A collection of problems in spectral analysis for self-adjoint and non self-adjoint operators

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Abstract

The overall aim of this dissertation is to investigate some problems in spectral analysis for self-adjoint and non self-adjoint operators which arise in different contexts of physics.

In the first part of this thesis we study the problem of localisation of complex eigenvalues of non Hermitian perturbations of self-adjoint operators realised by means of complex potentials. In particular, we focus our attention on two different operators. The first one is the Laplacian defined on the real half line. The other is a second order two dimensional operator which arises in the context of the physics of materials, in particular from the study of the Hamiltonian of a double layer graphene. For both we provide Keller-type estimates on the localisation of complex eigenvalues.

The existence of trapped waves solutions for a set of equations describing the dynamics of a stratified two layers fluid of different densities, confined in a ocean channel of fixed width and varying depth and subject to rotation is studied in the second chapter. The existence of these solutions is then recovered by proving the existence of points in the point spectrum of a two dimensional operator pencil. We prove that, under some smallness assumptions on the difference between the two fluid densities and some geometric assumption of the channel's shape, the problem has positive solution.

The last part of this dissertation focuses on existence of particular Wronskian type of solutions for the KdV equation of the type of complex complexitons. We study these solutions both from a dynamical point of view when seen evolving in time, and also for fixed values of time if regarded as potentials for a spectral problem for the Schrödinger operator.

A chi mi ha amato, mi ama e mi amerá.
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Perché questa tesi non sarebbe stata possibile altrimenti.

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Contents

Introduction

Dico paura, perché allora non avrei saputo definire con altra parola più vera il mio turbamento. Sebbene avessi letto libri e romanzi, anche d'amore, in realtà ero rimasto un ragazzino semi-barbaro; e forse, anche, il mio cuore, approfittava, a mia insaputa, della mia immaturità e ignoranza, per difendermi contro la verità?

E. Morante, L'Isola di Arturo

This dissertation presents some of the studies the author has undertaken during his doctoral studies. The aim is to include the original results while collocating them among the most relevant ones present in literature and to provide the reader with additional details. The collection includes works in spectral analysis which span a broad variety. They not only can be placed into distinct mathematical areas but they also differ in terms of research objectives.

In particular, the opportunity or not to settle the problem in the framework of the theory of self-adjoint operators draws a clear division into two different families of problems. Of course such distinction is yet too coarse and it requires some elucidations.

It is a fact that any closed and densely-defined operator H is self-adjoint if it satisfies any two of the following properties:

- *Normal* $HH^* = H^*H$,
- *Symmetric* $H\psi = H^*\psi$ for every function ψ in the domain of H,
- *Real spectrum* $\sigma(H) \subset \mathbb{R}$,

and conversely every self-adjoint operator satisfies all three of the properties above. Therefore every non self-adjoint operator can be classified according to the properties illustrated above, as having either only one or none of them.

In this dissertation we study, in particular, a certain type of operators for which none of the properties listed above hold. They are of the form

$$H = H_0 + V(x)$$

where H_0 is in general self-adjoint and $V(x)$ is a complex perturbation. Particular attention will be paid to determining the nature of the spectrum and the localisation of the eigenvalues in the complex plane. We note that in this context, the set of tools available in the complex case is very limited. Results such as the variational characterisation of the eigenvalue valid in the self-adjoint case and the spectral theorem, which extends to normal operators [26] and that can be furthermore generalised to symmetric operators [41], do not hold any more. In the words of E. B. Davies [33]:

> *"Studying non-selfadjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases."*

This dichotomy and the impossibility for the Hermitian and the non-Hermitian cases to cohabit under a unified theory, besides a range of different phenomena which are observed separately in each specific case, can be partially justified by an uneven success that they encountered in time, especially when used to describe mostly physic scenarios. In the last century there have been very few mathematicians attracted to the study of non self-adjoint perturbation problems from a mathematical perspective and among them, the most prominent names are those of Naimark and Pavlov. Their efforts were mainly devoted to the understanding of the structure of the spectrum of the Laplacian endowed with complex potentials and of the scattering theory for operators with dissipative terms.

While the mathematical community seemed to be not very engaged, new impulses to this field have come from other disciplines. A very simple yet meaningful example which shows how complex potentials naturally arise in physics is borrowed from optics. The following argument lacks rigour but provides an heuristic interpretation of a complex physical quantity.

Let us assume that we want to describe how an electromagnetic impulse

travels in a anisotropic waveguide with refractive index R_i. If a planar wave of frequency f propagates through a medium along the x-direction with velocity v, its electric field E is described in its generality by

$$E = E_0 \exp\{i2\pi f\,(t - (x/v))\}$$

and its velocity of propagation is related to c (see [75]), the speed of light in vacuum and the index R_i by

$$v = \frac{c}{R_i}.$$

What happens when the index R_i is complex. Substituting the above expression for the velocity in the one of E, a damping term appears. When considering I the intensity of the signal, which is proportional to the square of the modulus of the electric field, we have

$$I = const.|E_0|^2 e^{-2\pi f\,\mathrm{Im}(R_i)x/c}.$$

This simple computation shows how absorption phenomena in light propagation can be described by means of complex refractive indexes, which are in turn derived in mathematical terms as complex eigenvalues of an appropriate spectral problem where the perturbative term has then to be complex.

In fact, in the last twenty years an increasing interest in the use of complex potentials and eigenvalues has appeared in many other disciplines. A partial list of studies where complex potentials have already been used includes works on biological populations [127], improvements in numerical simulation [68] and dynamics of nuclear, atomic and molecular system in open physical models [8, 123, 128].

In studying the spectrum of non self-adjoint operator of the type $H = H_0 + V(x)$, and in particular its complex eigenvalues, two different types of results are usually sought: uniform bounds valid for all the eigenvalues or bounds for the sum of the their moments or more general functions of them. The former deals in its most inclusive form with the possibility of having bounds of the type

$$f(\lambda) \leq g(V(x)),$$

where λ stands for any eigenvalue of H and f and g are two positive functions while the latter, instead, can be interpreted as a generalisation of the Lieb-Thirring inequality

$$\sum_{\lambda \in \sigma_d} f(\lambda) \leq g(V(x)).$$

In this dissertation we will mainly focus our attention on problems of the first type, namely those concerning localisation results for complex eigenvalues. Nonetheless, some results of the second type will also be recalled. In particular, we will consider the special case when $f(\lambda) = |\lambda|^0$, which is related to the eigenvalue counting function.

A useful fact first appeared in [2] in 1999 is the extension of the validity of the *Birman-Schwinger principle* [13, 146] to the context of non self-adjoint operators. In its simplest formulation it provides a correspondence between any eigenvalue λ of the operator $H = H_0 + V(x)$ and the eigenvalue -1 of the operator $V|V|^{-1/2}(H_0 - \lambda)^{-1}|V|^{1/2}$, often referred to in literature as the *Birman-Schwinger Operator*. In the same paper, the most significant consequence which followed simply from an application of the Birman-Schwinger principle was the extension of a classical real Keller-type estimate for the Laplacian to complex potentials $V(x) \in L^1(\mathbb{R})$. In particular Davies proved that, under only the integrability assumption, any *non positive real* eigenvalue λ of the operator $-\partial_{xx}^2 + V(x)$ defined on the whole real line must lie in a disk, i.e.

$$|\lambda|^{1/2} \leq \frac{\|V\|_1}{2}.$$

The paper [2] can be considered in many respect a milestone in the context of spectral analysis of non self-adjoint perturbation. The Birman-Schwinger principle can be indeed regarded as the most powerful tool that mathematicians have now available for the analysis of the distribution of complex eigenvalues in combination with resolvent estimates. The first chapter of this thesis will be completely dedicated to the study of such problems.

We start with some classical results on spectral analysis which are needed to set up properly the spectral problem we want to study in the non self-adjoint context. This is really a preliminary section and the majority of the statements will be given without proof. Subsequently we present the main results, up to date in the literature on complex perturbations, mostly about the Schrödinger operator, showing the link

which exists between the geometry of the region where the complex eigenvalues lie in the complex plane and the regularity of the operator's kernel. It will be discussed, in particular, the use of the *uniform Sobolev estimates* first proved in 1987 by Kenig, Ruiz and Sogge and their more recent refinements, formulated in terms of Schatten norms, have produced progressively improved estimates on the localisation of complex eigenvalues. The two following sections in the same chapter are dedicated to original results.

In Section 3 we study a family of a one-dimensional Schrödinger operator defined on the half line endowed with an inverse square potential

$$H_{0,\nu} : u(x) \to \left(-\frac{d^2}{dx^2} + \frac{(\nu^2 - 1/4)}{x^2}\right) u(x).$$

The main results of this section are contained in Theorem 1.21 and Theorem 1.22. They both provide a description of the region in the complex plane where eigenvalues for different values of $\nu > 0$ of the perturbed problem lie, for a complex potential respectively in $L^1(0,\infty)$ and some weighted $L^p(0,\infty)$ spaces with $p > 1$.

The presence of weights in the estimates is directly connected to the classical decomposition of the multi-dimensional radial Laplacian operator into a direct sum of one-dimensional operators defined on the half line of the type of $H_{0,\nu}$, for opportune values of ν. As we will see in more detail at the end of that section, some numerical evidences suggests that the region $\mathcal{S}_d^r$, where complex eigenvalues lie for the radial Laplacian in $\mathbb{R}^d$ with $d \geq 2$, depends in fact not trivially on the angle that eventually the eigenvalue spans with the real positive axes. Therefore, we infer that the shape of $\mathcal{S}_d^r$ does not resemble any more a circle as it happens in Davies' example. Instead, for all integer dimensions, the shapes seem to look all alike the typical droplet shape, which was in fact firstly found in [62] in dimension one for the operator $H_{0,1/2}$ defined on the half positive line with the Dirichlet condition at the origin, as shown in Figure 1.

We note that shapes different from the circle one for the region where the complex eigenvalues might lie have been observed also in other works. We mention the work of Davies and Nath [36] which has been the first result showing consistently such characteristic and the paper of Enblom [47] where the operator $H_{0,1/2}$ is studied in the Banach space $L^p(0,\infty)$.

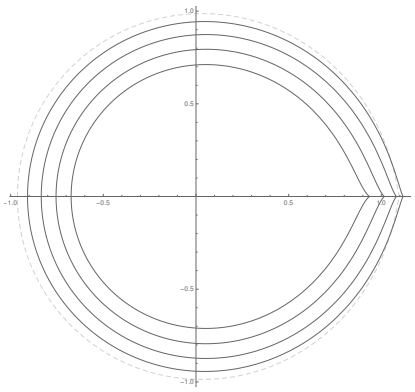


Figure 1: Continuous line: from the outer most, the plot of the regions $\mathcal{S}_d^r$ for $d = 3, 4, 5, 6$. Dashed line is the limit case, $d = 2$.

Another relevant example of such phenomenon is contained in Section 4. There, we study the complex perturbations of the second order differential operator defined on $L^2(\mathbb{R}^2)$ which comes from the formulation of the bilayer-graphene' Hamiltonian. This section is based on the results contained in [141]. In the recent past, graphene and agglomerations of it in several layers stacked together have reached a worldwide spotlight. Its promising applications in technology in fact made it one of the most studied and fashionable material of the last decade, attracting a number of researchers which is likely to increase in future. Thus the relevance of our study.

In particular, the operator of the bilayer graphene reads as

$$D_m = \begin{pmatrix} m & 4\partial_{\bar{z}}^2 \\ 4\partial_z^2 & -m \end{pmatrix}, \qquad \partial_{\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big), \quad \partial_z = \overline{\partial_{\bar{z}}}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

The main results of this section on the localisation of complex eigenvalues are in Theorem 1.25 and Theorem 1.27. Recently, improved results for the same operator have been produced by Cuenin [30] using an adaptation of the uniform Sobolev estimates for the operator D_m. As mentioned earlier, these uniform estimates have been initially used to improve the localisation results for the Laplacian. Differently to what happens for the two dimensional Schrödinger operator, where the lower end point in the uniform estimates has to be excluded, this need not to happen for the bilayer graphene operator. There results will be recalled in Section 1.4.4.

The final section contains a brief introduction on the problem of counting the number

of eigenvalues which are generated by complex perturbations.

We leave the field of complex perturbations of Hermitian operators to move to a more classical argument in spectral analysis.

The problem that will be addressed in the second chapter is the existence of trapped modes for a model of two layers of *Rotating Shallow Water* equation. From a physical point of view, such model is extensively used as a good approximation for the atmospheric and oceanic motion of fluids at mid-latitudes. It is employed specifically in models where the spatial scales of longitude and latitude are of several orders of magnitude bigger than the typical depth dimension. For what concerns the Earth's rotation effects, they come into play when the fluid motions evolve with a time scale that is comparable or, as in our case, bigger than the time scale of the Earth's rotation. For example, at mid-latitudes the Coriolis rotation frequency is of the order of $10^{-4}s^{-1}$. With scales of motion of 10^6m and maximal wind velocities of the order $10m/s$, the time scale of the fluid motion dynamic is 10^5s, of the order of days. Therefore, in this case, the effects of the earth's rotation are important in the model [114]. The time scale regime is then translated in our analysis in a constraint for the spectral parameter, as done in equation (2.21).

The principal result of this chapter is contained in Theorem 2.5 and its proof relies heavily on the one layer case [87] where the existence of trapped modes for a model of a single layer RSW equation was proved under simple geometric assumptions of the waveguide, in particular on the curvature which has to satisfy some integral conditions. Under the same assumptions and by means of similar techniques used for the single layer case, it is possible, in the particular regime where the difference of the densities in the two layers are small enough, to extend the result of existence of trapped waves also in the case of two layers.

The existence of trapped wave solutions of such problems is in fact linked to the existence of points in the discrete spectrum for a second order, self-adjoint, operator pencil restricted to an unbounded region $\mathcal{S} \subseteq \mathbb{R}^2$. In particular, we study the spectral

problem for the operator pencil $\mathcal{A}_\gamma$

$$\mathcal{A}_\gamma(\omega)\begin{pmatrix}\psi\\ h\end{pmatrix}=0,$$

where

$$\mathcal{A}_\gamma(\omega)\begin{pmatrix}\psi\\ h\end{pmatrix}:=\mathcal{L}_\gamma\begin{pmatrix}\psi\\ h\end{pmatrix}-\frac{1}{\omega}\mathcal{M}_\gamma\begin{pmatrix}\psi\\ h\end{pmatrix}$$

with $\mathcal{L}_\gamma$ and $\mathcal{M}_\gamma$ two matrix valued differential operators definined for $(\xi,\eta)\in\mathbb{R}\times[0,\delta]$

$$\mathcal{M}_\gamma:=\frac{\alpha(\eta)}{p(\xi,\eta)}\begin{pmatrix}f & H_1G\\ \frac{f^2}{G}\frac{1}{H_2} & f\frac{H_1}{H_2}\end{pmatrix}(-i\frac{\partial}{\partial\xi}),$$

$$\mathcal{L}_\gamma:=\begin{pmatrix}-\frac{1}{p}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{p}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(p\frac{\partial}{\partial\eta}\right)\right]+\alpha(\eta)\frac{\partial}{\partial\eta} & 0\\ 0 & -\frac{1}{p^2}\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\eta^2}+\frac{1}{p^3}\frac{\partial p}{\partial\xi}\frac{\partial}{\partial\xi}-\left(\frac{1}{p}\frac{\partial p}{\partial\eta}+\alpha(\eta)\frac{H_1}{H_2}\right)\frac{\partial}{\partial\eta}+\lambda^2(\eta)\end{pmatrix},$$

and $p(\xi,\eta)=1+\eta\gamma(\xi)$, where $\gamma(\xi)$ is the curvature of the waveguide.

We observe that the condition on the two densities of the fluids to be very close, which is fundamental in our proof, has already appeared naturally in the study of multilayer shallow water equations. The same smallness condition in fact has been imposed in the classical works of Allen [5], where trapped waves were studied for a two layer shallow water regime in straight channel and later extended by Mysak [125], to whom we refer for a compendium on stratified and multi layered shallow water models.

In particular, we consider in detail the case where the operator is defined on a region which is a non intersecting strip of infinite length and finite width. For historical reasons, usually the term adopted in literature to refer to such type of regions is *waveguide*. This terminology is rooted to problems formulated in the classical theory of acoustics and electromagnetism that originated from the study of the dynamics of waves in channels. In particular, the existence of points in the discrete spectrum when applied to such type of problems is related to existence of standing waves, namely to solutions of the original dynamical problem of the form

$$\Psi(x,y,t):=\psi(x,y)\,\zeta(t).$$

where $(x, y) \in \mathbb{R}^2$ and $\zeta(t)$ is a periodic function in time. The literature already available in this field is extensive and these type of problems turned out to be relevant in many different contexts. They have found applications, for example, to the design of optoelectronic circuits in two-dimensional photonic crystal waveguides [120] in engineering, in natural sciences have been applied to the description of the electromagnetic interaction between the ionosphere and the Earth [131] and more in fluid dynamics to the theory of inviscid fluids flowing in a channel [37, 49]. We have mentioned here only few examples of where the spectral theory for operators defined on waveguides is relevant with the aim of giving the reader a glimpse of the vastness of applications in the scientific production and the relevance of the topic. Despite being set up in various contexts, the majority of these problems when voided of their physical meanings reduce to the study of very similar operators and they only differ from the physical interpretation given to the spectral parameter.

Of course, in such settings, boundary conditions for the operator have to be introduced on $\partial\mathcal{S}$. In the examples cited above, the operator in question is often the Laplacian, or little modification of it, with imposed boundary conditions of type of Neumann or mixed Neumann-Dirichlet. What are the respective structures of the spectrum in these cases? Let us consider firstly the simplest case consisting in channel with constant null curvature (a straight channel). By separation of variables it is an easy computation to show that, regardless of the boundary conditions imposed, the spectrum of the Laplacian consists only of its continuous part which is typically a left-bounded interval of the real line which extends to positive infinity. Therefore, modification of this setting should be sought in order for the discrete spectrum to appear.

In literature, the existence of eigenvalues has been established under different circumstances. They vary from geometric assumptions, for example local deformations of the boundary [53], introduction of obstacles inside the channel in the case of acoustics waveguides [37] and sheared waveguides [21]. Further modifications can be introduced when modelling impurities by mean of potentials [51] which are usually of the type of delta interactions. We refer to the paper of Krejčiřík and Kříž [98] and to the reference therein for a richer presentation of the topic, a more detailed descriptions on the cases mentioned so far along with some additional modifications, and for methodologies which have not been mentioned above, responsible again for a non empty point spectrum.

Recent developments in the study of superconductivity in nano-materials and physics of crystals have lead the attention of the mathematicians community to the study of the case of pure Dirichlet boundary conditions too. Those, arise for example as natural conditions for the wavefunctions of Schrödinger operator when describing a model of two semiconductors made of different materials [74], interacting at theirs boundaries. In this case, by analogy of the names (acoustic-electromagnetic) given in the Neumann case, these waveguides are called *quantum waveguides*.

A very important result, firstly formulated by Exner and Šeba [52] and subsequently extended by a number of different authors, is the existence of discrete spectrum for the Dirichlet Laplacian under the sufficient condition of any non trivial shape of the waveguide. This purely geometric condition on the boundary and in particular on its curvature, does not emerge in fact for the Neumann cases. The differences between the Dirichlet and Neumann Laplacian are also in the nature of the eigenvalues: while in the former case they lie outside the essential spectrum, in the latter this does not happen [7, 49] and the stability of such point is a delicate matter. As an in between situation, the case of mixed boundary condition was studied is [98]. There, it was proven the existence of trapped modes in the case when the strip is bent in the 'direction of the Dirichlet boundary'.

The second chapter is organised as follows: in the first section we briefly recall some definitions and results for operator pencils. Subsequently we introduce the equation governing the fluid motion along with the physical and geometric constraints firstly for a straight channel and after for a general admissible geometry of the channel. In Section 6 we then introduce the operator pencil, we study its essential spectrum in Section 7 and finally produce our main result, Theorem 2.5, in Section 8 of the same chapter.

The third chapter deals with the existence of a particular class of complex solutions for the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

of the family of *Complex Complexiton* solutions, which are found by the so called

Wronskian method,

$$u(x,t) = -\frac{\partial^2}{\partial x^2} \ln W(\cosh(k_1 F(k_1,x,t)), \sinh(k_2 F(k_2,x,t)), \ldots, \\ \cosh(k_{2N-1} F(k_{2N-1},x,t)), \sinh(k_{2N} F(k_{2N},x,t))),$$

where $F(k,x,t) = (x - 4k^2 t)$ and $k_i \in \mathbb{C}$ for $i = 1, \ldots, 2N$.

We regard firstly these solutions as potentials for a spectral problem for the one dimensional Laplace operator. In particular these potentials are isospectral in time, and their shapes are exactly the shapes of the complexiton solutions for the KdV equation that we aim to study.

We study, for $t = 0$, the discrete spectrum which originates from such perturbations and we prove that any potential of the type of $u(x,0)$ is *reflectionless* for any complex choice of $2N$ distinct waveumbers. This result extends to any real time then from the isospectrality property.

We also consider the particular case when $k_{2i} = \overline{k_{2i-1}}$ for $i = 1, \ldots, N$

$$V(x,t) = -\frac{\partial^2}{\partial x^2} \ln W(\cosh(k_1 F(k_1,x,t)), \sinh(\overline{k_1} F(\overline{k_1},x,t)), \ldots, \\ \cosh(k_N F(k_N,x,t)), \sinh(\overline{k_N} F(\overline{k_N},x,t))),$$

These solutions appear to be in many respects the complex counterparts of the classical multisoliton solutions, which were initially introduced in [103] and [66] for two and N-real interacting solitons respectively. We study the main characterising properties of such complex solutions like localisation and boundedness for all real times, which in particular suggests the comparison between these solutions and their real multi-solitons counterparts. In fact, we will not be able to provide a complete proof for boundedness and localisation in the general case and we will leave it as an open conjecture, supporting it with numerical examples.

It is impossible to mention the KdV equation without briefly recalling its historical origin and its indissoluble bound with the developments of a theory for the solitons. It was, in fact, the 1834 when firstly J. S. Russell observed and brought up to the attention of the scientific community with the name of *Great Primary Wave of Translation* a certain type of water waves which were able to travel for very long distance

in a straight channel without disappearing or changing shape. The recognition of the shape of such waves in terms of hyperbolic functions $\eta(x,t) = a\cosh^{-2}(b(x-ct-x_0))$ which followed, was a result of works of Airy, Boussinesq and Rayleigh, but it was only in the 1895, when two Dutch mathematicians, Korteweg and his student de Vries formulated explicitly the equation which was then named after them, which describes the time evolution of a one dimensional small amplitude surface gravity wave in the shallow water context. They also proved that a class of solutions for this equation is given by the *cnoidal waves*, which are expressed in terms of the $cn(x;m)$ function*.

The KdV equation consists in fact of two qualitatively different terms. The first and non-linear one, is the term which appears in the inviscid Burger equation, whose solutions' main feature is to come to a shock in finite time while the other term is responsible for the dispersive effect. The balance of these two opposite tendencies, the former which shrinks while the latter that stretches out the shape of the solution, is at the origin of the soliton-like solution existence.

For a long time, the interest for the KdV equation was only limited to the field of fluid dynamics and the study of surface waves. The study of Zabusky and Kruskal [159] in the 1965, invested the KdV equation of new meanings, giving a decisive impulse to the study of non-linear and dispersive equations. The KdV equation was in fact found by Zabusky and Kruskal as the continuous limit of the Fermi-Ulam-Pasta lattice model† and it was in this context that the expression soliton was introduced firstly. They also pointed out the main features which characterise these solutions: they noted that these waves can only travel rightward and that the speed increases with the amplitude of the waves. They also "collide elastically", i.e. they restore the original shape after a short period of interaction whose only effect, probably due to the non-linear term, is that the faster wave is capable to reach a further position which would not in the absence of any interaction. The last phenomenon goes under the name of *phase shift*.

Despite the word soliton contains the suffix -*on* which is usually utilised in particle

*For details on $cn(x;m)$, the cosine elliptic function of modulus m we refer to Abramowitz [3]. We note here that for $m \to 1$, such functions are good approximations of the hyperbolic $\cosh^{-2}$

†The Fermi-Ulam-Pasta lattice is a one-dimensional system which consists of a sequence of springs of the same type whose elasticity property is ruled by a non linear version of the Hooke law. If the force exerted is supposed to be of the form $F = -k(\Delta + \alpha\Delta^p)$, where Δ is the displacement form the equilibrium position for each spring, then in the continuous limit one derives the KdV and the mKdV (modified-KdV) respectively for $p = 2$ and $p = 3$.
The mKdV reads as $u_t + 6uu_{xx} + u_{xxx} = 0$.

physics nomenclature and which suggests the interpretation of these objects in that context, in fact the question of what happens during the interaction of such waves is not come yet to a conclusive answer. Different hypothesis have been advanced to explain the phenomenon depending whether the waves are classically considered massive objects and so they bounce off each other elastically or, more recently, whether the solitons should be considered capable to cross each other by mean of exchange of energy [77], mass [121] or energy-mass [18]. We note that different interpretations correspond, in general, to different exact multi-soliton formulae proposed in literature. We refer to the papers [11] and [76] for a detailed introduction and a review of different definitions.

Since being brought to new life by Zabusky and Kruskal, the KdV equation has found application so far in a vastness of different fields, which spans water waves, ion-acustic waves in plasma (where u represents the density of the plasma) as well as in non-linear optics, biology and telecommunication in the study of signals' transmission through fibre optics. For more details, in particular on the latter application, and for other reference we refer to Turitsyn and Mikhailov [155].

We are interested in the case when the solutions of the KdV are complex. Separating the real and imaginary parts $u(x,t) = p(x,t) + iq(x,t)$ in the equation, we obtain the following coupled system for real quantities

$$\begin{cases} p_t + 6(pp_x - qq_x) + p_{xxx} & = 0, \\ q_t + 12(p_x q + pq_x) + q_{xxx} & = 0. \end{cases}$$

Complex versions of the KdV equation like the one just presented and more general versions of coupled KdV-like systems together with their solutions have been recently attracting interest from a variety of different disciplines. For example, in [121] a coupled KdV system has been used to address the problem of the soliton's collision mentioned earlier. Other generalisations of the complex KdV equation arise likewise in physically relevant systems such as the theory of water waves, e.g. in the case of irrotational systems [108, 107], or for two-wave modes in a shallow stratified liquid [69] and in the physics of plasma for the case of Bose-Einstein condensates [20]. Not only, the complex KdV equation has found relevant applications also in the physics of atmospheric systems [110], or in nano-physics for example in the model of growth

of a crystal structure [93]. More recent experimental observations of complex solitons with real energies in optical systems can be found in the context of $\mathcal{PT}$-symmetry and complex deformations of integrable equations [23, 27].

The third chapter is structured as it follows: we start by introducing the Lax's formulation of the KdV equation and the associated spectral problem. This gives us the chance to introduce the terminology related to the KdV's solutions, in particular we will recall the different types of solution obtained in literature with the method of the Wrosnkian. In the second section we introduce the class of solutions we aim to study and we discuss their boundeness and localisation properties in time, along with numerical simulations which explain the phase shift phenomenon also happening in the complex case. In the last section of the same chapter we study the spectral problem which arise for such complex perturbations of the Laplace operator.

In the last chapter we will summarise the results presented in this document and present new ideas for future works. We will also take the chance to gather some of the attempts tried which lead not to any satisfactory results. In particular, we will introduce in more detail the problem of counting the complex eigenvalues of complex perturbations of the operator $H_{0,\nu}$ and we will try to extend a result valid for the case $\nu = 1/2$ already present in literature. Subsequently, we will move to the topic of complex solutions of the KdV and there will be shown different approaches that can lead to a proof of the absence of singularities for the complexiton solutions introduced in the third chapter. All of them involve the study of a particular Wronskian.

1

Uniform Eigenvalue Bounds for complex perturbation

In this chapter we study the problem of localisation of complex eigenvalues for non-Hermitian perturbation of self-adjoint operators. The results here are obtained using the Birman-Schwinger principle in conjunction with estimates of the operator's kernels. The original results contained in Section 4 have been published in [141] while the new results contained in Section 3 have been submitted and collected in the paper [56].

1.1 The non self-adjoint problem

In this section we provide the tools needed in order to define the spectral problem for a non self-adjoint operator. These classical results are well known in literature and can be found in several textbooks. In order to preserve the consistency of the document and to facilitate the readability of it we privilege the approach provided in Kato's monograph [89] chapters III - VI. Useful references are as well the textbooks of Birman [15], Edmunds [44], Gesztesy [71] and Schmudgen [145]. We begin by recalling some basic definitions and properties of the spectral sets for closed operators. There follow

some stability results of the spectral sets and the definitions of sectorial operators and forms. We conclude this introductory section by discussing the Birman-Schwinger principle in its most general formulation.

1.1.1 The spectrum of a closed operator: terminology and essential properties

We start by fixing the notation and introducing very basic concepts of spectral sets and their fundamental properties. In the following, we will always assume T to be a closed operator defined on a Hilbert space $\mathcal{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle$.

Definition 1.1. An operator $(T, \mathcal{D}(T))$ defined on $\mathcal{H}$ is said to be *closed* if for any Cauchy sequence $\{u_n\}$ of functions in the operator's domain such that $u_n \to u \in \mathcal{D}(T)$, the sequence $\{Tu_n\}$ is also convergent and its limit is Tu.

Let's consider a linear operator $(T, \mathcal{D}(T))$ defined on a Hilbert space $\mathcal{H}$. Then the *range* and respectively the *kernel* of the operator T are the sets

$$\begin{aligned} \mathrm{Ran}(T) &:= \{Tu \in \mathcal{H} \mid u \in \mathcal{D}(T)\} \\ \mathrm{Ker}(T) &:= \{u \in \mathcal{D}(T) \in \mathcal{H} \mid Tu = 0\} \end{aligned}$$

We also recall the definition of the nullity and the deficiency number for a linear operator.

Definition 1.2. Given a closed linear operator T the *nullity* and the *deficiency* numbers of T are respectively the dimensions of the kernel $\mathrm{Ker}(T)$ and the dimension of the subspace $\mathcal{H}/\,\mathrm{Ran}(T)$, the cokernel of the operator T,

$$\begin{aligned} \mathrm{nul}(T) &:= \dim \mathrm{Ker}(T), \\ \mathrm{def}(T) &:= \dim \left(\mathcal{H}/\,\mathrm{Ran}(T)\right). \end{aligned}$$

The *index* of the operator T will be then defined as

$$\mathrm{ind}(T) := \mathrm{nul}(T) - \mathrm{def}(T).$$

We recall that the *numerical range* of an operator T is the (convex) set of the complex

plane defined as

$$\mathrm{Num}(T) := \{\langle Tu, u\rangle \mid u \in \mathcal{D}(T), \|u\| = 1\},$$

and introduce the following notation when referring to the resolvent of an operator T at the point z

$$R_H(z) = (T - z)^{-1}$$

Definition 1.3. A complex number $z \in \mathbb{C}$ is called a *quasi-regular point* for the operator T if there exists a number $c > 0$, which might depend on z, such that

$$\|(z - T)u\| \geq c\|u\| \qquad \text{for all } u \in \mathcal{D}(T).$$

The set of the quasi-regular points of T is often referred to as the *quasi-regularity domain* of T and it will be denoted by

$$\hat{\rho}(T) := \{z \text{ is a quasi-regular point of } T\}.$$

For any point $z \in \hat{\rho}(T)$ we define the function $d(T, z) := \mathrm{def}(T - z)$. If $z_0 \in \hat{\rho}(T)$ is such that $d(T, z_0) = 0$, then z_0 will be called a *regular point*.

We collect in the following some of the main properties of the regularity domain. For all the proofs of the following propositions we refer the interested reader to Schmudgen's textbook [145], Chapter 2.

Proposition 1.1. *Let T be a linear closed operator on $\mathcal{H}$ and $z \in \mathbb{C}$.*

(i) $z \in \hat{\rho}(T)$ if and only if $R_H(z)$ is a bounded operator, $R_H(z) \in \mathfrak{B}(\mathrm{Ran}(T - z))$, namely $T - z$ has a bounded inverse defined on the closed subspace $\mathrm{Ran}(T - z)$.

(ii) $\hat{\rho}(T)$ is a open subset of $\mathbb{C}$.

(iii) $\hat{\rho}(T)$ admits a decomposition into a union of open connected components $\hat{\rho}(T) = \cup_{n\in\mathbb{N}}\Delta_n$, and the function $d(T, z)$ is constant on each Δ_n.

(iv) If $z \in \mathbb{C} \setminus \overline{\mathrm{Num}(T)}$, then $z \in \hat{\rho}(T)$.

We finally recall the definitions of the resolvent set and the spectrum for a closed linear operator.

Definition 1.4. The *resolvent* set of a closed operator $(T, \mathcal{D}(T))$ is defined as the open set

$$\rho(T) := \{z \in \mathbb{C} \mid (z - T)^{-1} \text{ exists, is defined on the whole } \mathcal{H} \text{ and } \|(z - T)^{-1}\| < \infty\}.$$

Its complement in the complex plane is called the *spectrum* of the operator T and is clearly a closed set

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

We observe that from the definition above the closeness condition imposed to the operator is a necessary condition on the operator in order to have a meaningful definition of the resolvent set. In fact, it follows from the closed graph theorem that if T is not closed then the resolvent set reduces to the trivial case of the empty set.

The following provides a characterization of the resolvent set in terms of nullity and deficiency number. It is an immediate consequence of Proposition 1.1 and again of the Closed Graph theorem.

Proposition 1.2 (IV.5.2 Kato [89])**.** *Let T be a closed operator, then*

$$\rho(T) = \{z \in \mathbb{C} \mid \text{nul}(T - z) = \text{def}(T - z) = 0\}$$

Therefore from Proposition 1.2, the resolvent set of on operator T is the set of points $z \in \mathbb{C}$ such that $\text{def}(T - z) = 0$ and it coincides with the set of *regular points* defined in Definition 1.3.

Definition 1.5. An operator T is said to be compact if and only if T transforms every weakly convergent sequence into a strongly convergence sequence. The set $\mathfrak{S}_\infty$ of compact operators on the Hilbert space $\mathcal{H}$ is a closed subspace in the space of all bounded operators.

In this chapter we will focus solely on the study of the spectrum of non self-adjoint spectral problems, in particular on the localisation of complex eigenvalues which arise due to perturbations of the type of potential. A particularly important role in the theory of perturbed operator is played by the class of Fredholm operators, which will be defined in Definition 1.6. As it will be shown in the following, they provide a useful

tool for the characterisation of the part of the spectrum which stays stable under some appropriate small perturbations introduced in Definition 1.7.

Definition 1.6. An operator T is said to be *Fredholm* if its range $\mathrm{Ran}(T)$ is closed and both the nullity and deficiency numbers are finite, $\mathrm{nul}(T), \mathrm{def}(T) < \infty$.
Similarly, an operator T is said to be *semi-Fredholm* if its range $\mathrm{Ran}(T)$ is closed and at least one of the nullity and deficiency numbers are finite.

Definition 1.7. An operator T is said to be *relatively compact with respect to* T_0, or simply T_0-compact if $\mathcal{D}(T_0) \subset \mathcal{D}(T)$ and for every bounded pair of sequences $\{f_n\} \subset \mathcal{D}(T_0)$ and $\{T_0 f_n\}$, the set $\{T f_n\}$ is precompact, namely there exists a subsequence which converges in norm.

What follows is a useful characterisation for the T_0-compactness.

Proposition 1.3. *A linear operator T is T_0-compact if and only if $\mathcal{D}(T_0) \subset \mathcal{D}(T)$ and the composition $T(T_0 - z_0)^{-1}$ is compact for some $z_0 \in \rho(T_0)$.*

We note that the previous characterization extends easily to *any* $z_0 \in \rho(T_0)$ by means of the first resolvent identity

$$(T - z_1)^{-1} - (T - z_0)^{-1} = (z_0 - z_1)(T - z_1)^{-1}(T - z_0)^{-1}.$$

In other words, the T_0-compactness is equivalent to have T_0-boundedness with T-bound equal to zero. In the following we see how the T_0-compactness preserves both closability and the property of an operator of being (semi)-Fredholm.

Theorem 1.4 (IV.1.11 Kato [89])**.** *Let T and T_0 two linear operators defined on the Hilbert space $\mathcal{H}$ and let T to be T_0- compact. If T_0 is closable, $S = T + T_0$ is also closable, the closure of T_0 and S have the same domain and T is also S- compact. In particular S is closed if T is closed.*

Theorem 1.5 (IV.5.26 Kato [89])**.** *Let T_0 be a closed semi-Fredholm operator. If T is a T_0-compact operator then the closed operator $S = T + T_0$ is also semi-Fredholm with* $\mathrm{ind}(S) = \mathrm{ind}(T_0)$.

In the following, we recall the definition of the essential spectrum given in the general case of a closed operator. This is in fact the part of the spectrum which will remain

stable under the perturbative additional terms. We observe that in literature there have been suggested several different notions for the essential spectrum, all of them based on different assumptions of regularity of the operator $(T - z)$. We refer to the last section of Chapter 1 in Edmunds' classical book [44], where the main ones are presented and discussed in some details.

Definition 1.8. For a closed operator T defined in a Hilbert space $\mathcal{H}$ we call the *essential resolvent* set

$$\rho_{ess}(T) = \{z \in \mathbb{C} \mid \mathrm{Ran}(T - z) = \overline{\mathrm{Ran}(T - z)} \text{ and } (T - z) \text{ is semi-Fredholm } \}.$$

Its complement in the complex plane is called the *essential spectrum*

$$\sigma_{ess}(T) := \mathbb{C} \setminus \rho_{ess}(T).$$

The essential spectrum is therefore the set of all the points z in the complex plane such that either the range of the operator $T - z$ is not closed or the range is closed but the nullity and the deficiency numbers are both equal to infinity. Of course we have the following inclusion

$$\rho(T) \subset \rho_{ess}(T), \qquad \sigma_{ess}(T) \subset \sigma(T).$$

We finally state the main stability result for the essential spectrum.

Theorem 1.6 (IV.5.35 Kato [89])**.** *The essential spectrum is conserved under relatively compact perturbations. More precisely, let T_0 be closed and let T be a T_0-compact operator. Then T_0 and $T_0 + T$ have the same essential spectrum*

$$\sigma_{ess}(T_0) = \sigma_{ess}(T_0 + T).$$

Theorem 1.4 implies that the operator $T_0 + T$ is closed and so the stability of the essential spectrum is a direct consequence of Theorem 1.5.

Remark 1.1. From Definition 1.7 of relative compactness and from the second identity for the resolvent

$$R_{(T+T_0)}(z_0) - R_{T_0}(z_0) = R_{(T+T_0)}(z_0) T R_{T_0}(z_0).$$

it follows that a sufficient condition for the spectrum to be conserved is the existence of a point z_0 in the resolvent sets of both T and T_0, $z_0 \in \rho(T_0) \cap \rho(T)$ such that the difference $(T - z_0)^{-1} - (T_0 - z_0)^{-1}$ is a compact operator.

We conclude by noting that, from the following identity

$$R_{T_2}(z) - R_{T_1}(z) = (\xi - T_2)R_{T_2}(z)\Big(R_{T_2}(\xi) - R_{T_1}(\xi)\Big)(\xi - T_1)R_{T_1}(z),$$

it follows that if a point $z_0 \in \rho(T_0) \cap \rho(T)$ is such that the difference of the two resolvent operators exists and is compact, then the same is true for any other point $\xi \in \rho(T_0) \cap \rho(T)$.

For future reference we recall the Weyl's criterion for points in the essential spectrum.

Proposition 1.7 (Weidmann [157], Thm 7.24)**.** *Let T a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then $\lambda \in \sigma_{ess}(T)$ if and only if there exists a sequence $\psi_n \in \mathcal{D}(T)$, $\|\psi_n\|_{\mathcal{H}} = 1$, such that*

i) conveges weakly to 0: $\psi_n \rightharpoonup 0$ as $n \to \infty$,

ii) $\|(T - \lambda)\psi_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$.

Remark 1.2 ([40]). If $T \geq 0$, a weaker version of Proposition 1.7 holds for $\psi_n \in \mathcal{D}(T^{1/2})$ the quadratic form domain of T, where ii) is replaced by

ii') $\|(T - \lambda)\psi_n\|_{(\mathcal{D}(T^{1/2})^*} \to 0$ as $n \to \infty$

where

$$\|\psi\|_{(\mathcal{D}(T^{1/2})^*} = \sup_{\phi \in \mathcal{D}(T^{1/2})} \frac{\langle \phi, \psi \rangle}{\langle T\phi, \phi \rangle + \|\phi\|}.$$

It remains to discuss the part of the spectrum which is not included in the essential part. The essential resolvent set is an open set and it is also in general the union of countably many components Δ_n which are connected open sets. According to Theorem IV.5.7 in Kato [89] and Theorem 3.7.4 in Birman [15], the index $\mathrm{ind}(T - z)$ as well as both the deficiency $\mathrm{def}(T - z)$ and nullity $\mathrm{nul}(T - z)$ numbers are constant

on each component Δ_n

$$\begin{aligned}\operatorname{ind}(T-z)_{|z\in\Delta_n} &:= \mathfrak{i}_n,\\ \operatorname{def}(T-z)_{|z\in\Delta_n} &:= \mathfrak{d}_n\\ \operatorname{nul}(T-z)_{|z\in\Delta_n} &:= \mathfrak{n}_n\end{aligned}$$

except for some isolated values of $z_{n,j} \in \mathbb{C}$. In the very special case of $\mathfrak{d}_n = \mathfrak{n}_n = 0$ it happens that $\Delta_n \subset \rho(T)$ and $z_{n,j}$ are isolated points in the spectrum $\sigma(T)$. Since this particular situation is often encountered when studying perturbed operators (also in the non self-adjoint context), we will refrain from providing details for the more general case. We refer again to Kato's book [89] for a thorough examination and examples of operators whose essential spectrum reveals a very complicate form. It will not be the case of our analysis, which we will be always restricted to operators with essential spectrum contained in the real line.

Definition 1.9. Let T a closed operator, and let $z \in \mathbb{C}$ such that $z \in \sigma(T)$ is an isolated point. Then we define the *Riesz projection* of T with respect to z as

$$P_T(z) = \frac{1}{2\pi i}\int_\gamma (T-z)^{-1}$$

where the contour γ is a closed counterclockwise oriented curve which encloses the point z and no other points of the spectrum of T.

Definition 1.10. Let T be a closed operator. A point of the spectrum $z \in \sigma(T)$ is called an *eigenvalue of finite type* if it is isolated and if $\dim \operatorname{Ran}(P_T(z)) < \infty$. We also introduce two functions, $m_a(z)$ and $m_g(z)$, which count respectively the algebraic and the geometric multiplicity of each eigenvalue.

> The *algebraic multiplicity* of an eigenvalue z is the dimension of the range of the associated Riesz projection, $m_a(z) = \dim \operatorname{Ran}(P_T(z))$.
>
> The *geometric multiplicity* of an eigenvalue z is the number of linearly independent eigenfunctions, $m_g(z) = \dim \operatorname{Ker}(T-z)$.

From the fact that $\operatorname{Ran}(P_T(z)) = \bigcup_{n\in\mathbb{N}} \operatorname{Ker}((z-T)^n)$, it easily follows that in general

$$m_g(z) \le m_a(z).$$

The set

$$\sigma_d(T) := \{z \in \mathbb{C} \mid z \in \sigma(T) \text{ and } z \text{ is an eigenvalue of finite type}\}.$$

is called the *discrete spectrum* of T.

Proposition 1.8. *Let T be a closed operator. Consider an open and connected set $\Omega \subset \mathbb{C} \setminus \sigma_{ess}(T)$. Suppose furthermore that it has non-empty intersection with the resolvent set, $\Omega \cap \rho(T) \neq \emptyset$. Then the points in Ω which are also part of the spectrum must be eigenvalues of finite type*

$$\sigma(T) \cap \Omega \subset \sigma_d(T).$$

A proof of the previous proposition can be found in Gohberg [72] Theorem XVII.2.1. We note that a sufficient condition for the previous theorem for operators such that their essential spectrum is entirely contained in the real line is the existence in both the upper and the lower half complex plane of points of the resolvent $\rho(T)$. In our analysis, every operator which will be studied will have its essential spectrum entirely contained in the real line. What follows is a characterization of the essential spectrum in terms of $\dim \mathrm{Ran}(P_T(z))$.

Proposition 1.9. *Let T be a closed operator and let $z \in \sigma(T)$ be an isolated point. Then $z \in \sigma_{ess}(T)$ if and only if its algebraic multiplicity is $m_a(z) = \infty$.*

Remark 1.3. From the characterization of isolated points in the essential spectrum in Proposition 1.9 and the definition of the discrete spectrum given in Definition 1.10 it follows immediately that

$$\sigma_{ess}(T) \cap \sigma_d(T) = \emptyset.$$

While in general for selfadjoint operators T_0 it holds true that the decomposition of the spectrum falls into two disjoint components

$$\sigma(T_0) = \sigma_{ess}(T_0) \dot{\cup} \sigma_d(T_0)$$

this is not in general true in the non-selfadjoint case. See for instance the illustrative example provided by Kato [89] in IV.5.24. Nonetheless the disjoint decomposition of

the whole spectrum still follows in our case due to Proposition 1.8 and the fact that we will always verify the hypothesis of $\sigma_{ess}(T_0) \subset \mathbb{R}$.

