^{n+1} X_n$$

with **total differential** d^t, where

$$(d^t)_n : \bigoplus^{n+1} X_n \longrightarrow \bigoplus^{n} X_{n-1}$$

is given by:

$$d^t(x_1, \ldots, x_{n+1}) = (d^v x_1 + d^h x_2, d^v x_2 + d^h x_3, \ldots, d^v x_n + d^h x_{n+1}).$$

Remark 2.10.2 *The complex $C(X, d^f, d^b)$ has chains X_n and differential $d^h + d^v$. Recall that if (X, d^f, d^b) is a $\square$-set then we can construct a $\square$-module by replacing the sets X_n by $k[X_n]$, which are free modules over k, a commutative ring with a 1.*

2.10.2 The Homology Groups of the Diagonal of $\mathrm{Tot}(X)$.

Let $z = (z_1, \ldots, z_{n+1}) \in \bigoplus^{n+1} X_n$. Then

$$z \in Z_n(\mathrm{Tot}\, X) \Leftrightarrow (d^v z_1 + d^h z_2, d^v z_2 + d^h z_3, \ldots, d^v z_n + d^h z_{n+1}) = 0.$$

That is, if we have the following **cycle condition**:

$$d^v z_i + d^h z_{i+1} = 0, \text{ for } 1 \leq i \leq n. \tag{ix}$$

$$\begin{aligned} \text{Now} \quad z \in B_n(\mathrm{Tot}\, X) \quad \Leftrightarrow \quad & \text{there exists } (w_1, \ldots, w_{n+2}) \in \bigoplus^{n+2} X_{n+1} \text{ such that} \\ & (d^v w_1 + d^h w_2, \ldots, d^v w_n + d^h w_{n+1}) = (z_1, \ldots, z_{n+1}). \end{aligned}$$

That is, if we have the following **boundary condition**:

$$z_i = d^v w_i + d^h w_{i+1}, \text{ for } 1 \leq i \leq n+1. \tag{x}$$

Restrict the n-chains of the complex $\mathrm{Tot}\, X$ to their diagonal, that is, to elements of the form $(z, \ldots, z) \in \oplus^{n+1} X_n$. The resulting complex is called the **diagonal** of the total complex of X_{**}, and is denoted by $\mathrm{DTot}\, X$. Let $\phi_n : \bigoplus^{n+1} X_n \to X_n$, for $n \in \mathbb{N}$ be the module homomorphism given by $(z, z, \ldots, z) \mapsto z$, and let $\phi = \{\phi_n : n \in \mathbb{N}\}$.

Proposition 2.10.3 $\phi : (\mathrm{DTot}\, X, d^t) \to (X, d)$ *is a chain equivalence.*
In particular, we have:

$$H_n(\mathrm{DTot}\, X) \cong H_n(X) \text{ for all } n \geq 0.$$

Proof: We show that ϕ is a natural transformation of chain complexes:

$$\begin{aligned} \phi\, d^t(z, \ldots, z) &= \phi(d^v z + d^h z, \ldots, d^v z + d^h z) \\ &= d^v z + d^h z \\ &= dz \\ &= d\,\phi(z, \ldots, z) \end{aligned}$$

Also, ϕ has a natural inverse, so it determines an equivalence of complexes as required. □

Remark 2.10.4 *An interesting question is whether or not one can relate* $H_*(X, d^h)$, $H_*(X, d^v)$ *and* $H_*(\mathrm{Tot}(X))$. *In order to compute* $\mathrm{Tot}(X)$ *one could try to understand spectral sequences which correspond to a first quadrant bicomplex, which is in fact a shifted (with a -1 grading change), truncated (at both* $p < 0$ *and* $q < 0$*) complex.*

Chapter 3

Boundary Operators

3.1 Introduction

Given a rack X we associate six Δ-sets to it by defining six face maps, two of which are the usual front and back face maps of the $\square$-set of X. We give a method for obtaining face maps of Δ-sets by thinking of each face map as some product of morphisms in the terminating braid category, whence a corresponding isotopy of terminating braids will imply the existence of the Δ-set. Note that replacing the braid group at each vertex of the arcbraid category by a braid variant, such as the braid permutation group, BP_n, may be interesting.

A $\square$-set can be thought of as a 'nice' amalgamation of two Δ-sets. We show which pairs of the given Δ-sets associated to the rack X form $\square$-sets; one of these three $\square$-sets associated to X is the usual one. We describe the trunks, whose nerve is given by these $\square$-sets, and also the classifying spaces of these $\square$-sets.

3.2 Δ-sets Associated to a Rack.

Proposition 3.2.1 *Let X be a rack. Let the sets $X_n = X^n$ and let the face maps, d^α, for $\alpha \in \{0, \ldots, 5\}$, be given by:*

$$
\begin{aligned}
(x_1, \ldots, x_n) &\xrightarrow{d_i^0} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\
&\xrightarrow{d_i^1} (x_1^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \ldots, x_n) \\
&\xrightarrow{d_i^2} (x_1, \ldots, x_{i-1}, x_{i+1}^{\overline{x_i}}, \ldots x_n^{\overline{x_i}}) \\
&\xrightarrow{d_i^3} (x_1^{r_i}, \ldots, x_{i-1}^{r_i}, x_{i+1}^{r_i}, \ldots, x_n^{r_i}) \\
&\xrightarrow{d_i^4} (x_1^{x_i w_i}, \ldots, x_{i-1}^{x_i w_i}, x_{i+1}^{w_i}, \ldots, x_n^{w_i}) \\
&\xrightarrow{d_i^5} (x_1^{p_i}, \ldots, x_{i-1}^{p_i}, x_{i+1}^{p_i}, \ldots, x_n^{p_i}),
\end{aligned}
$$

$$
\begin{aligned}
\textit{where} \qquad r_i = r_i(x_1, \ldots, x_n) &= x_i^{x_{i+1} \cdots x_n}, \\
w_i = w_i(x_1, \ldots, x_n) &= x_i^{x_1 \cdots x_n}, \qquad \textit{and} \\
p_i = p_i(x_1, \ldots, x_n) &= \overline{x_i}^{\overline{x_{i-1}} \cdots \overline{x_1}}.
\end{aligned}
$$

Then (X, d^α) are Δ-sets.

Remark 3.2.2 *The definition of p_i given here is equivalent to the geometric definition given in Figure 3.7. This can be seen as follows:*

$$p_i = \overline{x_i^{\overline{x_{i-1}} \cdots \overline{x_1}}} \equiv \overline{x_1 \ldots x_i \overline{x_{i-1}} \ldots \overline{x_1}} \equiv x_1 \ldots x_{i-1} \overline{x_i} \ldots \overline{x_1} \equiv \overline{x_i}^{\overline{x_{i-1}} \cdots \overline{x_1}}.$$

Proof: For notational convenience, $\widehat{x_j}$ means omit the x_j entry or its obvious analogue in the sequence. We simply check that the face maps, d^α, satisfy the face relations of a Δ-object; that is, we check that $d_i^\epsilon d_j^\epsilon = d_{j-1}^\epsilon d_i^\epsilon$ for $1 \le i < j \le n$, and $\epsilon \in \{0, \ldots, 5\}$.

<u>The case $\epsilon = 0$</u>:

$$
\begin{aligned}
d_{j-1} d_i(x_1, \ldots, x_n) &= (x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n) \\
&= d_i d_j(x_1, \ldots, x_n).
\end{aligned}
$$

For $\epsilon = 1$:

$$
\begin{aligned}
d_{j-1}d_i(x_1,\ldots,x_n) &= (x_1^{x_ix_j},\ldots,x_{i-1}^{x_ix_j},x_{i+1}^{x_j},\ldots,x_{j-1}^{x_j},x_{j+1},\ldots,x_n)\\
&= (x_1^{x_jx_i^{x_j}},\ldots,x_{i-1}^{x_jx_i^{x_j}},x_{i+1}^{x_j},\ldots,x_{j-1}^{x_j},x_{j+1},\ldots,x_n)\\
&= d_id_j(x_1,\ldots,x_n).
\end{aligned}
$$

The second equality follows from the rack identity in the form $x_jx_i^{x_j} \equiv x_ix_j$.

For $\epsilon = 2$:

$$
\begin{aligned}
d_{j-1}d_i(x_1,\ldots,x_n) &= (x_1,\ldots,x_{i-1},x_{i+1}^{\overline{x_i}},\ldots,x_{j-1}^{\overline{x_i}},x_{j+1}^{\overline{x_i}\,\overline{x_j^{x_i}}},\ldots,x_n^{\overline{x_i}\,\overline{x_j^{x_i}}})\\
&= (x_1,\ldots,x_{i-1},x_{i+1}^{\overline{x_i}},\ldots,x_{j-1}^{\overline{x_i}},x_{j+1}^{\overline{x_jx_i}},\ldots,x_n^{\overline{x_jx_i}})\\
&= d_id_j(x_1,\ldots,x_n).
\end{aligned}
$$

Here we have used the following equivalence: $\overline{x_jx_i} \equiv \overline{x_ix_j^{x_i}} \equiv \overline{x_i}\,\overline{x_j^{x_i}}$.

The case $\epsilon = 3$:

$$
\begin{aligned}
d_{j-1}d_i(x_1,\ldots,x_n) &= (x_1^{r_is},\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_n^{r_is})\\
&= (x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_n)^{r_is}\\
&= (x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,x_n)^{r_jv}\\
&= d_id_j(x_1,\ldots,x_n).
\end{aligned}
$$

$$
\begin{aligned}
\text{Here}\qquad s &= r_{j-1}(x_1^{r_i},\ldots,\widehat{x_i},\ldots,x_n^{r_i})\\
&= x_j^{r_ix_{j+1}^{r_i}\cdots x_n^{r_i}}\\
&\equiv x_j^{x_{j+1}\cdots x_nr_i}\\
&\equiv \overline{r_i}r_jr_i,
\end{aligned}
$$

$$
\begin{aligned}
\text{and}\qquad v &= r_i(x_1^{r_j},\ldots,\widehat{x_j},\ldots,x_n^{r_j})\\
&= x_i^{r_jx_{i+1}^{r_j}\cdots\widehat{x_j}\cdots x_n^{r_j}}\\
&\equiv x_i^{x_{i+1}\cdots\widehat{x_j}\cdots x_nr_j}\\
&\equiv x_i^{x_{i+1}\cdots x_n}\\
&= r_i.
\end{aligned}
$$

So $r_i s \equiv r_j r_i \equiv r_j v$ as required.

<u>The case $\epsilon = 4$</u>:

$$\begin{aligned} d_{j-1}d_i(x_1,\ldots,x_n) &= (x_1^{x_i w_i x_j^{w_i} u},\ldots,x_{i-1}^{x_i w_i x_j^{w_i} u},x_{i+1}^{w_i x_j^{w_i} u},\ldots,x_{j-1}^{w_i x_j^{w_i} u},x_{j+1}^{w_i u},\ldots,x_n^{w_i u}) \\ &= (x_1^{x_j w_j x_i^{x_j w_j} k},\ldots,x_{i-1}^{x_j w_j x_i^{x_j w_j} k},x_{i+1}^{x_j w_j k},\ldots,x_{j-1}^{x_j w_j k},x_{j+1}^{w_j k},\ldots,x_n^{w_j k}) \\ &= d_i d_j(x_1,\ldots,x_n). \end{aligned}$$

$$\begin{aligned} \text{Here} \qquad u &= w_{j-1}(x_1^{x_i w_i},\ldots,x_{i-1}^{x_i w_i},x_{i+1}^{w_i},\ldots,x_n^{w_i}) \\ &= x_j^{w_i x_1^{x_i w_i}\ldots x_{i-1}^{x_i w_i} x_{i+1}^{w_i}\ldots x_n^{w_i}} \\ &\equiv x_j^{\overline{x_i} x_1 \ldots x_n w_i} \\ &\equiv x_j^{x_1\ldots x_n} \\ &\equiv w_j, \end{aligned}$$

$$\begin{aligned} \text{and} \qquad k &= w_i(x_1^{x_j w_j},\ldots,x_{j-1}^{x_j w_j},x_{j+1}^{w_j},\ldots,x_n^{w_j}) \\ &= x_i^{x_j w_j x_1^{x_j w_j}\ldots x_{j-1}^{x_j w_j} x_{j+1}^{w_j}\ldots x_n^{w_j}} \\ &\equiv x_i^{x_1\ldots x_n w_j} \\ &\equiv \overline{w_j} w_i w_j. \end{aligned}$$

Thus we see that:

- $w_i u \equiv w_i w_j \equiv w_j k$
- $w_i x_j^{w_i} u \equiv x_j w_i w_j \equiv x_j w_j k$
- $x_i w_i x_j^{w_i} u \equiv x_i x_j w_i w_j \equiv x_i x_j w_j k \equiv x_j w_j x_i^{x_j w_j} k.$

<u>The case $\epsilon = 5$</u>:

$$\begin{aligned} \text{We have} \qquad d_{j-1}d_i(x_1,\ldots,x_n) &= (x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots x_n)^{p_i q} \\ &= (x_1,\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots x_n)^{p_j t} \\ &= d_i d_j(x_1,\ldots,x_n). \end{aligned}$$

$$
\begin{aligned}
\text{Here} \qquad q &= p_{j-1}(x_1^{p_i}, \ldots, \widehat{x_i}, \ldots, x_n^{p_i}) \\
&= (\overline{x_j^{p_i}})^{\overline{x_{j-1}^{p_i}} \ldots \widehat{x_i} \ldots \overline{x_1^{p_i}}} \\
&\equiv \overline{x_j}^{p_i \overline{x_{j-1}}^{p_i} \ldots \widehat{x_i} \ldots \overline{x_1}^{p_i}} \\
&\equiv \overline{x_j}^{\overline{x_{j-1}} \ldots \widehat{x_i} \ldots \overline{x_1} p_i} \\
&\equiv \overline{x_j}^{\overline{x_{j-1}} \ldots \widehat{x_i} \ldots \overline{x_1} x_1 \ldots x_{i-1} \overline{x_i} \ldots \overline{x_1}} \\
&\equiv \overline{x_j}^{\overline{x_{j-1}} \ldots \overline{x_1}} \\
&\equiv p_j,
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \qquad t &= p_i(x_1^{p_j}, \ldots, \widehat{x_j}, \ldots, x_n^{p_j}) \\
&= \overline{x_i^{p_j}}^{\overline{x_{i-1}^{p_j}} \ldots \overline{x_1^{p_j}}} \\
&\equiv \overline{x_i}^{p_j \overline{x_{i-1}}^{p_j} \ldots \overline{x_1}^{p_j}} \\
&\equiv \overline{x_i}^{\overline{x_{i-1}} \ldots \overline{x_1} p_j} \\
&\equiv \overline{p_j} p_i p_j.
\end{aligned}
$$

So we have $p_i q \equiv p_i p_j \equiv p_j t$, as required. □

Note that we do not need to restrict X to be a rack in order for (X, d^0) to be a simplicial set; it only needs to be a collection of sets. However the Δ-sets given by (X, d^ϵ) for $\epsilon \in \{1, \ldots, 5\}$ require the use of the rack identity.

Remark 3.2.3 *The images of d_i^3, d_i^5 are equivalent to the image of d_i^0 acted on by $r_i, p_i \in X$ respectively and similarly, the images of d_i^2, d_i^4 are equivalent to the image of d_i^1 acted on by $\overline{x_i}, w_i \in X$ respectively.*

3.3 From Arcbraids to Δ-sets

We wish to form Δ-sets, (X, d), where X is some family of sets. So we look for a geometric condition which implies the face map relations: $d_i d_{j-1} = d_j d_i$ for $i < j$.

Let us represent a face map $d_i : X_n \to X_{n-1}$ as a product of morphisms in the terminating braid category, $\mathcal{T}$. So we require a collection of words $\{w_{i,n} \in B_n \,|\, 0 \leq i \leq n,\ n \geq 1\}$ such

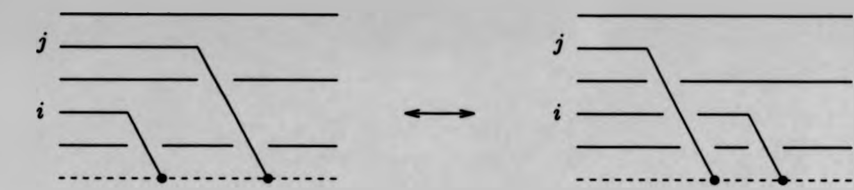

Figure 3.1: The face map relations for d^1 hold

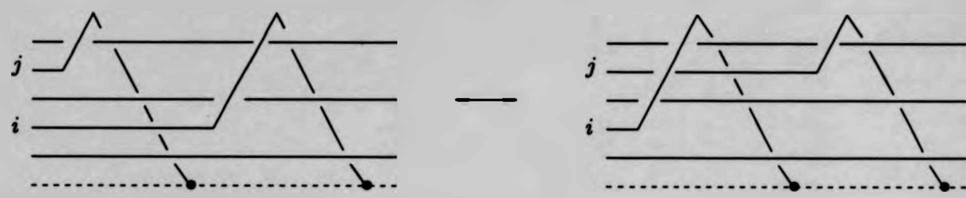

Figure 3.2: The face map relations for d^2 hold

that $w_{i,n}\, t \in \mathcal{T}$ represents the i-th face map from X_n to X_{n-1}. For notational convenience we will suppress the second index whenever possible.

Remark 3.3.1 *Replacing the braid group, B_n, by any braid variant, such as BP_n, the braid permutation group, may be of interest, since it would give rise to more possibilities for the face maps of the Δ-sets.*

Requiring the face map relations to hold gives rise to the following equation:

$$w_i\, t\, w_{j-1}\, t = w_j\, t\, w_i\, t \text{ for } i < j. \tag{i}$$

So the termination of the i-th and j-th strings in either order, after the appropriate braid (or braid-like) operation must yield isotopic arcbraids. Figures 3.1 and 3.2 are examples of terminating braids with this property.

