THE LONDON SCHOOL
OF ECONOMICS AND
POLITICAL SCIENCE

Asymmetric Ramsey properties of random graphs involving cliques and cycles

lseresearchonline@lse.ac.uk
https://eprints.lse.ac.uk/

We say that a graph G is a *Ramsey graph for the pair of graphs* (F, H) if, in every 2-edge-colouring of G, we can find either a copy of F in which all the edges have the first colour or a copy of H in which all the edges have the second colour. In this case, we write $G \to (F, H)$. When $F = H$, we simplify the notation by just writing $G \to F$. Ramsey's Theorem [7] implies that, for every pair of graphs (F, H), there exists a graph G such that $G \to (F, H)$.

A lot of research has been devoted to understand the structure of Ramsey graphs. For example, Erdős and Hajnal [1] asked to determine positive integers k for which there exists G containing no copy of K_{k+1} and such that $G \to K_k$. Folkman [2] proved that such G exists for all k. Nešetřil and Rödl [6] proved a more general result which states that, for every F, there exists G with the same clique number as F such that $G \to F$. Rödl and Rucinński [8] proved that the binomial random graph $\mathbf{G}_{n,p}$ with high probability (w.h.p.) is a Ramsey graph for F, for certain range of $p = p(F)$. More precisely, they showed the following.

Theorem 1 (Rödl, Ruciński, 1995). *Let F be a graph containing a cycle. Then there exist positive constants c and C such that, for $p = p(n)$, we have*

$$\lim_{n \to \infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to F\big] = \left\{ \begin{array}{ll} 0, & \text{if } p \leq cn^{-1/m_2(F)}; \\ 1, & \text{if } p \geq Cn^{-1/m_2(F)}, \end{array} \right.$$

where

$$m_2(F) = \max\left\{ \frac{e(F') - 1}{v(F') - 2} : F' \subseteq F, v(F') \geq 3 \right\}.$$

Therefore it is well understood when the random graph is a Ramsey graph for a fixed graph F. A natural generalisation of such a problem is to analyse for what values of $p = p(F, H)$ the random graph $\mathbf{G}_{n,p}$ is likely to be a Ramsey graph for a fixed pair of graphs (F, H). In this direction, Kohayakawa and Kreuter [3] conjectured the following.

Conjecture 2 (Kohayakawa, Kreuter, 1997). *Let F and H be graphs with $m_2(F) \geq m_2(H) > 1$. Then there exist positive constants c and C such that, for $p = p(n)$, we have*

$$\lim_{n \to \infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to (F, H)\big] = \begin{cases} 0, & \text{if } p \leq cn^{-1/m_2(F,H)}; \\ 1, & \text{if } p \geq Cn^{-1/m_2(F,H)}, \end{cases}$$

where

$$m_2(F, H) = \max\left\{ \frac{e(F')}{v(F') - 2 + 1/m_2(H)} : F' \subseteq F, e(F) \geq 1 \right\}$$

Kohayakawa and Kreuter [3] proved that the conjecture holds in the case where F and H are both cycles and Marciniszyn, Skokan, Spöhel and Steger [4] proved that it holds when F and H are both complete graphs.

Here we establish the validity of Conjecture 2 when F is a clique and H is a cycle by proving the following theorem.

Theorem 3. *For all $\ell, r \geq 3$, there exists $c > 0$ such that for $p = p(n) \leq cn^{-1/m_2(K_r, C_\ell)}$, we have*

$$\lim_{n \to \infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to (K_r, C_\ell)\big] = 0.$$

We then combine Theorem 3 with the result from Mousset, Nenadov and Samotij [5], who proved that, for any pair of graphs (F, H) as in the Conjecture 2, $\lim_{n \to \infty} \mathbb{P}\big[\mathbf{G}_{n,p} \to (F, H)\big] = 1$ for $p \geq Cn^{-1/m_2(F,H)}$.

2. Proof overview

In this section we shall give an overview of the proof of Theorem 3. Notice that we need to only consider case when $\ell, r \geq 4$; the remaining cases follow from [3] and [4].

