Existence of Periodic Travelling Waves to Reaction-Diffusion Equations with Excitable-Oscillatory Kinetics

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Widmung

Für meine Eltern,

Friedrich und Blandine Haaf

und zum Andenken an meinen Großvater

Wilhelm Haaf.

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H. H.

Declaration

The results in this thesis are, to the best of my knowledge, original, except where attributed to others.

Conventions and Notation

Conventions

Criticality of the Hopf Bifurcation

We call the Hopf bifurcation ***supercritical*** if the bifurcating periodic solutions exist on the side of the bifurcation curve for which the real parts of the critical eigenvalues of the linearization of the vector field at the rest point are positive. Similarly, it is called ***subcritical*** if the bifurcating periodic solutions exist on the side of the bifurcating curve for which the real parts of the critical eigenvalues are negative.

Order Symbols

In the definitions which follow $\phi(x;\epsilon)$ and $\psi(x;\epsilon)$ are real-valued functions of the variable x contained in some domain D and a small positive parameter ϵ. The behaviour of these functions as ϵ goes to zero can be compared by using the Landau order symbols $\mathcal{O}$, o and $\mathcal{O}_s$.

Large $\mathcal{O}$

We say $\phi = \mathcal{O}(\psi)$ for $\epsilon \to 0$ if there exist constants K and ϵ_0 such that $|\phi| \leq K|\psi|$ for $0 < \epsilon < \epsilon_0$ and uniformly in D.

Small o

We say $\phi = o(\psi)$ if $\lim_{\epsilon\to 0} \frac{\phi(x;\epsilon)}{\psi(x;\epsilon)} = 0$ uniformly in D, provided that $\psi \neq 0$.

Sharp $\mathcal{O}_s$

We say $\phi = \mathcal{O}_s(\psi)$ if $\phi = \mathcal{O}(\psi)$ and $\phi \neq o(\psi)$.

Notation & Abbreviations

TW	travelling wave
θ	propagation speed of travelling wave
$W^u(u_0)$, $W^s(u_0)$	unstable and stable manifold of the rest point u_0, resp.
$(\mathrm{FN};\theta,\varepsilon)$	travelling wave equations to the FitzHugh-Nagumo system
D	discriminant of a cubic equation, p. 27
β	perturbation parameter from the $\theta = \infty$ limit, $\beta = \frac{1}{\theta^2}$
(fast eqns.; β)	p. 38
(slow eqns.; β)	p. 38
$\mathcal{E}_\beta$	slow submanifold of (fast eqns.; β) for to $0 \ll \beta < 1$, p. 37
$S := S_\varepsilon$	slow submanifold of $(\mathrm{FN};\theta,\varepsilon)$ for $0 \ll \varepsilon < 1$, p. 44
S_i (i=1,2)	stable parts of S, p.44
Π_i (i=1,2)	projection of S_i on its third coordinate, p. 44
$u_i(w)$ (i=1,2), $\bar{u}(w)$	rest point of the fast system $(\mathrm{FN};\theta,0)$ for a fixed w
$\mathcal{F}_w(u)$	potential, p. 47
$H(u,v)$	Hamiltonian function, p. 48
$\Lambda^r(\theta,w)$	p. 51
$\theta^*(a)$	p. 54
B	block, p. 57
$b^\pm$	entrance- and exit set of B, p. 57
$T^\pm(u)$	time needed for u to hit $b^\pm$, p. 57
$\Phi^\pm$	p. 58
$D^\pm$	p. 58
Δ	subset of b_2^-, p. 58
δ_i (i=0,1)	lower and upper boundary of Δ
β_i (i=0,1)	connectedness components of the complement of Δ in b_2^-
$B_i(w)$ (i=1,2)	block of the fast system $(\mathrm{FN};\theta,0)$ for a fixed w, p. 62
B_i (i=1,2)	block of the full system $(\mathrm{FN};\theta,\varepsilon)$
$\Lambda_1(\theta,w)$	branch of $W^s(u_1(w),0)$, p. 63
$\Lambda_2(\theta,w)$	branch of $W^u(u_2(w),0)$, p. 63
$\Lambda^\varepsilon(\theta)$	branch of $W^u(0)$ with respect to $(\mathrm{FN};\theta,\varepsilon)$, p. 64
$\Lambda^0(\theta)$	corresponding branch for $(\mathrm{FN};\theta,0)$
Σ	subset of b_1^-, p. 68
ξ_i (i=0,1)	lower and upper boundary of Σ
α_i (i=0,1)	connectedness components of the complement of Σ in b_1^-

Chapter 0

Introduction

"Die Mathematiker sind eine Art Fransosen; redet man su ihnen, so übersetsen sie es in ihre Sprache, und dann ist es alsobald etwas gans Anderes."

Goethe

This thesis is about the dynamics of excitable and oscillatory systems.

So far there has been no general definition of excitability. We give a phenomenological description in that we call a system *excitable* if it exhibits an "all or none"-threshold behaviour. This means it has a stable rest state from which small disturbances get damped and rapidly die out. Disturbances, however, that exceed a certain threshold trigger the excitable medium into an abrupt and big excursion. This is followed by a spontaneous approach back towards the rest state during which it is typically refractory to further stimulation for some time before it recovers its full excitability. This sequence of events can be pictured by a phase plane diagram shown in Figure 0.1.

The best known physical example of an excitable system is a nerve cell, which gives rise to a neural action potential depicting the propagation of an electrical impulse along the nerve axon. A neural action potential is only developed if the external stimulus is beyond a certain threshold. Sub-threshold stimuli of the nerve cell do not show a significant response.
The gliding- and aggregation behaviour of the social amoebae "dictyostelium discoideum" shows a chemotactical[1] reaction to the substance cAMP, in which

[1] *Chemotaxis* is the chemically directed movement.

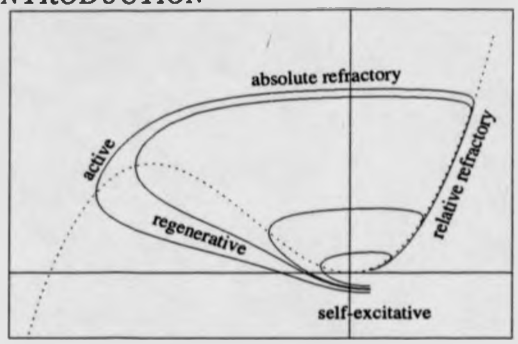

Figure 0.1: Physiological state diagram

waves of biochemical activity are observed during the aggregation of the amoebae in *slime molds*. Here, too, excitability is at play. A single "social amoeba" wanders about the substrate voraciously consuming bacteria. As they feed, the amoebae divide by fission. Eventually the population outruns its food supply. During the next few hours an internal process takes place by which these formerly independent cells become more alert and responsive to their neighbours. Depending on conditions a cell may become *spontaneously* active whereupon it emits a pulse at fixed time intervals, or it may *only* emit a pulse when triggered by its neighbours through a sufficiently high cAMP concentration. The latter explains, of course, the excitability feature of the system.

Other examples of excitable systems include certain chemical reactions, specifically the famous Belousov-Zhabotinsky reaction,[2] autocatalytic reactions etc.

Oscillatory behaviour simply means the existence of a spontaneously oscillating "pacemaker" so that a persistent wave pattern can develop. Mathematically, this corresponds to the existence of a stable limit cycle solution of the associated differential equation model.
Examples of biological oscillators are the so-called "circadian"[3] rhythms, the internal "biological clocks," which are supposed to underlie the persistent rhythm of physiological activity, compare [Win80]. A concrete example is given by the pacemaker neurons in the heart giving rise to the cardiac rhythm.

Nerve cells can under certain conditions also exhibit oscillatory behaviour, which

[2] More precisely, what has been called by Winfree [Win72] the Z reagent.
[3] Latin: "roughly daily".

Figure 0.2: Spiral wave in two dimensional excitable media

simply corresponds to spontaneous periodic "firing." The original reagent of the Belousov-Zhabotinsky reaction, discovered by Belousov, is oscillatory. The gliding and aggregation behaviour of social amoebae can also be oscillatory as implicit in the above description.

Our approach to the study of these phenomena is deterministic by use of reaction-diffusion equations. An alternative approach is by stochastic methods, specifically by cellular automata. We refer to the thesis of A. Stevens [Ste92] for an interesting cellular automaton simulation of the aggregation behaviour of myxobacteria, the "true" slime mold [Rei86], as well as an approximation of the chemotaxis equations as limit dynamics of moderately interacting stochastic processes.

We consider ***reaction-diffusion*** equations of the form

$$U_t = F(U) + D\,\nabla^2 U, \tag{0.1}$$

where $U = (u_1, \ \dots \ , u_n)$ is a vector of chemical concentrations or species in a population model etc.; $U = U(x, t)$, where t denotes time and the vector x is the spatial variable, which may have any number of dimensions.

$$U_t = F(U) \tag{0.2}$$

are the *kinetic equations*[4] which give rise to the *local dynamics* describing *homogeneous*, i.e. spatially constant solutions. Finally, D is a positive definite matrix of diffusion coefficients and the term $D\,\nabla^2 U$ is the standard model for diffusion, where

$$\nabla^2 = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$$

is the *Laplace* operator in the spatial coordinates. We remark that in population models diffusion is interpreted as migration of the species.

In a pioneering paper in 1952 [Tur52], Turing suggests that some patterns that occur in biology result from an interaction between a chemical reaction and diffusion. Turing concentrates on reaction-diffusion equations with linear reaction part and shows that they are capable of solutions which vary in space.

In general, however, a nonlinear kinetics is needed to stabilize spatially non-homogeneous patterns.

Generally speaking, the *pattern formation problem* for a reaction-diffusion equation is to find solutions, attractors that are non-homogeneous in space and stable as a process of time. One of the possible spatio-temporal scenarios are travelling waves, which we shall encounter later.

We focus on generic features captured by all excitable-oscillatory systems rather than giving a detailed exposition of numerous mathematical models.
It is common for dynamical models, including the celebrated model of Hodgkin and Huxley [HH52] for the propagation of nerve signals in a squid giant axon, to exhibit either excitable or oscillatory behaviour, depending on the choice of parameters.

In the following we concentrate on a particular nonlinear reaction-diffusion equation on the real line called the *FitzHugh-Nagumo* (FN) system

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \varepsilon u \end{aligned} \tag{0.3}$$

where $0 < \varepsilon \ll 1$ and f is the cubic nonlinearity

$$f(u) = u(u-a)(1-u). \tag{0.4}$$

[4]Also called *reaction equations* giving rise to the *reaction flow*.

In Chapter 1 we shall see that with respect to the kinetic equations $a > 0$ and ε small corresponds to the excitable and $a < 0$ to the oscillatory regime. Here the perturbation parameter ε needs to be small to get excitable behaviour.

The FitzHugh-Nagumo equations were originally formulated as a simplification of the four dimensional nonlinear system of the Hodgkin-Huxley equations, see FitzHugh [Fit61]. The most important change is the reduction of the number of "slow" variables from two to one. Since then they have become a central example in reaction-diffusion equations, because of their mathematical tractability and the rich structure they exhibit.

For us the FitzHugh-Nagumo equations serve as a simple representative of a class of excitable-oscillatory systems.

It is intuitively suggestive that for the excitable kinetics disturbances of the rest state propagate as "travelling pulses". A *travelling wave* (TW) is customarily taken to be a wave which travels without change of shape and with constant velocity in the direction of its propagation.
Mathematically, we mean by travelling waves bounded non-constant solutions to (0.3) of the form

$$(u(x,t), w(x,t)) = (u(z), w(z)), \quad \text{where } z = x + \theta\, t \quad \text{for} \quad \theta > 0.$$

Differentiating with respect to z and introducing $v = \dot{u}$ as a new variable we obtain the following system of three first order ODE's from (0.3)

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \theta v - f(u) + w, \\ \dot{w} &= \tfrac{\varepsilon}{\theta} u, \end{aligned} \tag{0.5}$$

where $\dot{} = \frac{d}{dz}$.

The travelling wave moves to the left with time if the wave speed is positive; it travels to the right if $\theta < 0$. For this reason we shall only consider positive values of the wave speed θ as otherwise waves simply travel in the opposite direction.
Two types of solutions to the travelling wave equations (0.5) are of particular interest: Periodic *wave trains* and *pulses*. Solutions of (0.5) which are periodic with respect to z correspond to (periodic) wave trains.

Similarly, pulses correspond to *homoclinic* solutions of the travelling wave equations which are bi-asymptotic to a rest point with respect to the travelling wave variable z. Thus, for the FitzHugh-Nagumo system (0.5) homoclinicity to its unique rest point at the origin means the existence of a solution $(u(z), v(z), w(z))$ such that

$$\lim_{|z| \to \infty} (u(z), v(z), w(z)) = (0,0,0).$$

Travelling wave solutions to reaction-diffusion equations represent an *asymptotic state*[5] to a wide class of initial value problems. This means an asymptotic equivalence class of solutions, which ultimately approach the same solution, neglecting transient effects. Stable asymptotic states, e.g. travelling waves, are important as they show up in applied contexts. The stability of travelling pulses to the FitzHugh-Nagumo system as solutions of the partial differential equation (0.3) is addressed in [Jon84].

We want to examine the behaviour of the travelling waves to (0.5) near the transition from excitable to oscillatory behaviour. Our aim is to show that both oscillatory and excitable kinetics support travelling waves, the former wave trains and the latter pulses, as is expected from the excitability. Pulse solutions appear as limits of families of wave trains solutions existing in the excitable regime when their wavelength goes to infinity. Roughly speaking, our analysis suggests that a distinction between the excitable and oscillatory regime for the dynamics of the travelling wave equations is fairly arbitrary.
The transition between the wave phenomena in the two regimes, however, depends on the value of the wave speed and can be very complicated.

The classic paper by Kolomogorov, Petrovsky and Piscounov [KPP37], published in 1937, was written at the beginning of the investigations of travelling waves in reaction-diffusion equations. The authors showed that the single reaction diffusion equation

$$u_t = u_{xx} + u(1-u)$$

admits for $\theta > 2$ a wave front[6] solution $u(x - \theta t)$. This reaction-diffusion equation, suggested by Fisher [Fis37], is meant to describe the spatial spread of an advantageous gene in a population.

[5] Compare Fife [Fif79].

[6] Waves which approach distinct rest states in the limit as $z \to \pm\infty$.

The organization of this thesis is as follows:
In Chapter 1 we give a thorough discussion of the periodic solutions to the kinetic equations (0.3) such as small amplitude periodic solutions from the Hopf bifurcation and relaxation oscillations as well as so-called "canard" and "phantom ducks" trajectories. Phantom ducks, introduced by Braaksma [Bra93], are closely related to the excitability feature of the system. It is interesting to recall that FitzHugh [Fit61] was already in 1961 observing canard trajectories in analog computer simulations of the Bonhoeffer-van der Pol equation.[7] He referred to them as "no man's land", as they are very hard to track. Canard type trajectories occur at the transition from excitable to oscillatory dynamics in the kinetic equations. The topic of this thesis is about the analogous behaviour when diffusion is added.
In Section 1.2 we compute under general assumptions the stability of relaxation oscillations and canard type limit cycles for a class of differential equations in the plane of which the kinetic equations of the FitzHugh-Nagumo system are a specific example.

In Chapter 2 we investigate the stability of the rest state, representing the trivial solution to travelling wave equations of the FitzHugh-Nagumo system. This involves determining the direction in which the periodic solution emanating from the Hopf bifurcation branches. Our analysis does not make use of any approximations and also incorporates the oscillatory regime with the kinetic equations in the limit as θ tends to infinity.

Chapter 3 deals with periodic travelling waves as perturbation from the infinite wave speed limit in the spirit of Kopell [Kop77]. It turns out that in the infinite wave speed limit the travelling wave equations correspond (up to a rescaling) to the kinetic equations discussed previously. It is then not surprising that if the diffusion coefficient[8] is small as compared to the wave speed, structurally stable periodic solutions of the kinetic equations perturb into periodic travelling waves. For large wave speeds the dynamics of the kinetic equations occurs on a two dimensional "slow submanifold". Since all periodic solutions to the kinetic equations are stable this holds for all types of periodic solutions. This implies, in particular, the existence of periodic travelling waves with canard profile living

[7] Basically the reaction part of the FitzHugh-Nagumo system.
[8] Which is here taken to be 1.

on a two dimensional invariant manifold.

In Chapter 4 we exploit the singular perturbation nature of the travelling wave equations of the FitzHugh-Nagumo system to formally construct "singular solutions." By separation of the time scales we are able to split the three-dimensional travelling wave equations into two lower dimensional systems corresponding to fast and slow time. The singular periodic and homoclinic travelling wave solutions, obtained by setting $\varepsilon = 0$, are then given as the piecewise smooth union of solution segments to the different systems. We recall work of Casten, Cohen and Lagerstrom [CCL75], who derived an explicit expression for the connecting orbits between saddles in the fast time system, forming part of the singular solutions. We generalize their work, which is exclusively for the excitable regime, to the oscillatory one and also extend it in another direction, too. In that we consider "degenerate" singular connections, that is orbits of the "fast" system, which connect a rest point of saddle type to one of saddle-node type.
We give a complete classification of all possible singular periodic and homoclinic solutions in the parameters a and θ connecting periodic travelling waves in the excitable regime with the homogeneous (spatially independent) oscillations of the kinetics equations, which exist for negative a in the limit as θ goes to ∞.

In the last chapter we deal with the persistence of the singular periodic and homoclinic travelling waves, whose existence we established in the previous chapter. The method of proof is of topological nature and goes back to Conley [Con75] and Carpenter [Car77]. It uses fairly sophisticated perturbation arguments. We begin by re-proving the results of Carpenter for the excitable regime, which serves as an introduction to the more complicated persistence result of the degenerate periodic solutions. We have also changed the proofs in that we have made use of the inherent symmetry of the cubic nonlinearity with respect to the inflection point, which is reflected in the construction of the blocks around the "slow submanifold". In order to demonstrate the persistence of the degenerate periodics we stretch the method of proof applied to the standard periodics to its very limits. Finally, we would like to point out that the persistence of the degenerate singular solutions is not covered by any of the known persistence proofs, neither of topological nor of asymptotic nature.

Chapter 1

The Kinetic Equations

1.1 Existence of Periodic Solutions

The *kinetic equations* to (0.3) are

$$\begin{aligned} \frac{du}{dt} &= f(u) - w, \\ \frac{dw}{dt} &= \varepsilon u, \end{aligned} \tag{1.1}$$

where $f(u) = u(u-a)(1-u)$ and $\varepsilon > 0$ is small. They are obtained by disregarding the diffusion term in (0.3) and describe *homogeneous*, i.e. spatially constant solutions to the original system (0.3).

We will discuss the dynamics of (1.1) by transforming them to a standardized form considered by W. Eckhaus [Eck83] and by applying results by him and B. Braaksma [Bra93] to establish the existence of periodic orbits. This will, in particular, show the existence of *canard* and *phantom duck* trajectories.

We now describe in detail the necessary coordinate changes.

Our first transformation is to shift the local minimum (u_{min}, w_{min}) of the cubic f to the origin by means of $\overline{u} = u - u_{min}$ and $\overline{w} = w - w_{min}$. We also transform the nonlinearity to $\overline{f}(\overline{u}) := f(\overline{u} + u_{min}) - w_{min}$, from which it is clear that $\overline{f}(0) = \overline{f}'(0) = 0$. Expanding $f(\overline{u}+u_{min})$ around u_{min} shows that $\overline{f}(\overline{u}) = \delta\overline{u}^2 - \overline{u}^3$, where $\delta := \sqrt{a^2 - a + 1}$. With respect to the coordinates $(\overline{u}, \overline{w})$ the system (1.1) reads

$$\begin{aligned} \frac{d\overline{u}}{dt} &= \overline{f}(\overline{u}) - \overline{w}, \\ \frac{d\overline{w}}{dt} &= \varepsilon(\overline{u} - \alpha), \end{aligned} \tag{1.2}$$

where $\alpha = -u_{min}$ and $u_{min} = \frac{1}{3}(a + 1 - \delta)$.

Thus $\alpha = \alpha(a)$, where

$$\alpha(a) := -\frac{1}{3}(a + 1 - \delta). \tag{1.3}$$

Next we set $\tilde{u} := -\bar{u}$, $\tilde{w} := \overline{w}$ and introduce a new nonlinearity by

$$\tilde{f}(\tilde{u}) \stackrel{def}{=} \bar{f}(-\tilde{u}).$$

Furthermore, we define functions g and h through

$$\tilde{f}'(\tilde{u}) = \tilde{u}\, g(\tilde{u}) \quad \text{and} \quad \tilde{f}'(\tilde{u}) = (\tilde{u} + \ell)\, h(\tilde{u}),$$

where $g(\tilde{u}) = 2\delta + 3u$, $\ell := -\tilde{u}_{min} = \frac{2}{3}\delta$ and $h(\tilde{u}) = 3\tilde{u}$.
Rescaling the time variable by $s = \varepsilon t$, we eventually obtain the equations in the form considered by Eckhaus

$$\begin{aligned} \varepsilon \tfrac{d\tilde{u}}{ds} &= \tilde{w} - \tilde{f}(\tilde{u}), \\ \tfrac{d\tilde{w}}{ds} &= -(\tilde{u} + \alpha), \end{aligned} \tag{1.4}$$

where the transformed cubic $\tilde{f}(\tilde{u}) = \delta\tilde{u}^2 + \tilde{u}^3$ depends also on the bifurcation parameter α, as δ can be expressed as a function of α.

However, we investigate (1.4) for a fixed $\delta > 0$, disregarding its relation with (1.1) for a while. The only rest point of (1.4) is $(-\alpha, \tilde{f}(-\alpha))$. The eigenvalues of the linearization of (1.4) at the rest point are given by

$$\lambda_\pm(\alpha) := \frac{1}{2\varepsilon}\left(-\tilde{f}'(-\alpha) \pm \sqrt{(\tilde{f}'(-\alpha))^2 - 4\varepsilon}\right). \tag{1.5}$$

Note, that for α sufficiently close to 0 or ℓ, the $\tilde{u}$-values of the local extrema the $\tilde{f}$, the eigenvalues become complex. The rest point is unstable for $0 < \alpha < \ell$ and stable for $\alpha < 0$ or $\alpha > \ell$.

For $\alpha = 0$ and $\alpha = \ell$ we have a pair of purely imaginary eigenvalues given by $\pm i\frac{1}{\sqrt{\varepsilon}}$. In order to make sure that they correspond to Hopf bifurcations we need to check that they cross the imaginary axis with non-zero speed. This is true in both cases as

$$\frac{d}{d\alpha}\mathrm{Re}\,\lambda_\pm(\alpha) = \frac{1}{2\varepsilon}\tilde{f}''(-\alpha) = \begin{cases} \frac{\delta}{\varepsilon} & \text{for} \quad \alpha = 0, \\ -\frac{\delta}{\varepsilon} & \text{for} \quad \alpha = \ell. \end{cases} \tag{1.6}$$

Thus, as the parameter α crosses zero from the left and ℓ from the right, the stable rest point becomes unstable and a branch of small amplitude periodic solutions bifurcates from the rest point in either case.

Under the change of coordinates $(\bar{u},\bar{w}) = (-\frac{1}{\sqrt{\varepsilon}}x, y)$ the system (1.4) transforms for $\alpha = 0$ to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{\varepsilon}} \\ \frac{1}{\sqrt{\varepsilon}} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{\varepsilon^2} \begin{pmatrix} \delta\sqrt{\varepsilon}x^2 - x^3 \\ 0 \end{pmatrix}, \tag{1.7}$$

where $\dot{} = \frac{d}{ds}$. Now we can apply the stability formula for two-dimensional systems given in [GH90] on page 152. The determining coefficient is easily computed to be $-\frac{3}{8}\frac{1}{\varepsilon^3}$ so that the periodic solutions bifurcating from $(0,0)$ are (strongly) stable limit cycles by Theorem 3.4.2 of [GH90]. To analyze the Hopf bifurcation at $\alpha = \ell$ we need to shift $(-\ell, \bar{f}(-\ell))$ to the origin which we combine with the above coordinate change, i.e. $(\bar{u},\bar{w}) = (-\ell - \frac{1}{\sqrt{\varepsilon}}x, \bar{f}(-\ell) + y)$, to obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{\varepsilon}} \\ \frac{1}{\sqrt{\varepsilon}} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{\varepsilon^2} \begin{pmatrix} \varepsilon(2\ell\delta - 3\ell^2)x + \sqrt{\varepsilon}(\delta - 3\ell)x^2 - x^3 \\ 0 \end{pmatrix}, \tag{1.8}$$

where $\dot{} = \frac{d}{ds}$.
Again, by an application of the stability formula the determining coefficient turns out to be the same as above so that the periodic solutions bifurcating from $(-\ell, \bar{f}(-\ell))$ are also (strongly) stable limit cycles. Thus, the Hopf bifurcations for $\alpha = 0$ and $\alpha = \ell$ are both supercritical.

