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# Atomic congestion games with random players: network equilibrium and the price of anarchy

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**Abstract**
In this paper, we present a new model of congestion games with finite and random number of players, and an analytical method to compute the random path and link flows. We study the equilibrium condition, reformulate it as an equivalent variational inequality problem, and establish the existence and non-uniqueness of the equilibria. We also upper bound the price of anarchy with affine cost functions to characterize the quality of the equilibria. The upper bound is tight in some special cases, including the case of deterministic players. Finally a general lower bound is also provided.

**Keywords** Congestion game · Network equilibrium · Price of anarchy

## 1 Introduction

Congestion games, first introduced by Rosenthal (1973), are a special class of non-cooperative games that illustrate self-interested resources sharing. There is a finite number of players and a finite set of resources. The strategies available to the players consist of subsets of resources. Each player chooses a strategy to minimize the total cost of his used resources. Since players' strategies may overlap, congestion effects occur and strategy interaction arises. The individual cost of using a resource depends

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on the total mass of the players using the resource. A typical application can be found in the internet or overlay networks built on top of the internet, where each network user (agent) chooses a path to send his traffic from his source to his destination and wishes to minimize his own travel cost. It is said to be a *Nash equilibrium* if the system arrives at such a steady state that no user can decrease his cost by unilaterally changing his path. It is well known that social cost normally is not minimized at Nash equilibria (e.g., "the prisoner's dilemma"). If all the players are fully coordinated by a central authority, the global objective function is to minimize the sum of users' costs (social cost), in which case is called a system optimum.

The price of anarchy (PoA), a term coined by Koutsoupias and Papadimitriou (1999), as a game-theoretical notion measures the system degradation resulting from lack of coordination. It is defined as the ratio between the social cost at a worst Nash equilibrium and that at a system optimum. Studies of the PoA started with networks of a simple structure, for example, parallel link networks in Koutsoupias and Papadimitriou (1999). For general network, the PoA was first studied for an extreme case of infinite number of users, known as non-atomic congestion games. In particular, Roughgarden and Tardos (2002, 2004) and Correa et al. (2008) proved that their upper bounds of the PoA depend only on the class of cost functions. They showed that the price of anarchy is exactly $4/3$ for affine cost functions, and they also showed bicriteria results for continuous and non-decreasing cost functions. These non-atomic works were also extended to capacitated network (Correa et al. 2004) and asymmetric cost functions (Perakis 2007). However, the model of classic congestion games with finite players known as atomic congestion games, is closer to reality in most application cases, especially when the number of players is relatively small. The upper bounds of the PoA for (atomic) congestion games were obtained by Awerbuch et al. (2005) and Christodoulou and Koutsoupias (2005) independently. The PoA with affine cost functions is bounded by 2.5 for unweighted demands and 2.618 for weighted demands, and both bounds were proved to be tight. Here with un-weighted demands, each player holds one unit of resource demand; while with weighted demands, each player has a weight which represents his resource demand (e.g., units of traffic). The PoA with polynomial cost functions of degree $m$ is bounded by $m^{\Theta(m)}$. Aland et al. (2011) improved the work of Awerbuch et al. (2005) and Christodoulou and Koutsoupias (2005), and established exact bounds of the PoA with polynomial cost functions.

All the above studies on deterministic players were built on an underlying assumption that every player knows the number of players in the system, which is not reasonable in many situations. Thus it is desirable to generalize the deterministic setting to the stochastic one with unknown number of players. There has been a growing interest in investigating uncertainty in traffic demand. Wang et al. (2014) and Correa et al. (2019) studied the PoA for non-atomic congestion games with stochastic demands. Cominetti et al. (2019) studied the atomic setting of Bernoulli congestion games in which each player participates in the game with an independent probability, and found that the upper bound of the PoA for deterministic models still hold. Cominetti et al. (2020) found such Bernoulli congestion games converge to a set of Poisson games in the sense of Myerson (1998), when players' participating probabilities tend to zero. In this paper we study a different model for atomic network congestion games with stochastic demands, an atomic counterpart of the model studied in Wang et al. (2014).

Instead of individual participation probabilities, we consider distributions of numbers of players as common knowledge, which follows the framework of population uncertainty proposed by Myerson (1998). Demand uncertainty has also been considered in other games such as resource allocation games Ashlagi et al. (2006).

The transition from deterministic to stochastic setting raises new conceptual and modelling issues and is particularly complicated when the number of players is finite. This is because every single player's traffic load is not negligible, and his own load will take an important role in the cost when switching from one strategy to another. The stochasticity of this study only comes from the random number of players, and the traffic hold by each player is still deterministic. We only focus on unweighed demand in our model and assume that each player holds one unit of traffic. Players' routing behaviors in an incomplete information environment is significantly different from the deterministic. The discreteness of the players' number and traffic flows also enhances the technical difficulty in analyzing the flow distributions, the network equilibrium and the price of anarchy. In this study, we take the number of players as random, and players only know the probabilities of possible numbers of players. We establish a more general model of atomic congestion games and provide an analytical method to determine random path and link flows under given distributions of random players. We incorporate random number of players into the notions of equilibrium and system optimum and establish their conditions in our new model. In addition, we also establish an upper bound on the price of anarchy for affine cost functions, and prove that the upper bound is tight in the special case.

The remainder of the paper is organized as follows. Section 2 introduces our new models for atomic congestion games with random players. Section 3 presents the equilibrium condition for the new model, and reformulates it into an equivalent Variational Inequality (VI) problem. The existence and non-uniqueness of equilibria are also discussed. Section 4 provides our upper bound for the price of anarchy for affine cost functions. Section 5 provides a lower bound of the price of anarchy. Section 6 discusses connections with existing results in the literature and makes some concluding remarks.

## 2 Atomic congestion games with random players

In this section, we present a mathematical model for atomic congestion games with random players.

### 2.1 Review of deterministic congestion games

Consider a deterministic congestion game. Let $J = \{1, 2, \ldots, n\}$ be the set of players, $\mathbf{E}$ be a set of facilities (resources), $\Omega_j \subseteq 2^{\mathbf{E}}$ be the strategy space of player $j$, and $c_e(\cdot)$ be a non-decreasing cost function associated with resource $e$. For a joint action, every player selects a set of resources, and the congestion on resource $e$ is the number of players whose action (strategy) contains resource $e$. Thus the more players use one resource, the higher cost they will pay for it. The cost of player $j$ is the individual cost of all resources that his has selected.

