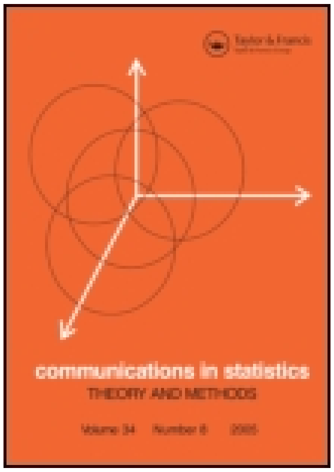

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Two-Dimensional Warranty Cost Analysis for Second-Hand Products

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In spite of the recent steady increase of the volume of the second-hand markets, often customers remain in doubt regarding the quality and durability of the second-hand products. Aiming to reduce and share this uncertainty, dealers offer warranty on their products. Offering warranty for second-hand products is a relatively new marketing strategy employed by dealers of used electronic equipment, furniture, automobiles, etc. Usually, for used products, the dealer's expected warranty cost is a function of product reliability, past age and usage, servicing strategy and conditions and terms of the warranty policy/contract. Sometimes the offered policy is limited by two parameters, typically the product age and usage after the sale. This type of policies is referred to as two-dimensional warranty policies. In this article, we develop statistical models for estimating the dealer's expected warranty cost for second-hand products sold with two-dimensional free repair/replacement warranty.

Keywords Expected warranty cost; Free repair/replacement; Second-hand product; Two-dimensional warranty; Usage rate.

Mathematics Subject Classification Primary 47N30; Secondary 60G20.

1. Introduction

Recently, due to the rapid technological development and fierce competition in the marketplace, product warranty is becoming increasingly important product attribute. A product warranty is an agreement offered by the manufacturer/dealer to a consumer to repair/replace a faulty item, or to partially or fully reimburse the consumer in the event of product failure during the warranty period. Murthy and

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Blischke (2000) advocated two important functions of warranty. Firstly, warranty assures protection (for customers—against product failure due to faulty design, manufacturing defects, etc., and for dealers/manufacturers—against consumers' product misuse, illegal claims, etc). Secondly, it promotes product quality and guarantees repair/replacement or refund in case of faulty product. Since the beginning of the 20th century, for many items the length of the warranty coverage has been gradually increasing. For example, in the early 1930s, the warranty period offered by the car manufacturers was 3 months, becoming 1 year in the 1960s, and currently it varies between 3–5 years (Kim et al., 2004). A longer warranty term signals better reliability and quality and attracts more customers, but at the same time it involves additional servicing costs. For example, American manufacturers spend over $25 billion—about 2% of their revenue—annually on warranty services (Manna et al., 2007). According to the 2002 General Motors annual report, the company had total revenues of US$186.7 billion and the future warranty cost on sold cars was estimated to be US$4.3 billion—about 2.3% of the revenue.

Warranty policy is a statement on the extent of the warranty coverage and the type of compensation provided to the customers in the case of faulty product. The compensation could be specified as a free replacement, lump sum payment, a pro-rata reimbursement, etc. Taxonomy of different warranty policies is available in Blischke and Murthy (1994) and Christozov et al. (2009). These policies can be broadly divided into two groups: one- and two-dimensional warranty policies. One-dimensional (1D) warranty policies are characterized by a warranty period, which is usually a time interval on the age of the item with no limitation imposed on its usage. In contrast, the two-dimensional (2D) warranty policies are depicted by a 2D-plane region with axes representing age and usage after the sale. For example, a typical automotive warranty could be specified as providing a free of charge service or repairs for a maximum of 5 years or 50,000 miles, whichever occurs first.

Recently, the importance of the used/second-hand product market as a part of the total market (new and used) has been on the rise. For instance, since 1999 in France, the used-to-new car ratio has increased from 2.4 to 2.6, whereas in UK, the total demand for used cars grew by 25% in volume and 68% in value from 1996 to 2006 (Car Internet Research Program II, 2008, pp. 3–5).

In spite of the increasing importance and share of the second-hand market, often customers of second-hand products encounter the following two problems.

1. Uncertainty regarding the quality, durability, and performance specifications of these products due to the lack of information on their past usage and maintenance history. In order to reduce the negative impact of this issue on the number of sales, the dealer often offers warranty. Through this warranty, the dealer and buyer share the risk of malfunctioning of the product.
2. Despite the fact that warranties for second-hand products are commonly used, the accurate pricing of warranties in many situations remains an unsolved problem for both the dealer and customer. This may seem surprising since the fulfilment of warranty claims may cost dealers a significant amount of money. Underestimating the true warranty costs can result in losses for the dealer. On the other hand, overestimating them will lead to uncompetitive prices, and as a result the amount of product sales will decrease. From the customer's perspective, the warranty is an investment aiming to reduce the risk of post-warranty early failures and the uncertainty of related servicing costs.

This article deals with statistical models for estimating the expected warranty cost for second-hand products sold with two-dimensional free repair/replacement warranty. The article proceeds as follows. In the next section, we carry out a comparison between warranty of new products and warranty of second-hand products. In Sec. 3, we present the model assumptions and notations used in the article. In Sec. 4, we give the model formulation. In Sec. 5, we derive the expected warranty cost per product to the dealer. In Sec. 6, a real-life example illustrates our findings. We conclude with brief summary of our results and outline some directions for future research in Sec. 7.

2. Warranty for New Product vs. Second-Hand Products

The literature on new product warranties has grown considerably with four important review articles, three books, and many journals and conference articles. Chukova et al. (1993), Thomas and Rao (1999), Murthy and Djamaludin (2002), and Karim and Suzuki (2005) are essential review articles on new product warranty. The article by Chukova et al. (1993) is a translation from Russian and it is the first warranty review article to appear in Eastern Europe. Thomas and Rao (1999) adopted a management perspective to warranty issues and focus on the works that address quantification of warranty costs and determination of warranty policies. The review article of Murthy and Djamaludin (2002) provided a list of 186 references on warranties for new products, which are classified into different categories. They also mentioned applications in some related areas such as law, accounting, economics, and sociology. Karim and Suzuki (2005) provided a recent survey of the literature on statistical models and methods for warranty analysis.