Our proof strategy is similar to [3] and [4]. We first show that if $\mathbf{G}_{n,p} \to (K_r, C_\ell)$, for some $p \leq cn^{-1/m_2(K_r, C_\ell)}$ then w.h.p. we are able to execute a procedure on $\mathbf{G}_{n,p}$ which, w.h.p., will find some subgraph of $\mathbf{G}_{n,p}$ which is either very dense or it is very large and has a tree-like structure. We then show that $\mathbf{G}_{n,p}$, for that range of p, w.h.p., does not contain such subgraphs. While the overall strategy is similar to [3] and [4], the analysis of the procedure in the first step heavily depends on the pair (K_r, C_ℓ). In this point, our work differs from previous work. In order to describe the procedure, we introduce some notation in the following.

Given a graph $G = (V, E)$, we denote by $\mathcal{G}(G)$ the hypergraph whose hyperedges correspond to copies of K_r and C_ℓ on G. More precisely, $V\big(\mathcal{G}(G)\big) = E(G)$ and $E\big(\mathcal{G}(G)\big) = \mathcal{E}_1 \cup \mathcal{E}_2$, where

$$\mathcal{E}_1 = \{E(F) : F \cong K_r, F \subseteq G\}$$
$$\mathcal{E}_2 = \{E(F) : F \cong C_\ell, F \subseteq G\}$$

Moreover, if $\mathcal{H}$ is a subhypergraph of $\mathcal{G}(G)$, we denote by $G(\mathcal{H})$ the underlying graph of G with edge set spanned by $\cup_{E \in E(\mathcal{H})} E$ and vertex set equal to $V(G)$. We also denote by $\mathcal{E}_i(\mathcal{H})$ the set of hyperedges of $\mathcal{H}$ belonging to $\mathcal{E}_i$. Then we have that $G \to (K_r, C_\ell)$ if, and only if, for every 2-colouring of the vertices of $\mathcal{G}(G)$, there exist a hyperedge $E \in \mathcal{E}_i(\mathcal{G})$, for some $i \in [2]$, such that every vertex in E has the colour i. We say that a hypergraph $\mathcal{H} \subseteq \mathcal{G}(G)$ is $\star$-critical if for any hyperedge $E \in \mathcal{E}_i(\mathcal{H})$, $i \in [2]$, and any hypervertex $e \in E$ there exists a hyperedge

$F \in \mathcal{E}_{3-i}(\mathcal{H})$ such that $E \cap F = \{e\}$. The following simple (though maybe not immediately obvious) lemma connects Ramsey graphs to $\star$-critical hypergraphs.

Lemma 4. *If $G \to (K_r, C_\ell)$, then there exist $\mathcal{H} \subseteq \mathcal{G}(G)$ which is $\star$-critical.*

For a simple graph H, let $\lambda(H) = v(H) - \frac{e(H)}{m_2(K_r,C_\ell)}$. Notice that the expected number of copies of H in $\mathbf{G}_{n,p}$, for $p \leq cn^{-1/m_2(K_r,C_\ell)}$, is at most $c^{e(H)} \cdot n^{\lambda(H)}$. In some sense, $\lambda(H)$ may be compared to the density of H. The following lemma, roughly speaking, states that $\star$-critical hypergraphs generated by $\mathbf{G}_{n,p}$ that do not have too may hyperedges must generate dense subgraphs in $\mathbf{G}_{n,p}$.

Lemma 5. *For all $\ell, r \geq 4$, there exist $\varepsilon_0, c > 0$ such that for $p = p(n) \leq cn^{-1/m_2(K_r,C_\ell)}$, the following holds w.h.p. If $\mathcal{H} \subseteq \mathcal{G}(\mathbf{G}_{n,p})$ is $\star$-critical and has at most $\ell r^2 \log n$ hypervertices, then $\lambda(G(\mathcal{H})) \leq -\varepsilon_0$.*

Algorithm 1, when applied to a $\star$-critical subhypergraph $\mathcal{G}_0 \subseteq \mathcal{G}(\mathbf{G}_{n,p})$, will create w.h.p. a sequence of subhypergraphs $\mathcal{H}_0 \subseteq \cdots \subseteq \mathcal{H}_i \subseteq \mathcal{G}_0$, each with a structure very close to a linear hypertree. The algorithm stops when the current hypergraph $\mathcal{H}_i$ is already too large or when the underlying graph $G(\mathcal{H}_i)$ is too dense. The first condition is quantified by the number of steps of the algorithm and the last condition is quantified by $\lambda(G(\mathcal{H}_i))$.