Congestion games are usually modelled with a network framework by grouping players into different types according to their strategy spaces. Consider a general network $G = (N, E)$, where $N$ and $E$ denote the set of nodes and links, respectively. Each link represents a resource. To each link $e \in E$, we associate a (link) cost function $c_e(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^+$, which is assumed to be nondecreasing in its argument, the link flow. A subset of nodes form a set of origin-destination (O-D) pairs, denoted by $I$. Every path, a sequence of connected links linking the origin to the destination, represents a strategy. Denote by $P_i$ the set of all the paths connecting O-D pair $i$. Each player with O-D pair $i$ holds one unit of load(traffic) and selects one path from $P_i$ to send his load from the origin to the destination. The demand of O-D pair $i$, denoted by $d_i \in \mathbb{N},\ i \in I$, is just the number of players holding strategy set $P_i$. The cost (or latency) of each player is the cost of the path he selected, which is the sum of the costs associated with all the links of the path. The congestion (load) on each resource is the flow on the link, which is the number of players using the link in their paths. The above network game can be denoted by a tuple of $(G, \mathbf{d}, \mathbf{c})$, where $\mathbf{d} = (d_i : i \in I)$ and $\mathbf{c} = (c_e(\cdot) : e \in E)$.

### 2.2 Congestion games with random players

As mentioned before, the assumption that every player knows the number of players is not reasonable in practice. We assume instead that a player does not know the exact number of other players, and only has knowledge of demand distributions from historical information, which can be collected and published by the central coordinator. Since each player holds one unit of load, the number of players for each O-D pair is also its traffic demand. In our new model, the demand for each O-D pair follows a discrete random distribution $D_i$, which is considered as common knowledge. We assume that there is at least one player for each O-D pair, i.e., $D_i \geq 1$, and demands of different O-D pairs are independent. The assumption $D_i \geq 1$ is realistic for many applications of network congestion games that usually have large demand for each O-D pair. For example, O-D pairs with rare commuters are usually omitted when modelling the transportation network. Technically, the assumption also guarantees that all O-D pairs are considered in every scenario. We denote such a congestion game with random players by a tuple $(G, \mathbf{D}, \mathbf{c})$, where $\mathbf{D} = (D_i : i \in I)$ is the vector of random demands.

As we know, to attain a Nash equilibrium, every single player needs to know all the others' routing choices to find his best strategy. But this assumption is not suitable for our stochastic model as the number of players is random and unknown. In this paper we consider mixed strategies, so that each player independently selects a probability distribution over all the paths between his O-D pair. Demand uncertainty in our stochastic model fits into the general framework of games under population uncertainty proposed by Myerson (1998) with O-D pairs as player types. Similar to Myerson (1998), we assume that players can only form perceptions about how other players make routing decisions depending on the information of which O-D pairs these players belong to and the common knowledge of demand distributions. In addition, it is difficult to perceive that two different individuals of the same O-D pair would behave differently given there is no attribute by which others can distinguish them from each other. Furthermore, demand uncertainty implies random numbers of players, which makes it impossible to specify strategies for individual players. All players from the same O-D pair are therefore assumed identical and treated symmetrically in our model, which means that players from the same O-D pair are assumed to adopt the same mixed strategy at equilibrium. Similar assumptions can be found in other games under population uncertainty such as resource selection games with unknown number of players (Ashlagi et al. 2006).

Now, let $p_k^i$ be the probability that path $k \in P_i$ is chosen. The set of mixed strategies of each player from O-D pair $i \in I$ is

$$\Omega_i = \left\{ \mathbf{p}^i = (p_k^i \geq 0 : k \in P_i) : \sum_{k \in P_i} p_k^i = 1 \right\}.$$

Let $\Omega = \prod_{i \in I} \Omega_i$. Then each vector $\mathbf{p} = (\mathbf{p}^i : i \in I) \in \Omega$ represents a strategy profile of players from all O-D pairs. Let random binary variables $\{X_{k,j}^i : 1 \leq j \leq D_i,\ k \in P_i,\ i \in I\}$ indicate whether player $j$ from O-D pair $i \in I$ chooses path $k$, i.e., $\mathbb{P}[X_{k,j}^i = 1] = p_k^i$ and $\mathbb{P}[X_{k,j}^i = 0] = 1 - p_k^i$. Every player has to choose one path for his traffic, i.e.,

$$\sum_{k \in P_i} X_{k,j}^i = 1, \quad \forall\, 1 \leq j \leq D_i. \tag{1}$$

The total load (flow) on path $k$ can be written as

$$F_k^i = \sum_{j=1}^{D_i} X_{k,j}^i, \quad k \in P_i,\ i \in I, \tag{2}$$

which is a *compound* random variable (Ross 2002). When demand $D_i$ is realized at $y$, the conditional path flow on $k \in P_i$ follows binomial distribution $B(y, p_k^i)$. Then the unconditional path flow $F_k^i$ in (2) can be identified by the *total probability theorem* with a given demand distribution. The mean path flow can be computed as

$$f_k^i = \mathbb{E}[D_i \cdot \mathbb{E}[X_{k,j}^i]] = p_k^i d_i, \quad k \in P_i,\ i \in I. \tag{3}$$

Given that demands of different O-D pairs are independent, the flows on paths connecting different O-D pairs are independent. However, the path flows from the same O-D pair are dependent due to flow conservation constraint (1).

Let $X_{e,j}^i$ be a random binary variable indicating whether player $j$ ($1 \leq j \leq D_i$, $i \in I$) chooses link $e \in E$, i.e., $X_{e,j}^i = \sum_{k \in P_i} \delta_{k,e}^i X_{k,j}^i$, where $\delta_{k,e}^i$ is the link-path incidence indicator, which is 1 if link $e$ is included in path $k$ and 0 otherwise. Define $p_e^i = \sum_{k \in P_i} \delta_{k,e}^i p_k^i$, then $\mathbb{P}[X_{e,j}^i = 1] = p_e^i$ for any $1 \leq j \leq D_i$. The link flow $V_e$ is a result of independent choices of all the players on link $e$:

$$V_e = \sum_{i \in I} \sum_{j=1}^{D_i} X_{e,j}^i, \ \forall \, e \in E. \tag{4}$$

Clearly $\sum_{j=1}^{D_i} X_{e,j}^i$ is also a compound random variable, which follows Binomial distribution $B(D_i, p_e^i)$ with $D_i$ itself a random variable. Thus the distribution of $\sum_{j=1}^{D_i} X_{e,j}^i$ can be identified given the distributions of $\mathbf{D}$ and the mixed strategy profile $\mathbf{p}$. The link flow in (4) is the sum of independent distributions $\sum_{j=1}^{D_i} X_{e,j}^i$ over all O-D pairs. From (2) and (4), we have the following conservations between link and path flows:

$$V_e = \sum_{i \in I} \sum_{k \in P_i} \delta_{k,e}^i F_k^i, \qquad \forall \, e \in E.$$

We can also write $v_e = \sum_{i \in I} \sum_{k \in P_i} \delta_{k,e}^i f_k^i$ for mean link flows. Given the link cost functions, the random path cost is simply the sum of the costs of those links that constitute the path, i.e.,

$$c_k^i(\mathbf{F}) = \sum_{e \in E} \delta_{k,e}^i c_e(V_e), \quad \forall \, k \in P_i, \ \forall \, i \in I,$$

where $\mathbf{F}$ is the vector of path flow, i.e., $\mathbf{F} = (F_k^i : k \in P_i, i \in I)$.

