Reversible Occurrence Nets and Causal Reversible Prime Event Structures[*]

Hernán Melgratti[1][**], Claudio Antares Mezzina[2][***], Iain Phillips[3],
G. Michele Pinna[4][†], and Irek Ulidowski[5]

[1] ICC - Universidad de Buenos Aires - Conicet, Argentina
[2] Dipartimento di Scienze Pure e Applicate, Università di Urbino, Italy
[3] Imperial College London, England
[4] Università di Cagliari, Italy
[5] University of Leicester, England

Abstract. One of the well-known results in concurrency theory concerns the relationship between event structures and occurrence nets: an occurrence net can be associated with a prime event structure, and vice versa. More generally, the relationships between various forms of event structures and suitable forms of nets have been long established. Good examples are the close relationship between inhibitor event structures and inhibitor occurrence nets, or between asymmetric event structures and asymmetric occurrence nets. Several forms of event structures suited for the modelling of reversible computation have recently been developed; also a method for reversing occurrence nets has been proposed. This paper bridges the gap between reversible event structures and reversible nets. We introduce the notion of reversible occurrence net, which is a generalisation of the notion of reversible unfolding. We show that reversible occurrence nets correspond precisely to a subclass of reversible prime event structures, the causal reversible prime event structures.

Keywords: Event Structures · Causality · Reversibility · Petri Nets

1 Introduction

Event structures and nets are closely related. Since the seminal papers by Nielsen, Plotkin and Winskel [21] and Winskel [29], the relationship among nets and event structures has been considered as a pivotal characteristic of concurrent systems. The ingredients of an event structure are a set of events and a number of relations that are used to express which events can be part of a configuration (the

snapshot of a concurrent system), modelling a consistency predicate, and how events can be added to reach another configuration, modelling the dependencies among the events. On the net side, the ingredients boil down to constraints on how transitions may be executed, and usually have a structural flavour.

Since the introduction of event structures there has been a flourish of investigations into the possible relations among events, giving rise to a number of different definitions of event structures. We first mention the classical *prime* event structures [29] where the dependency between events, called *causality*, is given by a partial order and the consistency is determined by a *conflict* relation. *Flow* event structures [6] drop the requirement that the dependency should be a partial order, and *bundle* event structures [18] are able to represent OR-causality by allowing each event to be caused by a member of a bundle of events. *Asymmetric* event structures [3] introduce the notion of weak causality that can model asymmetric conflicts. *Inhibitor* event structures [2] are able to faithfully capture the dependencies among events which arise in the presence of read and inhibitor arcs. In [4] event structures, where the causality relation may be circular, are investigated, and in [1] the notion of dynamic causality is considered. Finally, we mention the quite general approach presented in [27], where there is a unique relation, akin to a *deduction relation*. To each of the aforementioned event structures a particular class of nets corresponds. To prime event structures we have *occurrence* nets, to flow event structures we have *flow* nets, to bundle event structures we have *unravel* nets [7], to asymmetric and inhibitor event structures we have *contextual* nets [3,2], to event structures with circular causality we have *lending* nets [4], to those with dynamic causality we have *inhibitor unravel* nets [8] and finally to the ones presented in [27] *1-occurrence* nets are associated.

Recently a new type of event structure tailored to model *reversible* computation has been proposed [24,26]. In particular, in [24], *reversible* prime event structures have been introduced. In this kind of event structure two relations are added: the *reverse causality* relation and the *prevention relation*. The first one is a standard dependency relation: in order to reverse an event some other events must be present. The second relation, on the contrary, identifies those events whose presence *prevents* the event being reversed. This kind of event structure is able to model both causal-consistent reversibility [9,23,15] and out-of-causal-order reversibility [25,13]. Causal-consistent reversibility relates reversibility with causality: an event can be undone provided that all of its effects have been undone first. This allows the system to get back to a past state, which was possible to reach by just the normal (forward) computation. This notion of reversibility is natural in reliable distributed systems since when an error occurs the system tries to go back to a past consistent state. Examples of application of causal-consistent reversibility to model reliable systems include transactions [10,14] and rollback protocols [28]. Also, there are applications in program analysis and debugging [12,17]. The out-of-causal-order reversibility does not preserve causes, and this feature makes it suitable to model biochemical reactions where, for ex-

ample, a bond can be undone 'out-of-order' thus leading to a new state that was not present before.

Reversibility in Petri nets has been studied in [22,19] with two different approaches. In [22] reversibility in an acyclic Petri net is obtained by adding a new kind of tokens, called bonds, that keep track of the execution history. Bonds are rich enough to permit modelling of both the causal-consistent and out-of-causal order reversibility. In [19] a notion of *unfolding* of a P/T (place/transition) net, where all the transitions can be reversed, has been proposed. By resorting to standard notions of the Petri net theory [19] provides a causal-consistent reversible semantics for P/T nets. This exploits the well-known unfolding of P/T nets into occurrence nets [29], and is done by adding for each transition its reversible counterpart. We also note that a problem of making a Petri net reversible (meaning every computation is able to reach back to the initial state) has been solved by showing how to add a minimal number of additional transitions [5].

In this paper we study what kind of nets can be associated with reversible prime event structures. To this aim we introduce *reversible occurrence* nets, which are occurrence nets enriched with additional transitions (called *reversing* transitions) that undo the effects of executing the ordinary transitions.

Each reversing transition (event in the occurrence net dialect) is associated with a unique transition that produces the effects that the reversing transition undoes. A reversing event associated with an event e can be executed in a reversible occurrence net only when all the events caused by e have been previously reversed. If this is not possible then the reversing event cannot be executed. This means that some events in a reversible event structure may *prevent* the occurrence of a reversing event. A reversible occurrence net where the reversing events have been removed is just an occurrence net. This discussion suggests a natural way of relating reversible occurrence nets and reversible prime event structures: the causality relation is the one induced by the occurrence net while the prevention relation is induced by the inverse of causality: a reversing event associated with e is prevented by any event that causally depends on e. In this way we associate a reversible occurrence net with a *causal* reversible prime event structure [24,26], which is a subclass of reversible prime event structures.

We also show how to obtain a reversible occurrence net from a causal reversible prime event structure. The ingredients that are used are just the causality relation and the set of reversible events. We prove that the two formalisms have the same configurations. Hence, this gives us the precise correspondence between causal reversible prime event structures and reversible occurrence nets. We do not consider non-causal reversible prime event structures here; however, we hint at how this can be done in Section 5.

Structure of the paper. Section 2 reviews some preliminary notions for nets and event structures, including reversible prime event structures. Section 3 recalls the well-known relationship between prime event structures and occurrence nets. The core of the paper is Section 4 where we first introduce reversible occurrence nets and then we show how to obtain a reversible occurrence net from an occurrence

net. We then show how to associate a causal reversible prime event structure with a reversible occurrence net, and vice versa. Section 5 concludes the paper.

