Int J Adv Manuf Technol
DOI 10.1007/s00170-010-2994-7

ORIGINAL ARTICLE

Joint determination of price and upgrade level for a warranted second-hand product

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Received: 1 May 2010 / Accepted: 18 October 2010

Abstract An upgrade action is a pre-sale procedure that brings the second-hand item to an improved functional state and effectively reduces its age. This action is usually costly and adds directly to the sale price of the second-hand product, but it improves the product reliability and can reduce the warranty servicing cost. In the present paper, we propose a decision model to determine the optimal price and upgrade strategy of a warranted second-hand product to maximize the dealer's expected profit. The objective function includes both demand and cost functions, where purchase price from an end user, upgrade cost, and warranty cost are involved. We illustrate our finding using real data on second-hand electric device. Also, a sensitivity analysis is conducted to evaluate the effect of model parameters on the optimal solution.

Keywords Free replacement · Expected warranty cost · Pro rata warranty · Second-hand item · Upgrade action · Warranty length

1 Introduction

In the nowadays rapidly changing marketplace, the correct product pricing is a major decision-making challenge for manufacturers (or dealers) [1]. Product price is the easiest and most direct signal that customers use to evaluate the product, that's why the manufacturers treat product pricing as a competitive tool in their marketing strategy. In order to avoid discouraging consumers from buying, the product price should be not too high, and at the same time, in order to assure reasonable profit, this price should be not too low.

In addition to price, consumers may use product warranty to infer on the quality of a product. Warranty represents the commitment of the manufacturers to guarantee proper functionality of their products for a certain period of time after the sale. This specified time period is referred to as "warranty period". It plays a protective role for the buyer by acting as an insurance against early failures of the item due to design, manufacturing, or quality assurance problems. It also acts as a powerful advertising tool for the manufacturer to compete effectively in the marketplace. The car manufacturer Nissan is an example of using warranties as a means of differentiating his brand in the competitive automobile market. Recently, Nissan offered an outstanding 10 years/120,000 miles warranty for the cars (Nissan warranty information Booklet, 2009). Similarly, the computer manufacturers Gateway and Dell have used warranties to differentiate themselves in the computer market.

The warranty policy is a statement on the kind of compensation (repair, replacement, or money-back) the customer receives for a faulty item within the warranty period. A short warranty period may leave customers in doubt regarding the product quality, while a long warranty period may lead to unnecessary high servicing costs to the manufacturers.

In recent years, the concept of "warranty for second-hand products" has attracted significant attention among the researchers; see for example, Chattopadhyay and Murthy

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Published online: 20 January 2011

[2, 3], Chattopadhyay and Yun [4], Chukova et al. [5], and Shafiee et al. [6]. Offering warranty for second-hand products is a relatively new marketing strategy employed by dealers of a variety of products including automobiles, helicopters, home appliances, furniture, etc. The second-hand warranty contract is intended to assure the new buyer that the second-hand product will perform its intended functions for a pre-specified period of time or amount of usage.

Along with increasing the warranty period of second-hand products, the dealers are currently carrying out an effective procedure, called “upgrade action” before their release to the market. This action is usually costly and adds directly to the sale price of the second-hand product, but it improves the product reliability and can reduce the warranty servicing cost. Finding the optimal upgrade strategy/warranty policy for second-hand products is an interesting topic of research in reliability area, in which both, the dealer's costs and benefits need to be taken into account. Saidi et al. [7] investigate the optimal upgrade level to minimize the total mean cost, which is the sum of upgrade action cost and warranty servicing cost. Shafiee et al. [8] consider the outcome of the upgrade action as an uncertain process and present an optimization approach to achieve the optimal expected upgrade level.

Many researchers have investigated optimal marketing and technical strategies (such as price, quality, reliability, maintenance, and burn-in) for new products. For example, restricting our attention only on publications within the last 3 years, we mention the following studies. Ladany and Shore [9] develop a general model to determine the optimal warranty strategy based on a Cobb–Douglas-type demand function. Huang et al. [10] develop a model to determine the optimal product reliability, price, and warranty strategy for a general repairable product sold under a free repair/replacement warranty. Manna [11] considers the joint determination of price and warranty length to maximize the profit of manufacturer and proposes a methodology to extend the model for two-dimensional warranty. Matis et al. [12] develop a model to determine the optimal price, pro rata warranty length, and minimal/general repair strategy to maximize the manufacturer's profit. Fang and Huang [13] use a mathematical programming approach along with a Bayesian updating process to determine simultaneously the optimal price, production plan, and warranty policy. Zhou et al. [14] consider the joint determination of price and warranty policy for a repairable high-tech product over its effective lifetime. Wu et al. [15] develop a decision model for producers in the static demand market to determine the optimal price, warranty length and production rate. Lin et al. [16] investigate the optimal marketing and production strategy that consider price, warranty length, and production rate as simultaneous dynamic decision variables.

In contrast with research on new products, a literature review shows that only a few researchers have worked on the joint determination of optimal marketing strategies (i.e., sale price and warranty policy) and engineering issues (such as upgrade strategy and testing policies) for second-hand products to either maximize the dealer's expected profit or minimize the total mean cost. In the present paper, we propose a decision model to determine the optimal price and upgrade strategy of a warranted second-hand product to maximize the dealer's expected profit. When noticing the limitations of the previous research, this study adds to the literature by considering simultaneously three important aspects of business strategy: the upgrade, warranty, and pricing.

The rest of this paper is organized as follows. The assumptions and notations of the model are given in Section Model assumptions and notation. The upgrade action cost, expected warranty cost, and demand function are discussed in Section Components of the model. In Section The model, a stochastic model for the dealer's expected profit with three decision variables—the upgrade level, the length of the warranty period, and the sale price—is proposed. In Section Application to a real-life example, the model is applied to a real-life case study. In Section Conclusion and extensions, we conclude with a summary of our findings and offer some future research directions.