1.1.2 Sectorial forms and operators

We begin with recalling the property of sectoriality for quadratic forms and closed operators defined on a Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$. In the following, we will consider $S_{c,\theta}$ to be a sector in the complex plane with vertex $c \in \mathbb{R}$ and half-angle $\theta \in \mathbb{R}$

$$S_{c,\theta} = \{z \in \mathbb{C} \mid |\arg(z - c)| \leq \theta\} \subset \mathbb{C}.$$

Definition 1.11. A quadratic form $(q, \mathcal{D}(q))$ on $\mathcal{H}$ is said to be *sectorial* if there exists a real vertex $c \in \mathbb{R}$ and a half-angle $\theta \in [0, \pi/2)$ such that its numerical range is contained in the sector $S_{c,\theta} \subset \mathbb{C}$

$$\mathrm{Num}(q) := \{q(u) \mid u \in \mathcal{D}(q), \|u\|_{\mathcal{H}} = 1\} \subset S_{c,\theta}.$$

Definition 1.12. A closed operator $(T, \mathcal{D}(T))$ defined on $\mathcal{H}$ is said to be *m-sectorial* if T is sectorial and T is quasi accretive, that is, if there exists $c \in \mathbb{R}$ and $\theta \in [0, \pi/2)$ such that

$$\mathrm{Num}(T) \subset S_{c,\theta}$$

and such that the operator $T - z$ is invertible with a bounded inverse and operator norm which satisfies

$$\|(T - z)^{-1}\| \leq \frac{1}{|\operatorname{Re}(z)|} \qquad \text{if } \operatorname{Re}(z) < c.$$

Note 1.4. We note that any m-accretive operator is densely defined.

The following representation theorem establishes the link between sectorial forms and sectorial operators, providing the way to construct a m-sectorial operator starting from a sectorial form. This result will be important in the following sense: it provides the right interpretation of the spectral problem

$$H_0 u + V(x)u = \lambda u$$

for non self-adjoint operators which arise from complex potential perturbations in terms of the respective sectorial forms.

Theorem 1.10 (Kato [89] VI.2.1)**.** *Let $t[\cdot,\cdot]$ be a densely defined, closed, sectorial form in the Hilbert space $\mathcal{H}$. Then there exists an m-sectorial operator T such that*

i) $\mathcal{D}(T) \subset \mathcal{D}(t)$ and $t[u,v] = \langle Tu, v\rangle$ for every $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(t)$;

ii) $\mathcal{D}(T)$ is a core of t;[*]

iii) if $u \in \mathcal{D}(t)$, $w \in \mathcal{H}$ and $t[u,v] = \langle w, v\rangle$ holds for every v belonging to a core of t, then $u \in \mathcal{D}(T)$ and $Tu = w$.

The m-sectorial operator T is uniquely determined by condition (i) and in particular we note that this implies that the numerical range of the operator T is a dense subset of the numerical range of the sectorial form t.

Remark 1.5. It follows from the sectoriality property that the resolvent set of the operator which is generated from a quadratic form as in Theorem 1.10 covers the exterior of the numerical set

$$\mathbb{C} \setminus \mathrm{Num}(T) \subset \rho(T).$$

We continue the excursus on the sectorial forms with an approximation result.

Theorem 1.11 (Kato [89] VI.3.6)**.** *Let t be a densely defined, closed, sectorial form and let t_n be a sequence of forms with $\mathcal{D}(t_n) = \mathcal{D}(t)$ such that*

$$|t[u] - t_n[u]| \leq r_n \|u\|^2 + s_n \,\mathrm{Re}(t)[u] \qquad u \in \mathcal{D}(t),$$

where the constants $r_n, s_n > 0$ tend to zero as $n \to \infty$. Then the following holds

i) The forms t_n are closed and sectorial for sufficiently large n,

ii) If T and T_n denote the m-sectorial operators associated to t and to t_n then every $\lambda \in \rho(T)$ belongs to $\rho(T_n)$ for sufficiently large n and we have that the resolvent $(T_n - \lambda)^{-1}$ converges in norm to $(T-\lambda)^{-1}$ as n tends to infinity.

[*] see III.3 Kato [89] for the definition of the core of an operator.

Remark 1.6. Note that the previous result does not guarantee any convergence for the spectrum in the case of non Hermitian operators. We refer to Kato [89] section IV.3.2 for the lower semi-discontinuity of the spectrum of closed operators.

We conclude the subsection by recalling the property of *relative boundedness* for quadratic forms and the subsequent theorem which ensures the stability of the property for a form to be sectorial under the assumption of relatively boundedness.

Definition 1.13. Let $(q, \mathcal{D}(q))$ be a sectorial form on a Hilbert space $\mathcal{H}$. A form $(p, \mathcal{D}(p))$ on $\mathcal{H}$ is said to be *relatively bounded* with respect to q or simply *q-bounded* if $\mathcal{D}(p) \subset \mathcal{D}(q)$ and there exist two non negative constants $a, b \in \mathbb{R}$ such that

$$|p(u)| \leq a\|u\|^2 + b|q(u)|, \qquad u \in \mathcal{D}(q).$$

The infimum of all constants b for which a corresponding constant a exists such that the last inequality holds, is called the q-bound of p. Note that the same definition holds if the quadratic forms are replaced by linear operators.

Theorem 1.12 (Kato [89] V.1.33)**.** *Let $(q, \mathcal{D}(q))$ be a sectorial form and let $(p, \mathcal{D}(q))$ a q-bounded form with $b < 1$ in the sense of definition 1.13. Then the form $p + q$ is also sectorial.*

1.1.3 Non self-adjoint operator generated by complex perturbations

In this subsection we address the question of how to interpret the spectral problem generated by a complex perturbation. We make use of the results introduced in the last subsection to define the spectral problem for an appropriate class of potentials in terms of sectorial forms and we study firstly the stability of the essential spectrum under such perturbations. Let us consider the problem of complex perturbation of selfadjoint operators we want to study firstly in the case where the selfadjoint operator is bounded from below.

For simplicity, let $(H_0, \mathcal{D}(H_0))$ be a non negative selfadjoint operator on a Hilbert space $\mathcal{H}$. Let $V(x)$ a complex valued potential and consider its polar decomposition $V = U|V|$ where U is the partial isometry in the polar decomposition of V. Due to the positivity of the term $|V|$, we take the square root of it which yields the following

decomposition

$$\begin{aligned} V &= V_2 V_1, \\ V_1 &= \left(U|V|^{1/2}\right), \qquad V_2 = |V|^{1/2}. \end{aligned} \tag{1.1}$$

Regarding now V_1 and V_2 as two densely defined closed operators on $\mathcal{H}$, we require $\mathcal{D}(H_0) \subseteq \mathcal{D}(V_j)$ and the additional compactness assumption

$$V_j(H_0 + \lambda)^{-1} \in \mathfrak{S}_\infty \text{ for a } \lambda \in \rho(H_0) \tag{1.2}$$

for $j = 1, 2$. It is readily seen, see for example Frank's paper [58], that condition (1.2) implies that for every $\epsilon > 0$ there is a positive constant C_ϵ such that for all $u \in \mathcal{D}(H_0^{1/2})$

$$|(V_1 u, V_2 u)| \leq \epsilon \|H_0^{1/2} u\|^2 + C_\epsilon \|u\|^2,$$

namely that the form generated by the potential V is in fact relatively bounded with respect to the one which generates H_0 and that the bound is zero.

This in turn implies, by mean of Theorem 1.10, that the quadratic form

$$q(u) = \langle H_0 u, u\rangle + \langle V_1 u, V_2 u\rangle$$

with $\mathcal{D}(q) = \mathcal{D}(H_0^{1/2})$ generates an m-sectorial operator H with form domain $\mathcal{D}(q)$ such that its operator domain $\mathcal{D}(H)$ is a dense subspace of $\mathcal{D}(q)$ and $H = H_0 + V(x)$.

In the last part of this subsection we discuss briefly the explicit example for the case $H_0 = -\Delta$ on $\mathbb{R}^d$ where $d \geq 1$ and we provide a simple description of the class of admissible potentials. We refer to the papers o [39], [58] for a thorough and more general approach.

Of course $\sigma_{ess}(H_0) = [0, \infty)$ and the operator is non-negative, therefore bounded from below. We assume that $V(x) \in L^p(\mathbb{R}^d)$ with

$$p \in \begin{cases} [d/2, \infty) & d \geq 3, \\ (1, \infty) & d = 2, \\ [1, \infty) & d = 1. \end{cases} \tag{1.3}$$

The following theorem guarantees that the operator $H = H_0 + V$ is well defined and it follows as a direct consequence of Theorem 1.11 on sectorial forms.

Theorem 1.13. *Let $H_0 = -\Delta$ and consider $V(x) \in L^p(\mathbb{R}^d)$ as in (1.3) and let V_n a compactly supported $C^\infty(\mathbb{R}^d)$ sequence of potentials with $\|V - V_n\|_{L^p} \to 0$ as $n \to \infty$. Then the operator $H = H_0 + V$ is well defined through its quadratic form and every $\lambda \in \rho(H)$ belongs to $\rho(H_n)$, where $H_n = H_0 + V_n$, for sufficiently large n. Moreover,*

$$\|(H_0 + V - \lambda)^{-1} - (H_0 + V_n - \lambda)^{-1}\| \to 0 \qquad \text{for } \lambda \in \rho(H_0 + V).$$

In fact, under the same conditions on the potential V, it also implies the stability of the essential spectrum.

We introduce now the Neumann-Schatten class for operator $\mathfrak{S}_p$ which extends the definition of the class of compact operator $\mathfrak{S}_\infty$ given in Definition 1.5. We say that a compact operator T belongs to $\mathfrak{S}_p$ for $p \geq 1$ if

$$\|T\|_p^p := \operatorname{tr}(T^*T)^{p/2} = \sum_j s_j^p < \infty \tag{1.4}$$

where the s_j are the singular values of T, namely the eigenvalues of $\sqrt{T^*T}$.

Theorem 1.14 (Simon [147] Theorem 4.1)**.** *If $f, g \in L^p(\mathbb{R}^d)$ with $2 \leq p \leq \infty$, then $f(x)g(i\nabla) \in \mathfrak{S}_p$ and*

$$\|f(x)g(i\nabla)\|_p \leq (2\pi)^{-d/p}\|f(x)\|_p\|g(x)\|_p.$$

Corollary 1.15. *Let $V \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d \leq 3$, and $p > d/2$ if $d \geq 4$. Let H_0 to be the usual laplacian $-\Delta$. Then for $\lambda \in \mathbb{C} \setminus [0, \infty)$ we have*

$$\|V(x)(\lambda - H_0)^{-1}\|_p^p \leq C(p, d)\|V\|_{L^p}^p \frac{|\lambda|^{d/2-1}}{\operatorname{dist}(\lambda, [0, \infty))^{p-1}}$$

In particular the operator $V(x)(\lambda - H_0)^{-1}$ is compact if $V \in C^\infty(\mathbb{R}^d)$ and has compact support.

It follows the stability result on the essential spectrum which we promised earlier.

Theorem 1.16. *Let us consider $H_0 = -\Delta$ and $H = H_0 + V(x)$ where $V(x) \in L^p(\mathbb{R}^d)$ is a complex valued potential such that p satisfies* (1.3)*. Then*

$$\sigma_{ess}(H_0) = \sigma_{ess}(H)$$

and

$$\sigma(H) = \sigma_{ess}(H) \dot{\cup} \sigma_d(H).$$

The fact that $\sigma_{ess}(H_0) = \sigma_{ess}(H)$ follows by Theorem 1.6 in conjunction with Remark 1.1. Consider the sequence V_n defined in Theorem 1.13, the compactness for the difference of the resolvent operators is then easily deduced from

$$\left\| \left(R_{(H_0+V)}(\lambda) - R_{H_0}(\lambda)\right) - \left(R_{(H_0+V_n)}(\lambda) - R_{H_0}(\lambda)\right) \right\| = \left\| \left(R_{(H_0+V)}(\lambda) - R_{(H_0+V_n)}(\lambda)\right) \right\|.$$

We note that the right hand side of the previous identity tends to zero by Theorem 1.13. Furthermore, Corollary 1.15 and the second resolvent identity implies that for every $n \in \mathbb{N}$ the difference $(H_0 + V_n - \lambda)^{-1} - (H_0 - \lambda)^{-1}$ is compact and compactness is stable under norm convergence. The disjointness of the discrete and the essential part of the spectrum is a consequence of Proposition 1.8.

Remark 1.7. Note that the previous results hold for any lower order perturbation of a selfadjoint, bounded from below operator H_0 as seen in Laptev and Safronov [102].

1.1.4 The Birman-Schwinger principle

In the previous section we have provided the correct interpretation of the perturbed operator $H = H_0 + V$ via sectorial forms. This classical approach relies heavily on the selfadjointness of the initial operator H_0 and on its boundedness from below, which was indeed a necessary assumption in order to obtain the perturbed operator via the generated sectorial forms. In this case, a straightforward extension of the selfadjoint version of the *Birman-Schwinger principle* [13, 146] implies that to any eigenvalue $\lambda \in \mathbb{C} \cap \rho(H_0)$ of the operator $H = H_0 + V$ there corresponds an operator

$$B(\lambda) := V_1 R_0(\lambda) V_2, \tag{1.5}$$

called the *Birman operator* which has -1 as eigenvalue, where V_1 and V_2 Note thatin the definition of the Birman operator come from the polar decomposition introduced in (1.1). Indeed, let us rewrite $(H_0 + V)\psi = \lambda\psi$ as $(H_0 - \lambda)\psi = -V\psi$, so that it follows

$$\psi = -R_0(\lambda)V\psi.$$

Setting $\phi = V_1\psi$, the identity above can be rewritten as $V_1^{-1}\phi = -R_0(\lambda)V_2\phi$ or equivalently

$$-\phi = V_1R_0(\lambda)V_2\phi.$$

In a more general setting, the requirement for the operator H_0 to be bounded from below need not be satisfied, as for example happens for the Dirac operator. It might be the case that the operator H_0 is even not selfadjoint itself! Thus, these possibilities pose a challenge on how to interpret correctly the operator $H_0 + V$ and consequently how to state the correspondent Birman-Schwinger principle. In the following we recall two theorems given in [71], where the problem of defining the operator H, its Birman-Schwinger counterpart and the relative principle has been addressed in its maximum generality. These results, as noted in [71], generalise the respective results valid in the selfadjoint case proved respectively by Kato [88] and Konno and Kuroda [95].

Theorem 1.17. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $H_0 : \mathcal{H} \to \mathcal{H}$, $A : \mathcal{H} \to \mathcal{K}$ and $B : \mathcal{K} \to \mathcal{H}$ be closed densely defined operators. Suppose that $\rho(H_0) \neq \emptyset$ and that the following hold*

i) $AR_0(z) \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ and $\overline{R_0(z)B} \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$.

ii) For some $z \in \rho(H_0)$, the operator $AR_0(z)B$ has bounded closure

$$Q(z) := \overline{AR_0(z)B^*} \in \mathfrak{B}(\mathcal{K}).$$

iii) $-1 \in \rho(Q(z_0))$ for some $z_0 \in \rho(H_0)$.

Then there exists a closed densely defined extension H of $H_0 + BA$ whose resolvent $R(z) = (H - z)^{-1}$, $z \in \rho(H)$ is given by

$$R(z) = R_0(z) - \overline{R_0(z)B}(Id_{\mathcal{K}} + Q(z))^{-1}AR_0(z) \in \mathfrak{B}(\mathcal{H}), \qquad z \in \rho(H_0) \cap \rho(H),$$

with

$$\rho(H_0) \cap \rho(H) = \{z \in \rho(H_0) \mid -1 \in \rho(Q(z))\}.$$

The previous theorem allows to define the spectral problem also for an operator H_0 not bounded from below. The following provides the generalised version of the Birman-Schwinger principle for unbounded from below operators.

Theorem 1.18. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $H_0 : \mathcal{H} \to \mathcal{H}$, $A : \mathcal{H} \to \mathcal{K}$ and $B : \mathcal{K} \to \mathcal{H}$ be closed densely defined operators which satisfy the condition of Theorem 1.17. Assume furthermore that $\lambda_0 \in \rho(H_0)$ and that*

iv) The operator $Q(z) \in \mathfrak{S}_\infty$ for all $z \in \rho(H_0)$.

Then

$$Hf = \lambda_0 f, \quad 0 \neq f \in \mathcal{D}(H), \qquad \textit{implies} \qquad Q(\lambda_0)g = -g$$

where z_0 is fixed in the condition of Theorem 1.17 and $g = (\lambda_0 - z_0)^{-1}Af$. Conversely

$$Q(\lambda_0)g = -g, \quad 0 \neq g \in \mathcal{K} \qquad \textit{implies} \qquad Hf = \lambda_0 f,$$

where $f = \overline{R_0(\lambda_0)B^*g} \in \mathcal{D}(H)$. Moreover λ_0 and -1 have the same finite geometric multiplicity and the subspaces $\ker(H - \lambda_0)$ and $\ker(Id + Q(\lambda_0))$ are isomorphic. In particular, if $z \in \rho(H_0)$, then

$$z \in \rho(H) \qquad \textit{if and only if} \qquad -1 \in \rho(Q(z)).$$

Finally, the stability of the essential spectrum

$$\sigma_{ess}(H_0) = \sigma_{ess}(H)$$

also holds.

Remark 1.8. It follows that λ is an eigenvalue of H, then any norm of the corresponding Birman-Schwinger operator $B(\lambda)$ as in (1.5) has to be greater than one.

1.2 Literature review on the Laplace operator

In this section we provide an excursus on the most representative results present in literature about uniform bounds for complex eigenvalues which arise in the spectral problem of the type

$$-\Delta u + V(x)u = \lambda u, \tag{1.6}$$

where $V(x)$ is a complex-valued potential.

In order to better understand the non self-adjoint case we will briefly recall the classical results valid for real valued potential. Let $\lambda \in (-\infty, 0)$ be a negative eigenvalue of (1.6) defined in $L^2(\mathbb{R}^d)$. Then

$$|\lambda|^\gamma \leq L^1_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, dx, \qquad \gamma \in \begin{cases} [\frac{1}{2}, \infty) & d = 1, \\ (0, \infty) & d \geq 2. \end{cases} \tag{1.7}$$

The one dimensional case of (1.7) was proved firstly by Spruch and later independently by Keller [91], who also found an explicit formula for the optimal potential which gives the equality in (1.7). The constant $L^1_{\gamma,d}$ does not depend on the potential $V(x)$ and an explicit expression for $d = 1$ is known for any $\gamma \geq 1/2$, where in particular $L^1_{1/2,1} = 1/2$, while numerical values are known for $d = 2, 3$ [22]. We refer to the classical work of Lieb and Thirring [109] for the proofs of these facts and for a discussion on the relation between the constant $L^1_{\gamma,d}$ and the optimal constant $L_{\gamma,d}$ present in the Lieb-Thirring inequality. The latter provides an estimate of the sum of the eigenvalues moments of the type

$$\sum_j |\lambda_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, dx, \qquad \gamma \in \begin{cases} \left[\frac{1}{2}, \infty\right) & d = 1, \\ (0, \infty) & d = 2, \\ [0, \infty) & d \geq 3. \end{cases} \tag{1.8}$$

Of course, we immediately observe that $L^1_{\gamma,d} \leq L_{\gamma,d}$. For results on existence, uniqueness and stability for the optimal potential which attains the equality (1.7) in the case $d \geq 2$ we refer to the paper [22] and the references therein.

The question which we aim to address in this chapter is the following:

Is it possible to formulate a uniform bound on the moments of complex eigenvalues similar to the quantitative bound stated in Equation (1.7) **for real potentials, of the type**

$$|\lambda|^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, dx, \tag{1.9}$$

where $C_{\gamma,d}$ is a constant which depends only on the dimension and the exponent γ and $V(x)$ is a complex valued potentials?

Of course, it is natural to think that an attempt which should be tried at first should be to look for estimates which extend from the real case to the complex one without any modifications in the statements. The difficulties in proving such statements lie in the fact that none of the tools used in the self-adjoint case can be reproduced for the complex case. As we shall see in further examples in this chapter, in general the passage from real to complex potential is not always straight forward and major differences may appear between statements formulated in the two different contexts.

The first result in the literature on uniform bounds on the location of complex eigenvalues was obtained by Davies and collaborators [2] for the case of the Schrödinger operator for $d = 1$ and $\gamma = 1/2$. They proved that for any non positive eigenvalue $\lambda \in \mathbb{C} \setminus [0, \infty)$,

$$|\lambda|^{1/2} \leq \frac{1}{2} \int_{\mathbb{R}} |V(x)|\, dx. \tag{1.10}$$

As we can see, the optimal constant which appears above for complex potentials is the same which is obtained in the case of real potentials. Due to the simple Green's function formulation for the one dimensional Laplacian, the proof of inequality (1.10) is in fact very brief and it relies on the Birman-Schwinger principle, ref. Section 1.1.4 (in particular Remark 1.8) and on an immediate L^∞ estimate of the kernel of the Birman operator, as defined in (1.5),

$$1 \leq \int_{\mathbb{R}^2} |V(x)| \frac{e^{-2\,\mathrm{Re}(z)|x-y|}}{4|z|^2} |V(y)|\, dx\, dy \leq \frac{1}{4|z|^2} \int_{\mathbb{R}^2} |V(x)||V(y)|\, dx\, dy,$$

where $\lambda = -z^2$.

In the same paper it was also shown that an estimate similar to (1.10) holds for any eigenvalues $\lambda \in \mathbb{C}$, namely without the restriction for the eigenvalue to be off from the positive half line, with a different constant

$$|\lambda|^{1/2} \leq \frac{3}{2} \int_{\mathbb{R}} |V(x)|\, dx. \tag{1.11}$$

The bound stated above was proved under a stronger assumption for a potential $\|V(x)e^{\gamma x}\|_{L^1}$ with $\gamma > 0$. The same assumption also guarantees the finiteness of the number of eigenvalues which is proved in the same paper using simple results on number of zeroes of analytic functions and techniques of inverse scattering. It is not surprising that such result holds under this assumption. As we will see in Subsection 1.5.1 similar type of results on the finiteness of the eigenvalues had been previously obtained under very similar assumptions by Naimark, Blashak and Gaymov, to name just a few.

Different interesting results, valid for potentials with slower decaying rate at infinity, were obtained by Davies and Nath [36] for $V(x) \notin L^1(\mathbb{R})$. They considered potentials of the type $V(x) = W(x) + X(x)$ where $W(x) \in L^1(\mathbb{R})$ is complex and integrable and $X(x) \in L^\infty_0$ is bounded, measurable and vanishing at infinity and studied the location of the eigenvalues of the operator H_p, defined by the differential expression $-d^2/dx^2 + V(x)$ and acting on $L^p(\mathbb{R})$. They constructed two different methods to estimate the spectrum. The first one works only for H_2 and gives a description of the resolvent set. The second method, instead, works for any realisation H_p and it provides a region $\mathcal{R}$ which can be computed numerically, such that $\sigma(H_p) \subseteq \mathcal{R}$ for any $p \geq 1$. The methodology applied and the scope of this paper remain an isolated attempt in literature in the direction of studying the spectral properties of such slow decaying complex potential on the real line. Nonetheless this work marks some important features, such as for example the lack of boundedness for the region where the eigenvalues for complex perturbed operators might lie. A further evident novelty is about the shape of the region where the discrete spectrum lies. In particular, Figure 1.1 shows that the distance from the origin of the eigenvalues depends upon the angle in the complex plane of the eigenvalue itself and so it suggests that if an estimate like (1.9) exists, then the constant can depend as well on the phase of λ. The same picture also shows that a qualitative different behaviour of the spectrum should

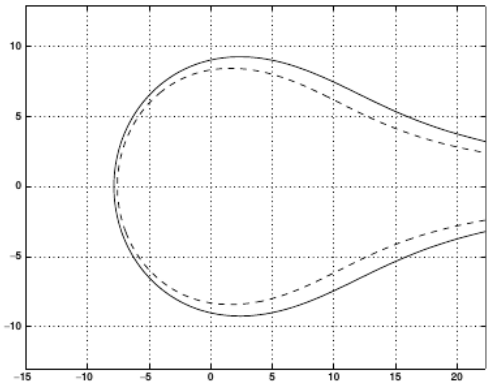


Figure 1.1: Comparison of the two methods. The continuous line is the contour of the region $\mathcal{R} \supseteq \sigma(H_s)$ whereas the dashed line describes $\sigma(H_2)$ obtained with the L^2 method. The operator is $H_s := -\frac{d^2}{dx^2} + c|x|^{-1/2}$, acting on $L^s(\mathbb{R})$ and the complex constant $c \in \mathbb{C}$ is such that$|c| = 1$.

be expected near the essential spectrum $[0, \infty)$.

A first analytical evidence of the special role that the essential spectrum plays is contained in a paper from Frank-Laptev-Lieb and Seiringer [60], where the authors prove two different versions of a Lieb-Thirring type inequality for a complex valued potential, for eigenvalues respectively in the left complex half plane and for eigenvalues inside the sector in the complex planes $\mathcal{C}_\chi := \{\lambda \mid |\operatorname{Im}(\lambda)| \leq -\chi \operatorname{Re}(\lambda), \chi > 0\}$ (see Theorem 1 and its corollary). In passing by we mention the recent paper of Someyama [148] where under the additional hypothesis for the potential of being *dilation analytic* (see assumption 2.1 in the paper) a version of the LT inequality, valid for any non positive eigenvalues, is obtained in terms of the $L^{\gamma+d/2}$-norm of the rotated potential $V(e^{\pm i\pi/4}x)$.
In the same paper [60], the authors derive bounds not only on the sum of the moments of the eigenvalues but also on single eigenvalues which we state in Theorem 1.19. These results represent a first attempt to a multidimensional generalisation of the Keller type estimates (1.10) proved in [2] in the one-dimensional case, yet they depend on the sign of the real part of λ and the statement has to be split into two cases again.

Theorem 1.19 (Frank-Laptev-Lieb-Seiringer [60])**.** *Let $\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$.*

For any eigenvalue with non-positive real part

$$|\lambda|^{\gamma} \leq 2^{\gamma/2+d/4} L^1_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, dx.$$

For any eigenvalue with non-negative real part

$$|\lambda|^{\gamma} \leq 2^{\gamma/2+d/4} L^1_{\gamma,d} \left(1 + \frac{2\,\mathrm{Re}(\lambda)}{|\,\mathrm{Im}(\lambda)|}\right)^{\gamma+d/2} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, dx.$$

where $L^1_{\gamma,d}$ is the constant defined in (1.7).

This theorem provides bounds for the real positive powers of the modulo of the eigenvalue λ in terms of the $(\gamma + d/2)$-norm of the potential $V(x)$ and a constant $C^1_{\gamma,d} := 2^{\gamma/2+d/4} L^1_{\gamma,d}$ independent from the particular potential shape. Additionally, for eigenvalues lying in the right half complex plane, the estimate involves also the term $\left(1 + \frac{2\,\mathrm{Re}(\lambda)}{|\,\mathrm{Im}(\lambda)|}\right)^{\gamma+d/2}$ which comprises real and imaginary part of λ. It turns out that this additional term does not guarantee any more the boundedness of the region in the complex plane where the eigenvalues might lie.

As already noted, the estimates provided by the LT inequality in [60] require for the eigenvalues in the right half complex plane to be away from the positive real half line, in particular those are valid for λ contained in the region $\mathcal{C}_\chi$. The eigenvalues which can possibly accumulate at the essential spectrum, require therefore a deeper investigation and were excluded at first. A detailed analysis of these eigenvalues was carried successively in a paper by Laptev and Safronov [102], where in Theorem 3 and Theorem 4, results on the accumulation rate of the eigenvalues to the essential spectrum of the operator $-\Delta + V(x)$ were provided. Furthermore, in the same paper the following uniform bound on complex eigenvalues was also proved.

Theorem 1.20. *Let $V(x) \in L^p(\mathbb{R}^d)$ a complex measurable function, where $p \geq 1$ if $d = 1$, $p > 1$ id $d = 2$ and $p \geq d/2$ if $d \geq 3$. Then every eigenvalue λ of the operator $-\Delta + V(x)$ lying in the right half complex plane satisfies the estimate*

$$|\,\mathrm{Im}(\lambda)|^{p-1} \leq |\lambda|^{d/2-1} C \int_{\mathbb{R}^d} |V(x)|^p\, dx. \tag{1.12}$$

As we can see, the estimate stated above generalises to higher dimensions the bound (1.10), which is reattained in (1.12) in the case $p = d = 1$. We observe that the presence of the term $|\operatorname{Im}(\lambda)|$ leaves (1.12) far from the optimal form as stated in (1.9), as well as it does not imply the boundedness of the region where the complex eigenvalues lie. While formula (1.12) is, in fact, not very satisfactory, this result is still worth mentioning as in its proof are consistently introduced general ideas which will be found in the subsequent literature on similar topic.

So far bounds for the Birman operator $B(\lambda)$ have been considered only in terms of its operator norm. The possibility of having improved results comes froom refined estimates formulated for operators in Neumann-Schatten classes $\mathfrak{S}_p$, in particular estimates of integral operators in terms of the Schatten norms $\|\cdot\|_p$. The main idea in the proof of Theorem 1.20 is to estimate the rightmost term in the following inequality

$$1 \leq \|B(\lambda)\| \leq \|B(\lambda)\|_p \leq \||V(x)|^{1/2}\,|-\Delta-\lambda|^{-1/2}\|_{2p}^2 \tag{1.13}$$

when $p > d/2$, by means of Simon's trace ideal estimate (see Theorem 1.14), valid for operators of the type $\beta(x)\alpha(i\nabla)$ and by the Czwikel's inequality valid for the weak Neumann-Schatten class for the case $p = d/2$. See Laptev's paper [102] and reference therein for more details on the latter case.

In order to obtain a finite region in the complex plane, Safronov [139] showed that a sufficient condition for the eigenvalues to be located in a ball of finite radius is that the potentials have to decay faster than a Coulomb potential

$$|V(x)| \leq \frac{L}{(1+|x|^2)^{p/2}}, \qquad L > 0, \quad 1 < p < 3.$$

This assumption, in particular, yields a bound for the operator norm of the Birman Schwinger operator.

Laptev and Safronov [102] conjectured the validity of the following generalisation of (1.10) to dimensions higher than one.

Conjecture 1.14. Let $d \geq 2$ and $0 < \gamma \leq d/2$ and let $V(x) \in L^{d/2+\gamma}(\mathbb{R}^d)$ be a complex valued potential. Then for any eigenvalue $\lambda \notin \mathbb{R}_+$ of the operator $-\Delta + V(x)$

$$|\lambda|^\gamma \leq C \int_{\mathbb{R}^d} |V(x)|^{d/2+\gamma}\,dx \tag{1.14}$$

where C depends only on γ and the dimension d.

It is interesting to note that the conjecture excludes real positive values of λ and furthermore forbids the case $\gamma > d/2$. The argumentation which supports such choice is quite heuristic and refers back originally to the Wigner-Von Neumann' example in combination with the possibility, at least in principle, to create complex eigenvalues by adding small enough complex perturbations to the Wigner potentials in order to push a real eigenvalue out from the essential spectrum. It turns out that, for values $\gamma > d/2$, the operator $H = -\Delta + V(x)$ may have arbitrary large positive eigenvalues even for a real potential $V(x)$. Therefore the possibility, at least in principle, that this property can be generalised to complex potentials justifies the exclusion of the range $\gamma > d/2$. For a review on the case $\gamma > d/2$ we refer to the paper of Arai [6]. More recently Frank and Simon [59] have constructed a sequence of potentials, which extends the Wigner-Von Neuman ones, which support again the exclusion of the case $\gamma > d/2$ from the formulation of the conjecture for the same reasons as just explained above. In particular, they proved that for any dimension $d \geq 1$, there is a sequence of radial potential V_n such that 1 is an eigenvalue of $\Delta + V_n$ in $L^2(\mathbb{R}^d)$ and such that $\|V_n\|_{L^p}$ tends to zero as $n \to \infty$ for any $p > d$.

To the best of our knowledge, Conjecture 1.14 has been proved in its original form, only in the case $0 < \gamma \leq 1/2$ and this result is due to Frank [57]. The proof relies upon an adaptation of the uniform Sobolev inequalities

$$\|(-\Delta - \lambda)^{-1}\|_{L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)} \leq C_{p,d} |\lambda|^{-\frac{d+2}{2} + \frac{d}{p}}, \quad \begin{cases} p \in \left[\frac{2d}{d+2}, \frac{2(d+1)}{d+3}\right] & d \geq 3 \\ p \in \left(1, \frac{6}{5}\right] & d = 2 \end{cases} \tag{1.15}$$

proved by Kenig, Ruiz and Sogge [92] in another context. We observe that the uniformity property comes from the fact that for sufficiently large value of $|\lambda|$ the estimates are independent of the spectral parameter. From their application, a substantial improvement of (1.13) follows from, where now the fundamental estimate reads as

$$1 \leq |(\phi, B(\lambda)\psi)| \leq C_{p,d} |\lambda|^{-\frac{d+2}{2} + \frac{d}{p}} \|V\|_{L^{\frac{p}{2-p}}} \|\phi\|_2 \|\psi\|_2$$

for $p = 2(2\gamma + d)/(2\gamma + d + 2)$ and any test functions $\psi, \phi \in L^2(\mathbb{R}^d)$. We observe that Frank's result has been successively extended by Frank and Simon [59] to any

eigenvalues $\lambda \in \mathbb{C}$ of $-\Delta + V$ in $L^2(\mathbb{R}^d)$.

We note that a consequence of the validity of the estimate (1.14) for $0 < \gamma \leq 1/2$ is that the potentials have to hold some integrability condition and in particular they need to decay at infinity faster than $|x|^{-2d/(d+1)}$. In fact, by introducing weighted Lebesgue spaces, a weaker formulation of (1.15) holds

$$\|(-\Delta - \lambda)^{-1}\|_{L^2(\omega^{-1}) \to L^2((\omega))} \leq C_{p,d,\alpha} \|\omega\|_{\mathcal{L}^{\alpha,p}} |\lambda|^{-1+\frac{\alpha}{2}}, \qquad \text{for } \begin{cases} 4/3 < \alpha < 2, \ d = 2 \\ \frac{2d}{d+1} < \alpha \geq 2, \ d \geq 3 \end{cases} \tag{1.16}$$

where $\|\cdot\|_{\mathcal{L}^{\alpha,p}}$ is the norm in the Morrey-Campanato space of functions $\mathcal{L}^{\alpha,p}(\mathbb{R}^d)$ which is defined as

$$\|V\|_{\mathcal{L}^{\alpha,p}} := \sup_{x,r} \left[r^\alpha \left(\frac{\int_{B_r(x)} |V(y)|^p \, dy}{r^d} \right)^{1/p} \right].$$

The advantage of using weighted uniform resolvent estimates is the possibility of including potentials which a slower decay than the one stated previously, namely up to $|x|^{-\rho}$ where $\rho > 1$. In particular for the specific choice $\omega = (1 + |x|^2)^\alpha$, one obtains from (1.16) the same condition found by Safronov [139].

For sake of completeness we mention that, in fact the L^p-norm in (1.14) has been substituted with weaker norms firstly in Frank [57] with the Morrey-Campanato norm and more recently by Lee and Seo [105] with a Kerman-Saywer norm. We refer to the respective papers for the definitions and details on their respective results.

We observe that the range of validity of the classical uniform Sobolev inequality (1.15) expressed in terms of p corresponds exactly to the case $\gamma \in (0, 1/2]$. Therefore, (1.15) will not provide any further help in order to prove the conjecture for $\gamma > 1/2$.

Following the same idea of modifying (1.15) to a general formulation of the type

$$\|(-\Delta - \lambda)^{-1}\|_{X' \to X} \leq C\gamma^\beta,$$

as done for example in (1.16) in order to allow in (1.14) slower decaying potentials with $X = L^2(\omega)$, in the following we collect two other different versions of the uniform Sobolev estimates which have been used, respectively, to prove the conjecture restricted to radial potential and to show that a modified version of the conjecture is

valid for any potential, both in the case $d \geq 2$ and $1/2 < \gamma < d/2$.

The former modified version was proved by Frank and Simon [59] for operators defined in the Banach space $X = L^p(\mathbb{R}_+, r^{d-1}; L^2(\mathcal{S}^{d-1}))$ and for values $\beta = -d/2 + d/p + 1$ and $2(d+1)/(d+3) < p < 2d/(d+1)$. The proof of it relies on estimates of one dimensional operators on the half real line and it is based on the decomposition of the radial Laplacian's resolvent into a direct sum of orthogonal operators, whose existence comes from the existence of spherical harmonics for the Laplacian restricted to the unitary sphere $\mathcal{S}^{d-1}$ of dimension $d-1$.

In particular Frank and Simon [59] proved that for any dimensions $d \geq 2$, any radially symmetric potential $V(|x|)$ and any non-positive eigenvalue λ of the operator $-\Delta + V(|x|)$ defined on $L^2(\mathbb{R}^d)$ satisfies

$$|\lambda|^\gamma \leq D_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, dx = D_{\gamma,d} \int_{\mathbb{R}_+} |V(r)|^{\gamma+d/2}\, r^{d-1}\, dr, \qquad 1/2 < \gamma < d/2, \tag{1.17}$$

where $D_{\gamma,d}$ is a positive constant independent from the potential $V(x)$. We refer to Subsection 1.3.4 for more details on the spherical harmonics decomposition and for a numerical results that shows the constant $D_{\gamma,d}$ depends in fact on the phase of the complex number λ.

A second refined version of the uniform Sobolev inequality (1.15), obtained by Frank [58] in a different paper, provides a bound for the Schatten norm of the Birman operator relative to $-\Delta + V(x)$ for any dimension $d \geq 1$ and $\gamma \geq 1/2$

$$\|B(\lambda)\|_{2(\gamma+d/2)} \leq C_{d,\gamma}\, \mathrm{dist}(\lambda, [0,\infty))^\alpha |\lambda|^\beta \|V\|^2_{2(\gamma+d/2)}, \tag{1.18}$$

where

$$\alpha = -1 + \frac{(d+1)/2}{\gamma + d/2}, \qquad \beta = -\frac{1}{2(\gamma+d/2)}.$$

This estimate represents a substantial improvement of the classical uniform estimate (1.15) and it was obtained by complex interpolation between two estimates for the Birman operator $B(\lambda)$, valid respectively in terms of its $d+1$ and ∞-Schatten norm. We refer to the paper of Frank and Sabin [63] for the description of the complex interpolation method and for the proof of the bounds mentioned above.

The so enhanced version of the Sobolev estimate (1.18) can be used to improve the

result of Laptev and Safronov in the chain (1.13)

$$1 \leq \|B(\lambda)\| \leq \|B(\lambda)\|_{2(\gamma+d/2)}$$

and therefore customarily derive the following bound for complex eigenvalues of the Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^d$, for $d \geq 1$ and $\gamma \geq 1/2$

$$|\lambda|^{1/2} \operatorname{dist}(\lambda, [0, \infty))^{\gamma - 1/2} \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma + d/2} \, dx. \tag{1.19}$$

We note that the previous formula is not quite close to the one (1.14) formulated by Laptev and Safronov in their conjecture and again the finiteness for the modulus of the eigenvalues is not guaranteed when approaching the positive half line. Nonetheless by observing that $\operatorname{Re}(\lambda) \leq |\lambda|$ it is possible to draw some conclusions on the accumulation property to the essential spectrum of the eigenvalues. Indeed, it follows that if λ_j is a sequence of eigenvalues of $-\Delta + V(x)$ in $L^2(\mathbb{R}^d)$ such that $\operatorname{Re}(\lambda) \to \infty$, then $\operatorname{Im}(\lambda) \to 0$ provided $V(x) \in L^{\gamma+d/2}(\mathbb{R}^d)$. A similar phenomenon was also pointed out earlier in Laptev and Safronov [102] under different yet similar assumption on the potential.

We ought mention that results concerning estimates in Schatten norms of the kernel of the Laplace operator for complex potentials have been also obtained by Demuth Hansmann and Katriel [38, 39] in a series of papers antecedent the works of Frank [58] and Frank and Sabin [63]. Demuth and collaborators considered the class of p-relatively compact potentials with respect to a general positive self-adjoint operator H_0 and derived, with the aid of results of the Jensen's type of inequality coming from complex analysis theory for zeroes of analytic functions in a disc (results introduced at first instance by Borichev Golinskii and Kupin [19]), estimates on the distribution and sum of moments of complex eigenvalues. We will not enter into details of these type of results. We are also aware of the fact that important results on the complex Lieb-Thirring inequality have been omitted in order to keep this dissertation focused on localisation's type of results. We refer the interested reader to the papers mentioned so far and to the references therein contained, in particular to Frank's paper [58] where the author compare his results with the ones obtained in [38, 39].

In conclusion we mention some additional results which have been obtained by employing newly derived uniform Sobolev estimate for different operators from the Laplacian.

In particular we mention the work of Mizutani [122] on the operator $H_0 = -\Delta + \frac{\sigma}{x^2}$ on $\mathbb{R}^d$ with $d \geq 3$ and $\sigma \geq -1/4$ where bounds similar to those obtained by Frank [58] have been deduced for potentials in the weak-Lebesgue class, extending (1.19) in the case $\sigma = 0$. We also mention the results on the fractional Laplacian, fractional Bessel and Dirac operators obtained by Cuenin [29] by proving uniform resolvent bounds in operator and Schatten norms. In a different paper, Cuenin [30] derived also uniform estimates for the two dimensional bilayer Graphene operator, extending the results previously obtained in [141] by the author together with Laptev and Safronov. We refer to Section 4 of this chapter for more details on it.

1.3 Second order operator on the half line with Hardy potential

In this section we study the problem of localisation of complex eigenvalues for a specific family of Schrödinger operators defined on $L^2(0,\infty)$ and subject to complex perturbation. The operators in question are defined via the differential expression

$$H_{0,\nu} : u(x) \to \left(-\frac{d^2}{dx^2} + \frac{(\nu^2 - 1/4)}{x^2}\right) u(x) \tag{1.20}$$

and are subject to Dirichlet boundary condition which are assumed to hold at the origin. The case $\nu = 1/2$ was investigated by Frank, Laptev and Seiringer [62]. Our aim is to extend the results valid for $\nu = 1/2$ to a larger class of operators corresponding to positive real $\nu > 0$. We will also study the case of potentials in weighted-L^p spaces. Using the information derived in the latter case we also show that, in fact, the region where complex eigenvalues of $-\Delta + V(|x|)$ defined in $L^2(\mathbb{R}^d)$ for $d \geq 2$ might appear has a different shape from the one of a ball centred at the origin. In particular, we provide an explicit formulation of the best constant found by Frank and Simon in [59] in the proof of the Conjecture1.14 for the case $1/2 < \gamma > d/2$.

1.3.1 The case $\nu = 1/2$

Due to an explicit formula for the Green's function for the operator $H_{0,\frac{1}{2}}$, the existence of a sharp bound for complex eigenvalues of the perturbed operator was established in [62] in terms of elementary functions. In fact, in the case $\nu = 1/2$, any eigenvalue $\lambda = |\lambda| e^{i\theta} \in \mathbb{C} \setminus [0,\infty)$ of the operator $H_{0,\frac{1}{2}} + V(x)$ on the half line subject Dirichlet boundary conditions at origin satisfies

$$|\lambda|^{1/2} \leq \left(\frac{1}{2} \sup_{y \geq 0} \left| e^{i \cot(\theta/2) y} - e^{-y} \right|\right) \int_0^\infty |V(x)|\, dx \tag{1.21}$$

for any integrable potential $V(x) \in L^1(0,\infty)$. Comparing the expression (1.21) with the analogous result (1.10) by Davies [2] valid for the operator $H_{0,1/2}$ defined on the whole line, we notice that the additional Dirichlet condition imposed at the origin implies that a different shape of the region, where eventually the eigenvalues might arise, is found. In contrast to what happens in the setting of the whole real line, when the bound is in fact expressed only in terms of the modulus of the complex eigenvalue

and where the optimal shape is a ball centred at the origin of constant radius 1/2, in the half line case an explicit dependence of such bound upon the phase of the complex eigenvalue is in place, and it implies a break in the symmetry and a different shape from the circle's one.

In particular, certain type of potentials of the form $V(x) = c\delta(x - b)$ realise the inequality (1.21) in fact as an equality. This can be in turn interpreted as a confirmation of the optimality of the result proved. It happens that in practice, the problem with such delta potentials can be solved explicitly and the constants $b \in \mathbb{R}$ and $c \in \mathbb{C}$ can be continuously varied so that the unique eigenvalue which is originated from the perturbed operator describes the continuous contour in figure 1.2. Later in this section, we will reproduce similar computations for the general case $\nu \geq 1/2$, proving again the sharpness of our results.