We now come to discuss periodic solutions with large amplitude. We begin with the following observation. For $\varepsilon = 0$ the cubic curve $\bar{w} = \bar{f}(\bar{u})$ consists entirely of rest points of (1.4). It is called *slow submanifold*. We refer to the outer branches of the slow submanifold, where $\bar{f}' > 0$ as its *stable* part and to the inner branch, where $\bar{f}' < 0$, as its *unstable* part. Eliminating time in (1.4) we obtain

$$(\bar{w} - \bar{f}(\bar{u}))\frac{d\bar{w}}{d\bar{u}} = \varepsilon\bar{u}. \tag{1.9}$$

For $\varepsilon = 0$ (1.9) implies that either $\bar{w} = \bar{f}(\bar{u})$ or that $\bar{w}$ is constant. Thus orbits are for small ε almost constant except near the curve $\bar{w} = \bar{f}(\bar{u})$.

This gives rise to the definition of a *singular solution* which consists of arcs on the outer branches of the curve $\bar{w} = \bar{f}(\bar{u})$ and horizontal fast flow segments at $\bar{w} = \bar{w}_{min}$ and $\bar{w} = \bar{w}_{max}$, where $\bar{w}_{min} = 0$, $\bar{w}_{max} = \bar{f}(-\ell)$, connecting the endpoints of these arcs with each other. In connection with canards we will also admit singular solutions, where the horizontal fast flow segment can jump at any $w \in [w_{min}, w_{max}]$.

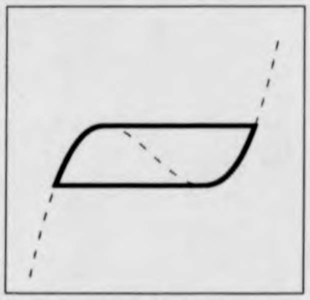

Figure 1.1: Singular relaxation oscillation

For sufficiently small $\varepsilon > 0$ there exist periodic solutions which approach the singular solution as $\varepsilon \to 0$, provided that the rest point is on the inner branch of the cubic curve. The character of this periodic solution is that of a *relaxation oscillation*, see Figure 1.1. This means that the velocity along the limit cycle is very far from being uniform, in that its velocity is along the horizontal segments very large compared with its velocity on the outer branches of the curve $\bar{w} = \bar{f}(\bar{u})$. This, of course, reflects the smallness of the parameter ε. Thus the flow jumps almost instantaneously, i.e. in a very short time interval, from one outer branch of the cubic curve to the other.

The existence of a periodic solution of relaxation oscillation type can be shown by a topological argument for any $\alpha \in (0, \ell)$, with $\alpha, \ell - \alpha \neq o(1)$. For this one constructs an annulus around the singular solution, which is rest point-free and whose diameter can be made arbitrarily small and yet is for sufficiently small ε positively invariant. Then by the Poincaré-Bendixson theorem for planar vector fields this "trapping region" will contain the limit cycle. Clearly, the limit cycle can then be made to approximate the singular solution as closely as desired by choosing a small enough annulus. For a detailed construction of the annulus we refer to Hale [Hal80], Thm. 1.7, p. 61. We remark that the cubic curve is there for simplicity taken to be symmetric with respect to the origin, which results in the standard van der Pol oscillator. It is, however, possible to build in the same way an annular region around the singular solution of the shifted cubic curve. Observe that if for $\alpha \notin [0, \ell]$ the stable rest point is on one of the outer branches of the cubic curve, and periodic solutions, which are obtained as perturbations

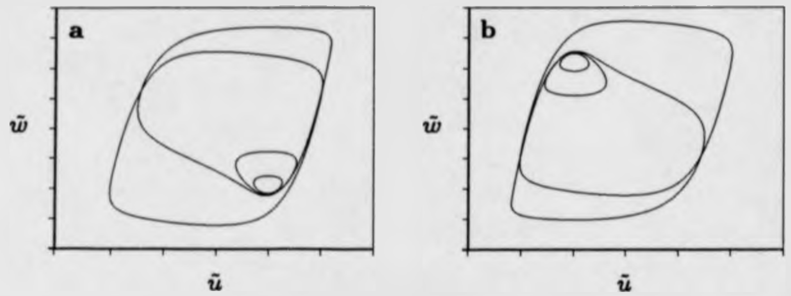

Figure 1.2: Hopf-canard -relaxation oscillation transition for the Eckhaus caricature: (a) for $\alpha \in (0, \frac{\ell}{2}]$, (b) for $\alpha \in [\frac{\ell}{2}, \ell)$

of the singular solutions, can not exist.

So far we have seen that there are small amplitude periodic solutions emanating from two Hopf bifurcation points existing to the right and left of $\alpha = 0, \ell$, respectively. We also have for $\alpha \in (0, \ell)$ attracting limit cycles, which are already for small $\alpha, \ell - \alpha$ of the type of fully developed relaxation oscillations.

The "missing" transitional medium size periodic solutions in this scenario are the so-called *canards*,[1] see Figures 1.2, 1.3. The transition from small to large amplitude limit cycles, which in practice appears to be discontinuous, can indeed shown to be continuous [CDD78], [Eck83]. Canards are specific to singularly perturbed differential equations[2] and have the defining property that they follow for some time the unstable part of the slow submanifold. They are confined to an exponentially small neighbourhood around some value $\alpha_c(\varepsilon) = \mathcal{O}(\varepsilon)$ and $\ell - \alpha_c(\varepsilon) = \mathcal{O}(\varepsilon)$, respectively. The name canard refers to their duck-shaped appearance for α slightly beyond α_c, compare Figure 1.3 (e).

We restrict our discussion of canards to the ones in the vicinity of 0, as those which exist near ℓ can be dealt with similarly. The above mentioned value $\alpha_c(\varepsilon)$ is given by

$$\alpha_c(\varepsilon) = \varepsilon \frac{g'(0)}{g(0)^3} + \mathcal{O}(\varepsilon^2). \tag{1.10}$$

[1] French: "Canard" not only means "duck", but also "false news".

[2] That is, differential equations which involve a small parameter. See p.43 for a formal definition.

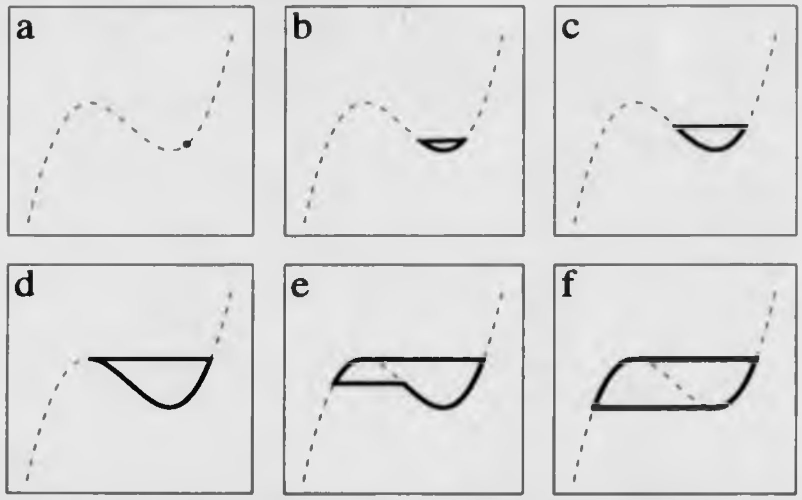

Figure 1.3: Canard transition: (a) excitable rest point for small $\alpha < 0$ (b) small amplitude limit cycle born in a Hopf bifurcation at $\alpha = 0$ (c) canard without head (d) emergence of the head for $\alpha = \alpha_c$ (e) canard with head (f) relaxation oscillation for $\alpha \gg \varepsilon$. Figure courtesy of B. Braaksma.

There are two "breeds" of canards, *sub-* and *supercritical* ones according to the direction of branching in the Hopf bifurcation. In (1.4) we encounter the simpler case of a supercritical canard[3] as $g(0) = 2\delta > 0$, $g'(0) = 3$ and therefore $\alpha_c > 0$. Here, canards exist while the rest point is unstable.
The exponentially small neighbourhood of $\alpha_c(\varepsilon)$ is given by

$$\left\{\alpha : \alpha = \alpha_c(\varepsilon) + \sigma\varepsilon^{\frac{1}{2}} \exp(-k^2/\varepsilon)\right\}, \tag{1.11}$$

where k determines the point at which the limit cycles leave the unstable part of the slow submanifold. For $\sigma < 0$ we have a canard limit cycle without head, which shrinks as k is being decreased. At $\alpha = \alpha_c(\varepsilon)$ the periodic solution passes through the local maximum of the cubic, a head is born, and finally for $\sigma > 0$ we obtain a canard type limit cycle with head, whose head shrinks again as k is increased.

Because of the variability of the rest point $(\alpha, \tilde{f}(-\alpha))$ along the curve, $\tilde{w} = \tilde{f}(\tilde{u})$, is the cubic's symmetry with respect to its inflection point reflected in the

[3]Termed subcritical by Eckhaus [Eck83].

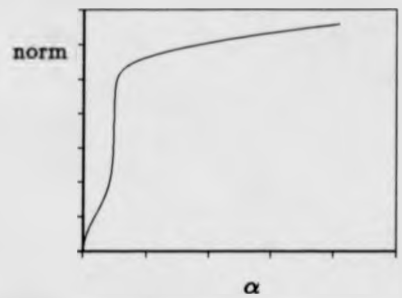

Figure 1.4: Bifurcation diagram: α versus the norm of the periodic solutions

dynamics of (1.4). This can easily be checked by means of the following coordinate transformation $(\hat{u}, \hat{w}) = (2\tilde{u}_{infl} - \tilde{u},\ 2\tilde{w}_{infl} - \tilde{w})$, describing the rotation of the $(\tilde{u}, \tilde{w})$ coordinates by π around the inflection point $(\tilde{u}_{infl}, \tilde{w}_{infl}) = (-\frac{1}{3}\delta, \frac{2}{27}\delta^3)$ of the cubic, for which (1.4) transforms to

$$\begin{aligned} \frac{d\hat{u}}{ds} &= \hat{w} - \tilde{f}(\hat{u}), \\ \frac{d\hat{w}}{ds} &= -\varepsilon(\hat{u} + \beta), \end{aligned} \tag{1.12}$$

where $\beta = \ell - \alpha$. Thus we obtain exactly the same equations again but with β replaced by α. This means that if for given fixed ε, $(\tilde{u}, \tilde{w})$ is a solution to (1.4) for $\alpha = \alpha_0$ then its image under the coordinate transformation $(\tilde{u}, \tilde{w}) \mapsto (\hat{u}, \hat{w})$ will also be a solution of the same equation for $\beta = \beta_0$, where $\beta_0 = \ell - \alpha_0$.

So the small amplitude periodic solutions growing in the Hopf bifurcation for $\alpha = \ell$ out of the rest point $(-\alpha, \tilde{f}(-\alpha))$ are identical to the ones for $\alpha = 0$, up to rotation by π round the inflection point of the cubic. The same holds for canard solutions and relaxation oscillations.

We summarize our findings about periodic solutions to (1.4) as follows:
There is a branch of periodic solutions parametrized by $\alpha \in (0, \ell)$, consisting of small amplitude periodic solutions emanating from a supercritical Hopf bifurcation at $\alpha = 0$ which grow for $\alpha = \mathcal{O}(\varepsilon)$ in a canard type fashion to large amplitude relaxation oscillations, which exist for sufficiently small ε. The fully developed relaxation oscillations attain at $\alpha = \frac{\ell}{2}$ their maximum amplitude before they shrink for $\alpha > \frac{\ell}{2}$ and eventually vanish at $\alpha = \ell$ in a (reverse) Hopf bifurcation. The approach $\alpha \nearrow \ell$ for $\ell - \alpha = \mathcal{O}(\varepsilon)$ again involves canard type limit cycles.

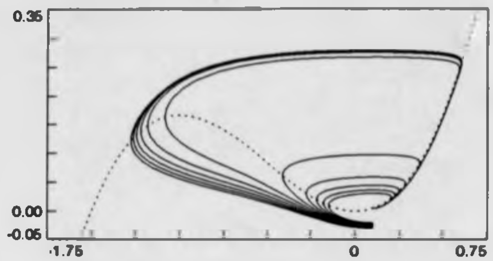

Figure 1.5: Phantom duck trajectories for a variety of initial conditions depicting the threshold behaviour ($\varepsilon = \frac{1}{100}$, $\alpha = -\frac{1}{10}$)

After having discussed the oscillatory regime of (1.4), corresponding to the existence of stable limit cycle solutions, we now come to discuss phantom ducks, which are closely related to the excitability feature of the system. *Phantom ducks* appear just before the Hopf bifurcation at $\alpha = 0$, when α is negative and the single rest point of (1.4) is stable. They are pictured in Figure 1.5. More precisely, they appear when the two small parameters $\varepsilon > 0$ and $\alpha < 0$ are related by

$$\alpha = \mathcal{O}(\sqrt{\varepsilon}),\ \varepsilon = o(\alpha).$$

Here, "phantom" refers to the fact that these duck-shaped trajectories are transient, i.e. they appear only once, before settling to rest.

For this choice of parameters a particular trajectory of (1.4) is identified as a threshold with respect to an "all or nothing" law and surrounded by a family of trajectories that we shall refer to as *phantom ducks*, see Figure 1.5.

The previous discussion of periodic solutions to (1.4) is also valid for the original kinetic equations (1.1), as (1.1) and (1.4) are related by a coordinate change. Thus we obtain qualitatively the same kind of periodic solutions for (1.1). In a sense this is to be expected since (1.1) and (1.4) have the same type of nonlinearity.

We proceed to give a brief discussion of (1.1). The eigenvalues of its linearization around its unique rest point at the origin are given by

$$\lambda_{\pm}(a) := \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4\varepsilon}\,\right). \tag{1.13}$$

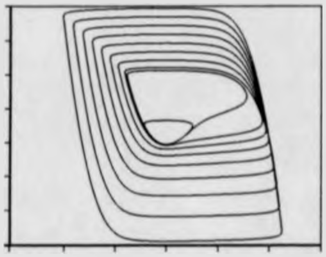

Figure 1.6: Hopf-canard-relaxation oscillation transition for FN kinetic equations

Thus the origin is stable for $a > 0$ and unstable for $a < 0$. There is a Hopf bifurcation for $a = 0$, with purely imaginary eigenvalues $\pm i\sqrt{\varepsilon}$. The transversality condition is satisfied as $\frac{d}{da}\mathrm{Re}\,\lambda_\pm(a) = -\frac{1}{2}$. Under the change of coordinates $(u, w) = (y, -\sqrt{\varepsilon}x)$ (1.1) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{\varepsilon} \\ \sqrt{\varepsilon} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ y^3 \end{pmatrix}, \qquad (1.14)$$

where $\dot{} = \frac{d}{dt}$. Applying the stability formula from [GH90], we compute the relevant coefficient to be $-\frac{3}{8}$. Thus the small amplitude periodic solutions bifurcating from the origin are stable. Thus, we have a supercritical Hopf bifurcation.
The existence of relaxation oscillations to (1.1) for $a < 0$ with $a \neq o(\varepsilon)$ can be shown in the same way as for (1.4). Furthermore, the existence of canard type limit cycles for negative a of the order $a = \mathcal{O}(\varepsilon)$ as transitional phenomenon between small amplitude limit cycles from the Hopf bifurcation at $a = 0$ and relaxation oscillations follows in analogy to our discussion for (1.4) from the fact that (1.1) and (1.4) are related by a coordinate change. The different types of periodic solutions are depicted in Figure 1.6. Phantom ducks to (1.1) are observed for positive a, just before the Hopf bifurcation, at $a = \mathcal{O}(\sqrt{\varepsilon})$, $\varepsilon = o(a)$.

Note that $\alpha(a) < \ell(a)$ for all a and that $\alpha(a)$ and $\ell(a)$ have for $a \to -\infty$ the same asymptotic behaviour governed by $-\frac{2}{3}a$. Since $\ell(a)$ is never attained by $\alpha(a)$, small amplitude periodic solutions and canard type trajectories existing in the vicinity of $\alpha = \ell$ do not show up in (1.4) with respect to the parametrization $\alpha = \alpha(a)$ and thus do not exist with respect to the original kinetic equations (1.1). Indeed, numerical pathfollowing confirms that periodic solutions grow as relaxation oscillations to infinite amplitude for $a \to -\infty$.

Henceforth, we refer to $\{a < 0\}$ *as the oscillatory regime of the kinetic equations* (1.1) *and define their excitable regime to be* $\{a > 0\}$, *assuming that* ε *is small.*

We summarize the previous analysis of (1.1) in the following proposition.

Proposition 1.1 *The kinetic equations* (1.1) *admit:*

(i) A branch of periodic solutions parametrized by a, *existing for all* $a < 0$. *Small amplitude periodic solutions emanate in a supercritical Hopf bifurcation for* $a = 0$ *from the origin and grow via canards for* $a = \mathcal{O}(\varepsilon)$ *to relaxation oscillations, which exist for sufficiently small* ε. *The amplitude of the relaxation oscillations is steadily increasing with* $|a|$ *and approaches infinity as* $a \to -\infty$.

(ii) Phantom ducks, existing for $a = \mathcal{O}(\sqrt{\varepsilon})$, $\varepsilon = o(a)$.

1.2 Stability of Canards and Relaxation Oscillations[4]

We consider the two dimensional system of singularly perturbed ODE's given by

$$\frac{dx}{dt} = \frac{1}{\varepsilon}(y - h(x)), \tag{1.15a}$$
$$\frac{dy}{dt} = -(x + \alpha), \tag{1.15b}$$

where h is a sufficiently smooth "cubic-like" function with a local minimum located at the origin, i.e. $h(0) = h'(0) = 0, h''(0) > 0$. Furthermore, we assume that $g'(0) \neq 0$, where g is defined through $h'(x) = xg(x)$. Observe that this last assumption is equivalent to $h'''(0) \neq 0$. We may want to think of $h(x) = x^2(x+\beta)$ for some $\beta > 0$ as a concrete example with a view towards an application to (1.4) and hence to the kinetic equations of the FitzHugh–Nagumo system.

We only need to determine the stability-type of canard trajectories and relaxation oscillations. The stability of small amplitude (i.e. $o(1)$–) periodic solutions, emanating from the Hopf bifurcation, has already been discussed in Section 1.1. In Appendix A we have shown that the stability-type of a limit cycle γ depends on the sign of its nontrivial Floquet exponent. In this section we compute this exponent, as given by the divergence integral (A.4) associated with the above equations (1.15a,1.15b).

We show that, except possibly for isolated values of α,

$$\oint_\gamma \operatorname{div}\, G^\varepsilon = \mathcal{O}_s(\frac{1}{\varepsilon}),$$

where the main contributions come from the stretches along the slow curve $y = h(x)$. The other parts of the limit cycle only contribute amounts of $o(\frac{1}{\varepsilon})$ to the divergence integral.

Recall that singular periodic solutions are obtained by piecing together trajectories of the fast dynamics and segments along the slow curve.

Let $(x^\varepsilon(t), y^\varepsilon(t))$ be a T^ε-periodic solution of (1.15a,1.15b). We divide the interval $[0, T^\varepsilon]$ into a finite number of subintervals

$$[T_0^\varepsilon, T_1^\varepsilon], [T_1^\varepsilon, T_2^\varepsilon], \ldots, [T_{n-1}^\varepsilon, T_n^\varepsilon]$$

[4]Joint work with B. Braaksma.

with $T_0^\varepsilon = 0, T_n^\varepsilon = T^\varepsilon$, such that each of the corresponding segments of the periodic solution converges to either a trajectory of the fast dynamics, to a piece along the slow curve or to a transitional trajectory from one of the former to the other. The segments to be considered can be characterized as follows.

- For trajectories in the fast field we have, cf. [Eck83], $y - h(x) \neq \mathcal{O}(\varepsilon)$ for all $t \in [T_{i-1}^\varepsilon, T_i^\varepsilon]$.

Our further characterization of segments is based on the quantity

$$z := y - h(x) + \varepsilon \frac{x + \alpha}{h'(x)} \tag{1.16}$$

which is a first-order approximation for the distance of the stable and unstable manifold from the slow curve. We distinguish two cases.

- $z = o(\varepsilon)$ for all $t \in [T_{i-1}^\varepsilon, T_i^\varepsilon]$. This corresponds to stretches so close to the slow curve that we can use the right hand side of (1.16) to obtain an estimate for the integral (A.4).

- $z = \mathcal{O}_s(\varepsilon)$ for some $t \in [T_{i-1}^\varepsilon, T_i^\varepsilon]$. These are short transitional segments connecting the fast field–parts of the limit cycle to trajectories along the slow curve.

Observe that the above characterization is complete, i.e. it covers all possible segments along the limit cycle.

With respect to the above dissection the integral (A.4) is, if we recall that div $G^\varepsilon = -\frac{1}{\varepsilon}\, h'$, given as

$$\oint_\gamma \operatorname{div}\, G^\varepsilon = -\frac{1}{\varepsilon} \sum_{i=1,\dots,n} \int_{T_{i-1}^\varepsilon}^{T_i^\varepsilon} h'(x^\varepsilon(t))\, dt. \tag{1.17}$$

We estimate the integrals on the right hand side of the above equation in different ways, depending on the part of the periodic solution along which we integrate. In cases (A) and (B) below we use the following transformation of the independent variable.

In segments where the periodic solution $(x^\varepsilon, y^\varepsilon)$ can be represented as a graph of some function Φ^ε, i.e.,

$$y^\varepsilon(t) = \Phi^\varepsilon(x^\varepsilon(t))$$

for $t \in [T_i^\varepsilon, T_{i+1}^\varepsilon]$, we have by putting $s := x^\varepsilon(t)$

$$-\frac{1}{\varepsilon}\int_{T_i^\varepsilon}^{T_{i+1}^\varepsilon} h'(x^\varepsilon(t))\,dt = -\frac{1}{\varepsilon}\int_{x^\varepsilon(T_i^\varepsilon)}^{x^\varepsilon(T_{i+1}^\varepsilon)} h'(s)\frac{dt}{ds}\,ds. \tag{1.18}$$

Using equation (1.15a) to compute $\frac{dt}{ds}$ we obtain

$$-\frac{1}{\varepsilon}\int_{T_i^\varepsilon}^{T_{i+1}^\varepsilon} h'(x^\varepsilon(t))\,dt = \int_{x_i^\varepsilon}^{x_{i+1}^\varepsilon} h'(s)/\{h(s) - \Phi^\varepsilon(s)\}\,ds, \tag{1.19}$$

where we have introduced the abbreviation $x_k^\varepsilon := x^\varepsilon(T_k^\varepsilon)$.

Now let us compute the contributions to the integral (A.4) of the various segments in the above decomposition.

(A) the fast field: $y^\varepsilon - h(x^\varepsilon) \neq \mathcal{O}(\varepsilon)$. This characterization of the fast field implies that the fast field trajectories are almost horizontal, they can therefore clearly be expressed as the graph of some function Φ^ε, cf. [Eck83]. Hence

$$\int_{x_i^\varepsilon}^{x_{i+1}^\varepsilon} h'(s)/\{h(s) - \Phi^\varepsilon(s)\}\,ds = o(\frac{1}{\varepsilon}), \tag{1.20}$$

since both $h'(x)$ and $|x_{i+1}^\varepsilon - x_i^\varepsilon|$ are bounded on the domain under consideration.

(B) the slow curve: $z = o(\varepsilon)$. Close to the slow curve the limit cycle can be given as the graph of some function Φ^ε with $\Phi^\varepsilon(x) = h(x) + \mathcal{O}(\varepsilon)$, cf. [Eck83]. Therefore the following ansatz is justified:

$$y^\varepsilon(t) = h(x^\varepsilon(t)) + \varepsilon\varphi^\varepsilon(x^\varepsilon(t)) \tag{1.21}$$

for some family of functions $\{\varphi^\varepsilon\}$, which is bounded for all x between x_j^ε and x_{j+1}^ε as $\varepsilon \to 0$. Differentiating the ansatz with respect to t gives

$$\frac{dy^\varepsilon}{dt} = \{h'(x^\varepsilon) + \varepsilon\varphi^{\varepsilon\prime}(x^\varepsilon)\}\,\frac{dx^\varepsilon}{dt}, \tag{1.22}$$

where the primes denote derivatives with respect to x. From equations (1.15a,1.15b) we have $\frac{dx^\varepsilon}{dt} = \varphi^\varepsilon(x^\varepsilon)$, $\frac{dy^\varepsilon}{dt} = -(x^\varepsilon + \alpha)$ and so

$$\{h'(x^\varepsilon) + \varepsilon\varphi^{\varepsilon\prime}(x^\varepsilon)\}\,\varphi^\varepsilon(x^\varepsilon) = -(x^\varepsilon + \alpha). \tag{1.23}$$

Hence $h'(x^0)\varphi^0(x^0) = -x^0 + \alpha$ and therefore

$$\varphi^\varepsilon(x^\varepsilon(t)) = -\frac{x^\varepsilon(t) + \alpha}{h'(x^\varepsilon(t))} + \mathcal{O}(\varepsilon). \tag{1.24}$$

For a periodic solution of (1.15a, 1.15b), solution segments of type (B) occur in either of two ways.