Although the concept of warranties for second-hand products has attracted significant attention among dealers, the academic research is very limited. Chattopadhyay and Murthy (1996) is the first article in this research area. Murthy and Chattopadhyay (1999) developed taxonomy for warranties for second-hand products. Chattopadhyay and Murthy (2000) discussed many aspects of one-dimensional free repair/replacement and pro-rata warranty policies for second-hand products. Chattopadhyay and Murthy (2001) provided three new cost sharing warranty policies for second-hand products and developed statistical models for their cost analysis. Chattopadhyay and Murthy (2004) developed simple models for reliability improvement of second-hand products sold with free repair warranty. Saidi-Mehrabad et al. (2010) developed two effective ways to decide on the optimal reliability improvement strategies for second-hand products sold under various warranty policies. Majority of these articles deal with one-dimensional (1D) policy, whereas the issues related to two-dimensional (2D) policy have received very limited attention.

The second-hand warranty is intended to assure the new buyer that the used item will perform its intended functions for a pre-specified period of time or amount of usage. Nowadays, many dealers of second-hand electronic equipment, furniture, automobiles, and home appliance offer warranty for their products. For example, in the early nineties the warranty period for used cars was 3 months/3,000 miles and currently it varies from 1 year/12,000 miles to 3 years/30,000 miles. In addition, some used car dealers offer warranty similar to the warranty of new cars (The Used Car Market Report, 2007).

The estimation of the warranty cost for second-hand products is quite involved due to the uncertainties regarding the age, usage, and past maintenance history of the items. The main factors which affect the warranty cost are:

(a) the age of the item—typically, the occurrence of a failure of an older item is more likely compare to occurrence of a failure of a newer item. Second-hand items, which have past their "use-by date" time, usually require significant amount of efforts and resources to be brought back to an acceptable working condition;
(b) the past usage of the item—the usage can be measured in different ways. For example, the usage can be measured as the number of copies produced by a photocopier, the distance travelled by a vehicle, the number of takeoffs and landings or the total hours flown for an aircraft, etc.;
(c) the warranty policy—by using an effective warranty servicing strategy the dealer can reduce the warranty costs;
(d) the maintenance strategy—by using an effective maintenance strategy the dealer can significantly reduce the warranty costs and prolong the lifetime of the second-hand item;
(e) the buyer's and the dealer's risk attitude.

Usually, the dealers of second-hand products need to estimate the warranty cost of their products and build it into the product price. Recently, in the context of rising second-hand market shares, the issue of estimating these costs has become essential. In addition, in many countries, such as the UK, the United States, and France, the law mandates that the second-hand dealers have to offer warranty on their products and provide a mechanism to ensure that related claims are effectively serviced.

3. Model Assumptions and Notations

In this section, we present the model assumptions and notations that will be used in the article.

3.1. *Assumptions*

In our further study, we make the following assumptions:

- The second-hand product is repairable.
- All repairs (during the past life of the item and the warranty coverage) are minimal. A minimal repair has no impact on the performance of the item. It simply brings the product from a "down" to an "up" state without affecting its performance.
- Failures are statistically independent.
- The reliability of a used item depends on its age and usage.
- Each failure results in a warranty claim and all claims are valid.
- All repairs/replacements are instantaneous.
- The second-hand products are not subjected to a pre-sale upgrade action.
- The repair cost is a random variable with known expected value.

3.2. *Notations*

The notations used in this article are given below.

Nomenclature			
A	past age of the second-hand item	$\Lambda_T(t) = \int_o^t \lambda_T(x)dx$	cumulative intensity function for product's age-wise failure
B	past usage of the second-hand item	Ω_i	two-dimensional warranty region
Ω_0	past life region $= [0, A] \times [0, B]$	t_0, u_0	warranty limits
(T, U)	two-dimensional random variable where $T =$ age and $U =$ usage	$H_{A,B}(a, b)$	joint cumulative distribution function (cdf) of the past age and usage
R	usage-rate	$h_{A,B}(a, b)$	joint probability density function (pdf) associated with $H_{A,B}(a, b)$
$F_R(r)$	cumulative distribution function of the usage rate	$E[C(\Omega_i; A, B)]$	the expected warranty cost
$\lambda_T(t)$	hazard rate for product's age-wise failure	C_r	the expected cost of each rectification action

4. Two-Dimensional Warranty

4.1. *Two-Dimensional Warranty Policies*

Warranty policy is a statement on the type and extent of the redress action offered by the manufacturer (or dealer) for their product. In general, there are two types of warranty contracts. In the first type—the extent of the contract depends on only one parameter, typically the age of the product, and it is characterized by the duration of the warranty period, which is an age dependent time interval. In the second type of contracts—the warranty coverage is given in terms of two parameters—typically the age and the usage of the product and the warranty coverage is characterized by a region Ω in 2D region. Different regions define different warranty policies; see Blischke and Murthy (1994) for a list of different possible shapes of Ω. In this article we consider two of these due to their simplicity in comprehension by the customers—the rectangular one and the L-shaped one, depicted in Fig. 1.

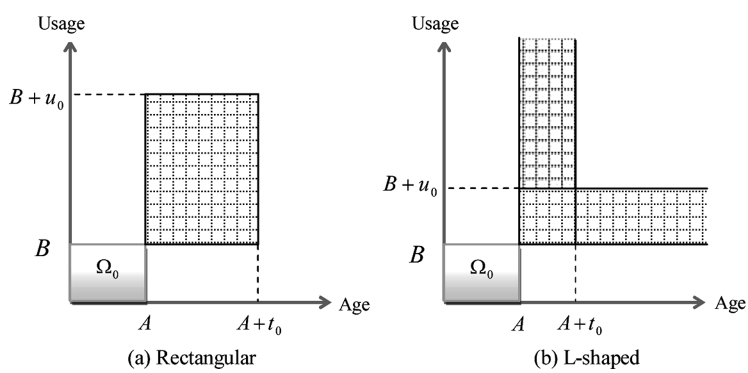

Figure 1. Two-dimensional warranty regions for second-hand products.