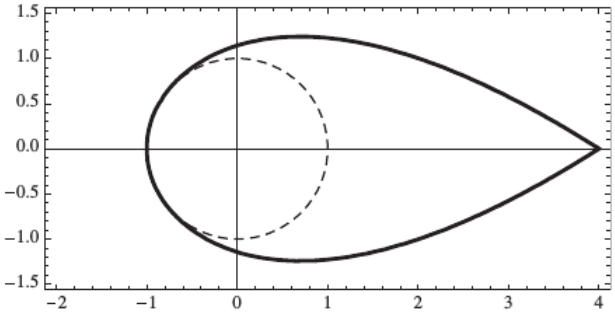


Figure 1.2: Continuous line: the plot of the maximal value of $4|\lambda|$ in the half-line case. Dashed line: the corresponding bound on the whole line. Ref. [62]

1.3.2 Uniform bounds

We start by introducing some notation. We define the following differential expression

$$H := H_{0,\nu} + V(x), \tag{1.22}$$

where $V(x)$ is a complex-valued potential and the associated eigenvalue problem

$$Hu = \lambda u.$$

In the following we will understand H as the operator generated by a sesquilinear form. We refer to Section 2.1.2 for the details on how this construction is realised. We note that the classical Hardy inequality guarantees the boundedness from below of the so obtained form. In particular it follows that

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_{0,\nu}) = [0, \infty),$$

and that the spectrum $\sigma(H)$ of the operator H is discrete in $\mathbb{C}\backslash\sigma(H_{0,\nu})$. (See Theorem 1.16 and its following Remark 1.7.) The green's function of the operator $H_{0,\nu}$ is

$$G_\nu(x,y,z) = \begin{cases} \frac{\pi}{2i}\sqrt{x}H_\nu^{(1)}(\sqrt{z}x)\sqrt{y}J_\nu(\sqrt{z}y) & \text{if } y \le x, \\ \\ \frac{\pi}{2i}\sqrt{x}J_\nu(\sqrt{z}x)\ \sqrt{y}\,H_\nu^{(1)}(\sqrt{z}y) & \text{if } y > x, \end{cases} \tag{1.23}$$

where $J_\nu(z)$ and $H_\nu^{(1)}(z)$ are respectively the Bessel and Hankel function of first kind of order ν. For details on the derivation of the previous formula we refer to the classical textbook of Titchmarsh [153] and to [50] for the case $0 < \nu < 1$, $\nu \neq 1/2$, when the Storum-Liuville problem in zero is of limit circle type. Here and after we consider the complex square root function with branch cut on $\mathbb{C} \setminus (0, \infty)$.

We note that, being $H_{0,\nu}$ selfadjoint, the Green's function is symmetric in its two variables

$$G_\nu(x,y,z) = G_\nu(y,x,z).$$

We introduce the polar decomposition relative to the potential $V(x)$ as

$$V(x) = V_1(x)V_2(x) := \left(V(x)|V(x)|^{-\frac{1}{2}}\right)|V(x)|^{\frac{1}{2}},$$

and so the Birman-Schwinger operator

$$B(\lambda) := V_1 R(\lambda) V_2, \tag{1.24}$$

where $R(\lambda)$ stands for the resolvent operator $(H_0 - \lambda)^{-1}$. We proceed now customarily by recalling the correspondence which holds between the eigenvalues of an operator and those of the relative Birman-Schwinger one. It holds that λ being an eigenvalue of the perturbed H is equivalent to the case of -1 being an eigenvalue of the operator

$B(\lambda)$. Therefore a necessary condition for λ to be an eigenvalues consists in the numerical radius of the Birman-Schwinger operator to be at least 1, thus that the following inequality

$$1 \leq \left| \int_0^\infty \int_0^\infty \bar{V}_2(x) G_\nu(x, y, \lambda) V_1(y) \bar{f}(x) g(y) \, dy \, dx \right|$$

must be satisfied for all f and g with unitary L^2-norm. It follows the main result regarding complex integrable potentials.

Theorem 1.21. *Let $\nu > 0$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$ such that $\lambda = |\lambda| e^{i\theta}$ where $\theta \in (0, 2\pi)$. Let λ be a complex eigenvalue of the operator $H_{0,\nu} + V(x)$ where $V(x) \in L^1(0, \infty)$ with Dirichlet boundary condition at the origin. Then there exists a constant $C(\nu, \theta)$ such that*

$$|\lambda|^{1/2} \leq C(\nu, \theta) \int_0^\infty |V(x)| \, dx, \tag{1.25}$$

where the constant $C(\nu, \theta)$, which depends upon the angle θ and the order ν, is

$$C(\nu, \theta) = \frac{\pi}{2} \sup_{0 \leq y \leq x < \infty} \left(\sqrt{x} |H_\nu^{(1)}(x e^{i\theta/2})| \sqrt{y} |J_\nu(y e^{i\theta/2})| \right). \tag{1.26}$$

Furthermore, for $\nu \geq 1/2$ the bound (1.25) is sharp, meaning that for given $m \in \mathbb{R}$ and $\theta \in (0, 2\pi)$, there exist $b > 0$ and $c \in \mathbb{C}$ such that $|c| = m$ and the unique eigenvalue of $H = H_{0,\nu} + c\delta(x - b)$, is

$$\lambda = |c|^2 \, C(\nu, \theta)^2 \, e^{i\theta}.$$

Remark 1.9. We observe that for the case $\nu = 1/2$, the formulation of the constant $C(\nu, \theta)$ given in (1.26) coincides with the expression in the round bracket of (1.21).

Proof of the Theorem 1.21. In the first instance we will consider the spectral parameter λ lying on the unit circle. Let $s = \lambda = e^{i\theta}$ where $\theta \in (0, 2\pi)$. By means of the Schur's inequality, we estimate the operator norm of $B(\lambda)$ as

$$\begin{aligned}\left|\iint B(\lambda)\bar{f}(x)g(y)\,dy\,dx\right| &\leq \left(\iint |\bar{V}_2(x)|^2\,|G_\nu(x,y,s)|\,|g(y)|^2\,dy\,dx\right.\\ &\qquad\left.\iint |\bar{V}_1(y)|^2\,|G_\nu(x,y,s)|\,|f(x)|^2\,dy\,dx\right)^{1/2}\\ &\leq \|f\|_{L^2}\|g\|_{L^2}\sup_{x\in(0,\infty)}\int_0^\infty |V(y)||G_\nu(x,y,s)|\,dy\\ &\leq \|f\|_{L^2}\|g\|_{L^2}\sup_{x,y\in(0,\infty)}|G_\nu(x,y,s)|\int_0^\infty |V(y)|\,dy.\end{aligned} \tag{1.27}$$

As previously observed, the Green's function $G_\nu(x,y,s)$ is symmetric in its variable x and y. Therefore it will be sufficient to consider the above supremum in the sector $0\leq y\leq x<\infty$. Thus, we define the constant

$$C(\nu,\theta)=\sup_{0\leq y\leq x<\infty}|G_\nu(x,y,s)|=\frac{\pi}{2}\sup_{0\leq y\leq x<\infty}\left(\sqrt{x}|H_\nu^{(1)}(xe^{i\theta/2})|\,\sqrt{y}|J_\nu(ye^{i\theta/2})|\right).$$

In order to estimate the quantity defined above, we need to recall the asymptotic behaviour of the Bessel's functions $J_\nu(z)$ and $H_\nu^{(1)}(z)$ near zero and infinity. We collect them in the following and we refer to Olver [132] for the details of the proof.

Let $z\in\mathbb{C}$, then:

$$\begin{aligned}J_\nu(z)&\sim\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} && \textit{for } z\to 0,\\ H_\nu^{(1)}(z)&\sim-\frac{i}{\pi}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} && \textit{for } z\to 0.\end{aligned} \tag{1.28}$$

and

$$\begin{aligned}J_\nu(z)&\sim\sqrt{\frac{2}{\pi}}\sqrt{\frac{1}{z}}\left(\cos(z-\frac{\nu\pi}{2}-\frac{\pi}{4})+e^{|\operatorname{Im}(z)|}O(1)\right) && \textit{for } z\to\infty,\\ H_\nu^{(1)}(z)&\sim\sqrt{\frac{2}{\pi}}\sqrt{\frac{1}{z}}e^{i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})} && \textit{for } z\to\infty,\end{aligned} \tag{1.29}$$

It follows that the finiteness of the constant $C(\nu,\theta)$ comes now immediately from the regularity of the Bessel's function and from the asymptotics reported in (1.28) and (1.29).

Let us consider now the general case, when the spectral parameter $\lambda \in \mathbb{C} \setminus [0,\infty]$ is given by $\lambda = |\lambda| e^{i\theta}$. From a rescaling argument, we immediately deduce that

$$\sup_{0\le y\le x<\infty} \frac{\pi}{2}\sqrt{x}|H_\nu^{(1)}(x\sqrt{\lambda})| \sup_{y\in(0,x)} \sqrt{y}|J_\nu(y\sqrt{\lambda})| = \frac{C(\nu,\theta)}{|\lambda|^{1/2}}$$

and this complete the proof of (1.25).

We note that for small values of the angle θ and for $\nu > 1/2$, the supremuum of the function $\sqrt{y}|J_\nu(ys)|$ for $y \in (0,x)$ is in general not attained at the end point x. This phenomenon can be easily understood by the fact that the Bessel function $J_\nu(x)$ with real argument is a bounded oscillating function and such oscillatory behaviour inherited by $\sqrt{y}|J_\nu(e^{i\theta}y)|$ for small values of the angle θ.

Let us now focus on the case $\nu \ge 1/2$ and the sharpness problem.

Let us recall now some well known facts about the Bessel and Hankel function of first order. We begin with a representation formula for the modified Bessel function $\mathcal{K}_\nu(z)$,

$$\mathcal{K}_\nu(z) = \frac{1}{\Gamma(\nu+1/2)}\left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\int_0^\infty e^{-t}t^{\nu-1/2}\left(1+\frac{t}{2z}\right)^{\nu-1/2} dt, \tag{1.30}$$

valid for $\mathrm{Re}(\nu) > -1/2\,, |\arg(z)| < \pi$ (see [62])
The followings provide the existing link between the Hankel functions and the modified Bessel function introduced above

$$\begin{aligned} H_\nu^{(1)}(z) &= \frac{2}{i\pi}e^{-i\nu\pi/2}\mathcal{K}_\nu(ze^{-i\pi/2}) \qquad \arg(z) \in \left(-\frac{\pi}{2},\pi\right), \\ H_\nu^{(2)}(z) &= -\frac{2}{i\pi}e^{-i\nu\pi/2}\mathcal{K}_\nu(ze^{i\pi/2}) \qquad \arg(z) \in \left(-\pi,\frac{\pi}{2}\right). \end{aligned} \tag{1.31}$$

By means of $J_\nu(z) = \frac{1}{2}\left(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right)$ and from the formulae above, it also follows a similar result for the Bessel function of first kind

$$J_\nu(z) = \frac{i}{2\pi}\left(e^{-i\nu\pi/2}\mathcal{K}_\nu(ze^{-i\pi/2}) - e^{i\nu\pi/2}\mathcal{K}_\nu(ze^{i\pi/2})\right) \qquad |\arg(z)| < \frac{\pi}{2}, \tag{1.32}$$

together with the following analytic continuation property

$$J_\nu(ze^{im\pi}) = e^{im\nu\pi} J_\nu(z), \qquad m \in \mathbb{Z}. \tag{1.33}$$

We use the above properties for Bessel and Hankel functions to re-write in a more convenient way the constant $C(\nu,\theta)$. We have that

$$\begin{aligned} |\sqrt{x}H_\nu^{(1)}(x)| &= \left| \sqrt{\frac{2}{\pi}} \frac{1}{\Gamma(\nu+\frac{1}{2})} e^{-xe^{i(\theta-\pi)/2}} \int_0^\infty e^{-t} t^{\nu-\frac{1}{2}} \left(1 + \frac{t}{2xe^{i(\theta-\pi)/2}}\right)^{\nu-\frac{1}{2}} dt \right| \\ &= \alpha(\nu) e^{-x\sin(\theta/2)} g(xe^{i(\theta-\pi)/2}), \end{aligned} \tag{1.34}$$

where, for convenience, we set the following quantities

$$\alpha(\nu) = \frac{\sqrt{2}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})},$$

$$g(xe^{i(\theta-\pi)/2}) = \left| \int_0^\infty e^{-t} t^{\nu-\frac{1}{2}} \left(1 + \frac{t}{2xe^{i(\theta-\pi)/2}}\right)^{\nu-\frac{1}{2}} dt \right|.$$

Obviously, the term $e^{-x\sqrt{|\lambda|}\sin(\theta/2)}$ is decreasing in x. Less obviously, for $\nu \geq 1/2$ also the term $g(x\sqrt{|\lambda|}e^{i(\theta-\pi)/2})$ appears to be decreasing in x from numerical simulations. We conclude then

$$\begin{aligned} C(\nu,\theta) &= \frac{\pi}{2} \sup_{0\leq y\leq x<\infty} \left(\sqrt{x}|H_\nu^{(1)}(xe^{i\theta/2})|\,\sqrt{y}|J_\nu(ye^{i\theta/2})|\right) \\ &= \frac{\pi}{2} \sup_{0\leq y<\infty} \left(\sqrt{y}|H_\nu^{(1)}(ye^{i\theta/2})|\,\sqrt{y}|J_\nu(ye^{i\theta/2})|\right) \\ &= \frac{\alpha(\nu)^2}{4} \sup_{0\leq y<\infty} \left[g(xe^{i(\theta-\pi)/2}) g(xe^{i(\theta+\pi)/2}) \left| \left(e^{-2x\sin(\theta/2)} \frac{g(xe^{i(\theta-\pi)/2})}{g(xe^{i(\theta+\pi)/2})} - 1 \right) \right| \right] \end{aligned} \tag{1.35}$$

In particular, the monotonicity of the term $g(xe^{i(\theta-\pi)/2})$ makes possible to prove the sharpness of the estimate (1.25) in the case $V(x) = c\delta(x-b)$. We proceed by standard arguments. Let us consider $u(x)$ the wave function solution of

$$H_{0,\nu}u(x) + c\delta(b-x)u(x) = \lambda u(x).$$

Let c_1 and c_2 be two complex constants such that

$$\begin{aligned} u(x) &= \sqrt{x}H_\nu^{(1)}(\sqrt{\lambda}x) && \text{for } x > b, \\ u(x) &= c_1\sqrt{x}J_\nu(\sqrt{\lambda}x) + c_2\sqrt{x}H_\nu^{(1)}(\sqrt{\lambda}x) && \text{for } x < b. \end{aligned} \tag{1.36}$$

Imposing the Dirichlet condition at the origin and the continuity condition at point b for $u(x)$, it immediately yields

$$c_1 = \frac{H_\nu^{(1)}(\sqrt{\lambda}b)}{J_\nu(\sqrt{\lambda}b)} \quad \text{and} \quad c_2 = 0.$$

We conclude by imposing the discontinuity jump at point b for the derivatives

$$\left[c_1\frac{d}{dx}\left(\sqrt{x}J_\nu(\sqrt{\lambda}x)\right) - \frac{d}{dx}\left(\sqrt{x}H_\nu^{(1)}(\sqrt{\lambda}x)\right)\right]_{x=b} = c\sqrt{b}\,H_\nu^{(1)}(\sqrt{\lambda}\,b)$$

which by means of the Wronskian of Bessel's functions formula

$$W\Big\{J_\nu(z), H_\nu^{(1)}(z)\Big\} = \frac{2i}{\pi z}$$

reduces to

$$\frac{2i}{\pi} = cbH_\nu^{(1)}(\sqrt{\lambda}\,b)J_\nu(\sqrt{\lambda}\,b). \tag{1.37}$$

One can then set $c = |c|e^{i\phi}$ and $l = \sqrt{\lambda}\,b$ to derive from (1.37) the following

$$\begin{aligned} \sqrt{|\lambda|} &= |c|\,|G_\nu(l, l, e^{i\theta/2})| \\ \theta/2 &= \phi + \arg(G_\nu(l, l, e^{i\theta/2})) \end{aligned}$$

and obtain respectively the values for l and ϕ such that $|G_\nu(l, l, e^{i\theta/2})| = C(\nu, \theta)$ and second identity above is verified. We finally find $b = l(|G_\nu(l, l, e^{i\theta/2})|)^{-1}$. We emphasize the need of equations (1.30)- (1.33) in order to derive the phase of c from equation (1.37). □

Remark 1.10. For $0 < \nu < 1/2$, the function $g(z)$ is not a decreasing function. The same is so true for $\sqrt{x}|H_\nu^{(1)}(zx)|$, which changes drastically its qualitatively behaviour near zero (see formula (1.28)). This in turn implies that the two supremuum in (1.26) needs not to be attained on the diagonal. In fact, numerical experiments show that

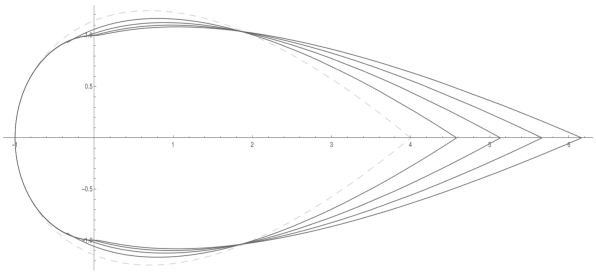


Figure 1.3: Continuous lines: plot of the numerical value of the constants $4C(\nu,\theta)$ as function of the angle, for different values of $\nu = 1, 3/2, 2, 5/2$. The dashed line is the case $\nu = 1/2$, plotted as reference.

for $\nu = 1/4$ and $\theta = \pi 10^{-3}$, the maximum of the function $G_\nu(x, y, e^{i\theta})$ is achieved at the point $(x, y) = (2.85212, 1.2624)$.

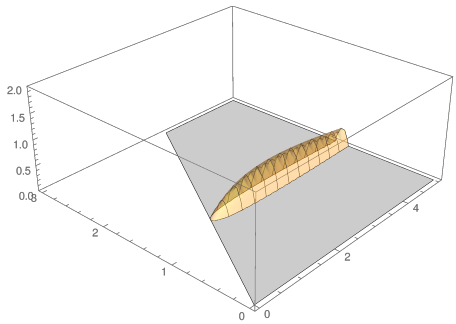


Figure 1.4: The plot of the positive part of the function $10^2[|G_\nu(x, y, e^{i\theta})| - \sup_x |G(x, x, e^{i\theta})|]$ over the grey-shaded region $\mathcal{R} = \{(x, y) \in [0, 5] \times [0, 3] \mid y \leq x\}$ for the following values $\nu = 1/4$, $\theta = 10^{-3}\pi$.

The absence of such monotone behaviour and the impossibility of expressing $C(\nu,\theta)$ as a supremuum of a single variable ultimately makes ineffective the use of delta potentials and poses the question whether the constant is sharp or not.

Despite what we have observed in Remark 1.10, Figure 1.4 suggests that the discrepancy between the value of the constant $C(\nu,\theta)$ as obtained in (1.26) and the value of the function $\sup_x |G(x, x, e^{i\theta})|$ is relatively small. Approximating the value of $C(\nu,\theta)$ by the latter function we obtain the following qualitative plots.

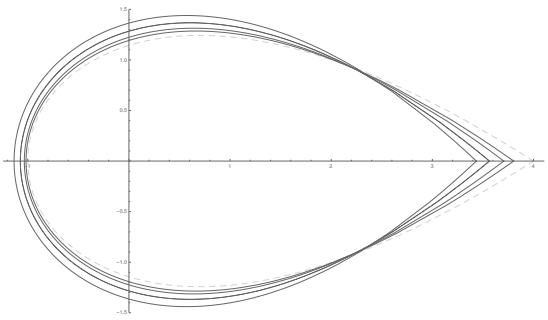


Figure 1.5: Continuous lines: plot of the numerical approximations of the constant $C(\nu, \theta)$ as function of the angle, for different values of $\nu = 1/3, 1/4, 1/8, 1/250$. The dashed line is the case $\nu = 1/2$ is plotted as reference. There is no appreciable difference between the plot we obtain for $\nu = 1/250$ and smaller values of this parameter.

1.3.3 The case $V(x) \in L^p$

In this section we suggest a different estimate type from the one used in the previous section, for the term at the top of the inequality chain (1.27). In particular we derive estimates valid for potentials in weighted $L^p(0,\infty)$ spaces; the weights, which might at first look artificial, are in fact justified from the decomposition of the multi-dimensional case of the radial Laplacian in spherical harmonics, as for example done in Frank and Simon [59]. Following the ideas adopted in the proof of the formula (1.17) we derive in Equation 1.43 similar estimates for the eigenvalues of the spectral problem for $H_{0,\nu}$ on the half line.

We proceed denoting by $\tilde{G}_\nu(x, e^{i\theta})$ the quantity $\tilde{G}_\nu(x, e^{i\theta}) := \sup_{y\in(0,\infty)} |G_\nu(x, y, \theta)|$

$$\tilde{G}_\nu(x, e^{i\theta}) = \frac{\pi}{2} \max \left(\sup_{y\in(0,x)} \sqrt{x}|H_\nu^{(1)}(xe^{i\theta/2})|\sqrt{y}|J_\nu(ye^{i\theta/2})|, \right. \\ \left. \sup_{y\in(x,\infty)} \sqrt{y}|H_\nu^{(1)}(ye^{i\theta/2})|\sqrt{x}|J_\nu(xe^{i\theta/2})| \right) \tag{1.38}$$

where $\theta \in (0, 2\pi)$. Let us now introduce a weight function $w(x)$ such that

$$\int_0^\infty |V(x)|\tilde{G}_\nu(x, e^{i\theta})\, dx \leq \left(\int_0^\infty (w(x)|V(x)|)^{\frac{d}{2}+\gamma}\, dx \right)^{\frac{2}{d+2\gamma}} \left(\int_0^\infty \tilde{G}_\nu(x, e^{i\theta})^\beta w(x)^{-\beta}\, dx \right)^{\frac{1}{\beta}} \tag{1.39}$$

where β satisfies $(\gamma + \frac{d}{2})^{-1} + \beta^{-1} = 1$. We furthermore assume the weight function to be a homogeneous function of degree α. For this reason let us introduce $w(x) = x^\alpha$ for any positive α such that $\alpha\left(\frac{d}{2} + \gamma\right) = d - 1$. This, together with the previous assumption made on β gives the following values

$$\alpha = 2(d-1)/(d+2\gamma) \tag{1.40}$$

and

$$\beta = \frac{d+2\gamma}{d+2\gamma-2}. \tag{1.41}$$

Let us finally introduce the constant $\tilde{C}(\nu, \theta)$ as

$$\tilde{C}(\nu,\theta) := \left(\int_0^\infty \tilde{G}_\nu(x, \theta e^{i\theta})^\beta w(x)^{-\beta}\, dx\right)^{\frac{1}{\beta}}. \tag{1.42}$$

We can now formulate the main results of this section in the following theorem.

Theorem 1.22. *Let $\nu > 0$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$ such that $\lambda = |\lambda| e^{i\theta}$ where $\theta \in (0, 2\pi)$. Let λ be a complex eigenvalue of the operator $H_{0,\nu} + V(x)$ with Dirichlet boundary condition at the origin for any real value of $d \geq 2(1-\gamma)$ such that $0 < \gamma < d/2$. Then there exists a constant $\tilde{C}(\nu, \theta)$ such that*

$$|\lambda|^\gamma \leq \tilde{C}(\nu,\theta)^{\frac{d}{2}+\gamma} \int_0^\infty x^{d-1} |V(x)|^{\frac{d}{2}+\gamma}\, dx, \tag{1.43}$$

where the constant $\tilde{C}(\nu, \theta)$, which depends upon the angle θ and the order ν, is defined in (1.42).

Proof. From the properties of the Bessel functions (1.28) and (1.29), we deduce that the function $\tilde{G}_\nu(x, s)$ defined in (1.38) is a bounded function on $x \in (0, \infty)$ which converges to zero at the origin with order one

$$\begin{aligned} \tilde{G}_\nu(x,s) &\sim O(x) \qquad && \text{for } x \to 0, \\ \tilde{G}_\nu(x,s) &\sim O(1) \qquad && \text{for } x \to \infty. \end{aligned}$$

Therefore, the condition $0 < \gamma < d/2$ provides the convergence of the integral in the second factor of right hand side of (1.39). From a homogeneity argument, it follows

that for general $\lambda \in \mathbb{C}$ we have

$$\left(\int_0^\infty \tilde{G}_\nu(x,\lambda)^\beta w(x)^{-\beta}\, dx \right)^{\frac{1}{\beta}} = \frac{\tilde{C}(\nu,s)}{|\lambda|^{\frac{1-\alpha}{2}+\frac{1}{2\beta}}},$$

and so the proof is complete. □

Remark 1.11. Note that the condition $d \geq 2(1-\gamma)$, or equivalently $d/2+\gamma \geq 1$, is essential in order to use the Holder inequality to obtain (1.39).

Remark 1.12. In the special case of $d = 1$, the constraint $0 < \gamma < d/2$ extends so that γ is allowed to take also the value $d/2$. In fact, following the notation introduced in the proof of Theorem 1.22, we have $\alpha = 0$ and $\beta = \infty$. We thus recover exactly the estimate proved in Theorem 1.21

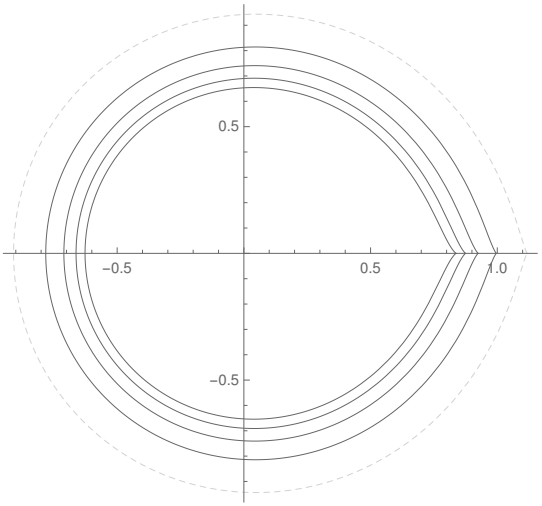


Figure 1.6: Continuous lines: from the outer most, plots of the numerical approximation value of the constant $\tilde{C}(\nu,\theta)$ as function of the angle, for different values of $\nu = 1, 3/2, 2, 5/2$ in (1.43), with $\gamma = 1/2$ and $d = 3$. The dashed line corresponds to the case $\nu = 1/2$.

Let us define $\mathcal{S}_{d,\nu}$ the region where the complex eigenvalues of H might lie according to (1.43). For fixed values of d and γ and for fixed value of the norm of the potential equal to one, numerical evidences suggest not only that the contour of the region $\mathcal{S}_{d,\nu}$ changes with varying the angle in the complex plane but, according to what would happen for a real potential, for greater values of ν the region $\mathcal{S}_{d,\nu}$ shrinks, concentrating around the origin. It seems, in fact, the case that the regions appear to

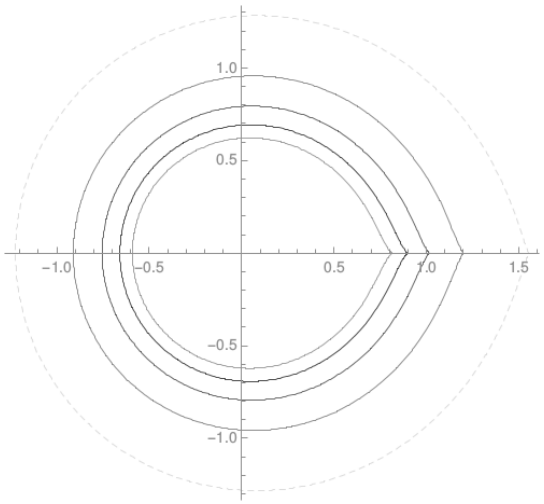


Figure 1.7: Continuous lines: from the outer most, plots of the numerical approximation value of the constant $\tilde{C}(\nu, \theta)$ as function of the angle, for different values of $\nu = 1, 3/2, 2, 5/2$ in (1.43), with $\gamma = 1/2$ and $d = 5$. The dashed line corresponds to the case $\nu = 1/2$.

be ordered in a monotonic inclusion order $\mathcal{S}_{d,\nu'} \subseteq \mathcal{S}_{d,\nu}$ for $\nu < \nu'$, as shown in Figure 1.6 and Figure 1.7.

In particular, the estimates obtained above for the half line problems provide some important insights on the shape of the region $\mathcal{S}_d^r \subseteq \mathbb{C}$, the region where lie the complex eigenvalues of the multi-dimensional Laplacian restricted to radial functions endowed with complex radial perturbations. This operator will be investigated in the next subsection.

1.3.4 The radial Laplacian in $\mathbb{R}^d$

As mentioned earlier, an interesting consequence which can be drawn from the properties of the complex spectrum of the perturbed operators $H_{0,\nu}$ is about the geometry of the region $\mathcal{S}_d^r$ in the complex plane where eigenvalues of the multi dimensional Laplacian can be found when a radial complex perturbation is introduced. A first result in this direction was produced by Frank and Simon [59] by proving the validity of the Laptev-Safranov's conjecture [102] (1.14) for radial potentials and for values of $\gamma \in (0, d/2)$ as shown in (1.17).

Let us consider the d-dimensional Laplacian in polar coordinates

$$\Delta_d f = \frac{1}{r^{(d-1)}} \frac{\partial}{\partial r} \left(r^{(d-1)} \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \Delta_{S^{(d-1)}} f$$

where $\Delta_{S^{(d-1)}}$ is the Laplace-Beltrami operator on $S^{(d-1)}$, the unitary $(d-1)$-sphere. It is a well known fact that the eigenspaces E_l of the Laplace-Beltrami operator on the d-sphere, with $l = 0, 1, 2, \ldots$ corresponding to the eigenvalue $c_{l,d} := l(l+d-2)$ yield the following direct sum decomposition

$$L^2(S^{(d-1)}) = \bigoplus_{l=0}^{\infty} E_l^d$$

where E_l are close, pairwise orthogonal and such that for each of them there exists a basis of orthonormal spherical harmonics $Y_{l,m}$ where the index m runs on an appropriate set of finite cardinality which depends on l. We observe that the subspaces $\tilde{E}_l = \left\{ f(x) = R(r) Y_{l,m}(\omega) \mid r \in (0,\infty), \omega \in S^{(d-1)} \right\}$ reduces the operator $-\Delta + V(|x|)$. It follows that

$$-\Delta_d + V(|x|) \; = \bigoplus_{l,m} \tilde{h}_l^d,$$

where

$$\tilde{h}_l^d := -\frac{\partial^2}{\partial r^2} - \frac{(d-1)}{r} \frac{\partial}{\partial r} + \frac{l(l+d-2)}{r^2} + V(r),$$

with $\mathcal{D}(\tilde{h}_l^d) = L^2((0,\infty), r^{2\xi} dr)$ and $\xi = \left(\frac{d-1}{2}\right)$. By mean of the unitary transformation $R(r) \to u(r) = r^{\xi} R(r)$, (see for example Teschl [151] for the case $d = 3$), and from the definition of $H_{0,\nu}$ given in (1.20), the above operators simplifies to

$$h_l^d := H_{0,N(d,l)} + V(|x|), \qquad \mathcal{D}(\tilde{h}_l) \subseteq L^2(0,\infty),$$

where

$$N(d,l) = \left| \sqrt{c_{l,d} + \xi(\xi - 1) + 1/4} \right|.$$

It finally follows the decomposition of the spectrum of the operator $-\Delta_d + V(|x|)$ in terms of

$$\sigma(-\Delta_d + V(|x|)) = \overline{\bigcup_l \sigma(h_l^d)}.$$

From the property $\mathcal{S}_{d,\nu'} \subseteq \mathcal{S}_{d,\nu}$ for $\nu < \nu'$, we in fact conclude that

$$\sigma_d(-\Delta_d + V(|x|)) \subseteq \mathcal{S}_{d,N(d,0)}.$$

This in turn implies that the shape of the region $\mathcal{S}_d^r$ where the complex eigenvalues of the Laplacian with complex radial perturbation and restricted to radial functions lie, depends in fact on the phase of the eigenvalue itself. In particular, for $d = 3$ the shape of $\mathcal{S}_3^r$ is determined by the contour corresponding to the value $\nu = 1/2$ in Figure 1.6, and for $d = 5$ by the one corresponding to $\nu = 3/2$ in Figure 1.7. In general,when $l = 0$, a little algebra shows that there is a simple formulation $N(d, 0) = \frac{d-2}{2}$.

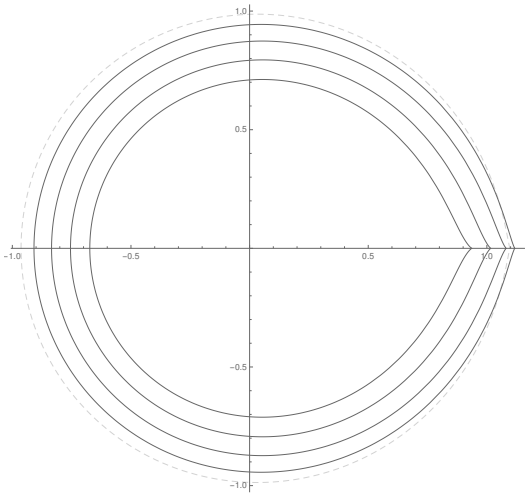


Figure 1.8: Continuous lines: from the outer most, plot of the contour of the regions $\mathcal{S}_d^r$ for $d = 3, 4, 5, 6$ obtained respectively for the values $\nu = 1/2, 1, 3/2, 2$. The dashed line is an approximation to the limit case $d = 2$ and $\nu = 0$.

1.4 Dirac-like operators: Bilayer Graphene

The operator $D = D_m + V$ defined in (1.45) below and which will be studied in the section arises from the Hamiltonian formulation of the energy of a system made of two layers of a bi-dimensional graphene's sheets, when a complex potential is applied. Again, we seek information on the region where complex eigenvalues lie. Hereafter we discuss the results obtained by Ferrulli Laptev and Safronov in [141]. Heuristically, one can think of the operator D_m as a sort of square root of the bi-harmonic operator $-\Delta^2$ as much as it is possible to interpret a Dirac operator as a square root of the classical Laplacian. Under this point of view, our results aim to extend the ones proved by Cuenin Laptev and Tretter [31], where an explicit formulation of the region where the eigenvalues might lie is given for the Dirac operator defined on the real line. From $D_m^2 = \Delta^2 + m^2$, see equation (1.48), we deduce that the spectrum of D_m is the set

$$\sigma(D_m) = \sigma_{ess}(D_m) = (-\infty, -m] \cup [m, \infty).$$

Our results show that the eigenvalues of D are located near the edges of the absolutely continuous spectrum, i.e. near the points $\pm m$. Since the spectrum of the unperturbed operator has two edges, our results resemble some of the theorems of the Cuenin's paper [31] related to the Dirac operator. However, the main difference between the two papers is that we study a differential operator on a plane, while the latter deals with operators on a line.

In the following we briefly recall the main result valid for the Dirac operator and then pass to the study of the graphene operator.

1.4.1 One dimensional Dirac case

Consider the operator $H = H_0 + V$ defined on $L^2(\mathbb{R}; \mathbb{C}^2)$ where V is a 2×2 matrix-valued function with entries in the Banach space $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ where H_0 is the one dimensional free Dirac operator:

$$H_0 = -ic\hbar \frac{d}{dx}\sigma_1 + mc^2\sigma_3, \quad \sigma_1 \coloneqq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.44}$$

where σ_1 and σ_3 are the Pauli matrices, c is a positive constant which usually stands for the speed of light, $\hbar$ is the reduced Planck constant, i the complex unit and m is the particle's mass which could take eventually complex values and the potential V might be a non-Hermitian matrix.

The operator H_0 can be understood in fact, by means of Fourier transform, as the square root of a Schrödinger's like operator. It is indeed unitarily equivalent to the operator

$$\begin{pmatrix} \sqrt{-\frac{d^2}{dx^2}+m^2} & 0 \\ 0 & -\sqrt{-\frac{d^2}{dx^2}+m^2} \end{pmatrix}$$

and so the spectrum of H_0 has only the essential component which reads as $\sigma(H_0) = \sigma_{ess}(H_0) = (-\infty, -m] \cup [m, \infty)$.

Theorem 1.23 (Cuenin, Laptev , Tretter [31]). *Let $V = (V_{ij})^2$ with $V_{ij} \in L^1(\mathbb{R})$ for $i, j = 1, 2$ be such that*

$$\|V\|_1 < 1.$$

Then every non-embedded eigenvalue $z \in \mathbb{C} \setminus \sigma(H_0)$ of H lies in a region $\mathcal{R}$ which is the disjoint union of two disks with same radius

$$z \in \mathcal{R} \coloneqq D(mx_0, |mr_0|) \cup D(-mx_0, |mr_0|)$$

where

$$x_0 \coloneqq \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1-\|V\|_1^2)} + \frac{1}{2}} \qquad r_0 \coloneqq \sqrt{\frac{\|V\|_1^4 - 2\|V\|_1^2 + 2}{4(1-\|V\|_1^2)} - \frac{1}{2}};$$

in particular the spectrum of the massless Dirac operator, corresponding to the case $m = 0$ with non Hermitian potential V is $\mathbb{R}$.

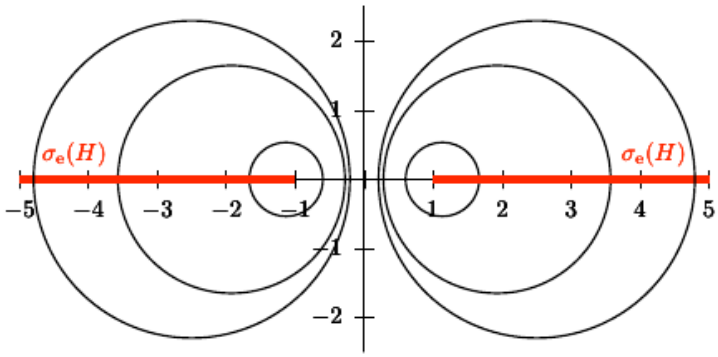


Figure 1.9: The Region $\mathcal{R}$ described in Theorem1.23 for different values of $\|V\|_1$ and fixed mass $m = 1$.

1.4.2 The Graphene operator

We consider now the graphene case. It turns out that the Hamiltonian of the two layers structure, near the Fermi points[†], can be approximated via a second order operator defined on the whole $\mathbb{R}^2$. The derivation of such operator follows after the use of the tight-binding methodology in order to describe a model for the free electrons in the graphene sheets.

From a a nano-material point of view, graphene is a two dimensional material made of a single layer of carbon atoms which are arranged in a honeycomb lattice structure. The structure is made possible by the bounds which are present between the various atoms. Those bounds are realised by a number of electrons that each carbon atom shares with its neighbours. In particular, out of six electrons of which every carbon atom is made of, only one does not participate into the formation of such bounds and is free to move and thus potentially to allow the graphene' sheet to conduct electricity. In fact, graphene is a *zero-gap* semiconductor, meaning that it cannot be fully categorized either as a perfect conductor (metals) nor as a perfect insulator. The zero gap terminology means that the conductance and valence bands, respectively the energy bands at which the material turns out to be a conductor or an insulator, have no separation. This implies that for any small variation of energy, off from the *Fermi*

[†]Points in the momentum space where the energy of the system is close to the Fermi level

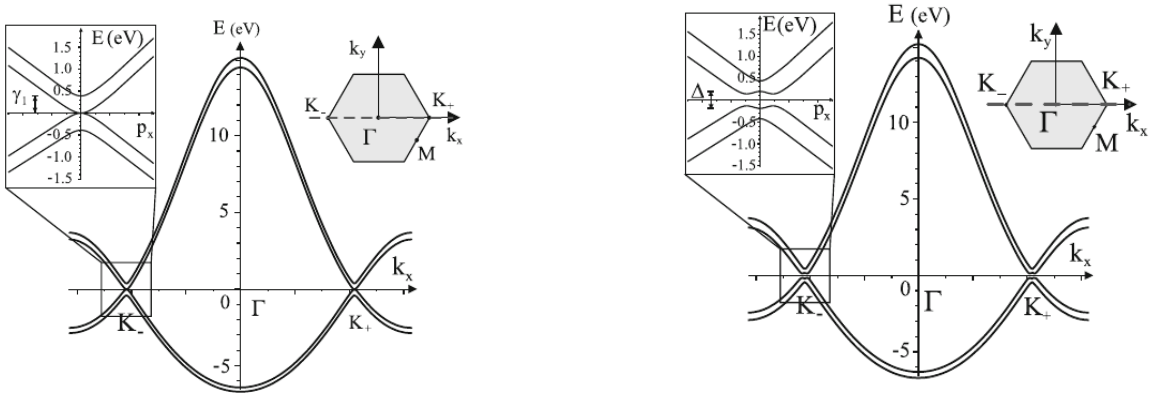
level[‡], a change in the nature of the material itself is achievable.

From a physical point of view, it turns out that, for energies sufficiently close to the Fermi level, the behaviour of the free (conducting) electrons can be described using a second order differential operator defined on $\mathbb{R}^2$ of the Dirac-like type.

In particular, the operator of the bilayer graphene reads as

$$D_m = \begin{pmatrix} m & 4\partial_{\bar{z}}^2 \\ 4\partial_z^2 & -m \end{pmatrix}, \qquad \partial_{\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big), \quad \partial_z = \overline{\partial_{\bar{z}}}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

While not engaging in details for the derivation of the operator defined above, we will try to explain why this operator falls into the class of the Dirac-like one, at least heuristically. For this purpose, a meaningful interpretation of the following figures can help.



(a) Case $m = 0$.

(b) Case $m \neq 0$.

Figure 1.10: Low-energy band structure. Ref. [137]

They describe the four (which reduce to only two, in the case of single layer) energy bands near the value of the Fermi level (In the plot the Fermi level corresponds to zero). In particular they show the dependence of the energy (kinetic) over the parameters k_x, k_y which can be interpreted as the physical momentum operator components. In mathematical terms, these two graphs can be interpreted as the plots of the symbols of the operators responsible for describing the Hamiltonian of the system. We can

[‡]The Fermi energy is the value for which there is a 50% of possibility to find any electron in the solid at the thermodynamic equilibrium with that energy level.

see in the zoom in, how for values of the energy near the Fermi level and for $m = 0$, the plot seems to have a linear behaviour, which re interpreted in terms of a symbol of an operator makes clear the connection with the Dirac one. We can also visualise the meaning of the zero gap given before. In fact the two bands collide in two points creating no gap. Gap which is in fact induced in the second plot, where $m \neq 0$. In practical experiences in laboratory, a *mass gap* can be introduced by physical strains or impurities of the lattice which alter the symmetry of the geometry.

We refer to the collection [137] and to Mc Cann's [119] for both the derivation of the Hamiltonian, for a detailed introduction of the tight-binding methodology used for the derivation of the operators in question and for further details on graphene's electronic properties.

1.4.3 Uniform bounds of the resolvent norm

Let us consider the following operators acting on the Hilbert space $L^2(\mathbb{R}^2; \mathbb{C}^2)$

$$D = D_0 + m\gamma_0 + V = D_m + V, \tag{1.45}$$

$$D_0 = \begin{pmatrix} 0 & 4\partial_{\bar{z}}^2 \\ 4\partial_z^2 & 0 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $\partial_{\bar{z}}$ and ∂_z denote the Wirtinger derivative operators on $\mathbb{R}^2$

$$\partial_{\bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big), \qquad (x_1, x_2) \in \mathbb{R}^2,$$
$$\partial_z = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big), \qquad (x_1, x_2) \in \mathbb{R}^2.$$

The potential V is a not necessary self-adjoint matrix-valued function

$$V(x) = \begin{pmatrix} V_{1,1}(x) & V_{1,2}(x) \\ V_{2,1}(x) & V_{2,2}(x), \end{pmatrix}$$

where the matrix elements are allowed to take complex values.

For $p \geq 1$ we consider the space $\mathcal{M}_{2,2}(\mathbb{C})$ of the 2×2 complex-valued matrices endowed

with the Frobenius norm $|\cdot|$ for the matrix $V(x)$

$$|V(x)| = \sqrt{\sum_{i,j=1,2} |(V(x))_{i,j}|^2} \tag{1.46}$$

and consider the space $L^p(\mathbb{R}^2; \mathcal{M}_{2,2}(\mathbb{C}))$ of the $\mathcal{M}_{2,2}(\mathbb{C})$-valued measurable function on $\mathbb{R}^2$ with finite L^p-norm

$$L^p(\mathbb{R}^2; \mathcal{M}_{2,2}(\mathbb{C})) = \left\{ V(x) : \|V(x)\|_p^p = \int_{\mathbb{R}^2} \left(\sum_{i,j=1,2} |(V(x))_{i,j}|^2 \right)^{p/2} dx < \infty \right\}.$$

The domain of D is the Sobolev space $\mathcal{H}^2(\mathbb{R}^2; \mathbb{C}^2)$. In the following we will refer to $D_m = D_0 + m\gamma_0$, where

$$D_m = \begin{pmatrix} m & 4\partial_{\bar{z}}^2 \\ 4\partial_z^2 & -m \end{pmatrix}, \tag{1.47}$$

as the free energy bi-dimensional two layers Graphene Hamiltonian's operator with mass m. For future reference, we list some elementary properties satisfied by D_0 and D_m.

$$\begin{aligned} D_m^2 &= \begin{pmatrix} m^2 + \Delta^2 & 0 \\ 0 & m^2 + \Delta^2 \end{pmatrix}, \\ D_0^2 - \mu^2 &= (D_0 + \mu)(D_0 - \mu), \\ (D_m - k)^{-1} &= (D_m + k)(D_m^2 - k^2)^{-1}. \end{aligned} \tag{1.48}$$

In view of the identities (1.48), we note that $|D_m|$ is unitarily equivalent to $\sqrt{\Delta^2 + m^2}$. Thus the spectrum of the operator D_m consists only of its essential part

$$\sigma(D_m) = \sigma_{ess}(D_m) = \{\pm(p^2 + m^2)^{1/2} \mid p \in \mathbb{R}\}.$$

In the following we will assume that $V(x)$ admits a factorisation

$$V(x) = B(x)A(x),$$

such that Theorem 1.17 and Theorem 1.18 hold for $H_0 = D_m$ and the perturbations $V(x)$ leave the essential spectrum of the perturbed operator D unchanged.