Define $\phi : B_{n-1} \hookrightarrow B_n$ to be the operation which adds one to the index of all the crossings; that is, we adjoin an extra string, with no crossings, below the given braid. Then, using the relations in the category $\mathcal{T}$ we can rewrite equation (i) as:

$$w_i\, \phi(w_{j-1})\, t^2 = w_j\, \phi(w_i)\, t^2. \tag{ii}$$

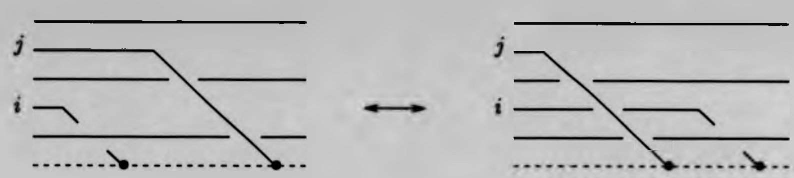

Figure 3.3: The face map relations for d^0 and d^1 hold

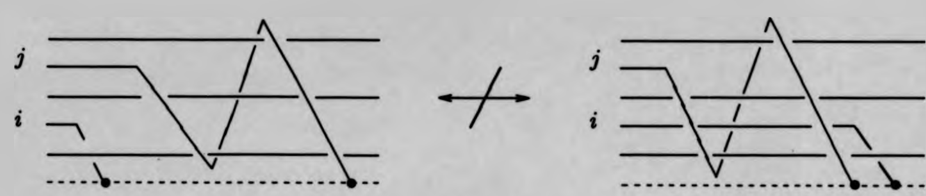

Figure 3.4: The face map relations for d^0 and d^4 do not hold

Note that the equivalent formulation in the braid group would yield the relation:

$$w_i\,\phi(w_{j-1})\,\sigma_1^k = w_j\,\phi(w_i) \text{ for some } k \in \mathbb{Z}. \tag{iii}$$

Remark 3.3.2 *In our calculations we have restricted to i-pure braids (braids which may be represented with all of the strands, except the i-th, as horizontal lines). There is no reason for such a restriction.*

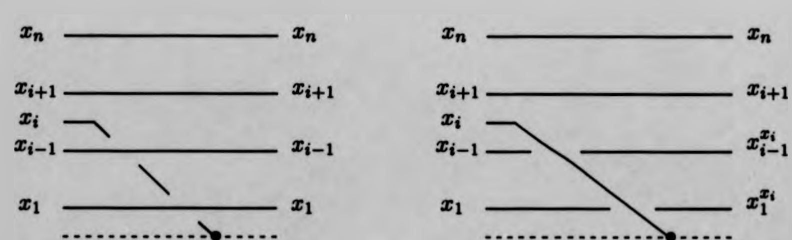

Figure 3.5: The face maps d_i^0 and d_i^1

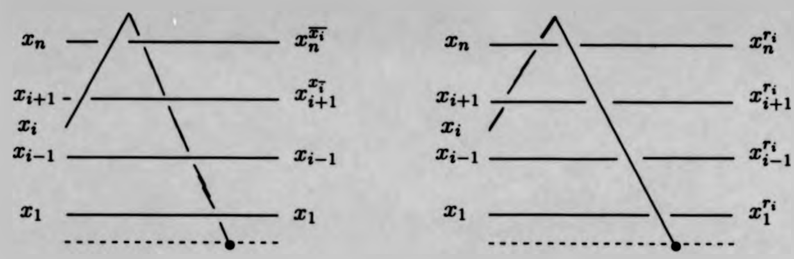

Figure 3.6: The face maps d_i^2 and d_i^3

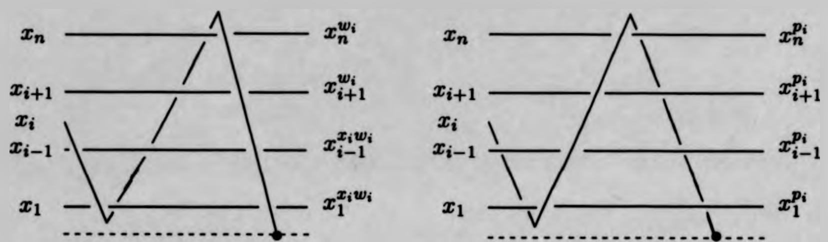

Figure 3.7: The face maps d_i^4 and d_i^5

3.3.1 Comparing Face Maps and Arcbraids

Figures 3.5, 3.6 and 3.7 give rise to some formulae describing the face maps in terms of arcbraids:

$$
\begin{aligned}
d_i^0 &= \overline{\sigma_{i-1}} \dots \overline{\sigma_1} t \\
d_i^1 &= \sigma_{i-1} \dots \sigma_1 t \\
d_i^2 &= \overline{\sigma_i} \dots \overline{\sigma_{n-1} \sigma_{n-1}} \dots \overline{\sigma_1} t \\
d_i^3 &= \sigma_i \dots \sigma_{n-1} \sigma_{n-1} \dots \sigma_1 t \\
d_i^4 &= \sigma_{i-1} \dots \sigma_1 \sigma_1 \dots \sigma_{n-1} \sigma_{n-1} \dots \sigma_1 t \\
d_i^5 &= \overline{\sigma_{i-1}} \dots \overline{\sigma_1 \sigma_1} \dots \overline{\sigma_{n-1} \sigma_{n-1}} \dots \overline{\sigma_1} t.
\end{aligned}
$$

For instance, we have

$$\begin{aligned} d_i^{\eta}(x_1,\ldots,x_n) &= (x_1,\ldots,\widehat{x}_i,\ldots,x_n) \\ &= (x_i^{\overline{x_{i-1}\ldots x_1}}, x_1,\ldots,\widehat{x}_i,\ldots,x_n)^t \\ &= (x_1,\ldots,x_i,\ldots,x_n)^{\overline{\sigma_{i-1}\ldots\sigma_1}t}. \end{aligned}$$

3.4 From Arcbraids to □-sets

In the same manner as in Section 3.3 we describe a condition in $\mathcal{T}$ which ensures that the family of sets, X, forms a □-set, i.e. that

$$d_i^{\epsilon} d_{j-1}^{\eta} = d_j^{\eta} d_i^{\epsilon}, \text{ for any } \epsilon, \eta \in \{f, b\}.$$

The case $\eta = \epsilon$ shows that (X, d^{η}) is a Δ-set and, for our examples, can be found in Section 3.3.

Let $v_{j,n}\,t$ and $w_{i,n}\,t$ be the arcbraids corresponding to d_j^{η} and d_i^{ϵ} respectively. Then our face relation condition becomes:

$$w_i\, t\, v_{j-1}\, t = v_j\, t\, w_i\, t \tag{iv}$$

Thus we get the arcbraid relation:

$$w_i\, \phi(v_{j-1})\, t^2 = v_j\, \phi(w_i)\, t^2 \tag{v}$$

Alternatively, we have the braid relation:

$$w_i\, \phi(v_{j-1})\, \sigma_1^k = v_j\, \phi(w_i) \text{ for some } k \in \mathbf{Z}. \tag{vi}$$

3.5 □-sets From a Rack

Proposition 3.5.1 *Let X be a rack, and let the sets X_n be X^n. Then (X, d^1, d^0), (X, d^2, d^0), and (X, d^1, d^3) are □-sets.*

Proof: By Proposition 3.2.1 it remains to check that $d_{j-1}^{\eta} d_i^{\epsilon} = d_i^{\epsilon} d_j^{\eta}$ for $\epsilon \neq \eta$ where ϵ, η are the relevant pair of values.

For (X, d^1, d^0) we have:

$$\begin{aligned} d^0_{j-1}d^1_i(x_1,\dots,x_n) &= (x_1^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,\widehat{x_j},\dots,x_n) &= d^1_i d^0_j(x_1,\dots,x_n) \\ \text{and}\quad d^1_{j-1}d^0_i(x_1,\dots,x_n) &= (x_1^{x_j},\dots,\widehat{x_i},\dots,x_{j-1}^{x_j},x_{j+1},\dots,x_n) &= d^0_i d^1_j(x_1,\dots,x_n). \end{aligned}$$

Secondly, for (X, d^2, d^0), we have:

$$\begin{aligned} d^0_{j-1}d^2_i(x_1,\dots,x_n) &= (x_1,\dots,x_{i-1},x_{i+1}^{\overline{x_i}},\dots,\widehat{x_j},\dots,x_n^{\overline{x_i}}) &= d^2_i d^0_j(x_1,\dots,x_n) \\ \text{and}\quad d^2_{j-1}d^0_i(x_1,\dots,x_n) &= (x_1,\dots,\widehat{x_i},\dots,x_{j-1},x_{j+1}^{\overline{x_j}},\dots,x_n^{\overline{x_j}}) &= d^0_i d^2_j(x_1,\dots,x_n). \end{aligned}$$

Finally, the case (X, d^1, d^3):

$$\begin{aligned} d^3_{j-1}d^1_i(x_1,\dots,x_n) &= (x_1^{x_i g},\dots,x_{i-1}^{x_i g},x_{i+1}^{g},\dots,\widehat{x_j},\dots,x_n^{g}) \\ &= (x_1^{r_j x_i^{r_j}},\dots,x_{i-1}^{r_j x_i^{r_j}},x_{i+1}^{r_j},\dots,\widehat{x_j},\dots,x_n^{r_j}) \\ &= d^1_i d^3_j(x_1,\dots,x_n) \end{aligned}$$

$$\begin{aligned} \text{and}\quad d^1_{j-1}d^3_i(x_1,\dots,x_n) &= (x_1^{r_i x_j^{r_i}},\dots,\widehat{x_i},\dots,x_{j-1}^{r_i x_j^{r_i}},x_{j+1}^{r_i},\dots,x_n^{r_i}) \\ &= (x_1^{x_j f},\dots,\widehat{x_i},\dots,x_{j-1}^{x_j f},x_{j+1}^{f},\dots,x_n^{f}) \\ &= d^3_i d^1_j(x_1,\dots,x_n), \qquad \text{where} \end{aligned}$$

$$\begin{aligned} g &= r_{j-1}(x_1^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n) &= x_j^{x_{j+1}\cdots x_n} && = r_j, \qquad \text{and} \\ f &= r_i(x_1^{x_j},\dots,x_{j-1}^{x_j},x_{j+1},\dots,x_n) &= x_i^{x_j x_{i+1}^{x_j}\cdots x_{j-1}^{x_j} x_{j+1}\cdots x_n} &= x_i^{x_{i+1}\cdots x_n} & = r_i. \end{aligned}$$

□

Remark 3.5.2 *One can check that no other pairs of d^ϵ, for $\epsilon \in \{0,\dots,5\}$, form a □-set. For example, Figure 3.4 shows that (X, d^0, d^4) is not a □-set.*

Corollary 3.5.3 *Let X be a rack, and let $(b, f) \in \{(1,0),(2,0),(1,3)\}$. Then the maps*

$$\begin{aligned} c^1 &= d^1 - d^0 \\ c^2 &= d^2 - d^0 \\ c^3 &= d^1 - d^3. \end{aligned}$$

are cubical boundary operators.

3.5.1 An Aid to Calculations

Let X be a rack and let $x, y, z \in X$. Then we have:

$$\begin{aligned}
c^1(x,y,z) &= ([d_1^1 - d_2^1 + d_3^1] - [d_1^0 - d_2^0 + d_3^0])(x,y,z) \\
&= [(y,z) - (x^y, z) + (x^z, y^z)] - [(y,z) - (x,z) + (x,y)] \\
&= -(x^y, z) + (x^z, y^z) + (x,z) - (x,y), \\
c^2(x,y,z) &= [(y^{\overline{x}}, z^{\overline{x}}) - (x, z^{\overline{y}}) + (x,y)] - [(y,z) - (x,z) + (x,y)] \\
&= (y^{\overline{x}}, z^{\overline{x}}) - (x, z^{\overline{y}}) - (y,z) + (x,z) \qquad \text{and} \\
c^3(x,y,z) &= [(y,z) - (x^y, z) + (x^z, y^z)] - [(y^{x^{yz}}, z^{x^{yz}}) - (x^{y^z}, z^{y^z}) + (x^z, y^z)] \\
&= (y,z) - (x^y, z) - (y^{x^{yz}}, z^{x^{yz}}) + (x^{y^z}, z^{y^z}).
\end{aligned}$$

$$\begin{aligned}
\text{Also,} \qquad c^1(x,y) &= ([d_1^1 - d_2^1] - [d_1^0 - d_2^0])(x,y) \\
&= (y - x^y) - (y - x) \\
&= x - x^y, \\
c^2(x,y) &= (y^{\overline{x}} - x) - (y - x) \\
&= y^{\overline{x}} - y \qquad \text{and} \\
c^3(x,y) &= (y - x^y) - (y^{x^y} - x^y) \\
&= y - y^{x^y}.
\end{aligned}$$

3.6 The Usual Procedure

Let X be a rack. We form a trunk $\mathcal{T}(X)$, the nerve of which is a $\square$-set, called the nerve of the trunk and is denoted by $\mathcal{N}\mathcal{T}(X)$. The geometric realisation of the nerve is the rack space BX. For convenience of notation in the following we will also refer to this trunk as $\mathcal{T}_1(X)$, and this space as B_1X. The homology groups of this space are precisely the homology groups of the complex $C(X, d^1, d^0; \mathbf{Z})$.

However, we may also form the chain complexes $C(X, d^2, d^0; \mathbf{Z})$ and $C(X, d^1, d^3; \mathbf{Z})$ from the rack X. We define the analagous construction of trunks $\mathcal{T}_2(X)$ and $\mathcal{T}_3(X)$ respectively, which give rise to B_2X and B_3X, which are spaces with the appropriate homology groups.

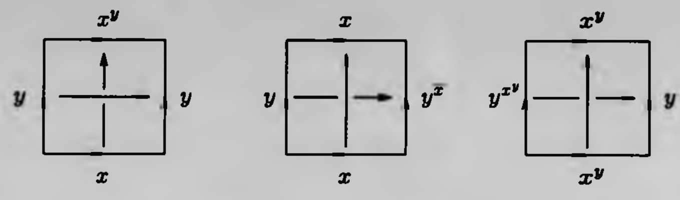

Figure 3.8: The preferred squares for $\mathcal{T}_1(X)$, $\mathcal{T}_2(X)$ and $\mathcal{T}_3(X)$ respectively.

3.7 The Corner Trunks $\mathcal{T}_i(X)$.

Let X be a rack, and let $x, y \in X$. The preferred square (x, y) for each of the trunks $\mathcal{T}_1(X)$, $\mathcal{T}_2(X)$ and $\mathcal{T}_3(X)$ is shown in Figure 3.8. These trunks have one vertex and an n-tuple of co-original edges is precisely an n-tuple of elements of X. So we have the nerves $\mathcal{N}\mathcal{T}_k(X) = X^n$ for $k \in \{1, 2, 3\}$ with the given face maps (i.e. the maps given in the description of the chain complex).

3.7.1 Remarks on the New Trunks

- The element $(x_1, \dots, x_n) \in X^n$ in the trunks $\mathcal{T}_1(X)$ and $\mathcal{T}_2(X)$ corresponds to the n-cube with ordered co-original edges $(x_1, \dots, x_n)$. However, for $\mathcal{T}_3(X)$, this element corresponds to the n-cube with ordered co-original edges $(x_1^{\overline{x_1} x_1 \dots x_n}, \dots, x_n^{\overline{x_n} x_1 \dots x_n})$. In the special case that X is a quandle this cube is just $(x_1, \dots, x_n)^{x_1 \dots x_n}$.

- Notice that if we reverse the lowest consecutive edge path for a cube in $\mathcal{T}_3(X)$ then we obtain the highest consecutive edge path for $\mathcal{T}_1(X)$.

- We refer the reader to [19, p 325] for the remark that the alternative theory based on consecutive edges rather than co-original edges coincides with the co-original theory for the case of racks, where we use the usual face maps. This is because the lowest consecutive edge path is identical to the co-original edge path for $\mathcal{T}_1(X)$. This property does not hold for the other trunks.

Chapter 4

Quasi $\square$-maps

4.1 Introduction

A $\square$-map between $\square$-sets is defined in [19]. Informally, it takes cubes to cubes in a manner which commutes with the face maps. We generalise this to quasi $\square$-maps which take cubes to cubes so that the set of all faces of each cube is preserved. We show that a bijective quasi $\square$-map induces a homeomorphism of classifying spaces. Then we apply this theory by defining bijective quasi $\square$-maps between the three $\square$-sets that we associated to a rack earlier.

We investigate the sign change induced in homology by the homeomorphisms, specifically considering the quasi $\square$-maps defined earlier. As an application we provide an explicit isomorphism between BX, the rack space of X, and BX^*, the rack space of X^*, the inverse of X. Finally, we make some remarks about the canonical class of a link, with the aim of providing a possibility for future research.

4.2 Quasi $\square$-maps.

The first two definitions given here may be found in [19, p 334]. A **map** of $\square$-sets $q : X \to Y$ is a family of maps $q_n : X_n \to Y_n$. A **$\square$-map** $q : X \to Y$, of $\square$-sets X and Y, is a family of maps $q_n : X_n \to Y_n$ which commute with the face maps, i.e.

$$q_{n-1}\, d_i^\epsilon = d_i^\epsilon\, q_n, \text{ for all } i, \epsilon. \qquad \text{(i)}$$

We wish to generalise the notion of a □-map. For notational convenience we will often suppress the subscripts where possible in the following material. Let $r = \{r_n | n \in \mathbb{N}\}$ be a family of bijections $r_n : \{1, \ldots, n\} \times \{f, b\} \to \{1, \ldots, n\} \times \{f, b\}$, for all $n \in \mathbb{N}$. Call r a **face bijection**. Recall that d_i^ϵ is the i-th ϵ face map, where we read front or back for $\epsilon = f$ or $\epsilon = b$ respectively. If P_1 and P_2 denote the projections onto the first and second factors respectively, then let $r(d_i^\epsilon)$ denote the $P_1\, r(i, \epsilon)$-th $P_2\, r(i, \epsilon)$ face map. Extend r to its obvious action on the face relations; explicitly, let $r(d_i^\epsilon d_j^\eta = d_{j-1}^\eta d_i^\epsilon)$ mean $r_{n-1}(d_i^\epsilon) r_n(d_j^\eta) = r_{n-1}(d_{j-1}^\eta) r_n(d_i^\epsilon)$. Finally, the set of all face relations for a □-set X is denoted by $\mathcal{F}(X)$.

Let $q : X \to Y$ be a map of □-sets. We call q a **quasi □-map** if there exists a face bijection r such that the following relations hold:

$$q_{n-1}\, d_i^\epsilon = r(d_i^\epsilon)\, q_n \qquad \text{and} \qquad \text{(ii)}$$

$$\mathcal{F}(Y) = r(\mathcal{F}(X)). \qquad \text{(iii)}$$

We refer to (ii) as **almost commuting** and to (iii) as the **compatibility relation.**

Remark 4.2.1 *A **trivial** quasi □-map is a quasi □-map where r is the identity map. This is an ordinary □-map.*

Let $q : X \to Y$ be a map of □-sets. Suppose that the family of maps $q_n : X_n \to Y_n$ satisfy one of the following families of relations:

$$q_{n-1}\, d_i^\alpha = d_{n+1-i}^\beta\, q_n, \text{ for } \alpha, \beta \in \{f, b\},\ \alpha \neq \beta. \qquad \text{(iv)}$$

$$q_{n-1}\, d_i^\alpha = d_{n+1-i}^\alpha\, q_n, \text{ for } \alpha \in \{f, b\}. \qquad \text{(v)}$$

Then q is said to be a **quasi □-map of type** (I) if (iv) is satisfied or **of type** (II) if (v) is satisfied. A **bijective quasi □-map** is a quasi □-map such that the maps $q_n : X_n \to Y_n$ are bijective for each n.

Lemma 4.2.2 *A quasi □-map of type (I) or of type (II) is a quasi □-map.*

Proof: We have $r(i, \alpha) = (n + 1 - i, \beta)$ where $\alpha \neq \beta$ for type (I) or $\alpha = \beta$ for type (II).