In order to specify the warranty regions, let us denote the age of the product at failure by T, and the cumulative usage of the product corresponding to T by U. Clearly, T and U are random variables. Usually, U is a covariate of T. We assume that on the XY plane, T is measured along the X-axis, and U is along the Y-axis. Then the two warranty regions are as follows:

- **Rectangular warranty region**–this warranty coverage is common in used car markets. It is characterized by the rectangular region: $\Omega_R(t_0, u_0) = \{(t, u) : t \in (A, A + t_0) \text{ and } u \in (B, B + u_0)\}$ for pre-specified limits t_0 and u_0 respectively on the age and usage of the used item.
- **L-shaped warranty region**–is defined by $\Omega_L(t_0, u_0) = \{(t, u) : t \in (A, A + t_0) \text{ or } u \in (B, B + u_0)\}$ for pre-specified limits t_0 and u_0. This coverage ensures warranty service for a minimum age of t_0 and a minimum usage of u_0.

It is well known that 1D regions are special cases of these two regions. Specifically, $\Omega_R(t_0, \infty) = \Omega_L(t_0, 0)$ and $\Omega_R(\infty, u_0) = \Omega_L(0, u_0)$ are 1D regions. In general, the pre-specified values of (t_0, u_0) differ from region to region, but for simplicity of notations, we will refer to these regions by Ω_R and Ω_L. When the pre-specified values of (t_0, u_0) are same for both the regions, then $\Omega_R \subseteq \Omega_L$.

In our study, we consider a free repair/replacement warranty policy (FRW). Under this policy all warranty repair/replacements within the warranty region are free of charge to the customer. The warranty ceases either at time limit t_0 or at usage limit u_0, whichever occurs first. For example, the car manufacturer General Motors (GM) offers FRW for its vehicles for 5 years or 100,000 miles, whichever occurs first (http://www.gm.com/explore/quality). The car company Hyundai also offers five years unlimited mileage free replacement warranty for its automobiles (http://www.hyundai.com.au/Warranty/default.aspx).

Second-hand products sold under failure-free warranties might include electronics such as large-screen color TVs, automobiles, refrigerators, or household appliances.

4.2. *Approaches to Modelling Warranty Claims*

Most products (for example, cars) can be viewed as systems comprising of several components. Whenever a component fails within the warranty period, it affects the performance of the system and results in a claim. These failures occur in an uncertain manner and can be characterized in terms of the reliability of the components.

There are three main approaches to modeling failures for products sold with two-dimensional warranties.

Approach 1. In this approach, the time to first failure is modelled by a bivariate distribution function $F(t, u)$. If failed items are replaced by new ones and replacement times are negligible, then failures over the warranty region occur according to a two-dimensional renewal process.

Approach 2. In this approach, the two measures—age and usage—are combined to provide a single composite scale z (e.g., $z = at + bu$) and the failure process is modelled using this new composite scale.

Table 1
A taxonomy for two-dimensional warranty cost modeling (since 1998)

Author/s	Year	Abstract	Approach #
Gertsbakh and Kordonsky	1998	– Develops an analysis of two-dimensional warranty cost based on a composite scale.	Approach 2
Chun and Tang	1999	– Assumes a linear function between age and usage to build models for warranty cost analysis.	Approach 3
Kim and Rao	2000	– Considers a simple bivariate exponential (BVE) distribution to describe the relationship between the two warranty variables.	Approach 1
Duchesne and Lawless	2000	– Provide a general framework and new results concerning time scale specification.	Approach 2
Yang and Nachlas	2001	– Provides a foundation for bivariate reliability modeling and shows how to define bivariate renewal models that can support maintenance planning.	Approach 1
Pal and Murthy	2003	– Presents an application of Gamble's bivariate exponential distribution in the context of estimating warranty costs of motor cycles under a new warranty policy.	Approach 1
Baik et al.	2004	– Extends the concept of minimal repair for the one-dimensional case to the two-dimensional case and carry out a study of the failures under minimal repair.	Approach 1
Rai and Singh	2005	– Presents a simple method to assess the impact of new time/mileage warranty limits on the number and cost of warranty claims for components/sub-systems of a new product.	Approach 3
Manna et al.	2006	– Proposes a methodology to determine the optimal region when customers' utility is measured by the length of warranty coverage time.	Approach 3
Baik et al.	2006	– Points out the reason for an error in the derivation of a result in Baik et al. (2004).	Approach 1

Chukova and Johnson	2006	– Focuses on a particular warranty repair strategy, related to the degree of the warranty repair, for non-renewing, two-dimensional, free of charge to the consumer warranty policy.	Approach 3
Manna et al.	2007	– Proposes a new methodology for the construction of failure model indexed by age and usage. Studies the effect of usege-rate on component failure.	Approach 3
Jung and Bai	2007	– Considers a method of estimating the bivariate lifetime distribution of products under two-dimensional warranty and investigates the properties of the estimators.	Approach 1
Majaske	2007	– Presents a non-homogenous Poisson process (NHPP) predictive model for automobile warranty claims consisting of two components: a population size function and a failure or warranty claim rate.	Approach 3
Manna et al.	2008	– Derives the warranty cost under rectangular two-dimensional policy for both repairable and non-repairable product.	Approach 3
Corbu et al.	2008	– Considers non-renewing, two-dimensional, free of charge to the consumer warranty policy. Derives a numerical procedure for computing 2D renewal function.	Approach 1
Jack et al.	2009	– Develops a repair–replace strategy for the manufacturer of a product sold with a two-dimensional warranty.	Approach 3

Approach 3. This approach assumes that the product usage rate R varies from customer to customer, but it is a constant for a given customer. Then, it can be assumed that the usage rate R is a random variable with given cumulative distribution function $F_R(r)$, $0 \leq r < \infty$.

In Table 1, we present a summary and brief overview of the recent publications on these approaches.

5. Model Formulation

In this section, using Approach 3, we evaluate the expected warranty cost of a second-hand product under non-renewing, two-dimensional, free of charge to the consumer warranty policy. We consider two types of warranty coverage represented by a rectangular or by L-shaped region.