It is an easy computation to see that the resolvent operator for D_m admits a decomposition in terms of the massless operator D_0

$$\begin{aligned}(D_m - k)^{-1} &= (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1}, \\ &= (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 + \mu)(\Delta^2 - \mu^2)^{-1},\end{aligned} \tag{1.49}$$

where

$$\mu^2 = k^2 - m^2. \tag{1.50}$$

The operator $(\Delta^2 - \mu^2)^{-1}$ is an integral operator with kernel $g_\mu(x, y)$

$$g_\mu(x, y) = \frac{i}{8\mu}\Big(H_0^{(1)}(\sqrt{\mu}r) - H_0^{(1)}(i\sqrt{\mu}r)\Big), \tag{1.51}$$

where $x, y \in \mathbb{R}^2$, $r = |x - y|$ is the usual Euclidean distance between two points in $\mathbb{R}^2$ and $H_0^{(1)}(z)$ is the Hankel function of first kind and zero order. This is a consequence of the fact that it is possible to express the fundamental solution of the Helmoltz equation for the bi-harmonic operator in terms of the fundamental solution of the Helmoltz equation for the Laplacian operator. Namely, the following decomposition holds true

$$(\Delta^2 - \mu^2)^{-1} = \frac{1}{2\mu}\Big((-\Delta - \mu)^{-1} - (-\Delta + \mu)^{-1}\Big).$$

Equation (1.51) simply follows after noting that $4^{-1}iH_0^{(1)}(\sqrt{\mu}r)$ is the kernel of the operator $(-\Delta - \mu)^{-1}$.

For the sake of clarity and to have a short notation at hand, we introduce the auxiliary function

$$G(z) = H_0^{(1)}(z) - H_0^{(1)}(iz),$$

so that the kernel (1.51) of the operator $\Delta^2 - \mu^2$ can be re-written as

$$g_\mu(x, y) = \frac{i}{8\mu}G(\sqrt{\mu}|x - y|). \tag{1.52}$$

Remark 1.13. We note here that in fact the points $z \in \mathbb{C}$ in which the function $G(z)$ will be evaluated are of the particular shape $z = \sqrt{\lambda}|x|$ where $|x| \in \mathbb{R}$ and $\lambda = \sqrt{k^2 - m^2} \in \mathbb{C}$. Even though in the following, $k \in \mathbb{C}$ will be the complex eigenvalue of the perturbed operator D and so it will be free to lie everywhere in the

complex plane, the presence of the double complex square root in the formulation of $G(z)$ let us to restrict from now on, without loss of generality, to the case where $z \in \mathcal{Q}$, where $\mathcal{Q} = \{z \in \mathbb{C} | \mathfrak{Re} z \geq 0, \mathfrak{Im} z \geq 0\}$ is the first quadrant of the complex plane.

The remark above is of some importance in view of the following lemma, since it restricts the domain of the function $G(z)$ to the 'good one'.

Lemma 1.24. *Consider $z \in \mathcal{Q}$ in the first quadrant of the complex plane. If $|z| < 1/2$ then the following estimates hold true for the function $G(z)$:*

(i) $|G(z)| \leq C$,

(ii) $|G'(z)| \leq C|z| \ln(|z|^{-1})$,

(iii) $|G''(z)| \leq C \ln(|z|^{-1})$.

If instead $|z| \geq 1/2$, then the asymptotic behaviour of $G(z)$ and its derivates is determined by the following

$$|G(z)| + |G'(z)| + |G''(z)| \leq \frac{C}{\sqrt{|z|}}.$$

All the constants are independent from $z \in \mathcal{Q}$.

Proof. We start proving the first statement which holds for $|z| < 1/2$. We recall the formula for the Hankel function $H_n^{(1)}(z)$

$$H_n^{(1)}(z) = J_n(z) + iY_n(z) \qquad z \in \mathbb{C}$$

where J_n and Y_n are the n-order Bessel functions of first and second kind respectively, so that

$$G(z) = J_0(z) - J_0(i\,z) + i\Big(Y_0(z) - Y_0(i\,z)\Big) \qquad z \in \mathbb{C}. \tag{1.53}$$

The behaviour of the function $G(z)$ for values of $|z| < 1/2$ is determined by analysing the expansion formula for the Hankel function near zero. We have, see for example

Olver [132] chapter 10,

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{z^2}{4})^k}{4!\Gamma(n+k+1)} \tag{1.54}$$

$$\begin{aligned} Y_n(z) = &-\frac{(\frac{z}{2})^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z^2}{4}\right)^k + \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_n(z) \\ &- \frac{(\frac{z}{2})^n}{\pi} \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{(-\frac{z^2}{4})^k}{k!(n+k)!} \end{aligned} \tag{1.55}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In particular, for $n = 0$ we obtain

$$\begin{aligned} J_0(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(k!)^2 2^{k2}} = 1 - \frac{z^2}{4} + \frac{z^4}{2^6} - \dots, \\ Y_0(z) &= \frac{2}{\pi} J_0(z) \left(\log(\frac{z}{2}) + \gamma\right) + \frac{2}{\pi} \left(\frac{\frac{1}{4}z^2}{1!} - (3/2) \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + (11/6) \frac{(\frac{1}{4}z^2)^3}{(3!)^2} - \dots \right), \end{aligned}$$

where γ is the Euler-Mascheroni constant. By means of the expansion formulae above we observe that in (1.53) the logarithmic singularity present at the origin disappear, and therefore it follows the boundedness of $|G(z)| < C$ for $|z| < 1/2$.

Similar arguments apply to the estimates of the first and second derivatives

$$\frac{d}{dz} G(z) = \frac{d}{dz} H_0^{(1)}(z) - i \frac{d}{dz} H_0^{(1)}(iz), \qquad z \in \mathbb{C}, \tag{1.56}$$

$$\frac{d^2}{dz^2} G(z) = \frac{d^2}{dz^2} H_0^{(1)}(z) + \frac{d^2}{dz^2} H_0^{(1)}(iz), \qquad z \in \mathbb{C}, \tag{1.57}$$

where

$$\frac{d}{dz} H_0^{(1)}(z) = -H_1^{(1)}(z), \tag{1.58}$$

$$\frac{d^2}{dz^2} H_0^{(1)}(z) = H_0^{(1)}(z) - \frac{1}{z} H_1^{(1)}(z). \tag{1.59}$$

Equation (1.54) and (1.55) for $n = 1$ read

$$J_1(z) = \sum_{k=0}^{+\infty} (-1)^k \frac{(z/2)^{1+2k}}{k!(k+1)!} \qquad z \in \mathbb{C}$$

$$Y_1(z) = J_1(z)\log(z) - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{1+2k}}{k!(k+1)!}\left[-2\gamma + 2\sum_{n=1}^{k}\frac{1}{n} + \frac{1}{k+1}\right] - \frac{2}{\pi z}$$
$$= J_1(z)\log(z) - g(z) - \frac{2}{\pi z},$$

where $g(z)$ is bounded. Equations (1.56) together with (1.58) and the formulas above imply that the behaviour of $|\frac{d}{dz}G(z)|$ is completely determined by the term

$$|J_1(z)\log(z)| \sim |z|\ln(|z|^{-1}).$$

The term $|\frac{d^2}{dz^2}G(z)|$ can be estimated using similar arguments and we omit the details.

On the other hand, the behaviour of $G(z)$ for $|z| > 1/2$ can be easily determined by observing that for $\nu > 0$ and $z \in \mathcal{Q}$, where we recall $\mathcal{Q} = \{z \in \mathbb{C} | \mathfrak{Re} z \geq 0, \mathfrak{Im} z \geq 0\}$, the following holds

$$\mathcal{K}_\nu(-iz) = \frac{\pi i}{2} e^{\nu i \pi/2} H_\nu^{(1)}(z),$$

where, as seen in the previous section in formula (1.30), the following integral representation formula holds

$$\mathcal{K}_\nu(z) = \frac{e^{-z}}{(\nu - 1/2)!}\sqrt{\frac{\pi}{2z}}\int_0^\infty e^{-t}t^{\nu-1/2}\big(1 + \frac{t}{2z}\big)^{\nu-1/2}dt, \qquad |\arg z| < \pi.$$

which can be found in Abramowitz [3]. □

Theorem 1.25. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator D, $1 < p < 4/3$ and set*

$$\omega(k,m) = \Big(\sqrt{\Big|\frac{k+m}{k-m}\Big|} + \sqrt{\Big|\frac{k-m}{k+m}\Big|} + 1\Big).$$

Then there exists a constant C_p which depends on p such that

$$\frac{C_p \int_{\mathbb{R}^2} |V(x)|^p dx}{|\mu|^{p-1}}\omega(k,m)^p \geq 1, \qquad \mu^2 = k^2 - m^2. \tag{1.60}$$

In particular, if $m = 0$:

$$|k|^{p-1} \leq C_p \int_{\mathbb{R}^2} |V(x)|^p dx, \qquad 1 < p < 4/3.$$

Proof. Let's first consider the case when the mass $m = 0$ and $\mu \in \mathbb{C}$ is a complex

number, and consider the operator $(D_0-\mu)^{-1}$. It is an integral operator with kernel the matrix $\rho_\mu(|x-y|)$

$$\rho_\mu(|x-y|)=\frac{i}{8\mu}\begin{pmatrix}\mu G(\sqrt{\mu}|x-y|) & \partial_{\bar z}^2 G(\sqrt{\mu}|x-y|)\\ \partial_z^2 G(\sqrt{\mu}|x-y|) & \mu G(\sqrt{\mu}|x-y|).\end{pmatrix} \tag{1.61}$$

It follows that its Frobenious norm (1.46) is

$$|\rho_\mu(|x-y|)|=\frac{1}{8|\mu|}\sqrt{2|\mu|^2|G(\sqrt{\mu}|x-y|)|^2+|\partial_{\bar z}^2 G(\sqrt{\mu}|x-y|)|^2+|\partial_z^2 G(\sqrt{\mu}|x-y|)|^2}, \tag{1.62}$$

where the second order derivative reads as

$$\begin{aligned}4\partial_{\bar z}^2 G(\sqrt{\mu}|x-y|)=\mu G''(\sqrt{\mu}|x-y|)\Big[\sum_{i=1,2}\frac{(x_i-y_i)^2}{|x-y|^2}\Big]\\ +2i\,\mu G'(\sqrt{\mu}|x-y|)^2\Big[\prod_{i=1,2}\frac{(x_i-y_i)}{|x-y|}\Big],\end{aligned} \tag{1.63}$$

and for which we have the following estimate

$$|\partial_{\bar z}^2 G(\sqrt{\mu}|x-y|)|^2\le C|\mu|^2\Big(|G''(\sqrt{\mu}|x-y|)|^2+|G'(\sqrt{\mu}|x-y|)|^4\Big). \tag{1.64}$$

As a consequence, if we denote by $\rho_\theta(|x-y|)$ the kernel of the operator $(D_0-e^{i\theta})^{-1}$ then

$$|\rho_\theta(r)|\le C\ln r^{-1},\qquad \text{if}\qquad r<1/2, \tag{1.65}$$

and

$$|\rho_\theta(r)|\le Cr^{-1/2},\qquad \text{if}\qquad r>1/2. \tag{1.66}$$

The positive constants in the inequalities (1.65) and (1.66) do not depend on $\theta\in(0,\pi/2)$. In particular, we deduce the following L^q-estimate for the kernel

$$M_q=\sup_\theta\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|\tilde\rho_\theta(|x-y|)|^q\,dy<\infty$$

which holds uniformly for $\theta\in(0,\pi/2)$ if and only if $q>4$. We observe that the latter condition simply follows from (1.66).

Let U be the partial isometry in the polar decomposition for $V = U|V|$ and consider its following factorization:

$$V = AB, \qquad A \coloneqq U|V|^{1/2} \qquad B \coloneqq |V|^{1/2.} \tag{1.67}$$

Let us estimate now the norm of the operator $T = A(D_0 - e^{i\theta})^{-1}B$ whose kernel is

$$\tau(x,y) = A(x)\rho_\theta(|x-y|)B(y).$$

For this purpose we estimate the sesqui-linear form of this operator

$$(Tu,v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bar{v}(x)A(x)\rho_\theta(|x-y|)B(y)u(y)\,dxdy,$$

so that, for $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
|(Tu,v)|^2 &= \Big|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bar{v}(x)A(x)\rho_\theta(|x-y|)B(y)u(y)\,dxdy\Big|^2 \\
&\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v(x)|^2|\rho_\theta(|x-y|)||V(y)|\,dxdy \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |V(x)||\rho_\theta(|x-y|)||u(y)|^2\,dxdy \\
&\le \Big(\sup_x \int_{\mathbb{R}^2} |\rho_\theta(|x-y|)||V(y)|\,dy\Big)^2 ||u||^2\,||v||^2 \\
&\le \Big(\int_{\mathbb{R}^2} |\tilde{\rho}_\theta(|x-y|)|^q\,dy\Big)^{2/q} ||V||_p^2\ ||u||^2\,||v||^2.
\end{aligned}$$

Thus the following estimates

$$||T|| \le M_q^{1/q}||V||_p, \qquad 1 < p < 4/3$$

holds uniformly for the angle $\theta \in (0, \pi/2)$.

Similarly, it follows the estimate for the sequilinear form for the operator $T_\mu = A(D_0 - \mu)^{-1}B$ for $\mu \in \mathbb{C} \setminus \sigma(D_0)$

$$
\begin{aligned}
|(T_\mu u, v)| &= \Big| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x) A(x) \rho_\mu(|x-y|) B(y) u(y)\, dxdy \Big| \\
&\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\bar{v(x)}||A(x)||\rho_\theta(\sqrt{|\mu|}|x-y|)||B(y)||u(y)|\, dxdy \\
&\leq \frac{1}{|\mu|^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\bar{v}(x/\sqrt{|\mu|})||A(x/\sqrt{|\mu|})||\rho_\theta(|x-y|)||B(y/\sqrt{|\mu|})||u(y/\sqrt{|\mu|})|\, dxdy \\
&\leq M_q^{1/q} \frac{1}{|\mu|^2} \|V(\cdot/\sqrt{|\mu|})\|_p \|u(\cdot/\sqrt{|\mu|})\| \ \|v(\cdot/\sqrt{|\mu|})\| \leq \frac{M_q^{1/q} \|V\|_p}{|\mu|^{(1-1/p)}} \|u\| \ \|v\|.
\end{aligned} \tag{1.68}
$$

Consequently,

$$
\|T_\mu\| \leq \frac{M_q^{1/q} \|V\|_p}{|\mu|^{(p-1)/p}}. \tag{1.69}
$$

Following a strategy similar to the one used for the term T_ν, and from the definition of the Frobenius norm given in (1.46) it is an easy computation to check that it holds also

$$
||A(m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} B|| \leq C||V||_p \frac{\omega(k,m)}{|\mu|^{(p-1)/p}}, \qquad 1 \leq p < 4/3, \tag{1.70}
$$

where we set

$$
\omega(k,m) = \Big(\sqrt{\Big|\frac{k+m}{k-m}\Big|} + \sqrt{\Big|\frac{k-m}{k+m}\Big|} + 1 \Big).
$$

In fact we observe that the kernel of the operator $(\Delta^2 - \mu^2)^{-1}$ is the matrix $iG(\sqrt{\mu}|x-y|)/(8\mu) I_{2\times 2}$, where $I_{2\times 2} \in \mathcal{M}_{2,2}(\mathbb{C})$ is the identity matrix, which has the same behaviour of $\rho_\theta(\sqrt{|\mu|}|x-y|)$ at infinity and is bounded near the origin.

We can finally state the estimate for the operator D_m. From equation (1.49) we have that the kernel splits in

$$
\begin{aligned}
(D_m - k)^{-1} &= (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1} \\
&= \begin{pmatrix} m + k - \mu & 0 \\ 0 & -m + k - \mu \end{pmatrix} (\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1}.
\end{aligned} \tag{1.71}
$$

The former representation with the estimates (1.69) and (1.70) leads to the final

$$||A(D_m - k)^{-1}B||^p \leq C\frac{\omega(k,m)^p}{|\mu|^{p-1}}\int_{\mathbb{R}^2}|V(x)|^p dx, \qquad \mu^2 = k^2 - m^2. \tag{1.72}$$

The statement of the theorem follows from the fact that if k is an eigenvalue of $D = D_m + V$, then $||A(D_m - k)^{-1}B|| \geq 1$ due to the Birman-Swinger principle. □

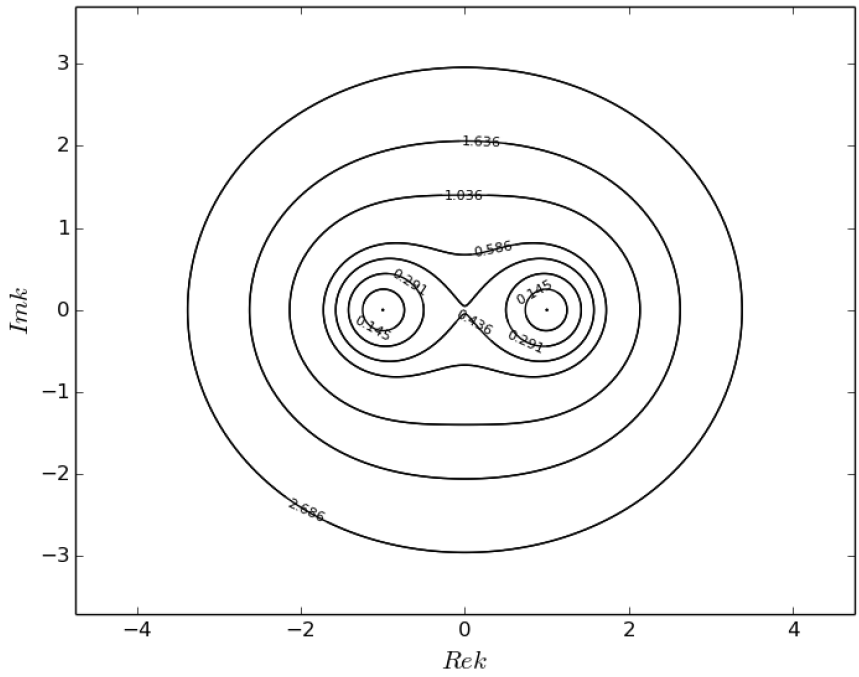


Figure 1.11: The region $\mathcal{R}$ of the complex plane which contains the discrete spectrum of the operator $D + V$ described in (1.60) for different values of $C\int_{\mathbb{R}^2}\|V\|^p$, where $m = 1$ and $p = 1.2$.

The previous theorem gives an estimate which can be refined if more information about the region where the eigenvalue k lies are available, in particular similarly to what happens in [60] if we consider eigenvalues away from the essential spectrum.

Let's fix the value of the mass $m \in \mathbb{R}$ and consider $\alpha \in \mathbb{R}$ such that $0 < \alpha < \pi$. We define

$$S^m_{-\alpha,\alpha} \coloneqq \{z + m^2 \in \mathbb{C} \mid -\alpha \leq \arg(z) \leq \alpha\}$$

a sector of the complex plane with vertex centred in m^2 with opening angle 2α. Let's suppose now that the eigenvalue $k \notin S^m_\alpha$ for some admissible value of α. It follows that the corresponding value of $\sqrt{\mu}$ lies in the sector $S_{\beta,\pi/2-\beta}$ which is completely inside

the in first quadrant of the complex plane and where $\beta = \alpha/4$ and $m = 0$ has been omitted. This in turn gives the possibility to improve the estimate for $|z| > 1/2$ in Lemma 1.24. We can indeed assume that the decay of the function $|G(z)|$ at infinity goes with an exponential rate, as well as does it for $|G'(z)|$ and $|G''(z)|$, and so we can drop off the constraint on the exponents and allow the estimate in the theorem to hold for any p and q real and conjugated.

Theorem 1.26. *Consider $m \in \mathbb{R}$ and $0 < \alpha < \pi$. Let $k \notin (\sigma(D_m) \cup S^m_\alpha)$ be an eigenvalue of the operator D. Then for every $p > 1$ it holds true the following:*

$$\frac{C\int_{\mathbb{R}^2}|V(x)|^p dx}{|\mu|^{p-1}}\Big(\sqrt{\Big|\frac{k-m}{k+m}\Big|}+\sqrt{\Big|\frac{k+m}{k-m}\Big|}+1\Big)^p \geq 1, \qquad \mu^2 = k^2 - m^2, \tag{1.73}$$

where the constant $C = C(\alpha)$ depends only on the angle spanned by α.
In particular, if $m = 0$, then

$$|k|^{p-1} \leq C\int_{\mathbb{R}^2}|V(x)|^p dx.$$

Moreover, if $\|V(x)\|_\infty < \infty$, then it holds true:

$$\frac{\Big(\sqrt{\Big|\frac{k-m}{k+m}\Big|}+\sqrt{\Big|\frac{k+m}{k-m}\Big|}+1\Big)}{|\mu|} \leq C\|V\|_\infty \tag{1.74}$$

Proof. The proof follows exactly the same steps of the proof of the Theorem 1.25 therefore we will note produce them again. We only note that under the more strict hypothesis on $k \notin (\sigma(D_m) \cup S^m_\alpha)$, we have that

$$M_q = \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|\tilde{\rho}_\theta(|x-y|)|^q\,dy = \int_{\mathbb{R}^2}|\tilde{\rho}_\theta(|x-y|)|^q\,dy < \infty$$

holds uniformly for $\theta \in [\epsilon, \pi/2 - \epsilon]$ for $\epsilon > 0$ and for all the real $1 \leq q < \infty$. □

In the spirit of the proof given for the previous theorems we present now a different type of estimate which holds true for the operator D.

Theorem 1.27. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator D. Let $\mu^2 = k^2 - m^2$. Then*

$$C\bigg(|\ln|\mu||\sup_{x\in\mathbb{R}^2}\int_{|x-y|<(2|\mu|)^{-1}}|V(y)|\,dy+\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(1+|\ln|x-y||\Big)|V(y)|dy\bigg)+$$
$$+C\int_{\mathbb{R}^2}|V(x)|dx\Big(\sqrt{\Big|\frac{k-m}{k+m}\Big|}+\sqrt{\Big|\frac{k+m}{k-m}\Big|}+1\Big)\geq 1. \tag{1.75}$$

Proof. Again, we use the representation formula (1.49)

$$(D_m-k)^{-1}=(m\gamma_0+k-\mu)(\Delta^2-\mu^2)^{-1}+(D_0-\mu)^{-1}.$$

The estimate for the operator $(m\gamma_0+k-\mu)(\Delta^2-\mu^2)^{-1}$ is derived as in the previous theorems and reads as in (1.70) with $p=1$.

We are left with the estimate of the norm of the operator $T_\mu = A(D_0-\mu)^{-1}B$ for $\mathrm{Im}\mu > 0$ where again A and B come from the factorization of the potential V in (1.67). The operator $(D_0-\mu)^{-1}$ is the integral operator with kernel $\rho_\mu(|x-y|) = \rho_\theta(\sqrt{|\mu||x-y|})$ where $\rho_\theta(r)$ satisfies (1.65) and (1.66) for $\mu = e^{i\theta}|\mu|$. As before, we consider the sesqui-linear form of this operator

$$(T_\mu u, v)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bar{v}(x)A(x)\rho_\mu(|x-y|)B(y)u(y)\,dxdy,$$

for which the following estimate holds

$$|(T_\mu u,v)|^2=\Big|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bar{v}(x)A(x)\rho_\mu(|x-y|)B(y)u(y)\,dxdy\Big|^2$$
$$\leq\Big(\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|\rho_\mu(\sqrt{|\mu|}|x-y|)||V(y)|\,dy\Big)^2||u||^2\,||v||^2$$

Therefore we deduce

$$||T_\mu||\leq\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|\tilde{\rho}_\mu(|x-y|))||V(y)|\,dy=\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|\rho_\theta(\sqrt{|\mu|}|x-y|))||V(y)|\,dy.$$

Let us define the region $\Sigma_\mu^x=\{y\in\mathbb{R}^2 \mid |\mu||x-y|<1/2\}\subseteq\mathbb{R}^2$. We can then split

the above integral into two complementary regions so to obtain

$$\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}|\tilde{\rho}_\theta(\sqrt{|\mu|}|x-y|))||V(y)|\,dy$$
$$=\sup_{x\in\mathbb{R}^2}\Big(\int_{\Sigma_\mu^x}|\tilde{\rho}_\theta(\sqrt{|\mu|}|x-y|))||V(y)|\,dy+\int_{\mathbb{R}^2\setminus\Sigma_\mu^x}|\tilde{\rho}_\theta(\sqrt{|\mu|}|x-y|))||V(y)|\,dy\Big)$$
$$\le C\Big(|\ln|\mu||\sup_{x\in\mathbb{R}^2}\int_{\Sigma_\mu^x}|V(y)|\,dy+\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(1+|\ln|x-y||\Big)|V(y)|dy\Big),$$

where we used (1.65) and (1.66) to obtain the last inequality.

The estimate above in turn implies

$$||T_\mu||\le C\Big(|\ln|\mu||\sup_{x\in\mathbb{R}^2}\int_{|x-y|<(2|\mu|)^{-1}}|V(y)|\,dy+\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(1+|\ln|x-y||\Big)|V(y)|dy\Big).$$

From

$$||A(D_m-k)^{-1}B||\le||A(m\gamma_0+k-\mu)(\Delta^2-\mu^2)^{-1}B||+||T_\mu||$$

we deduce

$$||A(D_m-k)^{-1}B||\le C\Big(|\ln|\mu||\sup_{x\in\mathbb{R}^2}\int_{|x-y|<(2|\mu|)^{-1}}|V(y)|\,dy+\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(1+|\ln|x-y||\Big)|V(y)|dy\Big)+$$

$$+C\int_{\mathbb{R}^2}|V(x)|dx\Big(\sqrt{\Big|\frac{k-m}{k+m}\Big|}+\sqrt{\Big|\frac{k+m}{k-m}\Big|}+1\Big).$$

Now the statement of our theorem follows from the fact that if k is an eigenvalue of $D=D_m+V$, then $||A(D_m-k)^{-1}B||\ge 1$ due to the Birman-Swinger principle. □

Remark 1.14. In view of (1.75), we note that if $m=0$, then for small V, the complex eigenvalues of D are situated in a circle of radius r which is roughly estimated by

$$r\approx\exp\Big(-\frac{C}{\int_{\mathbb{R}^2}|V|dx}\Big)\qquad\text{as}\qquad\int_{\mathbb{R}^2}|V(y)|dy,\int_{\mathbb{R}^2}|V(y)|\ln(x-y)dy\to 0.$$

contrarily to what happen to the one dimensional Dirac operator, as described in Theorem 1.23, where the spectrum has to be real.

1.4.4 Uniform estimates in Schatten class

In this last section we recall the results obtained by Cuenin [30] which extend the ones proved in Section 1.4.3.

The following result should be interpreted as the analogue for the bilayer graphene operator of the Laplacian uniform resolvent estimates (1.15).

Proposition 1.28 (Cuenin [30]). *Let $1 \leq q \leq 3/2$. There exists $C > 0$ such that for any $A, B \in L^{2q}(\mathbb{R}^2; \mathcal{M}_{2,2}(\mathbb{C}))$ and $z \in \rho(D_m)$ we have the inequality*

$$\|A(D_m - z)^{-1}B\|_{\mathfrak{S}^{\max(q,\alpha_{q,1/2,2})}} \leq C|k(z)|^{\frac{2}{q}-2}(|\zeta(z)| + |\zeta(z)|^{-1})\|A\|_{2q}\|B\|_{2q} \tag{1.76}$$

where

$$\alpha_{q,1/2,2} = \begin{cases} \frac{2(d-1-r)q}{d-q} & \text{if } \frac{d}{d-r} \leq q \leq 1+r \\ \frac{2rq_+}{2rq-d(q-1)} & \text{if } 1 \leq q \leq \frac{d}{d-r} \end{cases}$$

for $r > 0$, $1 \leq q \leq 1+r$ and where $2qr_+ = 2qr + \epsilon$ with $\epsilon > 0$ any positive real number.

Theorem 1.29 (Cuenin [30]). *Let $V(x) \in L^q(\mathbb{R}^2; \mathcal{M}_{2,2}(\mathbb{C}))$ with $1 \leq q \leq 3/2$ and consider*

$$k(z)^4 := z^2 - m^2, \qquad \zeta(z) := \frac{z+m}{k^2(z)}.$$

Then the following estimates hold for $D = D_m + V$ with some constant C_q independent of V, z and m:

i) If $1 < q \leq 3/2$, then any eigenvalue $z \in \sigma_p(D)$ satisfies

$$\frac{|k(z)|^{2q-2}}{(1 + |\zeta(z)| + |\zeta(z)|^{-1})^q} \leq C_q\|V\|_q^q. \tag{1.77}$$

ii) If $q = 1$, then there exists $c_1 > 0$ such that if $\|V\|_1 < c_1$, then

$$\sigma_p(D) \subset \left\{z \in \mathbb{C} \mid |z \pm m| \leq c_1 m\|V\|_1^2\right\} \qquad \text{if } m > 0 \tag{1.78}$$

whereas $\sigma_p(D) = \emptyset$ for $m = 0$.

The proof of the results in Theorem 1.29 follows customarily by means of the resolvent estimates 1.28 and the Birman-Schwinger principle, following the arguments given in

the proof of Frank [57] valid for the Schrödinger operator which we have already encountered in Section 1.2.

It is interesting to note how the range for the exponent $1 < q \leq 3/2$ is exactly the same as in Frank [57]

$$z^{q-1} \leq C_q \|V\|_q$$

that corresponds in terms of the notation previously introduced, to the case $0 < \gamma \leq 1/2$. As noted by Cuenin, it is indeed not surprising as in fact both $-\Delta$ and D_m are second order differential operators. Even more interesting is the difference which exists though for the case $q = 1$, namely $\gamma = 0$. We have already discussed in Section 1.2 that for the Laplacian an estimate like the previous one cannot hold due to the singularity of the fundamental solution which blows up logarithmically at the origin. Differently from the latter case, the estimates valid for the Graphene operator turn out to in fact to be totally legitimate for the operator D_m when $q = 1$.

1.5 Additional results

1.5.1 Number of eigenvalues and stability of the spectrum

In this last subsection we introduce the problem of studying the number of eigenvalues of complex perturbed self-adjoint operators. The scope of this section is not to produce an exhaustive text, rather to highlight the most important results on this topic which appeared close to the treatment of the localisation problem. We commence with a brief historical introduction.

Before the important paper [2] published in 1999, the literature available on this topic consisted of a scarce number of papers. A prosperous period, which turned out to be a fruitful one especially for problems related to the study of the number of eigenvalues. is the one which spans the 50's and 60's of last century. In the following we will review some of the most important results produced during that period. Starting from the works of Naimark on the spectrum of operators of the type

$$\begin{aligned} -y'' + q(x)\, y = 0, & x \in (0, \infty) \\ y'(0) - h\, y(0) = 0, & h \in \mathbb{C}. \end{aligned} \tag{1.79}$$

where $q(x) \in \mathbb{R}$ and the non self-adjointness condition was implemented by introducing a complex constant h in the formulation of the boundary condition at zero and eventually by considering at the same time complex valued potentials. Naimark [126] proved, under some smallness property of the potential of the type

$$\int_0^{+\infty} |q(x)| e^{\epsilon |x|}\, dx < \infty, \qquad \epsilon > 0, \tag{1.80}$$

the finiteness of the number of eigenvalues and the finiteness of their multiplicities. Similar results were later proved by Blashak [16] on the whole real line and Martirosjan [116] and Murtazin [124] for the two and three dimensional cases. Results on the finiteness of the eigenvalues and their multiplicities were also extended by Gasymov [67] to the case of potentials of the type $q(x) = \frac{l(l+1)}{x^2} + V(x)$, where $V(x)$ is a complex valued perturbation.

Would similar results hold under weaker conditions for the potential than the one

presented in (1.80)? A deep result was proved by Pavlov [133] later the same decade. He proved, using methods of inverse spectral theory, that under the milder assumption for the poteintail $q(x)$

$$\sup_{x\in(0,\infty)} |q(x)|e^{\epsilon\sqrt{x}} < \infty, \qquad \text{and} \qquad \int_0^\infty x|q(x)|\,dx < \infty$$

the problem (1.79) has yet finitely many eigenvalues (of finitely multiplicity). Pavlov showed also that this result is in fact sharp. He proved, in fact, that for any $\beta \in (0, 1/2)$ and any point $s \in (0, \infty)$ in the essential spectrum there exists a real potential $V(x)$ such that

$$\sup_{x\in(0,\infty)} |V(x)|e^{\epsilon x^\beta} < \infty$$

and such that the problem (1.79) has infinitely many eigenvalues accumulating at the point s. Even more interestingly, this result not only showed the sharpness of the exponent $1/2$, but shed some light on the differences which occur between the self-adjoint and non self-adjoint case. If we restrict to real potential, the Bargmann's estimate says that it is possible to control the number of the eigenvalues of a (self-adjoint) operator L by mean of a certain integral of the potential. More precisely [144], for $c > -1/4$ and $q \in L^1_{\text{loc}}(0, \infty)$ the number N of the negative eigenvalues of the operator

$$L := -y'' + \frac{c}{x^2} + q(x)$$

must satisfy

$$N\sqrt{4c+1} < \int_0^\infty q_-(r)r\,dr,$$

where $q_-(r) = \min\{0, q(r)\}$. In particular, from the Pavlov's result valid for $\beta \in (0, 1/2)$, one deduces the impossibility of extending the Bargmann's type of bounds on the number of eigenvalues valid for real potential in the self-adjoint case to the non self-adjoint one. Pavlov's type of results on the finiteness of eigenvalues were later extended by Pavlov himself to the three dimensional case and later by Davies [2] and Tunca-Baraimov [154] to the whole real line. We observe that the proofs given by the latter authors rely on topological arguments and they require the same decay's rate at infinity for the potential introduced by Pavlov. More recently, Kir [94] showed that

the operator

$$L := -y'' + \left(\frac{\nu^2 - 1/4}{x^2} + V(x) \right) y, \qquad x \in (0, \infty)$$

with boundary condition $\lim_{x\to 0} x^{-\nu-1/2} y(x) = 1$, where $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > 0$, such that the potential satisfies

$$\int_a^\infty |V(x)|\, dx < \infty \qquad \text{and} \qquad \int_0^a x|V(x)|\, dx < \infty$$

has at most a countable number of eigenvalues which can accumulate to a bounded interval in $\mathbb{R}_+$. If, additionally, there exists $\epsilon > 0$ such that

$$\int_0^\infty x\, V(x) e^{\epsilon x}\, dx < \infty \tag{1.81}$$

then the number of eigenvalues is finite as well as their multiplicities.

It has to be observed that the results mentioned so far have left, in fact, two fundamental questions unanswered. The first one regards the existence of any quantitative bound on the number of complex eigenvalues. In fact, while addressing the problem of the cardinality of the set of eigenvalues, the results mentioned so far do not provide any uniform bound on the number of them. The second question which is left open is whether the Pavlov's results about the cardinality of the set of eigenvalues and its accumulation points set can be extended to operators where the non self-adjointness property comes only from the perturbative term and not in the boundary condition.

The answer to the latter question for the operator $-\Delta + V(x)$ defined either on $\mathbb{R}^d$ or $R^d_+ := \{(x_1, \ldots, x_d) \mid x_d > 0\}$ was given by Bögli [17] showing the existence of bounded, decaying at infinity complex potentials, that have infinitely many eigenvalues in the lower half complex plane that accumulate at every point of the essential spectrum $[0, +\infty)$. We observe that such potentials are obtained via a constructive method. An interesting consequence which follows from Bögli's results is that the Lieb-Thirring inequality

$$\sum_{\lambda \in \sigma(H) \setminus [0,\infty)} |\lambda|^{p-d/2} \leq C_{d,p} \int_{\mathbb{R}^d} V(x)^p\, dx$$

which holds for real potentials for any $p \geq d/2$ if $d \geq 3$, $p > 1$ if $d = 2$ and $p \geq 1$ if

$d = 1$, cannot hold in the non self-adjoint case if $p > d$. It is again another evidence of the fact that one should not in principle expect that results valid for real potentials can be extended straightforwardly in the self-adjoint case for complex perturbations.

For what concerns the existence of quantitive bounds on the number of eigenvalues, a partial answer was given for odd dimensions by Frank, Laptev and Safronov [61] for the operator $-\Delta + V(x)$ defined either on the half line $[0, \infty)$ or $\mathbb{R}^d$ with d odd.

Theorem 1.30 ([61]). *Consider d odd. The number N of eigenvalues of $-\Delta + V$ in respectively $L^2(0, \infty)$ with Dirichlet boundary condition and $L^2(\mathbb{R}^d)$, counting the algebraic multiplicities, satisfies for any $\epsilon > 0$*

$$N \leq \frac{C_d}{\epsilon^2} \left(\int e^{\epsilon x} |V(x)|^{(d+1)/2} \, dx \right)^2 . \tag{1.82}$$

where, when $d = 1$, the constant $C_1 = 1$ and the integral has to be thought over the half line. The proof of the theorem is based on a *trace formula* approach. In particular it exploits the correspondence existing between the zeros of the n-order regularised determinant function of the Birman operator and the eigenvalues of the original problem. The estimate on the number of eigenvalues then follows after novel resolvent bounds in the Schatten classes of Birman operator, obtained by complex interpolation similarly as done by Frank and Sabin in [63] combined with results on bounds on the number of zeroes of analytic functions. For sake of completeness, we recall that similar results on the number of eigenvalues have been obtained by other authors for different type of operators. We mention the results of Stepin [150] for the Laplacian in any dimension and of Hulko on the discrete Laplacian [81] and discrete Dirac operator [82] and the results of Korotyaev and Safronov [96] on the three-dimensional Stark operator.

We conclude this subsection by recalling some results on the non existence of eigenvalues, which can be interpreted as a particular circumstance in terms of estimates on the number of eigenvalues.

We observe that for $d \geq 3$, the uniform Sobolev estimates (1.15) holds in particular for the left end point $p = 2d/(d + 2)$. An interesting consequence of this fact is that this allows to extend (1.14) to the case $\gamma = 0$ for $d \geq 3$, and interpret the inequality so obtained in terms of sufficient condition on the norm of the potential from which

it follows the absence of complex eigenvalue for sufficiently small potential,

$$D_{0,d} \int_{\mathbb{R}^d} |V(x)|^{d/2}\, dx < 1, \tag{1.83}$$

with $D_{0,d} = C_{p,d}^{d/2}$ where $C_{p,d}$ is the same as in (1.15). In particular $D_{0,3} = 4/3^{3/2}\pi^2$. This result, proved firstly by Frank [57] for $\lambda \in \mathbb{C} \setminus [0, \infty)$ and later extended by Frank and Simon [59] to any $\lambda \in \mathbb{C}$, opens up to the matter of existence of *Virtual level* in the non selfadjoint case. If $d \geq 3$, for the Schrödinger operator on the whole space $\mathbb{R}^d$ it is well known that no eigenvalues can appear outside the essential spectrum for small enough real (negative) potential. Therefore the previous result might be interpreted as a generalisation of this phenomenon for complex potentials.

An improved version of (1.83), where a refined constant is found, is proved by Fanelli and collaborators [54] for the Schrödinger operator again for $d \geq 3$. In particular, the result in the special case of $d = 3$ is proved under more general condition coming from a form subordination approach which allows to extend Frank's result for example to potentials of the Hardy type

$$|V(x)| \leq a \left(\frac{d-2}{2}\right)^2 \frac{1}{|x|^2}, \quad a < 1$$

for which the left term in (1.83) would be otherwise infinite. We conclude this chapter mentioning that similar stability property of the spectrum under complex perturbations have been proved by Cossetti [28] for Lame' operators.

2

Trapped modes for a two layer rotating shallow water model in a waveguide

In this chapter we prove the existence of trapped mode solutions for a model of two layers of rotating shallow water equation. In particular, these solutions are in fact connected to the existence of eigenvalues for a second order, self-adjoint operator pencil defined in a planar curved waveguide and subject to mixed boundary condition. The main result of this chapter given in Theorem 2.5 establishes the existence of points in the discrete spectrum under geometric assumptions on the curvature of the waveguide and on the depth profile. The results of this chapter will also be included in the forthcoming paper [55].

2.1 Spectrum of a self-adjoint operator pencil

Similarly to the previous chapter, we start by a very brief section where some classical results in the context of self-adjoint operators, in particular valid for operator pencils are recalled.

Let us consider T_1, T_2 two operators on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and let us assume

them to be self-adjoint for the rest of this section. Further, consider the spectral problem $T_1 f = \lambda T_2 f$ and the linear operator pencil A associated to such problem

$$A(\lambda)f := T_1 f - \lambda T_2 f, \qquad \mathcal{D}(A) = \mathcal{D}(T_1) \cap \mathcal{D}(T_2) \tag{2.1}$$

Definition 2.1. We say that $\lambda \in \mathbb{C}$ belongs to the spectrum of the pencil A, $\lambda \in \sigma(A)$, if for such value λ the operator $A(\lambda)$ is not invertible. If T_1 and T_2 are self-adjoint, it's easily seen that $\lambda \in \mathbb{R}$. Furthermore, the following extends to the pencil case the definition of essential and discrete spectrum given in the second chapter.

- $\lambda \in \sigma_{ess}(A)$, λ belongs to the essential spectrum of the pencil A, if for this value of λ the operator $A(\lambda)$ is not Fredholm.
- $\lambda \in \sigma_d(A)$, λ belongs to the discrete spectrum of the pencil A, if for this value of λ there exists a non trivial solution $\phi \in \mathcal{D}(A)$ of the problem $A(\lambda)\phi = 0$. Such λ will be called again an eigenvalue of the pencil.

In what follows we give a generalisation of the estimate on the spectrum of a operator pencil via the variational characterisation. See Theorem 4.5.2 in Davies [34] for the details of the proof in the case $T_2 = Id$.

Let us define, for any finite-dimensional subspace $L \subseteq \mathcal{D}(A)$ the following quantities

$$\lambda^s(L) := \inf_{f \in L} \frac{\langle T_1 f, f\rangle}{\langle T_2 f, f\rangle}, \qquad \lambda^i(L) := \sup_{f \in L} \frac{\langle T_1 f, f\rangle}{\langle T_2 f, f\rangle}.$$

Let us fix, for any $n \in \mathbb{N}$ the following Rayleigh-Ritz quantities

$$\lambda_n^i := \inf\{\lambda^i(L) \mid L \subseteq \mathcal{D}(A), \dim(L) = n\},$$
$$\lambda_n^s := \sup\{\lambda^s(L) \mid L \subseteq \mathcal{D}(A), \dim(L) = n\}.$$

Of course, the sequence (λ_n^i) is non-decreasing and (λ_n^s) in non-increasing, therefore we can consider respectively their limits

$$\lim_n \lambda_n^i = \lambda^i \qquad \lim_n \lambda_n^s = \lambda^s.$$

It then follows the following estimates on the upper and lower bound for the essential spectrum of a pencil operator.

Proposition 2.1 (Johnson, Levitin, Parnovski [87]). *Let A be the pencil operator as in (2.1) and λ_n^i, λ_n^s, λ^i, λ^n defined as above. Let's suppose $\sigma_{ess}(A) \neq \emptyset$. Then*

i) $\lambda^i = \inf \sigma_{ess}(A) < \infty$. Furthermore, every $\lambda_n^i < \lambda^i$ is an eigenvalue, each repeated according to its multiplicity.

ii) $-\infty < \sup \sigma_{ess}(A) = \lambda^s$. Furthermore, every $\lambda_n^s > \lambda^s$ is an eigenvalue, each repeated according to its multiplicity.

Remark 2.1. Note that in fact, Proposition 2.1 holds in a weaker form where $L \subseteq \mathcal{D}(A^{1/2})$ is a subspace of the form domain of A and the expressions $\langle T_i f, f\rangle$ are interpreted in terms of the quadratic forms associated to T_i. We refer to Section 1.1.2 in Chapter 1 for results on quadratic forms.

In view of Proposition 2.1, a simple criterion for the existence of point in the discrete spectrum follows. We present here only the case when the supremum of the essential spectrum is taken into account. An analogous version holds when the infimum is considered.

Proposition 2.2. *Suppose there exists a function $g \in \mathcal{D}(A^{1/2})$ such that*

$$\frac{\langle T_1 g, g\rangle}{\langle T_2 g, g\rangle} > \lambda^s.$$

Then $\sigma_d(A) \neq \emptyset$.

2.2 The Rotating Shallow Water model

In this section we introduce the fluid-dynamic problem from where the pencil spectral problem originates. We present the set of equations, specified for each layer, that describe in a rotating spatial reference frame the time evolution of the horizontal velocity components together with the incompressibility conditions for the two fluids which are subject to the shallow water regime. We start from a description of the waveguide geometry and the fluid layers' structure. It follows a list of assumptions: the majority of them are introduced because intrinsically related to the nature of the physical model we are considering, whereas some others should be considered as auxiliary mathematical simplification, necessary to perform a certain analysis.

2.2.1 Geometric assumptions

We will consider the fluid occupying a channel $\mathcal{C} \subseteq \mathbb{R}^3$. This region will be fully identified in terms of a planar strip $\mathcal{S} \subseteq \mathbb{R}^2$ contained in the plane spanned by the unit vectors $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ and by its topography. We denote by $\mathbf{e}_z = \mathbf{e}_\xi \times \mathbf{e}_\eta$, the vertical direction perpendicular to the plane (ξ, η) and we also assume that it coincides with the direction of the rotation axes. In particular, we will consider strips of constant width $\delta > 0$ along the direction $\mathbf{e}_\eta$ which extend indefinitely, both ways, along the longitudinal direction $\mathbf{e}_\xi$. The channel depth will be modelled by means of the function $H(\xi, \eta)$. The other two lateral boundaries of $\mathcal{C}$, respectively obtained for $\eta \in \{0, \delta\}$, are understood in our problem respectively as the coastal border and the vertical surface where the shallow water meets the open ocean.