The image under r of the face map relation $d_i^\epsilon d_j^\eta = d_{j-1}^\eta d_i^\epsilon$ is

$$d_{n-i}^\eta d_{n+1-j}^\epsilon = d_{n+1-j}^\epsilon d_{n+1-i}^\eta \qquad \text{for type (I) and}$$
$$d_{n-i}^\epsilon d_{n+1-j}^\eta = d_{n+1-j}^\eta d_{n+1-i}^\epsilon \qquad \text{for type (II).}$$

Since the inequalities $1 \le i < j \le n$ and $1 \le n+1-j < n+1-i \le n$ are equivalent, the compatibility relation holds. □

4.3 Properties of Quasi □-maps

Proposition 4.3.1 *Let X and Y be □-sets. A bijective quasi □-map $q : X \to Y$ induces a homeomorphism $Bq : BX \to BY$ of classifying spaces.*

Proof: First we show that $Bq : BX \to BY$ is bijective. Simply extend the bijections $q_n : X_n \to Y_n$ to a bijection $\bigsqcup_n X_n \times I^n \to \bigsqcup_n Y_n \times I^n$, and then just note that the equivalence relation generated by all the face maps is respected by q.

Next we show that Bq is continuous. Since Y is a □-set, BY is made up of standard cubes glued along their faces. Now Bq is bijective and it just rearranges the images of the faces of an n-cube by the almost commuting relation. Gluing coherence of individual cubes over a dimension span of two is due to the compatibility relation. Thus, it suffices to show that given two n-cubes A and B in BY with a common face C the preimages, in BX, consist of two n-cubes D and E with the correct common face F. So suppose we are given such cubes A, B and C, with $d_i^\alpha(A) = C = d_j^\beta(B)$. Then Bq bijective implies that there exists cubes D, E, F of the correct dimensions in BX with $Bq(D) = A$, $Bq(E) = B$, and $Bq(F) = C$. Call an element in X_n corresponding to a cube in the classifying space by the same name. Then we have $q_n(D) = A$, $q_n(E) = B$, and $q_{n-1}(F) = C$.

$$\text{So} \qquad d_i^\alpha q_n(D) = q_{n-1}(F) = d_j^\beta q_n(E). \qquad \text{(vi)}$$

Now, since q is a bijective quasi □-map,

$$\begin{aligned} q_{n-1}\, r^{-1}(d_i^\alpha(D)) &= d_i^\alpha\, q_n(D) && \text{by (ii)} \\ &= q_{n-1}(F) && \text{by (vi)} \\ &= d_j^\beta\, q_n(E) && \text{by (vi)} \\ &= q_{n-1}\, r^{-1}(d_j^\beta(E)) && \text{by (ii).} \end{aligned}$$

Since q_{n-1} is bijective, $r^{-1}(d_i^\alpha(D)) = F = r^{-1}(d_j^\beta(E))$. So the preimage consists of two n-cubes glued along the correct common face. Thus Bq is continuous.

Applying the same argument to the inverse map Bq^{-1} gives that Bq is a homeomorphism.

□

Remark 4.3.2 *The composite of two quasi □-maps is a quasi □-map. The composite of two quasi □-maps of type (I) is an ordinary □-map, as is the composite of two quasi □-maps of type (II).*

4.4 Applications

Let X be a rack. Recall that we can form the □-sets (X, d^1, d^0), (X, d^2, d^0) and (X, d^1, d^3), with face maps given by:

$$\begin{aligned} (x_1, \ldots, x_n) &\xrightarrow{d_i^0} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\ (x_1, \ldots, x_n) &\xrightarrow{d_i^1} (x_1^{x_i}, \ldots x_{i-1}^{x_i}, x_{i+1}, \ldots, x_n), \\ (x_1, \ldots, x_n) &\xrightarrow{d_i^2} (x_1, \ldots x_{i-1}, x_{i+1}^{\overline{x_i}}, \ldots x_n^{\overline{x_i}}), \qquad \text{and} \\ (x_1, \ldots, x_n) &\xrightarrow{d_i^3} (x_1^{r_i}, \ldots x_{i-1}^{r_i}, x_{i+1}^{r_i}, \ldots x_n^{r_i}), \end{aligned}$$

where $r_i = r_i(x_1, \ldots, x_n) = x_i^{x_{i+1} \ldots x_n}$.

We will exhibit a number of quasi □-maps between these □-sets. Figure 4.1 shows the braid representation of the map ψ_4 defined in Proposition 4.4.1.

Proposition 4.4.1 *There is a bijective quasi □-map of type (I)*

$$\begin{aligned} \psi : \quad & (X, d^2, d^0) \to (X, d^1, d^0) \qquad \textit{given by}: \\ \psi_n = \quad & (\overline{\sigma_{n-1}})(\overline{\sigma_{n-2}}\,\overline{\sigma_{n-1}}) \ldots (\overline{\sigma_1} \ldots \overline{\sigma_{n-1}}). \end{aligned}$$

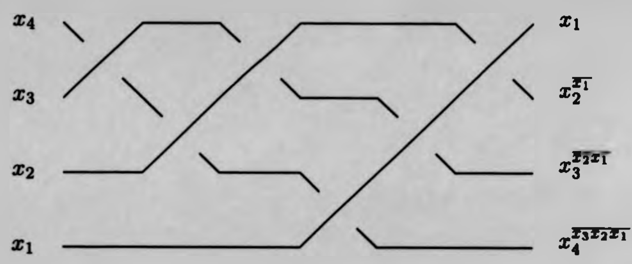

Figure 4.1: The map $\psi_4 = (\overline{\sigma_3})(\overline{\sigma_2\sigma_3})(\overline{\sigma_1\sigma_2\sigma_3})$.

Proof: Let $(x_1, \ldots, x_n) \in X_n$. Then

$$\begin{aligned}
\psi_{n-1}\, d_i^2(x_1, \ldots, x_n) &= \psi_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}^{\overline{x_i}}, \ldots, x_n^{\overline{x_i}}) \\
&= (x_n^{\overline{x_i}\,\overline{x_{n-1}}^{\overline{x_i}}\cdots\overline{x_{i+1}}^{\overline{x_i}}\,\overline{x_{i-1}}\cdots\overline{x_1}}, \ldots, x_{i+1}^{\overline{x_i}\cdots\overline{x_1}}, x_{i-1}^{\overline{x_{i-2}}\cdots\overline{x_1}}, \ldots, x_2^{\overline{x_1}}, x_1) \\
&= (x_n^{\overline{x_{n-1}}\cdots\overline{x_1}}, \ldots, \widehat{x}_i, \ldots, x_2^{\overline{x_1}}, x_1) \\
&= d^0_{n+1-i}(x_n^{\overline{x_{n-1}}\cdots\overline{x_1}}, \ldots, x_2^{\overline{x_1}}, x_1) \\
&= d^0_{n+1-i}\,\psi_n(x_1, \ldots, x_n) \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
\psi_{n-1}\, d_i^0(x_1, \ldots, x_n) &= \psi_{n-1}(x_1, \ldots, \widehat{x}_i, \ldots, x_n) \\
&= (x_n^{\overline{x_{n-1}}\cdots\widehat{x}_i\cdots\overline{x_1}}, \ldots, x_{i+1}^{\overline{x_{i-1}}\cdots\overline{x_1}}, x_{i-1}^{\overline{x_{i-2}}\cdots\overline{x_1}}, \ldots, x_2^{\overline{x_1}}, x_1) \\
&= (x_n^{\overline{x_{n-1}}\cdots\overline{x_1}\,x_i^{\overline{x_{i-1}}\cdots\overline{x_1}}}, \ldots, x_{i+1}^{\overline{x_i}\cdots\overline{x_1}\,x_i^{\overline{x_{i-1}}\cdots\overline{x_1}}}, x_{i-1}^{\overline{x_{i-2}}\cdots\overline{x_1}}, \ldots, x_2^{\overline{x_1}}, x_1) \\
&= d^1_{n+1-i}(x_n^{\overline{x_{n-1}}\cdots\overline{x_1}}, \ldots, x_2^{\overline{x_1}}, x_1) \\
&= d^1_{n+1-i}\,\psi_n(x_1, \ldots, x_n).
\end{aligned}$$

The inverse of ψ_n is given by $\psi_n^{-1} = (\sigma_{n-1}\ldots\sigma_1)\ldots(\sigma_{n-1}\sigma_{n-2})(\sigma_{n-1})$.

□

Proposition 4.4.2 *There is a bijective quasi □-map of type (I)*

$$\begin{aligned}
\phi : \quad & (X, d^1, d^3) \to (X, d^1, d^0) \qquad \textit{given by :} \\
\phi_n = \quad & (\sigma_1)(\sigma_2\sigma_1)\ldots(\sigma_{n-1}\ldots\sigma_1).
\end{aligned}$$

Proof: We have

$$\begin{aligned}
\phi_{n-1}\, d_i^1(x_1, \ldots, x_n) &= \phi_{n-1}(x_1^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \ldots, x_n) \\
&= (x_n, x_{n-1}^{x_n}, \ldots, x_{i+1}^{x_{i+2}\ldots x_n}, x_{i-1}^{x_i \ldots x_n}, \ldots, x_1^{x_i\, x_2^{x_i} \ldots x_{i-1}^{x_i}\, x_{i+1}\ldots x_n}) \\
&= (x_n, x_{n-1}^{x_n}, \ldots, \widehat{x_i}, \ldots, x_1^{x_2 \ldots x_n}) \\
&= d_{n+1-i}^0(x_n, x_{n-1}^{x_n}, \ldots, x_1^{x_2 \ldots x_n}) \\
&= d_{n+1-i}^0\, \phi_n(x_1, \ldots, x_n) \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
\phi_{n-1}\, d_i^3(x_1, \ldots, x_n) &= \phi_{n-1}(x_1^{r_i}, \ldots, \widehat{x_i}, \ldots, x_n^{r_i}) \\
&= (x_n^{r_i}, \ldots, x_{i+1}^{r_i\, x_{i+2}^{r_i} \ldots x_n^{r_i}}, x_{i-1}^{r_i\, x_{i+1}^{r_i} \ldots x_n^{r_i}}, \ldots, x_1^{r_i\, x_2^{r_i} \ldots \widehat{x}_i \ldots x_n^{r_i}}) \\
&= (x_n^{r_i}, \ldots, x_{i+1}^{x_{i+2} \ldots x_n r_i}, x_{i-1}^{x_{i+1} \ldots x_n r_i}, \ldots, x_1^{x_2 \ldots \widehat{x}_i \ldots x_n r_i}) \\
&= (x_n^{r_i}, \ldots, x_{i+1}^{x_{i+2} \ldots x_n r_i}, x_{i-1}^{x_i \ldots x_n}, \ldots, x_1^{x_2 \ldots x_n}) \\
&= d_{n+1-i}^1(x_n, x_{n-1}^{x_n}, \ldots, x_1^{x_2 \ldots x_n}) \\
&= d_{i+1-i}^1\, \phi_n(x_1 \ldots x_n).
\end{aligned}$$

The inverse of ϕ_n is given by $\phi_n^{-1} = (\overline{\sigma_1} \ldots \overline{\sigma_{n-1}}) \ldots (\overline{\sigma_1 \sigma_2})(\overline{\sigma_1})$. □

Proposition 4.4.3 *Let X be a rack and let X^* denote its inverse. There is a bijective quasi □-map of type (II)* $\quad p : (X, d^1, d^0) \to (X^*, d^2, d^0)$.

Proof: For convenience of notation we will assume that X has a finite generating set, G. Let $y_1, \ldots, y_m$ be the distinct elements of G. Then any element in X may be written as $y_i^{w_i}$, for some i, and some $w_i \in F(G)$, the free group with generating set G. The map $p : X \to X^*$ given by $y_i^{w_i} \mapsto y_i^{\overline{w_i}}$ is a natural bijection. Extend p to a map $X^n \to (X^n)^*$, for $n > 1$, given by

$$(y_1^{w_1}, \ldots, y_n^{w_n}) \mapsto (y_n^{\overline{w_n}}, \ldots, y_1^{\overline{w_1}}).$$

We check the almost commutativity relations:

$$\begin{aligned}
p_{n-1}\, d_i^0(y_1^{w_1}, \ldots, y_n^{w_n}) &= p_{n-1}(y_1^{w_1}, \ldots, \widehat{y_i^{w_i}}, \ldots, y_n^{w_n}) \\
&= (y_n^{\overline{w_n}}, \ldots, \widehat{y_i^{\overline{w_i}}}, \ldots, y_1^{\overline{w_1}}) \\
&= d_{n+1-i}^0(y_n^{\overline{w_n}}, \ldots, y_1^{\overline{w_1}}) \\
&= d_{n+1-i}^0\, p_n(y_1^{w_1}, \ldots, y_n^{w_n}) \qquad \text{and}
\end{aligned}$$

$$\begin{aligned} p_{n-1}\, d_i^1(y_1^{w_1}, \dots, y_n^{w_n}) &= p_{n-1}(y_1^{w_1 y_i^{w_i}}, \dots, y_{i-1}^{w_{i-1} y_i^{w_i}}, y_{i+1}^{w_{i+1}}, \dots, y_n^{w_n}) \\ &= (y_n^{\overline{w_n}}, \dots, y_{i+1}^{\overline{w_{i+1}}}, y_{i-1}^{\overline{w_{i-1}} y_i^{\overline{w_i}}}, \dots y_1^{\overline{w_1} y_i^{\overline{w_i}}}) \\ &= d_{n+1-i}^2(y_n^{\overline{w_n}}, \dots, y_1^{\overline{w_1}}) \\ &= d_{n+1-i}^2\, p_n(y_1^{w_1}, \dots, y_n^{w_n}). \end{aligned}$$

The inverse of p_n is given by the same formulae as p_n. □

Corollary 4.4.4 *There are homeomorphisms*

$$B_2(X) \cong B_1(X), \; B_3(X) \cong B_1(X) \;\text{ and }\; B_2(X) \cong B_1(X^*).$$

Proof: Follows from Propositions 4.3.1, 4.4.1, 4.4.2, and 4.4.3. □

4.5 Sign Changes in Homology

Let X and Y be □-sets arising from racks, which are also denoted by X and Y. The context will make clear which is meant. Let $q : X \to Y$ be a bijective quasi □-map. The induced homeomorphism $Bq : BX \to BY$ induces isomorphisms of homology groups; that is, $H_n(BX) \cong H_n(BY)$, for all $n \in \mathbb{N}$.

We wish to investigate the possible sign change of $Bq_* : H_n(BX) \to H_n(BY)$. Consider $BX^{(n)}$, the n-skeleton of BX. Each n-cube in $BX^{(n)}$ has a natural orientation induced from the orientation of the edges. The edges are all oriented away from a base point, which is a vertex, called b. Recall that the n ordered co-original edges at b are each labelled by an element of X, and that these n labels determine the labelling of the rest of the cube.

Remark 4.5.1 *In [19] the transverse orientation to the cells is taken, but, since we are interested only in the sign change this choice is unimportant here.*

A change in orientation gives rise to a sign change in homology. This can happen in two ways.

Firstly suppose that the base point is fixed, that is $q(b) = b$. Now $q : X^n \to Y^n$ is a map which rearranges the faces of the n-cube, so it may change the ordering of the co-original edges. Let $\tau(q)$ denote the permutation induced by this change of ordering. The orientation change in this case is given by $|\tau(q)|$ mod 2, where $|\tau(q)|$ means the number of transpositions in the word $\tau(q)$.

Now suppose that the base point is moved to some other vertex b', that is, $q(b) = b'$. The two vertices b and b' are joined by a sequence of edges $e_1, \ldots, e_k$. Movement of the base vertex across a single edge changes the orientation. So the orientation change given by this movement of the base vertex is k mod 2. Note that the number of transpositions and the number of edges described above are well defined modulo 2. Thus we have:

Lemma 4.5.2 *The sign change in homology induced by q is given by $(|\tau(q)| + k)$ mod 2, where $\tau(q)$ is the permutation of the co-original edges induced by q, and k is the number of edges between the base vertex and its image under q.* □

Alternatively we may calculate the sign change in homology in a strictly algebraic manner. Let C be the chain complex of a □-set, and let $q(C)$ denote the chain complex obtained from applying quasi □-map q. Then $q(C)$ differs from C by possible sign changes of the boundary formulae. Construct the natural transformation which takes C to $q(C)$, agreeing with signs.

4.5.1 Sign Changes for Quasi □-maps of Type (I) and (II)

Proposition 4.5.3 *The quasi □-maps ϕ_n and ψ_n induce a sign change in the n-th homology groups of the classifying spaces if and only if $n \equiv 1, 2$ mod 4. Similarly p_n induces a sign change if and only if $n \equiv 2, 3$ mod 4.*

Proof: First of all we show the effect of any quasi □-maps of type (I) or (II) on the base vertex b of an n-cube $x \in X_n$. Write $b = d_1^0 \ldots d_n^0(x)$, and $b' = q_0(b)$. Note that we change the variable in the definition of quasi □-maps from n to m in order to avoid confusion with the n corresponding to the n-cube and its n-th face map. Also, if $m = i$ then the quasi □-map relation becomes $q_{m-1}\, d_m^\alpha = d_1^\beta\, q_m$ where $\alpha \neq \beta \in \{0, 1\}$ for type (I) and $\alpha = \beta \in \{0, 1\}$ for type (II).

$$\text{So} \quad \begin{aligned} q_0 d_1^\alpha \dots d_n^\alpha(x) &= q_0 d_1^\alpha[d_2^\alpha \dots d_n^\alpha(x)] \\ &= d_1^\beta q_1 d_2^\alpha \dots d_n^\alpha(x) && \text{since } m = i = 1 \\ &= d_1^\beta q_1 d_2^\alpha[d_3^\alpha \dots d_n^\alpha(x)] \\ &= d_1^\beta d_1^\beta q_2 d_3^\alpha \dots d_n^\alpha(x) && \text{since } m = i = 2 \\ &= d_1^\beta \dots d_1^\beta q_n(x). \end{aligned}$$

Thus the base vertex is moved along n edges for type (I), since $\alpha = 0$ and $\beta = 1$, and is fixed for type (II), since $\alpha = 0 = \beta$.

Now we calculate the induced permutation from the quasi □-maps ψ, ϕ, and p. Recall that $\psi_n = (\overline{\sigma_{n-1}})(\overline{\sigma_{n-2}}\,\overline{\sigma_{n-1}})\dots(\overline{\sigma_1}\dots\overline{\sigma_{n-1}})$, and $\phi_n = (\sigma_1)(\sigma_2\sigma_1)\dots(\sigma_{n-1}\dots\sigma_1)$. They may be represented by braids, and so we already have the permutation representations. In both of these cases this is a half twist of an n-string braid, and the number of crossings involved is $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$. For p the permutation $\tau(p_n)$ is given by $(1, \dots, n) \mapsto (n, \dots, 1)$. The number of transpositions in $\tau(p_n)$ is also $\frac{n(n-1)}{2}$.