In the case of two-dimensional warranty, firstly we need to model the effect of the age and usage on the product degradation and model the failure process. In this study, using our assumptions in Sec. 3.1, we model the failure process by a non-homogeneous Poisson process (NHPP), i.e., we assume that all repairs are minimal. Secondly, we need to model the relationship between age and usage. Here, we assume that this relationship is a linear one, i.e.,

$$U = RT, \tag{1}$$

where R is a positive random variable, representing the usage rate, with cumulative distribution function $F_R(r)$.

Let $N(\Omega_i; A, B)$ be the number of warranty claims for any of the two warranty regions Ω_i $(i = R, L)$, given that at the time of the sale, the product age is A and product usage is B. Then the warranty cost is

$$C(\Omega_i; A, B) = \begin{cases} \sum_{j=1}^{N(\Omega_i; A, B)} C_{r_j} & \text{if } N(\Omega_i; A, B) > 0 \\ 0 & \text{if } N(\Omega_i; A, B) = 0 \end{cases}, \tag{2}$$

where C_{r_j} denotes the cost of the jth repair. The warranty cost $C(\Omega_i; A, B)$ is a two-dimensional random variable because the repair cost C_{r_j} and the number of failures over the warranty region Ω_i, $N(\Omega_i; A, B)$, are random variables. Note that at the point (A, B) the ownership of the product shifts from the end user to a new user. The new owner of the product might have a different time/usage pattern as it is shown in Fig. 2.

Let C_r denote the expected rectification cost. Then, the expected warranty cost for a second-hand product with past age A and past usage B at sale is given by

$$E[C(\Omega_i; A, B)] = C_r \cdot E[N(\Omega_i; A, B)]. \tag{3}$$

Blischke and Murthy (1994)) and Chun and Tang (1999) showed that, under Approach 3, by conditioning on the usage rate $R = r$, the computation of the expected warranty cost under 2D warranty can be reduced to estimating this cost in 1D settings. Based on this idea, we derive $E[C(\Omega_i; A, B)]$ for each of the regions Ω_R and Ω_L.

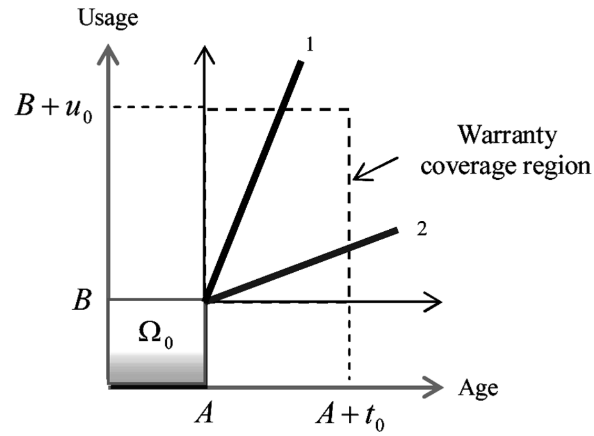

Figure 2. The usage rate varies from customer to customer.

- **Rectangular region (Ω_R)**–For any $(t, u) \in \Omega_i$, consider the two possible two cases: $r < \eta$, and $r \geq \eta$, where $\eta = \frac{u_0}{t_0}$. If the usage rate $r < \eta$, then the warranty expires at age $A + t_0$ and an estimate of the total usage is $B + rt_0$. On the other hand, if r is greater than η, the warranty expires at age $A + \frac{u_0}{r}$ when the usage limit u_0 is reached (see Fig. 3). Given $R = r$, we get

$$E\left[N(\Omega_R \mid r, A, B)\right] = \begin{cases} \Lambda_T(A + t_0) - \Lambda_T(A) & r < \eta \\ \Lambda_T\left(A + \dfrac{u_0}{r}\right) - \Lambda_T(A) & r \geq \eta \end{cases} \tag{4}$$

where $\Lambda_T(t) = \int_0^t \lambda_T(t)$ for $t \geq 0$. It then follows that

$$E\left[N\left(\Omega_R; A, B\right)\right] = \int_0^{\eta} \left[\Lambda_T(A + t_0) - \Lambda_T(A)\right] \times f_R(r)dr + \int_{\eta}^{\infty} \left[\Lambda_T\left(A + \frac{u_0}{r}\right) - \Lambda_T(A)\right] \times f_R(r)dr$$

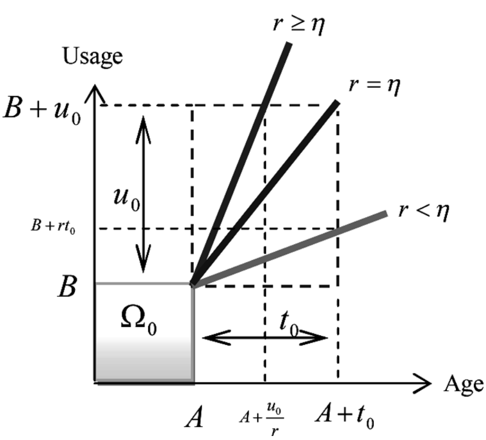

Figure 3. Warranty coverage region with $R = r$.

$$= [\Lambda_T(A+t_0) - \Lambda_T(A)] \times F_R(\eta) + \int_\eta^\infty \left[\Lambda_T\left(A+\frac{u_0}{r}\right) - \Lambda_T(A)\right] \cdot dF_R(r). \quad (5)$$

- **L-shaped** ($\mathbf{\Omega}_L$)–observe that

$$E\,[N(\Omega_L \mid r, A, B))] = \begin{cases} \Lambda_T\left(A+\frac{u_0}{r}\right) - \Lambda_T(A) & r < \eta \\ \Lambda_T(A+t_0) - \Lambda_T(A) & r \geq \eta \end{cases}. \quad (6)$$