(i) Firstly, a segment may be chosen along the stable part of the slow curve. Substituting $h(s) - \Phi^\varepsilon(s) = -\varepsilon\varphi^\varepsilon(s) = \varepsilon\frac{x^\varepsilon+\alpha}{h'(x^\varepsilon(t))} + \mathcal{O}(\varepsilon^2)$ in (1.19) yields

$$-\frac{1}{\varepsilon}\int_{T_j^\varepsilon}^{T_{j+1}^\varepsilon} h'(x^\varepsilon(t))\,dt = \frac{1}{\varepsilon}\int_{x_j^\varepsilon}^{x_{j+1}^\varepsilon} \frac{\{h'(s)\}^2}{s+\alpha}\,ds + \mathcal{O}(1). \tag{1.25}$$

It can be easily seen that integrals of the form

$$\int_{x_j^\varepsilon}^{x_{j+1}^\varepsilon} \frac{\{h'(s)\}^2}{s+\alpha}\,ds. \tag{1.26}$$

are negative and finite along these pieces. For example, in the case of $h(x) = x^2(x+\beta)$ we can take for x_j^ε and x_{j+1}^ε any two points of the limit cycle along the slow curve satisfying $0 < x_{j+1}^\varepsilon < x_j^\varepsilon$ and $-\beta < x_j^\varepsilon < x_{j+1}^\varepsilon < x_{max}$, respectively, where $h(-\beta) = 0$ and x_{max} denotes the x-coordinate of the local maximum of h.

(ii) Secondly, a segment may consist both of stretches along the stable part and stretches along the unstable part of the slow curve. Note, however, that this situation can only occur for canard type limit cycles. The critical parameter $\alpha_c(\varepsilon)$ in whose exponentially small neighbourhood (1.1) admits canard type limit cycles satisfies $\alpha_c(\varepsilon) = \mathcal{O}(\varepsilon)$ by (1.10). Hence equation (1.24) changes to

$$\varphi^\varepsilon(x^\varepsilon(t)) = -\frac{x^\varepsilon(t)}{h'(x^\varepsilon(t))} + \mathcal{O}(\varepsilon).$$

and we must consider

$$I(c) := \int_{A(c)}^{B(c)} \frac{\{h'(s)\}^2}{s}\,ds \tag{1.27}$$

where $A(c)$ and $B(c)$ denote the two largest roots of $h(s) = c$. Here we have implicitly assumed that c is such that there exist three real roots. This integral may take arbitrary values, depending on the specific choice of h and the value of c. For sufficiently small values of c, however, we can compute $I(c)$ from local data at the origin. Using Taylor's formula for h', we have

$$I(c) = \left[\frac{1}{2}h''(0)^2s^2 + \frac{1}{3}h''(0)h'''(0)s^3 + \mathcal{O}(s^4)\right]_{A(c)}^{B(c)}, \tag{1.28}$$

which can be rewritten to

$$I(c) = \left[h''(0)h(s) + \frac{1}{6}h''(0)h'''(0)s^3 + \mathcal{O}(s^4)\right]_{A(c)}^{B(c)}. \quad (1.29)$$

Now recall that, by definition, $h(A(c)) = h(B(c)) = c$. This shows that the first term in the above expression for $I(c)$ vanishes. Using $A(c) = \sqrt{\frac{2}{h''(0)}}\,\sqrt{c} + \mathcal{O}(c)$, $B(c) = -\sqrt{\frac{2}{h''(0)}}\,\sqrt{c} + \mathcal{O}(c)$ we obtain

$$I(c) = -\frac{2}{3}h'''(0)\sqrt{\frac{2}{h''(0)}}\,c\sqrt{c} + \mathcal{O}(c^2). \quad (1.30)$$

Thus, for sufficiently small values of c (which are independent of ε) the sign of $I(c)$ is determined by the sign of $h'''(0)$ or, equivalently, the sign of $g'(0)$.

We want to extend this to larger values of c. For this note that

$$\frac{dI}{dc} = \frac{\{h'(B)\}^2}{B}\frac{dB}{dc} - \frac{\{h'(A)\}^2}{A}\frac{dA}{dc}. \quad (1.31)$$

From $h(A) = c$ we obtain $h'(A)\frac{dA}{dc} = 1$ and similarly for B, so

$$\frac{dI}{dc} = \frac{h'(B)}{B} - \frac{h'(A)}{A} = [g(s)]_A^B. \quad (1.32)$$

Since $g'(0) \neq 0$, g is a strictly monotone function in a neighbourhood of the origin, and therefore $\frac{dI}{dc} \neq 0$ for small $c > 0$. Together with equation (1.30) we have that $I(c)$ has the sign of $-h'''(0)$ as long as g' does not change sign. If g' changes sign it can happen that $I(c)$ also changes sign for some c. This can, however, not be decided from local information near the origin, but depends on global features of the function h.

A short calculation shows that for $h(s) = s^2(s + \beta)$ we have $g'(s) \equiv 3$ for all s, independent of β. Hence in this case $I(c) < 0$ for all possible c-values.

(C) transitional trajectories: $z = \mathcal{O}_s(\varepsilon)$. We make a further distinction into the following two subcases of (C).

(i) $h'(x_k) = 0$ for some $x_k^\varepsilon = x(T_k^\varepsilon)$. This cares for the case when the flow is departing from the stable manifold close to the local extrema, reaching the fast field after time intervals of the order $T_{k+1}^\varepsilon - T_k^\varepsilon = \mathcal{O}(\varepsilon^{\frac{1}{3}})$ according to [Eck83], p. 471. Thus, evaluating the integral along the present interval with respect to time yields

$$-\frac{1}{\varepsilon}\int_{T_k^\varepsilon}^{T_{k+1}^\varepsilon} h'(x^\varepsilon(t))\,dt = \mathcal{O}(\varepsilon^{-\frac{2}{3}}) = o(\frac{1}{\varepsilon}). \tag{1.33}$$

(ii) $h'(x) \neq 0$ for all x in the interval. This takes care of the case when the flow reaches the stable manifold[5] coming from the fast field. We evaluate the corresponding integral with respect to time. Let us start at time $t = T_l^\varepsilon$ at a point $(x_l^\varepsilon, y_l^\varepsilon)$ with $y_l^\varepsilon - h(x_l^\varepsilon) = \mathcal{O}(1)$. This means that the starting point lies still in the fast field. In the following calculation we derive an estimate for the time it takes to enter an $o(\varepsilon)$-neighbourhood of the stable manifold. We show that for $T_{l+1}^\varepsilon = T_l^\varepsilon + \frac{2}{K}\,\varepsilon\,\log(\frac{1}{\varepsilon})$ we have $z(T_{l+1}^\varepsilon) = o(\varepsilon)$. Note that $y_l^\varepsilon - h(x_l^\varepsilon) = \mathcal{O}(1)$ at $t = T_l^\varepsilon$ implies $z(T_l^\varepsilon) = \mathcal{O}(1)$. Differentiation yields

$$\frac{dz}{dt} = \frac{dy}{dt} - (h'(x) - \varepsilon\ell(x))\frac{dx}{dt}, \quad \ell(x) := \frac{h'(x) - (x+\alpha)h''(x)}{(h'(x))^2}.$$

Multiplying through with ε and substituting $\varepsilon\frac{dx}{dt} = z - \varepsilon\frac{x+\alpha}{h'(x)}$ we obtain

$$\varepsilon\frac{dz}{dt} = -(h'(x) - \varepsilon\ell(x))z + \mathcal{O}(\varepsilon^2). \tag{1.34}$$

Now we set

$$K := \min\{h'(x) - \varepsilon\ell(x) \,:\, x = x^\varepsilon(t), t \in [T_l^\varepsilon, T_{l+1}^\varepsilon],\, 0 \leq \varepsilon \leq \varepsilon_0\}$$

for some sufficiently small $\varepsilon_0 > 0$. Since we assume that $h'(x) \neq 0$, throughout the interval, we have $K > 0$. We obtain the estimate

$$|z(t)| \leq |z(T_l^\varepsilon)|\exp\{-\frac{1}{\varepsilon}K(t - T_l^\varepsilon)\} + \mathcal{O}(\varepsilon(t - T_l^\varepsilon)) \tag{1.35}$$

for $t \in [T_l^\varepsilon, T_{l+1}^\varepsilon]$ and $\varepsilon \leq \varepsilon_0$.

[5] The case where the fast flow leaves the unstable manifold can be treated similarly, by a time reversal.

In particular we have $|z(T^\varepsilon_{i+1})| \leq |z(T^\varepsilon_i)|\varepsilon^2 + \mathcal{O}(\varepsilon^2 \log(\varepsilon))$ with the above value for T^ε_{i+1}, or $z(T^\varepsilon_{i+1}) = o(\varepsilon)$. Hence

$$-\frac{1}{\varepsilon}\int_{T^\varepsilon_i}^{T^\varepsilon_{i+1}} h'(x^\varepsilon(t))\,dt = \mathcal{O}(\log(\varepsilon)) = o(\frac{1}{\varepsilon}), \tag{1.36}$$

as $T^\varepsilon_{i+1} - T^\varepsilon_i = \mathcal{O}(\varepsilon\log(\varepsilon))$ and $h'(x)$ is bounded along the stretch from x^ε_i to x^ε_{i+1}.

Let us summarize our results. Pieces corresponding to stretches along the slow curve contribute amounts of $\mathcal{O}_s(\frac{1}{\varepsilon})$ to the divergence integral (A.4), except possibly for isolated values of α. All other trajectories only contribute amounts of order $o(1/\varepsilon)$. Hence, except possibly for isolated values of α,

$$-\frac{1}{\varepsilon}\int_0^{T^\varepsilon} h'((x^\varepsilon(t))\,dt = \mathcal{O}_s(\frac{1}{\varepsilon}) \tag{1.37}$$

or

$$\oint_\gamma \text{div } G^\varepsilon \to \pm\infty \text{ as } \varepsilon \to 0. \tag{1.38}$$

For stretches along the stable part of the slow curve the contributions are strictly negative, independent of h and α. This shows that relaxation oscillations are always stable.

For canard type limit cyles stretches along the unstable part of the slow curve give a positive contribution, which may cancel or even outweigh the negative contributions along the stable parts. This means that canards can be either stable or unstable. More precisely, for sufficiently small canard cycles we have shown that if $\varepsilon > 0$ is sufficiently small, the nontrivial Floquet exponent has the sign of $-h'''(0)$. Small canard cycles will therefore be asymptotically stable for $h'''(0) > 0$ and unstable for $h'''(0) < 0$. Note that small canard cycles have the same stability type as the limit cycles born in the Hopf bifurcation, cf. Section 1.1.

However, we can not *a priori* determine the stability for larger canard cycles. Depending on h several possibilities exist. In the case of a supercritical Hopf bifurcation the simplest possible scenario is that the Floquet exponent remains negative when α changes and the small canard cycle grows smoothly to a fully developed relaxation oscillation, without additional bifurcations. In fact, this scenario occurs for the Eckhaus caricature $h(x) = x^2(x + \beta)$.

In the case of a subcritical Hopf bifurcation the small cycles have a positive Floquet exponent, while relaxation oscillations always have a negative Floquet exponent. Now the simplest possible bifurcation scenario is that of a gradually shrinking relaxation oscillation and a growing unstable limit cycle, coalescing at the point where the nontrivial Floquet exponent changes sign and subsequently disappearing. Eckhaus also discusses this case in [Eck83]. Note that our integral $I(c)$ coincides with the function $\dot{Q}$ that he uses.

Chapter 2

Linear Stability Analysis and the Hopf bifurcation

2.1 Linear Stability Analysis

The travelling wave equations to the FitzHugh-Nagumo system are given by

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ a & \theta & 1 \\ \frac{\varepsilon}{\theta} & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ u^3 - (a+1)u^2 \\ 0 \end{pmatrix}, \tag{2.1}$$

where we have split the linear from the nonlinear part. As usual $\dot{}$ denotes differentiation with respect to the travelling wave variable $z = x + \theta t$. Note that the origin is the only rest point of (2.1).

The characteristic equation of the linearization of the TW - equations around the origin, which is given by the linear part of (2.1), is

$$\lambda^3 - \theta\lambda^2 - a\lambda - \frac{\varepsilon}{\theta} = 0. \tag{2.2}$$

We want to discuss the asymptotic stability of the unique rest point of (2.1) at the origin. This is determined by the the roots of the characteristic equation (2.2). The condition $D = 0$, where D[1] denotes the discriminant of the cubic equation (2.2), determines when the roots change from three distinct real roots to one real root and a pair of complex conjugate roots. In other words, when we

[1] A formula for D is given in Appendix B.

have three real roots, of which two are equal,

$$D = (\frac{a^2}{4} - \varepsilon)\theta^4 + (a^3 - \frac{9}{2}a\varepsilon)\theta^2 - \frac{27}{4}\varepsilon^2 = 0.$$

We may write the latter as the following bi-quadratic equation in θ

$$\theta^4 + \frac{2a(2a^2 - 9\varepsilon)}{a^2 - 4\varepsilon}\theta^2 - \frac{27\varepsilon^2}{a^2 - 4\varepsilon} = 0 \tag{2.3}$$

with roots

$$\theta^2_\pm(a) = \frac{1}{a^2 - 4\varepsilon}\left\{a(9\varepsilon - 2a^2) \pm 2\sqrt{(a^2 - 3\varepsilon)^3}\right\}. \tag{2.4}$$

Alternatively, we can transform (2.4) to get

$$\theta^2_\pm(a) = \frac{-27\varepsilon^2}{a(9\varepsilon - 2a^2) \mp 2\sqrt{(a^2 - 3\varepsilon)^3}}. \tag{2.5}$$

We are only interested in positive θ -values and will therefore only consider the following three branches of $\theta^2_\pm$, namely θ^2_+ on $(-\infty, -\sqrt{3\varepsilon}] \cup (2\sqrt{\varepsilon}, \infty)$ and θ^2_- on $(-2\sqrt{\varepsilon}, -\sqrt{3\varepsilon}\,]$, since all other branches of $\theta^2_\pm(a)$ are either not defined or take negative values.

Note that θ^2_+ and θ^2_- have poles at $a = -2\sqrt{\varepsilon}$ and $a = 2\sqrt{\varepsilon}$, respectively. However, $a = -2\sqrt{\varepsilon}$ is a removable singularity of θ^2_+ as $\theta^2_+(-2\sqrt{\varepsilon}) = \frac{27}{4}\sqrt{\varepsilon}$ by (2.5). Additionally, $\theta^2_-(a) > \theta^2_+(a)$ for all $a \in (-2\sqrt{\varepsilon}, -\sqrt{3\varepsilon})$ and $\theta^2_\pm(-\sqrt{3\varepsilon}) = 3\sqrt{3\varepsilon}$. In Appendix C we prove there is a cusp at $a = -\sqrt{3\varepsilon}$. All three positive branches of $\theta^2_\pm$ are strictly monotonically decreasing, where they are defined.

We expand (2.4) in order to study the asymptotic behaviour as $a \to \pm\infty$ to obtain

$$\theta^2_+(a) = \frac{1}{a^2 - 4\varepsilon}\left\{a(9\varepsilon - 2a^2) + 2\,|a|^3\left(1 - \frac{9}{2}\frac{\varepsilon}{a^2} + \frac{27}{8}\frac{\varepsilon^2}{a^4} + \ldots\right)\right\} \tag{2.6}$$

$$= \begin{cases} -4a + \ldots \to \infty \text{ for } a \to -\infty, \\ \frac{27}{4}\frac{\varepsilon^2}{a} + \ldots \to 0 \text{ for } a \to \infty, \end{cases} \tag{2.7}$$

where $\ldots$ denotes higher order terms in ε.

The system (2.1) has a curve of Hopf bifurcation points

$$\hat{a} = \frac{\varepsilon}{\theta^2}, \quad \text{for} \quad \hat{a} > 0, \tag{2.8}$$

along which we have a simple pair of purely imaginary eigenvalues, where $\hat{a} \stackrel{\text{def}}{=} -a$. On this curve we can factorise the characteristic polynomial to obtain for (2.2)

$$(\lambda^2 + \hat{a})(\lambda - \theta) = 0.$$

Hence the eigenvalues on the Hopf curve are given by

$$\lambda_{1,2} = \pm i\sqrt{\hat{a}} \quad \text{and} \quad \lambda_3 = \theta. \tag{2.9}$$

Note that the magnitude of the imaginary pair of eigenvalues is of order $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$, in accordance with a general result of Baer & Erneux [BE86] on singular Hopf bifurcation.

Differentiating (2.2) implicitly with respect to a, and changing from a to $\hat{a}$, gives

$$\lambda'(\hat{a}) = \frac{-\lambda}{3\lambda^2 - 2\theta\lambda + \hat{a}}.$$

With $\lambda = \pm i\sqrt{\hat{a}}$ in the latter we have

$$\operatorname{Re}\lambda'(\hat{a}) = \frac{1}{2}\frac{\theta}{\hat{a} + \theta^2} > 0. \tag{2.10}$$

Consequently, all the conditions of the Hopf bifurcation theorem are satisfied, whence periodic solutions emanate from the origin for $\hat{a} > 0$ with period $\frac{2\pi}{\sqrt{\hat{a}}}$ along the Hopf curve (2.8).

Let us now summarize the eigenvalue structure of the linearization at the origin in the following theorem.

Theorem 2.1 *The origin is as rest point of* (2.1) *always unstable. More precisely, the eigenvalue structure is as follows: (compare* Figure 2.1*)*

(i) In the interior of the region bounded by the branch of $\theta_+^2(a)$ on $a < -\sqrt{3\varepsilon}$ and $\theta_-^2(a)$ in $-2\sqrt{\varepsilon} < a < -\sqrt{3\varepsilon}$ there are three positive eigenvalues.

(ii) In the region to the left of the branch of $\theta_+^2(a)$ on $(2\sqrt{\varepsilon}, \infty)$, i.e. for $\theta > \theta_+^2(a)$ and $a > 2\sqrt{\varepsilon}$, there are three real eigenvalues of which two are negative.

(iii) In the complement of the regions defined above there exists a pair of complex conjugate eigenvalues and a single positive eigenvalue, which is divided by the Hopf curve $\theta = \sqrt{-\frac{\varepsilon}{a}}$ into two parts. In the subregion to the left of the Hopf curve the real parts of the complex conjugate pair of eigenvalues is positive and negative to the right.

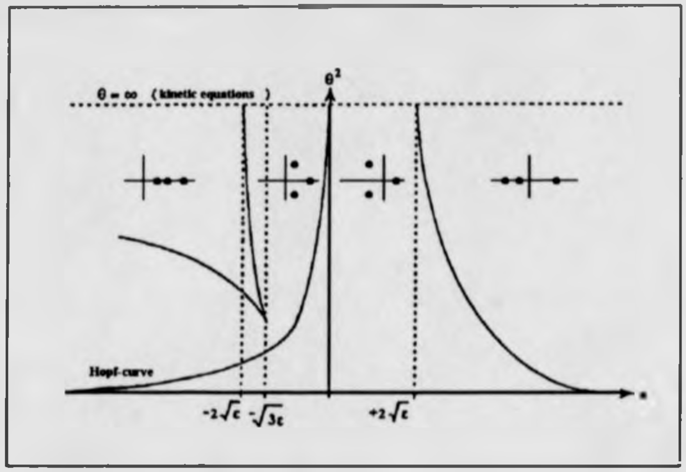

Figure 2.1: Eigenvalue structure of the linearization of (2.1) at the origin. Position of the eigenvalues in the complex plane is indicated by black dots.

2.2 Nonlinear Analysis of the Hopf bifurcation

We proceed to determine the direction of branching of the Hopf bifurcation from the nonlinear terms of the vector field using the results in [HKW81]. In order to carry out these calculations we have to find a basis, with respect to which the matrix of the linear part of (2.1) has the form

$$\begin{pmatrix} 0 & -\sqrt{\hat{a}} & 0 \\ \sqrt{\hat{a}} & 0 & 0 \\ 0 & 0 & \theta \end{pmatrix}.$$

We calculate a complex eigenvector of the linear part of (2.1) corresponding to the eigenvalue $i\sqrt{\hat{a}}$ to be

$$\begin{pmatrix} 1 \\ i\sqrt{\hat{a}} \\ -i\theta\sqrt{\hat{a}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{\hat{a}} \\ -\theta\sqrt{\hat{a}} \end{pmatrix}, \tag{2.11}$$

and a real eigenvector to the eigenvalue θ is given by

$$\begin{pmatrix} 1 \\ \theta \\ \hat{a} \end{pmatrix}. \tag{2.12}$$

We introduce a transformation matrix P whose columns are the imaginary and real part of (2.11) together with (2.12), i.e.,

$$P = \begin{pmatrix} 0 & 1 & 1 \\ \sqrt{\hat{a}} & 0 & \theta \\ -\theta\sqrt{\hat{a}} & 0 & \hat{a} \end{pmatrix}.$$

With respect to the new coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P^{-1} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

the TW equations (2.1) transform to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{\hat{a}} & 0 \\ \sqrt{\hat{a}} & 0 & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{(y+z)^3 - (1-\hat{a})(y+z)^2}{\hat{a}+\theta^2} \begin{pmatrix} \sqrt{\hat{a}} \\ -\theta \\ \theta \end{pmatrix} \tag{2.13}$$

Observe that all higher derivatives which involve x are zero. Also, the higher derivatives with respect to y and z are equal since the nonlinear part of (2.13) is a function of $y+z$. We are now in a position to apply the formulae in [HKW81] to determine the direction of branching.

We expand $\hat{a}$, which parametrizes the bifurcating branch of periodic solutions, in a Taylor series around a given but arbitrary point on the Hopf curve $\hat{a}_c := \hat{a}(\varepsilon, \theta)$

$$\hat{a} - \hat{a}_c = \mu_2 \epsilon^2 + \cdots \tag{2.14}$$

The determination of μ_2 is of particular interest, when it is not zero, in that it indicates where the periodic solutions occur, whether for $\hat{a} > \hat{a}_c$ or for $\hat{a} < \hat{a}_c$. It is given in [HKW81] on page 90 as

$$\mu_2 = -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re} \lambda'(0)}, \tag{2.15}$$

where $c_1(0)$ is some expression depending on higher order terms of the vector field.
Of the Floquet exponents of the periodic solution branch, for small ϵ, one will be close to the eigenvalue $\lambda_3 = \theta$ of the linearization at the origin on the Hopf curve, one is zero, of course, and the last one is $\beta = \beta_2\epsilon^2 + \cdots$ with $\beta_2 = -2\mu_2 \mathrm{Re}\, \lambda'(0) = 2\mathrm{Re}\, c_1(0)$. Thus the periodic solutions emanating from any point on the Hopf curve will for small ϵ always be unstable independent of the sign of β_2 since θ is positive.
Let the bifurcation parameter $\hat{a}$ be close to the Hopf curve such that the linear part of (2.1) at the origin possesses a pair of complex conjugate eigenvalues $\lambda(\hat{a})$ and $\bar{\lambda}(\hat{a})$.

A tedious routine calculation yields

$$-\mathrm{Re}\, c_1(0) = 2(1-\hat{a})^3 - 6(1-\hat{a})^2(\theta^2 + 2\hat{a}) + 3(\theta^2 + \hat{a})(\theta^2 + 4\hat{a}).$$

Using the relation $\theta^2 = \varepsilon/\hat{a}$ we can eliminate θ^2 from the above formula to obtain

$$-\mathrm{Re}\, c_1(0) = 8\hat{a}^5 + 8\hat{a}^4 + (2\varepsilon - 4)\hat{a}^3 + 17\varepsilon\hat{a}^2 - 4\varepsilon\hat{a} + 3\varepsilon^2. \tag{2.16}$$

We define

$$p(\hat{a}, \varepsilon) = -\mathrm{Re}\, c_1(0)$$

and look for roots of

$$p(\hat{a}, \varepsilon) = 0, \tag{2.17}$$

i.e., points where the direction of branching changes.
We can solve (2.17) explicitly for $\varepsilon = 0$ to obtain a triple root at $\hat{a} = 0$ and roots at $\hat{a}_{1,2} = \pm\frac{\sqrt{3}-1}{2}$. We can disregard the negative root $\hat{a}_2$, since $\hat{a}$ has to be positive for a Hopf bifurcation to occur. Since $\frac{\partial p}{\partial a}(\hat{a}_1, 0) \neq 0$, by use of the Implicit Function Theorem, the root $\hat{a}_1$ perturbs for small ε into a root $\hat{a}_1(\varepsilon)$.