## 3 Network equilibrium and the price of anarchy

In this section, we present the model of network equilibrium with random players, reformulate it with an equivalent VI problem, prove the existence and non-uniqueness of equilibria, and define the price of anarchy.

### 3.1 Network equilibrium formulation

At an equilibrium, there is no incentive for any player to change his strategy. Every path of any given O-D pair with positive probability must incur the same expected cost for every player from the O-D pair, since otherwise the expected cost of any of the players can be decreased by taking the lower-cost path with a higher probability. Given strategy profile $\mathbf{p}$ with the corresponding path flows $\mathbf{F}$, the expected cost of taking path $k \in P_i$ for a single player $j$ in O-D pair $i \in I$ can be expressed as the following conditional expectation

$$\mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1], \quad \forall \, 1 \leq j \leq D_i, \ i \in I. \tag{5}$$

Since at a Nash equilibrium of mixed strategies, each pure strategy involved (i.e., a path with positive probability) in the mixed strategy is a best response itself and yields the same expected cost, we arrive at the following definition of an equilibrium.

**Definition 1** *(Network equilibrium)* Strategy profile $\mathbf{p}$ with the corresponding path flows $\mathbf{F}$ is an equilibrium if and only if, for any $k, l \in P_i$, with $p_k^i > 0$,

$$\mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1] \leq \mathbb{E}[c_l^i(\mathbf{F}) \mid X_{l,j}^i = 1],$$

for arbitrary player $j$ $(1 \leq j \leq D_i, i \in I)$.

Let us calculate the conditional expectation in (5). We have

$$\begin{aligned}
\mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1] &= \mathbb{E}\left[\sum_{e \in E} \delta_{k,e}^i c_e(V_e) \mid X_{k,j}^i = 1\right] \\
&= \sum_{e \in E} \delta_{k,e}^i \mathbb{E}\left[c_e\left(\sum_{i' \in I} \sum_{j'=1}^{D_{i'}} X_{e,j'}^{i'}\right) \mid X_{k,j}^i = 1\right] \\
&= \sum_{e \in E} \delta_{k,e}^i \mathbb{E}\left[c_e\left(1 + \sum_{j'=1}^{D_i - 1} X_{e,j'}^i + \sum_{i' \neq i} \sum_{j'=1}^{D_{i'}} X_{e,j'}^{i'}\right)\right],
\end{aligned}$$

which implies that $\mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1]$ is independent of the choice of player $j$ of O-D pair $i$. Hence we can drop subscript $j$ by denoting $t_k^i(\mathbf{p}) = \mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1]$.

Let us use the corresponding lower-case letters to denote the means of random variables, e.g., $\mathbf{d} = (d_i : i \in I)$ for mean demands, $\mathbf{f} = (f_k^i : k \in P_i, i \in I)$ for the means of path flows, and $\mathbf{v} = (v_e : e \in E)$ for the means of link flows. Then we can reformulate the equilibrium condition as a variational inequality (VI) program as follows.

**Proposition 1** *A mixed strategy profile $\bar{\mathbf{p}}$ is an equilibrium if and only if it satisfies the following VI problem:*

$$(\mathbf{f} - \bar{\mathbf{f}})^T \mathbf{t}(\bar{\mathbf{p}}) \geq 0, \ \forall \mathbf{p} \in \Omega, \tag{6}$$

*where $\mathbf{t}(\bar{\mathbf{p}}) = (t_k^i(\bar{\mathbf{p}}) : k \in P_i, i \in I)$, $\mathbf{f}$ and $\bar{\mathbf{f}}$ are the mean flows corresponding to strategy profiles $\mathbf{p}$ and $\bar{\mathbf{p}}$, respectively.*

***Proof*** From the definition of the equilibrium, $\forall\, k, l \in P_i, \forall\, i \in I$, with $\bar{p}_k^i > 0$, we have

$$t_k^i(\bar{\mathbf{p}}) \leq t_l^i(\bar{\mathbf{p}}).$$

Let $\pi_i = \min_{l \in P_i} t_l^i(\bar{\mathbf{p}})$ for $i \in I$. The equilibrium condition is equivalent to

$$\bar{p}_l^i \left(t_l^i(\bar{\mathbf{p}}) - \pi_i\right) = 0, \quad \forall\, l \in P_i, \ \forall\, i \in I.$$

Multiplying both sides of the above by $d_i > 0$, we obtain

$$\bar{f}_l^i \left( t_l^i(\bar{\mathbf{p}}) - \pi_i \right) = 0, \quad \forall\, l \in P_i,\ \forall\, i \in I.$$

Summing up the above over all the paths, we get

$$\sum_{i \in I} \sum_{l \in P_i} \bar{f}_l^i \left( t_l^i(\bar{\mathbf{p}}) - \pi_i \right) = 0. \tag{7}$$

On the other hand, for any feasible strategy profile $\mathbf{p}$, as $f_l^i \geq 0$ for any $l \in P_I, i \in I$,

$$\sum_{i \in I} \sum_{l \in P_i} f_l^i \left( t_l^i(\bar{\mathbf{p}}) - \pi_i \right) \geq 0,$$

which together with (7) leads to

$$\sum_{i \in I} \sum_{l \in P_i} \left( \bar{f}_l^i - f_l^i \right) \left( t_l^i(\bar{\mathbf{p}}) - \pi_i \right) \leq 0. \tag{8}$$

From the feasibility of the mixed strategies we have

$$\sum_{l \in P_i} f_l^i \pi_i = \sum_{l \in P_i} \bar{f}_l^i \pi_i = \pi_i d_i,\ \forall\, i \in I.$$

Substituting the above into (8), we obtain

$$\sum_{i \in I} \sum_{l \in P_i} \left( \bar{f}_l^i - f_l^i \right) t_l^i(\bar{\mathbf{p}}) \leq 0,$$

which is (6).