Thus a change of orientation occurs if and only if $\frac{n(n-1)}{2} + n \equiv 1 \bmod 2$ for these quasi □-maps of type (I). That is, if $\frac{n(n+1)}{2} \equiv 1 \bmod 2$, or equivalently, if $n(n+1) \equiv 2 \bmod 4$. Substituting for $n \in \{4k, 4k+1, 4k+2, 4k+3\}$ gives the result.

Finally, a change of orientation occurs if and only if we have $\frac{n(n-1)}{2} \equiv 1 \bmod 2$ for the quasi □-map of type (II). □

Corollary 4.5.4 *A quasi □-map of type (I) induces a sign change in the n-th homology groups of the classifying spaces if and only if $n \equiv 1, 2 \bmod 4$. Similarly a quasi □-map of type (II) induces a sign change in the n-th homology groups of the classifying spaces if and only if $n \equiv 2, 3 \bmod 4$.*

Proof: Any two quasi □-maps of the same type will have the same induced permutation and the same movement of the base vertex. □

Remark 4.5.5 *These result are obtainable by purely algebraic means. For example, consider a pair of chain complexes C_1 and C_2, both of whose chain groups are X_n,* for $n \geq 0$, *with* *boundary maps $\delta_n : X_n \to X_{n-1}$ for C_1, and $(-1)^n\delta_n : X_n \to X_{n-1}$ for C_2. The natural transformation taking C_1 to C_2 may be described by the maps $f_n : X_n \to X_{n-1}$ where*

$$f_n = \begin{cases} +1 & \text{if } n \equiv 0,3 \bmod 4 \\ -1 & \text{otherwise.} \end{cases}$$

This is the required sign change for quasi □-maps of type (I) defined above.

Remark 4.5.6 *Compare the sign change (of the induced permutation) in Proposition 4.5.3 with the sign of the discriminant, D, of $P(z) = z^n + \sum_{i=1}^{n} a_i z^{n-i}$, in [24, p 147]. If $\alpha_1, \ldots, \alpha_n$ are the n roots of $P(z)$ counted with multiplicity, then $D = (-1)^{\frac{n(n-1)}{2}} \Pi_{k=1}^{n} P'(\alpha_k)$.*

4.6 Explicit Isomorphism in Homology

We remark that if L is a link then we can form $\Gamma(L)$, the fundamental rack of L, which is finitely generated. Let X be a rack, which we assume is finitely generated for convenience of notation, and let X^* denote its inverse. Fix a finite generating set $G = \{y_i : 1 \leq i \leq m\}$. Any element of X may be written in the form $y_i^{w_i}$, for some $w_i \in F(G)$. Let $z_n = (y_1^{w_1}, \ldots, y_n^{w_n})$ be a generator of $H_n(BX)$, and let $z'_n = (y_1^{\overline{w_1 y_2^{w_2} \ldots y_n^{w_n}}}, \ldots, y_{n-1}^{\overline{w_{n-1} y_n^{w_n}}}, y_n^{\overline{w_n}}) \in H_n(BX^*)$.

Theorem 4.6.1 *We have the following isomorphisms:*

$$H_n(BX) \cong H_n(BX^*), \text{ for all } n \geq 0, \text{ where}$$

$$z_n \mapsto \begin{cases} z'_n & \text{if } n \text{ is even} \\ -z'_n & \text{otherwise.} \end{cases}$$

Proof: The homeomorphisms $BX \cong B_2X \cong BX^*$ can be found in Corollary 4.4.4. We explicitly calculate the isomorphisms induced in homology. The bijective quasi □-maps

$$(X, d^1, d^0) \xrightarrow{p} (X^*, d^2, d^0) \xrightarrow{\psi} (X^*, d^1, d^0)$$

induce the following isomorphisms in homology:

$$H_n(BX) \xrightarrow{p_*} H_n(B_2X^*) \xrightarrow{\psi_*} H_n(BX^*).$$

Recall that we have $$p_*(y_1^{w_1},\dots,y_n^{w_n}) = (-1)^{\frac{n(n-1)}{2}}(y_n^{\overline{w_n}},\dots,y_1^{\overline{w_1}}).$$

Also $$\psi_*(y_1^{w_1},\dots,y_n^{w_n}) = (-1)^{\frac{n(n+1)}{2}}(y_n^{w_n \overline{y_{n-1}^{w_{n-1}}} \dots \overline{y_1^{w_1}}},\dots,y_2^{w_2\overline{y_1^{w_1}}},y_1^{w_1}).$$

So
$$\begin{aligned}
\psi_* \circ p_*(y_1^{w_1},\dots,y_n^{w_n}) &= (-1)^{\frac{n(n-1)}{2}}\psi_n(y_n^{\overline{w_n}},\dots,y_1^{\overline{w_1}}) \\
&= (-1)^{n^2}(y_1^{\overline{w_1}\,\overline{y_2^{\overline{w_2}}\dots y_n^{\overline{w_n}}}},\dots,y_{n-1}^{\overline{w_{n-1}}\,\overline{y_n^{\overline{w_n}}}},y_n^{\overline{w_n}}) \\
&= (-1)^{n^2}(y_1^{\overline{w_1 y_2^{w_2}\dots y_n^{w_n}}},\dots,y_{n-1}^{\overline{w_{n-1}y_n^{w_n}}},y_n^{\overline{w_n}})
\end{aligned}$$

□

4.7 The Canonical Class

Given a link L we can obtain the fundamental rack of the link, $\Gamma(L)$. Let $\overline{L}$ denote the mirror of the link L, and let $\Gamma(L)^*$ denote the inverse rack of $\Gamma(L)$.

Then, as in [17, p383], $$\Gamma(\overline{L}) \cong \Gamma(L)^*. \tag{vii}$$

Thus, we have shown that
$$\begin{aligned}
B\Gamma(L) &\cong B_2\Gamma(L) && \text{by Corollary 4.4.4} \\
&\cong B\Gamma(L)^* && \text{by Corollary 4.4.4} \\
&\cong B\Gamma(\overline{L}) && \text{by equation (vii).}
\end{aligned}$$

Given a link L, the **canonical class** of L is the element of $\pi_2(B\Gamma(L))$ determined by the diagram. This is dependent on the choice of representative of the isomorphism class of $\Gamma(L)$. The choice of representative is a labelling of the diagram, i.e. a choice of presentation of the rack $\Gamma(L)$. Note that $B\Gamma(L)$ is constructed independently of any such choice. The operation $rm : L \mapsto rm(L)$ determines an isomorphism of fundamental racks. Therefore, $B\Gamma(L) \cong B\Gamma(rmL)$. Also rm changes the sign of the canonical class determined by the diagram.

Remark 4.7.1 *It was shown that the pair* ($\Gamma(L)$, canonical class) *is a complete invariant of links in [18] .*

Now the rack space of any link is homeomorphic to the rack space of its mirror and its reverse. So one can investigate the change in the canonical element of π_2 (determined by the link diagrams) under the operations of mirroring and reversing. We leave such investigations to the interested reader.

Part II

The Jones Polynomial of Welded Links

Chapter 5

Codes, Links and Racks

5.1 Introduction

Given a classical link, one can obtain a signed Gauss code for the link. This is a type of description of the link. However, not all signed Gauss codes have a representation as a classical link. Therefore the class of links is extended to the class of virtual links [31]. These can be thought of as equivalence classes of signed Gauss codes under abstractly defined Reidemeister type moves.

We investigate the relationship between signed Gauss codes and the fundamental rack. Any two links whose codes differ by a permutation of consecutive o's have the same fundamental rack. Note that this operation of permuting consecutive o's is not a consequence of virtual isotopy, and in Section 5.3.3 we give the corresponding move on virtual link diagrams. In the other direction, we show that given a standard presentation of $\Gamma(L)$, the fundamental rack of a link, we can construct a representative of any virtual link, which has the same standard presentation of $\Gamma(L)$ obtainable from its diagram. Consequently, if we consider the class of non-split, framed virtual links, whose codes may differ by permutations of consecutive o's, then, in each equivalence class, there exists either:

- No classical representatives.
- One classical representative L.

- Two classical representatives, L, and its reverse mirror, rmL.

We wish to investigate the effects of link operations such as reversing, mirroring, changing crossings, and smoothing crossings on both codes and racks. So we give algorithms which compute the reverse, mirror and reverse mirror of a code. Thus, for alternating links we obtain standard presentations of $\Gamma(rL)$, $\Gamma(mL)$ and $\Gamma(rmL)$ from a standard presentation of $\Gamma(L)$. Now the reverse mirror operation applied to the code of the usual diagram of a trefoil knot changes exactly the sign of all of the crossings. We give a simple condition on the code of an alternating knot diagram for it to have this property. Furthermore, we show that this condition is satisfied for the code of any alternating classical knot whose diagram is in closed braid form.

One would like to obtain Jones polynomial information from the fundamental rack of a link. The Jones polynomial is definable via a Skein relation which relates the polynomial of the link to the polynomials of the same link, but with a single crossing changed or smoothed. These crossing change and smoothing operations are easily definable for codes. However, for the fundamental rack of a link we can only give algorithms to construct the racks of the links with a crossing changed or smoothed if we have an extra piece of information. This information is an unordered collection of crossing numbers corresponding to the set of consecutive o's in the code, strictly between ok and the next undercrossing, where k is the crossing to be changed or smoothed.

We finish the chapter with a remark which gives a connection with the material of Part I.

5.2 The Codes and Racks of Virtual Links

Let L be an oriented classical link, and let D be a diagram for L with n crossings. Note that the process for obtaining signed Gauss codes for virtual links, which are defined in Section 5.2.2, is identical to the classical case, which is given in Section 5.2.1.

5.2.1 A Signed Gauss Code for Links

A **signed Gauss code for** D is an expression encoding information from D, obtained as follows:

- Choose an ordering of the components of the link.
- Pick an **initial point** on each component. These are not allowed to be crossing points.
- Assign, to each of the crossing points, a unique integer between 1 and n, inclusive, called the **crossing number**.
- We "travel" along the first component, starting at the initial point and following the given orientation, writing down the following triple of information at each crossing we meet, stopping when we arrive back at the initial point:
  1. A u or an o describes whether we are on the under arc or the over arc at the crossing.
  2. The crossing number.
  3. The sign of the crossing. To obtain this we consider the switch, at the crossing, from the approaching underarc to the overarc. If we turn right as we switch then the sign is $+$, and it is $-$ if we turn left.
- Do this for each component in turn, separating each component's code by a /.

A **signed Gauss code for** L is a signed Gauss code for some diagram representing L. We will use the phrases code, Gauss code, and signed Gauss code interchangeably.

As shown in Figure 5.1, the Right Trefoil knot has code $o1+\ u2+\ o3+\ u1+\ o2+\ u3+$. Note that $u1,\ldots,un$ do not have to occur in a single component of L. The Right Hopf link has a signed Gauss code $u2+o1+/u1+o2+$. In this example, as shown in Figure 5.2, if we wish $u1$ and $u2$ to appear in ascending order then we can switch the crossing numbers 1 and 2.

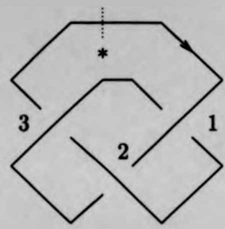

Figure 5.1: Computing the code of the Right Trefoil knot

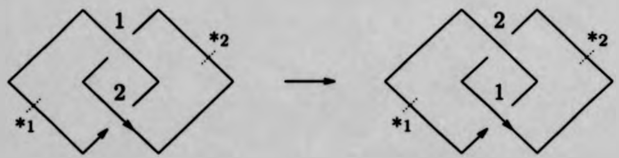

Figure 5.2: Putting the code of the Right Hopf link in standard form

Lemma 5.2.1 *Permuting the numbering in the Gauss code is the same as permuting the set of crossing numbers. A different initial point on a component gives a cyclic permutation of that component's code.* □

Identify Gauss codes that differ by cyclic permutations of their components. A signed Gauss code whose *ui* entries appear in ascending order is said to be **in standard form**.

In general, a **signed Gauss code** is defined to be a finite, ordered set of triples, together with an extra piece of seperating information, with the following properties:

- Each triple consists of an o or a u, a nonzero natural number and a sign.
- If k is a number appearing in the code then so is j for any $j < k$.
- Every number appears exactly twice in the set, once with an associated u and once with an associated o. The sign associated to a number is the same for both of its appearances.
- A / may be placed between any pair of consecutive triples. This is the extra piece of seperating information.

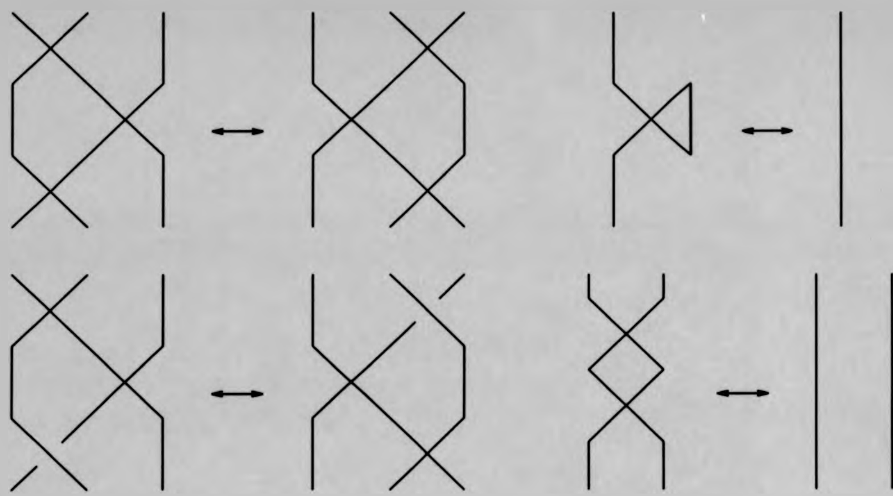

Figure 5.3: Extra isotopy moves for virtual link diagrams

5.2.2 Virtual Links

Details may be found in [31]. A Gauss code may or may not have a planar representation as a link diagram. If it does then it is said to represent a **classical** link. If not, then we represent it as a diagram in the plane which has **virtual crossings**, which are another type of crossing distinct from the usual ones. In this case the link is called **virtual**. The equivalence of virtual links is referred to as (virtual) isotopy, and may be defined via Reidemeister type moves; specifically, the isotopy moves are planar isotopy, R1)-moves, R2)-moves, R3)-moves and the moves shown in Figure 5.3. However, an intuitive method for the extra isotopy moves involving virtual crossings is to allow any arc which contains only virtual crossings (no classical crossings) to be moved to any other arc with the same property.

Recall that one can define abstract Reidemeister moves for Gauss codes of classical links. In [31] a virtual link is defined to be an equivalence class of signed Gauss codes under abstractly defined Reidemeister type moves for these codes. It was shown there that if two classical link diagrams are equivalent under the virtual Reidemeister moves then they are equivalent under the classical Reidemeister moves.

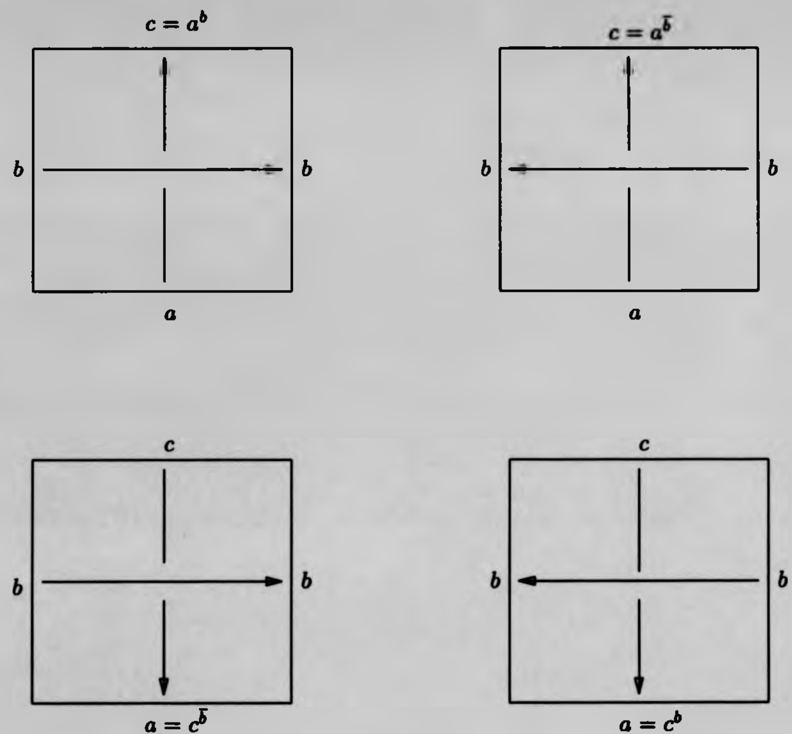

Figure 5.4: The relations corresponding to the four choices of orientation.

5.2.3 The Fundamental Rack of a Link

A choice of presentation for the fundamental rack of the link (see [17] for more information) is given by labelling each arc of D by a generator and reading off a relation among the generators at each crossing. The appropriate relation may be seen in Figure 5.4. Note that the fundamental rack of a virtual link depends only on the classical link crossings, that is, it is independent of the virtual crossings.

5.3 From Codes to Racks and Back Again

5.3.1 From Gauss Code to Fundamental Rack

In the following, we only consider link diagrams which contain no disjoint closed loops. The operation of adding a disjoint closed loop to a link diagram is undetectable by the Gauss code, but it alters the presentation of the fundamental rack of the link obtained from the diagram by adding an extra generator.

Proposition 5.3.1 *A signed Gauss code for an oriented link L gives rise to the fundamental rack of L without the need for a diagram.*

Proof: First of all we suppose that we have a diagram in order to describe the code and rack appropriately. Then we will give the algorithm to obtain the rack from the code, which does not require the use of a diagram.

Given a diagram D, with no disjoint closed loops, of an oriented link L, there is a bijection between the set of arcs of D and the set of crossings of D, which is given by mapping each arc to the crossing at its head. Without loss of generality we assume that the signed Gauss code has crossing numbers such that the head of the i-th arc is involved in crossing number i. Otherwise we renumber the code accordingly; that is, we renumber so that $u1, u2, \dots, un$ appear in order, where n is the maximal number appearing in the code.

<u>Case 1:</u> Suppose that L has one component.

To obtain $\Gamma(L)$ from D we take one generator for each arc; call them $g_1, \dots, g_n$. Then the head of the arc labelled g_i has crossing number i in the Gauss code. So the approaching underarc at crossing i is labelled by g_i and the leaving underarc is labelled by g_{i+1}. In order to obtain the relation in $\Gamma(L)$ given by crossing i we need to know the labelling on the overarc. As seen in Figure 5.5 this is given by g_j, where j is the crossing number associated to the first *underarc* to the right of oi in the Gauss code. Remember that we allow cyclic permutations of the components of the code, and if there are no u's in the component then we had a diagram with no crossings! If the sign of oi is $+$ (respectively $-$) then the i-th relation in $\Gamma(L)$ is given by $g_i^{g_j} = g_{i+1}$ (respectively $g_i^{\overline{g_j}} = g_{i+1}$).