From the assumption of constant width, in order to determine the geometry of the strip it suffices to introduce an infinite planar curve $\Gamma : \mathbb{R} \to \mathbb{R}^2$, the coastal profile, defined as

$$\Gamma := \{\xi \to (X(\xi), Y(\xi)), \, \xi \in \mathbb{R}\},$$

where the components $X(\xi)$ and $Y(\xi)$ are two smooth functions of the arc-length variable ξ, so that

$$|\Gamma'|^2 = X(\xi)'^2 + Y(\xi)'^2 = 1.$$

Associated to it, we also define the normal vector field $P(\xi)$ and the signed curvature $\gamma(\xi)$ defined at each point ξ

$$P(\xi) = (-Y'(\xi), X(\xi)'), \qquad \gamma(\xi) = X''Y' - Y''X'.$$

Let us fix now a planar transformation $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows

$$\Lambda : (\xi, \eta) \to \Gamma(\xi) + \eta P(\xi) = \begin{cases} X(\xi) - \eta Y'(\xi) \\ Y(\xi) + \eta X'(\xi), \end{cases} \tag{2.2}$$

Then, the strip $\mathcal{S}_\gamma$ associated to the coastal profile Γ is uniquely determined (see Kreyszig [99], II.20) by its curvature γ and is the image through the transformation

Λ of the straight band $\mathcal{S}_0 := \{(x, y) \mid x \in \mathbb{R}, y \in [0, \delta]\}$

$$\mathcal{S}_\gamma = \Lambda(\mathcal{S}_0).$$

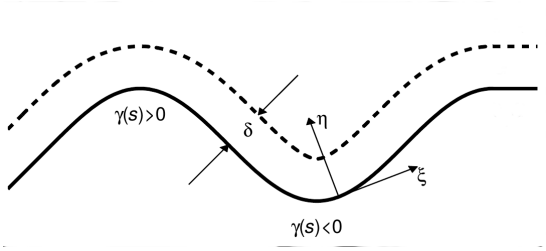


Figure 2.1: Coastal boundary: continuous line $\partial_c\mathcal{S} := \Lambda(\mathbb{R}, 0)$. Open ocean boundary: dashed line $\partial_o\mathcal{S} := \Lambda(\mathbb{R}, \delta)$.

We make the following assumptions.

[CPT] $\gamma(\xi)$ to be a smooth function with compact support contained in $[-R, R]$ for some positive $R > 0$.

[N-I] In order to avoid self-intersection of the channel we introduce the two quantities

$$\kappa^+ = \sup_{\xi\in[-R,R]} \gamma(\xi), \qquad \kappa^- = -\inf_{\xi\in[-R,R]} \gamma(\xi)$$

and we make the additional assumption that concerns the existence of a positive constant $0 \leq \Theta < 1$ such that

$$\kappa^\pm \leq \delta^{-1}\Theta. \tag{2.3}$$

[TOP] We assume a non flat topography of the channel. In particular we restrict to depth profile of the form

$$H(\xi, \eta) = H_1 + H_2(\eta), \qquad H_2(0) = 0 \tag{2.4}$$

where H_1 is a positive constant and $H_2(\eta)$ depends in fact only from the distance from the coast. We furthermore restrict our settings to the case of monotone increasing depth so that the depth profile derivative is non-negative

$$H'(y) = H_2'(y) > 0, \quad y \in [0, \delta). \tag{2.5}$$

In practical use of this model, the constant H_1 is chosen equal to the height of the first layer when the fluids are at rest.

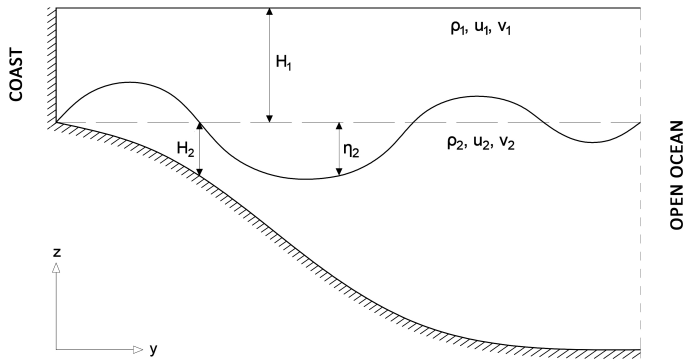


Figure 2.2: Cross section of the channel $\mathcal{C}$ in the plane (y, z). Assumption [TOP] implies the validity of the picture above for any value of $x \in \mathbb{R}$.

2.2.2 The fluid motion equations

We introduce now the set of equations that describe the dynamics of the fluid in the rotating shallow water regime. For simplicity, we do it in the case of straight strip, following the approach proposed in [104]. We refer to it also for their derivation and their physical interpretation (see Chapter III.16, equations 44-47).

In the following, for $i = 1, 2$, we have that $u_i(x, y), v_i(x, y)$ represent the two fluid velocity components in the i-th layer along the x and y direction (respectively ξ and η in curvilinear coordinates). We make the additional assumption on the fluids velocities.

[R-L] We restrict our study only to the *rigid lid* case. This approximation requires that variations of the free surface at the top of the fluids are negligible compared to any other physical quantity of the problem.

[H-V] We assume that for both fluids, their vertical velocity components can be neglected. Therefore the velocities will have only horizontal components.

The displacement of the two fluid layers interface from the equilibrium position H_1 will be modelled using the function $\eta_2(x, y)$. The subscripts in the following equations are introduced, coherently to what's done before, to distinguish the two layers

The equations for the horizontal velocities in first layer are

$$\partial_t u_1 - f v_1 + \frac{1}{\rho_1}\partial_x p_1^0 = 0 \tag{2.6a}$$

$$\partial_t v_1 + f u_1 + \frac{1}{\rho_1}\partial_y p_1^0 = 0, \tag{2.6b}$$

and respectively for the second layer we have

$$\partial_t u_2 - f v_2 + \frac{1}{\rho_2}\partial_x p_1^0 + G\partial_x \eta_2 = 0 \tag{2.6c}$$

$$\partial_t v_2 + f u_2 + \frac{1}{\rho_2}\partial_y p_1^0 + G\partial_y \eta_2 = 0. \tag{2.6d}$$

We conclude with the incompressibility condition of the two fluids

$$H_1(\partial_x u_1 + \partial_y v_1) - \partial_t \eta_2 = 0 \tag{2.6e}$$

$$\partial_x(H_2 u_2) + \partial_y(H_2 v_2) + \partial_t \eta_2 = 0. \tag{2.6f}$$

In the equations above

[PRS] p_1^0 denotes the value of the pressure at the bottom of the channel, therefore assumed to be independent from the depth $p_1^0 = p_1^0(x, y)$. In particular it is constant in time and its spatial derivatives are supposed to be very small.

[DNS] $0 < \rho_1 < \rho_2$ are two positive different constants that describe the density of the two fluids. They are also supposed to be constant in time.

[ROT] The *Coriolis constant* $f > 0$ which describes the rotation velocity of the reference frame.

[RGR] The *reduced gravity* constant $G > 0$, introduced in LeBlond [104] in equation (16.9) as

$$G = g\left(1 - \frac{\rho_1}{\rho_2}\right).$$

In the following, we will require $G \ll 1$ to be small.

Finally, we note the absence of the term η_1 in equations (2.6a), consequence of the rigid lid assumption [R-L].

2.3 The problem in the straight strip $\mathcal{S}_0$

In this section we aim to simplify the formulation of the problem and reduce the number of equations. In this section we also specialise the solutions into the form of trapped modes. We proceed firstly considering the mass conservation equations (2.6e) and (2.6f). Adding them yields

$$\operatorname{div}\big((H_1u_1 + H_2u_2, H_1v_1 + H_2v_2)\big) = 0, \tag{2.7}$$

so that the vector field in bracket results to be divergence free. Let us introduce then the function $\psi(x, y)$ which will referred to as the *Total Volume Stream Function*, defined by means of its derivatives

$$\begin{aligned} \psi_x &= H_1v_1 + H_2v_2 \\ \psi_y &= -(H_1u_1 + H_2u_2). \end{aligned} \tag{2.8}$$

We note that with such choice of ψ, equation (2.7) is automatically satisfied. Proceeding further, we obtain the second derivatives of the stream function

$$\begin{aligned} \psi_{xx} &= H_{1x}v_1 + H_{2x}v_2 + H_1v_{1x} + H_2v_{2x} \\ &= H_1v_{1x} + H_2v_{2x} \\ \psi_{yy} &= -(H_{1y}u_1 + H_{2y}u_2 + H_1u_{1y} + H_2u_{2y}) \\ &= -H_{2y}u_2 - (H_1u_{1y} + H_2u_{2y}), \end{aligned}$$

which combined together provide equation

$$(H_1 v_{1x} + H_2 v_{2x}) - (H_1 u_{1y} + H_2 u_{2y}) = \Delta\psi + H_{2y} u_2. \tag{2.9}$$

Since the height function H_2 is function of the only variable y, from now on we will use the prime symbol and H_2' to indicate the partial derivative of H_2 with respect to y.

We proceed with the derivation of the vertical vorticity equation in each layer; to this end we cross differentiate the velocity fields in the two layers by considering ∂_x(2.6b)- ∂_y(2.6a) and ∂_x(2.6d)- ∂_y(2.6c). We then get

$$(v_{1x} - u_{1y})_t + \frac{f}{H_1}\eta_{2t} = 0 \tag{2.10}$$

$$(v_{2x} - u_{2y})_t - \frac{f}{H_2}\big(\eta_{2t} + H_2' v_2\big) = 0. \tag{2.11}$$

Note that by direct calculation from equation (2.6e) and (2.6f) we can replace the divergence of the two velocity fields with the following quantities

$$(u_{1x} + v_{1y}) = \frac{\eta_{2t}}{H_1}, \quad (u_{2x} + v_{2y}) = \frac{-\eta_{2t} - H_2' v_2}{H_2}.$$

As done in [104] in equation 16.49a, we introduce the following auxiliary function

$$h = \eta_2 - \frac{1}{\rho_1 g} p_1^0.$$

Remark 2.2. We observe that, by the physical assumptions made in [PRS], it follows that for $\iota = x, y, t$, it holds

$$\partial_\iota h \simeq \partial_\iota \eta_2.$$

The problem will be fully addressed by studying the evolution in time of the two quantities h and ψ. We need therefore to find a convenient way to express each single velocity component in terms of the stream function ψ and the interface h. We start by introducing, at least formally, the differential operator

$$\mathcal{T} = \partial_{tt} + f^2.$$

Taking the derivative in time of equation (2.6a) and adding it to equation (2.6b) multiplied by the factor f, we deduce

$$\mathcal{T}u_1 = fv_{1t} - \frac{1}{\rho_1}p^0_{1xt} - fv_{1t} - \frac{f}{\rho_1}p^0_{1y}.$$

In a completely analogous way, we also derive the following identities

$$\begin{aligned}
\mathcal{T}v_1 &= -fu_{1t} - \frac{1}{\rho_1}p^0_{1yt} + fu_{1t} + \frac{f}{\rho_1}p^0_{1x},\\
\mathcal{T}u_2 &= fv_{2t} - \frac{1}{\rho_2}p^0_{1xt} - G\eta_{2xt} - fv_{2t} - \frac{f}{\rho_2}p^0_{1y} - fG\eta_{2y}\\
&= -\frac{1}{\rho_2}(p^0_{1xt} + fp^0_{1y}) - G(\eta_{2xt} + f\eta_{2y})\\
&= \frac{\rho_1}{\rho_2}\mathcal{T}u_1 - G(\eta_{2xt} + f\eta_{2y}),\\
\mathcal{T}v_2 &= -fu_{2t} - \frac{1}{\rho_2}p^0_{1yt} - G\eta_{2yt} + fu_{2t} + \frac{f}{\rho_2}p^0_{1x} + fG\eta_{2x}\\
&= -\frac{1}{\rho_2}(p^0_{1yt} - fp^0_{1x}) - G(\eta_{2yt} - f\eta_{2x})\\
&= \frac{\rho_1}{\rho_2}\mathcal{T}v_1 - G(\eta_{2yt} - f\eta_{2x}).
\end{aligned}$$

Remark 2.3. From the assumption [RGR] it follows that the values of the densities for the two layers are very close. This allows in our future computations to ignore the factor $\frac{\rho_1}{\rho_2}$ in the equations above involving u_2 and v_2.

The above observation on the densities, together with the Remark 2.2 yields

$$\mathcal{T}(u_2 - u_1) = -G(h_{xt} + fh_y), \tag{2.12a}$$

$$\mathcal{T}(v_2 - v_1) = -G(h_{yt} - fh_x). \tag{2.12b}$$

We use then the above identities together with the properties of the stream function given in (2.8) in order to express the quantities $\mathcal{T}u_1$ in terms of only the two quantities

ψ and h. We have

$$\begin{aligned}\mathcal{T}u_1 &= \frac{1}{H}\mathcal{T}(H_1u_1 + H_2u_2 - H_2u_2 + H_2u_1)\\ &= \frac{1}{H}\mathcal{T}(-\psi_y - H_2(u_2 - u_1))\\ &= -\frac{1}{H}\mathcal{T}\psi_y - \frac{H_2}{H}\mathcal{T}(u_2 - u_1)\\ &= -\frac{1}{H}\mathcal{T}\psi_y + \frac{H_2}{H}G(h_{xt} + fh_y),\end{aligned}$$

and analogously we deduce the whole suite

$$\mathcal{T}u_1 = \frac{1}{H}\Big[-\mathcal{T}\psi_y + H_2G(h_{xt} + fh_y)\Big] \tag{2.13a}$$

$$\mathcal{T}u_2 = \frac{1}{H}\Big[-\mathcal{T}\psi_y - H_1G(h_{xt} + fh_y)\Big] \tag{2.13b}$$

$$\mathcal{T}v_1 = \frac{1}{H}\Big[\mathcal{T}\psi_x + H_2G(h_{yt} - fh_x)\Big] \tag{2.13c}$$

$$\mathcal{T}v_2 = \frac{1}{H}\Big[\mathcal{T}\psi_x - H_1G(h_{yt} - fh_x)\Big] \tag{2.13d}$$

We now introduce the auxiliary function $\alpha(y)$ as the logarithmic derivative of the depth profile $H(y)$

$$\alpha(y) = \frac{H_2'(y)}{H(y)} = (\ln H)'(y). \tag{2.14}$$

We specify now the form of the solutions sought, by means of the following additional hypothesis on the time dependence of ψ and h. With use the following ansatz for the functions ψ and h:

$$\psi(x, y, t) = \psi(x, y)e^{-i\omega t} \qquad h(x, y, t) = h(x, y)e^{-i\omega t}. \tag{2.15}$$

2.3.1 Stream function equation

We proceed firstly in deriving the equation that is satisfied by the stream function ψ and we start considering equation (2.9); we observe that multiplying equation (2.10) and (2.11) respectively by H_1 and H_2, summing them up and then deriving in time,

it produces the left hand side of (2.9), so that it yields

$$\Delta\psi_t + H_2' u_{2t} - f H_2' v_2 = 0. \tag{2.16}$$

Using equations (2.13b), (2.13d) and (2.15) in (2.16), we get

$$i\omega\Delta\psi + i\omega H_2' u_2 + f H_2' v_2 = 0. \tag{2.17}$$

It follows from (2.13b) that

$$\begin{aligned} i\omega H_2' u_2 &= i\omega\, H_2' \frac{1}{H}\Big(-\psi_y - \frac{G\,H_1}{\mathcal{T}}(-i\omega h_x + f h_y)\Big) \\ &= i\omega\, H_2' \frac{1}{H}\Big(-\psi_y + \frac{G\,H_1}{\mathcal{T}}(i\omega h_x - f h_y)\Big), \end{aligned}$$

and from (2.13d) that

$$\begin{aligned} f\, H_2' v_2 &= f\, H_2' \frac{1}{H}\Big(\psi_x - \frac{H_1\, G}{\mathcal{T}}(-i\omega h_y - f h_x)\Big) \\ &= f\, H_2' \frac{1}{H}\Big(\psi_x + \frac{H_1\, G}{\mathcal{T}}(i\omega h_y + f h_x)\Big), \end{aligned}$$

Substituting this in the equation (2.17) we deduce

$$\begin{aligned} i\omega H\Delta\psi + H_2' \cdot \Big\{ & f\,\psi_x - i\omega\psi_y \\ & + h_x\Big((i\omega)^2 \frac{H_1\, G}{\mathcal{T}} + f^2 \frac{H_1\, G}{\mathcal{T}}\Big) \\ & + h_y\Big(i\omega f \frac{H_1\, G}{\mathcal{T}} - i\omega f \frac{H_1\, G}{\mathcal{T}}\Big)\Big\} = 0, \end{aligned}$$

which simplifies to

$$i\omega H\Delta\psi + f\, H_2'\psi_x - i\omega H_2'\psi_y + h_x\, H_2' \frac{f^2 - \omega^2}{\mathcal{T}} H_1\, G = 0.$$

From the identification between the time derivative and its spectral parameter ω, we note that $\mathcal{T} = (f^2 - \omega^2)$, so that the previous equation ultimately reduces to

$$-\Delta\psi + \alpha(y)\psi_y = -\frac{i}{\omega}\alpha(y)(f\psi_x + GH_1 h_x). \tag{2.18}$$

2.3.2 Fluid interface motion equation

We now consider the equation for the interface h. Starting from equation (2.11) and subtracting from it equation (2.10) we get

$$(v_2 - v_1)_{xt} - (u_2 - u_1)_{yt} - f\frac{H}{H_2 H_1}h_t = f\,\frac{H_2'}{H_2}v_2. \tag{2.19}$$

We study the left hand side (LHS) and the right hand side (RHS) of equation (2.19) separately. Recalling equations (2.12) and replacing the time derivative with $-i\omega$ we derive

$$(u_2 - u_1)_{yt} = -i\omega\frac{G}{\mathcal{T}}(i\omega h_{xy} - f h_{yy}),$$

$$(v_2 - v_1)_{xt} = -i\omega\frac{G}{\mathcal{T}}(i\omega h_{xy} + f h_{xx}),$$

and in view of them, the left hand side of (2.19) now reads as

$$LHS = -i\omega\, f\frac{G}{\mathcal{T}}\Delta h + i\omega f\frac{H}{H_2 H_1}h.$$

Further, we deduce the RHS from equation (2.13d),

$$RHS = f\frac{H_2'}{H_2}\frac{1}{H}\Big(\psi_x + \frac{i\omega}{\mathcal{T}}H_1 G\, h_y + \frac{f}{\mathcal{T}}G H_1\, h_x\Big)$$

so that the identity (2.19) finally reads as

$$-i\omega G\Delta h + i\omega(f^2 - \omega^2)\frac{H}{H_2 H_1}h = \frac{H_2'}{H_2 H}\Big[(f^2 - \omega^2)\psi_x + G(i\omega H_1 h_y + f H_1 h_x)\Big]. \tag{2.20}$$

We observe that in equation (2.20), the dependence on the spectral parameter ω is non linear. For this reason we linearise it, dropping all the terms that come with powers of the spectral parameter strictly higher than degree one. We note that, in fact, this is equivalent to consider the approximation

$$f^2 - \omega^2 \approx f^2, \tag{2.21}$$

namely we are seeking solutions whose time frequency is negligible with respect to the Earth rotation frequency.

Thus the non linear problem reduces to the linear pencil case

$$-\Delta h - \alpha(y)\frac{H_1}{H_2}h_y + \lambda^2 h = -\frac{i}{\omega}\frac{f}{G}\frac{\alpha(y)}{H_2}\Big(f\psi_x + H_1 G\, h_x\Big) \tag{2.22}$$

where

$$\lambda^2(y) := \frac{f^2}{G}\frac{1}{H_1}\frac{H(y)}{H_2(y)}. \tag{2.23}$$

2.4 The problem in the curved strip $\mathcal{S}_\gamma$

In this section we rewrite equations (2.18) and (2.22)

$$\begin{aligned}
&-\Delta\psi + \alpha(y)\psi_y = -\frac{i}{\omega}\alpha(y)(f\psi_x + GH_1 h_x),\\
&-\Delta h - \alpha(y)\frac{H_1}{H_2}h_y + \lambda^2(y)h = -\frac{i}{\omega}\frac{f}{G}\frac{\alpha(y)}{H_2}\Big(f\psi_x + GH_1\, h_x\Big),
\end{aligned}$$

valid in a straight strip, in the curvilinear coordinate system (ξ, η), more suitable for a generic geometry of the strip.

We start from pointing out here how the metric is affected by the change of the reference system. A simple calculation yields that the Jacobian of the transformation $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ introduced in (2.2) is

$$p(\xi, \eta) = 1 + \eta\gamma(\xi). \tag{2.24}$$

We continue now presenting the differential operators in the new curvilinear coordinate system. In particular, we have that the gradient of any given scalar field $f \in \mathbb{R}$ and the divergence of any vector field $\mathbf{v} \in \mathbb{R} \times \mathbb{R}$ read as

$$\nabla_{(x,y)} f = \frac{1}{p}\frac{\partial f}{\partial \xi}\mathbf{e}_\xi + \frac{\partial f}{\partial \eta}\mathbf{e}_\eta, \tag{2.25}$$

$$\mathrm{div}_{(x,y)}(\mathbf{v}) = \nabla_{(x,y)} \cdot (v_1, v_2) = \frac{1}{p}\left[\frac{\partial v_1}{\partial \xi} + \frac{\partial (p v_2)}{\partial \eta}\right]. \tag{2.26}$$

For future reference we list here the well known identities

$$\mathrm{div}(-\nabla f) = -\Delta f, \qquad \frac{1}{H}\nabla H \cdot \nabla f = \frac{H_2'}{H}\frac{\partial f}{\partial y}, \qquad \mathbf{e}_z \cdot \nabla f \times \nabla H = \frac{\partial f}{\partial x} H_2'$$

where we recall $\mathbf{e}_z = \mathbf{e}_\xi \times \mathbf{e}_\eta$ is the unitary vector perpendicular to the plane (ξ, η) and where the depth profile depends only on the transversal direction to the coast η.

Let us rewrite equation (2.18) in a compact form as

$$i\omega \left[\mathrm{div}(\nabla \psi) - \frac{1}{H}\nabla H \cdot \nabla \psi \right] + \frac{1}{H}\mathbf{e}_z \cdot \left[\nabla (f\psi + GH_1 h) \times \nabla H \right] = 0. \tag{2.27}$$

By the identities in (2.25) and (2.26), the expressions in the square brackets in (2.27) read in curvilinear coordinates respectively as

$$i\omega \left[\frac{1}{p}\left(\frac{\partial}{\partial \xi}\left(\frac{1}{p}\frac{\partial \psi}{\partial \xi} \right) + \frac{\partial}{\partial \eta}\left(p\,\frac{\partial \psi}{\partial \eta} \right) \right) - \frac{1}{H}\frac{\partial H}{\partial \eta}\frac{\partial \psi}{\partial \eta} \right],$$

and

$$\frac{1}{H}\mathbf{e}_z \cdot \left[\left(\frac{1}{p}\frac{\partial (f\psi + GH_1 h)}{\partial \xi}\mathbf{e}_\xi + \frac{\partial (f\psi + GH_1 h)}{\partial \eta}\mathbf{e}_\eta \right) \times \left(\frac{\partial H}{\partial \xi}\mathbf{e}_\xi + \frac{\partial (H)}{\partial \eta}\mathbf{e}_\eta \right) \right],$$

thus transforming equation (2.27) into

$$i\omega \left\{ \frac{1}{p}\left[\frac{\partial}{\partial \xi}\left(\frac{1}{p}\frac{\partial \psi}{\partial \xi} \right) + \frac{\partial}{\partial \eta}\left(p\,\frac{\partial \psi}{\partial \eta} \right) \right] - \frac{1}{H}\frac{\partial H}{\partial \eta}\frac{\partial \psi}{\partial \eta} \right\} + \frac{1}{p}\frac{1}{H}\frac{\partial (f\psi + GH_1 h)}{\partial \xi}\frac{\partial H}{\partial \eta} = 0.$$

Finally, by expanding all the derivatives and substituting the term $\alpha(\eta)$ as previously defined in equation (2.14)

$$\alpha(\eta) = \frac{\partial H}{\partial \eta}\frac{1}{H(\eta)} \tag{2.28}$$

we obtain

$$-\frac{1}{p^2}\frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{p^3}\frac{\partial p}{\partial \xi}\frac{\partial \psi}{\partial \xi} - \frac{1}{p}\frac{\partial p}{\partial \eta}\frac{\partial \psi}{\partial \eta} + \alpha(\eta)\frac{\partial \psi}{\partial \eta} - \frac{\partial^2 \psi}{\partial \eta^2} = -\frac{i}{\omega}\left[\frac{1}{p}\alpha(\eta)\frac{\partial}{\partial \xi}(f\psi + GH_1 h) \right]. \tag{2.29}$$

Remark 2.4. We emphasize here that in the case $G \equiv 0$, namely when the two densities

are the same $\rho_1 \equiv \rho_2$, equation (2.27) reduces to equation (1.8) in [86]

$$i\omega \left[\mathrm{div}(\nabla\psi) - \frac{1}{H}\nabla H \cdot \nabla\psi \right] + \frac{1}{H}\mathbf{e}_z \cdot \left[f\nabla\psi \times \nabla H \right] = 0.$$

We consider now the second equation (2.22): by simple combination of the formulae for the gradient in curvilinear coordinates (2.25) and the one for the divergence (2.26) we have that in particular

$$\Delta h = \frac{1}{p}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{p}\frac{\partial h}{\partial\xi} \right) + \frac{\partial}{\partial\eta}\left(p\frac{\partial h}{\partial\eta} \right) \right],$$

$$\nabla h \cdot \nabla H = \frac{\partial H}{\partial\eta}\frac{\partial h}{\partial\eta}.$$

By means of these identities, it is an easy computation to derive the general form in curvilinear coordinates for the equation (2.22)

$$-\frac{1}{p^2}\frac{\partial^2 h}{\partial\xi^2} - \frac{\partial^2 h}{\partial\eta^2} + \frac{1}{p^3}\frac{\partial p}{\partial\xi}\frac{\partial h}{\partial\xi} - \left(\frac{1}{p}\frac{\partial p}{\partial\eta} + \alpha(\eta)\frac{H_1}{H_2} \right)\frac{\partial h}{\partial\eta} + \lambda^2(\eta)h = -\frac{i}{\omega}\left[\frac{f}{G}\frac{1}{p}\frac{\alpha(\eta)}{H_2}\frac{\partial}{\partial\xi}(f\psi + GH_1 h) \right] \tag{2.30}$$

We conclude this section by pointing out two immediate identities which involve $p(\xi,\eta)$, the Jacobian of the coordinates transformation. The first shows the change in the volume metrics

$$d\mathcal{S}_\gamma = p(\xi,\eta)\, d\mathcal{S}_0 = p(\xi,\eta)\, d\xi d\eta, \tag{2.31}$$

while the second, is a simple property which will be useful in future computations

$$\frac{\partial p}{\partial\eta} = \frac{p-1}{\eta}. \tag{2.32}$$

2.5 Boundary Condition

In this section we discuss the boundary conditions that arise in the spectral problem we are going to solve. The two boundaries, as determined in Figure 2.1, are the coast line $\partial_c\mathcal{S}$ and the cross-section $\partial_o\mathcal{S}$ where the water regime changes from the shallow one to the open ocean one. For sake of clarity, we are going to present them in the

straight case geometry: we will observe, though, that the conditions derived will be independent from the particular location along the coast. Therefore the extension of the results to any admissible coastal shape will follow immediately.

In order to retrieve the information on the boundaries, we make the following assumptions on the nature of our solutions. As first instance we consider the *free wave* solutions to the equations (2.29) and (2.30), namely solutions of the form

$$\psi(x,y,t) = \mathrm{Re}(l(y)e^{i(kx-\omega t)}), \qquad h(x,y,t) = \mathrm{Re}(b(y)e^{i(kx-\omega t)}). \tag{2.33}$$

We note that those particular solutions do not belong to the space of integrable functions of the geometric domain but they are meaningful when considering the problem as a collection of different one-dimensional transversal problems, for any value $\xi \in \mathbb{R}$.

Remark 2.5. We note that for every choice of k and ω in (2.33) we get different corresponding boundary conditions. In practical problems, their values are determined from physical arguments which lie outside our research interests. We therefore replace them respectively with κ_0 and ω_0, remarking the fact that these are fixed real quantities.

2.5.1 The coast

The normal velocities of the fluids at the coast line must be null, meaning $v_1 = v_2 = 0$. From (2.8) we thus obtain that the longitudinal derivative of the stream function must be null as well

$$\psi_x(x,0) = 0.$$

Therefore, being the stream function defined up to a constant we set with no loss of generality the impermeability condition

$$\psi(x,y) = 0 \qquad \text{for } y = 0. \tag{2.34}$$

For what concern the interface, we note that $H_2(0) = 0$ holds true together with the fact that $\psi_x(x,0) = 0$. Therefore the interface equation (2.22) reduces by mean of

(2.33) to

$$-b_y(y) + \frac{f^2}{GH_2'(0)} b(y) = f\frac{\kappa_0}{\omega_0} b(y), \tag{2.35}$$

from which it follows the following Robin boundary condition

$$h_y(x,y) = a_0\, h(x,y) \qquad \text{for } y = 0, \tag{2.36}$$

where

$$a_0 = f\left(\frac{f}{GH_2'(0)} - \frac{\kappa_0}{\omega_0}\right). \tag{2.37}$$

2.5.2 The open ocean

The boundary condition for the stream function and the fluids interface at the cross section will be recovered by imposing a continuity matching condition between the solutions inside the channel (ψ, h) satisfying the shallow water regime and those outside the channel $(\tilde{\psi}, \tilde{h})$ which satisfy the open ocean regime. The latter is characterised by the condition of flat topography $H_2'(\delta) = 0$, which in turns implies

$$\alpha(\delta) = 0.$$

It immediately follows from this condition that the $\tilde{\psi}$, the stream function prolonged outside the channel, satisfies the Laplace equation $-\Delta\tilde{\psi} = 0$. By mean of (2.33), it transforms into $\tilde{l}_{yy}(y) - \kappa_0^2\tilde{l}(y) = 0$ from which we easily derive a solution

$$\tilde{l}(y) = Ce^{-|\kappa_0|(y-\delta)},$$

where C is any real constant. Thus, it follows that the stream function at the open ocean cross section satisfies the Robin boundary condition of the type

$$\psi_y(x,y) = -|\kappa_0|\psi(x,y) \qquad \text{for } y = \delta. \tag{2.38}$$

Similarly, the extended interface function $\tilde{h}$ satisfies

$$-\Delta\tilde{h} + \frac{f^2}{G}\frac{H(\delta)}{H_1 H_2(\delta)}\tilde{h} = 0,$$

from which it follows

$$-\tilde{b}_{yy}(y) + \kappa_0^2 \tilde{b}(y) + \lambda_\delta^2 b(y) = 0$$

where we set for convenience λ_δ to be the *internal Rossby radius* the following quantity

$$\lambda_\delta = \sqrt{\frac{f^2}{G} \frac{1}{H_1} \frac{H(\delta)}{H_2(\delta)}}.$$

Then, arguing as before, we deduce the following boundary condition for the interface function at the open ocean cross section

$$h_y(x,y) + \sqrt{\lambda_\delta^2 + \kappa_0^2} h(x,y) = 0 \qquad \text{for } y = \delta. \tag{2.39}$$

Remark 2.6. In view of (2.38) we will assume in the following $\kappa_0 > 0$.

2.6 The operator pencil

In this section we rewrite the equations (2.29) and (2.30) in a vectorial form in terms of a operator pencil spectral problem related to the spectral parameter ω introduced in (2.15). In the following we will provide the precise mathematical formalisation of the operators along with their spectrum properties.

2.6.1 The operators $\mathcal{A}_\gamma$, $\mathcal{L}_\gamma$ and $\mathcal{M}_\gamma$

We start from introducing the differential expressions of the two matrix valued operators $\mathcal{L}_\gamma$ and $\mathcal{M}_\gamma$ as

$$\mathcal{L}_\gamma := \begin{pmatrix} -\frac{1}{p}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{p}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(p\,\frac{\partial}{\partial\eta}\right)\right] + \alpha(\eta)\frac{\partial}{\partial\eta} & 0 \\ 0 & -\frac{1}{p^2}\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2} + \frac{1}{p^3}\frac{\partial p}{\partial\xi}\frac{\partial}{\partial\xi} - \left(\frac{1}{p}\frac{\partial p}{\partial\eta} + \alpha(\eta)\frac{H_1}{H_2}\right)\frac{\partial}{\partial\eta} + \lambda^2(\eta) \end{pmatrix} \tag{2.40}$$

and

$$\mathcal{M}_\gamma := \frac{\alpha(\eta)}{p(\xi,\eta)} \begin{pmatrix} f & H_1 G \\ \frac{f^2}{G}\frac{1}{H_2} & f\,\frac{H_1}{H_2} \end{pmatrix} \left(-i\frac{\partial}{\partial\xi}\right). \tag{2.41}$$

Equations (2.29) and (2.30) can be therefore now expressed by these two operators and written in a vectorial form as follow

$$\mathcal{A}_\gamma(\omega)\begin{pmatrix}\psi\\ h\end{pmatrix}=0, \tag{2.42}$$

where $\mathcal{A}_\gamma$ is the pencil operator

$$\mathcal{A}_\gamma(\omega)\begin{pmatrix}\psi\\ h\end{pmatrix}:=\mathcal{L}_\gamma\begin{pmatrix}\psi\\ h\end{pmatrix}-\frac{1}{\omega}\mathcal{M}_\gamma\begin{pmatrix}\psi\\ h\end{pmatrix} \tag{2.43}$$

In order to describe the domain of the operator defined in (2.43), we introduce the following weighted Lebesgue spaces

$$L^2(C;w_1,w_2):=\left\{(\psi,h)\big|\, C\subset\mathcal{S}_\gamma,\int_C|\psi|^2w_1<\infty,\int_C|h|^2w_2<\infty\right\} \tag{2.44}$$

endowed with the scalar product

$$\left\langle\begin{pmatrix}\psi_1\\ h_1\end{pmatrix},\begin{pmatrix}\psi_2\\ h_2\end{pmatrix}\right\rangle_{L^2(C;w_1,w_2)}=\iint_C\Big(\psi_1(\xi,\eta)\overline{\psi_2(\xi,\eta)}w_1(\xi,\eta)+h_1(\xi,\eta)\overline{h_2(\xi,\eta)}w_2(\xi,\eta)\Big)\,d\mathcal{S}_\gamma, \tag{2.45}$$

where w_1 and w_2 are two positive weight functions. Similarly, we consider the Sobolev space

$$H^2(C;w_1,w_2):=\big\{(\psi,h)\big|(D^i\psi,D^jh)\in L^2(C;w_1,w_2),|i|,|j|\le 2\big\}$$

endowed with the natural metric inherited from $L^2(C;w_1,w_2)$, where i and j are two positive multi index. We finally define the domain of the operator $\mathcal{A}_\gamma$ to coincide with the Hilbert space

$$\tilde{H}^2_{w_1,w_2}(\mathcal{S}_\gamma):=\big\{(\psi,h)\in H^2(\mathcal{S}_\gamma;w_1,w_2)\big|\textit{cond.}\,(2.34),(2.36),(2.38),(2.39)\ \textit{hold}\ \big\}. \tag{2.46}$$

In the following, we will study the spectrum of the operator $\mathcal{A}_\gamma$ using the variational principles introduced in the first section of this chapter. In particular, we will interpret equation (2.42) in its weak formulation and thus, we need to settle our problem in the context of the standard theory of quadratic forms. We firstly seek conditions on the

weights w_1 and w_2 so to guarantee the symmetry of $\mathcal{L}_\gamma$ and $\mathcal{M}_\gamma$ as operators defined on the domain $\tilde{H}^2_{w_1,w_2}$.

For future convenience, let us define two auxiliary operators

$$\mathcal{L}_{0,1} : \psi \to -\Delta\psi + \alpha(y)\partial_y\psi,$$
$$\mathcal{L}_{0,2} : h \to -\Delta h - \alpha(y)\frac{H_1}{H_2}\partial_y h + \lambda^2(y)h.$$

and consider

$$\mathcal{L}_0 = \begin{pmatrix} -\Delta + \alpha(y)\partial_y & 0 \\ 0 & -\Delta - \alpha(y)\frac{H_1}{H_2}\partial_y + \lambda^2(y) \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{0,1} & 0 \\ 0 & \mathcal{L}_{0,2} \end{pmatrix} \tag{2.47}$$

and

$$\mathcal{M}_0 = \alpha(y) \begin{pmatrix} f & GH_1 \\ \frac{f^2}{G}\frac{1}{H_2} & f\frac{H_1}{H_2} \end{pmatrix} (-i\partial_x). \tag{2.48}$$

2.6.2 Symmetry properties

Lemma 2.3. *The operators $\mathcal{L}_\gamma$, $\mathcal{M}_\gamma$ are symmetric operators on $\tilde{H}^2_{w_1,w_2}$ for the following choice of the weights*

$$w_1(\eta) = H^{-1}(\eta), \tag{2.49}$$
$$w_2(\eta) = \frac{G^2}{f^2}\frac{H_1H_2(\eta)}{H(\eta)}. \tag{2.50}$$

Proof. As the symmetry property does not depend on the particular shape of the channel, we will show it in the particular case of the straight strip. Therefore we will assume in the following that $p(\xi,\eta) = 1$ or equivalently $\gamma(\xi) = 0$.

We start the analysis from the operator $\mathcal{L}_0$. Given two pairs $(\psi_1, h_1), (\psi_2, h_2)$ in the domain $\tilde{H}^2_{w_1,w_2}$, we want to show

$$\langle \mathcal{L}_0 \begin{pmatrix} \psi_1 \\ h_1 \end{pmatrix}, \begin{pmatrix} \psi_2 \\ h_2 \end{pmatrix} \rangle_{L^2(\mathcal{S}_0;w_1,w_2)} = \langle \begin{pmatrix} \psi_1 \\ h_1 \end{pmatrix}, \mathcal{L}_0 \begin{pmatrix} \psi_2 \\ h_2 \end{pmatrix} \rangle_{L^2(\mathcal{S}_0;w_1,w_2)}. \tag{2.51}$$

Equation (2.51) can be formally rewritten in the compact form

$$\langle \mathcal{L}_0 \begin{pmatrix} \psi_1 \\ h_1 \end{pmatrix}, \begin{pmatrix} \psi_2 \\ h_2 \end{pmatrix} \rangle_{L^2(\mathcal{S}_0;w_1,w_2)} = \langle \mathcal{L}_{0,1}\psi_1, \psi_2 \rangle_{L^2(\mathcal{S}_0;w_1)} + \langle \mathcal{L}_{0,2}h_1, h_2 \rangle_{L^2(\mathcal{S}_0;w_2)} \tag{2.52}$$

where the definition of the space of weighted integrable functions $L^2(\mathcal{S}_\gamma; w_1, w_2)$ is given in (2.44).

By mean of Lemma 2.6 and from the boundary condition (2.34), (2.38)

$$\begin{aligned} \psi(x,y) &= 0 && \text{for} \quad y = 0, \\ \psi_y(x,y) &= -|\kappa_0|\psi(x,y) && \text{for} \quad y = \delta, \end{aligned}$$

we deduce

$$\begin{aligned} \langle \mathcal{L}_{0,1}\psi_1, \psi_2 \rangle_{L^2(\mathcal{S}_0;w_1)} = &\int_{\mathbb{R}}\int_0^\delta \nabla\psi_1 \cdot \nabla\psi_2 \, w_1(y)\, dxdy + \int_{\mathbb{R}}\int_0^\delta \nabla\psi_1 \cdot \nabla w_1(y)\psi_2 \, dxdy + \\ &\int_{\mathbb{R}}\int_0^\delta \partial_y\psi_1 \alpha(y)\psi_2 \, w_1(y)\, dxdy + \int_{-\infty}^{\infty} \kappa_0 \psi_1(x,\delta)\psi_2(x,\delta)w_1(\delta)dx. \end{aligned} \tag{2.53}$$

Similarly from Lemma 2.7 and from (2.36) (2.39)

$$\begin{aligned} h_y(x,y) &= a_0 h(x,y) && \text{for} \quad y = 0, \\ h_y(x,y) + \sqrt{\lambda_\delta^2 - \kappa_0^2}h(x,y) &= 0 && \text{for} \quad y = \delta. \end{aligned}$$

we have

$$\begin{aligned} \langle \mathcal{L}_{0,2}h_1, h_2 \rangle_{L^2(\mathcal{S}_0;w_1)} = &\int_{\mathbb{R}}\int_0^\delta \nabla h_1 \cdot \nabla h_2 \, w_2(y)\, dxdy + \int_{\mathbb{R}}\int_0^\delta \lambda^2(y) h_1 h_2 w_2 \, dxdy \\ &+ \int_{\mathbb{R}}\int_0^\delta \left(\nabla h_1 \cdot \nabla w_2(y) h_2 - \partial_y h_1 \alpha(y) \frac{H_1}{H_2(y)} h_2 \, w_2(y) \right) dxdy \\ &+ \sqrt{\lambda_\delta^2 + \kappa_0^2} w_2(\delta) \int_{-\infty}^{\infty} h_1(x,\delta)h_2(x,\delta)\, dx \\ &+ w_2(0)a_0 \int_{-\infty}^{\infty} h_1(x,0)h_2(x,0)\, dx \end{aligned} \tag{2.54}$$

From (2.53) we derive the sufficient conditions for the $\mathcal{L}_{0,1}$ symmetry are

$$\begin{aligned} \partial_x w_1 &= 0 \\ \partial_y w_1 &= -\alpha(y) w_1 \end{aligned} \tag{2.55}$$

from which it follows

$$w_1(y) = H^{-1}(y) > 0. \tag{2.56}$$

Similarly the conditions for $\mathcal{L}_{0,2}$ are

$$\begin{aligned} \partial_x w_2 &= 0 \\ \partial_y w_2 &= \alpha(y) \frac{H_1}{H_2} w_2. \end{aligned} \tag{2.57}$$

For what concerns the operator $\mathcal{M}_0$, we note that the weights do not depend on the variable x. On the other hand, the matrix M is not symmetric and this give rise to the extra mixing condition on the weights. Thus, in order to have a symmetric operator we have to impose

$$M_{12} w_1 = M_{21} w_2$$

which immediately implies

$$w_2 = \frac{G}{\lambda^2(y)} = \frac{G^2}{f^2} \frac{H_1 H_2}{H} = \frac{G^2}{f^2} H_1 H_2 w_1 \geq 0. \tag{2.58}$$

It is now easy to check that such choice for w_2 does satisfy the condition in (2.57). □

Remark 2.7. We note that from (2.4) it follows that $w_2(0) = 0$.

2.6.3 Quadratic form

We are now able to introduce formally the spectral problem. We start from the operator $\mathcal{L}_\gamma$ and we consider the quadratic form $Q_\mathcal{L}$ which originates from it

$$Q_\mathcal{L}(\psi, h) := \langle \mathcal{L}_\gamma \begin{pmatrix} \psi \\ h \end{pmatrix}, \begin{pmatrix} \psi \\ h \end{pmatrix} \rangle_{L^2(\mathcal{S}_\gamma; w_1, w_2)}. \tag{2.59}$$

Combining the results in Lemma 2.8 and Lemma 2.9, in particular equations (2.107)

and (2.108), and from the Remark 2.7 it follows that

$$
\begin{aligned}
Q_{\mathcal{L}}(\psi,h) &= \langle \mathcal{L}_\gamma \begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix} \rangle_{L^2(\mathcal{S}_\gamma;w_1,w_2)} \\
&= \int_{\mathbb{R}}\int_0^\delta \left[\frac{1}{p(\xi,\eta)}\left|\frac{\partial\psi}{\partial\xi}\right|^2 + p(\xi,\eta)\left|\frac{\partial\psi}{\partial\eta}\right|^2\right] w_1(\eta) + \left[\frac{1}{p(\xi,\eta)}\left|\frac{\partial h}{\partial\xi}\right|^2 + p(\xi,\eta)\left|\frac{\partial h}{\partial\eta}\right|^2\right] w_2(\eta) d\eta d\xi \\
&\quad + \int_{\mathbb{R}}\int_0^\delta p(\xi,\eta) G |h(\eta)|^2 \, d\eta \, d\xi + \kappa_0 w_1(\delta) \int_{\mathbb{R}} p(\xi,\delta)|\psi(\xi,\delta)|^2 \, d\xi \\
&\quad + \sqrt{\lambda_\delta^2 + \kappa_0^2} w_2(\delta) \int_{\mathbb{R}} p(\xi,\delta)|h(\xi,\delta)|^2 \, d\xi.
\end{aligned}
\tag{2.60}
$$

The respective quadratic form domain $\mathcal{D}(Q_{\mathcal{L}}) := \tilde{H}_0^1(\mathcal{S}_\gamma, \partial_c\mathcal{S}_\gamma; w_1, w_2)$ is then obtained considering the closure of

$$
\begin{aligned}
\tilde{C}_0^\infty(\mathcal{S}, \partial_c\mathcal{S}) := \Big\{ (\phi_1,\phi_2) \in C^\infty(F) \; | F \subseteq \mathcal{S}, \mathrm{supp}(\phi_1) \cap \partial_c\mathcal{S} \cap F = \emptyset, \\
\exists \, r_1, r_2 > 0 s.t. \phi_i(\xi,\eta) = 0 \; for \; |\xi| \geq r_i, i = 1,2 \Big\}
\end{aligned}
$$

with respect to the scalar product induced by the form

$$
\begin{aligned}
\langle \begin{pmatrix}\psi_1\\ h_1\end{pmatrix}, \begin{pmatrix}\psi_2\\ h_2\end{pmatrix} \rangle_{\tilde{H}_0^1(\mathcal{S}_\gamma, \partial_c\mathcal{S}_\gamma, (w_1,w_2))} &= \langle \begin{pmatrix}\psi_1\\ h_1\end{pmatrix}, \begin{pmatrix}\psi_2\\ h_2\end{pmatrix} \rangle_{L^2(\mathcal{S}_\gamma;w_1,w_2)} \\
&\quad + \langle \begin{pmatrix}\nabla\psi_1\\ \nabla h_1\end{pmatrix}, \begin{pmatrix}\nabla\psi_2\\ \nabla h_2\end{pmatrix} \rangle_{L^2(\mathcal{S}_\gamma;w_1,w_2)}.
\end{aligned}
$$

In principle, in order for the expression (2.43) to make sense as sum of forms, we should consider the quadratic form which arise from the operator $\mathcal{M}_\gamma$. As we shall see in the forthcoming sections, we will restrict our analysis to particular type of test functions, for which the operator $\mathcal{M}_\gamma$ reduces to a multiplication operator and the relative quadratic form to a simple integral with no derivatives involved. Therefore we omit the details and we conclude observing that from now on, we will understand the expression $\langle \mathcal{A}_\gamma \begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix} \rangle$ for $(\psi,h) \in \mathcal{D}(Q_{\mathcal{L}})$ in terms of the quadratic form of the operator $\mathcal{L}_\gamma$ defined in (2.60) with domain $\mathcal{D}(Q_{\mathcal{L}})$.

