Int J Adv Manuf Technol (2010) 46:253–265
DOI 10.1007/s00170-009-2084-x

ORIGINAL ARTICLE

Modeling and analysis of effective ways for improving the reliability of second-hand products sold with warranty

Mohammad Saidi-Mehrabad · Rassoul Noorossana · Mahmood Shafiee

Received: 2 February 2009 / Accepted: 29 April 2009 / Published online: 20 May 2009

Abstract Often, customers are uncertain about the performance and durability of the used/second-hand products. The warranties play an important role in reassuring the buyer. Offering the warranty implies that the dealer incurs additional costs to service any claims made by the customers. Reducing warranty costs is an issue of great interest to dealers. One way of improving the reliability and reducing the warranty servicing cost for second-hand items is through actions such as overhaul and upgrade which are carried out by the dealer or a third party. Improving actions allow the dealer to offer better warranty terms and to sell the item at a higher price. This paper deals with two effective approaches (virtual age approach and screening test approach) to decide on the reliability improvement strategies for second-hand products sold under various warranty policies (failure-free, rebate warranty, and a combination of free replacement and lump sum). A numerical example illustrates that from a dealer's point of view, it is beneficial to carry out an improvement action only if the reduction in the warranty servicing cost is greater than the extra cost incurred due to this improvement action.

M. Saidi-Mehrabad · R. Noorossana · M. Shafiee
Department of Industrial Engineering,
Iran University of Science and Technology,
Narmak Street, P.O. Box 16765-63, Tehran, Iran

M. Saidi-Mehrabad
e-mail: meharabad@iust.ac.ir

R. Noorossana
e-mail: rassoul@iust.ac.ir

M. Shafiee (✉)
Department of Industrial Engineering,
Iran University of Science and Technology,
P.O. Box 16846-13114, Tehran, Iran
e-mail: shafiee@iust.ac.ir

Keywords Free replacement · Reliability improvement · Second-hand product · Virtual age · Warranty policy

1 Introduction

The importance of the used/second-hand product market as a fraction of the total market (new + second-hand) has been growing significantly since the beginning of the twenty-first century. Second-hand products include products that have previously been used by an end user/consumer. Users change their products even if they are still in good condition. Some products such as computers and mobile phones have a short lifetime and technologies of these products are released to the market every day. As a result, the sale of new products is often tied to a trade-in, resulting in a market for second-hand products. For instance, in France, used car unit sales increased from 4.7 million to 5.4 million between 1990 and 2005, at the same time as new car sales declined from 2.3 million to 2.07 million units [1]. Table 1 looks at new car and used car sales by unit volume for eight countries in 2007. Used car market is most active in the UK, the USA, and France. In these markets, the used car business generates considerable economic income. In contrast, the market is weaker in Canada, Spain, and Japan.

Research shows that the process of buying a second-hand product is generally shorter compared with buying a new one. For example, in the US, buying a new car takes on average 6 months, whereas buying a used car takes just 2.31 months [2].

In spite of increasing the market share for second-hand products, often, customers are uncertain about the performance and durability of these products due to the lack of knowledge related to past usage and maintenance history. To reduce this uncertainty and increase sales, dealers are

Table 1 Used vehicle vs. new vehicle market sales—a national comparison

	UK	U.S	France	Germany	Italy	Canada	Spain	Japan
New car sales	2,567,000	16,995,000	2,070,000	3,320,000	2,262,383	1,583,000	1,517,490	5,852,067
Used car sales	7,701,308	44,136,000	5,400,000	6,650,000	4,586,894	2,300,000	2,080,754	5,984,800
Used car/new car ratio	3.0	2.6	2.6	2.0	2.0	1.5	1.4	1.0

offering warranty to customers. Offering the warranty implies that the dealer incurs additional costs to service any claims made by the customers. For example, according to Nasser et al. [3], on average, General Motors spends approximately \$3.5 billion per year (roughly 22.5 million warranty claims) paying dealerships to repair failed parts under warranty.

Reducing warranty costs is an issue of great interest to dealers. One way of improving the reliability and reducing the warranty servicing cost for second-hand items is through actions such as overhaul and upgrade which are carried out by the dealer or a third party. Determination of the optimal reliability improvement strategy is an optimization problem in which both costs and benefits from the dealer's viewpoint should be considered. On one hand, providing upgrade action is usually costly and adds directly to the sale price of the second-hand product, and on the other hand, upgrade action can reduce the warranty cost.

This paper deals with two effective ways (virtual age approach and screening test approach) to decide on the reliability improvement strategies for second-hand products sold under various warranty policies (failure-free, rebate warranty, and a combination of free replacement and lump sum). Our stochastic models make a useful contribution to the reliability literature as they develop applied reliability improvement strategies for the dealers of second-hand products. In our study, distribution is assumed to follow the Weibull distribution. It is well known that Weibull distribution is one of the distributions often used in modeling the failure times in reliability studies [4, 5]. It can provide a flexible model for an increasing failure rate as well as decreasing failure rate by appropriate selection of parameter values.