Proceeding similarly as in the case of Ω_R, we get:

$$\begin{aligned} E\,[N\,(\Omega_L; A, B)] &= \int_0^\eta \left[\Lambda_T\left(A+\frac{u_0}{r}\right) - \Lambda_T(A)\right] \times f_R(r)dr \\ &\quad + \int_\eta^\infty [\Lambda_T(A+t_0) - \Lambda_T(A)] \times f_R(r)dr \\ &= \int_0^\eta \left[\Lambda_T\left(A+\frac{u_0}{r}\right) - \Lambda_T(A)\right] \cdot dF_R(r) \\ &\quad + [\Lambda_T(A+t_0) - \Lambda_T(A)]\,(1 - F_R(\eta))\,. \end{aligned} \quad (7)$$

6. Joint Distribution of the Past Age and Usage

In order to attract more customers, the dealers must keep the product price as low as possible, but at the same time, in order to ensure reasonable profit they have to minimize the total expected servicing cost per product. Dealers of second-hand products buy and sell products with different ages and usages. The dealer's expected warranty costs for older products are higher than those of younger products. If these high warranty costs are built into the sale price, then older products become unattractive to the new buyers. To make the sale price of the older products attractive to the new buyers the dealer have to subsidize older products by charging less than the expected warranty cost and recover the subsidy from younger products by charging more than the expected warranty cost (Chattopadhyay and Murthy, 2000).

We model this by viewing (A, B) as a joint random with joint distribution function $H_{A,B}(a, b)$. Let $h_{A,B}(a, b) = \frac{d^2}{da.db} H_{A,B}(a, b)$ denote the density function associated with $H_{A,B}(a, b)$. On carrying out the expectation over (A, B), we have the total expected cost per product under warranty with limits t_0 and u_0, $E[C(\Omega_i; t_0, u_0)]$, given by

$$E\,[C\,(\Omega_i; t_0, u_0)] = \int_{(a,b)\in\Omega_0} E\,[C\,(\Omega_i; a, b)] \times h_{A,B}(a, b)da\,db. \quad (8)$$

Using $E[C(\Omega_i; t_0, u_0)]$ instead of $E[C(\Omega_i; A, B)]$ in the price decision implies that the dealer is charging equally for the warranty of all second-hand products regardless of their age and usage. This implies that buyers of younger products are subsidizing the buyers of older products. From the dealer's point of view, this is a better strategy, because determining the price using $E[C(\Omega_i; t_0, u_0)]$ implies that all products regardless of their age and usage will have comparable sale prices.

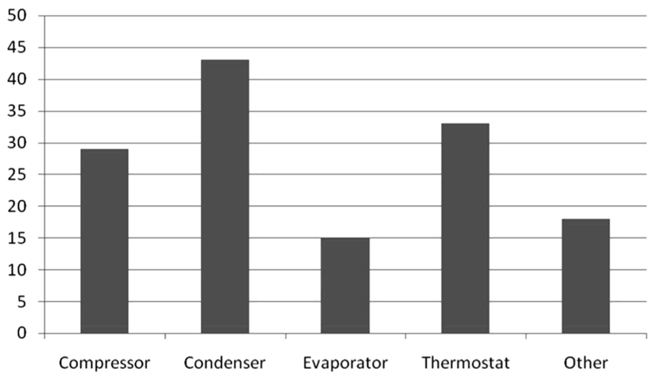

Figure 4. Failure frequencies from the warranty claims.

7. Application to a Real Life Example

This section presents an application of the proposed models to a problem faced by a second-hand refrigerator dealer. Statistical analysis was performed on data regarding a particular brand and model of used refrigerator. During the study period of $\tau = 95$ days (from September 20, 2008 until December 23, 2008), the total number of sold refrigerators (n) is 615, and the total number of warranty claims (f) is 138. Field data shows that failures occur mainly due to a several major parts of the refrigerator: the compressor ($f_1 = 29$ claims), the condenser ($f_2 = 43$ claims), the evaporator ($f_3 = 15$ claims), the thermostat ($f_4 = 33$ claims) and other parts such as door seals ($f_5 = 18$ claims). Figure 4 depicts the frequencies of the warranty claims due to the failures of these major parts.

The dealer considers grouping the components of the used refrigerator into two disjoint sets—I (included in warranty) and E (excluded in warranty). Under this policy, all failures of components from set I are rectified at no cost to the buyer over the warranty period. The costs of rectifying failed components belonging to set E are borne by the new buyer. The set I is identified as follows:

$$\{1.\ \text{Compressor},\ 2.\ \text{Condenser},\ 3.\ \text{Evaporator},\ 4.\ \text{Thermostat}\} \in \text{I}.$$

The dealer has decided to cover these parts by a rectangular or L-shaped free repair warranty. Due to the fact that the condensers have the highest failure frequency (see Fig. 4), the company desires to keep its average repair costs under control by effectively managing the repair process. The age of the condensers is measured in years and the unit for usage is 1,000 h, e.g., warranty with $t_0 = 2$ and $u_0 = 0.75$ implies that the coverage for maximum of two years and usage of 750 h.

Based on the available data, it turns out that the best-fit for the failure time of the condensers is two-parameter Weibull distribution with $\lambda_T(t) = \lambda\beta(\lambda t)^{\beta-1}$, $t \geq 0$ and $\Lambda_T(t) = (\lambda t)^{\beta}$. The method of maximum likelihood is followed for estimation of the model parameters. The observation on life T for ith failed condensers is denoted by t_i, $i = 1, 2, \ldots, f_2$. The maximum likelihood estimates (MLEs) for Weibull distribution are obtained by solving the following two equations (Lawless, 1982):

$$\frac{1}{\beta} + \frac{1}{f_2}\sum_{i=1}^{f_2} \ln t_i - \frac{\sum_{i=1}^{f_2} t_i^{\beta} \ln t_i + (n - f_2)\tau^{\beta} \ln\tau}{\sum_{i=1}^{f} t_i^{\beta} + (n - f_2)\tau^{\beta}} = 0 \tag{9}$$

$$\frac{\sum_{i=1}^{f_2} t_i^{\beta} + (n - f_2)\tau^{\beta}}{f_2} = \left(\frac{1}{\lambda}\right)^{\beta}. \tag{10}$$

The MLEs are given by $\beta = 1.1052$ and $\lambda = 0.0258$/year. Also, it turns out that the best-fit for the usage rate of the condensers is the Lognormal distribution with $f_R(r) = \frac{1}{r\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln r - \mu}{\sigma}\right)^2\right]$ and parameters of $\mu = 0.1519$ and $\sigma = 0.4472$. Therefore, the cdf of R is $F_R(r) = \Phi\left(\frac{\ln r - \mu}{\sigma}\right)$, where symbol $\Phi(\cdot)$ represents the cdf of standard normal variate.