2 Model assumptions and notation

In our modeling, we consider three decision variables linked to following three lifetime stages of a second-hand item.

1. *Past age*: The dealer purchases the second-hand item from an end user. Often, the dealer has knowledge of the age of the item. These are usually obtained from sources such as registration forms and/or log books [2]. Note that, at this point, the ownership shifts from the end user to the dealer.
2. *Upgrade action*: The upgrade action typically begins as soon as the item is received at the dealership. The item's reliability growth is achieved through an iterative process consisting of test, analysis, and rectification phases.
3. *Warranty period*: Warranty plays an important role in reassuring the buyers regarding their purchase decision and signals that the dealer stands by its product. Offering warranty incurs additional costs to the dealer due to servicing the warranty claims. These costs depend on the reliability, past age and usage of the item, servicing strategy, and terms of the warranty

contract. We consider a non-renewing warranty policy, i.e., the warranty period of length w is fixed and starts at the time of the sale. All post-warranty costs related to the item are borne by the buyer.

Also, we make the following assumptions:

1. The buyers are homogeneous in their risk attitudes towards a warranty price and uncertain repair costs.
2. The product demand decreases exponentially with respect to price and increases exponentially with length of the warranty period.
3. Failure occurrences are statistically independent.
4. Whenever a failure occurs within the warranty coverage, it results in an immediate legitimate warranty claim.
5. The time to repair/replace a faulty item is sufficiently small in relation to the mean inter-failure time, so the repairs are considered to be instantaneous.
6. The time lag between purchasing a second-hand product by the dealer and selling it to a new user is small and can be ignored.
7. Prior to the sale, the second-hand items are subjected to an upgrade action, which improves their reliability.
8. Unless it is otherwise stated, all rectification actions have a negligible impact on the failure rate of the item. Therefore, all repairs throughout product past life up to age x and all warranty repairs are assumed to be minimal.
9. The past age of the item is a random variable X with support $X \in [m,M]$ and distribution function $h(x)$.

Notations

The following notations are used in the paper.

Nomenclature

X	past age of the second-hand item
m	minimum of X (lower limit)
M	maximum of X (upper limit)
μ_x	average age of the second-hand item
ρ	parameter for the truncated exponential distribution used in the life distribution of products
L	expected lifetime of the new item
$H(t)$	distribution function of the past age
$h(t)$	density function associated with $H(t)$
p_0	sale price of the new item (with no warranty)
p_x	price of the second-hand item at sale
c_x	purchase price from an end user when the age is x
$R(t)$	expected warranty cost
w	warranty period
$d(p_x, w)$	demand of the second-hand product
$c(x)$	the total mean cost per unit of product of age x
γ	the effectiveness of the upgrade action ($0 \le \gamma \le 1$)
θ_1, θ_2	the parameters of Beta distribution
$c_u(x)$	upgrade action cost
u	upgrade level
c_m	cost of minimal repair to rectify a failed item
c, ψ, ζ	the parameters of the upgrade action cost
$\overline{c}$	the expected rectification cost during the warranty period
$c_r(x)$	the replacement cost of failed item with a used one of age x
$M(t)$	renewal function
$E[c_w(x)]$	the dealer's expected warranty cost
$F_1(t)$	cumulative failure distribution of the second-hand item after the sale
$F(t)$	cumulative failure distribution of the item
$f(t)$	density function associated with $F(t)$
$r(t)$	failure rate function of the item
$r(t,u)$	failure rate function after upgrade action

3 Components of the model

In order to maximize the expected profit, a dealer must consider several factors that influence the profit, such as the purchase price from an end user, upgrade action cost, product deterioration, customers' demand, sale price, and so on.

3.1 The cost function

The total expected cost to the dealer of a second-hand item includes the purchase price from an end user, the cost of the upgrade action, and the expected warranty cost.

3.1.1 The purchase price from an end user

We consider a product with useful lifetime duration equal to L. The useful lifetime is defined as the lifetime of the product in the market and is assumed to be terminated at any time due to the ownership change, technological, obsolescence, technical, or commercial reason. Denote by c_x the price that the dealer pays to an end user for a second-hand item of age x (>0). We assume that the purchase price function is a linearly decreasing function of the past age x, and it is modeled by

$$c_x = \begin{cases} p_0\left[1 - \frac{x}{L}\right] & \text{if} \quad x \in (0, L] \\ 0 & \text{if} \quad x > L \end{cases}, \tag{1}$$

where p_0 is the sale price of a new product “with no warranty”.

3.1.2 The cost of an upgrade action

In this paper, we consider the following upgrade strategies:

1. *Overhauling (perfect) repair/replacement with a new item* This action completely restores the item to "as good as new" condition, and the failure rate of the repaired item is as the failure rate of a new item (see Fig. 1, curve "a").
2. *Imperfect repair/replacement with a younger item* restores the item to the level, which is between "as good as new" and "as bad as old" (see Fig. 1, curve "b").
3. *Minimal repair* does not affect the performance of the item and restores the item to the level "as bad as old" (see Fig. 1, curve "c").

The effect of the upgrade action on the reliability of the product can be modeled using one of the following approaches:

(a) *Virtual age approach* will be used in the forthcoming section as a basic model for our analysis. It was introduced in Kijima et al. [17] and extended in Kijima [18]. It is assumed that the upgrade action reduces the age of the second-hand item by some proportion, which depends on the upgrade strategy.

(b) *Improvement factor approach* was introduced in Malik [19]. In this approach, the upgrade action results in a rejuvenation of the second-hand item, so that it effectively reduces the failure rate of the item. Shafiee et al. [20] use improvement factor approach and develop a cost/benefit model for investment made in reliability improvement program for second-hand items sold under warranty.