The rest of this paper is organized as follows. In Section 2, we carry out a review of the literature dealing with warranties and reliability improvement. Following this, we present the model assumptions and notations that will be used in the paper in Section 3. In Section 4, we deal with two stochastic models for estimating cost of providing improvement programs for repairable/non-repairable second-hand products. In this section, we also examine improvement level and warranty length to achieve a sensible trade-off between the improvement cost and reduction of the expected warranty cost. We consider the distribution of the past age to derive the total mean cost per product in Section 5. There are many topics that need further study and we discuss these in the last section.

2 Literature review

2.1 A brief review of literature on warranties for second-hand items

A significant amount of academic research has been conducted in modeling warranty policies and costs for new products. The review article of Murthy and Djamaludin [6] provides a list of 186 references. See also Thomas and Rao [7]. In contrast, a brief review of the literature shows that only few researchers have worked in the area of warranties for second-hand products. Chattopadhyay and Murthy [8] propose an early paper for this important research area. Murthy and Chattopadhyay [9] develop a taxonomy for warranties for second-hand products. Chattopadhyay and Murthy [10] discuss many important aspects of free repair/replacement warranty policies for second-hand products. Chattopadhyay and Murthy [11] provide three new cost sharing warranty policies for second-hand products. Also, some stochastic models have been developed and analyzed for the cost of these warranty policies. Chattopadhyay and Murthy [12] develop two simple models for reliability improvement of second-hand products. Saidi et al. [13] develop an analytical regression model to estimate the sale price of a second-hand product sold under two-dimensional free repair/replacement warranty. Chattopadhyay et al. [14] develop a stochastic model which assumes that buyers of the second-hand items are heterogeneous with a random risk-aversion parameter in their risk attitudes towards uncertain repair costs. They derive the optimal price and warranty duration based on a utility model which maximizes the dealer's expected profit.

2.2 A brief review of literature linking warranty and reliability improvement

Reliability improvement for new products has also received some attention in the reliability literature. A chronological review of models dealing with warranty and reliability improvement is shown in Table 2.

Majority of these articles deal with new products, but second-hand products have received very limited attention. When noticing the limitations of the previous research, this study develops two stochastic models in which the first one is based on Kijima's virtual age model [15] and the second one is more applicable in today's competitive markets.

3 Assumptions and notation

In our modeling, we consider three decision variables linked to the following three lifetime stages of a second-hand product (see Fig. 1).

Past age The dealer purchases the second-hand product from an end user. Often, the dealer has knowledge of the age of the product. These are usually obtained from sources such as registration forms and/or log books [10]. For instance, in the used car markets in France and Germany, about 10% to 11% of sold used vehicles have less than 1 year of age, while it is about 2% in the US market. Table 3 shows the share of total used car sales by vehicle age in four main countries.

Note that at the end of this stage, the ownership shifts from the end user to the dealer.

Reliability improvement Dealers of second-hand products must ensure that the products they supply are in compliance with the safety requirements and regulation. Sometimes, second-hand products have high failure rate just after the purchase by the new buyer. Low-quality second-hand product can be harmful for its user. Therefore, dealers should arrange some effective ways to certify the safety of the products and prevent any danger that may arise from the use of the products. To reduce possible damage from early failures, one way is to perform upgrade action before the warranty starts. This provides an opportunity to improve the reliability of the used items and to reduce the warranty servicing cost. Upgrade action allows the dealer to offer better warranty terms and sell the item at a higher price.

Warranty period The warranties play an important role in reassuring the buyers in their purchase decision. Nowadays, warranties for second-hand product are becoming increasingly important in consumer and commercial trans-

Table 2 A review of the literature linking warranty and reliability improvement

Author/s	Year	Abstract
Davis [24]	1952	Deals with the reliability characteristics of reconditioned bus engines and finds that the reliability improves after each reconditioning.
Malik [25]	1979	Introduces the main concept of "improvement factor" for modeling repairable systems.
Murthy and Nguyen [26]	1987	Develops three different models (models I–III) for determining optimal reliability improvement taking into account the impact of reliability on the expected warranty cost.
Kijima et al [27]	1988	Introduces the main concept of "virtual age" and calls it the "type I Kijima" where the improvement results in a reduction in the age of the system.
Kijma [15]	1989	Generalizes the "type I Kijima" and presents a second virtual age model "type II Kijima".
Jack and Dagpunar [28]	1994	Uses a virtual age model to determine the quality and the period between overhauls.
Zhang and Jardine [29]	1998	Proposes a failure rate model, where after an overhaul, the item performance is between as good as before and as good as after the previous overhaul.
Djamaludin et al. [30]	2001	Develops a framework to study preventive policies when the vendor offers an initial period of warranty and pays labor, materials and downtime costs if a failure occurs.
Hussain and Murthy [31]	2003	Develops a model for failure rate reduction, when the outcome of the improvement process is uncertain.
Kim et al. [32]	2004	Develops a strategy to determine maintenance policies over the warranty duration following Kijima's virtual age model.
Dimitrov et al. [33]	2004	Develops an age-dependent repair model and evaluates the expected warranty costs under different warranty scenarios.
Pascual and Ortega [34]	2006	Uses a virtual age model to determine optimal intervals between overhauls by minimizing global maintenance costs.
Chien [35] 2008	2008	Investigates the impact of an imperfect renewing free-replacement warranty on the age-replacement policy with an increasing failure rate.
Guida et al. [36]	2009	Proposes a Bayesian procedure to formalize the prior information available about the failure probability of an upgraded automotive component.