Next assume $\bar{\mathbf{p}}$ satisfies (6). We show that it also satisfies the equilibrium condition. First with the first order optimality condition we observe that $\bar{\mathbf{p}}$ is an optimal solution to the following linear program (LP):

$$\begin{aligned} \min\ & \mathbf{f}^T \mathbf{t}(\bar{\mathbf{p}}) \\ \text{s.t. } & \sum_{k \in P_i} f_k^i = d_i,\ \forall\, i \in I, \\ & f_k^i \geq 0, \qquad \forall\, k \in P_i,\ \forall\, i \in I. \end{aligned}$$

With LP duality we have

$$\begin{aligned} \max\ & \lambda^T \mathbf{d} \\ \text{s.t. } & \lambda_i \leq t_k^i(\bar{\mathbf{p}}),\ \forall\, k \in P_i,\ \forall\, i \in I. \end{aligned}$$

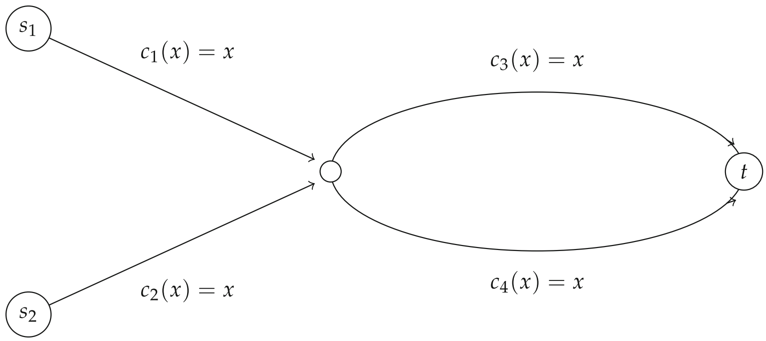


**Fig. 1** Multiple equilibria

Then the complementary slackness conditions lead us to

$$f_k^i(t_k^i(\bar{\mathbf{p}}) - \lambda_i) = 0, \quad \forall\, k \in P_i, \forall\, i \in I,$$

which implies satisfaction of the equilibrium condition. □

### 3.2 Existence and non-uniqueness of network equilibria

Given the equivalent VI condition in Proposition 1, the existence of an equilibrium can be guaranteed when link cost functions are continuous. Indeed, we can rewrite condition (6) in the following form by substituting $f_k^i = p_k^i d_i$ and $\bar{f}_k^i = \bar{p}_k^i d_i$:

$$(\mathbf{p} - \bar{\mathbf{p}})^T \mathbf{S}(\bar{\mathbf{p}}) \geq 0, \;\; \mathbf{p} \in \Omega, \tag{9}$$

where $\mathbf{S}(\mathbf{p})$ is a vector with the same dimension as $\mathbf{t}(\mathbf{p})$, obtained by replacing element $t_k^i(\mathbf{p})$ in $\mathbf{t}(\mathbf{p})$ with $t_k^i(\mathbf{p})d_i$ for every $k \in P_i$, $i \in I$. When link cost functions are continuous, the game admits at least one equilibrium, due to the fact that existence of a solution $\bar{\mathbf{p}} \in \Omega$ to VI problem (9) is guaranteed by the continuity of $\mathbf{S}(\mathbf{p})$ and the compactness of $\Omega$.

It is known that multiple mixed strategy equilibria can exist in simple instances of the deterministic atomic model with one single player per O-D pair (Awerbuch et al. 2005). These deterministic instances are apparently special cases of our stochastic model since the fact that there is only a single player in each O-D pair implies the assumption of an identical mixed strategy by all players of the same O-D pair. Therefore, our stochastic model allows multiple mixed strategy equilibria in general. The following example shows multiple mixed strategy equilibria of a stochastic instance with more than one player.

***Example 1*** Consider the network in Fig. 1 with two O-D pairs, i.e., from $s_1$ to $t$ and $s_2$ to $t$, denoted by $i = 1, 2$ respectively. There is only one player from $s_2$ to $t$, while the player number from $s_1$ to $t$ follows a random distribution $D$. Each player has two paths to choose from, paths 1 and 2 from $s_1$ to $t$ and paths 3 and 4 from $s_2$ to $t$, where path 1 consists of links 1 and 3, path 2 of links 1 and 4, path 3 of links 2 and 3, and path 4 of links 2 and 4. The cost function on each link is indicated in the figure. Let binary variable $X_{k,j}^i$ denote the random choice of player $j = 1, 2, \ldots, D$ on path $k$. Then the flows on paths 1 and 2 follow compound random distributions, i.e., $F_k^1 = \sum_{j=1}^{D} X_{k,j}^1$, $k = 1, 2$. The path flows on paths 3 and 4 follow binary distributions, since there is only one player from $s_2$ to $t$. Recalling $t_k^i(\mathbf{p}) = \mathbb{E}[c_k^i(\mathbf{F}) \mid X_{k,j}^i = 1]$, we have the following:

$$
\begin{aligned}
t_1^1(\mathbf{p}) &= \mathbb{E}\left[c_1^1(\mathbf{F}) \mid X_{1,j}^1 = 1\right] = \mathbb{E}\left[c_1(D) + c_3(F_1^1 + F_3^2) \mid X_{1,j}^1 = 1\right] \\
&= \mathbb{E}[D] + \mathbb{E}\left[\sum_{j=1}^{D-1} X_{1,j}^1 + 1 + F_3^2\right] = d + (d-1)p_1^1 + 1 + p_3^2, \\
t_2^1(\mathbf{p}) &= \mathbb{E}[D] + \mathbb{E}\left[\sum_{j=1}^{D-1} X_{2,j}^1 + 1 + F_4^2\right] = d + (d-1)p_2^1 + 1 + p_4^2, \\
t_3^2(\mathbf{p}) &= 1 + \mathbb{E}\left[c_3(F_1^1 + 1)\right] = p_1^1 d + 2, \\
t_4^2(\mathbf{p}) &= 1 + \mathbb{E}\left[c_4(F_2^1 + 1)\right] = p_2^1 d + 2.
\end{aligned}
$$

Apparently, any feasible strategy profile satisfying $p_1^1 \le p_2^1$ and $p_3^2 \le p_4^2$ makes $t_1^1(\mathbf{p}) \le t_2^1(\mathbf{p})$ and $t_3^2(\mathbf{p}) \le t_4^2(\mathbf{p})$, and hence is an equilibrium strategy according to Definition 1.