The asymptotic expansion of $\hat{a}_1(\varepsilon)$ for small ε is

$$\hat{a}_1(\varepsilon) = \hat{a}_1 + \alpha\varepsilon + \frac{\beta}{2}\varepsilon^2 + \ldots,$$

where the coefficients α and β are determined by

$$\alpha = -\frac{\frac{\partial p}{\partial \varepsilon}(\hat{a}_1, 0)}{\frac{\partial p}{\partial a}(\hat{a}_1, 0)} \approx 1.342$$

and

$$\beta = -\frac{\alpha^2 \frac{\partial^3 p}{\partial a^3}(\hat{a}_1, 0) + 2\alpha \frac{\partial^3 p}{\partial a \partial \varepsilon}(\hat{a}_1, 0) + \frac{\partial^2 p}{\partial \varepsilon^2}(\hat{a}_1, 0)}{\frac{\partial p}{\partial a}(\hat{a}_1, 0)} \approx 3.889$$

From Golubitsky-Schaeffer bifurcation theory [GS85] at p. 95, Prop. 9.2, it follows with $\hat{a}$ as a state variable and ε as a bifurcation parameter that the normal form of the bifurcation diagram in ε near the triple root of (2.17) at $(\hat{a}, \varepsilon) = (0, 0)$ is the pitchfork $-\hat{a}^3 - \varepsilon\hat{a}$ with three real roots for $\varepsilon < 0$ and a pair of complex conjugate roots and a real root for $\varepsilon > 0$.

An application of Descartes' rule of signs [Mur89], p. 704, to the fifth order equation (2.17) shows that for $\varepsilon > 0$ there is exactly one negative and an even number of positive roots.

For $\varepsilon > 0$ the pitchfork has two complex conjugate roots and so p must have exactly two positive roots, of which one stems from the triple root at $\varepsilon = 0$ and the other is $\hat{a}_1(\varepsilon)$.

In order to compute the first order term of the Taylor series expansion in ε of the unique positive branch of the pitchfork we make the ansatz

$$\hat{a}_3(\varepsilon) = \gamma\varepsilon + \mathcal{O}(\varepsilon^2).$$

This can be justified by an application of the Implicit Function Theorem to the function

$$\hat{p}(\gamma, \varepsilon) = \begin{cases} 3 - 4\gamma & \text{for} \quad \varepsilon = 0 \\ \frac{1}{\varepsilon^3} p(\gamma\varepsilon, \varepsilon) = 3 - 4\gamma + \varepsilon H(\gamma, \varepsilon) & \text{for} \quad \varepsilon \neq 0 \end{cases} \tag{2.18}$$

at $(\gamma, \varepsilon) = (\frac{3}{4}, 0)$, where H is determined by p. This gives $\gamma(\varepsilon) = \frac{3}{4} + \mathcal{O}(\varepsilon)$. Since p takes positive values for $\hat{a}$ between $\frac{3}{4}\varepsilon + \mathcal{O}(\varepsilon^2)$ and $\frac{\sqrt{3}-1}{2} + \mathcal{O}(\varepsilon)$, the Hopf bifurcation is subcritical in terms of $a = -\hat{a}$, the root of the cubic nonlinearity (0.4), for a between $-\frac{3}{4}\varepsilon + \mathcal{O}(\varepsilon^2)$ to $-\frac{\sqrt{3}-1}{2} + \mathcal{O}(\varepsilon)$.

Next, we can ask for which value(s) of ε, the roots $\hat{a}_1(\varepsilon)$ and $\hat{a}_3(\varepsilon)$ come together to form a double root of p? Using the fact that a double root annihilates $p'_\varepsilon(\hat{a})$ and that the latter is linear in ε, we can solve for ε as a function of $\hat{a}$, i.e.,

$$\varepsilon = -\frac{20\hat{a}^4 + 16\hat{a}^3 - 6\hat{a}^2}{3\hat{a}^2 + 17\hat{a} - 2}. \tag{2.19}$$

We substitute this expression for ε in p to obtain

$$24\hat{a}^6 - 146\hat{a}^5 + 934\hat{a}^4 + 253\hat{a}^3 - 457\hat{a}^2 + 164\hat{a} - 16 = 0. \tag{2.20}$$

There are two real roots satisfying (2.20) one of which is negative when substituted into (2.19); the other root which we require is $\varepsilon_0 \approx 0.1$. From this we conclude that for $\varepsilon > \varepsilon_0$ the branch of periodic solutions from the Hopf bifurcation is *always* supercritical.

We can now formulate the following proposition.

Proposition 2.2 *Criticality of the Hopf bifurcation in* (2.1).

(i) *There exists a unique value* $\varepsilon_0 = \varepsilon(\hat{a})$[1] ≈ 0.1, *where* $\hat{a}$ *is the unique real root of* (2.20) *for which* $\varepsilon(\hat{a})$ *is positive. Then for all* $\varepsilon > \varepsilon_0$ *the Hopf bifurcation is always supercritical independently of the value of the parameter* a, $a < 0$.

(ii) *Let* $0 < \varepsilon < \varepsilon_0$. *Then the Hopf bifurcation is subcritical for* a *between* $-\frac{3}{4}\varepsilon + \mathcal{O}(\varepsilon^2)$ *and* $-\frac{\sqrt{3}-1}{2} + \mathcal{O}(\varepsilon)$ *and supercritical otherwise.*

[1]Here $\varepsilon(\hat{a})$ denotes the R.H.S. of (2.19).

Chapter 3

Periodic Travelling Waves as Perturbations from $\theta = \infty$

3.1 Outline

We will show that the travelling wave equations of the FitzHugh-Nagumo system have for infinite wave speed a two dimensional manifold of rest points, which persists as a locally invariant manifold for sufficiently high wave speed independently of ε. Furthermore, the flow on this locally invariant manifold is a small perturbation of the reaction flow. It consists of the transition from small amplitude periodic orbits from a Hopf bifurcation to relaxation oscillations via canard type trajectories as the variable root a is decreasing from 0.

The main idea of the proof is to use for large values of θ the singular perturbation nature of the problem with respect to $\beta := \frac{1}{\theta^2}$ to look for solutions of the full system which are close to solutions of the reaction kinetics. We begin by reviewing some invariant manifold theory and its relation to the construction of invariant manifolds for singularly perturbed systems. Authors who have looked at perturbations from the $\theta = \infty$ limit include Kopell & Howard [KH73], Kopell [Kop77] and Schneider [Sch83]. Our presentation follows Kopell [Kop85].

3.2 Invariant Manifolds of Singularly Perturbed Systems

Consider a system of singularly perturbed ODE's of the form

$$\begin{aligned} \tfrac{dx}{dt} &= G_1(x,y;\beta) \\ \tfrac{dy}{dt} &= \tfrac{1}{\beta} G_2(x,y;\beta) \end{aligned} \tag{3.1}$$

where $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ and $0 < \beta \ll 1$. If $G_2(x,\overline{y}(x);0) = 0$, for some function $\overline{y}(x)$ defined on a compact subset K of $\mathbb{R}^m$, then the equation

$$\frac{dx}{dt} = G_1(x,\overline{y}(x);0) \tag{3.2}$$

is called the *reduced system* of (3.1). We want to investigate the relationship between the dynamics of (3.1) and (3.2). In the following we state a theorem which gives conditions under which, for β sufficiently small, (3.1) has a m-dimensional invariant submanifold $\mathcal{E}$, representable as the graph of some function $\overline{y}(x;\beta)$, such that, on this invariant manifold, the flow of

$$\frac{dx}{dt} = G_1(x,\overline{y}(x;\beta);\beta) \tag{3.3}$$

converges uniformly to (3.2) and $\overline{y}(x;\beta) \to \overline{y}(x)$ as $\beta \to 0$. Thus (3.1) contains a submanifold on which the flow of (3.3) is a regular perturbation of the flow of the reduced system (3.2). This has the consequence that every structurally stable feature of (3.2) such as a stable or unstable periodic orbit also exists for (3.1), provided that β is sufficiently small.

The main hypothesis of the theorem concerns a rescaled version of (3.1). With respect to the "stretched" time scale $\tau := \frac{1}{\beta}\, t$, (3.1) is equivalent to

$$\begin{aligned} \tfrac{dx}{d\tau} &= \beta\, G_1(x,y;\beta), \\ \tfrac{dy}{d\tau} &= G_2(x,y;\beta). \end{aligned} \tag{3.4}$$

If we now set $\beta = 0$, we see that $\mathcal{E}_0 = \{(x,\overline{y}(x)) : x \in K\}$ is a manifold of rest points for (3.4). We shall require that this manifold be a *normally hyperbolic* submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. For a manifold of rest points this means the following:

Definition 3.1 *Let $K \subset \mathbb{R}^m$ be a compact subset.*
Then $\mathcal{E}_0 := \{(x, \overline{y}(x)) : x \in K\}$ is a normally hyperbolic invariant manifold on K if, for each $x \in K$, all eigenvalues of the the matrix

$$\frac{\partial G_2}{\partial y}(x, \overline{y}(x); 0) \tag{3.5}$$

lie off the imaginary axis.

We remark that by compactness $|\mathrm{Re}\,\lambda(x, \overline{y}(x))|$ is then uniformly bounded away from 0, for each $\lambda(x, \overline{y}(x))$ an eigenvalue of (3.5), $x \in K$.
For a more general definition of normal hyperbolicity we refer to Hirsch, Pugh & Shub [HPS77].

Now we are in the position to state the following persistence result.

Theorem 3.1 *Suppose that $\mathcal{E}_0 = \{(x, \overline{y}(x)) : x \in K\}$ is a normally hyperbolic invariant submanifold of* (3.4) *with $\beta = 0$, on $K \subset \mathbb{R}^m$ compact. Also assume that the vector field $G = (G_1, G_2)$ is C^∞-smooth. Then for any positive integer r and $\beta > 0$ sufficiently small, there is a neighbourhood $\mathcal{N}$ of the graph $\mathcal{E}_0$ of $\overline{y}(x)$ and a C^r function $\overline{y}(x;\beta)$, such that if $(x(\tau), y(\tau))$ is a solution of* (3.4) *with $y(0) = \overline{y}(x(0);\beta)$ for $x(0) \in K$ and $(x(\tau), y(\tau)) \in \mathcal{N}$ for all $|\tau| \le \tau_0$, for some $\tau_0 < \infty$, then $y(\tau) = \overline{y}(x(\tau), \beta)$ for all $|\tau| \le \tau_0$. That is, the graph of $\overline{y}(x;\beta)$, $\mathcal{E}_\beta := \{(x, \overline{y}(x;\beta)) : x \in K\}$, is a locally invariant submanifold of* (3.4) *and hence also for* (3.1). *Furthermore, $\overline{y}(x;\beta) \to \overline{y}(x)$ uniformly in K as $\beta \to 0$.*

This theorem is contained in Theorem 9.1 of Fenichel's paper [Fen79], where $\mathcal{E}_\beta$ is constructed as a centre manifold. It is in general not unique.

We note that $\mathcal{E}_\beta$ is known as a *slow submanifold*, as the flow on $\mathcal{E}_\beta$ has a time derivative of order $\mathcal{O}(\beta)$ by (3.4).

3.3 Periodic Travelling Waves to the FitzHugh-Nagumo System

Our aim is to demonstrate the existence of periodic travelling waves to the FitzHugh-Nagumo equations for sufficiently high wave speed. We shall prove this by a perturbation argument from $\theta = \infty$.

Recall that the travelling wave equations to the FitzHugh-Nagumo system are given by

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \theta v - f(u) + w, \\ \dot{w} &= \tfrac{\varepsilon}{\theta} u, \end{aligned} \tag{3.6}$$

where $\dot{} = \frac{d}{dz}$ with $z = x + \theta t$ and $0 < \varepsilon \ll 1$.

We split the dynamics of (3.6) into a "slow" and a "fast" part. More precisely we introduce scalings by "squeezing" and "stretching" the travelling wave variable z.

We transform the phase space variables by setting $\hat{u} := u, \hat{v} := \theta v$ and $\hat{w} := w$; and introduce $\beta := \frac{1}{\theta^2}$. As mentioned before we want to consider the case that $\theta \gg 1$, or equivalently, that $0 < \beta \ll 1$. Then with respect to the squeezed travelling wave variable $\tau := \frac{1}{\theta} z$ we obtain the "slow equations" corresponding to (3.1)

$$\left.\begin{aligned} \frac{d\hat{u}}{d\tau} &= \hat{v}, \\ \beta\frac{d\hat{v}}{d\tau} &= \hat{v} - f(\hat{u}) + \hat{w}, \\ \frac{d\hat{w}}{d\tau} &= \varepsilon\hat{u}, \end{aligned}\right\} \tag{slow eqns.; β}$$

and with respect to the stretched travelling wave variable $\xi := \theta z = \beta\tau$ we obtain the "fast equations" corresponding to (3.4)

$$\left.\begin{aligned} \frac{d\hat{u}}{d\xi} &= \beta\hat{v}, \\ \frac{d\hat{v}}{d\xi} &= \hat{v} - f(\hat{u}) + \hat{w}, \\ \frac{d\hat{w}}{d\xi} &= \beta\varepsilon\hat{u}. \end{aligned}\right\} \tag{fast eqns.; β}$$

In the slow equations, in the limit as $\beta \to 0$, the middle equation is algebraic with no dynamics. In the fast equations, in the limit as $\beta \to 0$, $\hat{v} = f(\hat{u}) - \hat{w}$ describes a manifold of rest points, parametrized by $(\hat{u}, \hat{w}) \in \mathbb{R}^2$.

The reduced system is given by the slow equations at $\beta = 0$, which turns out to be the kinetic equations of the FitsHugh-Nagumo TW equations

$$\begin{aligned} \frac{d\hat{u}}{dt} &= f(\hat{u}) - \hat{w}, \\ \frac{d\hat{w}}{dt} &= \varepsilon\hat{u}, \end{aligned} \tag{3.7}$$

where we have replaced τ by t as $\tau = \frac{1}{\theta}x + t \to t$ and $0 < \varepsilon \ll 1$.

In Section 1.1 we have seen that (3.7) possess in the oscillatory regime a unique branch of stable periodic solutions. The branch consists of small amplitude periodic solutions emanating in a Hopf bifurcation from the origin which grow via canards for $a = \mathcal{O}(\varepsilon)$ to relaxation oscillations. The latter exist for all $a \neq o(\varepsilon)$, their amplitude is increasing with $|a|$ and approaches infinity as $a \to -\infty$.

From now on the variable root a of the cubic f is assumed to be in the oscillatory regime.

We proceed to show that the manifold of rest points of the fast equations for $\beta = 0$ persists as an invariant manifold for $\beta > 0$ sufficiently small.
For this we let K be a compact ball large enough to contain the fully developed relaxation oscillation of the reduced system (1.1) in its interior and define $\mathcal{E}_0 := \{(\hat{u}, \hat{v}, \hat{w}) : \hat{v} = \bar{v}(\hat{u}, \hat{w}),\ (\hat{u}, \hat{w}) \in K\}$, where $\bar{v}(\hat{u}, \hat{w}) := f(\hat{u}) - \hat{w}$. The manifold $\mathcal{E}_0$ is shown in Figure 3.1.
The verification of the normal hyperbolicity condition with respect to the fast equations for $\beta = 0$ on K is trivial, since

$$\frac{\partial G_2}{\partial \hat{v}}((\hat{u}, \hat{w}), \bar{v}(\hat{u}, \hat{w}); 0) \equiv 1.$$

Thus by Theorem 3.1 there is a nearby C^r-smooth two dimensional locally invariant manifold $\mathcal{E}_\beta$ for some integer $r > 0$ which can be represented as the graph of some function $\bar{v}(\hat{u}, \hat{w}; \beta)$ for sufficiently small $\beta > 0$, $\mathcal{E}_\beta = \{(\hat{u}, \hat{v}, \hat{w}) : \hat{v} = \bar{v}(\hat{u}, \hat{w}; \beta),\ (\hat{u}, \hat{w}) \in K\}$. This holds independently of ε. The flow on $\mathcal{E}_\beta$ is governed by

$$\begin{aligned} \frac{d\hat{u}}{dt} &= \bar{v}(\hat{u}, \hat{w}; \beta) \\ \frac{d\hat{w}}{dt} &= \varepsilon \hat{u} \end{aligned} \tag{3.8}$$

and thus is a small perturbation of the reaction flow as $\bar{v}(\hat{u}, \hat{w}; \beta) \to \bar{v}(\hat{u}, \hat{w})$ uniformly in K for $\beta \to 0$ again by Theorem 3.1.

Next we would like to show that the dynamics and in particular the periodic solutions on $\mathcal{E}_0$ persist for small $\beta > 0$.
In order to give a precise formulation under which a periodic solution of the reduced system (3.7) persists to a periodic solution of the full system for small $\beta > 0$ we introduce some more concepts from the stability theory of closed orbits.

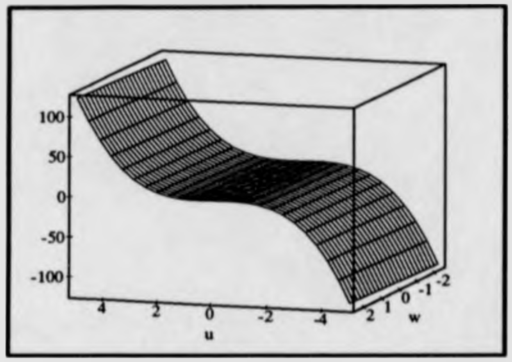

Figure 3.1: Graph of the slow submanifold for $\beta = 0$

3.3.1 Structural Stability of Closed Orbits to the Reduced System

We state a theorem under which closed orbits of the reduced system persist under small perturbations of the vector field. First we need to define a few concepts. A closed orbit γ is called *hyperbolic* if 1 is a simple Floquet multiplier and no other Floquet multiplier of γ lies on the unit circle of the complex plane. We call an asymptotically stable closed orbit γ a periodic *attractor*. Similarly, a periodic *repeller* is a periodic attractor when the time is reversed.
The precise formulation for the structural stability of closed orbits is then as follows:

Theorem 3.2 *Let $\dot{u} = G(u;\lambda)$ be a parameterized system of ODE's, where $(u;\lambda) \in W \times \Lambda$ with $W \subseteq \mathbb{R}^n, \Lambda \subseteq \mathbb{R}$ open, $0 \in \Lambda$ and $G(\cdot\,;\lambda)$ a C^1 vector field. Suppose that γ is a hyperbolic closed orbit of $\dot{u} = G(u;0)$ with minimal period $T > 0$. Then there exists a $\lambda_0 > 0$ such that for each λ, with $0 < \lambda \leq \lambda_0$ there exists a $\delta = \delta(\lambda) > 0$ so that, $\dot{u} = G(u;\lambda)$ has a unique closed orbit γ_λ which lies entirely in a δ-neighbourhood of γ and whose minimal period $T(\lambda) \to T$ as $\lambda \to 0$. In addition, $\delta(\lambda) \to 0$ as $\lambda \to 0$, i.e., the diameter of the neighbourhood around γ_λ goes with λ to zero.*

This is a reformulation of Theorem 4.1, p.226, in Hale's book [Hal80]. Actually, the hyperbolicity of the closed orbit is not strictly necessary for the persistence,

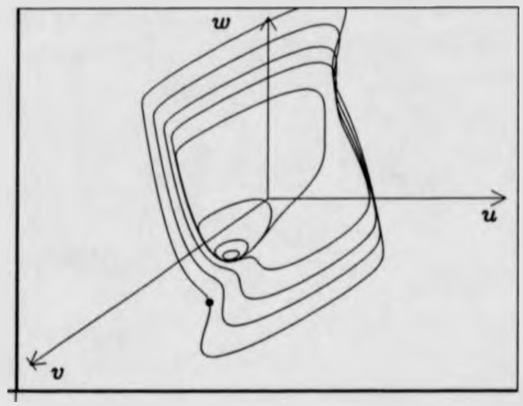

Figure 3.2: Hopf-canard-relaxation oscillation transition on slow submanifold

but merely the fact that 1 is a simple Floquet multiplier.
When γ is a periodic attractor the uniqueness of the perturbed periodic orbit γ_λ can be guaranteed. For if γ is a periodic attractor and $\lambda > 0$ is sufficiently small then γ_λ will also be a periodic attractor; hence, every trajectory that comes near γ_λ winds closer and closer to γ_λ as $t \to \infty$ and therefore can not be a closed orbit. Similarly, if γ is a periodic repeller, so is γ_λ, and again uniqueness holds. For hyperbolic closed orbits a weaker kind of uniqueness holds, as expressed in the above theorem.

In Section 1.2 we have shown that the periodic solutions to reduced system (3.7) are asymptotically stable. This includes the small amplitude periodic solutions from the Hopf bifurcation as well as the canard type trajectories and relaxation oscillations. Thus we can formulate the following corollary of Theorem 3.2.

Corollary 3.3 *The stable limit cycle solutions to the reduced equation* (3.7) *perturb into locally unique periodic solutions of* (slow eqns.; β) *for sufficiently small* β .

3.4 Conclusions

The travelling wave equations to the FitzHugh-Nagumo system w.r.t. the stretched travelling wave variable $\xi = \theta z$, the *fast equations* have for $\beta = 0$ a two dimensional manifold of rest points $\mathcal{E}_0$ which perturbs into a locally invariant two dimensional manifold $\mathcal{E}_\beta$ independently of ε.
The dynamics of the *slow equations* of (3.6), i.e. w.r.t. the squeezed travelling wave variable $\tau = \frac{1}{\theta}z$ at $\beta = 0$ is that of the kinetic equations on $\mathcal{E}_0$, discussed in section 1.1. It consists of the transition from small amplitude closed orbits emanating from a Hopf bifurcation to relaxation oscillations via canard type limit cycles as the variable root a is decreasing from 0. Each of these periodic solutions is a stable limit cycle.

For small β, on the other hand, the dynamics of the slow equations on $\mathcal{E}_\beta$ is a perturbation of the reaction flow. In particular, we obtain the existence of periodic solutions to the slow equations on the slow submanifold $\mathcal{E}_\beta$ as perturbation of those of the reduced system living on $\mathcal{E}_0$ for sufficiently small $\beta > 0$. They are shown in Figure 3.2.
This proves, in particular, the existence of canard type trajectories of (3.6) on the perturbed two dimensional invariant manifold $\mathcal{E}_\beta$.

With respect to the original time scale, equations (3.6), these periodic solutions exist for sufficiently high wave speed.

Chapter 4

Construction of Singular Solutions

4.1 Singular Periodic and Homoclinic Solutions

The travelling wave equations to the FitzHugh-Nagumo equations are given by

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \theta\, v - f(u) + w \\ \dot{w} &= \frac{\varepsilon}{\theta} u, \end{aligned} \tag{4.1}$$

where $\dot{} = \frac{d}{dz}$ with $z = x + \theta t$, $\theta > 0$ and $0 < \varepsilon \ll 1$ and f is the cubic nonlinearity $f(u) = u(u-a)(1-u)$. Throughout this chapter we shall refer to the travelling wave variable z as "time". We denote the local minimum, maximum and the inflection point of the cubic f by (u_{min}, w_{min}), (u_{max}, w_{max}) and (u_{infl}, w_{infl}), respectively.

Carpenter [Car77] as well as Casten, Cohen & Lagerstrom [CCL75] consider in their work exclusively the excitable regime. We extend the analysis to the oscillatory regime, when for negative a the projection of the rest point of the full system to the fast system moves to the inner branch of the cubic.

The equations (4.1) constitute an example of a *singularly perturbed* system with respect to ε in that they have two time scales; a *slow* time scale ξ and a *fast* time scale z. These are related by $\xi := \varepsilon z$. This difference in time scales, imposed by the smallness requirement on ε, can be exploited to formally construct approximate solutions, each piece of which satisfies some limiting version of the equations as the small parameter ε goes to zero.

With respect to the slow time scale equations (4.1) become

$$\begin{aligned} \varepsilon u' &= v \\ \varepsilon v' &= \theta\, v - f(u) + w \\ w' &= \tfrac{1}{\theta} u, \end{aligned} \tag{4.2}$$

where $' = \frac{d}{d\xi}$.

From (4.2) it is immediate that for small ε we have a *slow submanifold* S given by

$$S \stackrel{\text{def}}{=} \{(u,v,w) : w = f(u),\, v = 0\}, \tag{4.3}$$

see Figure 4.1. Observe that unless a point in phase space is close to this curve, u and v will change rapidly for small ε.