2   Preliminaries

We denote with $\mathbb{N}$ the set of natural numbers. Let A be a set, a *multiset* of A is a function $m : A \rightarrow \mathbb{N}$. The set of multisets of A is denoted by μA. We assume the usual operations on multisets such as union $+$ and difference $-$, and $k \cdot m$ stands for the scalar multiplication of m by k, i.e., $(k \cdot m)(a) = k \cdot m(a)$ for all $a \in A$. We write $m \subseteq m'$ if $m(a) \leq m'(a)$ for all $a \in A$. For $m \in \mu A$, we denote with $[\![m]\!]$ the multiset defined as $[\![m]\!](a) = 1$ if $m(a) > 0$ and $[\![m]\!](a) = 0$ otherwise. When a multiset m of A is a set, i.e. $m = [\![m]\!]$, we write $a \in m$ to denote that $m(a) \neq 0$, and often confuse the multiset m with the set $\{a \in A \mid m(a) \neq 0\}$. Furthermore we use the standard set operations like $\cap, \cup$ or $\setminus$. Given a set A and a relation $< \subseteq A \times A$, we say that $<$ is an *irreflexive partial order* whenever it is irreflexive and transitive. We shall write $\leq$ for the reflexive closure of a partial order $<$.

2.1   Petri nets

We review the notion of Petri net along with some auxiliary notions.

Definition 1. *A* Petri net *is a 4-tuple* $N = \langle S, T, F, \mathsf{m} \rangle$, *where S is a set of* places *and T is a set of* transitions *(with $S \cap T = \emptyset$), $F \subseteq (S \times T) \cup (T \times S)$ is the* flow *relation, and $\mathsf{m} \in \mu S$ is called the* initial marking.

Given a net $N = \langle S, T, F, \mathsf{m} \rangle$ and $x \in S \cup T$, we define the following sets: ${}^\bullet x = \{y \mid (y, x) \in F\}$ and $x^\bullet = \{y \mid (x, y) \in F\}$, which can be viewed as multisets. If $x \in S$ then ${}^\bullet x \in \mu T$ and $x^\bullet \in \mu T$; analogously, if $x \in T$ then ${}^\bullet x \in \mu S$ and $x^\bullet \in \mu S$. A multiset of transitions $A \in \mu T$, called *step*, is enabled at a marking $m \in \mu S$, denoted by $m \, [A\rangle$, whenever ${}^\bullet A \subseteq m$, where ${}^\bullet A = \sum_{x \in [\![A]\!]} A(x) \cdot {}^\bullet x$. A step A enabled at a marking m can *fire* and its firing produces the marking $m' = m - {}^\bullet A + A^\bullet$, where $A^\bullet = \sum_{x \in [\![A]\!]} A(x) \cdot x^\bullet$. The firing of A at a marking m is denoted by $m \, [A\rangle \, m'$. We assume that each transition t of a net N is such that ${}^\bullet t \neq \emptyset$, meaning that no transition may fire *spontaneously*. Given a generic marking m (not necessarily the initial one), a (step) *firing sequence* (shortened as fs) of $N = \langle S, T, F, \mathsf{m} \rangle$ starting at m is defined as: (i) m is a firing sequence (of length 0), and (ii) if $m \, [A_1\rangle \, m_1 \, \cdots \, m_{n-1} \, [A_n\rangle \, m_n$ is a firing sequence and $m_n \, [A\rangle \, m'$, then also $m \, [A_1\rangle \, m_1 \, \cdots \, m_{n-1} \, [A_n\rangle \, m_n \, [A\rangle \, m'$ is a firing sequence. Let us note that each step A such that $|A| = n$ can be written as $A_1 + \cdots + A_n$ where for each $1 \leq i \leq n$ it holds that $A_i = [\![A_i]\!]$ and $|A_i| = 1$, and $m \, [A\rangle \, m'$ then, for each decomposition of A in $A_1 + \cdots + A_n$, we have that $m \, [A_1\rangle \, m_1 \ldots m_{n-1} \, [A_n\rangle \, m_n = m'$. When A is a singleton, i.e. $A = \{t\}$, we write $m \, [t\rangle \, m'$. The set of firing sequences of a net N starting at a marking m is denoted by $\mathcal{R}_m^N$ and it is ranged over by σ. Given an fs $\sigma = m \, [A_1\rangle \, \sigma' \, [A_n\rangle \, m_n$, we denote with $start(\sigma)$ the marking m, with $lead(\sigma)$ the marking m_n and

with $tail(\sigma)$ the fs $\sigma'\,[A_n\rangle\,m_n$. $tail(\sigma)$ is defined only when σ has length greater than 0. Given a net $N = \langle S, T, F, \mathsf{m}\rangle$, a marking m is *reachable* iff there exists an fs $\sigma \in \mathcal{R}_\mathsf{m}^N$ such that $lead(\sigma)$ is m. The set of reachable markings of N is $\mathcal{M}_N = \{lead(\sigma) \mid \sigma \in \mathcal{R}_\mathsf{m}^N\}$. Given an fs $\sigma = m\,[A_1\rangle\,m_1 \cdots m_{n-1}\,[A_n\rangle\,m'$, we write $X_\sigma = \sum_{i=1}^n A_i$ for the multiset of transitions associated to fs. We call X_σ a *state* of the net and write $\mathsf{St}(N) = \{X_\sigma \in \mu T \mid \sigma \in \mathcal{R}_\mathsf{m}^N\}$ for the set of states of N.

Definition 2. *A net $N = \langle S, T, F, \mathsf{m}\rangle$ is said to be* safe *if each marking $m \in \mathcal{M}_N$ is such that $m = [\![m]\!]$.*

In this paper we consider safe nets $N = \langle S, T, F, \mathsf{m}\rangle$ where each transition can be fired, i.e. $\forall t \in T \; \exists m \in \mathcal{M}_N.\; m\,[t\rangle$, and every place is reachable (i.e., marked in at least one reachable marking).

2.2   Prime event structures

We now recall the notion of prime event structure [29].

Definition 3. *A* prime event structure *(PES) is a triple $P = (E, <, \#)$, where*

- *E is a countable set of* events,
- *$< \subseteq E \times E$ is an irreflexive partial order called the* causality relation, *such that $\forall e \in E.\ \{e' \in E \mid e' < e\}$ is finite, and*
- *$\# \subseteq E \times E$ is the* conflict relation, *which is irreflexive, symmetric and* hereditary *with respect to $<$: if $e \mathop{\#} e' < e''$, then $e \mathop{\#} e''$ for all $e, e', e'' \in E$.*

Intuitively, $e < e'$ models that e' can occur only after e, while $e \mathop{\#} e'$ indicates that e and e' are mutually exclusive. Given an event $e \in E$, $\lfloor e \rfloor$ denotes the set $\{e' \in E \mid e' \le e\}$. A set of events $X \subseteq E$ is left-closed if $\forall e \in X.\lfloor e \rfloor \subseteq X$. Given a set $X \subseteq E$ of events, we say that X is *conflict free*, written $\mathsf{CF}(X)$, iff for all $e, e' \in X$ it holds that $e \ne e' \;\Rightarrow\; \neg(e \mathop{\#} e')$. Given $X \subseteq E$ such that $\mathsf{CF}(X)$ and $Y \subseteq X$, then also $\mathsf{CF}(Y)$. When adding reversibility to PESes, conflict heredity may not hold. Therefore, we rely on a weaker form of PES by following the approach in [24].

Definition 4. *A pre-PES (pPES) is a triple $P = (E, <, \#)$, where*

- *E is a set of* events,
- *$\# \subseteq E \times E$ is an irreflexive and symmetric relation,*
- *$< \subseteq E \times E$ is an irreflexive partial order such that for every $e \in E.\ \{e' \in E \mid e' < e\}$ is finite and conflict free, and*
- *$\forall e, e' \in E.$ if $e < e'$ then not $e \mathop{\#} e'$.*

A pPES is a prime event structure in which conflict heredity does not hold, and since every PES is also a pPES the notions and results stated below for pPESes also apply to PESes.