### 3.3 Definition of the price of anarchy

Before addressing the price of anarchy for random players, let us introduce the system optimum, which is the optimal strategy profile $\mathbf{p}^*$ that minimizes the expected total cost in the network. At a system optimum the traffic can be considered as centrally coordinated and assigned so that the expected total social cost is at minimum. With deterministic demands, the system optimum can always be reached by an assignment with pure strategies, namely each player is allocated to a certain path (Awerbuch et al. 2005). However, with demand uncertainty (i.e., random numbers of players), such an assignment is no longer possible. What the central coordinator can do is to identify (mixed) routing strategies, one for each O-D pair, and use them to route players depending on which O-D pair they belong to. This setting follows naturally from our discussion in Sect. 2.2 that players from the same O-D pair are not distinguishable and they adopt the same mixed strategy at equilibrium.

**Definition 2** *(System optimum)* A strategy profile $\mathbf{p}$ is at a system optimum for congestion game $(G, \mathbf{D}, \mathbf{c})$ if and only if it solves the following minimization problem:

$$
\min_{\mathbf{p}\in\Omega} T(\mathbf{p}) \equiv \mathbb{E}[\sum_{e\in E} c_e(V_e)V_e]. \tag{10}
$$

The price of anarchy is the worst-case ratio between expected total cost at an equilibrium and at a system optimum, as formally defined below.

**Definition 3** *(Price of anarchy)* Given a stochastic instance $(G, \mathbf{D}, \mathbf{c})$, the corresponding PoA is defined as:

$$R(G, \mathbf{D}, \mathbf{c}) = \sup \left\{ \frac{T(\mathbf{p})}{T(\mathbf{q})} : \mathbf{p}, \mathbf{q} \in \Omega, \mathbf{p} \text{ an equilibrium; } \mathbf{q} \text{ a system optimum} \right\},$$

where $T(\cdot)$ is the expected total cost as social (system) objective function in (10).

Given any set $\mathcal{I}$ of stochastic instances, the PoA with respect to $\mathcal{I}$ is defined as

$$R(\mathcal{I}) := \sup_{(G,\mathbf{D},\mathbf{c}) \in \mathcal{I}} \text{PoA}(G, \mathbf{D}, \mathbf{c}).$$

## 4 Upper bound with affine cost functions

In this section, we consider affine cost functions, i.e., $c_e(x) = a_e x + b_e$, $a_e, b_e \geq 0$ for any $e \in E$, and upper bound the price of anarchy.

Denote $\theta_i$ as the ratio of the standard deviation to the mean of the demand in O-D pair $i \in I$, i.e., $\theta_i = \sigma_i / d_i$. Let $\overline{\theta} = \max_{i \in I} \theta_i$. We start by bounding the total expected cost in terms of $\overline{\theta}$, $\mathbf{f}^*$ and $\mathbf{t}$ by establishing the following technical lemma.

**Lemma 1** *Let $\overline{\mathbf{p}}$ and $\mathbf{p}^*$ be an equilibrium and a system optimum respectively. When cost functions are affine, we have*

$$T(\overline{\mathbf{p}}) \leq (1 + \overline{\theta}^2) \mathbf{f}^{*T} \mathbf{t}(\overline{\mathbf{p}}).$$

***Proof*** From law of total variance, we have

$$\text{Var}\left[\sum_{j=1}^{D_i} X_{e,j}^i\right] = \mathbb{E}\left[\text{Var}\left(\sum_{j=1}^{D_i} X_{e,j}^i \mid D_i\right)\right] + \text{Var}\left(\mathbb{E}\left[\sum_{j=1}^{D_i} X_{e,j}^i \mid D_i\right]\right) \tag{11}$$

Since

$$\text{Var}\left[\sum_{j=1}^{D_i} X_{e,j}^i \mid D_i = d_i\right] = \text{Var}\left[\sum_{j=1}^{d_i} X_{e,j}^i\right] = d_i p_e^i (1 - p_e^i),$$

and

$$\mathbb{E}\left[\sum_{j=1}^{D_i} X_{e,j}^i \mid D_i = d_i\right] = \mathbb{E}\left[\sum_{j=1}^{d_i} X_{e,j}^i\right] = d_i p_e^i,$$

we have

$$\mathrm{Var}\left[\sum_{j=1}^{D_i} X_{e,j}^i \mid D_i\right] = D_i p_e^i (1 - p_e^i), \quad \text{and} \quad \mathbb{E}\left[\sum_{j=1}^{D_i} X_{e,j}^i \mid D_i\right] = D_i p_e^i.$$

Substituting the above into (11), we have

$$\mathrm{Var}\left[\sum_{j=1}^{D_i} X_{e,j}^i\right] = \mathbb{E}\left[D_i p_e^i\right] + \mathrm{Var}\left[D_i p_e^i\right] = d_i p_e^i (1 - p_e^i) + \sigma_i^2 \left(p_e^i\right)^2.$$

Then the variance of the link flow can be written as

$$\begin{aligned}
\sigma_e^2 = \mathrm{Var}\left[V_e\right] &= \sum_{i \in I} \delta_e^i \mathrm{Var}\left[\sum_{j=1}^{D_i} X_{e,j}^i\right] \\
&= \sum_{i \in I} \delta_e^i \left(\sigma_i^2 - d_i\right) \left(p_e^i\right)^2 + \sum_{i \in I} \delta_e^i d_i p_e^i \\
&= \sum_{i \in I} \delta_e^i \left(\sigma_i^2 \left(p_e^i\right)^2 + p_e^i (1 - p_e^i) d_i\right),
\end{aligned} \tag{12}$$

where $\delta_e^i$ is the link-commodity indicator, which is 1 when link $e$ is involved in O-D pair $i \in I$, and 0 otherwise. Therefore, we can write the total cost as follows:

$$\begin{aligned}
T(\mathbf{p}) &= \sum_{e \in E} \mathbb{E}[c_e(V_e) V_e] = \sum_{e \in E} \left(a_e \mathbb{E}[V_e^2] + b_e v_e\right) = \sum_{e \in E} \left(a_e v_e^2 + a_e \sigma_e^2 + b_e v_e\right) \\
&= \sum_{e \in E} \left(a_e v_e^2 + b_e v_e\right) + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e \left(\sigma_i^2 \left(p_e^i\right)^2 + p_e^i (1 - p_e^i) d_i\right) \\
&= \sum_{e \in E} \left(a_e v_e^2 + b_e v_e\right) + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e (\theta_i d_i)^2 \left(p_e^i\right)^2 + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e p_e^i (1 - p_e^i) d_i \\
&\leq \sum_{e \in E} \left(a_e (1 + \overline{\theta}^2) v_e^2 + b_e v_e\right) + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e p_e^i (1 - p_e^i) d_i,
\end{aligned} \tag{13}$$

where the last two lines hold from $\theta_i = \sigma_i / d_i$ and $\overline{\theta} = \max_{i \in I} \theta_i$, respectively. Recall that, for any player $j$ from O-D pair $i \in I$,

$$t_k^i(\mathbf{p}) = \sum_{e \in k} \mathbb{E}[c_e(V_e) \mid X_{k,j}^i = 1] = \sum_{e \in k} \mathbb{E}[c_e(V_e + 1 - X_{k,j}^i)], \quad \forall\, k \in P_i.$$