So in a step $i \leq \log n$ with $\lambda(G(\mathcal{H}_i)) > -\varepsilon_0$, the Algorithm 1 will generate a hypergraph $\mathcal{H}' \not\subseteq \mathcal{H}_i$ with $v(\mathcal{H}') \leq \ell r^2$ and let $\mathcal{H}_{i+1} = \mathcal{H}_i \cup \mathcal{H}'$. Depending on how $\mathcal{H}'$ was generated, we may have to consider this step as degenerated and in this case we add $i + 1$ to the set DEG, which is an auxiliary set with the only purpose of tracking the degenerated steps. The way that we generate $\mathcal{H}'$ will depend on weather there is a hyperedge $E \in \mathcal{E}_1(\mathcal{G}_0)$ which intersects $G(\mathcal{H}_i)$ in at least two vertices and is not contained in $G(\mathcal{H}_i)$. This case distinction is done in line 4 of Algorithm 1. If such a hyperedge E exists, then $\mathcal{H}'$ will be simply $\{E\}$ and we consider this step degenerated. Otherwise, if we do not have such a hyperedge, then the procedure to generate $\mathcal{H}'$ is more intricate and we will not be able to describe it in detail here. But the idea is roughly the following. Since we have $e(\mathcal{H}_i) \leq \ell r^2 \log n$ and $\lambda(G(\mathcal{H}_i)) > -\varepsilon_0$, Lemma 5 implies that w.h.p. $\mathcal{H}_i$ is not $\star$-critical. Then, because we failed the condition on line 4 of the Algorithm 1 together with the fact that $\mathcal{G}_0$ is $\star$-critical, we will be able to show that there exist a hyperedge $F \in \mathcal{E}_2(\mathcal{G}_0)$ which intersects $G(\mathcal{H}_i)$ in an edge and is not contained in $G(\mathcal{H}_i)$. Then $\mathcal{H}'$ will be built as an extension of F. Finally, if $\mathcal{H}$ adds too many vertices to $\mathcal{H}_i$, then we consider this step degenerated.

In the following, we state claims that are sufficient to prove Theorem 3. We do not prove these claims here. While the proof of Claim 6 really depends on the fact that we are dealing with the pair (K_r, C_ℓ), the proofs of Claims 7 and 8 are general and follow the same argument of the corresponding lemmas in [3].

Claim 6. *For every $r, \ell \geq 4$, there exists $\delta > 0$ such that the following holds.*

  (i) If $i \in \mathrm{DEG}$, then $\lambda(G(\mathcal{H}_i)) \leq \lambda(G(\mathcal{H}_{i-1})) - \delta$.
  (ii) If $i \notin \mathrm{DEG}$, then $\lambda(G(\mathcal{H}_i)) = \lambda(G(\mathcal{H}_{i-1}))$.

In particular, $\lambda(G(\mathcal{H}_i)) \leq \lambda(K_r)$.

The following claim is actually a consequence of the previous claim.

Claim 7. *For every $r, \ell \geq 4$, there exists $M > 0$ such that for every output $(i, \mathcal{H}_i, \mathrm{DEG})$ of Algorithm 1, we have $|\mathrm{DEG}| \leq M$.*

Algorithm 1

Input: a $\star$-critical subhypergraph $\mathcal{G}_0 \subseteq \mathcal{G} = \mathcal{G}(\mathbf{G}_{n,p})$
Output: a triple $(i, \mathcal{H}_i, \mathrm{DEG})$ where $\mathcal{H}_i \subseteq \mathcal{G}_0$ and $\mathrm{DEG} \subseteq [i]$