Definition 5. *Let $P = (E, <, \#)$ be a pPES and $X \subseteq E$ such that $\mathsf{CF}(X)$. For $A \subseteq E$, we say that A is* enabled *at X if $A \cap X = \emptyset$ and $\mathsf{CF}(X \cup A)$, and $\forall e \in A$. if $e' < e$ then $e' \in X$. If A is enabled at X, then X can* reach $Y = X \cup A$, *and is written as $X \xrightarrow{A} Y$.*

Definition 6. *Let $P = (E, <, \#)$ be a pPES and $X \subseteq E$ such that $\mathsf{CF}(X)$. X is a* forwards reachable configuration *if there exists a sequence $A_1, \ldots, A_n$, such that $X_i \xrightarrow{A_i} X_{i+1}$ for all i, and $X_1 = \emptyset$ and $X_{n+1} = X$. We write $\mathsf{Conf}_{pPES}(P)$ for the set of all (forwards reachable) configurations of P.*

When a pPES is a PES we shall write $\mathsf{Conf}_{PES}(P)$ instead of $\mathsf{Conf}_{pPES}(P)$, with $\mathsf{Conf}_{PES}(P) = \mathsf{Conf}_{pPES}(P)$ holding. A PES can be obtained from a pPES.

Definition 7. *Let $P = (E, <, \#)$ be a pPES. Then $\mathsf{hc}(P) = (E, <, \sharp)$ is the* hereditary closure *of P, where $\sharp$ is derived by using the following rules*

$$\frac{e \mathbin{\#} e'}{e \mathbin{\sharp} e'} \qquad \frac{e \mathbin{\sharp} e' \qquad e' < e''}{e \mathbin{\sharp} e''} \qquad \frac{e' \mathbin{\sharp} e}{e \mathbin{\sharp} e'}$$

The following proposition relates pPES to PES [24].

Proposition 1. *Let $P = (E, <, \#)$ be a pPES. Then*

- $\mathsf{hc}(P) = (E, \leq, \sharp)$ *is a PES,*
- *if P is a PES, then $\mathsf{hc}(P) = P$, and*
- $\mathsf{Conf}_{pPES}(P) = \mathsf{Conf}_{PES}(\mathsf{hc}(P))$.

2.3   Reversible prime event structures

We now focus on the notion of *reversible prime event structure*. The definitions and the results in this subsection are drawn from [24]. In reversible event structures some events are categorised as *reversible*. In addition to the usual causality and conflict relations, reversible event structures incorporate two new ones that relate events and those representing the *actual* undoing of the reversible events. The undoing of events is represented by *removing* them (from a configuration), which is achieved by *executing* the appropriate *reversing* events.

Definition 8. *A reversible prime event structure (rPES) is the tuple $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ where $(E, <, \#)$ is a pPES, $U \subseteq E$ are the* reversible/undoable *events (with reverse events being denoted by $\underline{U} = \{\underline{u} \mid u \in U\}$ and disjoint from E, i.e., $\underline{U} \cap E = \emptyset$) and*

- $\rhd \subseteq E \times \underline{U}$ *is the* prevention *relation,*
- $\prec \subseteq E \times \underline{U}$ *is the* reverse causality *relation and it is such that $u \prec \underline{u}$ for each $u \in U$ and $\{e \in E \mid e \prec \underline{u}\}$ is finite and conflict-free for every $u \in U$,*
- *if $e \prec \underline{u}$ then not $e \rhd \underline{u}$,*
- *the* sustained causation *$\ll$ is a transitive relation defined such that $e \ll e'$ if $e < e'$ and $e \in U$, then $e' \rhd \underline{e}$, and*

– # *is hereditary with respect to* $\ll$: *if* $e \# e' \ll e''$, *then* $e \# e''$.

The ingredients of an rPES partly overlap with those of a PES: there is a causality relation ($<$) and a conflict one (#) and the two are related by the *sustained causation* relation $\ll$. The new ingredients are the *prevention* relation and the *reverse causality* relation. The prevention relation states that certain events should be absent when trying to reverse an event, e.g., $e \rhd \underline{u}$ states that e should be absent when reversing u. The reverse causality relation $e \prec \underline{u}$ says that $\underline{u}$ can be executed only when e is present.

Example 1. Let $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ where $E = U = \{a, b, c\}$, $a < b$ and $a \prec \underline{a}$, $b \prec \underline{b}$, $c \prec \underline{c}$, $c \prec \underline{a}$ with $b \rhd \underline{a}$ and no conflict. Then $a \ll b$ because $a < b$ and $b \rhd \underline{a}$. P states that b causally depends on a and that c is concurrent w.r.t. both a and b. Note that every event is reversible in P because $U = E$. As expected, the reverse causality relation $\prec$ is defined such that every reverse event requires the presence of the corresponding reversible event, i.e., $e \prec \underline{e}$ for all $e \in E$. Additionally, it also requires $c \prec \underline{a}$, i.e., a can be reversed only when c is present. The prevention relation states that a cannot be reversed when b is present, i.e., $b \rhd \underline{a}$.

Definition 9. *Let* $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ *be an* rPES *and* $X \subseteq yE$ *be a set of events such that* $\mathsf{CF}(X)$. *For* $A \subseteq E$ *and* $B \subseteq U$, *we say that* $A \cup \underline{B}$ *is enabled at* X *if*

– $A \cap X = \emptyset$, $B \subseteq X$ *and* $\mathsf{CF}(X \cup A)$,
– $\forall e \in A, e' \in E$. *if* $e' < e$ *then* $e' \in X \setminus B$,
– $\forall e \in B, e' \in E$. *if* $e' \prec \underline{e}$ *then* $e' \in X \setminus (B \setminus \{e\})$,
– $\forall e \in B, e' \in E$. *if* $e' \rhd \underline{e}$ *then* $e' \notin X \cup A$.

If $A \cup \underline{B}$ *is enabled at* X *then* $X \xrightarrow{A \cup \underline{B}} Y$ *where* $Y = (X \setminus B) \cup A$.

Example 2. Consider the rPES in Example 1. We have, e.g., $\emptyset \xrightarrow{\{a,c\}} \{a,c\} \xrightarrow{\{a\}} \{c\}$ and $\emptyset \xrightarrow{\{a\}} \{a\} \xrightarrow{\{b\}} \{a,b\} \xrightarrow{\{c,b\}} \{a,c\} \xrightarrow{\{b\}} \{a,b,c\}$. While $\emptyset \xrightarrow{\{a\}} \{a\}$ holds, $\emptyset \xrightarrow{\{a\}} \{a\} \xrightarrow{\{\underline{a}\}} \{\}$ does not hold; this is because $a \prec \underline{a}$ and $c \prec \underline{a}$ require that a and c are in the configuration ($\{a\}$) for $\{\underline{a}\}$ to be enabled. Also $\emptyset \xrightarrow{\{a,c\}} \{a,c\} \xrightarrow{\{b\}} \{a,c,b\}$ holds but $\{a,c,b\} \xrightarrow{\{\underline{a}\}} \{b,c\}$ does not hold since, given $b \rhd \underline{a}$, the presence of b prevents the execution of $\underline{a}$.

Reachable configurations are sets of events that can be reached from the empty set by performing events or undoing previously performed events.