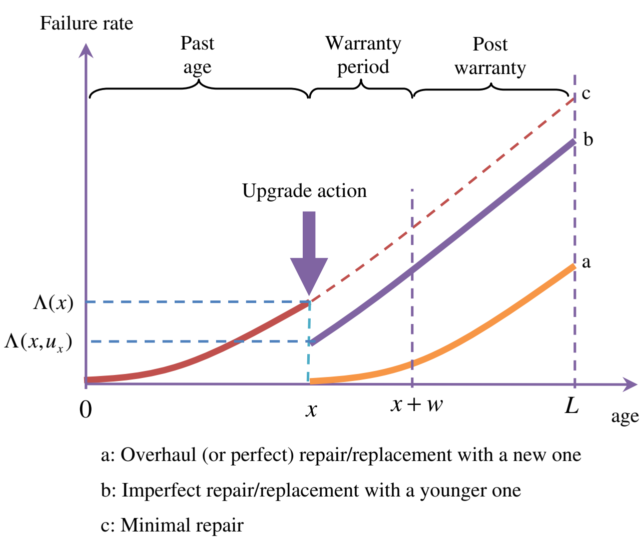

Fig. 1 Different types of upgrade actions versus failure rate change

(c) *Probabilistic approach* was introduced in Brown and Proschan [21] and extended in Block et al. [22]. In this approach, we assume that the second-hand item undergoes a "perfect repair" with probability p [$p(t)$] or an "imperfect repair" with probability $1-p$ [$1-p(t)$], where $p(t)$ is a continuous increasing function of t.

Readers can refer to Wu and Zuo [23] as a good source of references for linear and non-linear maintenance models. Let u be the level of the upgrade action applied to the second-hand item with past age of x. We model the effect of the upgrade action by the modification on the product failure rate function. We assume that after an upgrade action of level u, the failure rate function changes from $r(t)$ to

$$r(t,u) = r(t-\gamma u), \quad 0 \le u \le x \tag{2}$$

where $0\le\gamma\le1$ represents the effectiveness of the upgrade action. If the value of γ is close to one, then the upgrade action can effectively reduce the failure rate of the item and if the value of γ is close to zero, then the upgrade action does not have any significant effect on reliability.

In the real world, the upgrade action data cannot be treated as exact values due to human errors or some unexpected situations. The Beta distribution is a useful mean for describing probability functions in reliability analysis. Therefore, the effectiveness of the upgrade action can be considered as a Beta-distributed random variable, with parameters $\theta_1>0$ and $\theta_2>0$. This implies that the density of γ is given by

$$B(\gamma) = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)}\gamma^{\theta_1-1}(1-\gamma)^{\theta_2-1}, \tag{3}$$

and its expected value is

$$E[\gamma] = \frac{\theta_1}{\theta_1+\theta_2}. \tag{4}$$

We assume that the cost of an upgrade action $c_u(x)$ is a function of the upgrade level u and model it as in [24] by

$$c_u(x) = cu^{\psi}x^{\zeta}, \tag{5}$$

where the parameters $c>0$, $\psi>0$, and $\zeta>0$. Usually, these parameters are estimated by using past performance data of the product. Equation 5 implies that the cost of improvement increases as x and/or u increases. Note that $u=0$ implies no improvement and, as a result, $c_u(x)=0$, whereas $u=x$ implies a perfect repair.

3.1.3 Warranty cost

In this section, we present warranty cost models for repairable as well as non-repairable second-hand items sold under different warranty policies.

Repairable items

Policy I: Non-renewing free repair warranty (NRFRW)

Under a NRFRW with period w, the second-hand dealer rectifies, free of charge to the buyer, all warranty failures. After the expiration of the warranty period, all expenses are borne by the buyer. Second-hand products sold under NRFRW include large-screen color TVs, automobiles, refrigerators, household appliances, etc.

We assume that the dealer subjects the second-hand item of age x to an upgrade action of level u. Next, using our assumption 8, we derive the expected warranty costs. Assumption 8 means that the failure process of the item up to age x can be modeled by a non-homogeneous Poisson process (NHPP) with failure rate function $r(t) = f(t)/[1-F(t)]$. Also, it means that taking into account the effect of the upgrade action at age x on the failure rate function, the failure process over the warranty period is again a non-homogeneous Poisson process (NHPP) with intensity $r(t\text{-}\gamma u)$. Let us denote by $E[N_w(x)]$ the expected number of claims over the warranty period w for an item of age x at sale. Therefore, we have

$$E[N_w(x)] = \int_x^{x+w} r(t-\gamma u)dt, \tag{6}$$

and the dealer's expected warranty costs, $E[c_w(x)]$, is given by

$$E[c_w(x)] = \overline{c}\int_x^{x+w} r(t-\gamma u)dt, \tag{7}$$

where $\overline{c}$ is the expected rectification cost. Since γ is a random variable distributed according to Eq. (3), we have

$$E[c_w(x)] = \overline{c}\int_0^1 \left\{\int_x^{x+w} r(t-\gamma u)dt\right\} \times B(\gamma)d\gamma. \tag{8}$$

Non-repairable items

Denote by $F_1(t)$ [$f_1(t)$] the failure time cumulative [density] distribution function of a used item of age x subjected to an upgrade action of level u. For non-repairable items, the upgrade includes replacing the faulty item by a younger one of age $x-u$. Then, the cumulative distribution function can be expressed as follows:

$$\begin{aligned} F_1(t) &= P(T < x-u+t | T > x-u) \\ &= \frac{F(x-u+t)-F(x-u)}{1-F(x-u)}, \text{ for all } t \geq 0 \end{aligned} \tag{9}$$

In addition, we assume that any warranty repair improves the reliability of the used item to the level it was at the time of the sale. Denote by $N_w(x)$ the number of failures over the warranty period and by $M_w(x)$ its expected value. Therefore, the expected number of item replacements, $E[N_w(x)] = M_w(x)$, over the warranty period is the solution of the following renewal equation [25]

$$M_w(x) = F_1(w) + \int_0^w M(w-t)f_1(t)dt. \tag{10}$$

Policy I: Non-renewing free replacement warranty (NRFRW)

Under NRFRW, a faulty second-hand, non-repairable item is replaced, at no cost to the new buyer, by a used item, which is a copy of the initially sold second-hand item. The warranty cost function per replacement can be modeled by

$$R(t) = \begin{cases} c_r(x) & 0 \leq t < w \\ 0 & otherwise \end{cases},$$

where $c_r(x)$ is the expected replacement cost of the faulty item with a used one of age x. Next, in order to account for the item's age at failure, we prorate the expected cost $c_r(x)$ as follows:

$$c_r(x) = c_r\left[1-\frac{x}{L}\right], \quad 0 \leq x < L, \tag{11}$$

where $c_r(0)=c_r$ is the replacement cost with a new item.