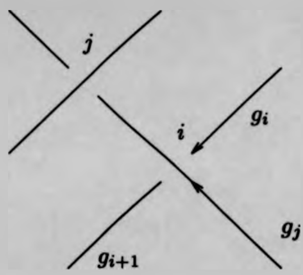

Figure 5.5: The i-th relation

Explicitly, given a Gauss code the algorithm to obtain $\Gamma(L) = [g_1, \dots, g_n : r_1, \dots, r_n]$ is:

- Change the numbers of the Gauss code that $u1, \dots, un$ appear in that order. Here n is the maximal number appearing in the Gauss code and corresponds to the number of generators of $\Gamma(L)$ taken.
- For each i, start at oi and find uj, the next occurrence of u to the right of oi, in that component of the code.
- The relation r_i is then given by $g_i^{g_j} = g_{i+1}$ if the sign of oi (or, equivalently ui) is $+$ or by $g_i^{\overline{g_j}} = g_{i+1}$ if the sign is $-$.

<u>Case 2:</u> Now suppose that L has more than one component.

The components are ordered. Apply the procedure above, taking a new generator for each component. In this case, if there are no u's in a component of the code then we have an unknotted component above the rest of the diagram (corresponding to a list of only o's in a component of the code). We assign an extra generator, g_k say, to such an unknotted component, and use this in the relevant relations. That is, if i is a number appearing in a component of the code which contains no u's then relation r_i is given by $g_i^{g_k} = g_{i+1}$ or $g_i^{\overline{g_k}} = g_{i+1}$, depending on the sign as usual.

□

For example, a two component link gives rise to a presentation of the form

$$\Gamma(L) = [g_1, \dots, g_n, h_1, \dots, h_m : r_1, \dots, r_n, s_1, \dots, s_m]$$

where, if we read g_1 for g_{n+1} and h_1 for h_{m+1} then $g_i^{f_i} = g_{i+1}$ and $h_i^{e_i} = h_{i+1}$ for all i, and $f_i, e_i \in \{g_1, \ldots, g_n, h_1, \ldots, h_m, \overline{g_1}, \ldots, \overline{g_n}, \overline{h_1}, \ldots, \overline{h_m}\}$.

A presentation of this form is called **standard**, as are presentations of the generalisation to links with a different number of components.

Corollary 5.3.2 *Let L_1 and L_2 be links (possibly virtual) whose Gauss codes differ only by a permutation of a sequence of consecutive o's. Then $\Gamma(L_1) \cong \Gamma(L_2)$.*

Proof: The order of consecutive o's in the code does not effect the algorithm given in Proposition 5.3.1. □

5.3.2 From Racks to Gauss Codes

In the following we only consider the one component case, but it easily generalizes. Note that for a link containing no unknots, the number of components of the Gauss code is equal to the number of cycles of relations in $\Gamma(L)$.

Proposition 5.3.3 *Let X be a rack with presentation given by*

$$X = [g_1, \ldots, g_n \mid g_i^{f_i} = g_{i+1} \text{ for } 1 \leq i \leq n]$$

where we read 1 for $n+1$, and $f_i \in \{g_1, \ldots, g_n, \overline{g_1}, \ldots, \overline{g_n}\}$ for each i.

Then we can construct all virtual links which have a representative diagram having X as presentation of its fundamental rack by a simple algorithm.

Proof: The i-th relation of X is $r_i = \{g_i^{f_i} = g_{i+1}\}$. This relates to the ui in the code. The algorithm to obtain a Gauss code from X is as follows:

- Write down $u1, \ldots, un$ in order.
- The sign of ui (and therefore of oi) is $-$ if and only if $f_i \in \{\overline{g_1}, \ldots, \overline{g_n}\}$.
- Place oi between $u(j-1)$ and uj, where $f_i \in \{g_j, \overline{g_j}\}$.
- If there is more than one o between consecutive u's then place the o's in any order.

Applying the algorithm to obtain the fundamental rack from the code gives us X again. □

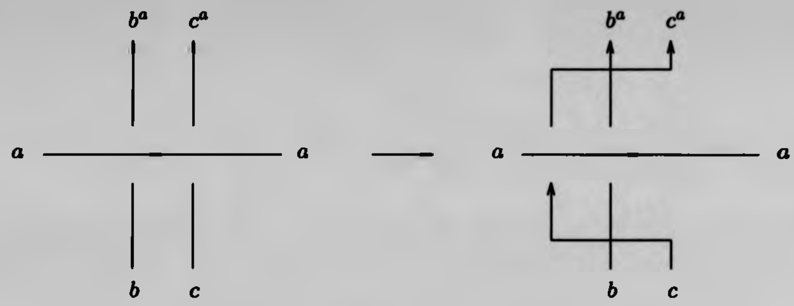

Figure 5.6: Transposition of two consecutive o's.

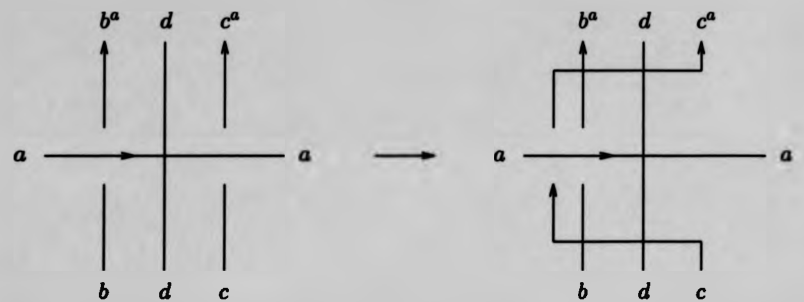

Figure 5.7: Transposition of two consecutive o's with a virtual crossing in between them.

5.3.3 Applications

Corollary 5.3.4 (to 5.3.1 and 5.3.3) *Let L be a classical link with an alternating diagram D. Let α denote the Gauss code of D, in standard form, and let $\Gamma(L)$ denote the standard presentation of the fundamental rack of L. Then the map $\alpha \mapsto \Gamma(L)$ defined in Proposition 5.3.1 is invertible with inverse given in Proposition 5.3.3.*

Proof: Since D is alternating, there are no consecutive o's in its Gauss code. □

A transposition of two consecutive o's can be realised, in the virtual category, by replacing the relevant part of the diagram as shown in Figure 5.6. The obvious replacement is made for diagrams which contain virtual crossings in between the two consecutive o's which are to be transposed. See Figure 5.7 for example. It is clear that this operation is independent of orientations of the arcs in the diagrams.

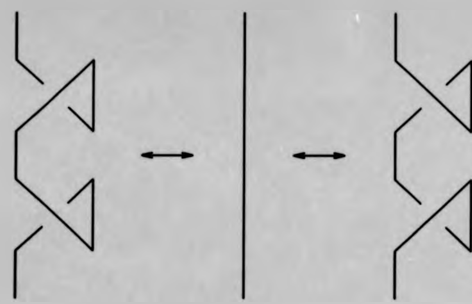

Figure 5.8: The double R1)-move

Let L be a link and let D be a diagram for L. The **mirror** of L, denoted by mL, is given by reversing the orientation of space. A diagram for mL is obtained by switching all over and under crossings in D. The **reverse** of L, denoted by rL, is given by reversing the orientation of the link itself.

As in [17] we consider an isotopy class of framed classical links to be an equivalence class of classical link diagrams under planar isotopy, R2)-moves, R3)-moves and the double R1)-move, which is shown in Figure 5.8. In the same manner, we consider an isotopy class of framed virtual links to be an equivalence class of virtual link diagrams under planar isotopy, R2)-moves, R3)-moves, the double R1)-move, and the virtual Reidemeister moves.

Proposition 5.3.5 *Let L be a non-split, framed classical link in S^3 and let α be a Gauss code for L in standard form. Suppose that the code of the framed virtual link L' is obtained from α by a permutation of consecutive o's. If L' is isotopic to a classical link K then either $K = L$ or $K = rmL$.*

Proof: Any virtual link L' whose code differs from α by a permutation of consecutive o's has $\Gamma(L) = \Gamma(L')$ since their classical crossings are the same. In [17] it was shown that for non-split, framed classical links the fundamental rack is a complete invariant up to the reverse mirror operation. If L' is isotopic to K then we have two framed classical links L and K with isomorphic fundamental racks. So either $K = L$ or $K = rmL$. □

5.4 Reverse and Mirror Operations

5.4.1 The Reverse and Mirror of Codes

Let α be a Gauss code in standard form which has maximal crossing number n. The **mirror** of α, denoted by $m\alpha$, in standard form, is obtained via the following algorithm:

- Switch all the o's and all the u's.
- Change all the signs in the code.
- Apply the permutation which takes the ui's in $m\alpha$ to $u1, \ldots, un$ in order.

Note that the last step, in this and the next algorithm, is only necessary if we wish to have the code in standard form.

Let $\pi \in S_n$ be the permutation which reverses the set of crossing numbers. That is,

$$\pi = \begin{cases} (1,2k)(2,2k-1)\ldots(k,k+1) & \text{if} \quad n = 2k \\ (1,2k-1)(2,2k-2)\ldots(k-1,k+1) & \text{if} \quad n = 2k-1 \end{cases} \quad \text{for some } k \in \mathbb{N}.$$

The following algorithm defines $r\alpha$, the **reverse** of α:

- Reverse the order of the code. Notice that this reverses the order of the components.
- Apply the permutation π to the crossing numbers.

Lemma 5.4.1 *Let α be the Gauss code of a link L. Then $m\alpha$ is a Gauss code for mL, and $r\alpha$ is a Gauss code for rL.* □

Corollary 5.4.2 (to 5.3.4) *Let L be an alternating classical link. Given $\Gamma(L)$ in standard form we can obtain $\Gamma(mL)$, $\Gamma(rL)$ and $\Gamma(rmL)$ in standard form without the need for a diagram.*

Proof: Since $\Gamma(L)$ gives rise to a code α, without the need for a diagram, we can apply the operations m and r to α, and then we can obtain $\Gamma(mL)$, $\Gamma(rL)$, and $\Gamma(rmL)$ as usual. □

5.4.2 The Reverse Mirror of a Code

In this section we give a simple algorithm to compute the code arising from the reverse mirror operation. As an interesting application we give a condition for an alternating classical knot diagram to have the same code as its reverse mirror, except for a sign change, generalising the following observation.

Example

Applying the reverse mirror operation to the code of the usual diagram of the Left Trefoil knot yields the same code, except for a change of all of the signs.

We have $\alpha = o2 - u1 - o3 - u2 - o1 - u3-$. Then the algorithms in Section 5.4.1 give $m\alpha = u1 + o3 + u2 + o1 + u3 + o2+$, $r\alpha = u1 - o3 - u2 - o1 - u3 - o2-$, and $rm\alpha = o2 + u1 + o3 + u2 + o1 + u3+$.

Let α be a Gauss code in standard form. Define $\mathbf{O}$ to be the ordered set of numbers associated to the o's and similarly define $\mathbf{U}$ for the u's. Let r be the operation which reverses such a set of numbers. Note that the notation r is also used for the operation that reverses a code. Finally we define ϕ_o to be the permutation which takes $\mathbf{O}$ to $(n, n-1, \ldots, 1)$, that is, it makes the o's descending.

Proposition 5.4.3 *Let α be a Gauss code in standard form. The following is a simple algorithm to compute $rm\alpha$ in standard form:*

- *Reverse the order of the code, replacing all of the o/u's by u/o's and switching all of the signs. Remove all the crossing numbers.*
- *Place $1, 2, \ldots, n$ next to the u's in order.*
- *Calculate $r\phi_o(\mathbf{U}) = r\phi_o(1, 2, \ldots, n)$. This gives the order of the crossing numbers associated to the o's.*

Proof: First note that the operations of changing all of the o/u's, of changing all signs, of reversing the code, and of applying a crossing number permutation can be done in any order.

In order for $rm\alpha$ to be in standard form the u's must appear in ascending order. These u's in $rm\alpha$ are obtained from the o's in α. The permutation ϕ_o takes the o's of α and writes them in descending order, which becomes ascending order on applying the reversing operation r. So applying $r\phi_o$ to the o's in α makes the u's in $rm\alpha$ appear in ascending order.

Since this is a crossing number permutation we also need to apply $r\phi_o$ to the u's in α so that we can obtain the order of the o's in $rm\alpha$. The u's in α were in ascending order since α is in standard form, so the order of the o's in $rm\alpha$ is given by $r\phi_o(\mathbf{U}) = r\phi_o(1, 2, \ldots, n)$. □

5.4.3 Application

Let $-\alpha$ denote the code α with all its signs changed.

Proposition 5.4.4 *Let α be the code of an alternating classical knot diagram, D, in standard form. Then $rm\alpha = -\alpha$ if and only if $r\phi_o(\mathbf{U}) = \mathbf{O}$.*

Proof: Since D is alternating, the o's and the u's in $rm\alpha$ are in the same places as in α. The numbers associated to the u's are the same since α and $rm\alpha$ are in standard form. Finally, $r\phi_o(\mathbf{U}) = \mathbf{O}$ says precisely that the order of the o's is the same in α as in $rm\alpha$. □

Proposition 5.4.5 *Let K be an alternating classical knot with alternating diagram D in closed braid form. Then the code of D satisfies the condition $r\phi_o(\mathbf{U}) = \mathbf{O}$.*

Proof: Let β be the braid diagram whose closure is D. The reverse mirror of β is given by reflection in the line through its endpoints (the points of β at level 1). Starting at the image of the start point of β it is clear that, in the closure, we meet every (reflected) crossing in the same order as in D with opposite sign. Thus $rm\alpha = -\alpha$ and Proposition 5.4.4 gives the result. □

Corollary 5.4.6 *Suppose α is the code of an alternating knot diagram which does not satisfy the condition $r\phi_o(\mathbf{U}) = \mathbf{O}$. Then the diagram is not in closed braid form.* □

5.4.4 Examples of the Reverse Mirror Algorithm

Example 1

Firstly we consider the left trefoil knot, 3_1, with code $o2-u1-o3-u2-o1-u3-$. We have $\mathbf{O}=(2,3,1)$, $\mathbf{U}=(1,2,3)$ and the permutation $\phi_o=\begin{pmatrix}2&3&1\\3&2&1\end{pmatrix}$. Therefore, computing gives: $r\phi_o(\mathbf{U})=r\phi_o(1,2,3)=r(1,3,2)=(2,3,1)=\mathbf{O}$. So the code for $rm3_1$ is given by $o2+u1+o3+u2+o1+u3+$, and $rm\alpha=-\alpha$ in this case.

Example 2

The code $u1+o5+u2+o4+u3+o7+u4+o2+u5+o1+u6+o3+u7+o6+$ describes the knot 7_2. We have $\mathbf{O}=(5,4,7,2,1,3,6)$, $\mathbf{U}=(1,2,3,4,5,6,7)$ and the permutation $\phi_o=\begin{pmatrix}5&4&7&2&1&3&6\\7&6&5&4&3&2&1\end{pmatrix}$. Thus $r\phi_o(\mathbf{U})=r\phi_o(1,2,3,4,5,6,7)=r(3,4,2,6,7,1,5)=(5,1,7,6,2,4,3)\neq\mathbf{O}$. So $u1-o5-u2-o1-u3-o7-u4-o6-u5-o2-u6-o4-u7-o3-$ is the code for $rm7_2$ and $rm\alpha\neq-\alpha$ in this case.

Remark 5.4.7 *For the knots in Rolfsen's table [37] with up to seven crossings, this condition holds for the knots $3_1,4_1,5_1$ and 7_1. It also holds for 8_{18} which is in closed braid form. Note that the hypothesis of Proposition 5.4.5 is not neccesary for the condition to hold; the condition holds for the knot 4_1 which is not in closed braid form.*

5.5 Smoothing and Crossing Changes

5.5.1 The Crossing Change of Codes and Racks

Given a link diagram, the **crossing change operation** changes an overarc to an underarc, or vice versa. Let α be a signed Gauss code in standard form, and suppose that k is a crossing number of α. Define $c_k(\alpha)$, the **k-th crossing change** of α, to be the code obtained by:

1. Changing the o and u to u and o, respectively, and the $+$ or $-$ to $-$ or $+$, respectively, for the crossing number k in the code.

2. Relabelling to obtain $c_k(\alpha)$ in standard form.

Lemma 5.5.1 *Let D be a link diagram and let α be a signed Gauss code of D. Then $c_k(\alpha)$ is a signed Gauss code of the diagram obtained from D by applying the crossing change operation to crossing number k.* □

Having changed the code we can now compute a presentation of the fundamental rack of the new link obtained by the crossing change. However, we cannot always compute the new fundamental rack from the old one without more information (since the presentation of the rack gives rise to a whole class of Gauss codes whose links are not necessarily isotopic). The best we can do is:

Proposition 5.5.2 *Let $k_1, \ldots, k_r$ be the (unordered) collection of crossing numbers which would correspond precisely to the crossing numbers of the o's strictly between ok and the next undercrossing in the Gauss code. Given a standard presentation of $\Gamma(L)$ together with $k_1, \ldots, k_r$ we can obtain the standard presentation of the fundamental rack of the link $c_k(L)$, which is the link obtained by changing crossing k in the diagram of L.*

Proof: Refer to Figure 5.9 whilst reading the algorithm below. Without loss of generality, suppose that L has two components and we have its presentation in standard form:

$$\Gamma(L) = [g_1, \ldots, g_n, h_1, \ldots, h_m : r_1, \ldots, r_n, s_1, \ldots, s_m]$$

where, if we read g_1 for g_{n+1} and h_1 for h_{m+1} then $g_i^{f_i} = g_{i+1}$ and $h_i^{e_i} = h_{i+1}$ for all i, and $f_i, e_i \in \{g_1, \ldots, g_n, h_1, \ldots, h_m, \overline{g_1}, \ldots, \overline{g_n}, \overline{h_1}, \ldots, \overline{h_m}\}$. Also, let $p_i \in \{g_i, h_i\}$.

For simplicity of notation we will assume that the signs of the crossings are all positive. We will indicate the places in which the sign changes. A sign change occurs in precisely the same places for negative crossings.

As indicated in the figure, the arc on the overcrossing at k is labelled by g_l; thus the first undercrossing in the Gauss code after the overcrossing at k would be called l.

1. Introduce a new generator g_{n+1}.

   The arc labelled g_l is split into two arcs by the crossing change; the one after crossing k will be labelled by g_{n+1}.