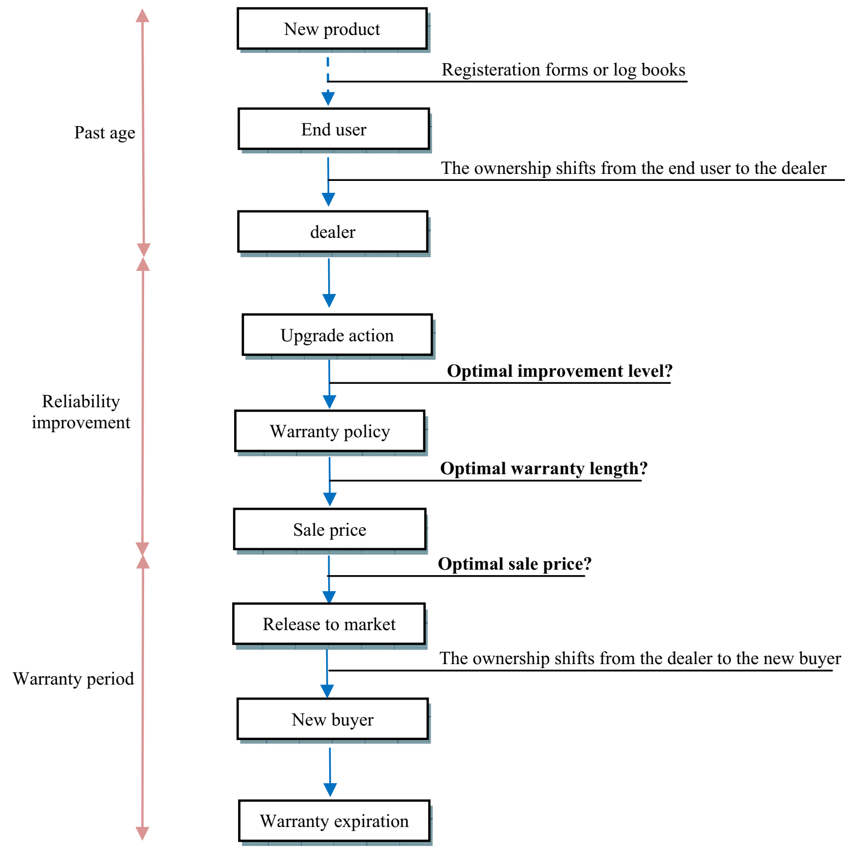

Fig. 1 Three decision variables in second-hand product process

actions. So, they are widely used and they serve many purposes including providing protection for both the dealer and the new buyer, being an indicator of second-hand product quality and reliability, and a promotional tool to gain reputation and assuring buyers against products which do not perform as promised. In response to this consumer demand for protection, public policy makers have begun enacting laws requiring dealers to offer warranties and effective service warranty claims [11].

Warranty for second-hand products is now being offered by dealers in some cases including automobiles, home appliance (television sets, air conditioners, refrigerators, washing machines, microwave ovens, and cloth dryers), personal computers, metal furniture (metal cupboard, shelves, office desks, and swivel chairs), gas and oil appliance, bathroom units, and kitchen systems. Most of the used car dealers, depending on the age and price of the car, have been offering 6 months to 3 years of warranty. For example, General Motors agrees to repair or provide replacement for the second-hand Buicks free of cost for a maximum period of 6 months [16].

The dealers offer different kinds of warranty policies for their used products. Warranty policy is a statement, in connection with the sale of a product, on the kind (e.g., free repair/replacement, lump sum payment, or a pro-rata reimbursement) and the extent (length of period) of compensation offered by the dealer in the event of failure. Offering the warranty implies that the dealer incurs additional costs to service any claims made by the customers. These costs are critically dependent on the product reliability [17], past age and usage, servicing strategy used by the dealer, and the terms of the warranty policy. Through effective warranty servicing strategies, the dealer can reduce the warranty costs. The warranty period, w, is fixed and starts from the purchase time by a new user. The buyer incurs the full replacement and maintenance cost on failures of the second-hand product after the original warranty period has expired.

In this section, we present the model assumptions and notations that will be used in the model formulation.