Then, the expected warranty cost per replacement, $E[R(t)]$, is

$$E[R(t)] = \int_0^w c_r(x)f_1(t)dt = c_r(x)F_1(w). \tag{12}$$

Policy II: *Pro rata warranty*

Under a pro rata warranty (PRW), the dealer replaces the faulty item at a charge to the new buyer, which is prorated to the age of the item at time of the failure. The PRW is usually offered for relatively cheap, non-repairable second-hand items. The cost function for warranty per replacement under the PRW policy can be modeled as

$$R(t) = \begin{cases} c_r(x)\left[1-\frac{t}{w}\right] & 0 \leq t < w \\ 0 & \text{otherwise} \end{cases},$$

and the expected warranty cost per replacement, $E[R(t)]$, is given by

$$\begin{aligned} E[R(t)] &= \int_0^w c_r(x)\left[1-\frac{t}{w}\right]f_1(t)dt \\ &= \frac{c_r(x)}{w}\int_0^w F_1(t)dt. \end{aligned} \tag{13}$$

Policy III: FRW/PRW

Under the FRW/PRW policy, prior to time w_f, (w_f<w), any faulty item is replaced free of charge, whereas after w_f, any failure in (w_f, w) results in a pro rata replacement. Typical examples of second-hand products sold with FRW/PRW are pumps, batteries, tires, and helicopters. The warranty cost function per replacement can be modeled as

$$R(t) = \begin{cases} c_r & 0 \leq t < w_f \\ kc_r(x)\left[1 - \alpha \frac{t-w_f}{w-w_f}\right] & w_f \leq t < w \\ 0 & \text{otherwise} \end{cases}, \tag{14}$$

where 0<k≤1 and 0≤α≤1 are proportionality coefficients. Low values of k and high values of α imply that the dealer will carry out lower warranty cost, compared with the cost for high values of k and low values of α. Then, the expected warranty cost per replacement, $E[R(t)]$, is given by

$$\begin{aligned} E[R(t)] &= \int_0^{w_f} c_r(x) f_1(t)dt + \int_{w_f}^{w} kc_r(x)\left[1 - \alpha \frac{t-w_f}{w-w_f}\right] f_1(t)dt \\ &= c_r(x)F_1(w_f) + \int_{w_f}^{w} kc_r(x)\left[1 - \alpha \frac{t-w_f}{w-w_f}\right] f_1(t)dt \end{aligned} \tag{15}$$

Policy IV: FRW/LSW

One special case of Eq. 14 is 0<k≤1 and α=0, which is the combination of free replacement and lump-sum warranty. Under the FRW/LSW policy, prior to time w_f, (w_f<w), any faulty item is replaced free of charge, whereas after w_f, any failure within (w_f, w) leads to a refund by a fraction of the replacement cost. The warranty cost function per replacement under the FRW/LSW policy can be modeled as

$$R(t) = \begin{cases} c_r(x) & 0 \leq t < w_f \\ kc_r(x) & w_f \leq t < w \\ 0 & \text{otherwise} \end{cases}.$$

Then, the expected warranty cost per replacement, $E[R(t)]$, is given by

$$\begin{aligned} E[R(t)] &= \int_0^{w_f} c_r(x) f_1(t)dt + \int_{w_f}^{w} kc_r(x) f_1(t)dt \\ &= c_r(x)\left[(1-k)F_1(w_f) + kF_1(w)\right]. \end{aligned} \tag{16}$$

Finally, we are able to summarize the above results and derive the expected warranty cost for non-repairable items. Using Wald's renewal equation [26], we obtain that the expected warranty cost for a second-hand item, subjected to an upgrade action of level u and warranted for duration of w, is given by

$$E[c_w(x)] = E[N_w(x)] \times E[R(t)], \tag{17}$$

where, $E[N_w(x)]$ is given by the solution of Eq. 10 and the expression of $E[R(t)]$ is given by Eqs. 12, 13, 15, or 16 depending on the offered warranty policy.

3.1.4 The total expected cost

We assume that the dealer of second-hand products sells products of ages within the interval [m,M]. Typically, the expected warranty costs for older products are higher than those for younger products and, usually, these warranty costs are included, in one way or another, in the product price. Therefore, older products are less appealing to the new buyers than the younger ones. To make the sale price of older products more attractive to the new buyers, the dealer can subsidize them by charging less than the expected warranty cost and recover this subsidy by charging for younger products more than their expected warranty cost. We model this by viewing the age X of the second-hand item, as a random variable with support [m,M] and a cumulative [density] distribution function $H(x)$[$h(x)$], with $H(m)$=0 and $H(M)$=1. Conditional on X=x, the expected warranty cost is $E[c_w(x)]$, given in Eq. 17. On carrying out the expectation over X, we obtain that the expected warranty cost per product, $c(w)$, is

$$c(w) = \int_m^M E[c_w(x)] \times h(x)dx. \tag{18}$$

Next, let $c(x, u, w)$ be the total expected cost per unit of product of age x, subjected to an upgrade action of level u and sold under warranty of length w. Therefore,

$$c(x,u,w) = c_x + c_u(x) + c(w), \tag{19}$$

where c_x is given by Eq. 1, $c_u(x)$ by Eq. 5, and $c(w)$ by Eq. 18.