2.6.4 Transversal problem

We introduce in this section an auxiliary problem, or more precisely a family of problems, which will be useful in establishing the nature of the essential spectrum of the pencil operator $\mathcal{A}_\gamma$. We are going to carry out the computations in the special case of a straight geometry, namely we will assume equivalently $p(\xi,\eta)\equiv 1$ on the curvature $\gamma(\xi,\eta)\equiv 0$ to be constant.

We proceed with a formal substitution

$$\psi(x,y)=e^{i\sigma x}\tilde{\psi}(y),\qquad h(x,y)=e^{i\sigma x}\tilde{h}(y),\qquad \sigma>0 \tag{2.61}$$

where σ will be the running index. Substituting the ansatz (2.61) into equation (2.18) and (2.22) and dropping the tildes for the sake of clarity, we deduce

$$\omega\Big(-\psi''(y)+\alpha(y)\psi'(y)+\sigma^2\psi(y)\Big)=\sigma\,\alpha(y)\Big(f\psi(y)+GH_1h(y)\Big) \tag{2.62}$$

$$\omega\Big(-h''(y)-\alpha(y)\frac{H_1}{H_2}h'(y)+\Big[\sigma^2+\lambda^2(y)\Big]h(y)\Big)=\sigma f\frac{\alpha(y)}{H_2}\Big(\frac{f}{G}\psi(y)+H_1h(y)\Big) \tag{2.63}$$

Using a notation similar to the one which has been used for the operator pencil problem, we define by mean of the equations (2.62) and (2.63) the *transversal spectral problem* as

$$\mathfrak{a}_{\sigma,0}(\omega)\begin{pmatrix}\psi\\ h\end{pmatrix}:=\omega\,\mathfrak{l}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}-\mathfrak{m}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}=0. \tag{2.64}$$

We note that the operators in the expression (2.64) have to be interpreted as defined on their relative quadratic form domains and the problem in its variational form. We avoid presenting once again them in full details as they might be considered as a special case of the ones introduced in the previous section. Note that $\mathcal{D}(\mathfrak{a}_\sigma)$ does not change for different values of σ.

In order to determine the essential spectrum for the general problem we need to

investigate the functional associated to the Rayleigh quotient

$$J_\sigma\begin{pmatrix}\psi\\ h\end{pmatrix} = \left(\frac{\langle \mathfrak{l}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle}{\langle \mathfrak{m}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle}\right)^{-1}. \tag{2.65}$$

In the trasversal problem, the operator $\mathfrak{m}_{\sigma,0}$ is simply a multiplication operator, therefore its quadratic form reads straightforwardly

$$\begin{aligned}\langle \mathfrak{m}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle &= \sigma\int_0^\delta \alpha(y)\big(f\psi(y)+GH_1h(y)\big)\bar\psi(y)w_1(y)\,dy\\ &\qquad + \sigma\int_0^\delta \frac{f\alpha(y)}{GH_2}\big(f\psi(y)+GH_1h(y)\big)\bar h(y)w_2(y)\,dy\\ &= \frac{\sigma}{f}\int_0^\delta \alpha(y)\big(f\psi(y)+GH_1h(y)\big)\overline{\big(f\psi(y)+GH_1h(y)\big)}w_1(y)\,dy\\ &= \frac{\sigma}{f}\int_0^\delta \alpha(y)\big|f\psi(y)+GH_1h(y)\big|^2 w_1(y)\,dy,\end{aligned} \tag{2.66}$$

where we used $w_2 = \frac{G^2}{f^2}H_1H_2w_1$ as in (2.58).

For what concerns the term involving $\mathfrak{l}_{\sigma,0}$, after some computations similar to the ones undertaken in the general case $\mathcal{L}_\gamma$ (for example in Lemma 2.8 and Lemma 2.9), it is readily seen that

$$\begin{aligned}\langle \mathfrak{l}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle &= \int_0^\delta \Big(|\psi'(y)|^2+\sigma^2|\psi(y)|^2\Big)w_1(y) + \Big(|h'(y)|^2+\sigma^2|h|^2(y)\Big)w_2(y)\,dy\\ &\qquad + G\int_0^\delta |h|^2\,dy + \kappa_0 w_1(0)|\psi(\delta)|^2 + \sqrt{\lambda_\delta^2+\kappa_0^2}\,w_2(\delta)|h(\delta)|^2.\end{aligned} \tag{2.67}$$

It follows straightforwardly that

$$\langle \mathfrak{m}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle > 0, \qquad \langle \mathfrak{l}_{\sigma,0}\begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle > 0.$$

For any fixed value of $\sigma > 0$, the spectrum of the transversal problem is purely discrete. We are interested in the behaviour of the maximum value of the spectrum

$$\omega_\sigma = \sup\left(\mathrm{spec}(\mathfrak{a}_\sigma)\right)$$

as a function of σ. In the formula above have used the formalism $\mathrm{spec}(\mathfrak{a}_\sigma)$ to refer to the spectrum of the operator $(\mathfrak{a}_\sigma)$, in order to avoid a repetition of the symbol σ which could have created some ambiguity in the notation. In particular

$$\omega_\sigma = \sup_{\tilde{\psi},\tilde{h}\in\mathcal{D}(\mathfrak{a}_\sigma)} J_\sigma\big([\tilde{\psi},\tilde{h}]^T\big)$$

corresponding to λ_1^s in the notation introduced for the variational principle presented in Proposition 2.1.

2.7 Essential spectrum

In this section we study the properties of the essential spectrum of the pencil operator $\mathcal{A}_\gamma$.

Proposition 2.4. *It holds*

$$\sigma_{\mathrm{ess}}(\mathcal{A}_\gamma) = \sigma_{\mathrm{ess}}(\mathcal{A}_0) = \overline{\bigcup_{\sigma\in\mathbb{R}} \mathrm{spec}(\mathfrak{a}_\sigma)} = \left[-\sup_{\sigma\in\mathbb{R}}\omega_\sigma,\ \sup_{\sigma\in\mathbb{R}}\omega_\sigma\right]. \tag{2.68}$$

We omit the proof of the previous proposition, as the major challenge in it would be the presence of long and tedious computations which are based, though, on standard arguments belonging to the spectral theory on waveguides.

We briefly mention that $\sigma_{\mathrm{ess}}(\mathcal{A}_0) \subseteq \sigma_{\mathrm{ess}}(\mathcal{A}_\gamma)$ follows after a Neumann-Dirichlet bracketing argument. The other inclusion, namely, $\sigma_{\mathrm{ess}}(\mathcal{A}_\gamma) \subseteq \sigma_{\mathrm{ess}}(\mathcal{A}_0)$ can be shown by separation of variables which allows to find a Weyl's sequence (ref. Proposition 1.7) of explicit functions in the curved strip for any point in the essential spectrum of the operator defined on the straight strip. We refer to [87], Lemma 3.5 and Lemma 3.6 and reference therein. We also refer to [98] for an illustrative case of the Laplacian. The other equalities follow from a separation of variables and the variational

characterisations of the spectrum.

Let us define then the supremum of the essential spectrum by Ω_*. It then follows that

$$\begin{aligned}\Omega_* \ :&= \sup \sigma_{\mathrm{ess}}(\mathcal{A}_0) = \sup_{\sigma\in\mathbb{R}} \omega_\sigma = \sup_{\sigma\in\mathbb{R}} \sup_{\psi,h\in\mathcal{D}(\mathfrak{a}_\sigma)} J_\sigma\big([\psi,h]^T\big) \\ &= \sup_{\psi,h\in\mathcal{D}(\mathfrak{a}_\sigma)} \sup_{\sigma\in\mathbb{R}} J_\sigma\big([\psi,h]^T\big).\end{aligned} \tag{2.69}$$

Let us consider $\sigma > 0$ and introduce now two auxiliary quantities $J^{(1)}$ and $J^{(2)}$, so that the expression for functional J_σ, definied in (2.65) can be reformulated in terms of them as follows

$$\left(\frac{\langle \mathfrak{l}_0 \begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle}{\langle \mathfrak{m}_0 \begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle}\right)^{-1} = \left(\sigma J^{(1)}\begin{pmatrix}\psi\\ h\end{pmatrix} + \frac{1}{\sigma} J^{(2)}\begin{pmatrix}\psi\\ h\end{pmatrix}\right)^{-1}, \tag{2.70}$$

where

$$J^{(1)}\begin{pmatrix}\psi\\ h\end{pmatrix} = f\frac{\int_0^\delta \psi(y)^2 w_1(y) + h(y)^2 w_2(y)\, dy}{\int_0^\delta \alpha(y)\left|f\psi(y) + GH_1 h(y)\right|^2 w_1(y)\, dy} \tag{2.71}$$

and

$$\begin{aligned}J^{(2)}\begin{pmatrix}\psi\\ h\end{pmatrix} = \Bigg(f\int_0^\delta \psi'(y)^2 w_1(y) + h'(y)^2 w_2(y) \ + Gh(y)^2 dy + \kappa_0 w_1(\delta)|\psi(\delta)|^2 \\ + \sqrt{\lambda_\delta^2 + \kappa_0^2}\, w_2(\delta)|h(\delta)|^2 \cdot \Bigg) \\ \Big(\int_0^\delta \alpha(y)\left(f\psi(y) + GH_1 h(y)\right)^2 w_1(y)\, dy\Big)^{-1}.\end{aligned} \tag{2.72}$$

We observe that the quantity defined in (2.71), is bounded away from zero. In fact, it is easy to show, similarly to how is done in [87], that there exists a positive constant $K > 0$ such that

$$J^{(1)}\begin{pmatrix}\psi\\ h\end{pmatrix} \geq K\frac{\sup_{y\in[0,\delta]} \chi(y)}{\inf_{y\in[0,\delta]} \alpha(y) w_1(y)} > 0,$$

where $\chi(y) = \min\left(w_1(y), \frac{H_2(y)w_1(y)}{H_1}\right) > 0$.

From (2.70), it follows immediately that $0 < \omega_\sigma < \infty$ and $\omega_\sigma \to +0$ as $\sigma \to \infty$ and that the value of σ which realises the internal sup in (2.69) is given by

$$\sigma_*\big([\psi, h]^T\big) = \sqrt{\frac{J^{(2)}([\psi, h]^T)}{J^{(1)}([\psi, h]^T)}}.$$

Maximising with respect to σ we obtain

$$J_\sigma\big([\psi, h]^T\big) \leq J_{\sigma_*}\big([\psi, h]^T\big)$$

where we recall $\sigma_* = \sigma_*([\psi, h])$ depends on the test functions.

Finally, maximising $J_{\sigma^*}\big([\psi, h]^T\big)$ with respect to the test functions, we obtain

$$\Omega_* = J_{\sigma_\bullet}([\psi_*, h_*]^T) < \infty \tag{2.73}$$

where the explicit expression for $\sigma_\bullet$ is given in (2.75). Being the operator $-i\partial_x$ a first order operator, it follows that

$$\operatorname{spec}(\mathfrak{a}_\sigma) = -\operatorname{spec}(\mathfrak{a}_{-\sigma})$$

thus we finally conclude

$$\sigma_{\mathrm{ess}}(\mathcal{A}_\gamma) = \sigma_{\mathrm{ess}}(\mathcal{A}_0) = [-\Omega_*, \Omega_*]\,. \tag{2.74}$$

If we interpret the inverse of ω as the quantity which is usually considered as the classical spectral parameter, equation (2.74) shows the presence of a band gap between the two components of the essential spectrum, a precondition needed for the existence of non embedded eigenvalues.

Remark 2.8. Let the maximising test functions be $(\psi_*, h_*) \in \mathcal{D}(\mathfrak{a}_\sigma)$. It follows by

(2.71) and (2.72) that the value of $\sigma_\bullet$ has the following explicit expression

$$\begin{aligned}\sigma_\bullet = \sigma_*([\psi_*, h_*]) &= \sqrt{\frac{J^{(2)}([\psi_*, h_*])}{J^{(1)}([\psi_*, h_*])}} \\ &= \Bigg(\int_0^\delta \psi_*'(y)^2 w_1(y) + h_*'(y)^2 w_2(y) + G h(y)_*^2 \, dy \\ &\qquad + \Big(k_0 w_1(\delta) |\psi_*(\delta)|^2 + \sqrt{\lambda_\delta^2 + k_0^2} |h_*(\delta)|^2 w_2(\delta) \Big) \Bigg)^{1/2} \\ &\qquad \cdot \left(\int_0^L \psi_*(y)^2 w_1(y) + h_*(y)^2 w_2(y) \, dy \right)^{-1/2}. \end{aligned} \tag{2.75}$$

Remark 2.9. Let ψ_* and h_* as in Remark 2.8. Then they solve equations (2.62) and (2.63) with $\omega = \Omega_*$ and $\sigma = \sigma_\bullet$,

$$\Omega_* \Big(-\psi_*''(y) + \alpha(y) \psi_*'(y) + \sigma_\bullet^2 \psi_*(y) \Big) = \sigma_\bullet \, \alpha(y) \Big(f \psi_*(y) + G H_1 h_*(y) \Big) \tag{2.76}$$

$$\Omega_* \Big(-h_*''(y) - \alpha(y) \frac{H_1}{H_2} h_*'(y) + \Big[\sigma_\bullet^2 + \lambda^2 \Big] h_*(y) \Big) = \sigma_\bullet f \frac{\alpha(y)}{H_2} \Big(\frac{f}{G} \psi_*(y) + H_1 h_*(y) \Big) \tag{2.77}$$

where $\sigma_\bullet$ is defined in (2.75) and Ω_* in (2.73)

$$\Omega_* = \frac{\langle \mathfrak{m}_{\sigma_\bullet} \begin{pmatrix} \psi_* \\ h_* \end{pmatrix}, \begin{pmatrix} \psi_* \\ h_* \end{pmatrix} \rangle}{\langle \mathfrak{l}_{\sigma_\bullet} \begin{pmatrix} \psi_* \\ h_* \end{pmatrix}, \begin{pmatrix} \psi_* \\ h_* \end{pmatrix} \rangle}. \tag{2.78}$$

Remark 2.10. We note here that all the quantities defined so far as well as all the test functions realising the maximum they do depend in fact upon the choice for the real quantities κ_0 and ω_0. In practical applications, these quantities are determined by physical considerations and the knowledge of the geometry of the problem.

2.8 Discrete spectrum

In this section we finally state and prove the main result which concerns the existence of discrete spectrum for the operator $\mathcal{A}_\gamma$ under some hypothesis on the topography and geometry of the channel.

Theorem 2.5. *Consider the pencil problem for $\mathcal{A}_\gamma$ defined in* (2.42) *with $\gamma \in C^\infty(\mathbb{R})$ such that* $\operatorname{supp}\gamma \Subset [-R, R]$ *for a real $R > 0$ and such that* (2.3) *is satisfied. Assume additionally that*

$$\alpha(\eta) > 0, \tag{2.79a}$$

$$\alpha'(\eta) \leq 0. \tag{2.79b}$$

where α is as defined in (2.14).

Then for sufficiently small values of G there exists a constant $C_\alpha > 0$, which depends only on the topography of the channel, such that $\sigma_{\mathrm{dis}}(\mathcal{A}_\gamma) \neq \emptyset$ whenever the curvature $\gamma(\xi)$ satisfies

$$\int_{\mathbb{R}} \gamma(\xi)\, d\xi > C_\alpha \int_{\mathbb{R}} \gamma(\xi)^2\, d\xi. \tag{2.80}$$

The proof for the existence of at least one eigenvalue proceeds similarly to the one holding for the single layer case. As the latter, it relies on a modification of a variational principle, based on the characterization of the supremum of the essential spectrum in terms of the Rayleigh quotient and a density argument.

It is important to note that all the results that we have obtained depend in fact on G. In other words, for a fixed channel γ, (2.59) defines rather a family of quadratic form indexed by G. This implies in turn that also ψ_*, h_* and thus $\sigma_\bullet$, Ω_* depend on G. As noted before, these quantities are well known for the case $G = 0$.

2.8.1 The one layer case

We have observed in Remark 2.4 that the case when only one layer is present can be seen as a special case of the two layers problem where $G \equiv 0$ in equation (2.18) and the operator $\mathcal{L}$ reduces to $\mathcal{L}_{0,1}$. In this particular setting, the statement of our

theorem 2.5 simply translate into the one stated in Theorem 4.1 in Johnson, Levitin and Parnovski [86].

2.8.2 Proof of the main result

Proof of Theorem 2.5. We begin by observing that, by mean of Proposition 2.2, the statement of Theorem 2.5 would follow if we were able to find a pair $(\tilde{\Psi}, \tilde{h}) \in \mathcal{D}(\mathcal{A}_\gamma)$ belonging to the form domain such that the respective Rayleigh quotient is strictly greater than Ω_*, namely the supremum of the essential spectrum

$$(\tilde{\Psi}, \tilde{h}) \in \mathcal{D}(\mathcal{A}_\gamma) \quad s.t. \quad \frac{\langle \mathcal{M}_\gamma \cdot \begin{pmatrix} \tilde{\Psi} \\ \tilde{h} \end{pmatrix}, \begin{pmatrix} \tilde{\Psi} \\ \tilde{h} \end{pmatrix} \rangle}{\langle \mathcal{L}_\gamma \cdot \begin{pmatrix} \tilde{\Psi} \\ \tilde{h} \end{pmatrix}, \begin{pmatrix} \tilde{\Psi} \\ \tilde{h} \end{pmatrix} \rangle} > \Omega_*. \tag{2.81}$$

Unfortunately, finding such pair $(\tilde{\Psi}, \tilde{h}) \in \mathcal{D}(\mathcal{A}_\gamma)$ is an unrealistic task, as it means to almost know the eigenfunctions themselves. We will bypass this difficulty using the technique introduced in [37] where the existence of discrete spectrum was proved for acoustic waveguides with obstacles, and later in [87] that allows to work with localised test functions which approximate the pair $(\tilde{\Psi}, \tilde{h}) \in \mathcal{D}(\mathcal{A}_\gamma)$ sought in (2.81). Heuristically we justify the use of the localisation argument by the fact that both in Davies' paper and in the shallow water regimes, the geometric perturbations which are responsible for the creation of point in the discrete spectrum are found within a compact subset of the strip $\mathcal{S}$. Therefore, it is in that part of the strip where modifications of the free solutions and respective conditions on the Rayleigh quotient have to been found.

Let us fix some notation. Let's define the portion of the strip $\mathcal{S}$ of length $2r$ as

$$\mathcal{S}_\gamma^r = \Lambda\left([-r, r] \times [0, \delta]\right) \tag{2.82}$$

where Λ is the planar transformation defined in (2.2). Further, consider a smooth cut-off function $\chi_r(\xi, \eta)$ such that $\chi_r(\xi, \eta) = \chi_r(\xi) \in C_c^\infty(\mathcal{S}_\gamma)$ and

$$\chi_r(\xi) \equiv 1 \qquad \text{for } \xi \in \mathcal{S}_\gamma^r, \tag{2.83}$$

for any $r > R$, where $R > 0$ has been introduced in [CPT].

The argument proceed as follows. Let us suppose that there exists a pair of functions (Ψ, Υ), may be not in the domain of the quadratic form, such that their restrictions

$$(\Psi_r, \Upsilon_r) = (\Psi(\xi, \eta)\chi_{1,r}(\xi), \Upsilon(\xi, \eta)\chi_{2,r}(\xi))$$

satisfy, for all $r > R$, the following inequality

$$\frac{\langle \mathcal{M}_\gamma \cdot \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix}, \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix} \rangle_{\mathcal{S}^r_\gamma}}{\langle \mathcal{L}_\gamma \cdot \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix}, \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix} \rangle_{\mathcal{S}^r_\gamma}} > \Omega_* \qquad \forall r > R, \tag{2.84}$$

where we used the abbreviation $\langle \cdot, \cdot \rangle_{\mathcal{S}^r_\gamma}$ for the the scalar product defined in (2.45) with $C = \mathcal{S}^r_\gamma$. This, in particular, implies that the quantity

$$a := \Omega_* \langle \mathcal{L}_\gamma \cdot \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix}, \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix} \rangle_{\mathcal{S}^r_\gamma} - \langle \mathcal{M}_\gamma \cdot \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix}, \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix} \rangle_{\mathcal{S}^r_\gamma} < 0$$

is negative for any $r > R$. Therefore, it follows that if the condition in (2.84) is satisfied, then having (2.81) with $(\tilde{\psi}, \tilde{h}) = (\Psi_r, \Upsilon_r)$ is equivalent to

$$\Omega_* \langle \mathcal{L}_\gamma \cdot \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix}, \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix} \rangle_{\mathcal{S}_\gamma \setminus \mathcal{S}^r_\gamma} - \langle \mathcal{M}_\gamma \cdot \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix}, \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix} \rangle_{\mathcal{S}_\gamma \setminus \mathcal{S}^r_\gamma} < -a, \tag{2.85}$$

where we observe that $(-a) > 0$ is a positive constant. Now the argument ends by noting that the last inequality comes from the arbitrary choice of the cut-off functions $\chi_{1,r}(\xi)$ and $\chi_{2,r}(\xi)$ and the value r sufficiently large. We refer to the proof of Proposition 4 in [37] for the details in the case of the Laplacian on a waveguide with obstacles.

We conclude by noting that the choice $(\tilde{\Psi}, \tilde{h}) = (\Psi_r, \Upsilon_r)$ for sufficiently big $r > 0$ would settle the argument and prove Theorem 2.5.

We are therefore left with proving (2.84). A suitable pair of candidates for the choice

of the functions Ψ and Υ is

$$\Psi(\xi,\eta)=\psi_*(\eta)e^{i\sigma_\bullet\xi},\qquad\qquad \Upsilon(\xi,\eta)=h_*(\eta)e^{i\sigma_\bullet\xi}, \tag{2.86}$$

where we recall the pair (ψ_*,h_*) realises (2.75)- (2.78).

Let us continue with some other preliminary considerations. From (2.78) we observe that

$$0<\Omega_*=\frac{\langle \mathfrak{m}_{\sigma_\bullet}\begin{pmatrix}\psi_*\\ h_*\end{pmatrix},\begin{pmatrix}\psi_*\\ h_*\end{pmatrix}\rangle}{\langle \mathfrak{l}_{\sigma_\bullet}\begin{pmatrix}\psi_*\\ h_*\end{pmatrix},\begin{pmatrix}\psi_*\\ h_*\end{pmatrix}\rangle}=\frac{\langle \mathcal{M}_0\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix},\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}\rangle_{\mathcal{S}_0^r}}{\langle \mathcal{L}_0\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix},\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}\rangle_{\mathcal{S}_0^r}}\qquad \text{for any } r>0 \tag{2.87}$$

follows since the phase factors in (2.86) simply cancel out.

Note 2.11. In the following, all the quadratic forms $\langle\cdot,\cdot\rangle$ have to be interpreted restricted to $\mathcal{S}_\gamma^r$, namely they all should be understood as $\langle\cdot,\cdot\rangle_{\mathcal{S}_\gamma^r}$. We omit the additional subscript, avoiding a heavy notation. We also recall the fact that in $\mathcal{S}_\gamma^r$ the functions (Ψ_r,Υ_r) and (Ψ,Υ) coincide by (2.83).

Moreover, we observe that the quadratic forms corresponding to the operators $\mathcal{M}_\gamma$ are independent from the curvature

$$\langle \mathcal{M}_0\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix},\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}\rangle=\langle \mathcal{M}_\gamma\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix},\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}\rangle \tag{2.88}$$

and are definite positive. Taking into account the relations in (2.87) and (2.88) we define the quantity

$$D_\gamma:=\langle \mathcal{L}_\gamma\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix},\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}\rangle-\langle \mathcal{L}_0\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix},\begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}\rangle \tag{2.89}$$

and note that the condition (2.84) is equivalent to have

$$D_\gamma<0,\qquad \forall r>R. \tag{2.90}$$

From Lemma 2.11 we have that D_γ decomposes into

$$D_\gamma = \mathcal{S}_2 + \mathcal{I}_1 \cdot \int_{-r}^{r} \gamma(\xi)\, d\xi, \tag{2.91}$$

where the following estimate

$$\mathcal{S}_2 \leq \mathcal{I}_2 \cdot \int_{-r}^{r} \gamma(\xi)^2\, d\xi.$$

is proved in Lemma 2.12. Therefore, letting $\mathcal{C}_\alpha := -\frac{\mathcal{I}_2}{\mathcal{I}_1}$ and observing that

$$D_\gamma \leq -\mathcal{I}_1 \cdot \Big(\mathcal{C}_\alpha \int_{-r}^{r} \gamma(\xi)^2\, d\xi - \int_{-r}^{r} \gamma(\xi)\, d\xi\Big),$$

we deduce that the condition $D_\gamma < 0$ is equivalent to ask $\mathcal{I}_1 < 0$ plus the additional condition on the curvature (2.80).

Substituting the explicit expression of $\sigma_\bullet$ given in (2.97) in the definition of $\mathcal{I}_1$ given in (2.110), we deduce that condition $\mathcal{I}_1 < 0$ is equivalent to

$$\begin{aligned}\int_0^\delta B(\eta)\, d\eta \cdot \Big(\delta M(\delta) + \int_0^\delta \eta\,(A(\eta) + N(\eta))\, d\eta\Big) \\ - \int_0^\delta \eta B(\eta)\, d\eta \cdot \Big(\delta M(\delta) + \int_0^\delta A(\eta) + N(\eta)\, d\eta\Big) < 0,\end{aligned} \tag{2.92}$$

where

$$A(\eta) := |\psi_*'(\eta)|^2 w_1 + |h_*'(\eta)|^2 w_2, \tag{2.93}$$

$$B(\eta) := |\psi_*(\eta)|^2 w_1 + |h_*(\eta)|^2 w_2, \tag{2.94}$$

$$N(\eta) := G|h_*(\eta)|^2, \tag{2.95}$$

$$M(\delta) := \kappa_0 |\psi_*(\delta)|^2 w_1(\delta) + \sqrt{\lambda_\delta^2 + \kappa_0^2}|h_*(\delta)|^2 w_2(\delta). \tag{2.96}$$

It follows that equation (2.75) can be re-written in terms of the quantities introduced above as

$$\sigma_\bullet^2 = \frac{\int_0^\delta A(\eta) + N(\eta)\, d\eta + M(\delta)}{\int_0^\delta B(\eta)\, d\eta}. \tag{2.97}$$

Let us define for convenience the following quantity

$$\mathcal{P}(\delta) := \frac{1}{2}\Big(\int_0^\delta \psi_*^2(\eta)\alpha(\eta)w_1(\eta) - \int_0^\delta h_*^2(\eta)\frac{\alpha(\eta)}{H_2}w_2(\eta)\Big). \tag{2.98}$$

From Lemma 2.15 and Lemma 2.14, and in particular by means of (2.117) and (2.118), the inequality in (2.92) transforms into

$$\begin{aligned}\int_0^\delta B(\eta)\,d\eta \cdot \left(-\frac{1}{2}B(\delta) - \mathcal{P}(\delta) + \frac{\sigma_\bullet}{\Omega_*}\frac{1}{f}\int_0^\delta \eta\frac{\alpha(\eta)}{H(\eta)}(f\psi_* + GH_1h_*)^2\,d\eta\right) \\ - \int_0^\delta \eta B(\eta)\,d\eta \cdot \left(\frac{\sigma_\bullet}{\Omega_*}\frac{1}{f}\int_0^\delta \frac{\alpha(\eta)}{H(\eta)}(f\psi_* + GH_1h_*)^2\,d\eta\right) < 0,\end{aligned} \tag{2.99}$$

where we observe that the terms which are multiplying $\sigma_\bullet^2$ cancel out.

Finally, the theorem will be proved if we show that (2.99) holds. For this purpose, let us break it into smaller parts, singling out terms which will be proved individually to be negative, eventually under some additional conditions. The first term which we will consider is the one involving $\mathcal{P}(\delta)$. We note that

$$2\mathcal{P}(\delta) = \int_0^\delta \psi_*^2(\eta)\alpha(\eta)w_1(\eta) - \int_0^\delta h_*^2(\eta)\frac{\alpha(\eta)}{H_2}w_2(\eta) = \int_0^\delta \frac{\alpha(\eta)}{f^2H}\left(f^2\psi_*^2(\eta) - G^2h_*^2(\eta)\right)\,d\eta. \tag{2.100}$$

Therefore a sufficient condition for the term involving $\mathcal{P}(\delta)$ to be negative is

$$f^2\psi_*^2(\eta) - G^2h_*^2(\eta) > 0. \tag{2.101}$$

Let us assume G small enough, then from (2.77) we have that

$$h_* \approx \frac{\sigma_\bullet}{\Omega_*}\frac{H_1}{H}\psi_* \tag{2.102}$$

thus, condition (2.101) is equivalent to

$$H^2f^2 - G\frac{\sigma_\bullet^2}{\Omega_*^2} > 0. \tag{2.103}$$

Note that, in fact, the quantity $\frac{\sigma_\bullet^2}{\Omega_*^2}$ depends ultimately on G, therefore condition

(2.103) might not hold. This is not the case as it is readily seen that the quantity $\frac{\sigma_\bullet^2}{\Omega_*^2}$ tends to the (finite) value coming form the single layer case when $G = 0$.
The second expression that we want to show to be negative is

$$\begin{aligned}\int_0^\delta B(\eta)\,d\eta \cdot \left(\frac{\sigma_\bullet}{\Omega_*}\frac{1}{f}\int_0^\delta \eta\frac{\alpha(\eta)}{H(\eta)}(f\psi_* + GH_1h_*)^2\,d\eta\right) \\ - \int_0^\delta \eta B(\eta)\,d\eta \cdot \left(\frac{\sigma_\bullet}{\Omega_*}\frac{1}{f}\int_0^\delta \frac{\alpha(\eta)}{H(\eta)}(f\psi_* + GH_1h_*)^2\,d\eta\right) \leq 0.\end{aligned} \tag{2.104}$$

We are not able to show inequality (2.104) directly, due to its complicated expression. Nonetheless, the validity of this inequality follows by a continuity argument. We noted in Remark 2.4 that when $G = 0$, the problem reduces to the single layer one. In this case, it was shown in [86] that inequality (2.104) holds (see Lemma 4.5 and Lemma 4.6 in the same paper), and so it extends, by continuity of the problem and in view of (2.102) to the case when G is a small positive constant. We emphasize that the condition (2.79a) is always verified by means of (2.5) and that together with (2.79b) are conditions needed for the existence of trapped modes in the single layer case. Inequality (2.99) is finally proved true after noting the following trivial inequality

$$-\frac{B(\delta)}{2}\int_0^\delta B(\eta)\,d\eta < 0.$$

Thus the theorem is proved. □

2.9 Auxiliary results

In this last section we collect for the reader's convenience, some technical and auxiliary results which have been used in the previous sections. The proofs are straightforward and will be occasionally omitted.

Lemma 2.6. *It holds*

$$\begin{aligned}\langle \mathcal{L}_{0,1}\psi_1, \psi_2\rangle_{L^2(\mathcal{S}_0;w_1)} = \int_{\mathbb{R}}\int_0^\delta \nabla\psi_1\cdot\nabla\psi_2\, w_1(y)\,dxdy + \int_{\mathbb{R}}\int_0^\delta \nabla\psi_1\cdot\nabla w_1(y)\psi_2\,dxdy \\ - \int_{-\infty}^{\infty}\partial_y\psi_1\psi_2 w_1\Big|_{y=0}^{y=\delta}dx + \int_{\mathbb{R}}\int_0^\delta \partial_y\psi_1\alpha(y)\psi_2\, w_1(y)\,dxdy\end{aligned}$$

Proof. It is a simple application of integration by part together with the decay property at infinity of ψ_1, ψ_2.

$$\begin{aligned}
\langle \mathcal{L}_{0,1}\psi_1, \psi_2\rangle_{L^2(\mathcal{S}_0;w_1)} &= \langle -\Delta + \alpha(y)\partial_y\psi_1, \psi_2\rangle_{L^2(\mathcal{S}_0;w_1)} \\
&= \int_{\mathbb{R}}\int_0^\delta \nabla\psi_1\cdot\nabla\psi_2\, w_1(y)\, dxdy + \int_{\mathbb{R}}\int_0^\delta \nabla\psi_1\cdot\nabla w_1(y)\psi_2\, dxdy \\
&\quad - \int_0^\delta \partial_x\psi_1\psi_2 w_1\Big|_{x=\infty}^{x=\infty} dy - \int_{-\infty}^{\infty} \partial_y\psi_1\psi_2 w_1\Big|_{y=0}^{y=\delta} dx \\
&\quad + \int_{\mathbb{R}}\int_0^\delta \partial_y\psi_1\alpha(y)\psi_2\, w_1(y)\, dxdy \\
&= \int_{\mathbb{R}}\int_0^\delta \nabla\psi_1\cdot\nabla\psi_2\, w_1(y)\, dxdy + \int_{\mathbb{R}}\int_0^\delta \nabla\psi_1\cdot\nabla w_1(y)\psi_2\, dxdy \\
&\quad - \int_{-\infty}^{\infty} \partial_y\psi_1\psi_2 w_1\Big|_{y=0}^{y=\delta} dx + \int_{\mathbb{R}}\int_0^\delta \partial_y\psi_1\alpha(y)\psi_2\, w_1(y)\, dxdy
\end{aligned} \tag{2.105}$$

□

Similarly, it follows

Lemma 2.7.

$$\begin{aligned}
\langle \mathcal{L}_{0,2}h_1, h_2\rangle_{L^2(\mathcal{S}_0;w_1)} &= \int_{\mathbb{R}}\int_0^\delta \nabla h_1\cdot\nabla h_2\, w_2(y)\, dxdy + \int_{\mathbb{R}}\int_0^\delta \nabla h_1\cdot\nabla w_2(y)h_2\, dxdy \\
&\quad - \int_{-\infty}^{\infty} \partial_y h_1 h_2 w_2\Big|_{y=0}^{y=\delta} dx + \int_{\mathbb{R}}\int_0^\delta \lambda^2(y) h_1 h_2 w_2\, dxdy \\
&\quad - \int_{\mathbb{R}}\int_0^\delta \partial_y h_1\alpha(y)\frac{H_1}{H_2(y)} h_2\, w_2(y)\, dxdy
\end{aligned} \tag{2.106}$$

Lemma 2.8. *It holds*

$$\langle \mathcal{L}_{0,1}\psi, \psi\rangle_{L^2(\mathcal{S}_0;w_1)} = \int_{\mathbb{R}}\int_0^\delta \frac{1}{p(\xi,\eta)}|\nabla\psi|^2\, w_1(y)\, d\xi d\eta + \int_{-\infty}^{\infty} \kappa_0|\psi(\xi,\delta)|^2 w_1(\delta)p(\xi,\delta)\, d\xi. \tag{2.107}$$

Proof. From equation (2.105) for the weight choice (2.55) we have that

$$\langle \mathcal{L}_{0,1}\psi_1, \psi_2 \rangle_{L^2(\mathcal{S}_0;w_1)} = \int_{\mathbb{R}} \int_0^\delta \nabla\psi_1 \cdot \nabla\psi_2 \, w_1(y) \, dxdy + \int_{-\infty}^{\infty} \kappa_0 \psi_1(x,\delta)\psi_2(x,\delta) w_1(\delta) \, dx.$$

Then the statement follows from (2.25) and (2.31)

$$\begin{aligned}\langle \mathcal{L}_{0,1}\psi, \psi \rangle_{L^2(\mathcal{S}_0;w_1)} &= \int_{\mathbb{R}} \int_0^\delta \frac{1}{p^2(\xi,\eta)} |\nabla\psi|^2 w_1(y) p(\xi,\eta) d\xi d\eta + \int_{-\infty}^{\infty} \kappa_0 |\psi(\xi,\delta)|^2 w_1(\delta) p(\xi,\delta) \, d\xi \\ &= \int_{\mathbb{R}} \int_0^\delta \frac{1}{p(\xi,\eta)} |\nabla\psi|^2 \, w_1(y) \, d\xi d\eta + \int_{-\infty}^{\infty} \kappa_0 |\psi(\xi,\delta)|^2 w_1(\delta) p(\xi,\delta) \, d\xi.\end{aligned}$$

□

Lemma 2.9. *It holds*

$$\begin{aligned}\langle \mathcal{L}_{0,2}h, h \rangle_{L^2(\mathcal{S}_0;w_2)} = &\int_{\mathbb{R}} \int_0^\delta \frac{1}{p(\xi,\eta)} |\nabla h|^2 w_2(y) \; d\xi d\eta + \int_{\mathbb{R}} \int_0^\delta G|h|^2 \, p(\xi,\eta) \, d\xi d\eta \\ &+ \sqrt{\lambda_\delta^2 + \kappa_0^2} w_2(\delta) \int_{-\infty}^{\infty} |h(\xi,\delta)|^2 p(\xi,\eta) \, d\xi \\ &+ w_2(0) a_0 \int_{-\infty}^{\infty} |h(\xi,0)|^2 \, d\xi\end{aligned} \tag{2.108}$$

Proof. From equation (2.106) for the weight choice (2.57) we have that

$$\begin{aligned}\langle \mathcal{L}_{0,2}h_1, h_2 \rangle_{L^2(\mathcal{S}_0;w_2)} = &\int_{\mathbb{R}} \int_0^\delta \nabla h_1 \cdot \nabla h_2 \, w_2(y) \, dxdy + \int_{\mathbb{R}} \int_0^\delta \lambda^2(y) h_1 h_2 w_2 \, dxdy \\ &+ \sqrt{\lambda_\delta^2 + \kappa_0^2} w_2(\delta) \int_{-\infty}^{\infty} h_1(x,\delta) h_2(x,\delta) \, dx \\ &+ w_2(0) a_0 \int_{-\infty}^{\infty} h_1(x,0) h_2(x,0) \, dx\end{aligned}$$

From the fact that from (2.58) we have

$$\lambda^2(y) = \frac{G}{w_2}.$$

Then the statement follows from (2.25) and (2.31)

$$\begin{aligned}\langle \mathcal{L}_{0,2}h, h\rangle_{L^2(\mathcal{S}_0;w_2)} &= \int_{\mathbb{R}}\int_0^\delta \frac{1}{p(\xi,\eta)^2}|\nabla h|^2 w_2(\eta)\, p(\xi,\eta)\, d\xi d\eta + \int_{\mathbb{R}}\int_0^\delta \frac{G}{w_2}|h|^2 w_2\, p(\xi,\eta)\, d\xi d\eta \\ &\quad + \sqrt{\lambda_\delta^2+\kappa_0^2}w_2(\delta)\int_{-\infty}^{\infty}|h(\xi,\delta)|^2 p(\xi,\eta)\, d\xi \\ &\quad + w_2(0)a_0\int_{-\infty}^{\infty}|h(\xi,0)|^2\, d\xi \\ &= \int_{\mathbb{R}}\int_0^\delta \frac{1}{p(\xi,\eta)}|\nabla h|^2 w_2(y)\ d\xi d\eta + \int_{\mathbb{R}}\int_0^\delta G|h|^2\, p(\xi,\eta)\, d\xi d\eta \\ &\quad + \sqrt{\lambda_\delta^2+\kappa_0^2}w_2(\delta)\int_{-\infty}^{\infty}|h(\xi,\delta)|^2 p(\xi,\eta)\, d\xi \\ &\quad + w_2(0)a_0\int_{-\infty}^{\infty}|h(\xi,0)|^2\, d\xi\end{aligned}$$

□

Lemma 2.10. *The quantity* $\langle \mathcal{L}_\gamma^r \begin{pmatrix}\psi\\ h\end{pmatrix}, \begin{pmatrix}\psi\\ h\end{pmatrix}\rangle_{L^2(\mathcal{S}_\gamma;w_1,w_2)}$ *for* (Ψ_r,Υ_r) *as in* (2.86) *reads as*

$$\begin{aligned}&\langle \mathcal{L}_\gamma \begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}, \begin{pmatrix}\Psi_r\\ \Upsilon_r\end{pmatrix}\rangle_{L^2(\mathcal{S}_\gamma;w_1,w_2)} = \\ &\quad = \int_{-r}^{r}\int_0^\delta \Big[\frac{1}{p}\sigma_\bullet^2|\psi_*(\eta)|^2 + p|\psi_*'(\eta)|^2\Big] w_1(\eta)\, d\eta d\xi + \Big[\frac{1}{p}\sigma_\bullet^2|h_*(\eta)|^2 + p|h_*'(\eta)|^2\Big] w_2(\eta)\, d\eta d\xi \\ &\qquad + \int_{-r}^{r}\int_0^\delta pG|h_*(\eta)|^2\, d\eta d\xi + \kappa_0 w_1(\delta)\int_{-r}^{r} p(\xi,\delta)|\psi_*(\xi,\delta)|^2\, d\xi \\ &\qquad\quad + \sqrt{\lambda_\delta^2+\kappa_0^2}w_2(\delta)\int_{-r}^{r} p(\xi,\delta)|h_*(\xi,\delta)|^2\, d\xi\end{aligned}$$

$$\langle \mathcal{L}_0 \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix}, \begin{pmatrix} \Psi_r \\ \Upsilon_r \end{pmatrix} \rangle_{L^2(\mathcal{S}_0;w_1,w_2)} =$$
$$= \int_{-r}^{r}\int_0^{\delta} \Big[\sigma_\bullet^2|\psi_*(\eta)|^2 + |\psi_*'(\eta)|^2\Big] w_1(\eta)\, d\eta d\xi + \Big[\sigma_\bullet^2|h_*(\eta)|^2 + |h_*'(\eta)|^2\Big] w_2(\eta)\, d\eta d\xi$$
$$+ \int_{-r}^{r}\int_0^{\delta} G|h_*(\eta)|^2\, d\eta d\xi + \kappa_0 w_1(\delta)\int_{-r}^{r}|\psi_*(\xi,\delta)|^2\, d\xi$$
$$+ \sqrt{\lambda_\delta^2+\kappa_0^2}\, w_2(\delta)\int_{-r}^{r}|h_*(\xi,\delta)|^2\, d\xi$$

Proof. The proof is a simple substitution. □

Lemma 2.11. *We can rewrite D_γ defined in* (2.10) *in terms of* (2.93), (2.94), (2.95) *and* (2.96) *as the following sum*

$$D_\gamma = \mathcal{S}_2 + \mathcal{I}_1 \cdot \int_{-r}^{r}\gamma(\xi)\, d\xi \tag{2.109}$$

where

$$\mathcal{I}_1 := \int_0^{\delta}\eta\Big(A(\eta)+N(\eta)\Big)\, d\eta \;-\; \sigma_\bullet^2\int_0^{\delta}\eta B(\eta)\, d\eta + \delta M(\delta), \tag{2.110}$$
$$\mathcal{S}_2 := \int_{-r}^{r}\int_0^{\delta}\frac{\eta^2\gamma^2(\xi)}{1+\eta\gamma(\xi)}\sigma_\bullet^2 B(\eta)\, d\eta\, d\xi. \tag{2.111}$$

Proof. From Lemma 2.11 and from the definition of D_γ we have

$$D_\gamma = \int_{-r}^{r}\int_0^{\delta}\Big[\Big(\frac{1-p}{p}\Big)\sigma_\bullet^2|\psi_*(\eta)|^2 + (p-1)|\psi_*'(\eta)|^2\Big] w_1(\eta)$$
$$+ \Big[\Big(\frac{1-p}{p}\Big)\sigma_\bullet^2|h_*(\eta)|^2 + (p-1)|h_*'(\eta)|^2\Big] w_2(\eta)\, d\eta d\xi$$
$$+ \int_{-r}^{r}\int_0^{\delta}(p-1)G|h_*(\eta)|^2\, d\eta d\xi$$
$$+ \kappa_0 w_1(\delta)\int_{-r}^{r}(p(\xi,\delta)-1)|\psi_*(\xi,\delta)|^2\, d\xi$$
$$+ \sqrt{\lambda_\delta^2+\kappa_0^2}\, w_2(\delta)\int_{-r}^{r}(p(\xi,\delta)-1)|h_*(\xi,\delta)|^2\, d\xi.$$

We also have the following identities

$$\frac{1}{p} - 1 = -\frac{\eta\gamma(\xi)}{1+\eta\gamma(\xi)} = -\eta\gamma(\xi) + \frac{\eta^2\gamma(\xi)^2}{1+\eta\gamma(\xi)},$$

$$p - 1 = \eta\gamma(\xi),$$

and in particular

$$p(\xi, 0) - 1 = 0 \qquad p(\xi, \delta) - 1 = \delta\gamma(\xi)$$

from which it follows that

$$\begin{aligned} D_\gamma = \int_{-r}^{r}\int_0^\delta \frac{\eta^2\gamma^2(\xi)}{1+\eta\gamma(\xi)}\sigma_\bullet^2\left(|\psi_*(\eta)|^2 w_1 + |h_*(\eta)|^2 w_2\right)\, d\eta\, d\xi \\ + \int_{-r}^{r}\int_0^\delta \eta\gamma(\xi)\Big[|\psi_*'(\eta)|^2 w_1 + |h_*'(\eta)|^2 w_2 + G|h_*(\eta)|^2 \\ - \sigma_\bullet^2\left(|\psi_*(\eta)|^2 w_1 + |h_*(\eta)|^2 w_2\right)\Big]\, d\eta\, d\xi \\ + \delta\Big(\kappa_0 w_1(\delta)|\psi_*(\delta)|^2 + \sqrt{\lambda_\delta^2 + \kappa_0^2}\, w_2(\delta)|h_*(\delta)|^2\Big)\int_{-r}^{r}\gamma(\xi)\, d\xi. \end{aligned}$$

Rearranging the terms in the above equality we finally get after (2.110) and (2.111)

$$\begin{aligned} D_\gamma &= \int_{-r}^{r}\int_0^\delta \frac{\eta^2\gamma^2(\xi)}{1+\eta\gamma(\xi)}\sigma_\bullet^2 B(\eta)\, d\eta\, d\xi \\ &\qquad + \int_{-r}^{r}\gamma(\xi)\, d\xi \cdot \left(\int_0^\delta \eta\left(A(\eta) + N(\eta) - \sigma_\bullet^2 B(\eta)\right) d\eta\, d\xi + \delta M(\delta)\right) \\ &= \mathcal{S}_2 + \int_{-r}^{r}\gamma(\xi)\, d\xi \cdot \mathcal{I}_1. \end{aligned}$$

□

Lemma 2.12. *Using the same argument used for the single layer case, the quantity* $\mathcal{S}_2 \geq 0$ *can be estimated by*

$$\mathcal{S}_2 = \int_{-r}^{r}\int_0^\delta \frac{\eta^2\gamma^2(\xi)}{1+\eta\gamma(\xi)}\sigma_\bullet^2 B(\eta)\, d\eta\, d\xi \leq \int_{-r}^{r}\gamma(\xi)^2\, d\xi \cdot \mathcal{I}_2 \tag{2.112}$$

with

$$\mathcal{I}_2 := \sigma_\bullet^2 \max\{1, \frac{1}{1-\Theta}\} \int_0^\delta \eta^2 B(\eta)\, d\eta. \tag{2.113}$$

Proof. The inequality in (2.112) follows after noting the following elementary facts

$$(1+\eta\gamma(\xi))^{-1} \le 1 \qquad \textit{if } \gamma(\xi) \ge 0,$$

$$(1+\eta\gamma(\xi))^{-1} \le (1-\Theta)^{-1} \qquad \textit{if } \gamma(\xi) \le 0,$$

where the latter follows from the fact that $1-\eta\xi \ge 1-\kappa^-\delta$ and the property of Θ in (2.3). □

We continue with a series of technical lemmas.