By using (5) and (7), we obtain that the corresponding expected warranty costs are expressed as follows:

$$E\left[C(\Omega_R; A, B)\right] = C_r\lambda^{\beta}\left[\left\{(t_0 + A)^{\beta} - A^{\beta}\right\} \cdot \Phi\left(\frac{\ln - \mu}{\sigma}\right) + \left\{(u_0 + B)^{\beta} - B^{\beta}\right\} \cdot \exp\left\{-\beta\mu + \frac{(\beta\sigma)^2}{2}\right\} \cdot \left\{1 - \Phi\left(\frac{\ln\eta - \mu}{\sigma} + \beta\sigma\right)\right\}\right] \tag{11}$$

and

$$E\left[C(\Omega_L; A, B)\right] = C_r\lambda^{\beta}\left[\left\{(t_0 + A)^{\beta} - A^{\beta}\right\} \cdot \left\{1 - \Phi\left(\frac{\ln\eta - \mu}{\sigma}\right)\right\} + \left\{(u_0 + B)^{\beta} - B^{\beta}\right\} \cdot \exp\left\{-\beta\mu + \frac{(\beta\sigma)^2}{2}\right\} \cdot \Phi\left(\frac{\ln\eta - \mu}{\sigma} + \beta\sigma\right)\right]. \tag{12}$$

Data indicates that repairs have average costs of \$100 d. Table 2 shows the expected warranty cost $E[C(\Omega_i; A, B)]$ for different values of A and B for second-hand refrigerators sold with a maximum coverage of 2 years and usage of 750 h on condensers.

Note that, as expected, the expected warranty cost increases with the increase of age/usage of the item.

Reliability and quality managers have noticed that about 21% of the sold refrigerators have less than 2 years of age and 400 h of usage. This is due to expiration of the original warranty. Therefore, the point $(A = 2, B = 0.4)$ is a change point at which 21% of the customers chose to change their used refrigerators for a new one. After this point, the new buyer has different time/usage pattern.

Table 2
$C(\Omega_i; A, B)$ for different combinations of A and B for second-hand refrigerators sold with 2D FRW policy

	Rectangular			L-shaped		
	$B = 0.5$	$B = 0.75$	$B = 1.0$	$B = 0.5$	$B = 0.75$	$B = 1.0$
$A = 2.0$	1.370	1.406	1.425	4.501	4.502	4.504
$A = 3.0$	1.371	1.408	1.437	4.609	4.609	4.611
$A = 4.0$	1.371	1.409	1.439	4.699	4.699	4.700
$A = 5.0$	1.371	1.409	1.440	4.776	4.776	4.776

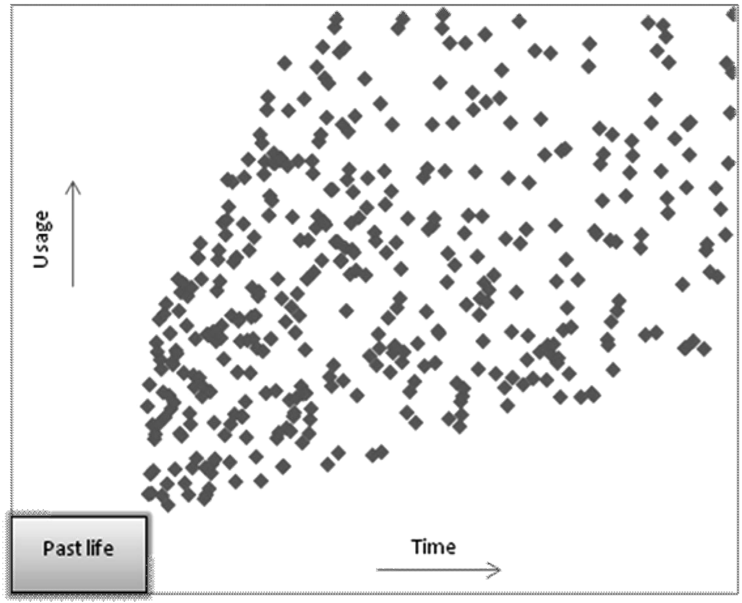

Figure 5. Usage pattern for the 615 buyers of second-hand refrigerators with $A = 2$ and $B = 0.4$.

Figure 5 shows scatter plot of the field data (T, U) for second-hand refrigerators with past age $A = 2$ and past usage $B = 0.4$.

Table 3 shows the expected warranty cost of the condensers $E[C(\Omega_i; A, B)]$ for different FRW limits t_0 and u_0 with past age of 2 years and past usage of 400 h.

These results reflect our findings from the statistical data analysis and our assumptions regarding the shape of the warranty regions, namely, Lognormal usage function, Weibull failure time, and rectangular or L-shaped regions. It is found that the dealer's preference is for the rectangular region against L-shaped region.

The correlation coefficient between time and usage is calculated as 0.749. In view of this correlation, the warranty cost analysis can be carries out solely based on the time dimension. Let us now consider the special case: $u_0 \to \infty$ for rectangular or $u_0 \to 0$ for L-shaped regions, that is, we are concerned with one-dimensional policy

Table 3
$E[C(\Omega_i; A, B)]$ for different combinations of FRW limits for a second-hand condenser with $A = 2$ and $B = 0.4$

	Rectangular			L-shaped		
	$u_0 = 0.5$	$u_0 = 0.75$	$u_0 = 1.0$	$u_0 = 0.5$	$u_0 = 0.75$	$u_0 = 1.0$
$t_0 = 1.0$	0.884	1.335	1.762	2.145	2.160	2.209
$t_0 = 1.5$	0.886	1.347	1.803	3.244	3.250	3.270
$t_0 = 2.0$	0.886	1.350	1.819	4.362	4.364	4.372
$t_0 = 2.5$	0.886	1.352	1.825	5.496	5.496	5.500