Using $c(w)$ instead of $E[c_w(x)]$ in Eq. 19, implies that the charges for warranty are independent of item's age. From dealer's point of view, this is a good strategy because it assures that the sale price of older products can be kept at a reasonable level. One possible, mathematically tractable form of $h(x)$ is

$$h(x) = \frac{\rho e^{-\rho x}}{e^{-\rho m} - e^{-\rho M}}. \tag{20}$$

Then, the corresponding cumulative function is

$$H(x) = \frac{e^{-\rho m} - e^{-\rho x}}{e^{-\rho m} - e^{-\rho M}},$$

with ρ>0, i.e., the past age X of the second-hand product has a truncated exponential distribution with parameter ρ>0, and support [m,M], see [27]. It is straightforward to show that the average age of the second-hand item is

$$\mu_x = \frac{\rho(me^{-\rho m} - Me^{-\rho M}) + (e^{-\rho m} - e^{-\rho M})}{\rho(e^{-\rho m} - e^{-\rho M})}.$$

Note that the average past age is a decreasing function of parameter ρ.

3.2 The demand function

We assume that the demand for second-hand product depends on the marketing strategies (i.e., the offered warranty coverage/service and the product price). We model the customer demand using Glickman and Burger's demand function [28], i.e., function, in which the product demand decreases exponentially with respect to price and increases exponentially with length of the warranty period. So, the demand function, $d\ (p_x,\ w)$, can be modeled as follows:

$$d(p_x, w) = k_1 p_x^{-a}(w + k_2)^b, \tag{21}$$

where w is the length of the warranty period; p_x is the sale price of the second-hand product; $k_1>0$ is a constant, an amplitude factor; $k_2\geq 0$ is a constant of time displacement: it allows for non-zero demand with no warranty offered, $a>1$ is a constant, parameter of the price elasticity; and $0<b<1$ is a constant, parameter of the displaced warranty length elasticity. Applications of this demand function can be found in the literature, see for example [9–12, 15, 16].

4 The model

We consider an optimization problem, such that the objective function—the dealer's expected profit, in terms of three decision variables: the upgrade level, warranty period, and sale price—is maximized. The optimization model with these three decision variables can be formulated as follows:

$$\max \quad J(u, w, p_x) = [p_x - c(x, u, w)] \times k_1 p_x^{-a}(w + k_2)^b, \tag{22}$$

where $J\ (u,\ w,\ p_x)$ is the dealer's expected profit from a second-hand item of past age x, subjected to an upgrade action of level u, sold under warranty characterized by a warranty period of length w, and $c(x,\ u,\ w)$ is given by Eq. 19.

The optimization problem in Eq. 22 can be solved by applying the maximum principle (for more see [29]). To apply the maximum principle, we first find the partial derivatives of the expected profit and then use the Hessian matrix (HM) to find the optimal values of the upgrade level, the length of the warranty period, and the sale price.

The partial derivatives of Eq. 22 with respect to u, w, and p_x yield

$$\frac{\partial}{\partial u} J(u, w, p_x) = -k_1 p_x^{-a}(w + k_2)^b \times \frac{\partial}{\partial u} c(x, u, w), \tag{23}$$

$$\frac{\partial}{\partial w} J(u, w, p_x) = \frac{k_1 a^{-a}}{(a-1)^{-a+1}} [c(x, u, w)]^{-a}(w + k_2)^{b-1}\left\{ b \times c(x, u, w) - (a-1)(w + k_2)\frac{\partial}{\partial w} c(w)\right\}, \tag{24}$$

and

$$\frac{\partial}{\partial p_x} J(u, w, p_x) = k_1 p_x^{-a-1}(w + k_2)^b [a \times c(x, u, w) - (a-1)p_x]. \tag{25}$$

Table 1 u^*, w^*, p_x^* and $J\left(u^*, w^*, p_x^*\right)$ for different combinations of x and the Beta parameters (θ_1 and θ_2)

	θ_1=2 and $\theta_2 = 3(E[\gamma] = 0.4)$.				θ_1=3 and $\theta_2 = 3(E[\gamma] = 0.)$				θ_1=3 and $\theta_2 = 2(\mathrm{E}[\gamma] = 0.6)$			
	u^*	w^*	p_x^*	$J\left(u^*, w^*, p_x^*\right)$	u^*	w^*	p_x^*	$J\left(u^*, w^*, p_x^*\right)$	u^*	w^*	p_x^*	$\left(u^*, w^*, p_x^*\right)$
$x=1$	0.2	12	1372.1	140.1	0.1	13	1367.2	142.0	0.0	14	1362.4	143.1
$x=2$	0.4	10	1220.3	127.5	0.3	11	1215.6	129.1	0.2	12	1211.1	130.8
$x=3$	0.8	7	1069.7	112.6	0.7	9	1065.2	113.9	0.6	10	1060.8	115.5
$x=4$	1.3	6	918.4	95.8	1.1	8	914.0	96.8	0.9	9	909.8	98.0
$x=5$	1.9	4	766.2	77.3	1.7	5	762.1	78.0	1.5	6	758.1	78.9
$x=6$	2.5	3	614.0	58.9	2.2	4	610.1	59.4	2.0	5	606.4	60.0
$x=7$	3.2	2	451.7	39.2	2.9	3	447.0	39.5	2.6	4	443.5	39.9

Table 2 Weibull distribution parameters and the expected rectification cost for motor and trigger, included in warranty

Component	Weibull		Mean time to first failure (year)	Type of rectification	Expected rectification cost
	Shape	Scale			
Motor	β_m=1.55	λ_m=0.312	2.875	Repair	\$50
Trigger	β_t=2.0	λ_t=0.354	2.503	Replacement with a used item	\$10