 1: $i \leftarrow 0$
 2: $\mathrm{DEG} \leftarrow \emptyset$
 3: Let $\mathcal{H}_0 = \{E_0\}$, where E_0 is any hyperdge from $\mathcal{E}_1(\mathcal{G}_0)$
 4: **while** $i \leq \log(n)$ and $\lambda(G(\mathcal{H}_i)) > -\varepsilon_0$ **do**
 5:     **if** there exists $E \in \mathcal{E}_1(\mathcal{G}_0)$ such that $E \nsubseteq G(\mathcal{H}_i)$ and $|V(E) \cap V(G(\mathcal{H}_i))| \geq 2$
        then
 6:         $\mathcal{H}' \leftarrow \{E\}$
 7:         $\mathrm{DEG} \leftarrow \mathrm{DEG} \cup \{i+1\}$
 8:     **else**
 9:         $\langle\langle$ Compute $\mathcal{H}' \rangle\rangle$
10:         **if** $\mathcal{H}'$ is degenerated **then**
11:             $\mathrm{DEG} \leftarrow \mathrm{DEG} \cup \{i+1\}$
12:         **end if**
13:     **end if**
14:     $\mathcal{H}_{i+1} \leftarrow \mathcal{H}_i \cup \mathcal{H}'$
15:     $i \leftarrow i+1$
16: **end while**

For all positive integers d and k, let $\mathcal{F}(d, k)$ be the family of all non-isomorphic graphs H such that $H = G(\mathcal{H}_i)$, where $\mathcal{H}_i$ comes from some possible output $(i, \mathcal{H}_i, \mathrm{DEG})$ of Algorithm 1 with $i \leq k$ and $|\mathrm{DEG}| \leq d$.

Claim 8. *For every $r, \ell \geq 4$, there exists $\alpha > 0$ such that for any $d, k \geq 1$, we have $|\mathcal{F}(d, k)| \leq k^{\alpha d}$.*

Proof of Theorem 3. From Claim 7, we have that after applying Algorithm 1 to some $\star$-critical subhypergraph $\mathcal{G}_0 \subseteq \mathcal{G}(\mathbf{G}_{n,p})$, we get, w.h.p., as an output $(i, \mathcal{H}_i, \mathrm{DEG})$ with $i \leq \log n$ and $|\mathrm{DEG}| \leq M$. In particular, $H \subseteq \mathbf{G}_{n,p}$, for some $H \in \mathcal{F}(M, \log n)$. Therefore

$$\mathbb{P}\big[\mathbf{G}_{n,p} \to (K_r, C_\ell)\big] \leq \mathbb{P}\big[\exists H \subseteq \mathbf{G}_{n,p} : H \in \mathcal{F}(M, \log n)\big] + o(1)$$
$$\leq \sum_{H \in \mathcal{F}(M, \log n)} \mathbb{P}\big[H \subseteq \mathbf{G}_{n,p}\big] + o(1)$$

The additional $o(1)$ term comes from the fact that Algorithm 1 will only generate an output with high probability.

Now for any $H \in \mathcal{F}(M, \log n)$, because of the condition in line 6 of the Algorithm 1, we have that either (i) $e(H) \geq \log n$ or (ii) $\lambda(H) \leq -\varepsilon_0$. In case (i), since $\lambda(H) \leq \lambda(K_r)$, we have

$$\mathbb{P}\big[H \subseteq \mathbf{G}_{n,p}\big] \leq c^{e(H)} n^{\lambda(H)} \leq c^{\log n} n^{\lambda(K_r)} \leq n^{-\varepsilon_0},$$

by choosing $c = c(\ell, r)$ small enough. In case (ii), we get

$$\mathbb{P}\big[H \subseteq \mathbf{G}_{n,p}\big] \leq c^{e(H)} n^{\lambda(H)} \leq n^{\lambda(H)} \leq n^{-\varepsilon_0}.$$

Therefore, by Claim 8, we get

$$\mathbb{P}\big[\mathbf{G}_{n,p} \to (K_r, C_\ell)\big] \leq |\mathcal{F}(M, \log n)| \cdot n^{-\varepsilon_0}$$
$$\leq (\log n)^{\alpha M} \cdot n^{-\varepsilon_0}$$
$$= o(1).$$

$\square$

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School of Mathematics and Statistics, University of New South Wales, Sydney, Australia
Email address: `a.liebenau@unsw.edu.au`

Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brazil
Email address: `leticiadmat@gmail.com`

Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brazil
Email address: `walner@impa.br`

Department of Mathematics, London School of Economics, London, UK
Email address: `j.skokan@lse.ac.uk`