Definition 10. *Let* $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ *be an* rPES *and let* $X \subseteq E$ *be a set of events such that* $\mathsf{CF}(X)$. *We say that* X *is a* (reachable) configuration *if there exist two sequences of sets* A_i *and* B_i, *for* $i = 1, \ldots, n$, *such that*

– $A_i \subseteq E$ *and* $B_i \subseteq U$ *for all* i, *and*

 – $X_i \xrightarrow{A_i \cup B_i} X_{i+1}$ *for all i with* $X_1 = \emptyset$ *and* $X_{n+1} = X$.

The set of configurations of P *is denoted by* $\mathsf{Conf}_{r\text{PES}}(\mathsf{P})$.

Example 3. The set of configurations of P defined in Example 1 is $\mathsf{Conf}_{r\text{PES}}(\mathsf{P}) = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ as illustrated by the sequences shown in Example 2.

As discussed in Section 1, rPESes accommodate different flavours of reversibility. Henceforth, we focus on causal-consistent reversibility [9,16], which is one of the most common models of reversibility in distributed systems, in which an event can be reversed only when all the events it has caused have already been reversed. In the setting of rPESes we consider these two forms of causal-consistent reversibility.

Definition 11. *Let* $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ *be an* rPES. *Then* P *is* cause-respecting *if for any* $e, e' \in E$, *if* $e < e'$ *then* $e \ll e'$. P *is* causal *if for any* $e \in E$ *and* $u \in U$ *the following holds:* $e \prec \underline{u}$ *iff* $e = u$, *and* $e \rhd \underline{u}$ *iff* $u < e$.

Example 4. The rPES P in Example 1 is a cause-respecting rPES. However P is not causal because of $c \prec \underline{a}$, which means that c has to be present for a to be reversed even if c does not causally depend on a. If we remove $c \prec \underline{a}$ then we obtain a causal rPES.

Example 5. An example of out-of-causal order reversibility can be obtained from the definition of the rPES P in Example 1 by replacing $b \rhd \underline{a}$ by $a \rhd \underline{b}$. Then, we have $\emptyset \xrightarrow{\{a\}} \{a\} \xrightarrow{\{b,c\}} \{a,b,c\} \xrightarrow{\{a\}} \{b,c\}$. Note that a can be reversed even in the presence of the event b, which causally depends on a.

Cause-respecting and causal rPESes enjoy the following useful properties [24].

Proposition 2. *Let* $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ *be an* rPES. *Let* X *be a left-closed and conflict-free set of events in* E *and let* $A, B \subseteq U$. *Then*

 – *if* P *is cause-respecting and* $X \xrightarrow{A \cup B} X'$, *then* X' *is also left-closed,*
 – *if* P *is cause-respecting and* $X \xrightarrow{B} X'$, *then* $X' \xrightarrow{B} X$,
 – *if* P *is causal and* $X \xrightarrow{A \cup B} X'$, *then* $X' \xrightarrow{B \cup A} X$.

Example 6. The above properties do not hold when an rPES is not cause-respecting or not causal. Consider the rPES in Example 5. We have that $\{a,b,c\} \xrightarrow{\{a\}} \{b,c\}$ but $\{b,c\}$ is not left-closed.

A particular rôle will be played by the configurations that can be reached without executing any reversible event.

Definition 12. *Let* $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ *be an* rPES *and* $X \in \mathsf{Conf}_{r\text{PES}}(\mathsf{P})$ *be a configuration.* X *is* forwards reachable *if there exists a sequence of sets* $A_i \subseteq E$, *for* $i = 1, \ldots, n$, *such that* $X_i \xrightarrow{A_i} X_{i+1}$ *for all i, with* $X_1 = \emptyset$ *and* $X_{n+1} = X$.

The set $\{b, c\}$ in Example 6 is a reachable configuration which is not forwards reachable. The configurations of a cause-respecting rPES are forwards reachable (see [24]).

Proposition 3. *Let* $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ *be a cause-respecting rPES, and let* X *be a configuration of* P. *Then* X *is forwards reachable.*

3 Occurrence nets and prime event structures

We review the notion of occurrence nets [21,29]. Given a net $N = \langle S, T, F, \mathsf{m} \rangle$, we write $<_N$ for the transitive closure of F, and $\leq_N$ for the reflexive closure of $<_N$. We say N is *acyclic* if $\leq_N$ is a partial order. For occurrence nets, we adopt the usual convention and refer to places and transitions respectively as *conditions* and *events*, and correspondingly use B and E for the sets of conditions and events ([29]). We will often confuse conditions with places and events with transitions.

Definition 13. *An* occurrence net *(*ON*)* $C = \langle B, E, F, \mathsf{c} \rangle$ *is an acyclic, safe net satisfying the following restrictions:*

- $\forall b \in B.\ {}^\bullet b$ *is either empty or a singleton, and* $\forall b \in \mathsf{c}.\ {}^\bullet b = \emptyset$,
- $\forall b \in B.\ \exists b' \in \mathsf{c}$ *such that* $b' \leq_C b$,
- *for all* $e \in E$ *the set* $\lfloor e \rfloor = \{e' \in E \mid e' \leq_C e\}$ *is finite, and*
- $\# \subseteq E \times E$ *defined as* $e \#_0 e'$ *iff* $e, e' \in E$, $e \neq e'$ *and* ${}^\bullet e \cap {}^\bullet e' \neq \emptyset$, $x \# x'$ *iff* $\exists y, y' \in E$ *such that* $y \#_0 y'$ *and* $y \leq_C x$ *and* $y' \leq_C x'$, *is an irreflexive and symmetric relation.*

The intuition behind occurrence nets is the following: each condition b represents the occurrence of a token, which is produced by the *unique* event in ${}^\bullet b$, unless b belongs to the initial marking, and it is used by only one transition (hence if $e, e' \in b^\bullet$, then $e \# e'$). On an occurrence net C it is natural to define a notion of *causality* among elements of the net: we say that x is *causally dependent* on y iff $y \leq_C x$. Occurrence nets are often the result of the *unfolding* of a (safe) net. In this perspective an occurrence net is meant to describe precisely the non-sequential semantics of a net (a semantics where concurrency is faithfully represented), and each reachable marking of the occurrence net corresponds to a reachable marking in the net to be unfolded. Here we focus purely on occurrence nets and not on the nets they are unfoldings of.

Definition 14. *Let* $C = \langle B, E, F, \mathsf{c} \rangle$ *be a* ON *and* $X \subseteq E$ *be a set of events. Then* X *is a* configuration *of* C *whenever* $\mathsf{CF}(X)$ *and* $\forall e \in X.\ \lfloor e \rfloor \subseteq X$. *The set of configurations of the occurrence net* C *is denoted by* $\mathsf{Conf}_{\mathrm{ON}}(C)$.

Given an occurrence net $C = \langle B, E, F, \mathsf{c} \rangle$ and a state $X \in \mathsf{St}(C)$, it is easy to see that it is *conflict free, i.e.* $\forall e, e' \in X.\ e \neq e' \Rightarrow \neg(e \# e')$, and *left closed, i.e.* $\forall e \in X.\ \{e' \in E \mid e' \leq_C e\} \subseteq X$.

The following propositions make clear the relations between prime event structures, occurrence nets, states of the occurrence nets and configurations of
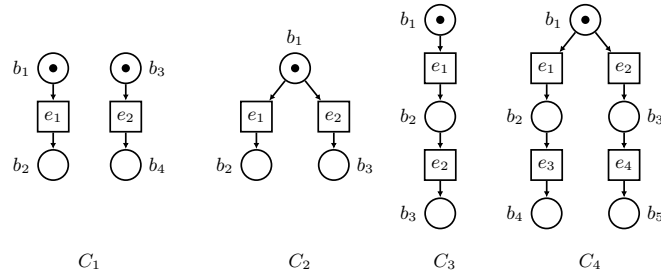
Fig. 1. Some occurrence nets

the prime event structures. Proofs are standard and can be found in papers investigating prime event structures and occurrence nets.