We introduce some notation at this point before we continue our discussion. Consider the subset $\{(u,0,w) \in S : f'(u) < 0\}$ of S consisting of two components, S_1 and S_2, with $(0,0,0) \in S_1$ when $a > 0$; or $(a,0,0) \in S_1$ when $a < 0$; and $(1,0,0) \in S_2$. Let Π_i be the image of S_i under the projection onto its third coordinate $(u,0,w) \mapsto w$. Then by the Implicit Function Theorem there exist uniquely determined smooth functions $u_1 : \Pi_1 \to (-\infty, u_{min})$ and $u_2 : \Pi_2 \to (u_{max}, \infty)$, such that $(u,0,w) \in S_i$ iff $w \in \Pi_i$ and $u = u_i(w)$. We can extend u_1 and u_2 continuously to functions on $(-\infty, u_{min}]$ and $[u_{max}, \infty)$, respectively. Defining $\Pi := \Pi_1 \cap \Pi_2$, $\Pi_- := \Pi \cap \{w : w < w_{infl}\}$ and
$\Pi_+ := \Pi \cap \{w : w > w_{infl}\}$, we have $\overline{\Pi} := \mathrm{cl}(\Pi) = [w_{min}, w_{max}]$ and moreover $u_1(w) < u_2(w)$ for $w \in \overline{\Pi}$. This allows us to decouple (4.1) for the limiting case $\varepsilon = 0$ into two lower dimensional problems.

With respect to the slow time ξ we define the one dimensional *slow flow* on the outer branches of the slow submanifold, S_1 and S_2, where the w-coordinate evolves according to

$$w' = \frac{1}{\theta}\, u_i(w), \quad \text{where} \quad u = u_i(w) \quad \text{for} \quad w \in \overline{\Pi}_i \tag{4.4}$$

with $' = \frac{d}{d\xi}$.

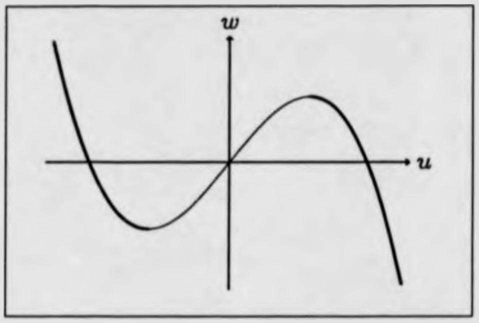

Figure 4.1: Slow submanifold S in (u,w)-space for $v=0$; S_1 and S_2 in bold

If one is off the slow submanifold the flow is more appropriately described in terms of the fast time scale z. For small ε the dynamics is approximately governed by the first two equations of (4.1) with w regarded as a constant. This gives rise to the definition of the *fast flow* by setting $\varepsilon = 0$ in (4.1) which results in the two dimensional system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \theta v - f(u) + w, \end{aligned} \tag{4.5}$$

for time z, where w is treated as an additional parameter, with $w \in \Pi$. We may view the vertical w–axis as the "base space" and the horizontal (u,v)-planes as the "fibres".

For later use we denote the vector field of the fast flow (4.5) by $F_w(u,v)$. Note that for a fixed $w \in \Pi$ (4.5) has three rest points $(u_1(w),0)$, $(\bar{u}(w),0)$ and $(u_2(w),0)$, which are roots of $f(u) - w = 0$, i.e.,

$$f(u) = w = (u - u_1(w))(u - \bar{u}(w))(u_2(w) - u), \tag{4.6}$$

with $u_1(w) < \bar{u}(w) < u_2(w)$.

It is easily checked that for $(u_1(w),0)$ and $(u_2(w),0)$, where $f' < 0$, we have hyperbolic rest points (or saddle points). For the one in between, where $f' > 0$, we have a spiral source, if $\theta^2 < 4f'(\bar{u}(w))$, or an unstable node, if $\theta^2 > 4f'(\bar{u}(w))$, depending on the value of w.

Thus, the right and left branch of the cubic curve, S_1 and S_2 respectively, consist of saddles and the inner branch of spiral source or unstable node points.

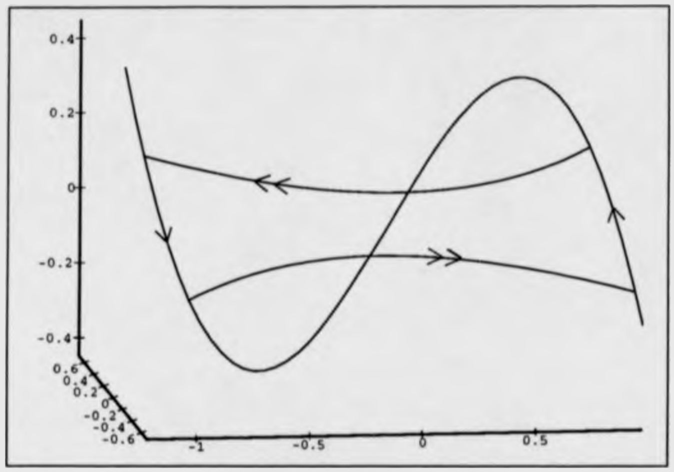

Figure 4.2: Singular periodic solution

Our aim is to construct a ***singular periodic*** solution of (4.1) by piecing together the appropriate solution segments which satisfy the fast or the slow equations. We define a singular periodic solution of (4.1) to be the piecewise smooth union of: (compare Figure 4.2)

(i) A heteroclinic solution of the fast equations (4.5) connecting $(u_1(\underline{w}), 0)$ to $(u_2(\underline{w}), 0)$ for some $\underline{w} \in \overline{\Pi}_-$ existing for some positive speed $\bar{\theta}$, say;

(ii) a solution segment of the slow equation (4.4) $(u_2(w), w)$ in the $\{v = 0\}$-plane from $\underline{w}$ to $\overline{w}$ for some $\overline{w} \in \overline{\Pi}_+$,

(iii) a heteroclinic solution of the fast equations (4.5) from $(u_2(\overline{w}), 0)$ to $(u_1(\overline{w}), 0)$, with the same speed $\bar{\theta}$ as at $\underline{w}$ and

(iv) a solution segment of the slow equation (4.4) $(u_1(w), w)$ in the $\{v = 0\}$-plane from $\overline{w}$ back to $\underline{w}$.

Clearly, a singular solution is not a proper solution of (4.1) for $\varepsilon = 0$. Also the tangent to the singular solution is discontinuous at the points $(u_1(\underline{w}), 0, \underline{w})$ and $(u_2(\overline{w}), 0, \overline{w})$.

Singular homoclinic solutions are defined similarly, with $\underline{w} = 0$. Note that for $a < 0$ we can not construct a singular homoclinic solution, as $u_1(0) < 0$ for all $a < 0$ and therefore the rest point of the fast system $(u_1(0), 0)$ can not be a projection of the origin in $\mathbb{R}^3$, the rest point of the full system.
For $a > \frac{1}{2}$ there are neither singular homoclinic solutions nor singular periodic solutions. The former can immediately be ruled out by the fact that $w_{infl} < 0$ for $a > \frac{1}{2}$, so that the way in which we constructed the singular solutions can not work. For the latter observe that the flow on S_1 is both for $\{u < 0\}$ and for $\{u > 0\}$ directed towards the origin in the (u, w)-space.

Thus singular homoclinic solutions can only exist for $0 \leq a \leq \frac{1}{2}$. We treat the case $a = 0$, which corresponds to a degeneracy, later.

It should be noticed that the relaxation oscillations of the kinetic equations are rather different from (singular) periodic travelling waves although the same cubic slow submanifold is involved in both cases. For the former the fast flow trajectories leave the slow submanifold at the local extrema of the cubic, and indeed no other fast flow trajectories leave it except on the middle branch where they all do. For the latter, however, some fast flow trajectories everywhere are going away from the slow submanifold.

4.1.1 Mechanical Interpretation

In order to work out the dependence between θ and w for which a saddle connection in the fast system exists, we make use of the following mechanical interpretation. We can rewrite the fast system (4.5) as a second order nonlinear differential equation

$$\ddot{u} - \theta\dot{u} + f(u) - w = 0 \tag{4.7}$$

describing a particle in a force field $f(u) - w$ with "negative friction" $-\theta\dot{u}$, as θ is positive. Note that the force is derivable from a potential with two local maxima

$$\mathcal{F}_w(u) = \int_0^u (f(s) - w)\, ds \tag{4.8}$$

where $w \in (w_{min}, w_{max})$. In Figure 4.3 the potential is shown for different choices of w.
In general the two local maxima will be of different height. The families of critical points of $\mathcal{F}_w$, parametrized by w, form the branches of the slow submanifold S.

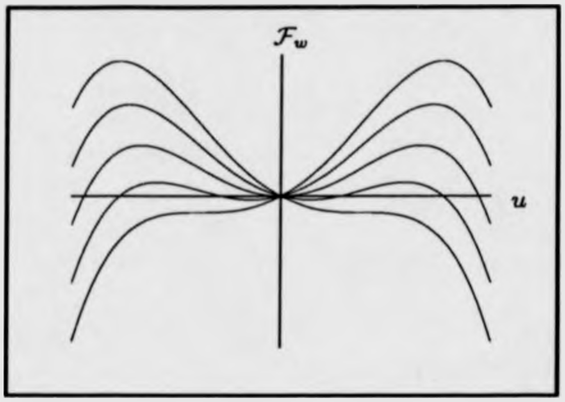

Figure 4.3: Potential $\mathcal{F}_w$ varying with $w \in [w_{min}, w_{max}]$, $a = -1$

Note that the local maxima correspond to saddles in this interpretation and the trajectories connecting the local maxima to saddle connections.

For $\theta = 0$ (4.5) forms a Hamiltonian system[1] with Hamiltonian function

$$H(u,v) \stackrel{def}{=} \frac{1}{2}v^2 + \mathcal{F}_w(u) \tag{4.9}$$

where $w \in (w_{min}, w_{max})$. Thus the phase portrait of (4.5) is determined by the level curves of the Hamiltonian. We choose the parameter w such that the two local maxima of $\mathcal{F}_w$ have the same height. This is determined by the condition

$$\int_{u_1(\hat{w})}^{u_3(\hat{w})} (f(s) - \hat{w})\, ds = 0 \tag{4.10}$$

the so-called *Maxwell line* value $w = \hat{w}(a)$. Because of the symmetry of the cubic we have $\hat{w} = w_{infl}$, the w-coordinate of the inflection point of the cubic, with $w_{infl} = \frac{1}{27}(1+a)(1-2a)(2-a)$. Thus for $\theta = 0$ and $w = \hat{w}$ there exist a pair of trajectories connecting the two local maxima in both ways. In other words, we have a *heteroclinic cycle*, i.e. a pair of saddle connections running in opposite directions shown in Figure 4.4.

[1] Compare Conley [Con75].

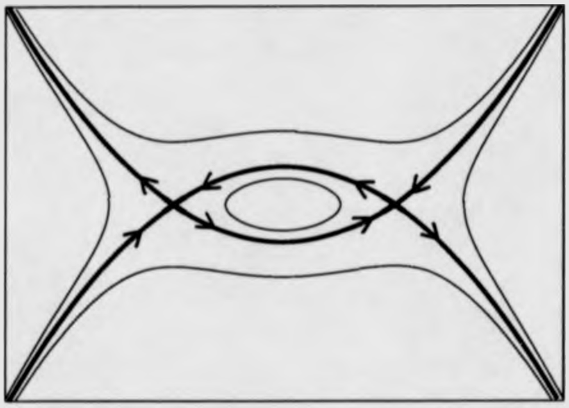

Figure 4.4: Phase portrait $\theta = 0, w = \hat{w}$

For positive θ the Hamiltonian is increasing along trajectories as the orbital derivative $\frac{dH}{dt} = \theta v^2$ is positive. Carpenter [Car77] proved the existence of some function $\theta(w)$ for $w \in \Pi$ such that the fast system admits a saddle connection from $(u_1(w), 0)$ to $(u_2(w), 0)$ at $\theta = \theta(w)$ if $\int_{u_1(w)}^{u_2(w)} (f(s) - w)\, ds \leq 0$ and a saddle connection in the other direction from $(u_2(w), w)$ to $(u_1(w), w)$ for $\theta = \theta(w)$ if $\int_{u_1(w)}^{u_2(w)} (f(s) - w)\, ds \geq 0$. The proof uses a shooting argument in θ applied to a branch of the unstable manifold of the respective rest point.

4.1.2 Derivation of the Saddle-Connection

We recall a result of Casten, Cohen & Lagerstrom [CCL75] who derived an explicit expression for the connecting orbit between $(u_1(w), 0)$ and $(u_2(w), 0)$ for $w \in \Pi_-$ and its corresponding wave speed $\theta(w)$. Note that (4.5) after eliminating the time variable z becomes

$$v \frac{dv}{du} = \theta\, v - f(u) + w, \tag{4.11}$$

since $\frac{\dot{v}}{\dot{u}} = \frac{dv}{du}$.
It is straightforward to check that

$$v = \lambda(u - u_1(w))(u_2(w) - u) \tag{4.12}$$

with $\lambda = \pm\frac{1}{\sqrt{2}}$ is a polynomial solution of (4.11) through $u_1(w)$ and $u_2(w)$, which exists for

$$\theta = \theta(w) = \lambda(u_1(w) + u_2(w) - 2\bar{u}(w)). \tag{4.13}$$

Since the curve $w = f(u)$, being a cubic, is symmetric about its inflection point (u_{infl}, w_{infl}) it follows that $\theta(w_{infl}) = 0$. Recall that we require θ to be non-negative. For $w \in \Pi_-$ we have $u_1(w) + u_2(w) - 2\bar{u}(w) > 0$ so we take $\lambda = \frac{1}{\sqrt{2}}$, but for $w \in \Pi_+$ λ must be given the negative value, $\lambda = -\frac{1}{\sqrt{2}}$, since $u_1(w) + u_2(w) - 2\bar{u}(w) < 0$.

There exists a uniquely determined $\overline{w} \in \Pi_+$ such that $(u_2(\overline{w}), \overline{w})$ is the point on the curve $w = f(u)$ which is symmetric to $(u_1(\underline{w}), \underline{w})$ with respect to the inflection point (u_{infl}, w_{infl}).

Because of the symmetry

$$\begin{aligned}
\theta(\overline{w}) &= -\frac{1}{\sqrt{2}}(u_1(\overline{w}) + u_2(\overline{w}) - 2\bar{u}(\overline{w})) \\
&= \frac{1}{\sqrt{2}}(u_1(\underline{w}) + u_2(\underline{w}) - 2\bar{u}(\underline{w})) \\
&= \theta(\underline{w}).
\end{aligned}$$

Observe that θ is continuous on $\overline{\Pi}$, but not differentiable as it does not have a unique tangent at w_{infl}. Moreover $\theta(w)$ is monotonically decreasing on $\overline{\Pi}_-$ and increasing, on $\overline{\Pi}_+$, being zero at the Maxwell line value $\hat{w} = w_{infl}$.

In the limit as $w \to w_{min}$ one of the humps becomes an inflectional plateau. For which value of the friction θ is there a trajectory connecting the local maxima to the inflectional plateau ? Is the limit of the friction $lim_{w \to w_{min}} \theta(w)$ finite ?

We postpone the answer to these questions to the next chapter, but introduce meanwhile the following notation.

For $w \in \Pi_-$ we denote the branch of the unstable manifold of the rest point $(u_2(w), 0)$ for (4.5) connecting it to $(u_1(w), 0)$ with respect to reversed[2] time

[2] We reverse time since we prefer to shoot away from the saddle node.

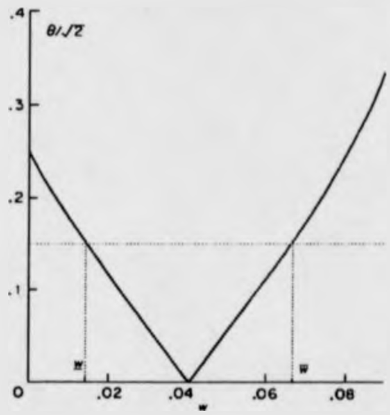

Figure 4.5: Typical graph of $\theta(w)$ for $a = \frac{1}{4}$

$\tau = -z$ by $\Lambda^r(w, \theta(w))$. Observe that the connecting orbit corresponding to $\underline{w}$ and $\overline{w}$ satisfies $\dot{u} = v \geq 0$ and $\dot{u} = v \leq 0$ respectively by (4.12). So for $\overline{w}$ and $\underline{w}$, u is an increasing and decreasing function of time z, respectively.

4.2 Degenerate Singular Periodic and Homoclinic Solution

We extend the theory to the oscillatory regime, where $\underline{w}$ can be taken down to $w_{min} = f(u_{min})$ and show that as $\tau \to \infty$, $\Lambda^r(w_{min}, \theta(w_{min}))$ tends to the rest point $(u_{min}, 0)$, being the merger of $(u_1(w), 0)$ and $(\bar{u}(w), 0)$ in the limit as $w \to w_{min}$, and, more importantly, that this connection between $(u_2(w_{min}), 0)$ and $(u_{min}, 0)$ exists for all $\theta \geq \theta^*$, for some $\theta^* > 0$.

We shall prove this in two steps. Firstly, we will consider a fixed $w \in \Pi_-$, and construct for all $\theta > \theta(w)$ a positively invariant region R with respect to time τ, in order to show that $\Lambda^r(w, \theta(w))$ tends to the rest point $(\bar{u}(w), 0)$ as $\tau \to \infty$. Secondly, we will consider the limit when w tends to w_{min} and the rest points $(u_1(w), 0)$ and $(\bar{u}(w), 0)$ merge in the single rest point $(u_{min}, 0)$.

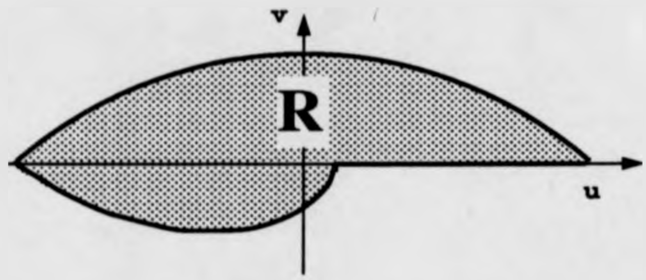

Figure 4.6: Trapping region R

Let us begin by prescribing the boundary of the positively invariant region R of the phase space of the (time reversed) fast flow, see Figure 4.6.

The upper boundary for non-negative v is given by the aforementioned polynomial solution through $(u_2(w), 0)$ and $(u_1(w), 0)$ with $\theta = \theta(w)$,
$v_1(u) := \frac{1}{\sqrt{2}}(u - u_1(w))(u_2(w) - u)$ for $u \in (u_1(w), u_2(w))$.

We have seen that for $\theta = 0$, (4.5) is a Hamiltonian system with the Hamiltonian function $H(u,v) = \frac{1}{2}v^2 + \mathcal{F}_w(u)$, where $\mathcal{F}_w$ denotes the potential (4.8). Using the fact that a Hamiltonian function is constant on orbits, we can give an explicit expression for the negative branch of its level curve corresponding to the orbit homoclinic to $(u_1(w), 0)$, viz. $v_2(u) := -\sqrt{2}\sqrt{\mathcal{F}_w(u_1(w)) - \mathcal{F}_w(u)}$ for $u \in (u_1(w), u^*(w))$. Moreover, $u^*(w)$ is given implicitly by $\int_{u_1(w)}^{u^*(w)} (f(s) - w)\, ds = 0$.

For $u \in [u^*(w), u_2(w))$ the u-axis of the (u,v)-space is the remaining part of the boundary.

We define $A_1(u,v) = v_1(u) - v$, $A_2(u,v) = v - v_2(u)$, $A_3(u,v) = v$; and the region R to be

$$R = \bigcap_{i=1}^{3} A_i^{-1}([0,\infty)). \tag{4.14}$$

Note that each boundary point $(u,v) \in \partial R$ satisfies $A_i(u,v) = 0$ for some i.

Lemma 4.1 *Let $w \in \Pi_-$ be fixed. Then R is a positively flow invariant region of the fast flow defined by* (4.5) *for each $\theta > \theta(w)$ and for reversed time τ.*

Proof: We show that the flow along the boundary of R is inward pointing, except at the points $(u_1(w), 0)$ and $(u_2(w), 0)$, where it is stationary. Denote by $F_w^r(u, v) = (-v, -\theta\, v + f(u) - w)^T$ the time reversed vector field (4.5), and let $v = v_1(u)$ for $u \in (u_1(w), u_2(w))$. Then

$$\langle \nabla A_1(u, v_1), F_w^r(u, v_1)\rangle = (\theta - \theta(w))v_1 > 0, \qquad (4.15)$$

since v_1 satisfies (4.11) for $\theta = \theta(w)$ and is positive. Remember, the negative branch of the homoclinic to $(u_1(w), 0)$, $v_2(u)$ is a solution to (4.11) for $\theta = 0$. Hence

$$\langle \nabla A_2(u, v_2), F_w^r(u, v_2)\rangle = -\theta v_2 > 0. \qquad (4.16)$$

The inequality follows from the fact that $v_2(u) < 0$ for $u \in (u_1(w), u^*(w))$. For the remaining part of the boundary, we have $v_3(u) = 0$ for $u \in [u^*(w), u_2(w))$, and therefore

$$\langle \nabla A_3(u, v_3), F_w^r(u, v_3)\rangle = f(u) - w > 0. \qquad (4.17)$$

This shows that R is a positively invariant region. ∎

Let for $w \in \Pi_-$, $\Lambda^r(w, \theta)$ denote the branch of the unstable manifold of the rest point $(u_2(w), 0)$ with positive half solution contained in $\{v \geq 0\}$.

Lemma 4.2 *$\Lambda^r(w, \theta)$ tends to $(\bar{u}(w), 0)$ for $\theta > \theta(w)$ as $\tau \to \infty$.*

Proof: By (4.1), $\Lambda^r(w, \theta)$ can not escape R for $\theta > \theta(w)$. The only boundary point of R to which $\Lambda^r(w, \theta)$ can possibly tend is $(u_1(w), 0)$. But the connection between $(u_2(w), 0)$ and $(u_1(w), 0)$ exists only for the unique value of $\theta = \theta(w)$. Furthermore, the slope of $\Lambda^r(w, \theta)$ at the rest point $(u_2(w), 0)$ is a decreasing function of θ, as can be seen from the linearization of the vector field F_w^r at $(u_2(w), 0)$. Therefore $\Lambda^r(w, \theta)$ is forced to tend to $(\bar{u}(w), 0)$ for $\theta > \theta(w)$ as $\tau \to \infty$. ∎

Finally, we consider the limit as w tends to w_{min}, i.e. when the saddle point $(u_1(w), 0)$ and the stable node $(\bar{u}(w), 0)$ of (4.5) become the saddle-node $(u_{min}, 0)$ of (4.5) for $w = w_{min}$. It is then clear from Lemma 4.2 that the heteroclinic connection between $(u_2(w_{min}), 0)$ and $(u_{min}, 0)$ exists for all $\theta \geq \theta(w_{min})$, where $\theta(w_{min}) = lim_{w \to w_{min}} \theta(w)$. We state this in the following proposition with respect to the non-reversed time x.

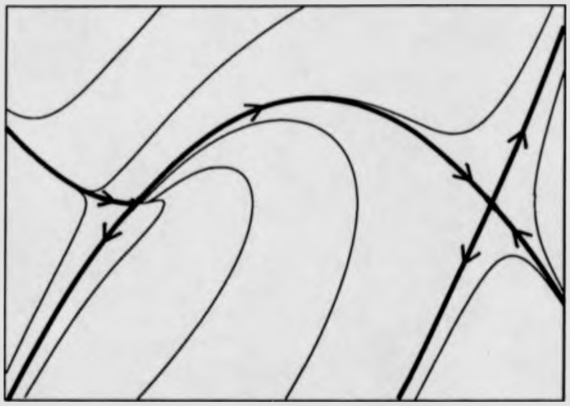

Figure 4.7: Connecting orbit from saddle-node to saddle for $\theta = \frac{1}{\sqrt{2}}$

Proposition 4.3 *The heteroclinic orbit of* (4.5) *for* $w = w_{min}$ *connecting the rest points* $(u_{min}, 0)$ *with* $(u_2(w_{min}), 0)$ *exists for all* $\theta \geq \theta(w_{min})$. *Equally, for* $w = w_{max}$ *the heteroclinic orbit connecting the rest points* $(u_{max}, 0)$ *with* $(u_1(w_{max}), 0)$ *exists for all* $\theta \geq \theta(w_{max})$. *In addition,* $\theta(w_{min}) = \theta(w_{max})$.