As cost functions are affine, we have

$$t_k^i(\mathbf{p}) = \sum_{e \in k} \left(a_e v_e + b_e + a_e (1 - p_k^i)\right).$$

Thus

$$
\begin{aligned}
\mathbf{f}^T\mathbf{t}(\mathbf{p}) &= \sum_{i\in I}\sum_{k\in P_i} f_k^i \cdot t_k^i(\mathbf{p}) \\
&= \sum_{i\in I}\sum_{k\in P_i} f_k^i \cdot \sum_{e\in k}\left(a_e v_e + b_e + a_e(1-p_k^i)\right) \\
&= \sum_{e\in E}(a_e v_e + b_e)v_e + \sum_{i\in I}\sum_{k\in P_i} p_k^i d_i \cdot \sum_{e\in k}\left(a_e(1-p_k^i)\right) \\
&= \sum_{e\in E}(a_e v_e + b_e)v_e + \sum_{e\in E} a_e\left(\sum_{k:e\in k}(1-p_k^i)p_k^i d_i\right) \\
&\geq \sum_{e\in E}\left(a_e v_e^2 + b_e v_e\right) + \sum_{e\in E}\sum_{i\in I}\delta_e^i a_e p_e^i(1-p_e^i)d_i,
\end{aligned}
\tag{14}
$$

where the third equality holds according to (3).

From (13) and (14), we can derive the following connection between $T(\mathbf{p})$ and $\mathbf{f}^T\mathbf{t}(\mathbf{p})$ for any feasible strategy profile $\mathbf{p}$:

$$
T(\mathbf{p}) \leq (1+\overline{\theta}^2)\mathbf{f}^T\mathbf{t}(\mathbf{p}).
$$

Together with Proposition 1, we have

$$
T(\overline{\mathbf{p}}) \leq (1+\overline{\theta}^2)\overline{\mathbf{f}}^T\mathbf{t}(\overline{\mathbf{p}}) \leq (1+\overline{\theta}^2)\mathbf{f}^T\mathbf{t}(\overline{\mathbf{p}}). \tag{15}
$$

The lemma can then be proved by setting $\mathbf{f} = \mathbf{f}^*$ in (15). □

In order to bound the PoA, we need the following two additional technical lemmas.

**Lemma 2** *(Christodoulou and Koutsoupias 2005, lemma 1) For every pair of non-negative integers $X$ and $Y$, it holds*

$$
X(Y+1) \leq \frac{5}{3}X^2 + \frac{1}{3}Y^2.
$$

**Lemma 3** *Let $\overline{\mathbf{p}}$ and $\mathbf{p}^*$ be an equilibrium and a system optimum respectively. Then*

$$
\mathbf{f}^{*T}\mathbf{t}(\overline{\mathbf{p}}) \leq \frac{5}{3}T(\mathbf{p}^*) + \frac{1}{3}T(\overline{\mathbf{p}}).
$$

***Proof*** First we have

$$
\begin{aligned}
\mathbf{f}^{*T}\mathbf{t}(\overline{\mathbf{p}}) &= \sum_{i\in I}\sum_{k\in P_i}\left(f_k^i\right)^* t_k^i(\overline{\mathbf{p}}) \\
&\leq \sum_{i\in I}\sum_{k\in P_i}\left(f_k^i\right)^* \sum_{e\in k}\mathbb{E}[c_e(\overline{V}_e+1)]
\end{aligned}
$$

$$= \sum_{e \in E} v_e^* \mathbb{E}[c_e(\overline{V}_e + 1)].$$

From the independence of $V_e$ and $\overline{V}_e^*$ for every $e \in E$, we obtain

$$\begin{aligned}
\mathbf{f}^{*T}\mathbf{t}(\bar{\mathbf{p}}) &\leq \sum_{e \in E} \mathbb{E}[V_e^* c_e(\overline{V}_e + 1)] \\
&= \sum_{e \in E} \mathbb{E}[V_e^*(a_e(\overline{V}_e + 1) + b_e)] \\
&\leq \sum_{e \in E} \left( a_e \mathbb{E}[\frac{5}{3} V_e^{*2} + \frac{1}{3} \overline{V}_e^2] + b_e \mathbb{E}[V_e^*] \right) \\
&\leq \sum_{e \in E} \frac{5}{3} \mathbb{E}[a_e V_e^{*2} + b_e V_e^*] + \frac{1}{3} \sum_{e \in E} \mathbb{E}[a_e \overline{V}_e^2] \\
&\leq \frac{5}{3} T(\mathbf{p}^*) + \frac{1}{3} T(\overline{\mathbf{p}}),
\end{aligned}$$

where the second inequality holds because of Lemma 2. □

**Lemma 4** *Let $\overline{\mathbf{p}}$ and $\mathbf{p}^*$ be an equilibrium and a system optimum respectively. Then we have*

$$\mathbf{f}^{*T}\mathbf{t}(\overline{\mathbf{p}}) \leq \sqrt{T(\overline{\mathbf{p}}) T(\mathbf{p}^*)} + T(\mathbf{p}^*).$$

***Proof*** For any feasible strategy profile $\mathbf{p}$, we have

$$\begin{aligned}
T(\mathbf{p}) &= \sum_{e \in E} \mathbb{E}[c_e(V_e) V_e] \\
&= \sum_{e \in E} \left( a_e v_e^2 + a_e \sigma_e^2 + b_e v_e \right) \geq \sum_{e \in E} \left( a_e v_e^2 \right). \qquad (16)
\end{aligned}$$