Proposition 4. *Let $C = \langle B, E, F, \mathsf{c}\rangle$ be an occurrence net and $X \in \mathsf{St}(C)$. Then $X \in \mathsf{Conf}_{\mathrm{ON}}(C)$.*

Occurrence nets and prime event structures are connected as follows ([29]).

Proposition 5. *Let $C = \langle B, E, F, \mathsf{c}\rangle$ be an occurrence net. Then $\mathcal{P}(C) = (E, \leq_C, \#)$ is a PES, and $\mathsf{Conf}_{\mathrm{ON}}(C) = \mathsf{Conf}_{\mathrm{PES}}(\mathcal{P}(C))$.*

Example 7. Figure 1 illustrates some (finite) occurrence nets (nets are depicted as usual). We can associate PESes to them as follows. The net C_1 has two concurrent events, which are neither causally ordered nor in conflict; hence $<$ and $\#$ are empty. The events e_1 and e_2 in C_2 are in conflict, i.e., $e_1 \# e_2$, while they are causally ordered in C_3, namely $e_1 < e_2$, but not in conflict. Finally, in C_4 we have $e_1 < e_3$ and $e_2 < e_4$ and $e_1 \# e_2$. Additionally, conflict inheritance give us $e_1 \# e_4$, $e_2 \# e_3$ and $e_3 \# e_4$.

Conversely, every PES can be associated with an occurrence net. With $\#(A)$ we denote the set of events A such that $\forall e, e' \in A.\ e \neq e' \ \Rightarrow \ e \# e'$.

Proposition 6. *Let $P = (E, <, \#)$ be a PES and let $\perp \notin E$ be a new symbol. Then $\mathcal{E}(P) = \langle B, E, F, \mathsf{c}\rangle$ defined as follows*

- $B = \{(a, A) \mid a \in E \cup \{\perp\} \wedge A \subseteq E \wedge \#(A) \wedge (a \neq \perp \ \Rightarrow \ \forall e \in A.\ a < e)\}$,
- $F = \{(b, e) \mid b = (a, A) \ \wedge \ e \in A\} \cup \{(e, b) \mid b = (e, A)\}$, *and*
- $\mathsf{c} = \{(a, A) \mid (a, A) \in B \ \wedge \ a = \perp\}$.

is an occurrence net, and $\mathsf{Conf}_{\mathrm{PES}}(P) = \mathsf{Conf}_{\mathrm{ON}}(\mathcal{E}(P))$.

4 Reversible occurrence nets and causal reversible prime event structures

We now introduce the notion of *reversible occurrence nets*. A similar notion has been proposed in [19] for adding causal-consistent reversibility to Petri nets by

making reversible every event in the unfolding of the net. In this work we deal with a generalised version of reversible occurrence nets in which transitions may be irreversible, i.e., we do not require every transition of a net to be undoable. The intuition behind reversible occurrence nets is the following: we add special transitions (events in the classical occurrence net terminology) to an occurrence net which, when executed, *undo* the execution of other (standard) transitions. When we remove these special transitions from a reversible causal net we obtain a standard occurrence net.

Definition 15. *A reversible occurrence net (* RON*) is a tuple $R = \langle B, E, U, F, \mathsf{c} \rangle$ where $\langle B, E, F, \mathsf{c} \rangle$ is a safe net such that*

- $U \subseteq E$ *and* $\forall u \in U.\ \exists!\ e \in E \setminus U$ *such that* $^{\bullet}u = e^{\bullet}$ *and* $u^{\bullet} = {}^{\bullet}e$,
- $\forall e, e' \in E.\ {}^{\bullet}e = {}^{\bullet}e' \ \wedge \ e^{\bullet} = e'^{\bullet} \ \Rightarrow \ e = e'$,
- $\bigcup_{e \in E}({}^{\bullet}e \cup e^{\bullet}) = B$, *and*
- $C_{E \setminus U} = \langle B, E \setminus U, F', \mathsf{c} \rangle$ *is an occurrence net, where F' is the restriction of F to the transitions in $E \setminus U$.*

The events in U are the reversing ones and we often say that a reversible occurrence net R is reversible *with respect to* U. We write $\overline{E}$ for the set of events $E \setminus U$ and $C_{\overline{E}}$ instead of $C_{E \setminus U}$. The first condition in Definition 15 implies that each reversing event $u \in U$ is associated with a unique event e that causes the effects that u is intended to *undo*; hence e here is a *reversible* event. Moreover, the second condition ensures that there is an injective mapping $h : U \to E$ that associates each event $u \in U$ with a different event $e \in E$ such that $^{\bullet}e = u^{\bullet}$ and $e^{\bullet} = {}^{\bullet}u$, in other words, each reversible event has exactly one reversing event. The third requirement guarantees that all conditions (places) of the net appear at least in the pre or the postset of some event (transition), i.e., there are no isolated conditions. The last condition ensures that the net obtained by deleting all reversing events is an occurrence net.

Example 8. We present some reversible occurrence nets in Figure 2. The reversing events are drawn in red, and their names are underlined. The events e_1 and e_2 in R_1 are both reversible, while e_1 is the only reversible event in R_2. In R_3 the events e_1, e_3 and e_4 are the reversible ones.

We prove that the set of reachable markings of a reversible occurrence net is not influenced by performing reversing events.

Proposition 7. *Let $R = \langle B, E, U, F, \mathsf{c} \rangle$ be an* RON*. Then $\mathcal{M}_R = \mathcal{M}_{C_{\overline{E}}}$.*

Proof. One direction is trivial, namely $\mathcal{M}_{C_{\overline{E}}} \subseteq \mathcal{M}_R$. For the other direction, we first observe that $\neg(\mathsf{c}\, [e\rangle\,)$ holds for all $e \in U$. This is because $C_{\overline{E}}$ is an occurrence net, and this implies that $\forall b \in \mathsf{c}.\ {}^{\bullet}b$ is either $\emptyset$ or it contains elements in E^r, and $\forall e \in U.\ {}^{\bullet}e \cap b^{\bullet} = \emptyset$. Now we show that if an event $u \in U$ is executed then the corresponding event $h(u)$ has been executed before. W.l.o.g. we assume that all the events executed before u are the events in $E \setminus U$. Consider the fs $\sigma\, [u\rangle\, \sigma'$, then we have $lead(\sigma)\, [u\rangle\,$, which means that $^{\bullet}u \subseteq lead(\sigma)$, but the
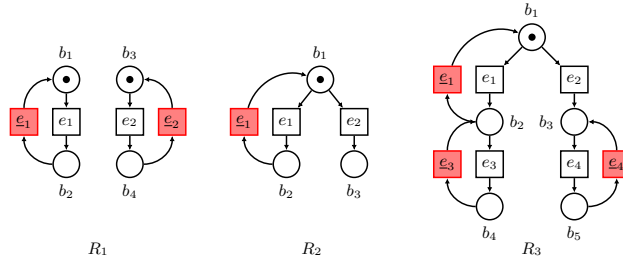
Fig. 2. Some reversible occurrence nets

conditions ${}^\bullet u$ have been produced by the execution of a unique event, namely $h(u)$. Now we prove that $\sigma\,[u\rangle\,m$ can be reached without executing both u and $h(u)$. Consider the marking $lead(\sigma)$, as $\sigma\,[u\rangle$ we know that $h(u)^\bullet \subseteq lead(\sigma)$. Now σ can be rewritten as $\sigma''\,[h(u)\rangle\,\sigma'''$ and $h(u)$ is concurrent with all the events in σ''', which means that σ can be rewritten as $\hat\sigma\,[h(u)\rangle\,lead(\sigma)$. Now we have $m = lead(\hat\sigma)$ which implies that each reachable marking can be reached executing the events in $E \setminus U$ only, hence $\mathcal{M}_R \subseteq \mathcal{M}_{C_{\overline{E}}}$.