Table 4
The expected warranty cost for a second-hand condenser with $A = 2$ and $B = 0.4$ sold under 1D FRW policy

	Rectangular $u_0 \to \infty$	L-shaped $u_0 \to 0$
$t_0 = 1.0$	2.142	2.142
$t_0 = 1.5$	3.244	3.244
$t_0 = 2.0$	4.362	4.362
$t_0 = 2.5$	5.495	5.495

having no limitation on usage. Hence, Eqs. (11) and (12) can be re-written as:

$$E\left[C(\Omega_R; A, B)\right] = E\left[C(\Omega_L; A, B)\right] = C_r\lambda^\beta\left[(t_0 + A)^\beta - A^\beta\right] \cdot \left[1 - \Phi\left(\frac{\ln r_0 - \mu}{\sigma}\right)\right]. \quad (13)$$

Table 4 shows the expected warranty cost of the condensers with past age of 2 years and past usage of 400 h sold under one-dimensional FRW policy.

In order to derive the expected warranty cost per product we need a bivariate probability distribution function for the past age and past usage (A, B) of the refrigerators, called onwards the past area variables. Multivariate Pareto distributions, such as Beta Stacy distribution, can be considered as candidates for the joint distribution of the past area variables. Beta Stacy distribution is due to Mihram and Hultquist (1967) and its application in warranty analysis is proposed in Murthy et al. (1995).

Therefore, we can consider the joint pdf of (A, B) as

$$h_{A,B}(a, b) = \frac{(c/\varphi) \cdot k^{\alpha c}}{\Gamma(\alpha) \cdot B(\theta_1, \theta_2)} a^{\alpha c - \theta_1 - \theta_2}\left(\frac{b}{\varphi}\right)^{\theta_1 - 1}\left(a - \frac{b}{\varphi}\right)^{\theta_2 - 1} \cdot \exp[-(ka)^c], \quad (14)$$

where $B(\theta_1, \theta_2)$ is the beta function, $a > 0$, $0 < b < a\varphi$, and α, c, κ, ρ, θ_1, $\theta_2 > 0$.

The best way of obtaining the distribution parameters is by observing the past history of the sold refrigerators. There are two main approaches for estimating the parameters of Beta Stacy distribution: Bayesian approach (for more see, Bulla et al., 2009) and Monte Carlo simulation. Monte Carlo analysis has been successfully applied in a number of reliability studies (Hall and Strutt, 2003). In Monte Carlo simulation, each trial would includes an estimation of the parameters α, c, k, φ, θ_1, θ_2 for every observed set of past area variables a and b. A sample of 10,000 Monte Carlo simulations produced the following results: $\alpha = 2.6$, $c = 3.0$, $k = 5$, $\varphi = 1.1$, $\theta_1 = 1.2$, $\theta_2 = 1.1$. Also, to validate the distribution model, a goodness-of-fit test is carried out based on the chi-square method. The observed χ^2 test value is found to be equal to 7.07, whereas the critical value of χ^2, at 5% level of significance is 18.31.

Table 5 shows the expected warranty cost per unit of the product for different combinations of FRW limits.

On comparing Table 5 with Table 3, we see that for $t_0 = 2$ years and $u = 1.0$, $E[C(\Omega_R; t_0, u_0)]$ is 1.864, and $E[C(\Omega_R; A = 2, B = 0.4)] = 1.819$. Also, we can

Table 5
$E[C(\Omega_i; t_0, u_0)]$ for different values of the FRW limits for second-hand condensers with joint past age and usage distribution as Eq. (14)

	Rectangular			L-shaped		
	$u_0 = 1.0$	$u_0 = 1.5$	$u_0 = 2.0$	$u_0 = 1.0$	$u_0 = 1.5$	$u_0 = 2.0$
$t_0 = 2.0$	1.864	2.763	3.528	4.420	4.520	4.778
$t_0 = 3.0$	1.875	2.845	3.769	6.701	6.729	6.828
$t_0 = 4.0$	1.877	2.864	3.850	9.041	9.049	9.086
$t_0 = 5.0$	1.878	2.866	3.875	11.423	11.425	11.439

obtain $E[C(\Omega_R; A = 6, B = 0.4)] = 1.950$. This implies that the shortfall of 0.86 for a 6-year old item is partly subsidized by the 0.45 excess from a 2-year old item.

8. Conclusions and Topics for Future Research

The use of warranty is a relatively new strategy for dealers of second-hand items. This article proposes statistical models for estimating the expected warranty costs for two particular types of two-dimensional warranty for second-hand products. These are illustrated by a real-life example. The models developed in this article can be useful for dealers of second-hand products such as automobiles, home appliances personal computers, electronic equipment, and used helicopters.

There are many open questions and number of opportunities for future research related to warranty of second-hand products, such as:

(a) developing a taxonomy for two-dimensional warranties for second-hand products;
(b) developing several alternative warranty regions, so every new buyer can choose the one that suits him/her best. In the literature this type of policy is referred as *flexible* warranty policy;
(c) determining the optimal price and warranty policy for second-hand products to maximize the dealer's total expected profit;
(d) developing models (using Approach 1) to incorporate the past usage and maintenance history of a second-hand product;
(e) modeling the failures and warranty costs for three- or higher-dimensional warranties. These are offered for used acquisitions (used fleet of tanks or jet fighter engines) and can be used for managerial decisions in purchasing second-hand government acquisitions.

The authors are currently studying some of these.

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References

Baik, J., Murthy, D. N. P., Jack, N. (2004). Two-dimensional failure modeling with minimal repair. *Nav. Res. Logist.* 51(3):345–362.

Baik, J., Murthy, D. N. P., Jack, N. (2006). Erratum: two-dimensional failure modeling with minimal repair. *Nav. Res. Logist.* 53:115–6.

Blischke, W. R., Murthy, D. N. P. (1994). *Warranty Cost Analysis*. New York: Marcel Dekker.

Bulla, P., Muliere, P., Walker, S. (2009). A Bayesian nonparametric estimator of a multivariate survival function. *J. Statist. Plann. Infer.* 139:3639–3648.