Table 3 Share of total used vehicle sales by vehicle age [37]

	US (%)	Canada (%)	France (%)	Germany (%)
<1 year	2	3	10	11
1–5 year	28	41	31	30
5+ year	70	56	59	59

3.1 Assumptions

1. The product has been maintained or repaired through minimal repairs during its past life all the way until age of x.
2. At age x, the item is subjected to an upgrade action with upgrade level u, which affects the performance of the item.
3. The second-hand product is subjected to three types of upgrade actions: imperfect overhaul, replacement with a younger item, and replacement with a new one; each action has its own associated costs.
4. The product is repairable or not and the dealer rectifies all failures occurring during the warranty period.
5. Whenever a failure occurs, it results in an immediate claim.
6. All warranty claims are valid.
7. The mean time to repair is negligible in front of the mean time between failures so that it can be approximated as being zero.
8. Failures are statistically independent.
9. The dealer sells products with upgrade actions. The effect of upgrade actions is to improve the reliability of the item prior to its sale.
10. Since dealers of second-hand products deal with products of different ages in an interval $[m, M]$, we consider the past age as a random variable by a distribution function $H(x)$.
11. The setup cost and the improvement action cost per unit of time are considered as fixed parameters.

3.2 Notations

The notations given in nomenclature are required for the purpose of this paper.

Nomenclature

x	past age of the second-hand product
m	minimum of x(lower limit)
M	maximum of x(upper limit)
$H(x)$	distribution function of past age
ρ	parameter for the truncated exponential distribution used in the life distribution of products
L	expected lifetime of the new item
P_0	sale price of the new item
c_x	purchase price from an end user when the age is x
$R(t)$	refund function
w	warranty period
u	upgrade level for a second-hand item
τ	screening test time for a second-hand item
$C_u(x)$	upgrade action cost for a second-hand product with past age x and upgrade level u
$F(t)$	cumulative failure distribution of the item
$f(t)$	density function associated with $F(t)$
$r(t)$	intensity function for product failure

4 Reliability improvement strategies

Reliability improvement is considered as part of the second-hand product process which can be used to reduce warranty cost and extend the lifetime of second-hand product. Improvement action typically begins with the arrival of the second-hand item at the dealer. The reliability growth is achieved through an iterative process of test, analyze, and rectify cycles (see Fig. 2).

For examples:

- GMC carries out some upgrade actions and inspections on air intankes, exhaust tips, door handles, body colors, wheels, lightning, air bag systems, and brake systems for used Chevrolets (www.Chevrolet.com).
- Bell helicopter company has applied upgrade action as a way of improving the performance of the used components such as gearbox, rotor drive shafts, wheels, and so on (www.bellhelicopter.com).
- Most of the used refrigerator dealers carry out some inspections on water filters, ice makers, water valves, kitchen-aid filter, etc. before their release to the market.

There are many interesting questions relating to the idea of reliability improvement procedure and warranty for second-hand products. For example:

1. Should reliability improvement actions be used for second-hand products?
2. If so, what improvement action should be taken (overhaul imperfect or screening test)?
3. What should be the optimal reliability improvement level?

A proper evaluation of the effect of the reliability improvement on warranty requires realistic models and an appropriate set of analytical tools. We consider two different models for reliability improvement.

Model I—virtual age approach This approach is most popular in reliability literature and is proposed by Kijima [15]. This is a model of imperfect repair which brings the item to a state somewhere between as good as new and as bad as old. Here, the effect of improvement is to (1) control the deterioration process, (2) reduce the likelihood of a failure over the warranty period, (3) restore the second-hand items to a better functional state, and (4) prolong the remaining life of a used item. This is usually applied to repairable used items such as high-definition television, projection, and liquid crystal display TVs.