A consequence of the above proposition is the following corollary, which establishes that each marking can be reached by using just *forward events*.

Corollary 1. *Let $C = \langle B, E, U, F, \mathsf{c}\rangle$ be an* RON *and σ be an* fs*. Then, there exists an* fs *σ' such that $X_{\sigma'} \subseteq \overline{E}$ and $lead(\sigma) = lead(\sigma')$.*

Definition 16. *Let $R = \langle B, E, U, F, \mathsf{c}\rangle$ be an* RON*, and $X \subseteq \overline{E}$ be a set of forward events. Then, X is a configuration of R whenever $\mathsf{CF}(X)$ and $\forall e \in X.\ \lfloor e \rfloor \cap \overline{E} \subseteq X$. The set of configurations of R is usually denoted with $\mathsf{Conf}_{\mathrm{RON}}(R)$.*

A configuration of a reversible occurrence net R with respect to U is a subset of $E \setminus U$; consequently, the reversing events (i.e., the ones in U) that may have been executed to reach a particular marking are not considered as part of the configuration. Observe that, differently from occurrence nets, $\mathsf{St}(R) \neq \mathsf{Conf}_{\mathrm{RON}}(R)$ because the former may contain also reversing events. However, as a consequence of Corollary 1, there is no loss of information.

Proposition 8. *Let $R = \langle B, E, U, F, \mathsf{c}\rangle$ be an* RON*. Then $X \in \mathsf{Conf}_{\mathrm{RON}}(R)$ iff $X \in \mathsf{Conf}_{\mathrm{ON}}(C_{\overline{E}})$.*

We show how to construct a reversible occurrence net from an occurrence net, once we have identified the events to be *reversed*.

Definition 17. *Let $C = \langle B, E, F, \mathsf{c}\rangle$ be an occurrence net and $U \subseteq E$ be the set of reversible events. Define $\mathsf{R}(C) = \langle B, \hat{E}, U \times \{r\}, \hat{F}, \mathsf{c}\rangle$ be the net where $\hat{E}$ and $\hat{F}$ are defined as follows:*

- *$\hat{E} = E \times \{\mathtt{f}\}\ \cup\ U \times \{\mathtt{r}\}$, and*

$$- \; \hat{F} \; = \; \{(b, (e, \mathtt{f})) \mid (b, e) \in F\} \quad \cup \quad \{((e, \mathtt{f}), b) \mid (e, b) \in F\} \quad \cup$$
$$\{(b, (e, \mathtt{r})) \mid (e, b) \in F\} \quad \cup \quad \{((e, \mathtt{r}), b) \mid (b, e) \in F\}.$$

The mapping $h : U \times \{\mathtt{r}\} \to E \times \{\mathtt{f}\}$ is defined as $h(e, \mathtt{r}) = (e, \mathtt{f})$.

The construction above simply adds as many events (transitions) as those to be reversed. The preset of each added event is the postset of the corresponding event to be reversed, and its postset is defined as the preset of the event to be reversed. The events in $U \times \{\mathtt{r}\}$ are the reversing events.

Proposition 9. *Let $C = \langle B, E, F, \mathsf{c} \rangle$ be an occurrence net, $U \subseteq E$ be the set of reversible events, and $\mathsf{R}(C) = \langle B, \hat{E}, U \times \{\mathtt{r}\}, \hat{F}, \mathsf{c} \rangle$ be the net in Definition 17. Then, $\mathsf{R}(C)$ is a reversible occurrence net with respect to $U \times \{\mathtt{r}\}$.*

Proof. We just have to prove that $\mathsf{R}(C)$ is a safe net; the other conditions are satisfied by construction. First we observe that if $b \notin \mathsf{c}$ and $^\bullet b$ is not a singleton in $\mathsf{R}(C)$ then $^\bullet b$ contains at most one event of the form $(e, \mathtt{f})$, and it contains at least one of the form $(e', \mathtt{r})$, and these are originated by the events in $b^\bullet$ in C. In the case $b \in \mathsf{c}$ and $^\bullet b$ is not empty, then again $^\bullet b$ contains only events of the form $(e', \mathtt{r})$, and these are originated by the events in $b^\bullet$ in C. Assume it is not, and assume that $b \in B$ is the condition which receives a token when it is already marked. As C is an occurrence net, if the condition is marked then the event $e \in E$ such that $b \in e^\bullet$ has been executed and none of the events $e' \in E$ such that $e' \in b^\bullet$ (if any) have yet been executed. Thus in $\mathsf{R}(C)$ the event $(e, \mathtt{f})$ has been executed and none of the events $(e', \mathtt{f}) \in b^\bullet$ has been executed yet. To be marked again an event of the form $(e'', \mathtt{r}) \in {}^\bullet b$ should have occurred, but this is impossible as none of the events $(e', \mathtt{f}) \in b^\bullet$ have been executed, and among these also $(e'', \mathtt{f})$, contradicting the fact that the condition b is marked again.

Example 9. Consider the occurrence net C_1 in Figure 1, and assume that both events are reversible. The net R_1 in Figure 2 is $\mathsf{R}(C_1)$ (after renaming events with the convention that $(e, \mathtt{f})$ is named as e and $(e, \mathtt{r})$ as $\underline{e}$). The RON R_3 in Figure 2 is $\mathsf{R}(C_4)$, with C_4 in Figure 1 and the set of reversible events $U = \{e_1, e_2, e_4\}$.

From RON *to* rPES*:* As is usually done for occurrence nets, we now associate each reversible occurrence net with a reversible prime event structure. Given an RON $R = \langle B, E, U, F, \mathsf{c} \rangle$, we denote the set of events $\{e' \mid e <_R e'\}$ by $\lceil e \rceil$. Observe that this set is not necessarily conflict-free.

Proposition 10. *Let $R = \langle B, E, U, F, \mathsf{c} \rangle$ be a reversible occurrence net with respect to U, then $\mathcal{C}_r(R) = (E', U', <, \#, \prec, \rhd)$ is its associated rPES, where*

- *$E' = \overline{E}$ and $U' = h(U)$,*
- *$<$ is $<_{C_{\overline{E}}}$, and $\#$ is the conflict relation defined on the occurrence net $C_{\overline{E}}$,*
- *$e \rhd \underline{e'}$ whenever $e \in \lceil e' \rceil$, $e \prec \underline{e'}$ whenever $e = e'$, and $\ll = <$.*

Proof. First of all it is quite clear that $(E', <, \#)$ is a pPES (if we close $<$ reflexively we get indeed a PES), as it is obtained by $C_{\overline{E}}$. The relation $\prec \subseteq E' \times U'$

satisfies the requirement that $e \prec \underline{e}$ and that $\{e' \mid e' \prec \underline{e}\}$ is finite for each $e \in U'$ as it contains just e. If $e \prec \underline{e}$ then not $e \rhd \underline{e}$ as $e \notin \lceil e \rceil$. The sustained causation relation $\ll$ coincides with the relation $<$ and so the conflict relation is inherited along this relation. Furthermore, for $e \in U'$, if $e < e'$ for some e', then we have that $e' \rhd e$, as required. We can then conclude that $\mathcal{C}_r(R)$ is an rPES.