The optimal solution satisfies the following necessary conditions:

$$\frac{\partial}{\partial u}J(u,w,p_x) = J_u = 0\,,\ \frac{\partial}{\partial w}J(u,w,p_x) = J_w = 0,\ \frac{\partial}{\partial p_x}J(u,w,p_x) = J_{p_x} = 0\,.$$

Therefore, u^* and w^* are the solutions of the following equations:

$$\frac{\partial}{\partial u}c(x,u,w) = 0\,, \tag{26}$$

$$(a-1)(w+k_2)\frac{\partial c(w)}{\partial w} = b \times c(x,u,w)\,, \tag{27}$$

and the optimal sale price is given by

$$p_x^* = \frac{a}{a-1}c\left(x,u^*,w*\right)\,. \tag{28}$$

Using similar arguments as those given in Shafiee [30], we can show that the HM is a negative definite matrix, which assures that the dealer's maximum expected profit is reached at $\left(u^*,w^*,p_x^*\right)$. The solution of Eq. 26–28 is quite involved and our next step in analyzing model/Eq. 22) will be to develop an approach to find $\left(u^*,w^*,p_x^*\right)$. This approach will be reported in our extended paper.

4.1 A numerical example

Consider a dealer who sells repairable second-hand products of age distributed as given in (20) with m=1, M=7 and ρ=0.2 (This implies that the average age of the second-hand item is 3.4 years). Assume that the product expected lifetime is L=10 years, and the product is subject to random failures with the time to failure following the Weibull distribution with $r(t) = \lambda\beta(\lambda t)^{\beta-1}$. We assume that u, w, and p_x are decision variables to be determined so that the dealer's expected profit is maximized. Let p_0=\$1,500 (with no warranty), λ=0.461, β=1.5, c_m=10, c=10, ψ=0.9, ζ= 0.3, $\overline{c} = 15$, a=2.2, b=0.8, k_1=1,000, and k_2=0.12. Table 1 gives the optimal upgrade level u^*, the optimal warranty length w^* (in month), the optimal sale price $p_x^*$and the corresponding expected profit $J\left(u^*,w^*,p_x^*\right)$ for different combinations of x and the Beta parameters (θ_1 and θ_2).

5 Application to a real-life example

This section presents an application of our model to a problem faced by a dealer of second-hand electric drills. The price of a new cordless drill with no warranty is p_0= \$350. This price is high enough to prompt customers to consider purchasing a second-hand drill. Also, it should be noted that drills are not affected by fashion, so their age and functionalities are far more important than their models. The expected lifetime of a new electric drill is 8 years. We assume that the depreciation is linear with respect to product age.

Some of the components of the drill are non-repairable, such as its trigger and other plastic parts. Their rectification requires a perfect repair. However, some of its more expensive internal parts, like the motor, can be returned to a good working condition by an imperfect repair. For these, the second-hand warranty will be less than that of a new drill.

The dealer groups the components of the second-hand drills into two disjoint sets—a set I, which consists of components covered by warranty, and a set E that consists of components that are not covered by the warranty. Under

Table 3 Summary of the parameter values

Price for a new drill (with no warranty)	p_0=\$350
Warranty policy for motor	FRW
Warranty policy for trigger (and plastic parts)	FRW, PRW, FRW/PRW, FRW/LSW
The terms of warranty (year)	w_f=0.25, w=0.5, 1.0, 1.5, k=0.5
The parameters of upgrade action	$\widehat{c} = 4.210$, $\widehat{\psi} = 0.638$, $\widehat{\zeta} = 0.126$, θ_1=0.523, θ_2=4.71
Demand function for used electric drill	$d(p_x,w) = k_1 \times p_x^{-4.2}(w+0.12)^{0.85}$
Distribution of the past age	ρ=0.23, m=1.0, M=5.0

Table 4 u^* and p_x^* for second-hand electric drills with different past ages and different warranty policies

		x=1.0		x=2.0		x=3.0		x=4.0		x=5.0	
		u^*	p_x^*	u^*	p_x^*	u^*	p_x^*	u^*	p_x^*	u^*	p_x^*
Policy I	w=0.5	0.0	336.2	0.0	263.3	0.0	190.9	0.7	119.1	0.9	46.8
	w=1.0	0.0	352.4	0.4	281.1	0.9	209.2	1.4	137.2	2.0	64.7
	w=1.5	0.8	374.7	1.1	300.6	1.5	228.3	1.9	156.0	2.4	83.3
Policy II	w=0.5	0.0	332.5	0.0	260.9	0.0	187.5	0.5	116.1	0.7	43.5
	w=1.0	0.0	349.6	0.3	278.2	0.8	206.1	1.2	134.3	1.7	60.9
	w=1.5	0.7	370.1	1.0	297.7	1.3	225.0	1.6	152.6	2.0	79.8
Policy III	w=0.5	0.0	334.7	0.0	261.5	0.0	188.7	0.6	116.9	0.7	44.3
	w=1.0	0.0	351.0	0.3	279.1	0.8	207.7	1.3	135.9	1.8	62.3
	w=1.5	0.7	371.5	1.0	298.8	1.3	226.2	1.7	153.5	2.1	80.4
Policy IV	w=0.5	0.0	335.8	0.0	262.6	0.0	189.3	0.7	117.9	0.8	45.2
	w=1.0	0.0	351.9	0.4	280.5	0.9	208.6	1.3	136.6	1.9	63.5
	w=1.5	0.8	372.6	1.1	299.1	1.5	227.5	1.8	154.4	2.2	81.9

this policy, the dealer rectifies all components from set I at no cost to the buyer over the warranty period. The costs of rectifying failed components from set E are borne by the new buyer. If a component from set I fails during the warranty period, it affects the performance of the electric drill and results in a warranty claim. After the expiration of warranty, the buyer incurs all of the product post-warranty repair/replacement and maintenance costs. The two drill components that were warranted were the motor, which is reparable, and the trigger, which is non-repairable.