We can compute $\theta(w_{min})$ in terms of a, the root of the cubic f. Set $u_2 = u_2(w_{min})$ then from (4.13) we have $\theta(w_{min}) = \frac{1}{\sqrt{2}}(u_2 - u_{min})$. Expanding $f(u_2)$ around u_{min} we get after some algebraic manipulations $u_2 - u_{min} = \frac{1}{2}f''(u_{min}) = \sqrt{a^2 - a + 1}$, since $f(u_2) = w_{min} = f(u_{min})$, $f'(u_{min}) = 0$ and $u_{min} = \frac{1}{3}\{a + 1 - \sqrt{a^2 - a + 1}\}$. Thus, $\theta(w_{min}) = \frac{1}{\sqrt{2}}\sqrt{a^2 - a + 1}$. Define

$$\theta^*(a) := \begin{cases} \frac{1}{\sqrt{2}}(1 - 2a) & \text{for} \quad 0 < a \leq \frac{1}{2}, \\ \frac{1}{\sqrt{2}}\sqrt{a^2 - a + 1} & \text{for} \quad a \leq 0. \end{cases} \tag{4.18}$$

We call singular periodic and homoclinic solutions *degenerate* if their fast flow segments consist of a saddle–node to saddle connection or vice versa, rather than simply saddle connections. See Figure 4.2.
The conclusion of the previous analysis with respect to the original, non–reversed, time z is summarized in the following theorem.

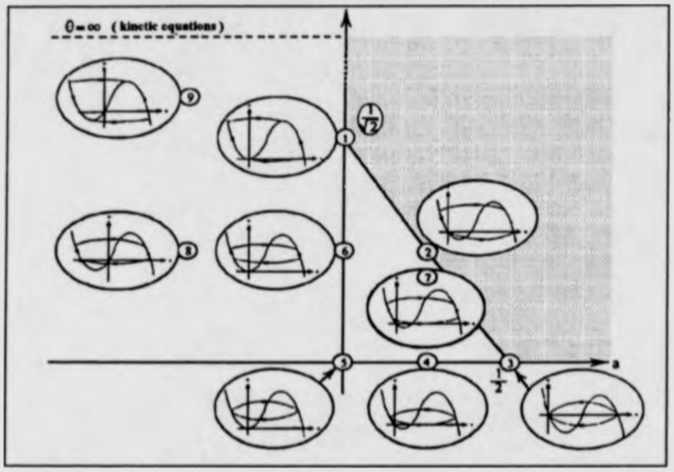

Figure 4.8: Two parameter family of singular TW solutions in (a, θ)-parameter space

Theorem 4.4 *The travelling wave equations of the FitzHugh-Nagumo system* (4.1) *admit singular solutions for the following choice of parameters:*

(i) Singular periodic solutions exist for all $\theta \in [0, \theta^*(a))$ *and* $a < \frac{1}{2}$.

(ii) Singular homoclinic solutions exist for $\theta = \theta^*(a)$ *and* $0 < a \leq \frac{1}{2}$.

(iii) Degenerate singular periodic solutions exist for all $\theta \geq \theta^*(a)$ *and* $a < 0$.

(iv) Degenerate singular homoclinic solutions exist for all $\theta \geq \theta^*(0) = \frac{1}{\sqrt{2}}$ *at* $a = 0$.

It should be pointed out that diagram 9 in Figure 4.8 is qualitatively true for all $\theta \geq \frac{1}{\sqrt{2}}$. In the previous chapter we have seen that as θ tends to infinity the periodic travelling waves in the oscillatory regime, which are obtained as perturbation from the corresponding singular solutions, tend to homogeneous oscillations. Thus we have shown that there exists a continuous two-parameter family of singular periodic travelling waves connecting the excitable ones with the homogeneous oscillations existing in the limit as θ goes to ∞.

Chapter 5

Persistence of Singular Solutions

We show that the singular solutions, which we have constructed in the previous chapter, perturb into genuine solutions, close to the singular ones, for small positive ε. We prove that all but the degenerate singular homoclinic solutions persist. For the latter we provide reasons for their non-persistence.

There exist a number of methods for systems of singularly perturbed ODE's which achieve this, e.g., [JK] & [JKL91], [Lan80], [MR80], [Sch92], [Smo82], [Szm91]. All of them use in one way or another invariant manifold theory, except [MR80], which uses asymptotic expansions. Our proof is based on the work of G. Carpenter [Car77], which is inspired by ideas of C. Conley outlined in [Con75]. Basically it consists of two steps:

(a) Defining hypotheses on the dynamics of the ODE's under which topological methods (Wazewski's principle, Brouwer degree or, alternatively, the Conley index) can be applied to prove the existence of homoclinic and periodic solutions.

(b) Using the associated singular solution to construct the machinery, specifically (isolating) blocks, needed to apply the results of (a).

Though this method has wide applicability, we merely use it as a tool to demonstrate the existence of homoclinic and periodic solutions to the FitzHugh-Nagumo travelling wave equations,

$$\left.\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \theta v - f(u) + w, \\ \dot{w} &= \tfrac{\varepsilon}{\theta} u, \end{aligned}\right\} \qquad (\mathrm{FN};\theta,\varepsilon)$$

where $f(u) = u(u-a)(1-u)$. Unlike in Carpenter work [Car77], here a, the root of the cubic, may also take non-positive values.

5.1 Preliminaries

To develop the requisite machinery, we need a number of concepts. Suppose we are given a system of ODE's,

$$\dot{u} = G(u), \tag{5.1}$$

with G of class C^1 and $u \in \Omega \subseteq \mathbb{R}^N$, Ω open and connected. We assume that this system of ODE's generates a global flow $\phi : \Omega \times \mathbb{R} \to \Omega$, where global means that we assume solutions to exist for all time. Henceforth we shall write $u \cdot t$ for $\phi(u,t)$ and $\gamma_+(u)$ for the positive semi-orbit of u under the flow, i.e. $\gamma_+(u) \stackrel{def}{=} u \cdot [0,\infty)$.
A set $B \subset \Omega$ will be called a *block* for ϕ if:

(I) There exist N functions $f_1, \ldots, f_N$ from $\mathbb{R}^N$ into $\mathbb{R}$ such that $B \stackrel{def}{=} \bigcap_{i=1}^N f_i^{-1}([0,\infty))$ is homeomorphic to the unit cube in $\mathbb{R}^N$.

(II) $\langle \nabla f_i(u), G(u)\rangle \neq 0$ for $u \in f_i^{-1}(0) \cap B$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^N$.

Note that property (II) means that the trajectories cannot be tangent to the boundary of B. The property of being a block is preserved under perturbations of the flow.
The *entrance set* of a block B is the set $b^+ \subset \partial B$, such that for each $u \in b^+$ we have $f_i(u) = 0$ and $\langle \nabla f_i(u), G(u)\rangle < 0$ for some i. This means that trajectories point inward into B on b^+. The *exit set* of a block, b^-, is defined in a similar way, with the last inequality reversed. The corners of the block are contained in both the entrance- and the exit set.

Let B be a block. We define the time it takes to reach various portions of ∂B for an arbitrary point $u \in \Omega$ by

$$T^{\pm}(u) \stackrel{def}{=} \begin{cases} 0 & \text{if } u \in b^{\pm}, \\ \sup\{t > 0 : u \cdot (0,t) \cap b^{\pm} = \emptyset\} & \text{if } u \notin b^{\pm}. \end{cases} \tag{5.2}$$

If $T^{\pm}(u)$ is finite, we denote by $\Phi^{\pm}(u)$ the point in $b^{\pm}$ into which u is mapped by the flow after time $T^{\pm}(u)$, that is, $\Phi^{\pm}(u) = u \cdot T^{\pm}(u)$. We shall also need the sets D^+ and D^-, where $D^+ = \{u \in \Omega : \ 0 < T^+(u) < \infty, \ \Phi^+(u) \notin b^-\}$ (excluding corners). D^- is defined in similar way with all pluses replaced by minuses and vice versa. Thus $D^{\pm}$ is the set of points in $\Omega \backslash B$, trajectories of which intersect $b^{\pm}$ transversely.
The following result is Lemma 1.3 of [Car77].

Lemma 5.1 *If B is a block, then $T^{\pm}$, $\Phi^{\pm}$ are continuous on $D^{\pm}$.*

Consider now a parametrized system of ODE's

$$\dot{u} = G(u, \lambda), \tag{5.3}$$

$(u, \lambda) \in \Omega \times \Lambda \subseteq \mathbb{R}^N \times \mathbb{R}^k$, where Ω, Λ are open and connected, and G is of class C^1 as a mapping of u and λ. If B is a block for (5.3) for $\lambda = \lambda_0$, then it will remain a block for values of λ close to λ_0. The same will be true for $b^{\pm}$. Below we shall denote dependence on λ by subscripts. We often drop the subscripts, provided that there is no confusion involved.

5.2 Homoclinic Solutions

The hypotheses *HOM* used in [Car77] for the existence of a homoclinic solution are as follows:

(A) There exist two blocks, B_1 and B_2, where B_1 is the one that contains the rest point $\bar{u}$.

(B) For all $\lambda \in \Lambda$, $\bar{u}$ is a rest point of (5.3), and if $\gamma_+(u) \subset B_1$, this means that $u \in W^s(\bar{u})$ (lies on the stable manifold of the rest point $\bar{u}$). That is, the flow is "gradient-like" , meaning that nothing can enter B_1 without eventually hitting $\bar{u}$. Furthermore, for no u is $\gamma_+(u)$ contained in B_2.

(C) There exists an open subset Δ of $b_2^- \cap D_1^+$ (points in the exit set of the block B_2 which are going to enter B_1 transversely) such that $b_2^- \backslash \Delta$ consists of two components, β_0 and β_1. Let $\delta_i \equiv \beta_i \cap \mathrm{cl}(\Delta)$ ($i = 0, 1$). Then $\delta_0 \cup \delta_1 \subset D_1^-$. This means that all points on the lower and upper boundary of Δ will

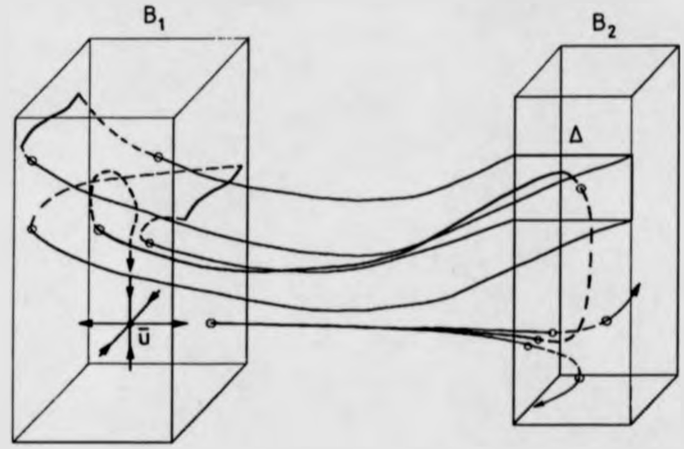

Figure 5.1: Hypotheses for the existence of homoclinic solutions. b_1^+: front, back, top and bottom face of B_1; b_2^-: front, back and top face of B_2

enter and then leave B_1 and $\Phi_1^-(\delta_0)$ and $\Phi_1^-(\delta_1)$ are contained in different components of b_1^- (this means that they leave through different components of the exit set of B_1). This is to hold for all values of the parameter in some small interval. What changes as we change the parameter is the behaviour of the unstable manifold $W^u(\bar{u})$ of $\bar{u}$. This is given by

(D) There exists a path $\Gamma = \{(u_s; \lambda_s) : s \in [-1,1]\} \subset D_2^+ \times \Lambda$, such that $u_s \in W^u(\bar{u})$, i.e., lies for all s on the unstable manifold of the rest point $\bar{u}$ of (5.3,λ_s) and $\Phi_2^-(u_{-1}) \in \beta_0$ and $\Phi_2^-(u_1) \in \beta_1$. Note that this in conjunction with condition (C) means that $\Phi_2^-(\Gamma)$ has to intersect both δ_0 and δ_1. See Figure 5.1 to clarify the situation.

Under these assumptions (5.3) has a homoclinic solution. Take our curve Γ and follow it along the flow till it exits B_1 again. The curve is connected; its image on b_1^- is not. Wazewski's principle stated in Appendix D.1 now clinches the existence proof, since the curve lies on the unstable manifold (of the product flow).
We shall, however, in the proof of the following proposition not explicitly make use of Wazewski's principle.

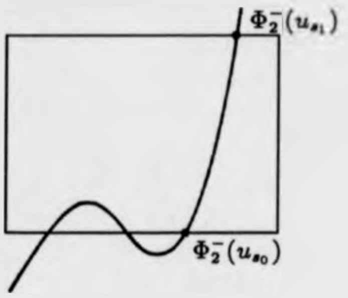

Figure 5.2: Intersection of $\Phi_2^-(\Gamma)$ with Δ

Proposition 5.2 *The above hypotheses imply that* $(5.3, \lambda_s)$ *admits a homoclinic solution for some* λ_s, *with* $s \in [-1, 1]$.

Proof: If Γ is a path in D_2^+ then it is also a path in D_2^-, since by hypothesis (B) no positive semi orbit is contained in B_2. Then by Lemma 5.1, the continuity of $\Phi^{\pm}$ on $D^{\pm}$, $\Phi_2^-(\Gamma)$ is a path and hence connected in b_2^-, whose endpoints are contained in β_0 and β_1, respectively, by hypothesis (D). By restricting the domain of the path Γ to some closed subset, say, $[s_0, s_1]$ of the index-set $[-1, 1]$, we may assume that $\Upsilon \stackrel{\text{def}}{=} \Phi_2^-(\Gamma\big|_{[s_0,s_1]}) \subseteq \mathrm{cl}(\Delta)$ and therefore in D_1^+, where the endpoints of Υ (corresponding to λ_{s_0} and λ_{s_1}) are contained in δ_0 and δ_1, respectively. This is shown inf Figure 5.2.
Were Υ also contained in D_1^- then its image under Φ_1^- would be connected by Lemma 5.1. However, Υ is mapped by Φ_1^- to distinct components of b_1^- by hypothesis (C) and can therefore not be contained in D_1^-. Thus there exists an orbit passing through some point of Γ, for some $s \in (s_0, s_1)$, which enters B_1, but does not leave it for positive time. This orbit is by hypothesis (B) on the stable manifold of the rest point $\bar{u}$. This completes the proof. ∎

Our next task is to see when these assumptions are satisfied for $(\mathrm{FN}; \theta, \varepsilon)$. To be able to construct a (non-degenerate) singular homoclinic solution, the rest point of the fast system $(0,0)$ (corresponding to the unique rest point of the full system at the origin) must be of saddle type. This only holds for $0 < a < \frac{1}{2}$, where a denotes the root of the cubic f.

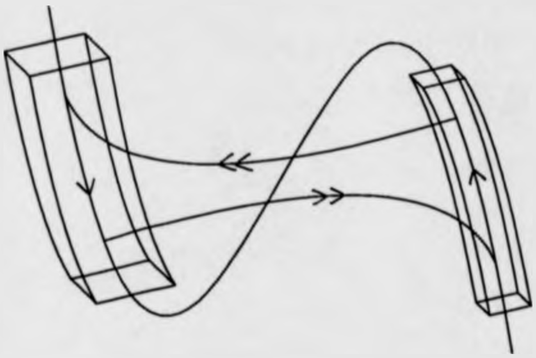

Figure 5.3: Blocks around S_1 and S_2 in the proof of homoclinic solutions

There are other homoclinic solutions as well, which exist for values of θ of the order $\sqrt{\varepsilon}$ as $\varepsilon \to 0$, whose existence can be proven using connectedness arguments from plane topology, compare [Has76]. There is also an analytical proof of slow homoclinics given in [dO92]. However, these *slow homoclinics* are not perturbations of singular solutions. Also, the fact that no singular homoclinic solutions can be constructed for $a < 0$, does, of course, not mean that the system does not admit any homoclinic solutions in this range.

Theorem 5.3 *Let $a \in (0, \frac{1}{2})$ be fixed. Then there exist some $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ (FN; $\theta_\varepsilon, \varepsilon$) admits a homoclinic orbit to the origin $(0,0,0)$ for some $\theta_\varepsilon > 0$. Moreover, θ_ε tends to $\bar{\theta} = \theta^*(a)$ for $\varepsilon \to 0$; i.e., the singular solution perturbs into homoclinic solutions of the full system for nonzero ε.*

Proof: Before we begin to verify the hypotheses *HOM* one by one, we recall some facts about the fast flow. Both $(u_1(0), 0) = (0, 0)$ and $(u_2(0), 0) = (1, 0)$ are hyperbolic rest points, saddles, of the fast system (FN; $\theta, 0$) for $\underline{w} = 0$. For fixed a, there exists a unique $\bar{\theta} = \theta^*(a)$ for which there is a heteroclinic solution running from the former to the latter. Also, $\underline{w}$ uniquely determines $\overline{w}$, for which there is a heteroclinic solution from $(u_2(\overline{w}), 0)$ to $(u_1(\overline{w}), 0)$ in the fast system (FN; $\bar{\theta}, 0$).

A: We construct blocks for the fast and the full system. We start by choosing w_1 such that $w_{min} < w_1 < \underline{w} = 0$ and define w_2 symmetrical with respect to w_{infl}, i.e., $w_2 - w_{infl} = w_{infl} - w_1$. For $i = 1,2$ and each $w \in [w_1, w_2]$ we define $B_i(w) = \bigcap_{j=1}^{4} f_{i,j;w}^{-1}([0,\infty))$ and $B_i = \bigcup_{w \in [w_1, w_2]} B_i(w) \times \{w\}$, where for some $c_i > 0$:

$$\begin{aligned}
f_{i,1;w}(u,v) &= -v - (\theta+1)(u - u_i(w)) + (\theta+1)\, c_i,\\
f_{i,2;w}(u,v) &= -v + (\theta+1)(u - u_i(w)) + (\theta+1)\, c_i,\\
f_{i,3;w}(u,v) &= v + (\theta+1)(u - u_i(w)) + (\theta+1)\, c_i,\\
f_{i,4;w}(u,v) &= v - (\theta+1)(u - u_i(w)) + (\theta+1)\, c_i.
\end{aligned}$$

Note that $B_i(w)$ can be more conveniently expressed as

$$B_i(w) = \{(u,v) : |v \pm (\theta+1)(u - u_i(w))| \leq (\theta+1)\, c_i\} \quad (i = 1,2).$$

Clearly, $B_i(w)$ is homeomorphic to the unit square and B_i to the unit cube. The blocks B_1 and B_2 are depicted in Figure 5.3.

We proceed to show that the appropriate flow cannot be tangent to any point on the boundary of $B_i(w)$ and B_i. For a fixed $w \in [w_1, w_2]$, we denote the vector field corresponding to (FN; θ, 0) by F_w. For example, we have for sufficiently small $c_1 > 0$

$$\langle \nabla f_{1,1;w}, F_w \rangle = (2\theta+1)(\theta+1)(v - u_1(w) - c_1) + f(u) - w < 0,$$

if $(u,v) \in f_{1,1;w}^{-1}(0)$. So $f_{1,1;w}^{-1}(0) \cap B_1(w) \subseteq b_1^-(w)$. The calculation for the other faces are similar.

For a given $w \in (w_{min}, w_{max})$, we denote the supremum of the diameters for which $B_i(w)$ is a block for the fast system (FN; θ, 0) by $c_i^*(w)$. Note that $c_1^*(w)$ goes to zero for w_1 tending to w_{min}. Similarly, $c_2^*(w)$ approaches zero as w_1 tends to w_{max}.

A computation analogous to the one for the fast system, with $f_{i,j}(u,v,w) \stackrel{\text{def}}{=} f_{i,j;w}(u,v)$, shows that B_i is a block around the slow submanifold S_i of the full system (FN; θ, ε), for some $c_i := c_i(w_i)$, with $0 < c_i(w_1) < c_1^*(w)$ and sufficiently small $\varepsilon > 0$. Observe that both the bottom and top face of the block B_1 are not contained in its exit set, which is therefore disconnected.

B: Note that the positive semi orbit of a point in B_1 can only be contained in B_1 if it is on the stable manifold of the origin. This holds by inspection of the slow flow. For the same reason no positive semi orbit is contained in B_2.

C: We construct the set Δ and show that it is contained in D_1^+ and that its lower and upper boundaries, δ_0 and δ_1, get mapped by Φ_1^- to distinct components of b_1^-. We show this in C.1 for the fast system and generalize it in C.2 to the full system. We now proceed to define $\Delta = \bigcup_{|w-\overline{w}|<\beta} \Delta(w) \times \{w\}$ for small β, where

$$\Delta(w) = \{(u,v) \in b_2^-(w) : -(\theta+1)c_2 < v < 0\}$$

for fixed w. Clearly, Δ is an open set and contained in b_2^-.

With respect to the fast system (FN; θ, 0) we define $\Lambda_1(\theta, w)$ to be the branch of $W^s(u_1(w), 0)$ beginning in $\{v < 0\}$ and $\Lambda_2(\theta, w)$ to be the branch of $W^u(u_2(w), 0)$ also beginning in $\{v < 0\}$. We remark that the connecting orbit satisfies $\Lambda_1(\bar{\theta}, \overline{w}) = \Lambda_2(\bar{\theta}, \overline{w})$.

C.1: Let $c_1 > 0$ be chosen such that B_1 is a block for sufficiently small $\varepsilon > 0$.

We define $\beta_+ > 0$ and $\beta_- < 0$ to be the values of β for which $\Lambda_2(\bar{\theta}, \overline{w} + \beta)$ passes through the the corners of the block $B_1(\overline{w} + \beta)$, $(u_1(\overline{w} + \beta) + c_1,\ 0)$ and $(u_1(\overline{w} + \beta), -(\theta+1)c_1)$, respectively.

In the following we implicitly make use of the fact that the trajectories of (FN; $\bar{\theta}$, 0) depend monotonically on w. Meaning that the intersection point of trajectories of (FN; $\bar{\theta}$, 0) parametrized by w with two suitably chosen lines, $\{v = 0\}$ and $\{u = u_2(w)\}$ depends monotonically on w, for w close to $\overline{w}$. This is a consequence of the fact that $\Lambda_1(\bar{\theta}, \overline{w})$ and $\Lambda_2(\bar{\theta}, \overline{w})$ pass with non-zero "speed" through the heteroclinic connection for $w = \overline{w}$ which can be shown in terms of a Melnikov integral that is non-zero. The Melnikov integral is an explicit expression for $\frac{\partial Q}{\partial w}(\overline{w})$, where $Q(w)$ is a measure for the "distance" between $\Lambda_1(\bar{\theta}, w)$ and $\Lambda_2(\bar{\theta}, w)$. In [Den91] a formula for $\frac{\partial Q}{\partial w}(\overline{w})$ is derived (in a different context), applied to (FN; θ, 0) and shown to be positive.

Then the block $B_2(w)$ is for each $|w - \overline{w}| \leq \hat{\beta}$, for some fixed $\hat{\beta}$ satisfying $0 < \hat{\beta} < \min\{\beta_+, |\beta_-|\}$ constrained by: (compare Figure 5.4)

(a) The intersection point of $\Lambda_1(\bar{\theta}, \overline{w} - \hat{\beta})$ with $\{v = 0\}$ and the v-coordinate of the backward orbit of (FN; $\bar{\theta}$, 0) through the point $(u_1(\overline{w} - \hat{\beta}),\ -(\theta+1)c_1)$ at $u = u_2(\overline{w} - \hat{\beta})$.

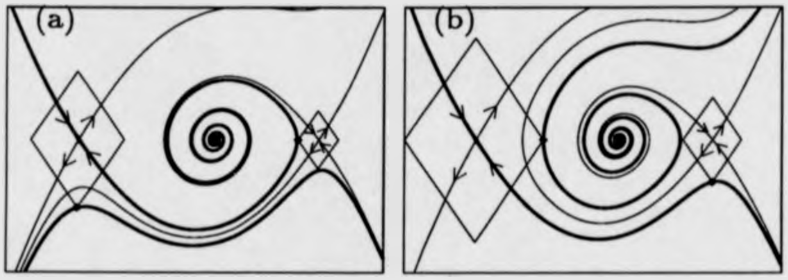

Figure 5.4: Verification of the mapping condition on Δ: Phase portraits of the fast flow for $\bar{\theta} = \theta(\overline{w})$ at (a) $w = \overline{w} - \hat{\beta}$, (b) $w = \overline{w} + \hat{\beta}$

(b) The v-coordinate of $\Lambda_1(\bar{\theta}, \overline{w} + \hat{\beta})$ at $u = u_2(\overline{w} + \hat{\beta})$ and the intersection point of the backward orbit of $(\mathrm{FN}; \bar{\theta}, 0)$ through the point $(u_1(\overline{w} + \hat{\beta}) + c_1,\ 0)$ with $\{v = 0\}$.

We set $c_2 = c_2(\overline{w} + \hat{\beta})$, for some $c_2(\overline{w} + \hat{\beta})$ with $0 < c_2(\overline{w} + \hat{\beta}) < c_2^*(\overline{w} + \hat{\beta})$ Then for all w, with $|w - \overline{w}| \leq \hat{\beta}$, Δ is contained in D_1^+ under the family of fast flows $(\mathrm{FN}; \bar{\theta}, 0)$ parametrized by w. Also note that the lower and the upper boundaries of Δ, δ_0 and δ_1 at $w = \overline{w} - \hat{\beta}$ and $w = \overline{w} + \hat{\beta}$, respectively, leave the exit set of the block $B_1(\overline{w} \mp \hat{\beta})$ through $\{v < 0\}$ and $\{v > 0\}$, respectively.