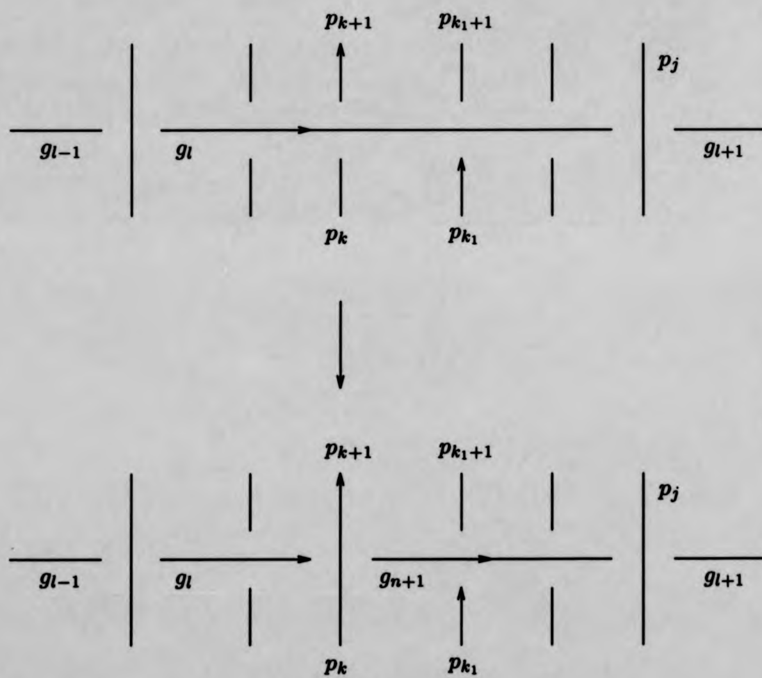

Figure 5.9: Crossing change k.

2. The relations $p_{k_t}^{g_l} = p_{k_t+1}$ are replaced by the relations $p_{k_t}^{g_{n+1}} = p_{k_t+1}$.
   All of the arcs which both passed under the arc labelled g_l and occured after the crossing k now pass under the arc labelled g_{n+1}; these crossings would correspond to the consecutive o's between ok and ul in the Gauss code.

3. The relation $g_l^{p_j} = g_{l+1}$ becomes $g_{n+1}^{p_j} = g_{l+1}$.
   This is the final crossing which involved g_l and now involves g_{n+1}.

4. The relation $p_k^{g_l} = p_{k+1}$ becomes the relation $g_l^{\overline{p_k}} = g_{n+1}$
   (and $p_k^{\overline{g_l}} = p_{k+1}$ would become $g_l^{p_k} = g_{n+1}$).
   This is the crossing change.

5. The final step (of relabelling) is dependent on the number of components involved in crossing k.

   - Case 1: Two components.
     Map the relevant generators and relations one step along in the following chain:
     $$g_{n+1} \mapsto g_{l+1} \mapsto g_{l+2} \mapsto \ldots \mapsto g_{n-1} \mapsto g_n \mapsto g_{n+1}$$
     and for the other component $h_m \mapsto h_{m-1} \mapsto \ldots \mapsto h_{k+1} \mapsto h_k$.
     Note that we replaced the p's by h's since we know there are two components involved in crossing k.
   - Case 2: One component.
     Map the relevant generators and relations one step along in the following chain:
     $$g_{n+1} \mapsto g_{l+1} \mapsto g_{l+2} \mapsto \ldots \mapsto g_{k-1} \mapsto g_k \mapsto g_{k+1}$$
     where $g_n \mapsto g_1$ if it occurs in the above sequence of g's.

□

5.5.2 The Smoothing of Codes and Racks

Let α be a signed Gauss code in standard form. We define $s_k(\alpha)$, the k-**th smoothing** of α. See Figure 5.12 for the representative diagram. Let $\beta, \gamma, \delta,$ and ϵ represent the relevant parts of the code α. We fix the signs in the code, and ignore them in the description. There are two cases depending on the number of components involved in crossing k.

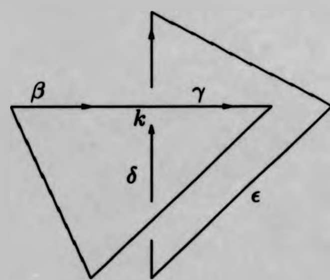

Figure 5.10: Two components involved in a smoothing

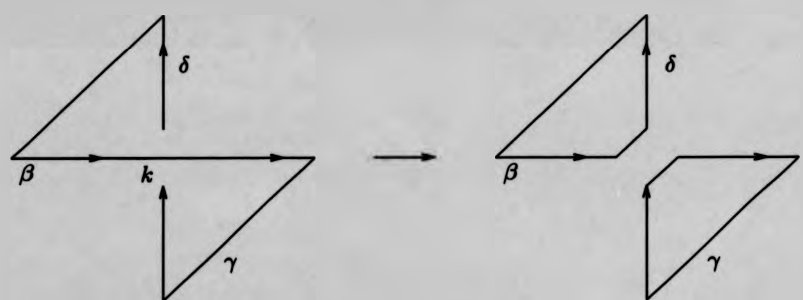

Figure 5.11: One component involved in a smoothing

- Case 1: Two components.
  See Figure 5.10. Suppose that $\alpha = \beta \; ok \; \gamma/\delta \; uk \; \epsilon$. Then let $s_k(\alpha) = \beta \; \epsilon \; \delta \; \gamma$.
- Case 2: One component.
  See Figure 5.11. Suppose that $\alpha = \beta \; ok \; \gamma \; uk \; \delta$. Then let $s_k(\alpha) = \delta \; \beta/\gamma$.

We require the same extra information as for the crossing change case in order to compute the presentation of the fundamental rack of the smoothed link in standard form from the presentation of the fundamental rack of the original link in standard form. We have:

Proposition 5.5.3 *Let $k_1, \ldots, k_r$ be the (unordered) collection of crossing numbers which would correspond precisely to the crossing numbers of the o's strictly between ok and the*

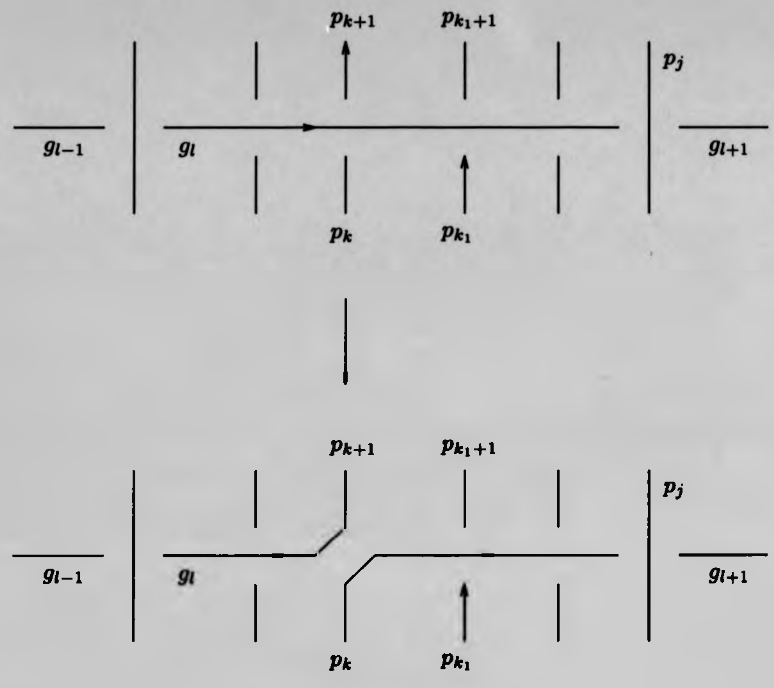

Figure 5.12: Smoothing crossing k.

next undercrossing in the Gauss code. Given a standard presentation of $\Gamma(L)$ *together with* $k_1, \ldots, k_r$ *we can obtain the standard presentation of the fundamental rack of the link* $s_k(L)$*, which is the link obtained by smoothing crossing* k *in the diagram of* L*.*

Proof: Refer to Figure 5.12 whilst reading the algorithm below. Without loss of generality, suppose that L has two components and we have its presentation in standard form:

$$\Gamma(L) = [g_1, \ldots, g_n, h_1, \ldots, h_m : r_1, \ldots, r_n, s_1, \ldots, s_m]$$

where, if we read g_1 for g_{n+1} and h_1 for h_{m+1} then $g_i^{f_i} = g_{i+1}$ and $h_i^{e_i} = h_{i+1}$ for all i, and $f_i, e_i \in \{g_1, \ldots, g_n, h_1, \ldots, h_m, \overline{g_1}, \ldots, \overline{g_n}, \overline{h_1}, \ldots, \overline{h_m}\}$. Also, let $p_i \in \{g_i, h_i\}$.

For simplicity of notation we will assume that the signs of the crossings are all positive. We will indicate the places in which the sign changes. A sign change occurs in precisely the same places for negative crossings.

As indicated in the figure, the arc on the overcrossing at k is labelled by g_l; thus the first undercrossing in the Gauss code after the overcrossing at k would be called l.

1. The relations $p_{k_t}^{g_l} = p_{k_t+1}$ are replaced by the relations $p_{k_t}^{p_k} = p_{k_t+1}$.
2. The relation $g_l^{p_j} = g_{l+1}$ becomes $p_k^{p_j} = g_{l+1}$.
3. The relation $p_k^{g_l} = p_{k+1}$ is removed.
4. The final step (of relabelling) is dependent on the number of components involved in crossing k.

   - Case 1: Two components.

     This becomes a single component after smoothing. We insert the generators of the second component into the relevant place. Formally we map the relevant generators and relations as follows:

     Fix $g_1, \dots, g_l$. Then map $h_{k+1} \mapsto g_l, h_{k+2} \mapsto g_{l+1}, \dots, h_m \mapsto g_{l+(m-k-1)}$, $h_1 \mapsto g_{l+(m-k)}, \dots, h_k \mapsto g_{l+m-1}$, and $g_{l+1} \mapsto g_{l+m}, \dots, g_n \mapsto g_{n+m-1}$.

   - Case 2: One component.

     This becomes two components after smoothing.

     First suppose that $1 \leq l < k \leq n$. Fix $g_1, \dots, g_{l-1}$.

     If $k = n$ then map $g_l \mapsto g_1$, else

     if $k \neq n$ then map $g_{k+1} \mapsto g_l, g_{k+2} \mapsto g_{l+1}, \dots, g_n \mapsto g_{l+(n-k-1)}$.

     Finally map $g_k \mapsto h_1, g_{l+1} \mapsto h_2, g_{l+2} \mapsto h_3, \dots, g_{k-1} \mapsto h_{k-l}$.

     Now suppose that $1 \leq k < l \leq n$. Fix $g_1, \dots, g_k$.

     If $l \neq n$ then map $g_{l+1} \mapsto g_{k+1}, \dots, g_n \mapsto g_{n+k-l}$.

     Finally, map $g_l \mapsto h_1, g_{k+1} \mapsto h_1, g_{k+2} \mapsto h_2, \dots, g_{l-1} \mapsto h_{l-k-1}$.

   Note that the number of generators remaining after smoothing is reduced by one. This leaves $n + m - 1$ generators remaining in the first case, and $n - 1$ in the second case.

□

Remark 5.5.4 (Connection with Part I) *Let L be a link and let L' be obtained from L by some operation, such as mirroring, reversing, reverse mirroring, crossing changing* *or smoothing. If $B\Gamma(L) \cong B\Gamma(L')$, then we can, algorithmically, see the change in the canonical class determined by the link diagram. Corollary 4.4.4 in Chapter 4 shows that $B\Gamma(L) \cong B\Gamma(mL) \cong B\Gamma(rL) \cong B\Gamma(rmL)$.*

Chapter 6

The Jones Polynomial of Welded Links

6.1 Introduction

Given a link, L, [17] shows how to obtain its fundamental rack, $\Gamma(L)$, in standard form, from a diagram for L. In Chapter 5 we showed how to associate a class of Gauss codes to $\Gamma(L)$ in standard form. Two codes in this class are equivalent if they differ by permutations of consecutive o's. Recall that a Gauss code corresponds to a virtual link. A welded link can be thought of as the closure of a welded braid, defined in [15]; note that, as in the case of classical links and braids, the welded analogue of an R1)-move holds for welded links, but is not defined for welded braids. We show, in Section 6.2, that the extra move on virtual link diagrams, corresponding to a permutation of consecutive o's in the code, is equivalent to the extra isotopy relation required for welded links. Thus a welded link corresponds to such a class of code. In fact, extending the class to allow the analogue of isotopy, we have the following are equivalent:

- Virtual links which have a representative diagram having the standard presentation of $\Gamma(L)$.
- Virtual links which differ by the introduction or removal of pairs of virtual crossings

seperated by an overarc.

- The class of signed Gauss codes whose members differ by the analogue of isotopy for codes or by a permutation of consecutive o's.
- Welded links.

The Jones polynomial [28] is an invariant of oriented (classical) links, which may be obtained via the state sum invariant of unoriented (classical) links, called the Bracket polynomial [30]. These polynomials are well defined for virtual links and one can apply the usual method of computation to all of the classical crossings in a virtual link diagram, leaving a number of diagrams containing circles with only virtual crossings, which can then be removed.

Let L be a link with diagram D. Given P, a local part of D containing m classical crossings, together with the information of how the ends of P are joined up after smoothing all of the crossings in the complement of P, we can compute the factor that P contributes to the corresponding 2^m states of the Bracket polynomial. This is because computing the Bracket polynomial of P gives rise to a number of terms, where the bracket part of each term consists of some arcs (with no crossings) which are parts of some circles after smoothing all of the classical crossings in the complement of P (i.e. fixing a particular state of the complement of P). Then, joining the ends in the given manner allows us to calculate the contribution of P to the 2^m term summand of the polynomial, since any extra circles added occur in all of the terms coming from P. The Bracket polynomial of L can be written as a certain sum over all possible states of $D - P$. For each state of $D - P$ we can compute a factor of the corresponding state summand.

We take P to consist of an arc overcrossing two other arcs. A computation of the fifteen possible ways to join up the ends of the arcs gives rise to all of the possible factors obtainable from P. Also consider P_v, which is P together with the introduction of two virtual crossings, each of which joins one pair of underarcs, one on each side of the overcrossing. The factors of P_v are just the factors of P after a suitable permutation of the joining of the endpoints. We compare each factor of a summand of the polynomial of the link containing P with the corresponding factor of a summand of the polynomial of the link containing P_v. The factors are either the same or they differ by a use of one of the relations $A^6 + A^{-4} - A^2 - 1 = 0$ or

$A^{-6} + A^4 - A^2 - 1 = 0$. Let I be the ideal generated by $A^6 + A^{-4} - A^2 - 1$. Note that I contains the ideal generated by $A^{-6} + A^4 - A^2 - 1$.

Thus, if we consider each state's summand in the quotient ring $\mathbb{Z}[A, A^{-1}]/I$ then this summand does not change under the introduction (or removal) of two new virtual crossings, which corresponds to a permutation of consecutive o's in the code. The usual Bracket polynomial is a sum over all states of these summands, where the summands are Laurent polynomials. We define the Welded Bracket polynomial to be the sum over all states of these summands, where the summands live in the quotient ring. Equivalently, the Welded Bracket polynomial of a welded link, L, is given by considering a diagram for L as a virtual link diagram, computing its usual Bracket polynomial and then obtaining its image in the quotient ring.

Let D be a diagram of a welded link L. We have defined the Welded Bracket polynomial, $\langle D\rangle^*(A) \in \mathbb{Z}[A, A^{-1}]/I$, which is a regular, welded isotopy invariant of welded links. This gives rise to a welded isotopy invariant of welded links, called the Welded f-polynomial, $\mathcal{F}_L(A) = (-A)^{-3w(D)}\langle D\rangle^*(A)$, where $w(D)$ is the writhe of D. Finally, the Welded Jones polynomial, $\mathcal{V}_L(t)$, is a welded isotopy invariant given by the change of variable $A = t^{-\frac{1}{4}}$. These new invariants are nontrivial; for example, we show that the unlink with two components, the Left Hopf link and the Left Hopf link with one of its classical crossings replaced with a weld all have different Welded Jones polynomials.

In [29] Jones notices that for a classical knot, K, the Jones polynomial, $V_K(t)$, is a Laurent polynomial in t, and satisfies the relation: $1 - V_K(t) = (1-t)(1-t^3)W_K(t)$, where $W_K(t)$ is some Laurent polynomial in t. He ignores this so called "extraneous information" and records the values of $W_K(t)$ in his table. Jones also states that the Jones polynomial of a classical link is either a Laurent polynomial in t or $t^{\frac{1}{2}}$ times a Laurent polynomial in t. Neither of these theorems are true for virtual knots or links; in fact, we give examples of virtual links with any number of components whose Jones polynomial is neither a Laurent polynomial in t nor $t^{\frac{1}{2}}$ times a Laurent polynomial in t.

We show that it is precisely this "extraneous information" that causes the Welded Jones polynomial, $\mathcal{V}_K(t)$, to be equal to one for all classical knots. We define a nontrivial, welded isotopy invariant of classical knots called the Welded W-polynomial, $\mathcal{W}_K(t)$. Recall that the

Jones polynomial of the connected sum of two classical knots, K_1 and K_2, is the *product* of the individual Jones polynomials; that is $V_{K_1 \# K_2}(t) = V_{K_1}(t)V_{K_2}(t)$. It is amusing to note that the Welded W-polynomial of the connected sum of two classical knots, K_1 and K_2, is the *sum* of the individual Welded W-polynomials; that is $\mathcal{W}_{K_1 \# K_2}(t) = \mathcal{W}_{K_1}(t) + \mathcal{W}_{K_2}(t)$. Thus the connected sum of n copies of the Right Trefoil knot has $\mathcal{W}$-polynomial equal to n, for instance.

6.2 Welded Links

Welded links/braids are virtual links/braids modulo the extra isotopy relation which involves passing a strand over a weld, as shown in Figure 6.1. Notice that the term **weld** is now used instead of virtual crossing. Refer to [15] for an algebraic definition of welded braids.

Figure 6.1: Extra isotopy move

Proposition 6.2.1 *The operation of permuting two consecutive o's in the signed Gauss code is equivalent to the use of this extra isotopy move.*

Proof: Figure 6.2 shows the introduction or removal of two virtual crossings either side of an arc which crosses over two others. This is equivalent to the permutation of two consecutive o's in the code, and is denoted by (II) here. Let the extra isotopy move be denoted by (I). We show in Figure 6.3 how to obtain (II) using (I) and virtual isotopy, and Figure 6.4 shows how to obtain (I) from (II) and virtual isotopy. □

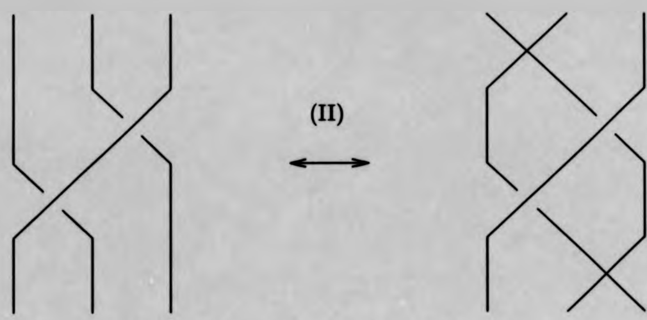

Figure 6.2: Permuting consecutive o's

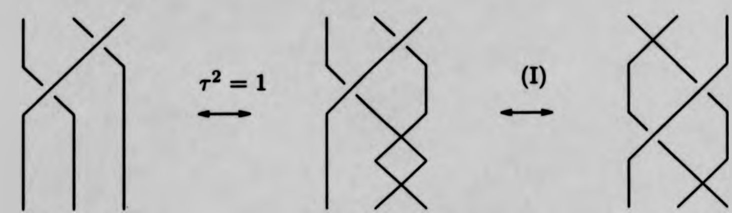

Figure 6.3: From (I) to (II)

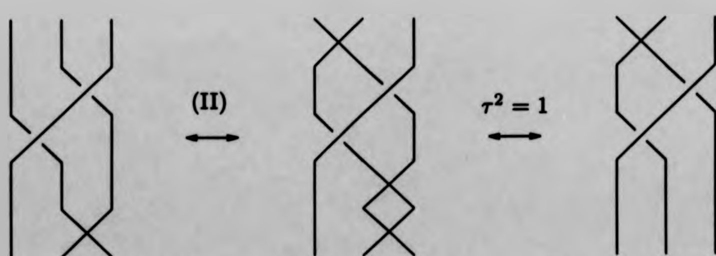

Figure 6.4: From (II) to (I)

6.3 Computing the Polynomials

6.3.1 The Bracket Polynomial

Let L be an oriented link and let D denote a diagram of L. Recall [30] that the Bracket polynomial, $\langle D\rangle \in \mathbb{Z}[A, A^{-1}]$ is a regular isotopy invariant of the unoriented version of L. A method of computation is to smooth all of the classical crossings in the diagram in the manner shown in Figure 6.5; note that we multiply by the coefficient A if we smooth the approaching underarcs rightwards, and by A^{-1} if we smooth leftwards. The value of the unknot is 1, and after smoothing, each further circle give rise to a factor of $d = -A^2 - A^{-2}$.