Our data is based on a sample of 1,546 warranty claims collected over more than 1 year, from September 2007 until October 2008. These claims include 719 motor claims and 827 trigger claims. We suggest two-parameter Weibull distribution for the failure time of the motor and for the trigger. The evaluation of this proposed distribution requires the examination of the "goodness of fit". This was achieved by using Statistica™ (a statistical package). The "goodness of fit" was then determined using the Chi-square and Kolomogorov–Smirnov (K–S) methods. Both of the tests indicate that the Weibull distribution provides the first-best fit. Also, we used the maximum likelihood approach to estimate the model parameters, which are given in Table 2.

The dealer's aim is to satisfy the customers' demand and expectations and to improve the quality of second-hand electric drills by providing an overhaul/imperfect repair of a faulty motor and a replacement of a faulty trigger. An upgrade action includes a series of reliability improvement stages, such as component cleaning, reconditioning, and electrical test, which result in rejuvenation of the item. As a result, after the upgrade action, the intensity function for motor's failure is

$$r(x-\gamma u+t)=\lambda_m\beta_m[\lambda(x-\gamma u+t)]^{\beta_m-1},\quad t\geq 0 \tag{29}$$

and the cumulative distribution function for trigger's time to failure is

$$F_1(t)=\frac{\exp\left[-(\lambda_t(x-u+t))^{\beta_t}\right]-\exp\left[-(\lambda_t(x-u+t))^{\beta_t}\right]}{\exp\left[-(\lambda_t(x-u+t))^{\beta_t}\right]},\quad t\geq 0. \tag{30}$$

Based on a market survey, the marketing department of the dealership estimated the parameters of the demand function. Using these estimations from the simulated data and the logarithmic regression, we specify the demand function as follows:

$$d(p_x,w)=k_1\times p_x^{-4.2}(w+0.12)^{0.85}. \tag{31}$$

We consider a number of possible warranty options on the motor and the trigger. Specifically, we consider a free repair warranty on the motor and several warranty policies on the trigger, such as free replacement, pro rata, a

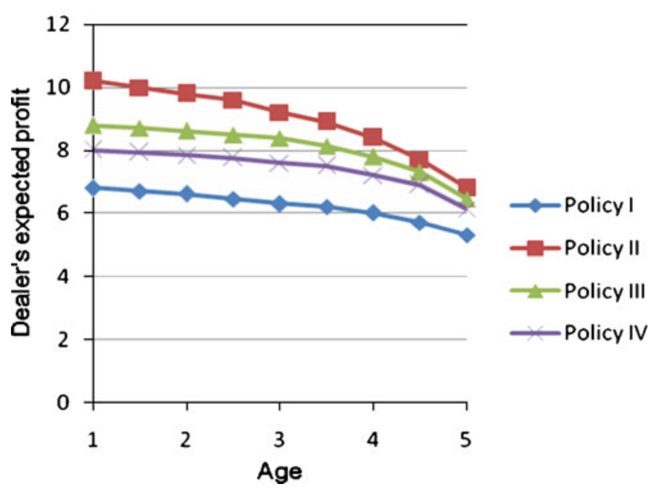

Fig. 2 $J\left(u^*,\mathrm{w}=0.5,p_x^*\right)$ for different warranty policies

Table 5 The optimal upgrade level and optimal sale price p_x^* for second-hand electric drills with different past ages: sensitivity analysis for the price elasticity and the warranty length elasticity

		x=1.0		x=2.0		x=3.0		x=4.0		x=5.0	
		u^*	p_x^*	u^*	p_x^*	u^*	p_x^*	u^*	p_x^*	u^*	p_x^*
a=4.1	b=0.80	0.0	335.0	0.0	263.4	0.0	190.0	0.5	118.6	0.7	46.1
	b=0.85	0.0	335.3	0.0	263.8	0.0	190.5	0.5	119.1	0.7	46.6
	b=0.90	0.0	333.8	0.0	264.3	0.0	191.1	0.5	119.7	0.7	47.3
a=4.2	b=0.80	0.0	331.9	0.0	260.3	0.0	186.8	0.5	115.3	0.7	42.7
	b=0.85	0.0	332.5	0.0	260.9	0.0	187.5	0.5	116.1	0.7	43.5
	b=0.90	0.0	333.1	0.0	261.6	0.0	188.2	0.5	117.0	0.7	44.4
a=4.3	b=0.80	0.0	328.5	0.0	256.7	0.0	183.1	0.5	111.6	0.7	38.9
	b=0.85	0.0	329.3	0.0	257.7	0.0	184.2	0.5	112.9	0.7	40.1
	b=0.90	0.0	330.0	0.0	258.4	0.0	184.9	0.5	113.7	0.7	41.0

combination of free replacement and pro rata, and a combination of free replacement and lump-sum with w_f= 0.25. We specify the remaining model parameters as follows:

(a) the length of the warranty period w as 0.5, 1, or 1.5 years;
(b) The parameters of the upgrade action: a logarithmic regression model for Eq. 5 produced the following results: $\widehat{c} = 4.210$, $\widehat{\psi} = 0.638$ and $\widehat{\zeta} = 0.126$.
(c) Due to lack of historical data on the effectiveness of the upgrade action, we use the judgement of the reliability and quality engineers to estimate the parameters of the degree of the restorability γ of the motor as $\widehat{\theta}_1 = 0.523$ and $\widehat{\theta}_2 = 4.71$. Then, the expected restorability is 0.099.

After considering the past age data and using Monte Carlo simulation, we estimated the parameter of the truncated exponential distribution of the past age X as $\widehat{\rho} = 0.23$/year, with m=1.0 and M=5.0. Also, to validate the distribution model, a chi-square goodness-of-fit test was carried out.