In addition, we have

$$\begin{aligned}
T(\mathbf{p}) &= \sum_{e \in E} \left( a_e v_e^2 + b_e v_e \right) + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e \left( \sigma_i^2 \left( p_e^i \right)^2 + p_e^i (1 - p_e^i) d_i \right) \\
&= \sum_{e \in E} \left( a_e v_e^2 + b_e v_e \right) + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e \sigma_i^2 \left( p_e^i \right)^2 + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e p_e^i d_i - \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e \left( p_e^i \right)^2 d_i \\
&= \sum_{e \in E} (a_e v_e + b_e v_e) + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e \sigma_i^2 \left( p_e^i \right)^2 + \sum_{e \in E} a_e \left( v_e^2 - \sum_{i \in I} \delta_e^i \left( p_e^i \right)^2 d_i \right) \\
&= \sum_{e \in E} (a_e v_e + b_e v_e) + \sum_{e \in E} \sum_{i \in I} \delta_e^i a_e \sigma_i^2 \left( p_e^i \right)^2 + \sum_{e \in E} a_e \left( \left( \sum_{i \in I} \delta_e^i p_e^i d_i \right)^2 - \sum_{i \in I} \delta_e^i \left( p_e^i \right)^2 d_i \right),
\end{aligned}$$

where the first equality holds from (12). As $d_i \geq 1$ for any $i \in I$, we have

$$
\begin{aligned}
T(\mathbf{p}) &\geq \sum_{e\in E}(a_e v_e + b_e v_e) + \sum_{e\in E}\sum_{i\in I}\delta_e^i a_e \sigma_i^2 \left(p_e^i\right)^2 \\
&\quad + \sum_{e\in E} a_e \left(\left(\sum_{i\in I}\delta_e^i p_e^i d_i\right)^2 - \sum_{i\in I}\delta_e^i \left(p_e^i\right)^2 d_i^2\right) \\
&\geq \sum_{e\in E}(a_e v_e + b_e v_e) + \sum_{e\in E}\sum_{i\in I}\delta_e^i a_e \sigma_i^2 \left(p_e^i\right)^2 \\
&\geq \sum_{e\in E}(a_e v_e + b_e v_e), 
\end{aligned} \tag{17}
$$

where the second inequality is from AM-GM inequality. Consequently, we obtain

$$
\begin{aligned}
\mathbf{f}^{*T}\mathbf{t}(\overline{\mathbf{p}}) &= \sum_{i\in I}\sum_{k\in P_i}\left(f_k^i\right)^* t_k^i(\overline{\mathbf{p}}) \leq \sum_{i\in I}\sum_{k\in P_i}\left(f_k^i\right)^* \sum_{e\in k}\mathbb{E}[c_e(\overline{V}_e + 1)] \\
&= \sum_{e\in E} v_e^* \mathbb{E}[c_e(\overline{V}_e + 1)] = \sum_{e\in E} v_e^* \left(a_e \overline{v}_e + a_e + b_e\right) \\
&= \sum_{e\in E} a_e \overline{v}_e v_e^* + \sum_{e\in E}\left(a_e v_e^* + b_e v_e^*\right) \leq \sqrt{T(\overline{\mathbf{p}})T(\mathbf{p}^*)} + T(\mathbf{p}^*),
\end{aligned}
$$

where the last inequality holds because of (16) and (17). □

With all the preparations, we are now arriving at an upper bound of the PoA.

**Theorem 1** (Upper bound) *Let $\mathcal{I}$ be the set of atomic congestion games with random players and affine cost functions. Then*

$$
R(\mathcal{I}) \leq \begin{cases} \dfrac{A(A+2)+\sqrt{A(A+4)}}{2}, & \text{if } \overline{\theta} \geq \sqrt{2}, \\ \min\left\{\dfrac{A(A+2)+\sqrt{A(A+4)}}{2}, \dfrac{5(1+\overline{\theta}^2)}{2-\overline{\theta}^2}\right\} & \text{if } \overline{\theta} < \sqrt{2}, \end{cases}
$$

*where $A = 1 + \overline{\theta}^2$.*

***Proof*** From Lemma 1 and 3, we have

$$
T(\overline{\mathbf{p}}) \leq (1+\overline{\theta}^2)\left(\frac{5}{3}T(\mathbf{p}^*) + \frac{1}{3}T(\overline{\mathbf{p}})\right),
$$

which is equivalent to

$$
\left(2-\overline{\theta}^2\right)T(\overline{\mathbf{p}}) \leq 5\left(1+\overline{\theta}^2\right)T(\mathbf{p}^*).
$$

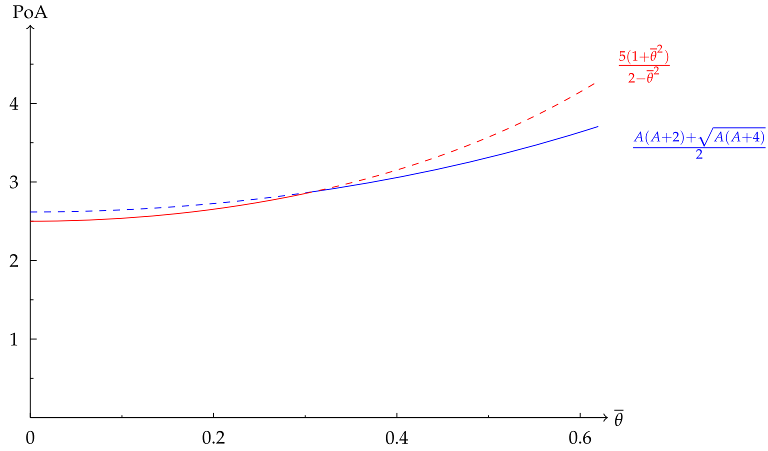


**Fig. 2** Upper bound of the PoA

Thus when $2 - \overline{\theta}^2 > 0$, we have

$$\frac{T(\overline{\mathbf{p}})}{T(\mathbf{p}^*)} \leq \frac{5(1+\overline{\theta}^2)}{2-\overline{\theta}^2}. \tag{18}$$

Similarly from Lemmas 1 and 4, we have

$$T(\overline{\mathbf{p}}) \leq (1+\overline{\theta}^2)\left(\sqrt{T(\mathbf{p})T(\mathbf{p}^*)} + T(\mathbf{p}^*)\right),$$

from which we can derive another upper bound for the whole range of $\overline{\theta}$ as

$$\frac{T(\overline{\mathbf{p}})}{T(\mathbf{p}^*)} \leq \frac{(\overline{\theta}^2+1)(\overline{\theta}^2+3)+\sqrt{(\overline{\theta}^2+1)(\overline{\theta}^2+5)}}{2}. \tag{19}$$

The theorem is proved by combining (18) and (19). □

Figure 2 illustrates the upper bound of the PoA in Theorem 1.

***Remark 1*** When demands return to deterministic, i.e., $\overline{\theta} \to 0$, the bound in Theorem 1 matches the deterministic bound $5/2$ for un-weighted atomic congestion games in (Christodoulou and Koutsoupias 2005; Awerbuch et al. 2005), which implies that the bound in Theorem 1 is tight in the special case.