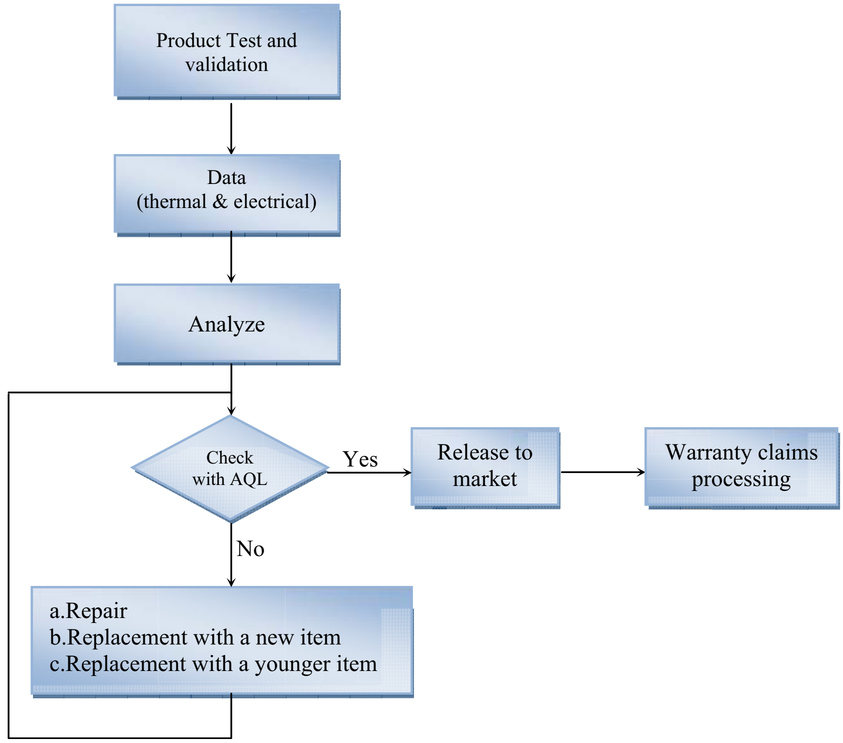

Fig. 2 Reliability improvement procedure for used items

Model II—screening test approach Here, the improvement involves screening test of non-repairable items for a short time, called test time, which is carried out by operating the products under electrical or thermal conditions and has been applied as a way of enhancing second-hand product quality and reliability. The type of the tests used depends on the type of faults to be detected and should be selected for maximum effectiveness on weak items without causing damage to good items. Based on the analysis of the test run and failure mode, the items that fail during the test procedure will be scrapped; only those which survive the test procedure will be considered to be of good quality and released to the market. This is usually applied to non-repairable second-hand items such as circuit boards, memory components, and DVD drives.

4.1 Model I—virtual age approach

The total mean cost of the product includes the purchase price from an end user, reliability improvement cost, and the warranty cost.

4.1.1 Purchase price from an end user

First, the dealer pays c_x to the end user for purchasing a second-hand product with age x. This price can be investigated from the market or estimated from the depreciation rate of the product. The purchase price to dealer for an item of age x is a function of the age of the item and is modeled by [12]:

$$c_x = k_0 P_0 \left(1 - \frac{x}{L}\right) \tag{1}$$

where $0<k_0<1$ and models the immediate loss in resale value subsequent to the sale of a new item, P_0 is the sale price of new item, and L is the expected useful life of the new product.

4.1.2 Reliability improvement cost

Let u be the upgrade level and T be the random variable describing the failure time of the product. We assume that cumulative failure distribution of the product is modeled as $F(t)$ with density function $f(t) = \frac{\mathrm{dF(t)}}{\mathrm{dt}}$ and the claims occur according to a non-homogeneous Poisson process (NHPP) with intensity function $r(t) = f(t)/[1 - F(t)]$ (for more discussions on NHPP, see [18]). The function $r(t)$ is an increasing function of t indicating that the number of claims (in a statistical sense) increases with age and/or usage.

Here, the effect of upgrade action is to make the used item effectively younger. One of the well-known reliability growth models is due to Duane [19] in which the improvement effort leads to a reduction in the failure rate. The failure rate after improvement for a period u is $r(t,u)$ and the associated reliability improvement cost is $c_u(x)$ with $0 \le u \le x$.

We consider two cases for failure rate after improvement:

Case I: $r(t,u) = r(t-u)$ Note that in this case, u=0 implies minimal repair (a repair has no impact on the reliability of the used item), $u \in (0,x)$ implies imperfect overhaul (a repair contributes to some noticeable reliability improvement) or replacement of the failed item with a younger one, and u=x implies complete repair (a repair restores the used item back to new) or replacement with a new item. In the context of used car market, a typical example of minimal repair would be changing the tires, rectifying the ignition or wiring system, or any repair of the engine that does not change the overall performance of the car, whereas a typical complete repair would be a transmission replacement or an engine replacement.

By modifying Chattopadhyay and Murthy's model [12] for second-hand products, we can model the cost of improvement which is a function of the age (x) and the improvement level u(>0) as follows:

$$E[c_u(x)] = c_s + c_u u^{\psi} x^{\zeta} \tag{2}$$

where c_s is the setup cost of the upgrade action per unit of the item (e.g., upgrade action overheads), c_u is the expected upgrade cost per unit of time (e.g., tool cost or labor cost per unit of time), and the parameters ψ and ζ are greater than zero. Each of the parameters, c_s, c_u, ψ, and ζ can be considered as a non-negative random variable with a specific distribution. The best way of estimating these parameters is by observing the past history of the product. Expression 2 implies that the cost of improvement increases as x and/or u increases.

Case II: $r(t,u) = r(t) + p \times [r(t-x) - r(t)]$ In this approach, we assume that the second-hand product undergoes "complete repair" condition with a constant probability of p and "minimal repair" condition with a probability of $1-p$. Therefore,

$$r(t,u) = p \times r(t-x) + (1-p) \times r(t)$$

or,

$$r(t,u) = r(t) + p \times [r(t-x) - r(t)] \tag{3}$$

where p depends on the level of improvement (u). As such, it is a decision variable constrained to $0 \le p \le 1$ and to be selected optimally. Note that p=0 implies no improvement and p=1 implies restoring the item back to new. The cost of improvement is given by 2, with p replacing u.