Table 3 summarizes the values of the model parameters. We examine and compare the following four warranty policies:

Policy I Motor: non-renewing free repair warranty, Trigger: non-renewing free replacement warranty
Policy II Motor: non-renewing free repair warranty, Trigger: non-renewing pro rata warranty
Policy III Motor: non-renewing free repair warranty, Trigger: non-renewing FRW/PRW
Policy IV Motor: non-renewing free repair warranty, Trigger: non-renewing FRW/LSW

Table 4 shows the optimal upgrade level u^* and optimal price p_x^* for second-hand electric drills with different past ages, distributed as given in Eq. 20, and various warranty policies.

Based on the Table 4, we can provide the buyer with a list of options (upgrade strategy, warranty policy, and sale price) to choose from when making a decision regarding the purchase of the second-hand electric drill. From Table 4, it is easy to see that (a) the optimum upgrade level increases as w increases, (b) the optimum sale price increases as u and/or

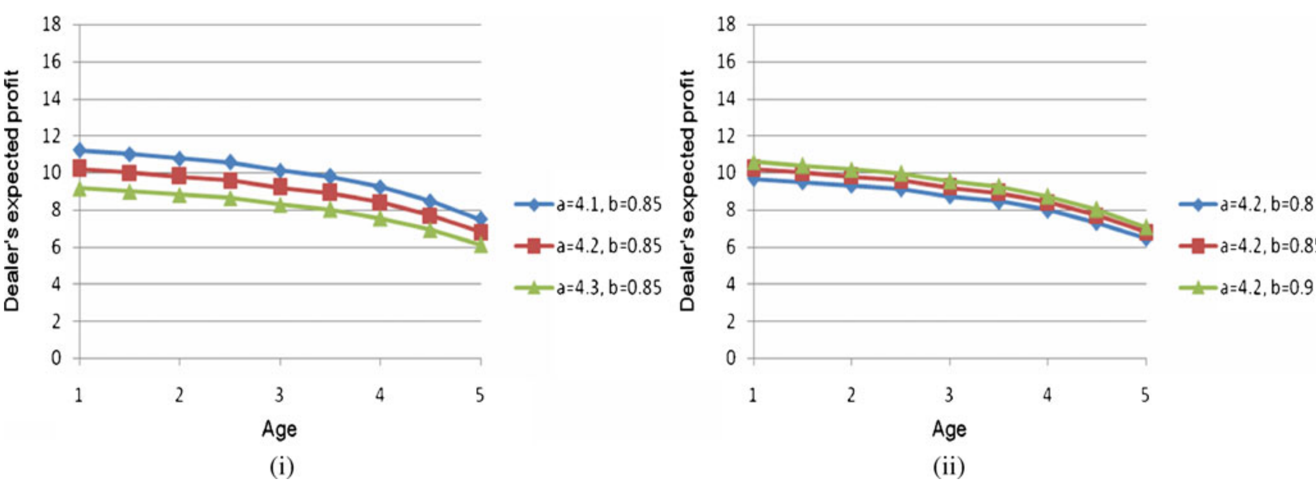

Fig. 3 $J\left(u^*, w^*, p_x^*\right)$ for a range of past age and (*i*) different price elasticity, (*ii*) different warranty length elasticity, under *policy II*

w increases, and (c) the price of a second-hand electric drill under *policy I* is higher compared with any of the other policies.

The "best" warranty policy, from dealer's viewpoint, is the policy that assures maximum expected profit. As expected, the maximum expected profit is assured by *Policy II*, whereas *Policy I* leads to minimum expected profit. Figure 2 compares the $J\left(u^{*}, w, p_{x}^{*}\right)$ under *Policies I–IV* for a second-hand item sold under different warranty policies with w=0.5. So, under the considered model parameters the best warranty policy is to offer 6-month non-renewing free replacement warranty on the motor and 6-month non-renewing pro rata warranty on the trigger.

Next, we conduct a sensitivity analysis to evaluate the effect of model parameters—the price elasticity (a) and the warranty length elasticity (b)—on the optimal solution of the model. In Table 5, we show the results of this sensitivity analysis for *Policy II* with 6-month warranty coverage.

On comparing Table 5 with Table 4, we see that (1) the sale price increases/decreases as the price elasticity decreases/increases, (2) an increase/decreases in warranty length elasticity does not have any significant effect on the price, and (3) a variation in warranty length elasticity and/or price elasticity does not have any effect on optimal upgrade level.

Figure 3 presents the corresponding expected profit, $J\left(u,^{*} w^{*}, p_{x}^{*}\right)$ for a range of past ages, different price elasticity and warranty length elasticity under *policy II*.

From Fig. 3 we see that (1) the dealer's expected profit increases/decreases as the price elasticity decreases/increases, and (2) an increase/decreases in warranty length elasticity does not have any significant effect on the price.

6 Conclusion and extensions

In this paper, we develop a stochastic model from dealer's point of view, to determine price and upgrade level for a warranted repairable/non-repairable second-hand item. The objective function—the dealer's total expected profit—is assumed to depend on the sale price, upgrade action cost, warranty costs, and demand function. The dealer's total expected profit is derived for four types of warranties: free repair warranty, pro rata warranty, FRW/PRW, and FRW/LSW and illustrated by a real-life case study. To observe the effect of model parameters on the optimal solution, a sensitivity analysis for the price elasticity and warranty length elasticity is conducted.

There is a wide-ranging scope for future research in the area of second-hand products. Some possible directions are as follows:

- Developing a game–theoretic model of upgrade level–warranty–price for second-hand items to maximize the dealer's total expected profit;
- In this study, the heterogeneity in customers' purchasing decisions is not considered, i.e., under given product price and warranty length, we assume that all customers have the same preference for the used product. In practice, this might not be the case, and addressing different risk attitudes towards the second-hand product is of importance.
- Applications of the proposed model for two types of market (static demand market and dynamic demand market) will be considered in our future work.
- Technological advance leading to improvement of product reliability is not considered in this paper and will be reported in our extended paper.

Acknowledgment The authors would like to thank the referees for their valuable comments and suggestions that appreciably improved the quality of this paper. The authors also want to thank Dr. S. Chukova for her valuable comments.

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