A **state** S of D is a choice of smoothing of all of the crossings in D; this may be referred to as a 'choice of splitting markers'. Let $\langle D|S\rangle$ denote the power of A obtained as a coefficient by this choice of smoothing; that is, each smoothing of a single classical crossing contributes a single power of A or A^{-1} corresponding to whether the crossing was smoothed 'rightwards' or 'leftwards'. Finally, let s denote the number of circles in the diagram after the smoothing of state S of D. Then, the state sum form of the Bracket polynomial is given by:

$$\langle D\rangle = \sum_{\text{states } S} \langle D|S\rangle d^{s-1}.$$

Note that in the case of virtual links, we simply smooth all the classical crossings as usual, leaving a number of circles, possibly joined by some virtual crossings which can be isotoped away leaving disjoint circles, as required. A proof of invariance of the Bracket polynomial under the required virtual version of the Reidemeister moves can be found in [31].

Let L_+, L_- and L_0 denote oriented links which are identical outside a small region, and have a single positive, negative or oriented smoothed crossing inside the region, respectively. A simple calculation yields the **Skein relation:**

$$A\langle L_+\rangle - A^{-1}\langle L_-\rangle = (A^2 - A^{-2})\langle L_0\rangle. \tag{i}$$

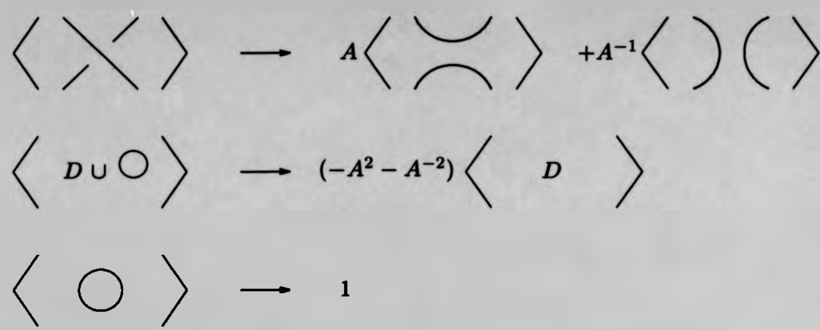

Figure 6.5: Defining the Bracket polynomial

6.3.2 The Jones Polynomial

Let L be an oriented link with diagram D. Recall that the **writhe** of D, denoted $w(D)$, is the number of positive crossings minus the number of negative crossings in D. The **f-polynomial** [31] is an isotopy invariant, and $f_L(A)$ can be obtained from $\langle D \rangle$ by multiplying by the factor $(-A)^{-3w(D)}$. Setting $A = t^{-\frac{1}{4}}$ in the f-polynomial gives rise to the **Jones polynomial**, $V_L(t) \in \mathbf{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. Thus, the Skein relations for the f-polynomial and the Jones polynomial are given by:

$$\begin{aligned} A^4 f_{L_+}(A) - A^{-4} f_{L_-}(A) &= (A^{-2} - A^2) f_{L_0}(A) \qquad \text{and} \\ t^{-1} V_{L_+}(t) - t V_{L_-}(t) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) V_{L_0}(t). \end{aligned}$$

6.4 The Welded Bracket Polynomial

6.4.1 Rewriting the Bracket polynomial

Let D be a virtual link diagram with a local part, P, consisting precisely of an arc overcrossing two other arcs, as shown in Figure 6.6. It is clear how the following material generalises to any local part; the choice of P here is for clarity.

Let a **state**, T, of $D - P$ be a choice of splitting markers in the complement of P. As usual, we may refer to the smoothing in a particular state as the state itself. Let P_1, P_2, P_3 and P_4 denote the $2^2 = 4$ states of P where P_1 has both crossings smoothed rightwards and P_4

has both crossings smoothed leftwards; they can be seen in Figure 6.6.

Now, each state S of D may be written uniquely as $S = T \cup P_i$ for some state T of $D - P$ and some state P_i of P, where $\cup$ has the obvious meaning. Let s_i denote the number of circles left in state $T \cup P_i$ after smoothing. We can write:

$$\langle D \rangle = \sum_{\text{states } T \text{ of } D-P} \langle D - P|T \rangle \left(\langle P|P_1 \rangle d^{s_1 - 1} + \ldots + \langle P|P_4 \rangle d^{s_4 - 1} \right).$$

Fix state T of $D - P$. After smoothing, the ends of the arcs of each P_i are joined up in the same manner in each of the four states $T \cup P_i$, because this is determined by T alone. In subsection 6.4.2 we compute t_i, the number of circles in state $T \cup P_i$ which intersect the arcs of P_i. The circles in state $T \cup P_i$ which do *not* intersect P_i arise from state T only and so the number of them, denoted by t, is the same for all of the four states $T \cup P_i$. Note that the number t remains the same if P is replaced by any other local region.

Now we can write:

$$\langle D \rangle = \sum_{\text{states } T \text{ of } D-P} \langle D - P|T \rangle d^t \left(\langle P|P_1 \rangle d^{t_1 - 1} + \ldots + \langle P|P_4 \rangle d^{t_4 - 1} \right).$$

Thus, for our local part P, we have:

$$\langle D \rangle = \sum_{\text{states } T \text{ of } D-P} \langle D - P|T \rangle d^t \left(A^2 d^{t_1 - 1} + d^{t_2 - 1} + d^{t_3 - 1} + A^{-2} d^{t_4 - 1} \right),$$

where t_i is the number of circles obtained from P_i, whose ends are joined up according to the state T of $D - P$.

Note that, for each fixed state T of $D - P$, the term $\langle D - P|T \rangle\, d^t$ is dependent only on T.

Let P_v be P together with the introduction of two virtual crossings, as shown in Figure 6.8. For any fixed state T of $D - P$, we investigate how the term of $\langle D \rangle$ in parenthesis above changes when we replace P by P_v.

6.4.2 Factors of Summands

Let P be the local part of the link diagram considered in Figure 6.6, that is, it consists of one arc passing over two others. Let the notation $[ab]$ mean that we join the end of the arc labelled by a to the end of the arc labelled by b. Figure 6.7 gives an example of joining of the ends of the four terms in the polynomial in the manner $[ab][cf][de]$. The following table gives, for each manner of joining the ends, the number of circles obtained for the four terms in the Bracket polynomial. Note that the columns entitled "Number of circles in P_1" and "Number of circles in P_4" correspond to the 'bracket part' of the terms with coefficients A^2 and A^{-2} respectively.

Joining of ends	$t_1 =$ Number of circles in P_1	$t_2 =$ Number of circles in P_2	$t_3 =$ Number of circles in P_3	$t_4 =$ Number of circles in P_4	$t_1+t_2+t_3+t_4 =$ Total number of circles in P
$[ab]\,[cf]\,[de]$	2	1	3	2	8
$[ab]\,[ce]\,[df]$	1	1	2	2	6
$[ab]\,[cd]\,[ef]$	1	2	2	3	8
$[ac]\,[bd]\,[ef]$	1	2	1	2	6
$[ac]\,[be]\,[df]$	2	1	1	1	5
$[ac]\,[bf]\,[de]$	1	1	2	1	5
$[ad]\,[bc]\,[ef]$	2	3	1	2	8
$[ad]\,[be]\,[cf]$	3	2	2	1	8
$[ad]\,[bf]\,[ce]$	2	2	1	1	6
$[ae]\,[bc]\,[df]$	1	2	1	1	5
$[ae]\,[bd]\,[cf]$	2	1	2	1	6
$[ae]\,[bf]\,[cd]$	1	1	1	2	5
$[af]\,[bc]\,[de]$	1	2	2	1	6
$[af]\,[bd]\,[ce]$	1	1	1	1	4
$[af]\,[be]\,[cd]$	2	1	1	2	6

Let P_v be the local part of the virtual link diagram, obtained from P by adding virtual crossings, one of which involves the points d and e, and the other involves b and c, as shown in Figure 6.8. This figure shows the ammended calculation of the bracket polynomial of P_v.

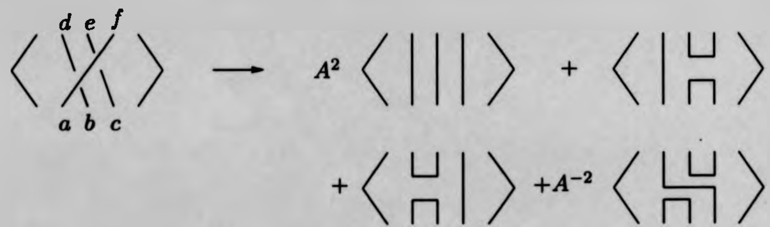

Figure 6.6: The Bracket polynomial of an arc overcrossing two other arcs

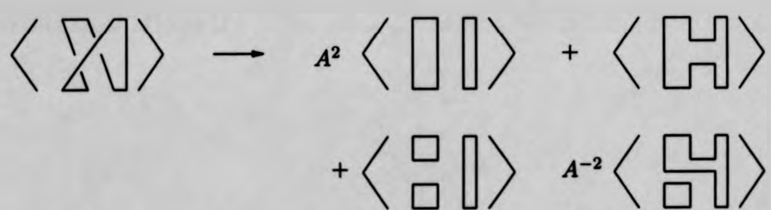

Figure 6.7: Joining the ends in the manner $[ab][cf][de]$

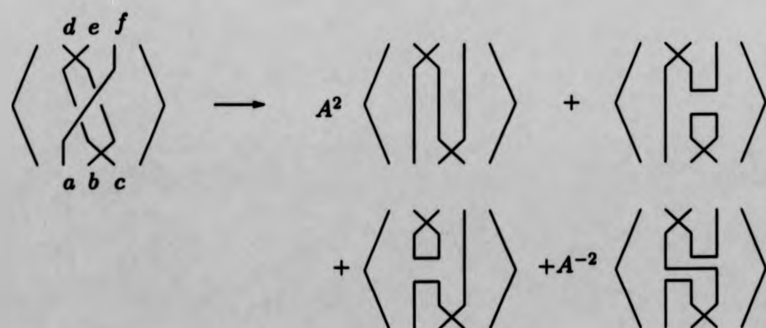

Figure 6.8: The Bracket polynomial of the virtual replacement

Lemma 6.4.1 *The contribution to the 4 term summand of the Bracket polynomial obtained from P_v is the same as that obtained from P, provided that we apply the permutation $(de)(bc)$ to the joining of the ends of P.*

Proof: After smoothing all of the classical crossings in the diagram containing P_v we will be left with a number of circles and two extra virtual crossings. These virtual crossings can then be removed. The number of circles is the same as that obtained from the diagram containing P, after the required permutation, because we are joining up the ends of the given arcs in the same manner. □

The following table gives the result of applying the permutation $(de)(bc)$ to the endpoints of the arcs. Note that the number of circles in the new joining of ends columns are the values for P_v. Also, the permutation $(de)(bc)$ has order 2 and so takes the right hand side of columns back to the left as well; that is, we can interchange all of the labels P and P_v throughout the table. Note that the "Number of circles in P_v" column corresponds to both the original "Joining of ends" of P_v and the "New joining of ends" of P. As an example, Figure 6.9 shows the joining $[ab][cf][de]$ of P_v and Figure 6.10 shows the joining $[ac][bf][de]$ of P. The number of circles in each is the same, and can be found at the top right of the table.

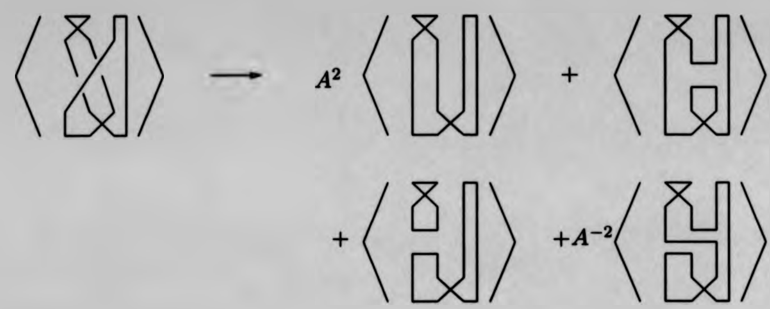

Figure 6.9: Joining the ends of the virtual replacement in the manner $[ab][cf][de]$

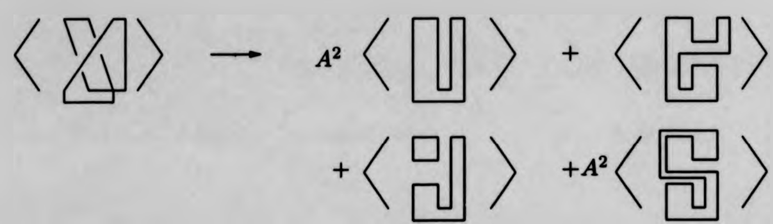

Figure 6.10: Joining the ends in the manner $[ac][bf][de]$

Joining of ends	Number of circles in P	Total number of circles in P	New joining of ends	Number of circles in P_v	Total number of circles in P_v
$[ab]\,[cf]\,[de]$	2 1 3 2	8	$[ac]\,[bf]\,[de]$	1 1 2 1	5
$[ab]\,[ce]\,[df]$	1 1 2 2	6	$[ac]\,[bd]\,[ef]$	1 2 1 2	6
$[ab]\,[cd]\,[ef]$	1 2 2 3	8	$[ac]\,[be]\,[df]$	2 1 1 1	5
$[ad]\,[bc]\,[ef]$	2 3 1 2	8	$[ae]\,[bc]\,[df]$	1 2 1 1	5
$[ad]\,[be]\,[cf]$	3 2 2 1	8	$[ae]\,[bf]\,[cd]$	1 1 1 2	5
$[ad]\,[bf]\,[ce]$	2 2 1 1	6	$[ae]\,[bd]\,[cf]$	2 1 2 1	6
$[af]\,[bc]\,[de]$	1 2 2 1	6	$[af]\,[bc]\,[de]$	1 2 2 1	6
$[af]\,[bd]\,[ce]$	1 1 1 1	4	$[af]\,[bd]\,[ce]$	1 1 1 1	4
$[af]\,[be]\,[cd]$	2 1 1 2	6	$[af]\,[be]\,[cd]$	2 1 1 2	6

Proposition 6.4.2 *The introduction of virtual crossings corresponding to a permutation of consecutive o's in the signed Gauss code may only change each 4 term summand of the Bracket polynomial by a factor of either $A^6 + A^{-4} - A^2 - 1$ or $A^{-6} + A^4 - A^{-2} - 1$.*

Proof: The contribution to the summand of the Bracket polynomial is given by:

$$A^2 d^{t_1 - 1} + d^{t_2 - 1} + d^{t_3 - 1} + A^{-2} d^{t_4 - 1}$$

where $d = -A^2 - A^{-2}$. So if the t_i are all fixed, except perhaps for a switch of t_2 and t_3, then the factor is fixed. Looking at the previous table, this is true for all entries except those whose total number of circles is either 5 or 8. Those with total number of circles 5 or 8 switch, under the permutation change, to having 8 or 5 respectively. We analyse the changes in the polynomial and show that these changes correspond precisely to the given factors.

Case 1: 2132 $\leftrightarrow$ 1121 and 2312 $\leftrightarrow$ 1211

$$\begin{aligned} A^2 d + 1 + d^2 + A^{-2} d &= A^2(-A^2 - A^{-2}) + 1 + (-A^2 - A^{-2})^2 + A^{-2}(-A^2 - A^{-2}) \\ &= -A^4 - 1 + 1 + A^4 + 2 + A^{-4} - 1 - A^{-4} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{and } A^2 + 1 + d + A^{-2} &= A^2 + 1 + (-A^2 - A^{-2}) + A^{-2} \\ &= 1. \end{aligned}$$

Case 2: 1223 $\leftrightarrow$ 2111

$$\begin{aligned} A^2 + d + d + A^{-2} d^2 &= A^2 + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-2}(-A^2 - A^{-2})^2 \\ &= A^2 - A^2 - A^{-2} - A^2 - A^{-2} + A^{-2}(A^4 + 2 + A^{-4}) \\ &= A^2 - A^2 - A^{-2} - A^2 - A^{-2} + A^2 + 2A^{-2} + A^{-6} \\ &= A^{-6} \end{aligned}$$

$$\begin{aligned} \text{and } A^2 d + 1 + 1 + A^{-2} &= A^2(-A^2 - A^{-2}) + 1 + 1 + A^{-2} \\ &= -A^4 - 1 + 1 + 1 + A^{-2} \\ &= -A^4 + 1 + A^{-2}. \end{aligned}$$

Case 3: $3221 \leftrightarrow 1112$

$$\begin{aligned} A^2d^2 + d + d + A^{-2} &= A^2(-A^2 - A^{-2})^2 + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-2} \\ &= A^2(A^4 + 2 + A^{-4}) - A^2 - A^{-2} - A^2 - A^{-2} + A^{-2} \\ &= A^6 + 2A^2 + A^{-2} - A^2 - A^{-2} - A^2 - A^{-2} + A^{-2} \\ &= A^6 \end{aligned}$$

$$\text{and } \quad \begin{aligned} A^2 + 1 + 1 + A^{-2}d &= A^2 + 1 + 1 + A^{-2}(-A^2 - A^{-2}) \\ &= A^2 + 1 + 1 - 1 - A^{-4} \\ &= A^2 + 1 - A^{-4}. \end{aligned}$$

□

6.4.3 The Welded Bracket Polynomial and the Welded Jones polynomial

Let D be a diagram of a welded link, L, and let I denote the ideal of $\mathbf{Z}[A, A^{-1}]$ generated by $A^6 + A^{-4} - A^2 - 1$. With the notation of Section 6.3.1, define the **Welded Bracket polynomial** of L to be:

$$\begin{aligned} \langle D \rangle^* &= \sum_{\text{states } S} [\langle D|S \rangle d^{s-1}/I] \\ &= \left[\sum_{\text{states } S} \langle D|S \rangle d^{s-1} \right] /I. \end{aligned}$$

If $p(A)$ is a Laurent polynomial in A then we call $p(A^{-1})$ the **involution** of $p(A)$.