C.2: By the classical theorem of continuous dependence of the flow on parameters, there exists a $\tau_1 = \tau(c_1, \hat{\beta}) > 0$ and an $\varepsilon_1 = \varepsilon(c_1, \hat{\beta}, \tau_1) > 0$ such that $\Delta \subset D_1^+$, $\Phi_1^+(\delta_0 \cup \delta_1; \theta, \varepsilon) \subset D_1^-$; and $\Phi_1^-(\delta_0; \theta, \varepsilon)$ and $\Phi_1^-(\delta_1; \theta, \varepsilon)$, respectively, leave b_1^- through $\{v < 0\}$, respectively $\{v > 0\}$, under the flow of the full system $(\mathrm{FN}; \theta, \varepsilon)$ for all $|\theta - \bar{\theta}| \leq \tau_1$ and $0 < \varepsilon < \varepsilon_1$.

D: Recall from the linear stability analysis that dim $W^u(0) = 1$ for $(\mathrm{FN}; \theta, \varepsilon)$ with $\theta \gtrsim 0$ and $\varepsilon > 0$. Let $\Lambda^\varepsilon(\theta)$ be the branch of $W^u(0)$ beginning in $\{v > 0\}$ and define $\Lambda^0(\theta)$ to be the corresponding branch of $W^u(0,0)$ of the fast system $(\mathrm{FN}; \theta, 0)$ at $w = 0$. Note that $\Lambda^0(\bar{\theta})$ stands for the singular

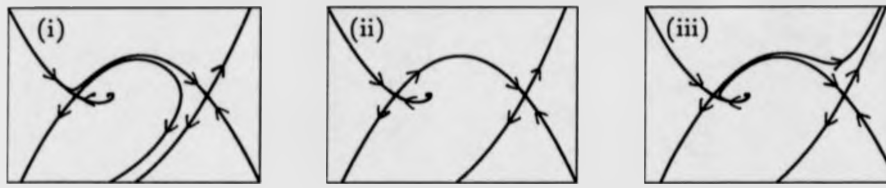

Figure 5.5: Shooting argument in θ for $w = 0$: (i) $\theta < \bar{\theta}$, (ii) $\theta = \bar{\theta}$, (ii) $\theta > \bar{\theta}$

connection between the saddles $(0,0)$ to $(0,1)$. There exists an $\varepsilon_2 > 0$ and a $\tau_2 = \tau_2(\varepsilon_2) > 0$ such that for fixed ε, with $0 < \varepsilon < \varepsilon_2$, $\Lambda^\varepsilon(\theta) \cap B_2 \neq \emptyset$ for θ with $|\theta - \bar{\theta}| \leq \tau_2$, by continuous dependence on parameters from the singular connection for $\theta = \bar{\theta}$ and $w = 0$. We define the path for a fixed ε, with $0 < \varepsilon < \varepsilon_2$, to be

$$\Gamma_\varepsilon = \{U_s \stackrel{\text{def}}{=} (u_s, v_s, w_s) \in \partial B_1 : U_s \in \Lambda^\varepsilon(\bar{\theta} + s\tau_2) \text{ for } s \in [-1,1]\}.$$

This determines, for a fixed ε, a unique path.

Clearly, $\Gamma_\varepsilon \subset D_2^+$ by construction. We prove the conditions on the end-points of the path by a shooting argument in s applied to $\Lambda^0(\bar{\theta} + s\tau_2)$ and extend it by continuous dependence on parameters to $\varepsilon > 0$. For this it is sufficient to state that the singular connection breaks up for $\theta \neq \bar{\theta}$ and that for $s < 0$, $\Lambda^0(\bar{\theta} + s\tau_2)$ undershoots $W^s(1,0)$, the stable manifold of the saddle $(1,0)$ at $w = 0$, i.e., $\Lambda^0(\bar{\theta} + s\tau_2) \cap \beta_0 \neq \emptyset$, together with the fact that for $s > 0$ it overshoots $W^s(1,0)$, i.e., $\Lambda^0(\bar{\theta} + s\tau_2) \cap \beta_1 \neq \emptyset$. This is illustrated in Figure 5.5. Analytically this "break up" of the saddle connection follows again from the fact that the Melnikov integral $\frac{\partial Q}{\partial \theta}(\bar{\theta}) \neq 0$ at $w = 0$ proved in [Den91], where $Q(\theta)$ serves here as "distance" between the appropriate branches of $W^u(0,0)$ and $W^s(1,0)$.

Conclusion: Thus, all the hypotheses can be satisfied for small enough $0 < \varepsilon < \varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$ and $|\theta - \bar{\theta}| \leq \tau$, with $0 < \tau < \tau_0 := \min\{\tau_1, \tau_2, \varepsilon\}$. Therefore Proposition 5.2 implies that (FN; $\theta_\varepsilon, \varepsilon$) admits a homoclinic solution with $\theta_\varepsilon = \bar{\theta} + s\tau$ for some s. Clearly, by the choice of τ_0, $\theta_\varepsilon \to \bar{\theta}$ for $\varepsilon \to 0$. ∎

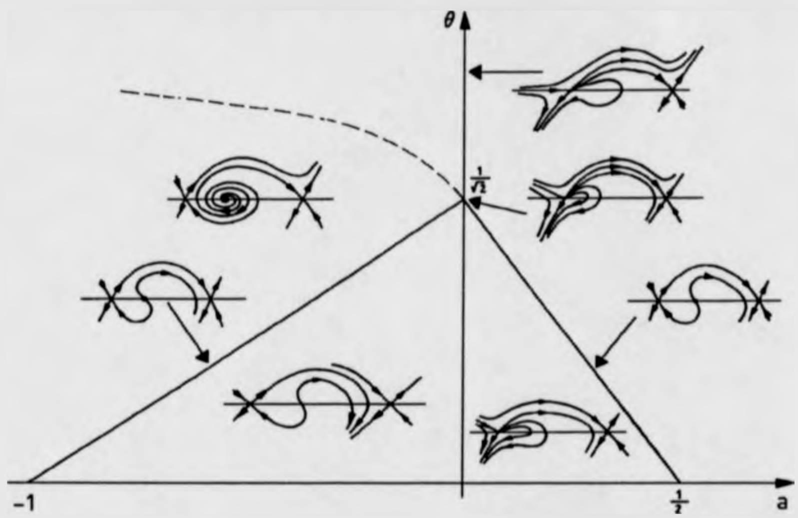

Figure 5.6: Various phase portraits of the fast system in (a,θ)-parameter space for $w = 0$

We remark that this proof can be generalized to case when the dimension of the slow submanifold is bigger than one. Furthermore, it does not imply local uniqueness of homoclinic solutions which comes out of Langer's [Lan80] proof, using invariant manifold theory á la Hirsch, Pugh & Shub [HPS77].

5.3 Reasons for the Non-Persistence of Degenerate Singular Homoclinic Solutions

Rather than giving a formal non-existence proof we explain why the persistence proof for singular homoclinic solutions does not work for the degenerate singular homoclinic solutions existing along the line $\{(a,\theta) : a = 0,\ \theta \geq \frac{1}{\sqrt{2}}\}$.

Examining the hypotheses on the existence of homoclinic solutions shows that this depends exclusively on whether or not the path condition (D) can be fulfilled. In light of the locus of degenerate singular homoclinic solutions in (a,θ)-parameter space we parametrize the curve Γ on the unstable manifold of the origin w.r.t. the full system by a rather than θ.

Recall that the path condition requires that for $|a| < \tau$, for some $\tau > 0$, the endpoints of the path

$$\Gamma_\epsilon = \{U_s \in \partial B_1 : U_s \in \Lambda^\epsilon(s\tau)^1 \text{ for } s \in [-1,1]\},$$

get mapped by Φ_2^- to distinct components of $b_2^- \backslash \Delta$ for each given fixed $\theta \geq \frac{1}{\sqrt{2}}$. With respect to the fast system (FN; $a, 0$) for $w = 0$ this amounts to show that $\Lambda^0(a)$ under- and overshoots the appropriate branch of the stable manifold of the R.H.S. saddle $(1,0)$ for negative and positive values of a, respectively.
However, the locus in (a,θ)-space across which this happens, i.e. where $\Lambda^0(a)$ connects to the the saddle at $(1,0)$, is given by $\theta = \theta^*(a)$ for $0 \leq a \leq \frac{1}{2}$ and $\theta = \frac{1}{\sqrt{2}}(a+1)$ for $-1 \leq a \leq 0$, compare Figure 5.6. Thus, for any fixed $\theta > \frac{1}{\sqrt{2}}$, $\Lambda^0(a)$, does for all $a \neq 0$ overshoot the corresponding branch of the stable manifold of the saddle $(1,0)$ independently of the sign of a. Hence the path condition can not be satisfied this way.
The bigger the value of θ the more unlikely the persistence of the degenerate singular homoclinic solutions becomes. In particular, in the limit as θ goes to infinity the full system tends to the two dimensional kinetic equations, for which the origin has a two dimensional unstable manifold for $a < 0$ and a two dimensional stable manifold for $a > 0$.

5.4 Periodic Solutions

We state our results concerning the existence of periodic solutions in two theorems, for singular *non-degenerate* and *degenerate* periodic solutions.

An existence proof for periodic travelling waves by means of the Conley index is sketched in Smoller [Smo82], Chapter 24. We are not using the Conley index here because it does not give any additional information, and requires a great deal of machinery. On the other hand, most of the work needed to pursue a proof along Carpenter's lines was already done in the homoclinic case.

As in the homoclinic case, first we set out the hypotheses needed for periodic solutions to exist, and then we verify them for the FitzHugh–Nagumo equations. The hypotheses *PER* are:

[1] $\Lambda^\epsilon(a)$ is defined similarly to $\Lambda^\epsilon(\theta)$.

(A) There exist two blocks, B_1 and B_2, such that:

(B) There are no positive semi orbits contained either in B_1 or in B_2.

(C) There exist subsets Δ of $b_2^- \cap D_1^+$ and $\Sigma$of $b_1^- \cap D_2^+$ such that $b_2^- \backslash \Delta$ consists of two components, β_0 and β_1 and $b_1^- \backslash \Sigma$ consists of two components, α_0 and α_1. In addition, if $\delta_i := \beta_i \cap \mathrm{cl}(\Delta)$ and $\xi_i := \alpha_i \cap \mathrm{cl}(\Sigma)$, then $\Phi_1^-(\delta_i) \subseteq \mathrm{int}(\alpha_i)$ and $\Phi_2^-(\xi_i) \subseteq \mathrm{int}(\beta_i)$ for $i = 0, 1$. As before, this is to hold for all values of the parameter in some small interval. (Here, there is no condition that varies as a parameter is varied, as periodic solutions exist for all a whole interval of parameter values.)

(D) There exist homeomorphisms $h_i : b_i^- \to [0,1] \times [-1,2]$ for $i = 1,2$ such that $h_1(\Sigma) = [0,1] \times (0,1)$, $h_1(\xi_i) = [0,1] \times \{i\}$ $(i = 0,1)$ and $h_2(\Delta) = [0,1] \times (0,1)$, $h_2(\delta_i) = [0,1] \times \{i\}$ $(i = 0,1)$.

The above conditions are illustrated in Figure 5.7. The above hypotheses mean that one can set up a return map from Σ through B_2 into itself (conjugated to the homeomorphisms of hypothesis (D)), so that the Lemma D.2 can be invoked to show the existence of a fixed point giving rise to a periodic orbit.

Proposition 5.4 ([Car77], Thm. 1.9) *The above hypotheses imply that $\dot{u} = G(u)$ admits a periodic solution.*

Proof: Recall that $\mathrm{cl}(\Delta)$ is contained is contained in D_1^+ by hypothesis (C) and hence in D_1^- by hypothesis (B). In order to set up the return map we will have to restrict the image of Σ under Φ_2^- to $\mathrm{cl}(\Delta)$, so that the composition with Φ_1^- is defined. We achieve this by means of $\varphi_2 : \mathrm{cl}(\Sigma) \to \mathrm{cl}(\Delta)$, such that

$$\varphi_2(u) = \begin{cases} h_2^{-1}((F_1(u),0)) & \text{if } -1 \le F_2(u) < 0, \\ \Phi_2^-(u) & \text{if } 0 \le F_2(u) \le 1, \\ h_2^{-1}((F_1(u),1)) & \text{if } 1 < F_2(u) \le 2, \end{cases} \tag{5.4}$$

with $F_1(u)$ and $F_2(u)$ denoting the coordinate functions of
$h_2 \circ \Phi_2^- : \mathrm{cl}(\Sigma) \to [0,1] \times [-1,2]$. Note that φ_2 is continuous, because the projections are continuous. Next we define the conjugated return map

$$\varphi := h_1 \circ \Phi_1^- \circ f_2 \circ h_1^{-1} : [0,\ 1] \times [0,\ 1] \to (0,\ 1) \times [-1,\ 2].$$

The conditions on the lower and upper boundaries of Σ and Δ imply that φ satisfies the hypothesis of Lemma D.2 and hence possesses a fixed point. Note

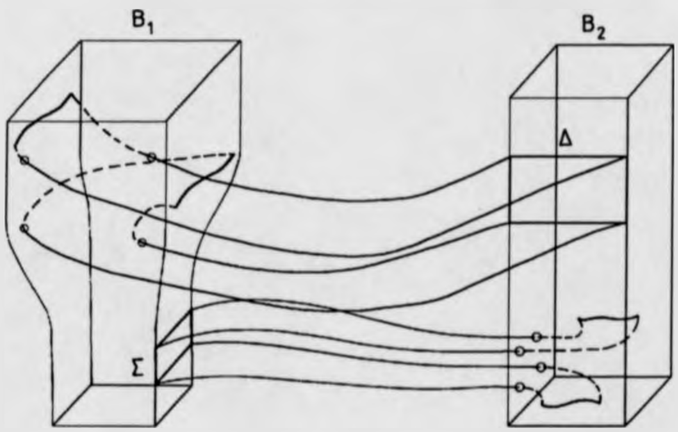

Figure 5.7: Hypotheses for the existence of periodic solutions. b_1^+: front, back, top face of B_1; b_2^-: front, back and top face of B_2

that if $F_2(u) \notin [0,1]$ for some $u \in \Sigma$ then $\Phi_1^- \circ \varphi_2(u) \notin \Sigma$ and so $h_1(u)$ can not be a fixed point of φ. ∎

5.4.1 The Non-degenerate Case

We have changed Carpenter's original construction slightly in that we have chosen the blocks around the stable parts of the slow submanifold symmetrically. Our proof accommodates the periodic solutions in the oscillatory regime as well, that is, the ones which exist for negative values of a.

Now we can formulate the following theorem, which requires that there be only one slow variable.

Theorem 5.5 *Let a, the variable root of the cubic f, be less than $\frac{1}{2}$. Then for each $\theta \in (0, \theta^*(a))$ there exists $\varepsilon_\theta > 0$, so that for all $\varepsilon \in (0, \varepsilon_\theta)$ (FN; θ, ε) admits a periodic solution.*

Proof: Firstly, recall that singular periodic solutions exist for an interval of w- and hence θ-values, for a given fixed a. Secondly, note that the difference between the cases $a < 0$ and $a > 0$ lies in the fact that for $a < 0$, $\underline{w}$ can be taken to be negative (down to $w_{min} = f(u_{min})$). We define

$$w_0 = \begin{cases} 0 & \text{for} \quad a \geq 0, \\ w_{min} & \text{for} \quad a < 0. \end{cases} \tag{5.5}$$

Thus $\underline{w}$ may take values in $(w_0, w_{infl}]$. As in the preceding proof for homoclinic solutions $\bar{\theta} = \theta(\overline{w})$; and $\overline{w}$ is obtained by symmetry, such that there is a homoclinic connection in (FN; $\bar{\theta}, 0$) from $(u_2(\overline{w}), 0)$ to $(u_1(\overline{w}), 0)$.

A: In order to define blocks compatible with the mapping properties of Σ and Δ under the flow, we introduce a subdivision on $\Pi_1 \cap \Pi_2$, such that

$$\left.\begin{array}{ll} \text{for} \quad a \leq 0: & w_{min} = w_0 < w_1 \\ \text{for} \quad a > 0: & w_{min} < w_0 = 0 < w_1 \end{array}\right\} < \underline{w} < w_4 < w_{infl} <$$

$$< w_5 < \overline{w} < w_8 < w_{max},$$

which is symmetric around w_{infl}. That is, $w_{infl} - w_1 = w_8 - w_{infl}$ and $w_{infl} - w_4 = w_5 - w_{infl}$; w_0 acts merely as a dummy variable. In a construction similar to that in the homoclinic case we define two families of blocks $\{B_i^s : s \in (0,1]\}$, which are symmetric to each other with respect to the inflection point of the cubic and set $B_i = B_i^s$ after s has been chosen. Similarly to the previous proof, we set

$$B_i^s(w) = \{(u,v) : |v \pm (\bar{\theta}+1)(u - u_i(w))| \leq (\bar{\theta}+1)c_i^s(w)\} \tag{5.6}$$

and

$$B_i^s = \bigcup_{w \in [w_1, w_8]} B_i^s(w) \times \{w\} \text{ for } i = 1, 2, \tag{5.7}$$

where the diameter $c_i^s(w)$ of B_i^s is a monotonic C^1–function of w for $w \in [w_1, w_8]$. For $i = 1$, we want it to satisfy

$$c_1^s(w) = \begin{cases} sc & \text{for} \quad w \in [w_1, w_4], \\ c & \text{for} \quad w \in [w_5, w_8], \end{cases} \tag{5.8}$$

for some fixed $c > 0$ chosen as in the homoclinic proof and $s \in (0,1]$ to be determined in (C); $c_2^s(w)$ is defined symmetrically.

The blocks B_1^s and B_2^s are shown in Figure 5.8.

Note that for each $s \in (0,1]$ and $w \in [w_1, w_8]$, $B_i^s(w)$ is a block for the fast system (FN; $\theta, 0$) if $c > 0$ is sufficiently small. If, in addition, $\varepsilon > 0$ is small

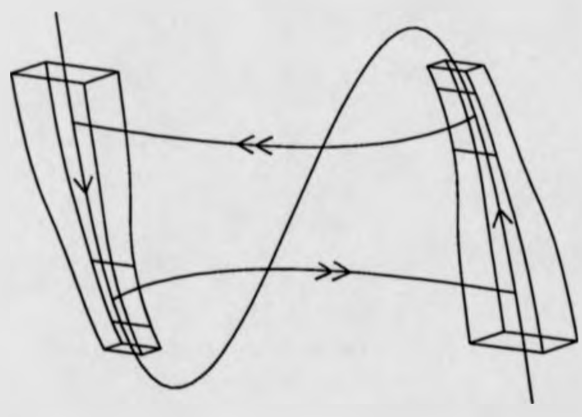

Figure 5.8: Symmetric blocks around S_1 and S_2 in the proof of periodic solutions

enough, then for each s, B_1^s and B_2^s are blocks for the full system (FN; θ, ε). Observe that, the bottom face of B_1 and the top face of B_2 are contained in their respective exit sets.

B: That no positive semi orbit is contained in either B_1 or B_2 follows immediately by an inspection of the slow flow.

C: For $\hat{\beta}(c) > 0$ determined as in the homoclinic proof we set $w_{6/7} = \overline{w} \mp \hat{\beta}$. Then we define $\Delta = \bigcup_{w \in (w_6, w_7)} \Delta(w) \times \{w\}$, where

$$\Delta(w) = \{(u, v) \in b_2^-(w) : -(\theta + 1)c_2^s(w) \leq v \leq 0\}$$

and $s \in (0, 1]$ is small enough, so that Δ is contained in D_1^+. We define Σ symmetric to Δ. That is, for $w_{2/3} = \underline{w} \mp \hat{\beta}$, $\Sigma = \bigcup_{w \in (w_2, w_3)} \Sigma(w) \times \{w\}$, with $\Sigma(w) = \{(u, v) \in b_1^-(w) : 0 \leq v \leq (\theta + 1)c_1^s(w)\}$. Then $\Sigma \subseteq D_2^+$ and $b_1^- \backslash \Sigma$ and $b_2^- \backslash \Delta$ consist each of two components. Furthermore, δ_0 and δ_1, as well as, ξ_0 and ξ_1, leave the blocks B_1 and B_2 through distinct components of their respective exit sets. Also, $\Phi_1^-(\delta_0 \cup \delta_1; \theta, \varepsilon) \subseteq \{w > w_5\}$ and $\Phi_2^-(\xi_0 \cup \xi_1; \theta, \varepsilon) \subseteq \{w < w_4\}$ for ε and $|\theta - \bar{\theta}|$ small enough.

D: The conditions concerning the homeomorphisms are clearly satisfied.

Conclusion: Thus all the hypotheses *PER* are satisfied. Hence, Proposition 5.4 implies the theorem. ■

5.4.2 The Degenerate Case

"Da mußt er mit dem frommen Heer
durch ein Gebirge, wüst und leer.
Daselbst erhob sich große Not,
viel Steine gab's und wenig Brot."
L. Uhland

Our aim in this section is to prove the persistence of degenerate singular periodic solutions, as defined in the previous chapter.
We want to apply the proof of Proposition 5.4 for which we shall define sets $\bar{\bar{B}}_1$, $\bar{\bar{B}}_2$, $\bar{\bar{\Sigma}}$ and $\bar{\bar{\Delta}}$ similar to those of the previous theorem. In the course of the proof it shall become clear that the as yet undefined sets $\bar{\bar{B}}_1$ and $\bar{\bar{B}}_2$ are not blocks in the proper sense of the definition, as tangencies on their boundaries are unavoidable. Thus the program for our proof will be to adapt the standard definition of B_i and $B_i(w)$ for $i = 1, 2$, as given in the previous proofs, such that the mapping conditions on the sets $\bar{\bar{\Sigma}}$ and $\bar{\bar{\Delta}}$ and their respective upper and lower boundaries are satisfied.
We state the persistence result for the degenerate singular periodic solutions in the following theorem, whose proof is similar to that of the non-degenerate case.

Theorem 5.6 *Let a, the variable root of the cubic f, be negative. Then, for each $\theta \geq \theta^*(a)$ there exists $\varepsilon_\theta > 0$, such that for all $\varepsilon \in (0, \varepsilon_\theta)$ $(\mathrm{FN}; \theta, \varepsilon)$ admits a periodic solution.*

Proof: We shall only examine those parts of the proof where the degeneracy affects the argument.
Recall that for the degenerate singular periodic solutions there is no one-to-one correspondence between the w and the θ–values any more, as for $\underline{w} = w_{min}$ and $\overline{w} = w_{max}$ degenerate singular periodic solutions exist for all $\theta \geq \theta^*(a)$ provided that $a < 0$.
To start the actual proof, we define a subdivision on an extension of the set $\overline{\Pi}$, being symmetric around w_{infl},

$$w_1 < w_2 < \underline{w} = w_{min} < w_3 < w_4 < w_{infl} <$$
$$< w_5 < w_6 < w_{crit} < \overline{w} = w_{max} < w_7 < w_8,$$

where

(a) there is no further restriction on the choice of w_1 and w_8, as the bottom face of $\bar{B}_1$ is for all $w_1 < w_{min}$ an exit set. Similarly, is the top face of $\bar{B}_2$ for all $w_8 > w_{max}$ an exit set.

(b) $w_{6,7} = w_{max} \mp \beta$, where $\beta > 0$ will be determined later and

(c) w_{crit} is the w–level at which the node in the fast flow enters the standard block $B_2(w)$. Without loss of generality we may assume that $w_{crit} > w_6$.

Next, we set $\bar{B}_i = \bigcup_{w \in [w_1, w_8]} \bar{B}_i(w) \times \{w\}$, where the sets $\bar{B}_i(w)$ will be defined in the following. Because of the symmetry, it suffices to specify the changes to the set $\bar{B}_2$ containing the set $\bar{\Delta}$.
Here $\bar{\Delta} = \bigcup_{w \in (w_6, w_7)} \bar{\Delta}(w) \times \{w\}$, where $w_{6,7} = w_{max} \mp \beta$ for some β with $0 < \beta < \beta_-$ and β_- as determined in the homoclinic proof.
For $w \in [w_1, w_{crit})$ we set $\bar{B}_2(w) = B_2(w)$, with $B_2(w)$ as defined in (5.6) of the previous proof. Note that $\bar{B}_2(w)$ is then a block for the fast system.
However, for $w \geq w_{crit}$, we need to amend this standard definition in order to satisfy the condition $\bar{\Delta}(w) \subseteq (\bar{D}_1^+)(w)$.