***Remark 2*** The bounding technique is extended from (Awerbuch et al. 2013) by considering the randomness of player number. With stochastic demand, we involve variance of $\sigma_e = \sum_{i\in I}\delta_e^i\left(\sigma_i^2\left(p_e^i\right)^2 + p_e^i(1-p_e^i)d_i\right)$, $e \in E$, in computing the expected total cost, which significantly increase the difficulty of the bounding work.

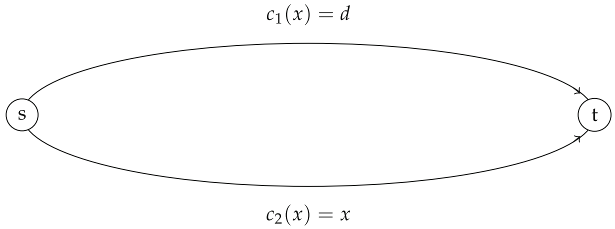


**Fig. 3** Two-link network

## 5 Lower bound

In this section, we provide a lower bound of the price of anarchy with affine cost functions by the following example.

***Example 2*** Consider a two-link network in Fig. 3. A random number $D$ of players have to move from node $s$ to $t$. They can choose either the upper link with constant cost of $d = \mathbb{E}[D]$, or the lower link with cost of $x$.

Apparently, all the players will choose the lower link at the equilibrium, thus the expected total cost is $\mathbb{E}[D^2] = d^2 + \sigma^2$, where $\sigma$ is the standard derivation of the demand. At the same time, we can find the optimal strategy by minimizing the expected total cost, which can be written as

$$\begin{aligned} T[\mathbf{p}] &= \mathbb{E}[V_1 d] + \mathbb{E}[V_2^2] = v_1 d + v_2^2 + \sigma^2 (p_2)^2 + p_2(1-p_2)d \\ &= (d^2 + \sigma^2 - d)p_2^2 - (d^2 - d)p_2 + d^2 \end{aligned}$$

The optimal solution is attained at

$$\mathbf{p}^* = \left(1 - \frac{d^2 - d}{2(d^2 + \sigma^2 - d)}, \frac{d^2 - d}{2(d^2 + \sigma^2 - d)}\right),$$

and the minimal expected total cost is

$$T(\mathbf{p}^*) = \frac{d\left(d^2\left(4\theta^2 + 3\right) - 2d - 1\right)}{4\left(d\theta^2 + d - 1\right)}.$$

Thus we can compute the PoA as

$$PoA = (d^2 + \sigma^2)/T(\mathbf{p}^*) = \frac{4d\left(\theta^2 + 1\right)\left(d\theta^2 + d - 1\right)}{d^2\left(4\theta^2 + 3\right) - 2d - 1}, \tag{20}$$

where $\theta = \sigma/d$. The PoA in (20) is increasing with $d$ by its derivative, thus the following supremum is a lower bound of the PoA for general instances with affine cost functions

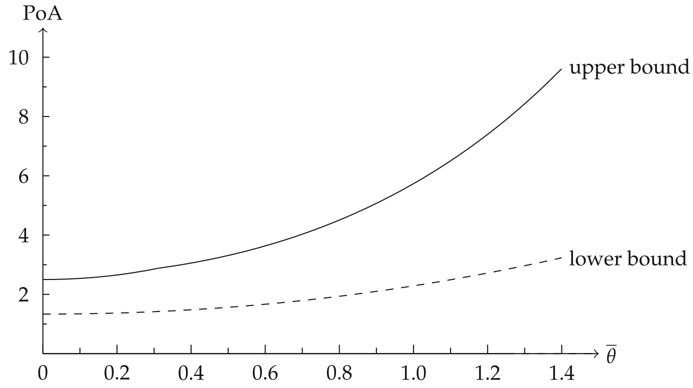


**Fig. 4** Upper bound and lower bound of the PoA

$$\lim_{d\to\infty} \frac{4d\left(\theta^2+1\right)\left(d\theta^2+d-1\right)}{d^2\left(4\theta^2+3\right)-2d-1} = \frac{4\left(\theta^2+1\right)^2}{4\theta^2+3}.$$

**Theorem 2** *Let $\mathcal{I}$ be the set of atomic congestion games with random players and affine cost functions. Then we have*

$$R(\mathcal{I}) > \frac{4\left(\theta^2+1\right)^2}{4\theta^2+3}.$$

Figure 4 illustrates both the upper and lower bound in Theorem 1 and 2. The lower bound goes up as the variation rate of players' number $\theta$ increases, and approaches to infinity when $\theta \to \infty$. Thus the equilibrium can be extremely inefficient when the variation of players' number is large.

***Remark 3*** Theorems 1 and 2 show that the bounds of PoA in our model depend on the coefficients of variation and increase with the level of uncertainty, which are quite intuitive, given that when the variation is high, it will be more difficult to balance individual costs and the social cost. In contrast, Cominetti et al. (2019, 2020) consider the Bernoulli congestion games and the PoA in their model is *maximized* to $5/2$ as uncertainty decreases. This striking difference on the behavior of PoA bounds comes from the fact that the two models are very different. We focus on network congestion games with uncertain number of players for each O-D pair and follow the framework of population uncertainty proposed by Myerson (1998) with the assumption that players of a same O-D pair have the same (mixed) strategy at equilibrium. In contrast, in the model of Cominetti et al. (2019), players are *not* assumed to join the game, instead they do so independently with known probabilities (common knowledge). They use Bayesian Nash equilibrium concept to transform the stochastic model into an equivalent deterministic game with (unconditional) expected costs, while our approach is to define the equilibria using the conditional expected costs, i.e., players only pay attention to the cost when (not *if*) they participate in the game.

## 6 Conclusions

In this paper, we have presented a general model for atomic congestion games with random number of players. The notion of mixed strategies has been adopted to model equilibrium and system optimum with random players to describe players' and coordinator's behaviors.

Based on our reformulation of the equilibrium condition as a variational inequality problem, we have proved the existence and non-uniqueness of equilibria for our new model. We have provided upper bounds of the price of anarchy for affine cost functions, which are proved to be tight in some special cases including the deterministic case and the extreme case of infinite number of players.

Our bounding approach can be extended to become applicable for a wider range of cost functions, such as polynomial cost functions. However, the actual analysis, which would not provide additional insights, is much more complicated, since each link flow is a sum of independent compound random variables and its higher moments are very complex. Even though the lower bound in this study goes to infinity when $\overline{\theta} \to \infty$, there still exists a gap between the upper and lower bound when $\overline{\theta}$ is small, which suggests that better lower bound is worth investigating further in future study.

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