Car Internet Research Program II (CIRP II) (2008). *The Anatomy and Physiology of the Used Car Business, A Global Analysis With Particular Reference to Four Key Markets: U.S., Canada, France, German.*

Chattopadhyay, G. N., Murthy, D. N. P. (1996). Modelling warranties for second-hand products. *Proc. Second-hand Australia Japan workshop on Stochastic Models in Engineering Technology and Management*. Gold Coast, Australia, July, pp. 91–99.

Chattopadhyay, G. N., Murthy, D. N. P. (2000). Warranty cost analysis for second-hand products. *Math. Comput. Model.* 31:81–88.

Chattopadhyay, G. N., Murthy, D. N. P. (2001). Cost sharing warranty policies for second-hand products. *Int. Trans. Operat. Res.* 8.1:47–60.

Chattopadhyay, G. N., Murthy, D. N. P. (2004). Optimal reliability improvement for used items sold with warranty. *Int. J. Reliab. Appl.* 5.2:47–57.

Christozov, D., Chukova, S., Mateev, P. (2009). On two types of warranties: warranty of malfunctioning and warranty of misinforming. *Asia Pac. J. Operat. Res.* 26(3):1–22.

Chukova, S., Dimitrov, B., Rykov, V. V. (1993). Warranty analysis. *J. Soviet Math.* 67:3486–3508.

Chukova, S., Johnston, M. R. (2006). Two-dimensional warranty repair strategy based on minimal and complete repairs. *Math. Comput. Model.* 44:1133–1143.

Chun, Y. H., Tang, K. (1999). Cost analysis of two-attribute warranty policies based on product usage rate. *IEEE Trans. Eng. Manage.* 46(2):201–209.

Corbu, D., Chukova, S., Sullivan, J. O. (2008). Product warranty: modelling with 2D-renewal process. *Int. J. Reliab. Safefy* 2(3):209–220.

Duchesne, T., Lawless, J. F. (2000). Alternative time scales and failure time models. *Lifetime Data Anal.* 6:157–79.

Gertsbakh, I. B., Kordonsky, K. B. (1998). Parallel time scales and two-dimensional manufacturer and individual customer warranties. *IIE Trans.* 30:1181–9.

Hall, P., Strutt, J. (2003). Probabilistic physics-of-failure models for component reliability using Monte Carlo simulation and Weibull analysis: a parametric study. *Reliab. Eng. Syst. Safety* 80:233–42.

Jack, N., Iskandar, B. P., Murthy, D. N. P. (2009). A repair–replace strategy based on usage rate for items sold with a two-dimensional warranty. *Reliab. Eng. Syst. Safety* 94:611–617.

Jung, M., Bai, D. S. (2007). Analysis of field data under two-dimensional warranty. *Reliab. Eng. Syst. Safety* 92:135–143.

Karim, M. R., Suzuki, K. (2005). Analysis of warranty claim data: a literature review. *Int. J. Qual. Reliab. Manage.* 22(7):667–686.

Kim, C. S., Djamaludin, I., Murthy, D. N. P. (2004). Warranty and discrete preventive maintenance. *Reliab. Eng. Syst. Safety* 84(3):301–9.

Kim, H. G., Rao, B. M. (2000). Expected warranty cost of two-attribute free-replacement warranties based on a bivariate exponential distribution. *Comput. Indust. Eng.* 38:425–434.

Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*. New York: John Wiley & Sons.

Majeske, K. D. (2007). A non-homogeneous Poisson process predictive model for automobile warranty claims. *Reliab. Eng. Syst. Safety* 92:243–251.

Manna, D. K., Pal, S., Sinha, S. (2006). Optimal determination of warranty region for 2D policy: a customers' perspective. *Comput. Indust. Eng.* 50:161–174.

Manna, D. K., Pal, S., Sinha, S. (2007). A use-rate based failure model for two-dimensional warranty. *Comput. Indust. Eng.* 52:229–240.

Manna, D. K., Pal, S., Sinha, S. (2008). A note on calculating cost of two-dimensional warranty policy. *Comput. Indust. Eng.* 54:1071–1077.

Mihram, G. A., Hultquist, R. A. (1967). A bivariate warning-time/failure-time distribution. *J. Amer. Statist. Assoc.* 62:589–599.

Murthy, D. N. P., Blischke, W. R. (2000). Strategic warranty management: a life-cycle approach. *IEEE Trans. Eng. Manage.* 47:40–54.

Murthy, D. N. P., Chattopadhyay, G. N. (1999). Warranties for second-hand products. *Proc. Ninth Int. Conf. Flexible Automat. Intell. Manufactur. (FAIM)*, Tilburg, Netherland, June, pp. 1145–1159.

Murthy, D. N. P., Djamaludin, I. (2002). New product warranty: a literature review. *Int. J. Prod. Econ.* 79:231–260.

Murthy, D. N. P., Iskandar, B. P., Wilson, R. J. (1995). Two-dimensional failure-free warranty policies: two-dimensional point process models. *Operat. Res.* 43(2):356–366.

Pal, S., Murthy, G. S. R. (2003). An application of Gumbel's bivariate exponential distribution in estimation of warranty cost of motorcycles. *Int. J. Qual. Reliab. Manage.* 20:488–502.

Rai, B., Singh, N. (2005). A modeling framework for assessing the impact of new time/mileage warranty limits on the number and cost of automotive warranty claims. *Reliab. Eng. Syst. Safety* 88:157–169.

Saidi-Mehrabad, M., Noorossana, R., Shafiee, M. (2010). Modeling and analysis of effective ways for improving the reliability of second-hand products sold with warranty. *Int. J. Adv. Manuf. Technol.* 46:253–265.

The Used Car Market Report (2007), BCA group; Produced in association with Sewells Information and Research. http://www.sewells.co.uk; accessed 3 November 2008.

Thomas, M. U., Rao, S. S. (1999). Warranty economic decision models: a summary and some suggested directions for future research. *Operat. Res.* 47(6):807–820.

Yang, S. C., Nachlas, J. A. (2001). Bivariate reliability and availability modeling. *IEEE Trans. Reliab.* 50:26–35.