4.1.3 Warranty cost

This section derives the expected warranty cost for a repairable second-hand product with past age of x sold under upgrade level u and warranty period w for free replacement warranty (FRW) policy. Under a FRW with period w, the dealer agrees to rectify a failed unit, free of charge to the new buyer, during [0, w] after sale. Second-hand products sold under failure-free warranties might include electronics such as large-screen color TVs, automobiles, refrigerators, and household appliances.

Let$E[N_w(x)]$ be the density number of claims over the warranty period w when the product age is x at sale.

Case I: $r(t,u) = r(t-u)$ The expected number of claims over the warranty period is given by:

$$E[N_w(x)] = \int_x^{x+w} r(t-u)\mathrm{dt}. \tag{4}$$

Case II: $\Lambda(t) = p \times r(t-x) + (1-p) \times r(t)$ The expected number of claims over the warranty period is given by:

$$\begin{aligned} E[N_w(x)] &= \int_x^{x+w} [p \times r(t-x) + (1-p) \times r(t)]\mathrm{dt} \\ &= p \int_0^w r(t)\mathrm{dt} + (1-\mathrm{p}) \int_\mathrm{x}^{\mathrm{x+w}} \mathrm{r(t)dt}\ . \end{aligned} \tag{5}$$

Then, the dealer's expected warranty costs, $E[c_w(x)]$, for a product of age x at sale is given by:

$$E[c_w(x)] = \overline{c}E[N_w(x)] \tag{6}$$

where $\overline{c}$ is the expected cost of each rectification.

4.1.4 Total mean cost

For reliability improvement given by model I, total mean cost is:

$$c(x,u,w) = c_x + E[c_u(x)] + E[c_w(x)]. \tag{7}$$

A longer upgrade level is usually costly and adds to the sale price of the second-hand product but leads to a lower warranty cost for the dealer. This is worthwhile only if the cost of improvement is less than the reduction in the warranty servicing cost. As shown in Fig. 3, the optimal u* can be obtained by minimizing, $c(x, u, w)$, the total mean cost per item (if it is an interior point of the admissible region), if not, is either zero (implying no reliability improvement) or =$u_{\max}$ (implying maximum reliability improvement).

Example 1 Consider a dealer who sells a second-hand electrical device with a free-repair warranty. The product lifetime is 10 years, and the product is subject to random failure with the time to failure following a Weibull distribution function with $F(t) = 1 - \exp\left[-(\lambda t)^{\beta}\right]$. There-

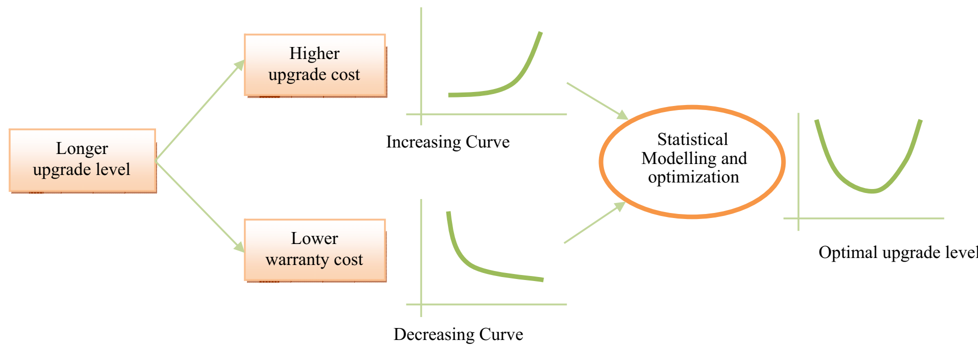

Fig. 3 Block diagram of determining the optimal upgrade level

fore, $r(t)$ is given by $r(t) = \lambda\beta(\lambda t)^{\beta-1}$, with $\beta>1$ and $\lambda>0$. Note that for $\beta>2$, then $r(t)$ is increasing convex. Let $r(t,u) = r(t-u)$. Table 4 lists the numerical values of the example.

Table 5 gives the u^* and the corresponding expected total cost, $c(x, u^*, w)$, for a range of w and x.

For w=0.5 to 1.5, u^*=0. This implies that if the warranty is less than or equal to 1.5 years, it is not worth subjecting the item to any improvement process since the reduction in the expected warranty cost is less than the cost of the improvement process. As w increases, $u^*>0$, this age reduction is more for higher values of x. Total mean costs for 3- to 6-year items sold with a 2-year warranty are shown in Fig. 4. It shows that upgrade decision is worthwhile compared to selling without upgrade.

4.2 Model II—screening test approach

The total mean cost of the product includes the purchase price from an end user, screening cost, and the warranty cost.