Lemma 2.13. *We have*

$$\begin{aligned}\int_0^\delta \eta A(\eta)\, d\eta = &-\delta M(\delta) - \frac{1}{2}\Big(\psi_*^2(\delta)w_1(\delta) + h_*^2(\delta)w_2(\delta)\Big) - \frac{1}{2}\int_0^\delta \psi_*^2(\eta)\alpha(\eta)w_1(\eta)\\ &+ \frac{1}{2}\int_0^\delta h_*^2(\eta)\alpha(\eta)\frac{H_1}{H_2}w_2(\eta) - \sigma_\bullet^2\int_0^\delta \eta(\psi_*^2 w_1 + h_*^2 w_2)\\ &+ \frac{\sigma_\bullet}{\Omega_*}\frac{1}{f}\int_0^\delta \eta\frac{\alpha(\eta)}{H(\eta)}(f\psi_* + GH_1h_*)^2\, d\eta - \int_0^\delta \eta N(\eta)\, d\eta.\end{aligned} \tag{2.114}$$

Proof. We start from

$$\int_0^\delta \eta A(\eta)\, d\eta = \int_0^\delta \eta|\psi_*'(\eta)|^2 w_1\, d\eta + \int_0^\delta \eta|h_*'(\eta)|^2 w_2\, d\eta.$$

Expanding the addends by making use of iterated integration by part, we have

$$\begin{aligned}\int_0^\delta \eta|\psi_*'(\eta)|^2 w_1\, d\eta &= \delta\psi_*(\delta)\psi_*'(\delta)w_1(\delta) - \int_0^\delta \psi_*(\eta)\Big(\eta\psi_*'(\eta)w_1(\eta)\Big)'\\ &= -\kappa_0\delta\psi_*^2(\delta)w_1(\delta) - \int_0^\delta \psi_*(\eta)\psi_*'(\eta)w_1(\eta)\\ &\quad + \int_0^\delta \eta\psi_*(\eta)\Big(-\psi_*''(\eta) + \psi_*'(\eta)\alpha(\eta)\Big)w_1(\eta)\end{aligned}$$

so that plugging (2.76) into the above expression we deduce

$$\int_0^\delta \eta|\psi_*'(\eta)|^2 w_1\,d\eta = -\psi_*^2(\delta)w_1(\delta)(\delta\,\kappa_0+1/2) - \frac{1}{2}\int_0^\delta \psi_*^2(\eta)\alpha(\eta)w_1(\eta) + \\ + \int_0^\delta \eta\psi_*(\eta)\Big(\frac{\sigma_\bullet}{\Omega_*}\alpha(\eta)(f\psi_*(\eta)+GH_1h_*(\eta)) - \sigma_\bullet^2\psi_*(\eta)\Big)w_1(\eta)d\eta \tag{2.115}$$

where we have used the fact that

$$\int_0^\delta \psi_*\psi_*' w_1\,d\eta = \frac{1}{2}\psi_*^2(\delta)w_1(\delta) + \frac{1}{2}\int_0^\delta \psi_*^2\alpha(\eta)w_1(\eta)\,d\eta.$$

Similarly, by mean of equations (2.77) and (2.58) we also deduce

$$\begin{aligned}\int_0^\delta \eta|h_*'(\eta)|^2 w_2\,d\eta &= -\delta\sqrt{\kappa_0^2+\lambda_\delta^2}h_*^2(\delta)w_2(\delta) - \int_0^\delta h_*(\eta)h_*'(\eta)w_2(\eta) \\ &\quad - \int_0^\delta \eta h_*(\eta)\Big(h_*''(\eta)+h_*'(\eta)\alpha(\eta)\frac{H_1}{H_2}\Big)w_2(\eta) \\ &= -h_*^2(\delta)w_2(\delta)(\delta\sqrt{\kappa_0^2+\lambda_\delta^2}+1/2) + \frac{1}{2}\int_0^\delta h_*^2(\eta)\alpha(\eta)\frac{H_1}{H_2}w_2(\eta) \\ &\quad + \int_0^\delta \eta h_*(\eta)\Big(\frac{\sigma_\bullet f}{\Omega_*}\,\frac{\alpha(\eta)}{H_2}(\frac{f}{G}\psi_*(\eta)+H_1h_*(\eta)) - \sigma_\bullet^2 h_*(\eta)\Big)w_2(\eta)\,d\eta \\ &\quad - \int_0^\delta \eta N(\eta)\,d\eta,\end{aligned} \tag{2.116}$$

So the statement follows by summing the two quantities above. □

Lemma 2.14. *We have that*

$$\int_0^\delta \eta\,(A(\eta)+N(\eta))\,d\eta + \delta M(\delta) = -\frac{1}{2}B(\delta) - \sigma_\bullet^2\int_0^\delta \eta B(\eta)\,d\eta \\ + \frac{\sigma_\bullet}{\Omega_*}\frac{1}{f}\int_0^\delta \eta\frac{\alpha(\eta)}{H(\eta)}(f\psi_*+GH_1h_*)^2\,d\eta - P(\delta) \tag{2.117}$$

Proof. It simply follows from the above lemma and the definition of $\mathcal{P}(\delta)$ in (2.98). □

Lemma 2.15. *We have that*

$$M(\delta)+\int_0^\delta A(\eta)+N(\eta)\,d\eta=-\sigma_\bullet^2\int_0^\delta B(\eta)\,d\eta+\frac{\sigma_\bullet}{\Omega_*}\frac{1}{f}\int_0^\delta\frac{\alpha(\eta)}{H(\eta)}(f\psi_*+GH_1h_*)^2\,d\eta \tag{2.118}$$

Proof. It easily follows from the following identities

$$\begin{aligned}\int_0^\delta|\psi_*'(\eta)|^2w_1\,d\eta=&-\psi_*^2(\delta)w_1(\delta)\kappa_0\\&+\int_0^\delta\psi_*(\eta)\Big(\frac{\sigma_\bullet}{\Omega_*}\alpha(\eta)(f\psi_*(\eta)+GH_1h_*(\eta))-\sigma_\bullet^2\psi_*(\eta)\Big)w_1(\eta)\,d\eta,\end{aligned}$$

$$\begin{aligned}\int_0^\delta|h_*'(\eta)|^2w_2\,d\eta=&-h_*^2(\delta)w_2(\delta)\sqrt{\kappa_0^2+\lambda_\delta^2}\\&+\int_0^\delta h_*(\eta)\Big(\frac{\sigma_\bullet f}{\Omega_*}\,\frac{\alpha(\eta)}{H_2}(\frac{f}{G}\psi_*(\eta)+H_1h_*(\eta))-\sigma_\bullet^2h_*(\eta)\Big)w_2(\eta)\,d\eta\\&-\int_0^\delta N(\eta)\,d\eta,\end{aligned}$$

□

3

Complex soliton-like solution for the Korteweg-de Vries equation

This chapter is dedicated to the study of a certain class of complex solutions of the Korteweg-de Vries equation. The study of such solutions obtained via the Wronskian method, also known as Darboux-Crum method in scattering theory, will be carried both in terms of their time evolution and also scattering properties when regarded at a fixed time as potentials for a spectral problem.

It will be shown by means of numerical simulations that the solutions introduced in (3.21) may be considered in many respects as the natural extension of the classical real multisoliton solutions firstly introduced in [66]. We will see in fact that many characterising qualitative aspects of the dynamics present in the real solutions are also found in their complex counterparts. We furthermore characterise in Theorem 3.4 the discrete spectrum of the spectral problem for the Laplacian where the solutions defined in (3.21) at time zero play the role of complex potentials. We also study the reflection scattering property of such perturbed operators.

3.1 Lax's formulation and solution's nomenclature

We commence with introducing the Lax's formulation of the KdV equation in terms of two linear operators and after that we will explain the terminology used to indicate the solutions.

We recall that the KdV equation reads, regardless of whether the variables are real or complex, as

$$u_t + 6uu_x + u_{xxx} = 0, \qquad t, x \in \mathbb{R}. \tag{3.1}$$

A major breakthrough in the study of the equation (3.1) and more generally in the context of integrable systems was given by Lax [103] in the 1968. He showed in fact that the non-linear KdV equation could be viewed as a system of two linear equations. Let us consider the two operators

$$L\phi := \left(-\partial_{xx}^2 - u(x,t)\right)\phi, \tag{3.2}$$

$$M\phi := \left(u_x(x,t) - (2u(x,t) + 4\lambda)\partial_x\right)\phi \tag{3.3}$$

and the following problem

$$L\phi = \lambda\phi \tag{3.4}$$

where the function $u(x,t)$ plays in equation (3.2) the role of a time dependent potential and $\lambda = \lambda(t)$ is a (time dependent) parameter. We do not regard (3.4) as a spectral problem as it might be a misleading terminology since no integrability is required at this stage. As showed in Ablowitz [1], the time-dependence of the associated solutions of (3.4) reads as

$$\phi_t = M\phi. \tag{3.5}$$

Then from a simple manipulation of the terms introduced above and taking into account the identities in (3.1),(3.2) and (3.4) it follows that

$$\begin{aligned}
\phi_{txx} &= \left[u_x(\lambda - u) + u_{xxx} + 6uu_x\right]\phi - (4\lambda + 2u)(\lambda - u)\phi_x, \\
\phi_{xxt} &= \left[(\lambda - u)u_x - u_t + \lambda_t\right]\phi - (\lambda - u)(4\lambda + 2u)\phi_x.
\end{aligned}$$

Therefore, from purely algebraic arguments we have that the compatibility of the two

equations above $\phi_{txx} = \phi_{xxt}$ and $u(x,t)$ being solution of the equation (3.1) follows if and only if

$$\partial_t \lambda = 0. \tag{3.6}$$

The property (3.6) is referred as the isospectrality property of the KdV equation while the pair of operators introduced in (3.2) and (3.3) is called the *Lax pair*.

Remark 3.1. In the following we will use the notation $u(x,t)$ when interested in time behaviour of the solutions of (3.1) whereas we will prefer the notation $V(x,t)$ or simply $V(x)$ when interested in seeing them as potentials in a spectral problem.

Let us consider for the moment the situation when $u(x,t) \equiv 0$ and solutions of the problem (3.4) are in the form of a travelling function $\phi(x,t) = \phi(kx - \omega t)$. Then from (3.3), we deduce that M coincides with $-4\partial_{xxx}$, which in turn implies that the dispersion relation (3.5) has to be nonlinear and of the form $\omega = 4k^3$.

Any real solution of the problem (3.4) then reads as a linear combination of the two independent solutions for arbitrary real constant C_1 and C_2

$$\begin{aligned}
&\phi(x,t) = \phi_0 = C_1 x + C_2 && \text{if } \lambda = 0,\\
&\phi(x,t) = \phi_\lambda = C_1 \cos\left(\frac{1}{2}\sqrt{\lambda}(x-\lambda t)\right) + C_2 \sin\left(\frac{1}{2}\sqrt{\lambda}(x-\lambda t)\right) && \text{if } \lambda > 0,\\
&\phi(x,t) = \phi_\lambda = C_1 \cosh\left(\frac{\sqrt{|\lambda|}(x-|\lambda| t)}{2}\right) + C_2 \sinh\left(\frac{\sqrt{|\lambda|}(x-|\lambda| t)}{2}\right) && \text{if } \lambda < 0.
\end{aligned} \tag{3.7}$$

The Lax formalism, combined with some algebraic properties of the Wrosnkian determinant, makes very easy to construct a wide class of solutions for the KdV equation which go under the name of Wrosnkian solutions. A glimpse on this method will be given in the next couple of paragraphs.

These solutions were firstly studied by Satsuma [143] and Freeman and Nimmo [64] and later extended by Matveev [117] to the so called generalised Wronskian solutions. Since the technique employed to derive such solutions for the KdV equation in its most generality is nowadays widely used to derive solutions for many other integrable system*, a detailed construction of such solutions is not included in this thesis. We

*We say that an equation of a system is integrable if there exists a Lax pair associated to it

present directly their most generic formulation and refer to the papers [111, 113] for a thorough derivation. See also Proposition 3.3 in Section 3.

Proposition 3.1 (Ma [111])**.** *Given* $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ *the function*

$$u(x,t) = 2\partial_{xx}^2 \Big[\ln W(\phi_1, \ldots, \frac{1}{(k_1-1)!}\partial_{\lambda_1}^{k_1-1}\phi_1, \ldots \\ \ldots, \phi_N, \ldots, \frac{1}{(k_N-1)!}\partial_{\lambda_N}^{k_N-1}\phi_N)\Big], \tag{3.8}$$

is solution of (3.1)*, where* $\phi_j(x,t)$ *is the solution corresponding to* λ_j *defined in* (3.7) *and where the function* $W(\cdot, \ldots, \cdot)$ *is the Wronskian determinant*

$$W(f_1, \ldots, f_N)(x) := \det\left[\left(\frac{d}{dx^{i-1}} f_j\right)_{ij}\right], \qquad x \in \mathbb{R} \tag{3.9}$$

We note that in fact the Wrosnkian need not be positive and that $(\partial_{xx} \ln) f$ is a short notation to indicate $\frac{f''f - f'^2}{f^2}$. Depending on the sign of the spectral parameters, the following particular cases arise.

Definition 3.1. The following terminology is adopted for $u(x,t)$ in (3.8). If

$\lambda_1, \ldots, \lambda_N > 0$, then $u(x,t)$ is called a *positon*, and if

$\lambda_1, \ldots, \lambda_N < 0$, then $u(x,t)$ is called a *negaton*

of order $(k_1 - 1, \ldots, k_N - 1)$ to indicate the order of derivatives with respect to the parameter λ. In particular, from the above formulation it follows that the classical N*-soliton* solution is a special case of N-negaton of order $(0, \ldots, 0)$:

$$u(x,t) = 2\partial_{xx}^2 \ln W(\phi_{\lambda_1}, \ldots, \phi_{\lambda_N}).$$

where ϕ_{λ_i} is as in (3.7) with $C_2 = 0$.

We observe that if $\lambda = 0$ it is possible to obtain the so called *rational solutions* of the form

$$u(x,t) = 2\partial_{xx}^2 \ln W(l_1, \ldots, l_N),$$

where we have that

$$\sinh\left(\frac{1}{2}k(x-k^2t\right) = \sum_{i=0}^{\infty} l_i(x,j)k^{2i+1}.$$

For details and explicit examples we refer again to [111, 113].

What happens if instead of real λ, we consider a complex parameter in (3.4)? Heuristically, it is easy to realise that the complexification of the (spectral) parameter leads to a mix between trigonometric and hyperbolic functions in the solutions of the (spectral) problem.

Similar to the case of real parameters, solutions of (3.1) which originate from complex eigenvalues are called *Complexiton solutions*. We shall differentiate between real and complex complexitons and point out immediately that it is possible to derive real solution of the KdV equation by mean of an *ad hoc* combinations of complex wavenumbers. These solutions are described by a generalisation of the formula (3.8) to the pairs of real and complex parts of the relative eigenfunctions. We refer to [113] (see section 4.3) and references therein for a comprehensive exposition on *real complexitons* and for their general formulation.

One of the first example in literature of a solution for the real KdV equation obtained via complex wavenumbers appeared in 1984 in the paper of Jaworski [83]. Its formula is given by an extension of the Miura's N-soliton formulae, initially introduced in [66], to a pair of conjugated complex wavenumbers k_1, $k_2 = \overline{k_1}$ by setting

$$u_J(x,t) := 2\left[\ln\det(I+C)\right]_{xx},$$

where I is the 2×2 identity matrix and $C = C(x,t) = (c_{mn}(x,t))_{mn}$ with

$$c_{mn}(x,t) = \frac{2(k_m k_n)^{1/2}}{k_m + k_n} e^{-(k_m+k_n)x+4(k_m^3+k_n^3)t}$$

for $m,n = 1,2$. Let us consider $\lambda \in \mathbb{C}$. We adopt the following notation for the

fundamental solutions of equation (3.4),

$$\begin{aligned}\Psi_k(x,t) &= \cosh\left(\frac{k}{2}(x-k^2t)\right),\\ \Phi_k(x,t) &= \sinh\left(\frac{k}{2}(x-k^2t)\right),\end{aligned} \tag{3.10}$$

where now $k \in \mathbb{C}$ is a complex wave number, and $\lambda = -\frac{k^2}{4}$ and consider

$$u(x,t) = 2\partial_{xx} \ln W(\Psi_k(x,t), \Psi_{\overline{k}}(x,t)), \tag{3.11}$$

where $\overline{k} = \mathrm{Re}(k) - i\,\mathrm{Im}(k)$ stands for the complex conjugate of k. Then, by simply showing the explicit formula it is possible to see that $u_J(x,t)$ coincides with (3.11). Thus, the Javorsky solution is a particular case of a real complexiton solution. Such solution was described by Jaworski as *breathers-like* in analogy with the breather solutions[†] found by Wadati [156] for the mKdV equation. We point out that Jaworski's solution (3.11) and, in general, complexiton solutions obtained using the generalised Wronskian method presented in Ma [111] are real and singular. As the main purpose of this chapter is to study a particular type of complex solutions of the KdV equation, we will not provide any further details on the real case. We end this section by briefly mentioning the fact that real complexiton solutions have been found as well for the Toda lattice [111] and for the Boussinesq equation [112].

3.2 Non singular Complex Complexiton

In what follows we present a class of complex solutions which stay finite at all time and that for many reasons can be understood as the respective complex version of the real multi-soliton solutions. Besides the finiteness, these solutions appear to be localised and travelling. In analogy with the breather solutions, though, their shape is not fixed in time, therefore they can be considered as travelling waves only up to spatial oscillations. Furthermore, the solutions also exhibit the phase shift phenomenon typical of the interaction of several solitons.

[†]Breathers have been defined by Lamb [100] as localised oscillatory solutions (wave-packet) whose envelope and oscillatory part move at different velocities. We also note that in [4] the characterising property used to define a breather is as in (3.14).

In the last decade a growing interest in complex soliton-like solutions for integrable systems has been registered, in particular for the complex KdV equation. However the question of existence of non singular solutions seems to be little studied. Among the few results, we mention the work [160] where an example of non-singular complex complexiton solution was given without a formal proof. In fact, the example in [160] has a very intricate formulation, which after some analysis turns out to be equivalent to the one we give below in (3.13)[‡]. Recently, other examples of non singular complex solution of the KdV equation have been produced. In [27] a new type of solution has been introduced under the name of *regularised degenerate multi-solitons*. Those are extensions to complex wave-numbers of the so-called degenerate solutions obtained via the Wronskian method (see for instance Matveev [118]) where a translation method is applied in order to avoid a blow up. Further, in [158] non singular complex solutions for the KdV equation have been obtained by operating a Miura's transform to non singular complex solution of the mKdV equation (modified KdV) .

Let us consider $\lambda = -\frac{k^2}{4}$. With the notation introduced in (3.10) we define

$$u_1(x,t) = 2\partial_{xx}(\ln \Psi_k(x,t)) \tag{3.12}$$

and

$$u_2(x,t) = 2\partial_{xx}(\ln W(\Psi_k(x,t), \Phi_{\overline{k}}(x,t))). \tag{3.13}$$

Both (3.12) and (3.13) are solutions of the KdV equation (3.1), but as we shall see in the following, are qualitatively different. For future reference, we explicitly re-write the formulae of the solutions (3.12) and (3.13) as

$$u_1(x,t) = \frac{k^2}{2\cosh^2 f},$$

$$u_2(x,t) = \begin{cases} 0, & \text{if } \operatorname{Im}(k)\operatorname{Re}(k) = 0, \\ \dfrac{(\overline{k}^2 - k^2)(\overline{k}^2 \cosh^2 f + k^2 \sinh^2 \overline{f})}{2\left(\overline{k}|\cosh f|^2 - k|\sinh f|^2\right)^2} & \text{otherwise,} \end{cases}$$

where we set for brevity

$$f = f(x,t) := \frac{k}{2}(x - k^2 t).$$

[‡]The results in [160] were known to the author only on a subsequent time once the formulation and studies of the class of complexitons was mostly concluded.

3.2.1 Periodicity and spatial localisation

We start by deriving a periodicity formula for the two complexitons introduced in equations (3.12) and (3.13). For the sake of simplicity, we restrict our analysis to the single complex soliton $u_1(x,t)$ with $k = \alpha + i\beta$; the same arguments can be reproduced to extend the result to the case of $u_2(x,t)$. As observed earlier, from the fact that the wave number is now a complex quantity, we shall not expect the shape of the solution to remain unchanged in time, but should expect a space-time periodicity. Namely, in what follows we determine two real quantities X and T, eventually depending on k, such that

$$u_i(x+X,t) = u_i(x,t+T). \tag{3.14}$$

We observe that in order to find such quantities, it suffices to derive them for (3.10), the fundamental solutions of the spectral problem. Setting for convenience

$$\begin{aligned} 2f_1(x,t) &= \alpha x - (\alpha^3 - 3\alpha\beta^2)t \\ 2f_2(x,t) &= \beta x - (3\alpha^2\beta - \beta^3)t, \end{aligned} \tag{3.15}$$

and using the summation formulae for the hyperbolic functions, we obtain

$$\cosh\left(\frac{k}{2}(x-k^2t)\right) = \cosh(f_1(x,t))\cos(f_2(x,t)) + i\sinh(f_1(x,t))\sin(f_2(x,t)). \tag{3.16}$$

Observing that the trigonometric and hyperbolic terms in (3.16) have respectively the same argument and that equation (3.14) has to be satisfied for all $t, x \in \mathbb{R}$, we are left with solving the equation

$$\cosh(f_1(x+X,0))\cos(f_2(x+X,0)) = \cosh(f_1(x,T))\cos(f_2(x,T)).$$

This reduces after simple manipulation and by means of (3.15) to the system

$$\begin{cases} X = -(\alpha^2 - 3\beta^2)T \\ X = -(3\alpha^2 - \beta^2)T + 2\pi/\beta, \end{cases} \tag{3.17}$$

which has the solution

$$T(k) = \frac{\pi}{|k|^2 \operatorname{Im}(k)}, \qquad X(k) = \frac{-\pi \operatorname{Re}(k^3)}{|k|^2 \operatorname{Im}(k) \operatorname{Re}(k)}. \tag{3.18}$$

We are now in the position to formulate our main result which describes the motion of real and complex parts of $u_1(x,t)$ and $u_2(x,t)$.

Proposition 3.2. *Let $u_1(x,t)$ and $u_2(x,t)$ be the complex solutions of the KdV equation given by (3.12) and (3.13) with wave number $k = \alpha + i\beta \in \mathbb{C}$. Then, for $j = 1, 2$, the following statements hold:*

(i) For the values of k situated on the two lines in the complex plane $\alpha = \pm\sqrt{3}\beta$ (that is, $\arg k = \pm\pi/6 + \pi m$, $m \in \mathbb{Z}$) both the real and imaginary part of $u_j(x,t)$ are localised, oscillating and non-travelling waves centred at the origin.

(ii) For the values of k situated in the region of the complex plane defined by $\alpha^2 - 3\beta^2 > 0$ both the real and imaginary part of $u_j(x,t)$ are localised oscillating waves travelling to the right.

(iii) For the values of k situated in the region of the complex plane defined by $\alpha^2 - 3\beta^2 < 0$ both the real and imaginary part of $u_j(x,t)$ are localised oscillating waves travelling to the left.

In addition, u_2 is non-singular for all real x and t, whereas u_1 has countably many singular points in the space-time plane.

Proof. The proof of the first statement is immediate once we observe that a non-travelling periodic wave corresponds to having the spatial periodicity parameter $X(k) = 0$ in (3.18), which happens when $\alpha = \pm\sqrt{3}\beta$. The other two statements simply follow from the observation that, once we have expressed (3.12) or (3.13) as a travelling wave $u(x - ct)$, its velocity c is given by mean of (3.18) by

$$c(k) = -\frac{X(k)}{T(k)} = \frac{\operatorname{Re}(k^3)}{\operatorname{Re}(k)} = (\alpha^2 - 3\beta^2).$$

Thus, it remains to analyse the eventuality of when solutions u_j are singular. We find the blow-up points for u_1 by a direct calculation (see e.g. [158] for a different

approach which uses the Miura's trasnformation). To identify the time and location for the blow-up points for $u_1(x,t)$ we simply note that they occur when $\cosh(f)=0$, namely $f(x,t)=i\left(\frac{\pi}{2}+m\pi\right)$, with $m\in\mathbb{Z}$. This condition is equivalent to

$$\begin{cases} f_2(x,t)=\pi/2\pm m\pi \\ f_1(x,t)=0 \end{cases} \quad m\in\mathbb{Z}, \qquad \Longleftrightarrow \qquad \begin{cases} x_m=(\alpha^2-3\beta^2)t_n \\ t_m=\frac{\pi/2+\pi m}{2\beta|k|^2} \end{cases} \quad m\in\mathbb{Z}. \tag{3.19}$$

We turn our attention now on $u_2(x,t)$: the absence of singular points comes form from its definition, specifically from the fact that the a necessary and sufficient blow-up condition for consists in

$$\overline{k}|\cosh f|^2-k|\sinh f|^2=0,$$

which is equivalent to $\arg(k)=\frac{m\pi}{2}$ where $m\in\mathbb{Z}$. We note that the previous condition is in fact equivalent to require the Wronskian to be null. The localisation of u_1 and u_2 is immediate from their explicit formulae. □

Remark 3.2. It is important to remark that, from (3.15), the spatial oscillations move at a speed determined by f_2 which is in general different from the one of the envelope of the wave, in fact determined by the expression f_1.

Remark 3.3. We note that Proposition 3.2 provides a simple proof of the results claimed in [160].

Unfortunately, it is not straightforward to generalize the results on localisation and non-singularity to the case of general solutions, involving several pairs of complex conjugated wavenumbers. In the following we formulate our conjecture. For $i=1,\dots,N$ let $k_i\in\mathbb{C}$ be N distinct complex numbers with non zero real and imaginary part such that $k_i\neq\pm k_j,\pm\overline{k_j}$ for $i\neq j$. Consider

$$W_N(x,t):=W(\Psi_{k_1},\Phi_{\overline{k}_1},\Psi_{k_2},\Phi_{\overline{k}_2},\dots,\Psi_{k_N},\Phi_{\overline{k}_N}) \tag{3.20}$$

the Wronskian associated with the N pairs $\{(k_i,\overline{k}_i)\}_{i=1,\dots,N}$.

Conjecture 3.2. The complex multisoliton $V_N(x,s)$ defined as

$$V_N(x,t):=2\partial_{xx}\ln W_N(x,t) \tag{3.21}$$

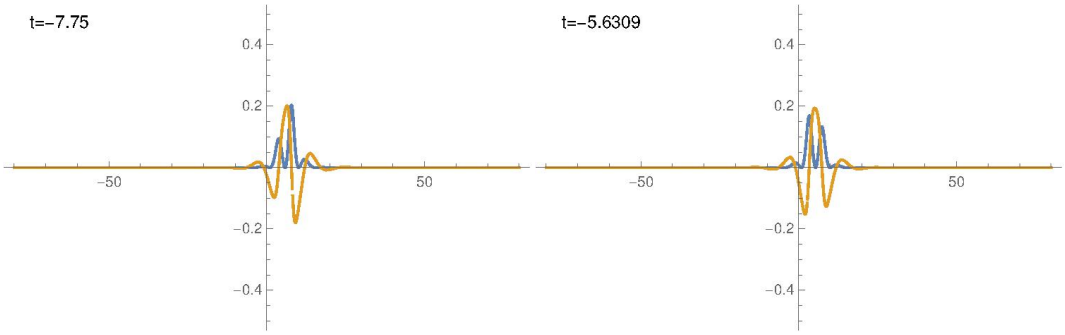
is a non singular complex solution of (3.1) for all real x and t.

Remark 3.4. We note that a positive answer to the conjecture would follow by proving that $W_N(x,t) \neq 0$ for all x, t.

Remark 3.5. In the formulation of Conjecture 3.2 the condition introduced on the wavenumbers $k_i \neq \pm k_j, \pm\overline{k_j}$ for $i \neq j$ is justified by a following simple argument. If $k_i = \pm k_j$, then from the parity of the hyperbolic cosine it follows that $W_N \equiv 0$. On the other hand, if $k_i = \pm\overline{k_j}$, then we obtain a generalisation of the Javorsky's solution (3.11), which is singular.

3.2.2 Numerical examples

In the following we gather a sequence of snapshots obtained from a simulation conducted with the software Mathematica. In the first sequence, shown in Figure 3.2, we plot the real and imaginary parts of the function $u_2(x,t)$ at different times for the value $k = 1/\sqrt{3}+i$. As predicted in Proposition 3.2, the support of the solution moves to the left.



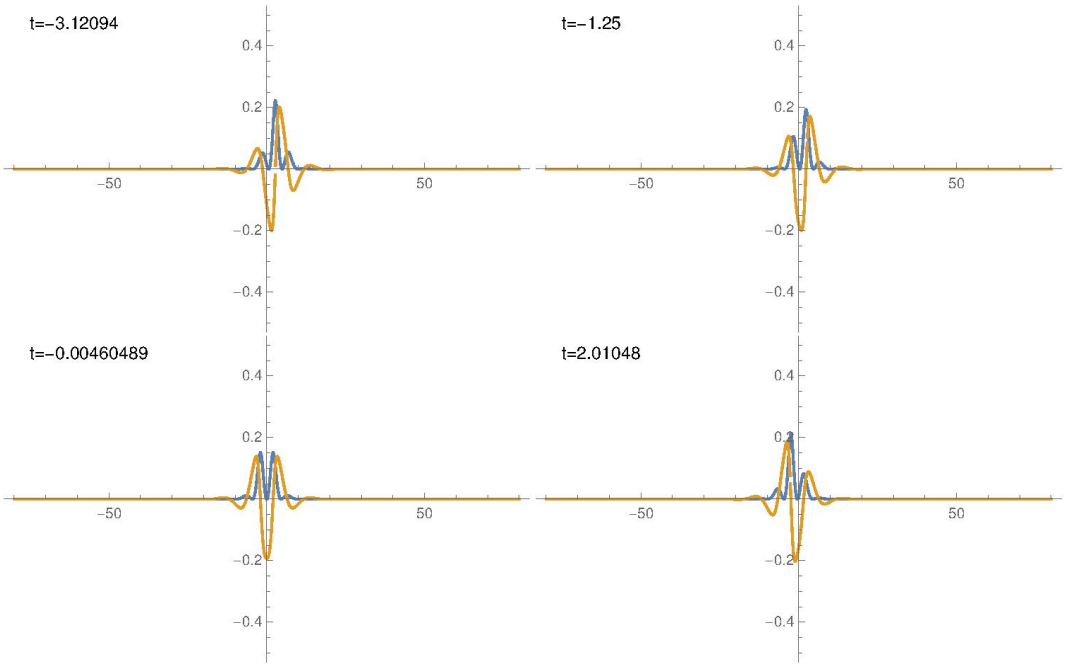
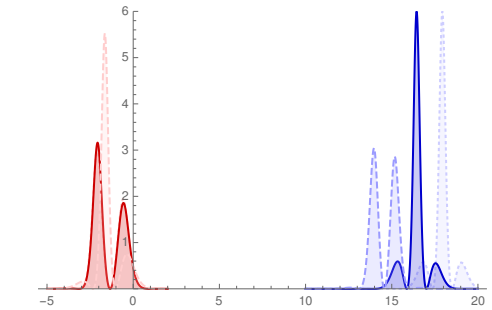


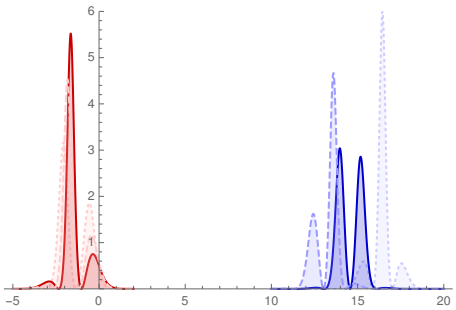
Figure 3.2: The blue line corresponds to $\mathrm{Re}(u_2(x,t))$. The orange line corresponds to $\mathrm{Im}(u_2(x,t))$.

In the second sequence of plots we examine only the real part of the function $V_N(x,t)$ defined in (3.21) for $N = 2$, $k_1 = \sqrt{3} + i$ and $k_2 = \sqrt{3} + 3/2i$. We can observe at once the localisation property, the phase shift phenomenon and the fact that the solution doesn't blow up at any time. Each snapshots contains three plots taken at three different times to which correspond three different lines. The continuous line refers to the middle time, the second in the list in the caption. The dotted and the dashed ones refer respectively to the first and third time listed in the description.

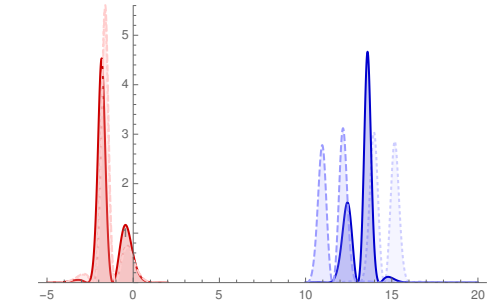
We observe that the pictures always split into two parts: one, which is coloured in blue, is originated from the presence of the wavenumber k_2, the other in red comes from the presence of k_1. The blue one is moving leftward over time whereas the red portion isn't, except for the time interval over which it happens that the two parts interact, (see Figures 3.4a and 3.4b) producing a purple tone. These peculiarities have been in fact introduced in Proposition 3.2 for each single component, that we might shortly address as single-soliton element, of which consists the multisoliton solution. In particular, we can observe from the motion of the red single-soliton the *phase shift* phenomenon.



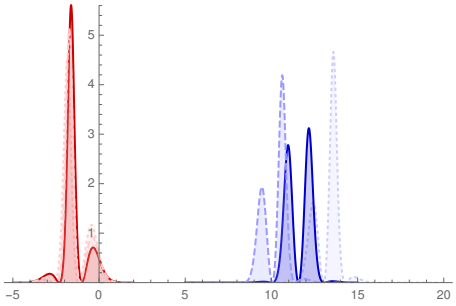
(a) $t = -4.4, -4, 3.5$



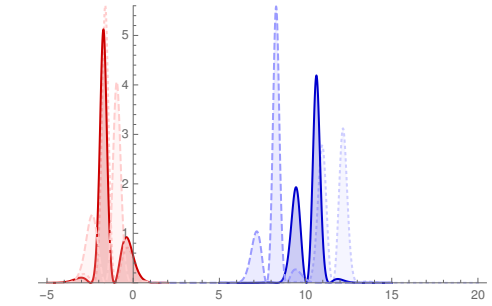
(b) $t = -4, -3.5, -3.15$



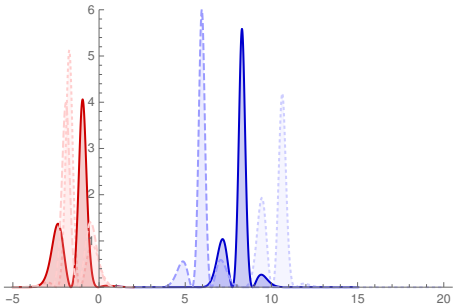
(c) $t = -3.5, -3.15, -2.71$



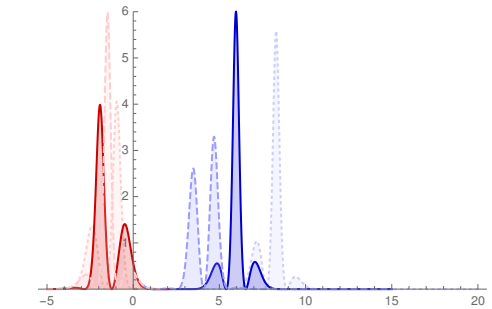
(d) $t = -3.15, -2.71, -2.34$



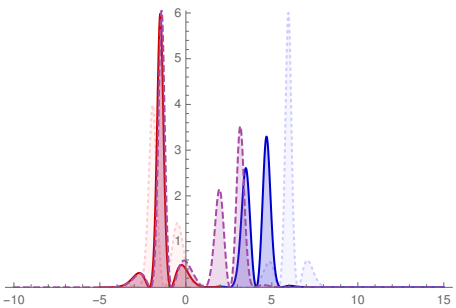
(e) $t = -2.71, -2.34, -1.78$



(f) $t = -2.34, -1.78, -1.21$

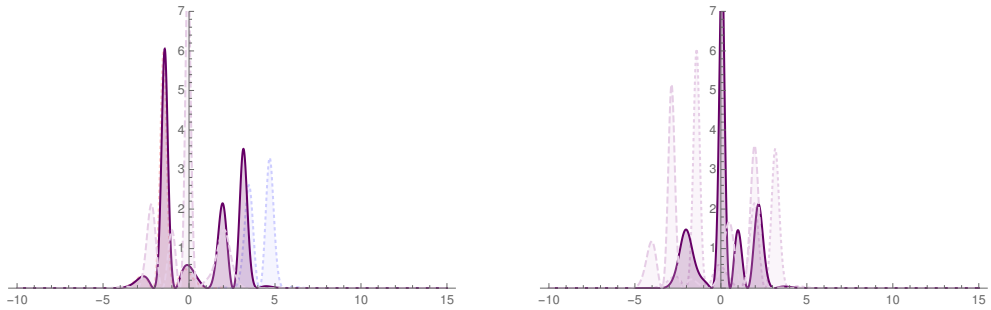


(g) $t = -1.78, -1.21, -0.72$



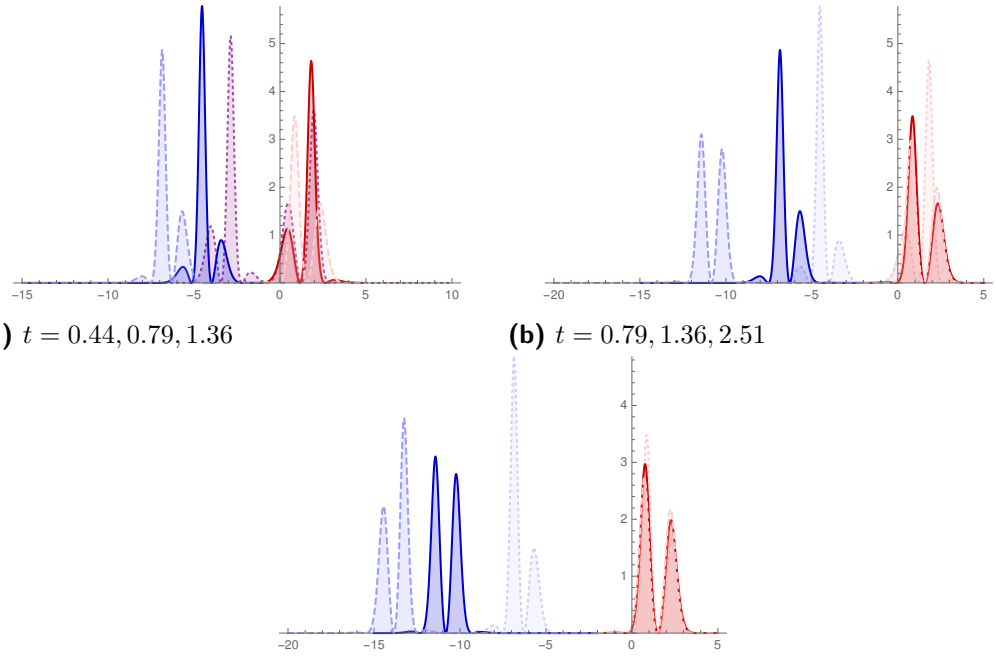
(h) $t = -1.21, -0.72, -0.32$

Figure 3.3: Real part of V_N before the interaction



(a) $t = -0.72, -0.32, -0.07$ **(b)** $t = -0.32, -0.07, 0.44$

Figure 3.4: Real part of V_N during the interaction



(a) $t = 0.44, 0.79, 1.36$ **(b)** $t = 0.79, 1.36, 2.51$

(c) $t = 1.36, 2.51, 3.28$

Figure 3.5: Real part of V_N after the interaction

We produce two more pictures, Figure 3.6 and Figure 3.7 which describe the same situation studied in the second sequence of plots, now evolving over a continuous interval of time. In this case the phase shift is even more evident, especially when observed from above.

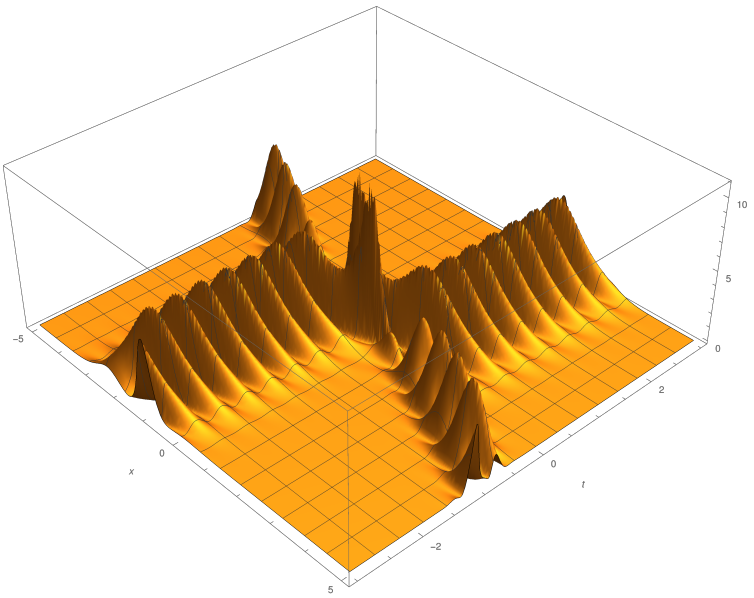


Figure 3.6: The real part of $V_N(x,t)$ defined in (3.21) for $k_1 = \sqrt{3}+i$ and $k_2 = \sqrt{3}+3/2i$. The time interval is $t \in [-3,3]$.

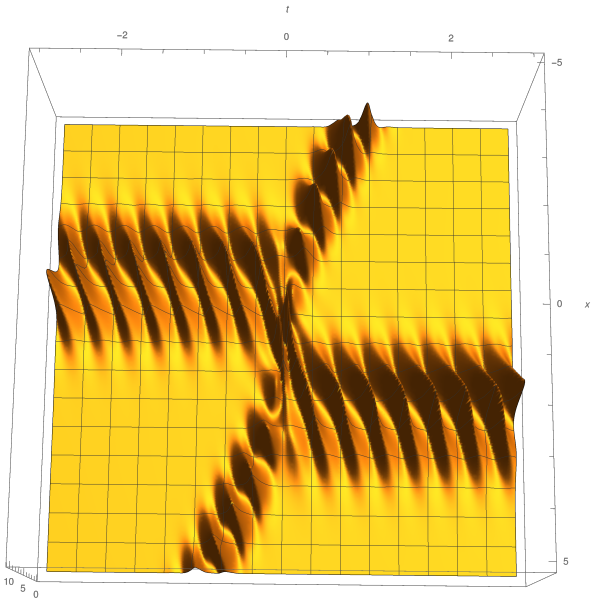


Figure 3.7: The real part of $V_N(x,t)$ defined in (3.21) for $k_1 = \sqrt{3}+i$ and $k_2 = \sqrt{3}+3/2i$ this time seen from above. The time interval is $t \in [-3,3]$. This view emphasize the phase-shift phenomenon.