Example 10. Consider the reversible occurrence net R_3 in Figure 2. The associated rPES has the events $\{e_1, e_2, e_3, e_4\}$ and the reversible events $\{e_1, e_3, e_4\}$. The causality relation of the associated pPES is $e_1 < e_3$, $e_2 < e_4$, the conflict relation is *generated* by $e_1 \# e_2$, and it is inherited along $\ll$, which coincides with $<$. The reverse causality stipulates that $e_1 \prec \underline{e_1}$, $e_3 \prec \underline{e_3}$ and $e_4 \prec \underline{e_4}$ and finally $e_3 \rhd \underline{e_1}$, as to be allowed to undo e_1 it is necessary to undo e_3 first.

The following result states that the rPES associated to a reversible occurrence net is causal, hence cause-respecting.

Proposition 11. *Let* $R = \langle B, E, U, F, \mathsf{c} \rangle$ *be a reversible occurrence net with respect to* U *and* $\mathcal{C}_r(R) = (E', U', <, \#, \prec, \rhd)$ *be the associated rPES. Then* $\mathcal{C}_r(R)$ *is a causal rPES.*

Proof. Easy inspection of the construction in Proposition 10. The sustained causality $\ll$ clearly coincides with $<$. If $e \prec \underline{e'}$ then $e' = e$ and by construction if $e \rhd \underline{e'}$ then $e' < e$ as $e \in \lceil e' \rceil$.

We show that each configuration of an RON is a configuration of the corresponding rPES, and vice versa.

Theorem 1. *Let* $R = \langle B, E, U, F, \mathsf{c} \rangle$ *be a reversible occurrence net with respect to* U *and* $\mathcal{C}_r(R) = (E', U', <, \#, \prec, \rhd)$ *be the associated rPES. Then* $X \subseteq E'$ *is a configuration of* R *iff* X *is a configuration of* $\mathcal{C}_r(R)$.

Proof. As $\mathcal{C}_r(R)$ is a cause-respecting and causal rPES we have that each configuration is forward reachable, and the forward reachable configurations are precisely those conflict-free and left-closed of the pPES $\mathcal{C}_r(R) = (E', <, \#)$, which correspond to the configurations of the occurrence net $R_{\overline{E}}$.

We stress that a reversing event in a reversible occurrence net is enabled at a marking when the conditions in the postset of the event to be reversed are marked. This may happen only when all the events that causally depend on the event to be reversed have either been executed and reversed or have not been executed at all. Thus every RON enjoys *causally consistent* reversibility [9,15], and consequently cannot implement the so called *out-of-causal order* reversibility [13]. In contrast, rPESes are able to model *out-of-causal order* reversibility (as illustrated in Example 5).

Proposition 12 below formalises what are called *mixed-reverse* transitions in [11], namely a correspondence between the steps in a reversible occurrence net and the sequences of reachable configurations of the associated rPES. We now introduce some auxiliary notation. Let $R = \langle B, E, U, F, \mathsf{c} \rangle$ be an RON, and $X \subseteq E$ be a configuration of R, we write $\mathsf{mark}(X)$ to denote the marking reached after executing the events in X; this marking can be expressed as $(\mathsf{c} \cup X^{\bullet}) \setminus {}^{\bullet}X$.

Proposition 12. *Let $R = \langle B, E, U, F, \mathsf{c} \rangle$ be a reversible occurrence net and $\mathcal{C}_r(R) = (E', U', <, \#, \prec, \rhd)$ be its associated rPES. Let $X \in \mathsf{Conf}_{\mathrm{RON}}(R)$ and $A \subseteq E$ be a set of events such that $\mathsf{mark}(X)\,[A\rangle$. Then $\hat{A} \cup \underline{B}$ is enabled at X in $\mathcal{C}_r(R)$, where $\hat{A} = \{e \in A \mid e \notin U\}$ and $\underline{B} = \{e \in A \mid e \in U\}$.*

Proof. By Theorem 1 we know that $X \in \mathsf{Conf}_{r\mathrm{PES}}(\mathcal{C}_r(R))$. We have to check that $\hat{A} \cup \underline{B}$ is enabled at X. As $\mathsf{mark}(X)\,[A\rangle$ we know that ${}^{\bullet}A \subseteq \mathsf{mark}(X)$, hence $A \cap X$ should be equal to $\emptyset$. Furthermore for any $e \in A \cap U$, as $\mathsf{mark}(X)\,[\{e\}\rangle$, we have that $h(e) \in X$ (otherwise the conditions enabling e would not have been produced), and then we have that $B = \{h(e) \mid e \in \underline{B}\} \subseteq X$. Finally, as $\mathsf{mark}(X)\,[A\rangle$, we have that $\mathsf{CF}(X \cup \hat{A})$ holds. Consider now $e \in \hat{A}$, and $e' < e$. Clearly $e' \in X \setminus B$. Assume the contrary, then $e' \in B$ and there exists an $\underline{e}' \in A \cap U$ such that $h(\underline{e}') = e'$, but then we have that $\neg\mathsf{mark}(X)\,[A\rangle$. Consider now $e \in B$ (which means that $\underline{e} \in A \cap U$) and $e' \prec \underline{e}$. As $\mathcal{C}_r(R)$ is a causal rPES, we know that $e' = e$ and $e \in X \setminus (B \setminus \{e\})$. Take now $e \in B$ and $e' \rhd \underline{e}$. This means that $e' \in \lceil e \rceil$ which implies that $e \notin X$, and also that $e \notin \hat{A}$. By Definition 9 we can conclude that $\hat{A} \cup \underline{B}$ is enabled at X. Finally we observe that $\mathsf{mark}(Y) = \mathsf{c}'$ where $\mathsf{mark}(X)\,[A\rangle\,\mathsf{c}'$ and $X \xrightarrow{\hat{A} \cup \underline{B}} Y$.

From rPES to RON*:* Correspondingly to what is usually done when relating nets to event structures, we show that if we focus on causal rPESes then we can relate them to reversible occurrence nets. The construction is indeed quite standard (see [29,4] among many others), but we do need a further observation on causal rPES.

Proposition 13. *Let $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ be a causal rPES. Then, $\#$ is inherited along $<$, i.e. $e \# e' < e'' \Rightarrow e \# e''$.*

Proof. In general we have that, given an rPES, $(E, \ll, \#)$ is a PES [24]. But in a causal rPES we have that $\ll$ is indeed the transitive closure of $<$.

A consequence of this proposition is that the conflict relation is fully characterized by the causality relation.

The same intuition underlying the introduction of reversible occurrence net can be used in associating a net to a causal rPES like the one used to associate an occurrence net to a PES.