4.2.1 Purchase price from an end user

The purchase price to dealer for an item with past age x is the same as 1.

4.2.2 Screening cost

Let τ be the test time. The value of τ depends on the level of improvement. A longer test time leads to better reliability and provides higher customer/user peace of mind and a lower warranty cost for the dealer, but leads to additional costs to the dealer, which depends on the duration of the test. As such, it is a decision variable to be selected optimally. The cost of testing each component comprised a fixed setup cost and a variable cost of the test time (τ) and is given by a linear function as follows:

$$c_\tau(x) = c_s + c_\tau \tau \tag{8}$$

where c_s is the setup cost of the screening test per unit of the item and c_τ is the screening test cost per unit of time (e.g., tools and labor cost). This implies that the cost of screening test increases as τ increases.

By modifying Nguyen and Murthy's general model for non-repairable products [20], we can obtain the screening cost for a second-hand product with past age x as follows:

$$\begin{array}{ll} c_s + c_\tau(t-x) & t \in [x, x+\tau) \text{ if the item fails at age } t \text{ during the test} \\ c_s + c_\tau\tau & t \in [x+\tau, +\infty) \text{ if the item survives the test} \end{array}$$

And the expected screening cost for a second-hand product with age x, $c_\tau(x)$, is:

$$c_\tau(x) = \int_x^{x+\tau} [c_s + c_\tau(t-x)] f(t)\mathrm{d}t + \int_{x+\tau}^{\infty} (c_s + c_\tau\tau) f(t)\mathrm{d}t$$

therefore,

$$\begin{aligned} c_\tau(x) &= c_s \int_x^{x+\tau} f(t)\mathrm{d}t - c_\tau x[F(x+\tau) - F(x)] \\ &\quad + c_\tau \int_x^{x+\tau} t f(t)\mathrm{d}t + c_\tau\tau[1 - F(x+\tau)] \\ &= c_s[1 - F(x)] - c_\tau x[F(x+\tau) - F(x)] \\ &\quad + c_\tau\left[tF(t)|_x^{x+\tau} - \int_x^{x+\tau} F(t)\mathrm{d}t\right] + c_\tau\tau[1 - F(x+\tau)] \\ &= c_s[1 - F(x)] + c_\tau \int_x^{x+\tau} [1 - F(t)]dt \\ &= c_s\overline{F}(x) + c_\tau \int_x^{x+\tau} \overline{F}(t)dt \end{aligned} \tag{9}$$

where $\overline{F}(t) = 1 - F(t)$ is the survival function.

Table 4 Summary of the example

New product's price	P_0=2,500
The value of k	k=0.975
Warranty policy	FRW
The terms of warranty (year)	$w = 0.5, 1.0, 1.5\ , 2.0 \quad \overline{c} = 100$
The parameters of upgrade experiment	$\psi = 0.95,\ \zeta = 0.6,\ c_s = 10,\ c_u = 50$

Table 5 $c(x,u^*, w)$ and u* for a range of w and x

	x = 3.0	x = 4.0	x = 5.0	x = 6.0
w = 0.5	1,788.10	1,578.49	1,371.40	1,166.48
	u^*=0	u^*=0	u^*=0	u^*=0
w = 1.0	1,886.69	1,712.52	1,543.21	1,378.10
	u^*=0	u^*=0	u^*=0	u^*=0
w = 1.5	2,002.68	1,865.17	1,734.69	1,610.32
	u^*=0	u^*=0	u^*=0	u^*=0
w = 2.5	2,066.43	1,973.48	1,895.29	1,828.59
	u^*=3.0	u*=3.5	u^*=3.7	u^*=3.8

Differentiating 9 with respect to τ yields:

$$\frac{\partial}{\partial\tau}c_\tau(x) = c_\tau \times \overline{F}(x+\tau) > 0. \quad (10)$$

It also shows that the reliability improvement cost for a non-repairable second-hand product increases with test time.

The probability that a second-hand product survives at the test time is:

$$P(t > x+\tau | t > x) = \frac{\overline{F}(x+\tau)}{\overline{F}(x)}.$$

Therefore, the expected reliability improvement cost for a non-repairable second-hand product with past age x and testing time τ, $E[c_\tau(x)]$, is:

$$E([c_\tau(x)]) = \frac{\overline{F}(x)}{\overline{F}(x+\tau)} \times c_\tau(x) \quad (11)$$

4.2.3 Warranty cost

Let the used item fail after a period t subsequent to the sale and denote, respectively, $f_\tau(t)$ and $F_\tau(t)$ as the failure time