Theorem 6.4.3 *The Welded Bracket polynomial of a welded link is well defined. In fact, this is the largest definable quotient of the classical Bracket polynomial which is a regular welded isotopy invariant of welded links.*

Proof: Given a welded link diagram, compute the polynomial of the virtual link obtained by replacing the welds with virtual crossings. This polynomial modulo the given relations is well defined on the class of virtual links that differ by a permutation of consecutive o's. These are precisely welded links. Finally we note that there is no need to quotient by the involution of this relation, since $(A^{-6} - 1)(1 - A^4) = A^{-2}(A^6 - 1)(1 - A^{-4})$. □

Lemma 6.4.4 *This invariant is non trivial. In particular the Hopf link has invariant* $-A^{-4} - A^4 = A^6 - A^4 - A^2 - 1 \neq 1$. □

Corollary 6.4.5 (to 6.4.3) *The Jones polynomial of a welded link is well defined modulo the relation* $(t^{-\frac{3}{2}} - 1)(1 - t) = 0$.

Proof: The writhe is unaffected by the introduction of virtual crossings. The change in the relation under the subsitution $A = t^{-\frac{1}{4}}$ is as given. □

Define $\mathcal{V}_L(t)$, the **Welded Jones polynomial** of a welded link L, to be the Jones polynomial of the virtual replacement for L (replacing welds with virtual crossings) modulo the relation $(t^{-\frac{3}{2}} - 1)(1 - t) = 0$. Note that we will slightly abuse notation and let I also denote the ideal of $\mathbf{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ generated by $(t^{-\frac{3}{2}} - 1)(1 - t)$.

6.4.4 Computations

Regard the coefficients of the Bracket polynomial of a virtual link as a bi-infinite sequence in $\mathbf{Z}$, called a **Bracket coefficient sequence**, where only a finite number of the entries are nonzero, and the n-th term in the sequence is the coefficient of A^n. For $n \in \mathbf{Z}$, let $r_n = (\ldots, 0, -1, 0, 0, 0, 1, 0, 1, 0, 0, 0, -1, 0, \ldots)$, where the first nonzero entry is the n-th term in the sequence, which corresponds to the coefficient of A^n in the Bracket polynomial. Two coefficient sequences, u and v, are said to be **equivalent** if they differ by any number of shifts of r_n; that is, if $u = v + \sum_{i=1}^{k} c_i r_{n_i}$, where $c_i, n_i \in \mathbf{Z}$ and $k \in \mathbf{N}$. In a similar manner we obtain the **f-coefficient sequence** and the **Jones coefficient sequence**, noting that the n-th term in the Jones coefficient sequence corresponds to the coefficient of $t^{-\frac{n}{4}}$.

Lemma 6.4.6 *Two Bracket coefficient sequences are equivalent if and only if the Welded Bracket polynomials are equal.*

Proof: The sequence r_n corresponds to the polynomial $-A^n + A^{n+4} + A^{n+6} - A^{n+10}$. This is equal to $-A^{n+4}(A^{-4} - 1 - A^2 + A^6)$, which is zero in the quotient ring. □

Corollary 6.4.7 *Two f-coefficient sequences are equivalent if and only if the Welded f-polynomials are equal. Two Jones coefficient sequences are equivalent if and only if the Welded Jones polynomials are equal.* □

For computations involving coefficient sequences we will underline the 0-th entry, which is the coefficient A^0 or t^0, for clarity. Also note that we may abuse notation slightly in computations, letting the n-th term in the Jones coefficient sequence correspond to the coefficient of $t^{\frac{n}{4}}$ instead of the coefficient of $t^{-\frac{n}{4}}$. It will be clear which convention is used.

Proposition 6.4.8 *Let L be a welded link. The $\mathcal{F}$-polynomial of L and the $\mathcal{V}$-polynomial take the value $(-2)^{n-1}$ when evaluated at $1 \in \mathbb{C}$, where n is the number of components of L.*

Proof: Let L also denote the virtual link obtained by replacing the welds with virtual crossings. The Skein relations, at $1 \in \mathbb{C}$, imply that classical crossing changes do not effect either the f-polynomial or the V-polynomial of the virtual version of L. Thus, at $1 \in \mathbb{C}$, these polynomials take the same value as the unlink with n components, which is $(-2)^{n-1}$.

Now notice that the sum of the entries of a coefficient sequence (which is the sum of the coefficients of the polynomial) is unchanged by equivalence because the sum of the entries of r_n is zero. Summing the entries in a coefficient sequence is the same as evaluating the polynomial at $1 \in \mathbb{C}$. Therefore these welded polynomials of the welded link L take the value $(-2)^{n-1}$ at $1 \in \mathbb{C}$. □

If we wish to distinguish between welded polynomials then we need to pick representatives in the quotient rings. For the Welded f-polynomial pick the unique representative whose coefficient sequence is zero for $n < -4$ or $n > 4$. Thus the chosen representative of the Welded f-polynomial is of the form $a_{-4}A^{-4} + a_{-3}A^{-3} + \ldots + a_3A^3 + a_4A^4$, where $a_j \in \mathbb{Z}$ for all j. This gives rise to the chosen representative of the Welded Jones polynomial, which is of the form $a_{-4}t + a_{-3}t^{\frac{3}{4}} + \ldots + a_3t^{-\frac{3}{4}} + a_4t^{-1}$.

Let RH denote the Right Hopf link, as shown in Figure 6.11, let WRH denote the Right Hopf link with one of its classical crossings replaced by a weld, as shown in Figure 6.12, and let U_n denote the unlink with n components.

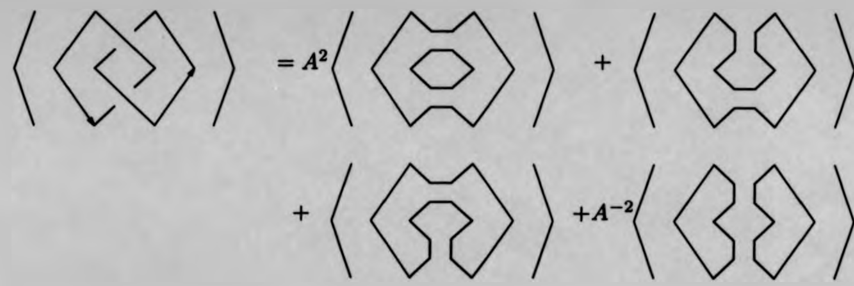

Figure 6.11: The Bracket polynomial of the Right Hopf link, RH

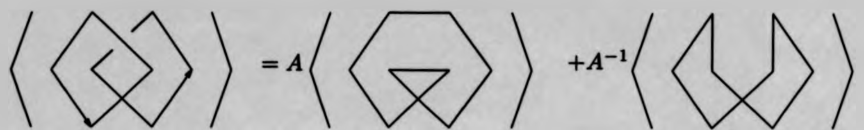

Figure 6.12: The Bracket polynomial of the welded link, WRH

Theorem 6.4.9 *The welded links RH, WRH and U_2 are not welded isotopic and are distinguished by their Welded Jones polynomials:*

$$\begin{aligned} \mathcal{V}_{RH}(t) &= -t - 2t^{\frac{1}{2}} + t^{-1}, \\ \mathcal{V}_{WRH}(t) &= -t^{\frac{1}{2}} - t^{-1} \\ \text{and} \quad \mathcal{V}_{U_2}(t) &= -t^{\frac{1}{2}} - t^{-\frac{1}{2}}. \end{aligned}$$

Proof: Using Figures 6.11 and 6.12 we see that:

$$\begin{aligned} \langle RH \rangle^* &= A^2 d + 2 + A^{-2} d &&= -A^4 - A^{-4}, \\ \langle WRH \rangle^* &= A + A^{-1} \\ \text{and} \quad \langle U_2 \rangle^* &= d &&= -A^2 - A^{-2} \end{aligned}$$

Thus the chosen representatives of the $\mathcal{F}$-polynomial are given by:

$$\begin{aligned} \mathcal{F}_{RH} &= (-A)^{-6}(-A^4 - A^{-4}) &&= -A^{-2} - A^{-10} &&= -A^{-4} - 2A^{-2} + A^4, \\ \mathcal{F}_{WRH} &= (-A)^{-3}(A + A^{-1}) &&= -A^{-2} - A^{-4} \\ \text{and} \quad \mathcal{F}_{U_2} &= -A^2 - A^{-2} \end{aligned}$$

Note that using coefficient sequences in order to obtain the chosen representative of the $\mathcal{F}$-polynomial makes the computation straightforward; for example:

$$\begin{aligned}
-A^{-2}-A^{-10} &= (\ldots,0,-1,0,0,0,0,0,0,0,0,-1,0,\underline{0},0,0,0,0,0,\ldots) \\
&\mapsto (\ldots,0,0,0,0,0,-1,0,-1,0,-1,0,\underline{1},0,0,0,0,0,\ldots) \\
&\mapsto (\ldots,0,0,0,0,0,0,0,-1,0,-2,0,\underline{0},0,0,0,1,0,\ldots) \\
&= -A^{-4}-2A^{-2}+A^{4}.
\end{aligned}$$

Substituting $A = t^{-\frac{1}{4}}$ yields:

$$\begin{aligned}
V_{RH}(t) &= -t-2t^{\frac{1}{2}}+t^{-1}, \\
V_{WRH}(t) &= -t^{\frac{1}{2}}-t^{-1} \\
\text{and} \quad V_{U}(t) &= -t^{\frac{1}{2}}-t^{-\frac{1}{2}}.
\end{aligned}$$

□

6.5 A Welded Isotopy Invariant of Classical Knots

6.5.1 Jones' theorems for classical knots

The following two theorems were given by Jones in [28] and [29]:

Theorem 6.5.1 *If the (classical) link L has an odd number of components, then $V_L(t)$ is a Laurent polynomial over the integers. If the number of components is even, then $V_L(t)$ is $\sqrt{t}$ times a Laurent polynomial.*

Theorem 6.5.2 *If K is a (classical) knot then $1 - V_K(t) = W_K(t)(1-t)(1-t^3)$ for some Laurent polynomial $W_k(t)$.*

In his table of polynomials of classical knots, Jones records $W_K(t) = (1-V_K(t))/(1-t)(1-t^3)$, ignoring the apparently "extraneous information" recorded in $V_K(t)$.

Let $L_1\#L_2$ denote the connected sum of two classical links. Jones gives the formula: $V_{L_1\#L_2}(t) = V_{L_1}(t)V_{L_2}(t)$. For the W-polynomial of classical knots, he records the formula: $W_{K_1\#K_2}(t) = W_{K_1}(t) + W_{K_2}(t) - (1-t)(1-t^3)W_{K_1}(t)W_{K_2}(t)$.

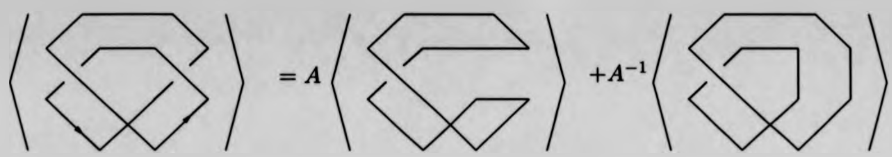

Figure 6.13: The Bracket polynomial of the virtual knot $vL3_1$

Theorems 6.5.1 and 6.5.2 do not hold for virtual links. In fact, we give examples of virtual links with an arbitrary number of components for which cannot by written as either a Laurent polynomial over the integers or $t^{\frac{1}{2}}$ times a Laurent polynomial.

Let $vL3_1$ denote the Left Trefoil with one classical crossing replaced by a virtual crossing, and let vRH denote the Right Hopf link with one classical crossing replaced by a virtual crossing. Let U_n denote the unlink with n components and let $\sqcup$ denote disjoint union.

Proposition 6.5.3 *There exist virtual links with an arbitrary number of components which are neither a Laurent polynomial in t over the integers, nor $t^{\frac{1}{2}}$ times a Laurent polynomial. In fact, for any n,*

$$\begin{aligned} V_{vL3_1 \sqcup U_n} &= (-t^{-\frac{5}{2}} + t^{-\frac{3}{2}} + t^{-1})(-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^n \\ \text{and} \qquad V_{vRH \sqcup U_n} &= (-t^{\frac{1}{2}} - t^{-1})(-t^{\frac{1}{2}} - t^{-\frac{1}{2}})^n. \end{aligned}$$

Proof: As seen in Figure 6.13 the f-polynomial of $vL3_1$ is given by:

$$\begin{aligned} f_{vL3_1}(t) &= (-A)^6[A(-A^3) + A^{-1}(A + A^{-1})] \\ &= (-A)^6[-A^4 + 1 + A^{-2}] \\ &= -A^{10} + A^6 + A^4. \end{aligned}$$

Thus $V_{vL3_1}(t) = -t^{-\frac{5}{2}} + t^{-\frac{3}{2}} + t^{-1}$.

It was shown in Theorem 6.4.9 that $V_{vRH} = -t^{\frac{1}{2}} - t^{-1}$. Note that the the image of this in the quotient ring would give the Welded Jones polynomial of WRH.

We have given links with one or two components which cannot be written as a Laurent polynomial or as $t^{\frac{1}{2}}$ times a Laurent polynomial. Now it suffices to note that this property is unchanged by taking the disjoint union with unknots, because this simply multiplies the polynomial by a factor of $d = -t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. □

6.5.2 The Welded W-polynomial

The "extraneous information" recorded in $V_K(t)$ for classical knots makes the Welded Jones polynomial of a classical knot trivial.

Proposition 6.5.4 *If K is a classical knot then the Welded Jones polynomial of K is trivial; that is, $\mathcal{V}_K(t) = 1$.*

Proof: Theorem 6.5.2 says that for classical knots we have: $1 - V_K(t) = W_K(t)(1-t)(1-t^3)$ for some Laurent polynomial $W_k(t)$. Recall that $\mathcal{V}_K(t) = V_K(t)/I$, where I is the ideal generated by $(1-t)(t^{-\frac{3}{2}} - 1)$. We show that $(1-t)(1-t^3) = 0$ is a consequence of the relation $(1-t)(t^{-\frac{3}{2}} - 1) = 0$. Then Theroem 6.5.2 implies that $\mathcal{V}_K(t) = 1$.

Rewrite $(1-t)(t^{-\frac{3}{2}} - 1) = 0$ as $t^{-\frac{3}{2}}(1-t) = (1-t)$. Applying the relation twice gives $t^{-3}(1-t) = (1-t)$. Multiplying by t^3 gives $(1-t) = t^3(1-t)$, which, on rearranging, gives the required result. □

For a classical knot, K, we define the **Welded W-polynomial**, to be:

$$\mathcal{W}_K(t) = W_K(t)/I \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]/I$$

where $W_K(t) = (1 - V_K(t))/(1-t)(1-t^3)$ and I is the ideal generated by $(1-t)(t^{-\frac{3}{2}} - 1)$.

Let U_1 denote the unknot, and let $3_1, 4_1$ and 5_1 be the knots given by Jones in [29], with polynomials:

$$\begin{aligned} W_{U_1}(t) &= 0, \\ W_{3_1}(t) &= 1, \\ W_{4_1}(t) &= -t^{-2}, \\ \text{and} \qquad W_{5_1}(t) &= 1 + t + t^3. \end{aligned}$$

Theorem 6.5.5 *$\mathcal{W}_K(t)$ is a nontrivial, welded isotopy invariant of classical knots. In fact,*

$$\begin{aligned} \mathcal{W}_{U_1}(t) &= 0, \\ \mathcal{W}_{3_1}(t) &= 1, \\ \mathcal{W}_{4_1}(t) &= -t^{-1} - t^{-\frac{1}{2}} + t^{\frac{1}{2}}, \\ \text{and} \qquad \mathcal{W}_{5_1}(t) &= -t^{-1} - t^{-\frac{1}{2}} + 2 + t^{\frac{1}{2}} + 2t. \end{aligned}$$

Proof: Let K_1 and K_2 be two classical knots which are welded isotopic. Then the summands of $V_{K_1}(t)$ are either the same as, or differ from the summands of $V_{K_2}(t)$ by factors in the ideal generated by I. Now Theorem 6.5.2 implies that $1 - V_{K_1}(t) = W_{K_1}(t)(1-t)(1-t^3)$ and $1 - V_{K_2}(t) = W_{K_2}(t)(1-t)(1-t^3)$. Therefore the summands of $W_{K_1}(t)$ are either the same as, or differ from the summands of $W_{K_2}(t)$ by factors in the ideal generated by I.

It remains to compute the representatives in the quotient ring:

$$\begin{aligned} -t^{-2} &= (\ldots,0,-1,0,0,0,0,0,0,0,\underline{0},0,0,0,\ldots) \\ &\mapsto (\ldots,0,0,0,0,0,-1,0,-1,0,\underline{0},0,1,0,\ldots) \\ &= -t^{-1} - t^{-\frac{1}{2}} + t^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{and} \quad 1 + t + t^3 &= (\ldots,0,0,0,0,0,\underline{1},0,0,0,1,0,0,0,0,0,0,0,1,0,\ldots) \\ &= (\ldots,0,0,0,0,0,\underline{1},0,-1,0,1,0,1,0,1,0,0,0,0,0,\ldots) \\ &= (\ldots,0,0,0,-1,0,\underline{1},0,0,0,2,0,1,0,0,0,0,0,0,0,\ldots) \\ &= (\ldots,0,-1,0,-1,0,\underline{2},0,1,0,2,0,0,0,0,0,0,0,0,0,\ldots) \\ &= -t^{-1} - t^{-\frac{1}{2}} + 2 + t^{\frac{1}{2}} + 2t. \end{aligned}$$

□

Proposition 6.5.6 *The $\mathcal{W}$-polynomial of the connected sum of two classical knots, K_1 and K_2, is the sum of the polynomials of the individual knots. That is, $\mathcal{W}_{K_1 \# K_2} = \mathcal{W}_{K_1} + \mathcal{W}_{K_2}$.*

Proof: Jones [28] gives the formula for the W-polynomial of classical knots, K_1 and K_2: $W_{K_1 \# K_2}(t) = W_{K_1}(t) + W_{K_2}(t) - (1-t)(1-t^3)W_{K_1}(t)W_{K_2}(t)$. In the quotient ring we have $(1-t)(1-t^3) = 0$, so the formula reduces to to $\mathcal{W}_{K_1 \# K_2} = \mathcal{W}_{K_1} + \mathcal{W}_{K_2}$. □

Corollary 6.5.7 *The connected sum of n copies of the Right Trefoil knot has $\mathcal{W}$-polynomial equal to n.* □

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