Note that for $w = w_{crit}$ the node $(\bar{u}(w), 0)$ enters $\bar{B}_2(w)$ at its left corner $(u_2(w) - c_2^s(w), 0)$ and therefore the map $\Phi_1^+((u_2(w) - c_2^s(w), 0); \theta, 0)$[2] is not defined. We can, however, continuously extend $\Phi_1^+(\cdot\,; \theta, 0)$ to the node $(\bar{u}(w), 0)$ where we define it to be the intersection of the branch of $W^s(u_1(w), 0)$ starting from the node $(\bar{u}(w), 0)$ with the face $\bar{f}_{1,4;w}^{-1}(0)$.

Analogously to the homoclinic proof, $\bar{B}_2(w)$ is given as

$$\bar{B}_2(w) = \bigcap_{j=1}^{4} \bar{f}_{2,j;w}^{-1}([0, \infty)). \tag{5.9}$$

Our strategy will be to determine $\bar{B}_2(w)$ in terms of $\tilde{K}(w)$, the intersection of the backward fast flow of certain subintervals $\bar{H}(w)$ of $H(w) = f_{1,4;w}^{-1}(0) \cap B_1(w)$ with $K(w) = f_{2,3;w}^{-1}(0) \cap B_2(w)$, at different distinguished w–levels. Note that $\Phi_1^+(\tilde{K}(w); \theta, 0) = \bar{H}(w)$ by definition.

[2]Since $\Phi_1^+(u; \theta, 0) = u \cdot T_1^+(u; \theta, 0)$ and the time map $T_1^+((u_2(w) - c_2^s(w), 0); \theta, 0)$ approaches infinity when for $w \to w_{crit}$ the corner becomes a rest point, the node.

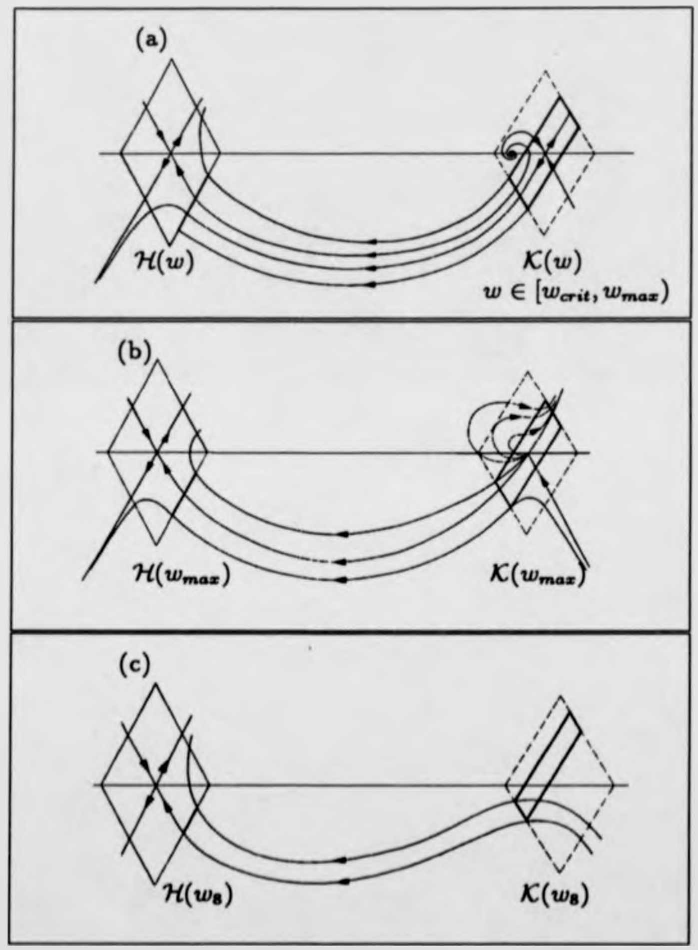

Figure 5.9: Construction of the $\mathcal{K}(w)$ (in bold) at different w-levels

As we require, for $j = 1,3$; $\tilde{f}^{-1}_{2,j;w}(0) := f^{-1}_{2,j;w}(0)$ and for $j = 2,4$; the lines $\tilde{f}^{-1}_{2,j;w}(0)$ to be parallel to the corresponding lines of $B_2(w)$ and to go through the lower and upper endpoints of $\tilde{K}(w)$, respectively, the set $\tilde{B}_2(w)$ is uniquely determined and satisfies by construction the above stated property. We illustrate this approach in the following sequence of figures.

Fig. 5.9 (a): Consider $w \in [w_{crit}, w_{max})$, for which the node is inside the standard set $B_2(w)$. Here, we choose a subinterval $\tilde{H}(w)$ of $H(w)$ around the intersection point of $W^s(u_1(w), 0)$ with $H(w)$, which contains this intersection point as an inner point and is small enough for its backward fast image $\tilde{K}(w)$ to be contained in $K(w)$.

Fig. 5.9 (b): At the saddle node, for $w = w_{max}$, we choose $\tilde{H}(w)$ similar to the last case, except that the distance between the lower end point of the subinterval and the intersection point with $W^s(u_1(w_{max}), 0)$ is decreased. Again, $\tilde{H}(w)$ must be chosen small enough for $\tilde{K}(w)$ to be contained in $K(w)$.

Fig. 5.9 (c): Finally, for the upper boundary of $\tilde{\Delta}$, at $w_7 = \text{proj}_v\ \tilde{\delta}_1$, we choose the subinterval of H_{w_8} such that its lower interval endpoint equals the intersection point with $W^s(u_1(w_8), 0)$ and the upper end point taken as in the last instance. We set $u_2(w) = u_2(w_{max})$ for all $w \in (w_{max}, w_8]$ then this construction does so also apply to all $w \in (w_7, w_8]$. Note, in particular, we have at $w_7 = \text{proj}_v(\tilde{\delta}_1)$ that $\Phi_1^-(\tilde{\delta}_1) \subseteq \{v > 0\}$ as required.

As the construction of the $\tilde{K}(w)$ can be made smooth in w, we require $\tilde{K}(w)$ to be a C^1-smooth function of w on (w_{crit}, w_8).
Finally, we investigate the tangencies of $\tilde{B}_1$ and $\tilde{B}_2$. The tangencies, which can easily be characterized in the fast flow, must occur on the face $\tilde{f}^{-1}_{2,2;w}(0) \cap \tilde{B}_2(w)$ for some interval starting at the w-level at which the node enters the set $\tilde{B}_2(w)$ and terminating at w_{max}. In particular, they do not occur for any w on the faces $\tilde{K}_v = f^{-1}_{2,3;w}(0) \cap \tilde{B}_2(w)$, which make up the set $\tilde{\Delta}$. Hence, the maps $\Phi_1^{\pm}$ and $\Phi_2^{\pm}$ on Σ and $\tilde{\Delta}$, respectively, are continuous and the return map of Proposition 5.4 is well defined and continuous. ∎

Appendix A

Brief Review of Floquet Theory

Let $G: W \to \mathbb{R}^n$ be a C^1 vector field on an open set $W \subset \mathbb{R}^n$; and let the flow of the differential equation

$$\dot{u} = G(u) \tag{A.1}$$

be denoted by ϕ_t.

Let γ be a non-trivial closed orbit of the flow. We call γ *orbitally asymptotically stable* if for every open set $U_1 \subset W$ with $\gamma \subset U_1$ there is an open set U_2, with $\gamma \subset U_2 \subset U_1$, such that $\phi_t(U_2) \subset U_1$ for all $t > 0$ and

$$\lim_{t \to \infty} d(\phi_t(u), \gamma) = 0$$

for all $u \in U_2$. Here $d(x, \gamma)$ stands for the minimal distance from x to a point on γ.

The following theorem is reminiscent of the eigenvalue characterization of an asymptotically stable rest point. However, it is not as convenient to use since it requires information about the solutions of (A.1) and not merely about the vector field. Nevertheless it is of great theoretical importance for stability questions.

Theorem A.1 *Let γ be a closed orbit of* (A.1) *of period T and $p \in \gamma$. Suppose that $n-1$ eigenvalues of the map $D\phi_T(p) \in \mathcal{L}(\mathbb{R}^n)$ are in the interior of the unit disc in the complex plane. Then γ is orbitally asymptotically stable.*

Several remarks on this theorem are in order. Firstly, 1 is always an eigenvalue of $D\phi_T(p)$, since

$$D\phi_T(p)G(p) = G(p).$$

Secondly, the eigenvalues of $D\phi_T(p)$ do not depend on the particular point p on the closed orbit γ. For if, $q \in \gamma$ is a different point, then it can be easily seen, by an application of the chain rule, that $D\phi_T(q)$ is similar to $D\phi_T(p)$. Next, we put this in relation to Floquet theory. Differentiating $\frac{\partial}{\partial t}\phi_t(p) = G(\phi_t(p))$ with respect to the phase space variable and interchanging the order of differentiation we obtain,

$$\frac{\partial}{\partial t} D\phi_t(p) = DG(\phi_t(p))D\phi_t(p) \tag{A.2}$$

and $D\phi_0(p) = I$. Thus, $D\phi_t(p)$ is a fundamental matrix solution of the linearization of (A.1) about the T-periodic solution $\phi_t(p)$, i.e.,

$$\dot{\xi} = DG(\phi_t(p))\xi \tag{A.3}$$

and $D\phi_T(p)$ is the *monodromy operator*, i.e. the time-T-map of the solution of the T-periodic linear *variational* equation (A.3). The eigenvalues λ of the monodromy operator $D\phi_T(p)$ are called *Floquet multipliers* of (A.3) and any ρ such that $\lambda = \exp\{\rho T\}$ is called a *Floquet exponent.* Note that the Floquet exponents are not uniquely defined, but the multipliers are. Since all the Floquet multipliers of (A.3) are independent of $p \in \gamma$, we can talk about the *Floquet multipliers of the closed orbit* γ. We say that 1 is a *simple* Floquet multiplier of γ if it is an algebraically simple eigenvalue of the monodromy operator of (A.3). Theorem A.1 is usually stated in the terms of Floquet theory. For an exposition on this subject we refer to the book of Hale [Hal80].

Finally, we remark that the nontrivial Floquet exponent for closed orbits of planar vector fields can be expressed in terms of some curve integral.

Proposition A.2 *Let γ be a nontrivial closed orbit of period T through $p \in \gamma$ of the planar differential equation $\dot{u} = G(u)$, where $G \in C^1$. Then the periodic orbit γ is (orbitally asymptotically) stable if and only if the nontrivial Floquet exponent*

$$\oint_\gamma \operatorname{div} G < 0 \tag{A.4}$$

and unstable for the reversed inequality.

Proof: Recall Liouville's formula [Hal80], p. 82, Lemma 1.5, in which the monodromy operator $D\phi_T(p)$ with $D\phi_0(p) = I$ of the T-periodic linear system (A.3) satisfies the relation

$$\det\ \phi_T(p) = \exp\left\{ \int_0^T \operatorname{tr}\ DG(\phi_t(p))\,dt \right\}. \tag{A.5}$$

As 1 is always a Floquet multiplier the other one must equal the right hand side of equation (A.5), since the determinant is the product of the Floquet multipliers. Stability requires that the multiplier different from 1 must be contained in the interior of the unit disc by Theorem A.1, this is equivalent to $\oint_\gamma \operatorname{div} G < 0$, since $\operatorname{tr} DG = \operatorname{div} G$ and by a property of the exponential function. ∎

Thus in order to determine the stability of a limit cycle in the plane we need to show that the curve integral of the divergence of the vector field along the closed orbit is negative.

Appendix B

Cubic Equations

Consider the cubic equation

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0,$$

with real coefficients a_0, a_1 and a_2. Set

$$\begin{aligned} q &= \tfrac{1}{3}a_1 - \tfrac{1}{9}a_2^2 \\ r &= \tfrac{1}{6}(a_1 a_2 - 3a_0) - \tfrac{1}{27}a_2^3 \\ D &= q^3 + r^2 \\ &= \tfrac{1}{4}a_0^2 + \tfrac{1}{27}a_1^3 - \tfrac{1}{6}a_0 a_1 a_2 - \tfrac{1}{108}a_1^2 a_2^2 + \tfrac{1}{27}a_0 a_2^3. \end{aligned}$$

Then, if

$D > 0$ there is one real and a pair of complex conjugate roots;
$D = 0$ there are three real roots of which two are equal;
$D < 0$ there are three distinct real roots.

Moreover if

$$\begin{aligned} s_1 &= \left[r + \sqrt{D}\right]^{\frac{1}{3}} \\ s_2 &= \left[r - \sqrt{D}\right]^{\frac{1}{3}} \end{aligned}$$

then the roots are given by

$$\begin{aligned} z_1 &= s_1 + s_2 - \tfrac{1}{3}a_2 \\ z_2 &= -\tfrac{1}{2}(s_1 + s_2) - \tfrac{1}{3}a_2 + \tfrac{i\sqrt{3}}{2}(s_1 - s_2) \\ z_3 &= -\tfrac{1}{2}(s_1 + s_2) - \tfrac{1}{3}a_2 - \tfrac{i\sqrt{3}}{2}(s_1 - s_2). \end{aligned}$$

Note that

$$\begin{aligned} s_1^3 + s_2^3 &= 2r \\ s_1\, s_2 &= -q \end{aligned}$$

and so $(s_1 + s_2)^3 = 2r - 3q(s_1 + s_2)$.

Appendix C

Cusp Calculation

We show that the function

$$p(a,\theta) := (a^2 - 4\varepsilon)\theta^4 + 2(2a^3 - 9\varepsilon)\theta^2 - 27\varepsilon^2, \tag{C.1}$$

which is obtained by multiplying (2.3) through with the term $a^2 - 4\varepsilon$, has for fixed $\varepsilon > 0$ at $z_0 := (-\ell, \sqrt{3\ell})$ a *cusp point*[1], where $\ell := \sqrt{3\varepsilon}$.
We show below that z_0 is a degenerate critical point of p, i.e. $p(z_0) = 0$, $Dp(z_0) = (0,0)$ and $D^2p(z_0)$, the Hessian of p at z_0, has each a zero and a non-zero eigenvalue; additionally we show that the third derivative of p at z_0 satisfies a nondegeneracy condition, which we will state later. Under these conditions it is a straightforward exercise in singularity theory to prove that p around z_0 is equivalent to

$$u^3 + v^2$$

after a smooth change of coordinates $(a,\theta) \mapsto (u,v)$.
We shall, however, make use of this result only later. Meanwhile, assuming the above conditions, we may write p as

$$p(z) = Q(z,z) + C(z,z,z) + h.o.t., \tag{C.2}$$

where the quadratic form Q is given by

$$Q(z,z) := \langle A(z - z_0), z - z_0\rangle$$

with $A = \frac{1}{2}D^2p(z_0)$ and the cubic form by

$$C(z,z,z) := \frac{1}{6}D^3p(z_0)(z - z_0, z - z_0, z - z_0).$$

[1] More precisely, the zero set of p at z_0 is locally a cusp.

We introduce coordinates (x, y) according to $\mathbb{R}^2 \ni z - z_0 = x\phi + y\psi$, where $\phi \in \operatorname{Ker} A$, $\phi \neq 0$ and ψ is an eigenvector to the nontrivial eigenvalue of A, say $A\psi = \lambda\psi$ for some $\lambda \neq 0$. Then the quadratic term satisfies

$$Q(z, z) = \kappa y^2, \quad \text{where} \quad \kappa := \lambda||\psi||^2. \tag{C.3}$$

Exploiting the multilinearity of C we write (C.2) as

$$\kappa y^2 + \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 + \ldots,$$

where $\alpha = C(\phi, \phi, \phi)$, $\beta = 3C(\phi, \phi, \psi)$ etc. and subsequently as

$$\kappa y^2(1 + \gamma' x + \delta' y)^2 + \alpha(x + \beta' y)^3 + \ldots,$$

with $\beta' = \frac{\beta}{3\alpha}$, $\gamma' = \frac{1}{2\kappa}(\gamma - \frac{\beta^2}{3\alpha})$ and $\delta' = \frac{1}{2\kappa}(\delta - \frac{\beta^3}{27\alpha^2})$.
With respect to the (nonlinear) coordinate change

$$\begin{aligned} \tilde{x} &= x + \beta' y \\ \tilde{y} &= y(1 + \tfrac{\gamma'}{2}x + \tfrac{\delta'}{2}y) \end{aligned} \tag{C.4}$$

the Taylor series of p around z_0 is then given by

$$\kappa y^2 + \alpha x^3 + \text{terms of degree at least } 4. \tag{C.5}$$

Having carried out these calculation we now appeal to the previously stated result in singularity theory which tells us that if α and κ are non-zero, then the terms of degree less than or equal to three are 3-determined, so there is a change of coordinates that transforms away the higher order terms.
Thus for (C.1) at z_0 to be a cusp point it is sufficient to show that both α and κ are non-zero. We have

$$p(z_0) = 0 \quad \text{and} \quad \frac{\partial p}{\partial a}(z_0) = 0 = \frac{\partial p}{\partial \theta}(z_0).$$

Furthermore, the Hessian of p at z_0 is given by

$$-6\varepsilon \begin{pmatrix} 27 & 6\sqrt{3\ell} \\ 6\sqrt{3\ell} & 4\ell \end{pmatrix}. \tag{C.6}$$

Note that the determinant of (C.6) is zero and therefore 0 is an eigenvalue, the other nontrivial eigenvalue is given by the trace, $\lambda = -6\varepsilon(27 + 4\ell)$. The eigenvector corresponding to the trivial eigenvalue is $\phi = (2\ell, -3\sqrt{3\ell})^T$ and the one corresponding to λ is given by $\psi = (9, 2\sqrt{3\ell})^T$. We can now compute κ to be

$$\kappa = -6\varepsilon(27 + 4\ell)\sqrt{81 + 12\ell}.$$

Thus it only remains to be shown that $\alpha = D^3p(z_0)(\phi, \phi, \phi)$ is non-zero. In order to do this we have to compute the third derivative of p at z_0 which is completely determined by the following terms:

$$\frac{\partial^3 p}{\partial a^3}(z_0) = 72\ell, \frac{\partial^3 p}{\partial a^2\theta}(z_0) = -24\ell\sqrt{3\ell}, \frac{\partial^3 p}{\partial a\theta^2}(z_0) = -180\varepsilon, \frac{\partial^3 p}{\partial \theta^3}(z_0) = -24\varepsilon\sqrt{3\ell}.$$

Finally, a computation shows that

$$\alpha = -96 \cdot 64\,\varepsilon^2,$$

which settles the argument.

Appendix D

Topological Techniques

D.1 Wazewski's principle

Suppose that B is a block. Then the reason why these concepts are of interest is the following. $D^+ \backslash (D^- \cup B)$ is the set of points from outside B that enter it but do not ever leave it. We concentrate on assumptions that force this set to be nonempty. From Lemma 5.1, we derive the weak form of the Wazewski Principle, which in our notation reads:

Corollary D.1 *Let $\Sigma \subset D^+$ be a set, such that trajectories intersect it only once. If $\Sigma \subset D^-$, then Σ is homeomorphic to $\Phi^-(\Sigma)$.*

Compare Figure D.1. The bijectivity of $\Phi^-(\Sigma)$ follows from the global existence of the trajectories and the well known fact that two trajectories to (5.1) can not cross. The continuity of its inverse is easily established by considering the time reversed map.

Compare the above statement with the one in [Dun81]. We refer the interested reader to [Con76] for a formulation of the Wazewski Principle in its full power, relating it to homotopy theory.

Provided there are trajectories, being asymptotic to a rest point in B, we can apply the principle of Wazewski to give a non-constructive proof of homoclinic solutions. Let us assume that the unstable manifold of the rest point intersects Σ. Then the restatement as an existence proof is as follows: Suppose $\Sigma \subset D^+$ is connected, but $\Phi^-(\Sigma)$ is not, so that they are not homeomorphic. This means

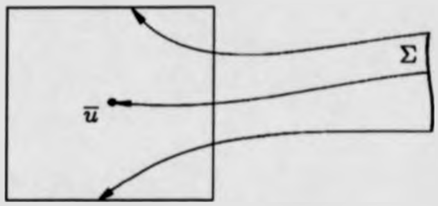

Figure D.1: Existence of an homoclinic orbit

that $\Sigma \backslash D^-$ is not empty, that is, there exist trajectories which stay in B for all positive time.

D.2 Brouwer degree

We need different tools to find periodic solutions. This is because we can not use invariant manifolds of rest points. The method we shall employ relies on a fixed point theorem. Equivalently, this means that the Brouwer degree of some mapping will be non-zero. Suppose that Ψ is an open bounded set in $\mathbb{R}^k$. Let $F \in C^1(\Psi, \mathbb{R}^k) \cap C^0(\mathrm{cl}(\Psi), \mathbb{R}^k)$ and let y be a regular value[1] of F with $y \notin F(\partial\Psi)$. Then the degree of a point y in $\mathbb{R}^k$ relative to Ψ, denoted by $\deg(F, \Psi, y)$, is given by

$$\deg(F, \Psi, y) = \sum_{x \in F^{-1}(y)} \operatorname{sgn} \det DF(x). \tag{D.1}$$

Note that since $\mathrm{cl}(\Psi)$ is compact and since y is a regular value of F the sum has by the Inverse Function Theorem at most finitely many terms. From this definition the concept of degree is extended to singular values by a prominent theorem of differential topology, Sard's lemma, and to continuous functions by a density argument. Roughly speaking, the degree is a measure for the number of zeros of F in $\mathrm{cl}(\Psi)$. The book of Amann [Ama90] is a good reference on the subject.

The Brouwer degree does have the following crucial properties: Consider continuous mappings F, $G : \mathrm{cl}(\Psi) \to \mathbb{R}^k$, then:

[1]This means that the Jacobian of F is nonsingular on the set $F^{-1}(y) \subset \Psi$.

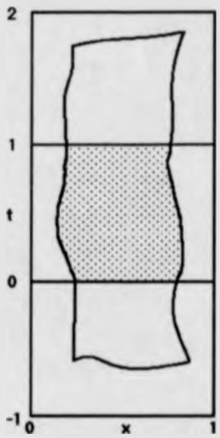

Figure D.2: Existence of a fixed point in the shaded region

(i) (Dependence on boundary values only): If $F|_{\partial\Psi} = G|_{\partial\Psi}$ and $y \notin F(\partial\Psi) = G(\partial\Psi)$, then $\deg(F,\ \Psi,\ y) = \deg(G,\ \Psi,\ y)$.

(ii) (Solution property): If $\deg(F,\ \Psi,\ y) \neq 0$, then $F(\Psi)$ is a neighbourhood of y in $\mathbb{R}^k$.

(iii) (Homotopy invariance): If $\{F_s\}_{s\in[0,1]}$ is a continuous family of mappings such that $F_s(x) \neq x$ for all $x \in \partial\Psi$ and $s \in [0,1]$ then $\deg(F_s - I,\ \Psi,\ 0)$ is independent of $s \in [0,1]$. In particular, we have that

$$\deg(F_0 - I,\ \Psi,\ 0) = \deg(F_1 - I,\ \Psi,\ 0).$$

The properties of the Brouwer degree allow us to prove the following lemma which will be needed later.

Lemma D.2 ([Car77], lemma p.359) *Let* $\varphi : [0,\ 1] \times [0,\ 1] \to (0,\ 1) \times [-1,\ 2]$ *be a continuous map such that*

$$\varphi([0,\ 1] \times \{0\}) \subseteq (0,\ 1) \times [-1,\ 0)$$

and

$$\varphi([0,\ 1] \times \{1\}) \subseteq (0,\ 1) \times (1,\ 2].$$

Then φ has a fixed point, that is, there exist $(\bar{x}, \bar{t})$, such that $\varphi(\bar{x}, \bar{t}) = (\bar{x}, \bar{t})$.

Proof: Take $\Psi = (0, 1) \times (0, 1)$. Set $\varphi_0 \equiv \varphi$ and define $\varphi_1(x,t) = (\frac{1}{2}, 2t - \frac{1}{2})$ on Ψ. Then it is straightforward to check that φ_0 is fixed–point homotopic to φ_1 by $\varphi_s \equiv (1-s)\varphi_0 + s\varphi_1$, since $\varphi_s(u) \neq u$ on $\partial\Psi$ for all s, i.e. no fixed points of φ_s leave through the boundary of Ψ. Furthermore, $(\frac{1}{2}, \frac{1}{2})$ is a fixed-point of φ_1 and from (D.1) we can explicitly compute $\deg(\varphi_1 - I,\ \Psi,\ 0) = -1$. The fixed–point homotopy invariance of the degree implies that $\deg(\varphi_0 - I,\ \Psi,\ 0) = -1$, and therefore $\varphi_0 = \varphi$ has a fixed point, by the solution property of the degree. Refer to Figure D.2.∎

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