Definition 18. *Let $\mathsf{P} = (E, U, <, \#, \prec, \rhd)$ be a causal rPES, and $\perp \notin E$ be a new symbol. Define $\mathcal{E}_r(\mathsf{P})$ as the Petri net $\langle B, \hat{E}, F, \mathsf{c} \rangle$ where*

- $B = \{(a, A) \mid a \in E \cup \{\perp\} \wedge A \subseteq E \wedge \#(A) \wedge a \neq \perp \Rightarrow \forall e \in A.\ a \ll e\}$,
- $\hat{E} = E \times \{\mathtt{f}\} \cup U \times \{\mathtt{r}\}$,
- $F = \{(b, (e, \mathtt{f})) \mid b = (a, A) \wedge e \in A\} \cup \{((e, \mathtt{f}), b) \mid b = (e, A)\} \cup \{(b, (e, \mathtt{r})) \mid b = (e, A)\} \cup \{((e, \mathtt{r}), b) \mid b = (a, A) \wedge e \in A\}$, *and*
- $\mathsf{c} = \{(a, A) \mid (a, A) \in B \wedge a = \perp\}$.

In essence the construction above takes the PES associated to an rPES and constructs the associated occurrence net, which is then *enriched* with the reversing events (transitions). The result is a reversible occurrence net.

Proposition 14. *Let* $P = (E, U, <, \#, \prec, \rhd)$ *be a causal* rPES. *Then* $\mathcal{E}_r(P) = \langle B, \hat{E}, U \times \{r\}, F, c\rangle$ *as defined in Definition 18 is a reversible occurrence net with respect to* $U \times \{r\}$.

Proof. By construction $\mathcal{E}_r(P)_{E \times \{f\}}$ is a occurrence net. The other requirements can be easily checked. For each (e, r) there exists a unique event (e, f), and if two events share the same preset and postset they are clearly the same event. Each condition $b \in B$ is clearly related to an event in $E \times \{f\}$ hence in $\hat{E} \setminus (E' \times \{r\})$.

Theorem 2. *Let* P *be a causal* rPES. *Then* X' *is a configuration of* $\mathcal{E}_r(P)$ *iff* X *is a configuration of* P, *where* $X' = \{(e, f) \mid e \in X\}$.

Proof. Let $P = (E, U, <, \#, \prec, \rhd)$. Consider $X \in \mathsf{Conf}_{r\mathrm{PES}}(P)$. As P is a cause-respecting and causal rPES we have that X is forward reachable, hence X is a configuration of the pPES $(E, <, \#)$, which we denote with P, and then $X' = \{(e, f) \mid e \in X\}$ is a configuration also of the occurrence net associated to this event structure as, by Proposition 1, we have that $\mathsf{Conf}_{p\mathrm{PES}}(P) = \mathsf{Conf}_{\mathrm{PES}}(\mathsf{hc}(P))$. For the converse it is enough to observe that, up to renaming of events, $\mathcal{C}_r(\mathcal{E}_r(P))$ is indeed P.

Clearly, if we start from a reversible occurrence net, we get an rPES from which a reversible occurrence net can be obtained having the same states (up to renaming of events).

Corollary 2. *Let* R *be a* RON. *Then* $\mathsf{St}(\mathcal{E}_r(\mathcal{C}_r(R))) = \mathsf{St}(R)$.

Example 11. Consider the rPES with four events $\{e_1, e_2, e_3, e_4\}$ such that $e_1 < e_3$ and $e_2 < e_4$, e_1 is in conflict with e_2 and this conflict is inherited along $<$. Furthermore, let e_1 and e_3 be reversible, and $e_3 \rhd \underline{e}_1$. The construction in Definition 18 gives the net below.
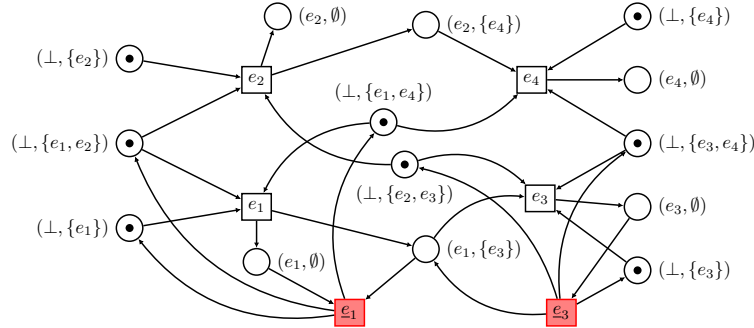

5   Conclusions and future work

The constructions we have proposed to associate a reversible occurrence net to a causal reversible prime event structure, and vice versa, are certainly driven by the classical ones (see [29]) for relating occurrence nets and prime event structures.

The consequence of this approach is that the causality relation, either the one given in an rPES or the one induced by the flow relation in the occurrence net obtained ignoring the reversing events, is the one driving the construction. One of the other two relations of an rPES is substantially ignored (and we obtain from a RON a causal rPES where the reverse causality relation just says that an event can be reversed only after it has occurred) whereas the second (prevention) is tightly related to the causality relation: b is caused by a precisely when b prevents undoing of a. The notion of reversible occurrence net we have proposed suggests this construction, so the problem of finding which kind of net would correspond to, for example, a cause-respecting or even an arbitrary rPES remains open and certainly deserves to be investigated. It is however interesting to observe that the construction in Definition 18 gives a reversible occurrence net even when the rPES one started with is not a causal rPES. Consider the rPES with two events $\{e_1, e_2\}$ such that $e_1 < e_2$ and where the conflict and the prevention relations are empty. The only reversible event is e_1 and $e_1 \prec \underline{e_1}$. The set $\{e_2\}$ is a reachable configuration: we can remove e_1 from a reachable configuration $\{e_1, e_2\}$ by performing the event $\underline{e_1}$. This is an example of out-of-causal order computation. Given this rPES, our construction produces the following RON, which does not have $\{e_2\}$ among its configurations.
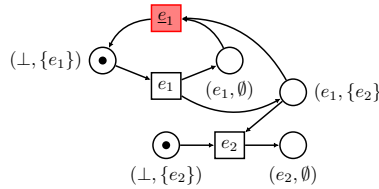
The constructions we have proposed are somehow the more adherent to what is usually done, based on the interpretation that *causality* implies that the event causing some other event somehow produces something that is used by the latter. This is not the only interpretation of what causality could mean. In fact, causality is often confused with the observation that two causally related events appear ordered in the same way in each possible execution, and when we talk about ordered execution, it should be stressed that this can be achieved in several ways, for instance using *inhibitor* arcs. Consider the net C:
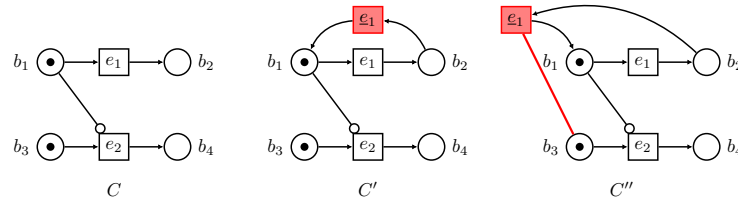
Here the event e_2 can be executed only after the event e_1 has been executed. However, e_1 does not produce a token (resource) that must be used by e_2. If we simply make the event e_1 reversible but do nothing to prevent reversing of e_1

before e_2 is reversed, then we would obtain the net C'. We could do better in C'' where we model prevention using so-called *read arcs* [20]. Hence, using inhibitor or read arcs seem feasible ways forward to capture more precisely the new relations of rPESes, including prevention. A similar approach has been already pursued in [8] to model so-called *modifiers* that are able to change the causality pattern of an event. This suggests that, for arbitrary rPESes, we need to find relations different from the flow relation to capture faithfully (forward and reverse) causal and prevention dependencies. This will be the subject of future research.

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