3.3 Spectral properties of Wronskian potentials

In this last section we regard the solutions $V(x,t)$ defined in (3.28) as potentials for the spectral problem introduced in (3.4) below and in particular our interest will be focused on the structure of the point spectrum. Furthermore, we will see that such potentials are in fact *Reflectionless*[§], namely their reflection coefficients, defined as in (3.3) are identically zero. This is again a sign of the strict connection which holds between the complex solutions introduced in (3.21) with the classical real multisoliton ones.

The interest in real reflectionless potentials dates back to the 1930s with the works of Epsteim [48] and Eckart [43] in the context of quantum mechanics on the potential of the Pöschl-Teller type

$$-\frac{c}{2ma^2}\frac{\nu(\nu+1)}{\cosh^2(x/a)}.$$

It is a remarkable feature of such potentials that they do not reflect at any real energy if ν is a positive integer number (see Lekner [106]). The literature on this topic is extensive, therefore we mention only the works of Kay and Moses [90] and Gardner, Greene, Kruskal and Miura [66] as the most representative ones for our discussion. In the former case, the results on the Pöschl-Teller potentials were extended in a optics context while in the latter the study of such potentials was carried in the context of inverse scattering theory, and in particular in the formulation of the real multisoliton solution for the KdV equation. We refer then to Ablowitz's textbook [1] and the extensive references contained therein for more details.

Lately, a renewed interest in reflectionless potentials has been brought up by a thriving use in physical models of complex potentials. See for example the work of Horsley-Longhi [79] on the deformation of reflectionless potentials and the works [142], [24] respectively on the reflection property and the spectrum structure of the Pöschl-Teller potential with $\nu \in \mathbb{C}$. As mentioned in the introduction, those potentials arise typically in the study of effective Hamiltonians in optics or electromagnetism problems when there are regions where the waves get amplified or dissipated. They appear as

[§]Since the potentials studied in this section are complex, this might lead to some ambiguity in the terminology. We refer to the reflectionless property in the usual way as it is done for real potentials (see for example Ablowitz [1]) therefore for scattering waves of the form e^{ikx} for $k \in \mathbb{R}$. See (3.27) for the definition of the transmission and reflection coefficients.

well in the study of meta-materials [85]. Also, such potentials have become very popular after [10, 9] where complex potentials with real spectrum in the context of $\mathcal{PT}$ theory where firstly introduced.

In the previous sections we have presented solutions of the KdV equation which were generated using Wronskian method. In particular, this method is a simple combination of the Lax's formulation of the KdV equation in terms of a pair of compatible linear operators L and M which we defined respectively in (3.2) and (3.3) and the Darboux-Crum method. The latter, firstly introduced by Darboux and subsequently extended by Crum, provides a scheme which generates, starting from a collection of generalised-eigenfunctions, a sequence of new potentials and relative eigenfunctions. We briefly recall it. Since it relies on purely algebraic arguments, we will not include a proof. For the details we refer to Matveev [118].

Proposition 3.3 (Matveev [118])**.** *Let us consider the following spectral problem for a general potential $Q(x)$*

$$-\frac{\partial^2}{\partial x^2}u(x) - Q(x)u(x) = \lambda u(x) \tag{3.22}$$

and a set of $\{\phi_{\lambda_i}\}_{i=1,\dots,n+1}$ solutions to it associated to $n+1$ distinct values $\{\lambda_i\}_{i=1,\dots,n+1}$. Let us consider the function

$$\phi_{\lambda_{n+1}}[n](x) = \frac{W(\phi_{\lambda_1}, \phi_{\lambda_2}, \dots, \phi_{\lambda_{n+1}})}{W(\phi_{\lambda_1}, \phi_{\lambda_2}, \dots, \phi_{\lambda_n})} \tag{3.23}$$

where $W(\cdot, \dots, \cdot)$ is the Wronskian operator, defined in (3.9)*.*

Then $\phi_{\lambda_{n+1}}[n](x)$ is solution of the new spectral problem

$$-\frac{\partial^2}{\partial x^2}\phi_{\lambda_{n+1}}[n](x) - Q_{New}(x;n)\phi_{\lambda_{n+1}}[n](x) = \lambda_{n+1}\phi_{\lambda_{n+1}}[n](x) \tag{3.24}$$

where

$$Q_{New}(x;n) = 2\frac{\partial^2}{\partial x^2}\ln W(\phi_{\lambda_1}, \phi_{\lambda_2}, \dots, \phi_{\lambda_n}). \tag{3.25}$$

In the following we are going to study the scattering properties of solutions of a spectral problem. We therefore introduce hereby some notation which will be useful in the following.

Definition 3.3. Let $k \in \mathbb{R}$ and consider two pairs of solutions of (3.4) ($\psi(x)$, $\tilde{\psi}(x)$) and ($\phi(x)$, $\tilde{\phi}(x)$) such that they posses the following asymptotic behaviour at infinity

$$\begin{cases} \psi(x,k) \sim e^{ikx} \\ \tilde{\psi}(x,k) \sim e^{-ikx} \end{cases} \qquad \text{as } x \to +\infty, \tag{3.26a}$$

$$\begin{cases} \phi(x,k) = a(k)\tilde{\psi}(x,k) + b(k)\psi(x,k) \sim e^{-ikx} \\ \tilde{\phi}(x,k) = -\tilde{a}(k)\psi(x,k) + \tilde{b}(k)\tilde{\psi}(x,k) \sim e^{ikx} \end{cases} \qquad \text{as } x \to -\infty. \tag{3.26b}$$

We set

$$\begin{aligned} \tau(k) &= a(k)^{-1} \\ \rho(k) &= b(k)a(k)^{-1} \end{aligned} \tag{3.27}$$

which are called respectively the *transmission* and *reflection* coefficients [¶]. The functions ($\psi(x)$, $\tilde{\psi}(x)$) and ($\phi(x)$, $\tilde{\phi}(x)$) are often in literature indicated with the name of Jost solutions.

We are now in the position of formulating the main result of this section:

Theorem 3.4. *Let $k_1, k_2, \dots, k_M \in \mathbb{C}$ be $M = 2N$ distinct complex numbers with non null both real and immaginary parts, be $N \in \mathbb{N}$ and let*

$$V(x) = -\frac{\partial^2}{\partial x^2} \ln W(\cosh(k_1 x), \sinh(k_2 x), \dots, \cosh(k_{2N-1} x), \sinh(k_{2N} x)). \tag{3.28}$$

Consider the problem

$$-\frac{\partial^2}{\partial x^2} u(x) - V(x)u(x) = \lambda u(x)$$

Then $V(x)$ is reflectionless, namely $|\rho(k)| \equiv 0$ for any $k \in \mathbb{R}$. and the problem has $2N$-complex eigenvalues $\lambda_i = -\frac{k_i^2}{4}$ for $i = 1, \dots, 2N$.

Proof. To begin with, we prove the statement for the simple case $N = 1$ and without loss in generality we assume for the rest of the proof

$$\mathrm{Re}(k_1) \geq \mathrm{Re}(k_2) > 0. \tag{3.29}$$

[¶]Those are part of the scattering data, used in order to reconstruct solutions of initial data problem for the KdV by mean of the inverse scattering method. When the potential is real, they satisfy $|\tau(k)|^2 + |\rho(k)|^2 = 1$. We refer to Ablowitz [1] for further details.

In order to prove the reflectionless property we start by introducing the so-called Jost solutions. Let $\kappa \in \mathbb{C}$ and consider the generalised eigenfunction obtained by the Darboux-Crum method as introduced in Proposition 3.3

$$g(x,\kappa) = \frac{W(\Psi_{k_1}, \Phi_{k_2}, e^{\kappa t})}{W(\Psi_{k_1}, \Phi_{k_2})}.$$

From the multilinearity property of the Wronskian and the determinant formula for the Vandermonde matrix (4.17), we expand the numerator and the denominator of the previous expression as a sum of exponentials

$$W(\Psi_{k_1}, \Phi_{k_2}) = (k_2 - k_1)\left(e^{(k_1+k_2)x} + e^{-(k_1+k_2)x}\right) + (k_2 + k_1)\left(e^{(k_1-k_2)x} + e^{-(k_1-k_2)x}\right) \tag{3.30}$$

and

$$\begin{aligned} W(\Psi_{k_1}, \Phi_{k_2}, e^{\kappa t}) = (k_2 - k_1)\Big(c_1(\kappa)e^{(k_1+k_2)x}e^{\kappa x} + c_2(\kappa)e^{-(k_1+k_2)x}e^{\kappa x}\Big) \\ + (k_2 + k_1)\Big(c_3(\kappa)e^{(k_1-k_2)x}e^{\kappa x} + c_4(\kappa)e^{-(k_1-k_2)x}e^{\kappa x}\Big) \end{aligned} \tag{3.31}$$

where

$$\begin{cases} c_1(\kappa) = (\kappa - k_1)(\kappa - k_2) \\ c_2(\kappa) = (\kappa + k_1)(\kappa + k_2) \\ c_3(\kappa) = (\kappa - k_1)(\kappa + k_2) \\ c_4(\kappa) = (\kappa + k_1)(\kappa - k_2) \end{cases} \tag{3.32}$$

Let us consider $\kappa = ik$ where $k \in \mathbb{R}$. Then from (3.31) and (3.30) it follows that

$$g(x, \pm ik) \sim c_1(\pm ik)e^{\pm ikx} \qquad x \to +\infty,$$

$$g(x, \pm ik) \sim c_2(\pm ik)e^{\pm ikx} \qquad x \to -\infty.$$

Then the following choice for the Jost solutions

$$\psi(x,k) = \frac{g(x,ik)}{c_1(ik)}, \quad \tilde{\psi}(x,k) = \frac{g(x,-ik)}{c_1(-ik)}$$

as in Definition 3.3, yields

$$a(k) = \frac{c_1(-ik)}{c_2(-ik)} = \frac{\prod_{j=1,2}(-ik - k_j)}{\prod_{j=1,2} -ik + k_j}, \qquad b(k) = 0,$$

and so the potential is reflectionless. Unlike the case of real potentials, we observe that there is in general no *conservation of energy*. Indeed, for $k \in \mathbb{R}$, $(k \neq 0)$ and $k_1, k_2 \in \mathbb{C}$ such that $k_1 \neq \bar{k}_2$, we have in general that $|a(k)| \neq 1$, where the inequality tends to an equality only in the asymptotic regime

$$\lim_{k \to \pm\infty} \frac{1}{|a(k)|} = 1.$$

Exceptionally, if $k_2 = \bar{k}_1$, then a simple calculation shows that for all $k \in \mathbb{R}$

$$\frac{1}{|a(k)|} = 1,$$

and the potential is said to be *non-dissipative*.

Let us now turn our attention to the spectrum. We have already observed that $g(x, \kappa)$ is a solution of (3.4) with $\lambda = -\frac{\kappa^2}{4}$. What is left to be proved is that $g(x, \kappa)$ is actually an eigenfunction for such λ. Let us rewrite $g(x, \kappa)$ as

$$g(x,\kappa) = \frac{e^{(k_1+k_2)x}}{e^{(k_1+k_2)x}} \left(\frac{c_1(\kappa) + c_2(\kappa)e^{-2(k_1+k_2)x} + \frac{k_2+k_1}{k_2-k_1}\left(c_3(\kappa)e^{-2k_2x} + c_4(\kappa)e^{-2k_1x}\right)}{1 + e^{-2(k_1+k_2)x} + \frac{k_2+k_1}{k_2-k_1}\left(e^{-2k_2x} + e^{-2k_1x}\right)} \right) e^{\kappa x} \tag{3.33}$$

or

$$g(x,\kappa) = \frac{e^{-(k_1+k^2)x}}{e^{-(k_1+k^2)x}} \left(\frac{c_2(\kappa) + c_1(\kappa)e^{2(k_1+k_2)x} + \frac{k_2+k_1}{k_2-k_1}\left(c_3(\kappa)e^{2k_1x} + c_4(\kappa)e^{2k_2x}\right)}{1 + e^{2(k_1+k_2)x} + \frac{k_2+k_1}{k_2-k_1}\left(e^{2k_2x} + e^{2k_1x}\right)} \right) e^{\kappa x} \tag{3.34}$$

Let us consider for example the case $\kappa = k_1$. In order to study the behaviour of $g(x, \kappa)$ for $x \to +\infty$ we consider the factorisation introduced in (3.33) and after noting that $c_1(k_1) = c_3(k_1) = 0$ we easily deduce the exponential decay. On the other hand, from the assumptions (3.29) and by simply looking at (3.34) we deduce the decay at minus infinity. The case $\kappa = k_2$ follows similarly.

Let us now consider the case $N > 1$. Again, from the assumption $\mathrm{Re}(k_1) \geq \mathrm{Re}(k_2) \geq$

$\ldots, \mathrm{Re}(k_{2N}) > 0$ we only have to address the case for $x \to \infty$. Similar considerations done in order to obtain (3.33) and (3.34) lead us to claim that

$$g(x,\kappa) \sim \frac{e^{(k_1+\cdots+k_{2N})x}}{e^{(k_1+\cdots+k_{2N})x}} \frac{\sum c(\kappa, k_1, \ldots, k_{2N}) e^{\Omega(k_1,\ldots,k_{2N})}}{(K + o(\exp(-x)))} e^{\kappa x}, \tag{3.35}$$

where K is a constant and the summation runs over a set of cardinality 2^{2N}. In particular, for any $j \in \{1, 2, \ldots, 2N\}$, we have that $\Omega(k_1, \ldots, k_{2N})$ can be expressed as following

$$\Omega(k_1, \ldots, k_{2N}) = \begin{cases} -\left(2k_j + q_j(k_1, \ldots, k_{2N})\right) & (\sharp) \\ -q_j(k_1, \ldots, k_{2N}) & (\flat) \end{cases} \tag{3.36}$$

where $q_j(k_1, \ldots, k_{2N})$ are homogeneous polynomials of degree 1 of the form

$$q_j(k_1, \ldots, k_{2N}) = (2k_1)^{\frac{\pm 1+1}{2}} + \cdots + (2k_{j-1})^{\frac{\pm 1+1}{2}} + (2k_{j+1})^{\frac{\pm 1+1}{2}} + \cdots + (2k_{2N})^{\frac{\pm 1+1}{2}}$$

such that $\mathrm{Re}(q_j) \geq 0$. The coefficients $c(\kappa, k_1, \ldots, k_{2N})$ appearing in (3.35), which are the generalisation of those introduced in (3.32) for $N = 1$, are as well products of terms $\prod_{j=1}^{2N}(\kappa \pm k_j)$ where, for any $j \in \{1, 2, \ldots, 2N\}$ the sign which has to be considered is plus if k_j appears in $\Omega(k_1, \ldots, k_{2N})$ (case $\sharp$) or minus otherwise (the case $\flat$). In particular,

$$c(\kappa = k_j, k_1, \ldots, k_{2N}) \begin{cases} \neq 0 & (\sharp) \\ = 0 & (\flat). \end{cases} \tag{3.37}$$

We conclude from (3.35),(3.36) and (3.37) that $g(x, k_j)$ decays exponentially at plus infinity. Indeed, when in the case $(\sharp)$, the term $\Omega(k_1, \ldots, k_{2N})$ is enough to compensate $e^{k_j x}$ whereas when in the case $(\flat)$, the divergent term $e^{k_j x}$ disappear by mean of (3.37). In a very similar fashion of what done in the case $N = 1$, the reflectionless property of the potential follows as well, where now

$$a(k) = \frac{\prod_{j=1}^{2N}(ik + k_j)}{\prod_{j=1}^{2N}(-ik + k_j)}, \qquad b(k) = 0.$$

Again, in the special case $k_{2j+1} = \overline{k}_{2j}$ for $j = 1, \ldots, N$, we have $|a(k)| = 1$ for any real k.

Remark 3.6. We conclude that the potentials $V_N(x,t)$ introduced in (3.21) are reflec-

tionless and non-dissipative for all real time $t \in \mathbb{R}$.

□

4

Conclusion

E se io che sono scienziato-ricercatore posso anche non saperlo,
le rondini no, le rondini sanno quello che fanno.
Ecco, alzo lo sguardo dalle mie mani al cielo.
Non è mai nero il cielo dell'Hoggar, neppure quando non c'è luna.
Viola, turchino cupo, blu dell'oltremare scuro, ma non nero.
Troppe stelle.
M. Maggiani, Il viaggiatore notturno

We have arrived at the last chapter of the dissertation, end of this journey. Each argument studied in this document will be hereby summarised and accompanied with some conclusive considerations. Furthermore, this space will be taken up in order to present some open questions which the author hasn't been able to answer during the duration of his doctoral studies. The aim of this chapter is then to show some of the solution's attempts tried which hadn't been very successful and to introduce the ideas on which them rely. The character of this chapter is intended to be less formal and more prone to a direct exposition.

In the first chapter of this thesis we discussed the localisation problem of complex eigenvalues for non-Hermitian operator defined via complex perturbation of self-adjoint operators. In Subsection 1.1.3 and Subsection 1.1.4 we have presented the correct interpretation which should be given to the expression $Hu = \lambda u$, via the theory of forms for semi-bounded operators and via some more general theorems valid for a broader class of operators. We have also introduced the Birman-Schwinger principle which has been at the basis of all results of the chapter. Subsequently we have commented upon the main results regarding complex perturbation of the Schrödinger operator, in particular showing the connection existing between resolvent estimates and localisation of eigenvalues. In the third section of the same chapter we have introduced the problem of localisation of eigenvalues for the operator defined as in (1.20)

$$H_{0,\nu} : u(x) \to \left(-\frac{d^2}{dx^2} + \frac{(\nu^2 - 1/4)}{x^2}\right) u(x)$$

on $L^2(0,\infty)$, and extended to any $\nu \in \mathbb{R}$ the results proved by Frank et al. [62] for $H_{0,1/2}$. In particular, we showed

$$|\lambda|^{1/2} \leq C(\nu,\theta) \int_0^\infty |V(x)|\, dx,$$

and by mean of numerical experiments we found a qualitative accordance between the shape of region where the complex eigenvalue lie in the case $\nu = 1/2$ and the shape obtained for a general real value of $\nu > 0$, reporting in all cases the typical drop-shaped form (see Figure 1.3 and Figure 1.5). We also proved the sharpness of our result for the case $\nu \geq 1/2$ by mean of complex delta potentials. This methodology unfotunately does not provide any positive answer when $0 < \nu < 1/2$, leaving the sharpness property unsettled. Furthermore, we have proved the following type of estimate

$$|\lambda|^\gamma \leq \tilde{C}(\nu,\theta)^{\frac{d}{2}+\gamma} \int_0^\infty x^{d-1} |V(x)|^{\frac{d}{2}+\gamma}\, dx.$$

For particular values of ν then, the estimate above has been useful to draw information on the localisation for the complex eigenvalues of the radial Laplacian subject to radial potential. In particular we have seen that for the d-dimensional radial Laplacian, the

eigenvalue lie in a region which is the same determined for $H_{0,\frac{d-2}{2}}$.

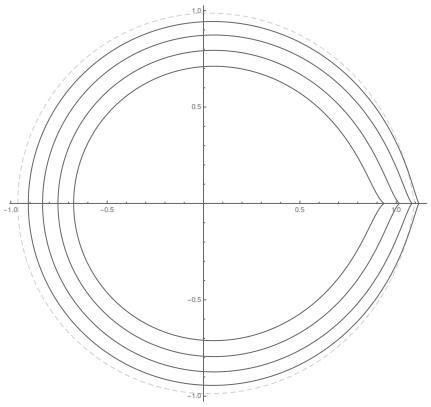


Figure 4.1: Continuous lines: from the outer most, plot of the contour of the regions $\mathcal{S}_d^r$ for $d = 3, 4, 5, 6$ obtained respectively for the values $\nu = 1/2, 1, 3/2, 2$. The dashed line is an approximation to the limit case $d = 2$ and $\nu = 0$.

Future work on number of eigenvalues

For $\nu = 1/2$, Frank Laptev and Safronov [61] have proved a quantitative estimate on the number of eigenvalues of $H_{0,1/2}$ that we have presented in Theorem 1.30. It is the author's belief that such result should hold also in the general case $\nu \in \mathbb{R}$.

As seen in Section 1.5.1, under the stronger assumptions (1.81), Kir has proved the finiteness of the eigenvalue's number for any $\nu \in \mathbb{C}$ such that $\mathrm{Re}(\nu) > 0$. In what follows we repeat pedantically the method proposed in Frank et al. [61] , introducing the needed modifications in order to deal with the case $\nu \in \mathbb{R}_+$.

We begin with recalling a general results on the number of zeros of analytic functions.

Proposition 4.1 (Frank et al. [61])**.** *Let $\eta < 0$. Let $b(k)$ be an analytic function in $\{\mathrm{Im}(k) > \eta\}$ such that for every $\eta' > \eta$ it holds true*

$$b(k) = 1 + o(|k|^{-1}) \quad \text{as } |k| \to \infty \qquad \text{in } \{\mathrm{Im}(k) > \eta'\}. \tag{4.1}$$

Suppose, moreover , that for η' sufficiently close to η, it also holds

$$\ln |b(k)| \leq A(\eta')|k|^{-\beta} \qquad \text{on } \{\mathrm{Im}(k) = \eta'\} \tag{4.2}$$

where $A(\eta') \geq 0$ is positive and $\beta > 1$.

Then the zeros k_j of $b(k)$ in the upper half plane $\{\mathrm{Im}(k) \geq 0\}$, repeated according to multiplicities, satisfy

$$\big|\{j \mid \mathrm{Im}(k_j) \geq 0\}\big| \leq c_\beta A|\eta|^{-\beta} \tag{4.3}$$

where

$$A = \lim_{\eta' \to \eta} A(\eta'), \quad c_\beta = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(1+t^2)^{\beta/2}}\, dt, \quad c_2 = \frac{1}{2}.$$

In particular, we will apply the previous result to the function $a(k)$, which will be the *regularized second order determinant* of the Birman-Schwinger operator $B(\lambda)$

$$a(k) = \det_2\big(1 + B(k^2)\big) = \prod_j \big((1 + \lambda_j(B(k^2)))\exp\big(-\lambda_j(B(k^2))\big)\big), \tag{4.4}$$

where we recall $B(\lambda)$ has been defined in (1.24). Providing a detailed introduction to the theory of regularised determinant would require a number of results and definitions. We will not enter in further details for which we refer to Frank et al. [61] and reference therein and we will restrict to the following estimates which are the only ones needed in our analysis.

Lemma 4.2 (Frank et al. [61])**.** *Suppose $B(k^2)$ be Hilbert-Schmidt. Then*

$$\ln|a(k)| \leq \frac{1}{2}\|B(k^2)\|_2^2; \tag{4.5}$$

Moreover, let $0 \leq \theta < 1$ such that $\|B(k^2)\| \leq \theta$. Then there exists a constant $\Gamma(\theta)$ so that

$$|\log a(k)| \leq \Gamma(\theta)\|B(k^2)\|_2^2. \tag{4.6}$$

Remark 4.1. We observe that there is a one to one correspondence between the zeros of the function $a(k)$ and the eigenvalues of the operator H defined in (1.22). In particular the order of the zeros coincide with the algebraic multiplicity of the eigenvalues.

In the first chapter we have recalled the link between the modified Bessel function

$\mathcal{K}_\nu(z)$,

$$\mathcal{K}_\nu(z) = \frac{1}{\Gamma(\nu+1/2)}\left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\int_0^\infty e^{-t}t^{\nu-1/2}\left(1+\frac{t}{2z}\right)^{\nu-1/2} dt,$$

for $\mathrm{Re}(\nu) > -1/2\,, |\arg(z)| < \pi$. with the Hankel functions

$$\begin{aligned} H_\nu^{(1)}(z) &= \frac{2}{i\pi}e^{-i\pi/2\nu}\mathcal{K}_\nu(ze^{-i\pi/2}) \qquad \arg(z) \in \left(-\frac{\pi}{2}, \pi\right), \\ H_\nu^{(2)}(z) &= -\frac{2}{i\pi}e^{-i\pi/2\nu}\mathcal{K}_\nu(ze^{i\pi/2}) \qquad \arg(z) \in \left(-\pi, \frac{\pi}{2}\right) \end{aligned}$$

and Bessel function

$$J_\nu(z) = \frac{i}{2\pi}\left(e^{-i\nu\pi/2}\mathcal{K}_\nu(-iz) - e^{i\nu\pi/2}\mathcal{K}_\nu(iz)\right), \qquad |\arg(z)| < \frac{\pi}{2}.$$

We use the above properties for Bessel and Hankel functions to re-write them in a more convenient way for our analysis. We have the following.

Lemma 4.3. *Consider $z \in \mathbb{C}$ with $\mathrm{Im}(z) < 0$. Then there exist two functions $\tilde{J}_\nu(z)$ and $\tilde{H}_\nu^{(1)}(z)$ such that*

$$|H_\nu^{(1)}(z)| = \sqrt{\frac{2}{\pi}}\frac{1}{\Gamma(\nu+1/2)}\frac{e^{-\mathrm{Im}(z)}}{\sqrt{|z|}}\tilde{H}_\nu^{(1)}(z) \tag{4.7}$$

$$|J_\nu(z)| = \sqrt{\frac{2}{\pi}}\frac{1}{\Gamma(\nu+1/2)}\frac{e^{-\mathrm{Im}(z)}}{\sqrt{|z|}}\tilde{J}_\nu(z) \tag{4.8}$$

with the following asymptotic behaviour at the origin and at infinity

$$\begin{aligned} \tilde{H}_\nu^{(1)}(z) &\sim O(\frac{1}{z^{\nu-1/2}}) \ \textit{for } z \to 0, \\ \tilde{J}_\nu^{(1)}(z) &\sim O(z^{\nu-1/2}) \ \textit{for } z \to 0, \\ \tilde{H}_\nu^{(1)}(z) &\sim O(1) \qquad \textit{for } z \to \infty, \\ \tilde{J}_\nu^{(1)}(z) &\sim O(1) \qquad \textit{for } z \to \infty. \end{aligned} \tag{4.9}$$

Proof. By mean of the analytic continuation formulae for Bessel functions

$$\sin(\nu\pi)H_\nu^{(1)}(ze^{im\pi}) = -\sin((m-1)\nu\pi)H_\nu^{(1)}(z) - e^{-i\nu\pi}\sin(m\nu\pi)H_\nu^{(2)}(z),$$

$$J_\nu(ze^{im\pi}) = e^{im\nu\pi} J_\nu(z),$$

where $m \in \mathbb{Z}$. Formulae (4.7) and (4.8) then follow from the representation formulae for $J_\nu(z)$ and $H_\nu^{(1)}(z)$ in terms of the modified Bessel function $\mathcal{K}_\nu(z)$ recalled above. The asymptotics claimed in (4.9) can be obtained after a direct comparison with the ones stated in (1.28) and (1.29). □

In the following statement we speculate what should now represent the extension of Theorem 1.30 to any $\nu \geq 0$.

Conjecture 4.1. Let $\nu \geq 0$, then for any $\epsilon > 0$ the number N of eigenvalues of the operator $H = H_{0,\nu} + V(x)$ with Dirichlet boundary condition, counted with algebraic multiplicities, satisfies the following

$$N \leq \frac{1}{\epsilon^2} \Lambda(\nu) \left(\int_0^\infty e^{\epsilon x} |V(x)| \, dx \right)^2, \tag{4.10}$$

where $\Lambda(\nu)$ is a positive constant which depends only on ν.

The entire proof would rely on the observation made in Remark 4.1, namely on the fact that there is a one to one correspondence between the eigenvalues of H and the zeros of the function $a(k)$ defined in (4.4) in combination with Proposition 4.1. In the following, we present the steps that would lead to a validation for the condition (4.2) for $a(k)$ by means of the results in Lemma 4.2 and estimates on the Hilbert-Schmidt norm of the Birman operator $B(k^2)$

$$\|B(k^2)\|_2^2 = \int_0^\infty \int_0^\infty |V(x)||G_\nu(x, y, k^2)|^2 |V(y)| \, dy \, dx. \tag{4.11}$$

Let us consider the case $k \in \mathbb{C}$, with fixed imaginary part $\mathrm{Im}(k) < 0$. From (4.11)

and (1.23) we have that

$$\|B(k^2)\|_2^2 \leq \frac{1}{\Gamma(\nu+1/2)^4}\frac{1}{|k|^2}\left(\int_0^\infty |V(x)|e^{-2\,\mathrm{Im}(k)x}\tilde{H}_\nu^{(1)}(kx)^2\int_0^x |V(y)|e^{-2\,\mathrm{Im}(k)y}\tilde{J}_\nu^{(1)}(ky)^2 + \right.$$
$$\left.\int_0^\infty |V(x)|e^{-2\,\mathrm{Im}(k)x}\tilde{J}_\nu^{(1)}(kx)^2\int_x^\infty |V(y)|e^{-2\,\mathrm{Im}(k)y}\tilde{H}_\nu^{(1)}(ky)^2\right)$$
$$\leq \frac{1}{|k|^2}\Lambda(\nu)\left(\int_0^\infty |V(x)|e^{-2\,\mathrm{Im}(k)x}\right)^2 \tag{4.12}$$

where, similarly to what's been done for the uniform bound, we define

$$\Lambda(\nu) = \frac{1}{\Gamma(\nu+1/2)^4}\sup_{\theta\in(\pi,2\pi)}\sup_{x\in(0,\infty)}\tilde{H}_\nu^{(1)}(xe^{i\theta})^2\sup_{y\in(0,x)}\tilde{J}_\nu(ye^{i\theta})^2 \tag{4.13}$$

The statement then would follow by taking $\mathrm{Im}(k) = -\frac{\epsilon}{2}$ with $\epsilon > 0$ and from equations (4.9). We note that, in fact, the same estimates combined with (4.6) yields assumption (4.1).

Remark 4.2. Note that the in fact, the supremum in (4.13) implies that the function $\tilde{H}_\nu^{(1)}(xe^{i\theta})^2\sup_{y\in(0,x)}\tilde{J}_\nu(ye^{i\theta})^2$ has to be evaluated over the whole lower half-complex plane. The finiteness of the bound (4.13) is yet unknown.

Remark 4.3. We observe that the validity of an estimate like (4.10) formulated in Theorem 4.1 would provide an estimate on the number of eigenvalues for the radial Laplacian in any dimension greater than one. In particular, if for odd dimensions quantitative results on the number of complex eigenvalues have been obtained by Frank et al. [61] even for the non-radial Laplacian, in the case of even dimensions nothing is known at the moment and Theorem 4.1 might be helpful to get closer to new results also in the latter case.

Trapped modes for a two layer RSW model

The subject of the second chapter of this dissertation is the existence of eigenvalues outside the essential spectrum for self-adjoint, second order differential operator pencil. In particular we considered operators that arise in the study of trapped waves in the context of a system of two layer shallow water model in a channel embedded

in a rotating reference system. The channel is assumed to be of constant width but variable depth, which depends only on the distance form the coastal line. Our results extend those proved by Johnson et al. [87] valid for the single layer case.

As in the case of a single layer, we have made several auxiliaries assumptions. Among the most important ones we have considered the rigid lid approximation, which means that we have supposed the top of the water volume to be unchanged through the time and fixed to the shape of a straight surface. This has in particular allowed to consider only certain type of waves, called of Class 2 according to the classification given in Johnson et al. [87]. These waves, which distinguish from the ones of Class 1, are 'slow in time' and typically their characteristic time is of the same order or bigger compared to the rotation's period one.

Waves of Class 2 vanish in absence of depth change or rotation. It is then the simultaneous presence of non trivial topography for the bottom combined with the effects coming from the rotation of the frame of reference and the non trivial geometry of the channel that guarantee the existence of such slow waves and in particular, from a spectral theory point of view, produce points in the discrete spectrum.

Differently to the case of a single layer, when it suffices to study the problem only in terms of the Volume Flux Stream-function, in the presence of two distinct layers of fluids at different densities, it is also needed to study the displacement function of the interface between the two fluids from the position of equilibrium when the fluids are at rest. This, in particular, gives rise to a system of two coupled equations which cannot be studied separately. Therefore the main difficulty in studying this problem.

The proof of the main result contained in Theorem 2.5 relies, similarly to what happens in the single layer case, on a specific estimate (2.90) which involves the quadratic forms of the operators in question. A thorough analysis of its terms combined with the results which hold in the single layer case, yields the validation of (2.90) also in the case of two layers when the difference between the two densities is sufficiently small.

The third chapter has been dedicated to the study of a certain class of complex solutions of the KdV equation, generated using the Wronskian method, which belong to the class of complex complexitons. This particular class of solutions represents to many extent the complex counterpart to the classical real multi-soliton solutions found by Zabusky and Kruskal [159]. They are defined by mean of the following formula

$$V_N(x,t) := 2\partial_{xx} \ln W_N(x,t)$$

where

$$W_N(x,t) := W(\Psi_{k_1}, \Phi_{\overline{k}_1}, \Psi_{k_2}, \Phi_{\overline{k}_2}, \dots, \Psi_{k_N}, \Phi_{\overline{k}_N})$$

is the Wronskian associated with the N pairs $\{(k_i, \overline{k}_i)\}_{i=1,\dots,N}$ and

$$\Psi_k(x,t) = \cosh\left(\frac{k}{2}(x-k^2t)\right),$$
$$\Phi_k(x,t) = \sinh\left(\frac{k}{2}(x-k^2t)\right),$$

for $k_i \in \mathbb{C}$, N distinct complex numbers with non null real and imaginary part such that $k_i \neq \pm k_j, \pm\overline{k_j}$ for $i \neq j$.

Solutions of the type of $V_N(x,t)$ have been found in literature, to the best of the author's knowledge, only in a single paper by Zhang et al. [160]. It was there claimed, without any proof, the property of finiteness at all times. As we have seen in the third chapter, this is not the only feature which characterise the solutions $V_N(x,t)$. They in fact reproduce the phase-shift phenomenon along with the possibility of modelling fully elastic collisions. Despite the very simple and elegant formulation which defines these solutions, we have been able to prove all these properties only for the case $N = 1$. In the following are gathered some attempts made in order to generalise these results to a generic value of $N \in \mathbb{N}$, in particular we will consider the problem of absence of singularities.

As observed in the Remark 3.4, the absence of singularities is equivalent to the condition $W_N(x,t) \neq 0$ for all real x and t. Let us give a bit more details on what it means.

In order to simplify the expressions, let us first redefine $k/2 \to k$ with no loss of generality. We also denote

$$F = F(k, x, t) := (x - 4k^2 t).$$

which, specified of a particular k_i will be denoted with

$$F_i = F(k_i, x, t) := (x - 4k_i^2 t).$$

Note that $F(\overline{k}, x, t) = \overline{F(k, x, t)}$ and $\partial_x F_k = 1$. In what follows we use the following short notation

$$\Psi(k_i F_i) = \cosh(k_i F_{k_1}), \qquad \Phi(k_i F_i) = \sinh(k_i F_{k_1})$$

With abuse of notation, we will indicate with W_N either the Wronskian of a $2N \times 2N$ complex matrix itself defined as

$$W_N = \begin{vmatrix} \Psi(k_1 F_1) & \Phi(\overline{k_1 F_1}) & \Psi(k_2 F_2) & \dots & \dots & \Psi(k_N F_N) & \Phi(\overline{k_N F_N}) \\ k_1 \Phi(k_1 F_1) & \overline{k}_1 \Psi(\overline{k_1 F_1}) & k_2 \Phi(k_2 F_2) & \dots & \dots & k_N \Phi(k_N F_N) & \overline{k}_N \Psi(\overline{k_N F_N}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_1^{2N-2} \Psi(k_1 F_1) & \overline{k}_1^{2N-2} \Phi(\overline{k_1 F_1}) & k_2^{2N-2} \Psi(k_2 F_2) & \dots & \dots & k_N^{2N-2} \Psi(k_N F_N) & \overline{k}_N^{2N-2} \Phi(\overline{k_N F_N}) \\ k_1^{2N-1} \Phi(k_1 F_1) & \overline{k}_1^{2N-1} \Psi(\overline{k_1 F_1}) & k_2^{2N-1} \Phi(k_2 F_2) & \dots & \dots & k_N^{2N-1} \Phi(k_N F_N) & \overline{k}_N^{2N-1} \Psi(\overline{k_N F_N}) \end{vmatrix} \tag{4.14}$$

or the matrix itself, whose determinant is the Wronskian stated above, whose $i-$th row reads

$$\left[W_N\right]_i = \begin{cases} \left(k_1^{2i-2} \Psi(k_1 F_1), \overline{k}_1^{2i-2} \Phi(\overline{k_1 F_1}), \dots\dots, k_N^{2i-2} \Psi(k_N F_N), \overline{k}_N^{2i-1} \Phi(\overline{k_N F_N})\right), & \text{if } i \text{ is odd}, \\ \left(k_1^{2i-1} \Phi(k_1 F_1), \overline{k}_1^{2i-1} \Psi(\overline{k_1 F_1}), \dots\dots, k_N^{2i-1} \Phi(k_N F_N), \overline{k}_N^{2i-1} \Psi(\overline{k_N F_N})\right), & \text{if } i \text{ is even}. \end{cases}$$

We also note that $W_N = A_N \circ B_N$ is the Hadamard product of two block matrices

$$A_N = \begin{pmatrix} \Psi(k_1F_1) & \Phi(\overline{k_1F_1}) & \dots & \dots & \Psi(k_NF_N) & \Phi(\overline{k_NF_N}) \\ \Phi(k_1F_1) & \Psi(\overline{k_1F_1}) & \dots & \dots & \Phi(k_NF_N) & \Psi(\overline{k_NF_N}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi(k_1F_1) & \Phi(\overline{k_1F_1}) & \dots & \dots & \Psi(k_NF_N) & \Phi(\overline{k_NF_N}) \\ \Phi(k_1F_1) & \Psi(\overline{k_1F_1}) & \dots & \dots & \Phi(k_NF_N) & \Psi(\overline{k_NF_N}) \end{pmatrix} = \begin{pmatrix} \hat{A_1} & \dots & \dots & \hat{A_N} \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \hat{A_1} & \dots & \dots & \hat{A_N} \end{pmatrix}$$

and

$$B_N = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 & 1 \\ k_1 & \overline{k}_1 & \dots & \dots & k_N & \overline{k}_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_1^{2N-2} & \overline{k}_1^{2N-2} & \dots & \dots & k_N^{2N-2} & \overline{k}_N^{2N-2} \\ k_1^{2N-1} & \overline{k}_1^{2N-1} & \dots & \dots & k_N^{2N-1} & \overline{k}_N^{2N-1} \end{pmatrix} = \begin{pmatrix} 1 & \dots & \dots & 1 \\ \hat{B}_1 & \dots & \dots & \hat{B}_N \\ \hat{B}_1^{\circ 2} & \dots & \dots & \hat{B}_N^{\circ 2} \\ \vdots & \dots & \dots & \vdots \\ \hat{B}_1^{\circ N-1} & \dots & \dots & \hat{B}_N^{\circ N-1} \end{pmatrix}$$

which is a block-Vandermonde matrix, where

$$\hat{B_i}^{\circ j} = \underbrace{\hat{B}_i \circ \hat{B}_i \circ \cdots \circ \hat{B}_i}_{j- \text{ times}}.$$

The blocks are

$$\hat{A}_j = \begin{pmatrix} \Psi(k_jF_j) & \Phi(\overline{k_jF_j}) \\ k_j\Phi(k_jF_j) & \overline{k_j}\Psi(\overline{k_jF_j}) \end{pmatrix}, \qquad \hat{B}_j = \begin{pmatrix} k_j^2 & \overline{k_j}^2 \\ k_j^2 & \overline{k_j}^2 \end{pmatrix}.$$

Proceeding by induction, let us suppose $W_N \neq 0$. From the determinant formulae for block matrix it follows that

$$W_{N+1} = W_N \cdot \det\left(\hat{B}_{N+1}^{\circ N} \circ \hat{A}_{N+1} - P_N A_N Q_N\right) \tag{4.15}$$

where P_N and Q_N are two blocks matrix of dimensions respectively $1 \times N$ and $N \times 1$ of the shape

$$P_N = \begin{pmatrix} \hat{B}_1^{\circ N} \circ \hat{A}_1 & \dots & \dots & \dots & \hat{B}_N^{\circ N} \circ \hat{A}_N \end{pmatrix}$$

and

$$Q_N = \begin{pmatrix} \hat{A}_{N+1} & \hat{B}_{N+1} \circ \hat{A}_{N+1} & \dots & \dots & \dots & \hat{B}_{N+1}^{\circ N-1} \circ \hat{A}_N \end{pmatrix}^T$$

Of course, formula (4.15) has the great advantage of expressing the determinant at the inductive step as a product. Unfortunately, the second factor is far from being an easy computations, mainly for the reason that there is a sum of matrices (which does not get along with the determinant function) and for the explicit form of the matrix $P_N A_N Q_N$ which is very intricate.

Another possible approach which might be successful in proving that W_N is never zero consists in using perturbation theory arguments. Let us restrict to the case of $N = 2$ and $t = 0$ for the sake of simplicity and clarity. Then we have

$$\begin{aligned} W_2 &= \begin{vmatrix} \Psi(k_1F_1) & \Phi(\overline{k_1F_1}) & \Psi(k_2F_2) & \Phi(\overline{k_2F_2}) \\ k_1\Phi(k_1F_1) & \overline{k}_1\Psi(\overline{k_1F_1}) & k_2\Phi(k_2F_2) & \overline{k}_2\Psi(\overline{k_2F_2}) \\ k_1^2\Psi(k_1F_1) & \overline{k}_1^2\Phi(\overline{k_1F_1}) & k_2^2\Psi(k_2F_2) & \overline{k}_2^2\Phi(\overline{k_2F_2}) \\ k_1^3\Phi(k_1F_1) & \overline{k}_1^3\Psi(\overline{k_1F_1}) & k_2^3\Phi(k_2F_2) & \overline{k}_2^3\Psi(\overline{k_2F_2}) \end{vmatrix} \\ &= \prod_{i=1}^{2} \left(\frac{|e^{k_i x}|^2}{2} \right)^4 \cdot \begin{vmatrix} (1+e^{-2k_1x}) & (1-e^{-2\overline{k_1}x}) & (1+e^{-2k_2x}) & (1-e^{-2\overline{k_2}x}) \\ k_1(1-e^{-2k_1x}) & \overline{k}_1(1+e^{-2\overline{k_1}x}) & k_2(1-e^{-2k_2x}) & \overline{k}_2(1+e^{-2\overline{k_2}x}) \\ k_1^2(1+e^{-2k_1x}) & \overline{k}_1^2(1-e^{-2\overline{k_1}x}) & k_2^2(1+e^{-2k_2x}) & \overline{k}_2^2(1-e^{-2\overline{k_2}x}) \\ k_1^3(1-e^{-2k_1x}) & \overline{k}_1^3(1+e^{-2\overline{k_1}x}) & k_2^3(1-e^{-2k_2x}) & \overline{k}_2^3(1+e^{-2\overline{k_2}x}) \end{vmatrix} \\ &= \prod_{i=1}^{2} \left(\frac{|e^{k_i x}|^2}{2} \right)^4 \cdot \det\left(B_2 + B_2 \circ C_2\right) \end{aligned} \tag{4.16}$$

where C_2 is the singular matrix

$$C_2 = \begin{pmatrix} e^{-2k_1x} & -e^{-2\overline{k_1}x} & e^{-2k_2x} & -e^{-2\overline{k_2}x} \\ -e^{-2k_1x} & e^{-2\overline{k_1}x} & -e^{-2k_2x} & e^{-2\overline{k_2}x} \\ e^{-2k_1x} & -e^{-2\overline{k_1}x} & e^{-2k_2x} & -e^{-2\overline{k_2}x} \\ -e^{-2k_1x} & e^{-2\overline{k_1}x} & -e^{-2k_2x} & e^{-2\overline{k_2}x} \end{pmatrix}$$

and B_2 is the Vandermonde matrix with entries $k_1, \dots, \overline{k}_N$. From the fact that B_N

and in particular B_2 in non singular [*] , what are the conditions on $B_2 \circ C_2$ such that the perturbation leaves the matrix non singular?
The following facts hold true.

- If A is non singular then if $\|A^{-1}E\|_p < 1$, then $A + E$ is non singular.
- $\|A \circ B\|_2^2 \leq \|A^*A\|_2 \|B^*B\|_2$ (See [78])
- $1 \leq \|A^{-1}\|\|A\|$

Then

$$\|B_2^{-1}(B_2 \circ C_2)\|_2 \leq \|B_2^{-1}\|_2 \|(B_2 \circ C_2)\|_2 \leq \|B_2^{-1}\|_2 \|B_2\|_2 \sqrt{\|C_2^* C_2\|_2} < 1$$

would be enough to be proven.

We conclude by observing that a *complexification* approach similar to the one introduced in [18], namely the study of the function $W_N(z,t)$ for $z \in \mathbb{C}$ and in particular the study of the dynamics of its poles in the complex plane, might provide elucidations on the dynamics of the real and complex part of the solitons studied in this thesis and provide as well an interpretation of the phase shift phenomenon happening also in the complex case.

[*] Let $V_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \dots & \dots \\ \alpha_1^n & \alpha_2^n & \dots & \alpha_n^n \end{pmatrix}$, then $\det(V_n) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i)$ (4.17)

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