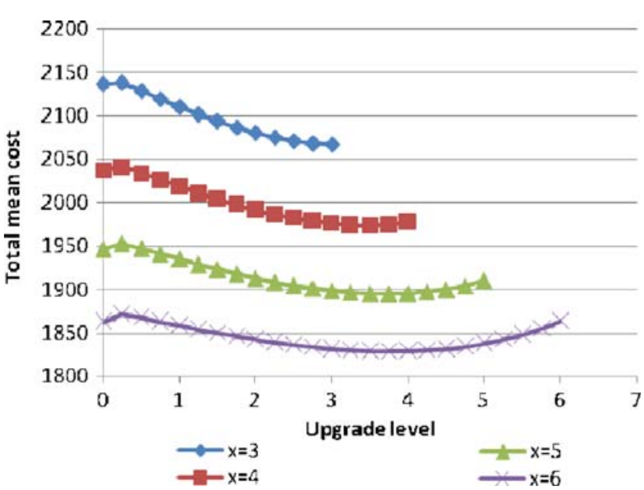

Fig. 4 Total mean cost for 3- to 6-year items sold with a 2-year warranty

probability density function and cumulative distribution function after the sale. We assume that the failure distribution of a used item of age is given by:

$$F_\tau(t) = \frac{F(x+\tau+t) - F(x+\tau)}{1 - F(x+\tau)} \quad (12)$$

For non-repairable products, the expected free replacement frequency follows by a renewal process constituting a renewal equation [21]. We assume that the dealer is required to provide a new product at no cost to the buyer from the time of purchase. Then, free replacements are provided until a product having a life of at least w is found. By assuming that the failure time of the products is independently and identically distributed, the number of replacements required to satisfy the warranty conditions is a random variable which has a geometric distribution. The probability distribution of $N_w(x)$ is:

$$P[N_w(x) = n] = \begin{cases} F_\tau(w) & n = 0 \\ 1 - F_\tau(w) & n = 1 \end{cases}$$

Then, the expected number of replacements for a non-repairable second-hand product, during the time interval [0, w], after sale is:

$$E[N_w(x)] = \frac{F_\tau(w)}{1 - F_\tau(w)} \quad (13)$$

This section derives the expected warranty cost for a second-hand product with age x sold with test time τ and warranty period w for the three types of warranty—the failure-free policy, the rebate warranty, and a combination of free replacement and lump sum policy.

Policy I: free replacement warranty Under the FRW policy, if the item fails before the expiration warranty w, it is replaced by a new one at no cost to the new buyer. The cost function for warranty per replacement under the FRW policy may be expressed as:

$$R(t) = \begin{cases} c_r & 0 \le t < w \\ 0 & \text{otherwise} . \end{cases}$$

where c_r is the expected replacement cost per failure during the warranty length.

Then, the expected warranty cost per replacement, $E[R(t)]$, is:

$$E[R(t)] = \int_0^w c_r f_\tau(t)\mathrm{d}t = \mathrm{c_r}F_\tau(w). \quad (14)$$

Policy II: pro-rata warranty Under a pro-rata warranty, the dealer agrees to replace the item that fails during the warranty period at a charge to the new buyer that is prorated to the age of the failed item. The pro-rata warranty (PRW) is sometimes offered on relatively cheap products such as batteries, tires, ceramics, and so on.

The cost function for warranty per replacement under the PRW policy may be expressed as:

$$R(t) = \begin{cases} c_r\left[1 - \frac{t}{w}\right] & 0 \leq t < w \\ 0 & \text{otherwise} \end{cases}$$

Then, the expected warranty cost per replacement, $E[R(t)]$, is:

$$E[R(t)] = \int_0^w c_r\left[1 - \frac{t}{w}\right] f_\tau(t)\mathrm{d}t = \frac{\mathrm{c_r}}{\mathrm{w}}\int_0^{\mathrm{w}} \mathrm{F}_\tau(\mathrm{t})\mathrm{dt} \tag{15}$$

Policy III: FRW/LSW Under this policy, the dealer provides a replacement item, covered by the same warranty, free of charge to the customer for an item that fails before reaching age w_f and during the time interval [w_f, w] the lump sum amount of replacement cost is offered to the buyer. The cost function for warranty per replacement under the FRW/lump sum rebate warranty (LSW) policy may be expressed as:

$$R(t) = \begin{cases} c_r & 0 \leq t < w_f \\ kc_\mathrm{r} & w_\mathrm{f} \leq t < w \\ 0 & \text{otherwise} \end{cases}$$

where $0<k\leq1$ is the proportionality coefficient of c_r (see Fig. 5).

Then, the expected warranty cost per replacement, $E[R(t)]$, is given by:

$$\begin{aligned} E[R(t)] &= \int_0^{w_f} c_r f_\tau(t)\mathrm{d}t + \int_{w_\mathrm{f}}^{w} kc_\mathrm{r} f_\tau(t)\mathrm{d}t \\ &= \mathrm{c_r}[(1-\mathrm{k})\mathrm{F}_\tau(\mathrm{w_f}) + \mathrm{kF}_\tau(\mathrm{w})] \end{aligned} \tag{16}$$

According to Wald's renewal equation, the expected warranty cost for a non-repairable second-hand product, under testing time τ and warranty length w, is equal to:

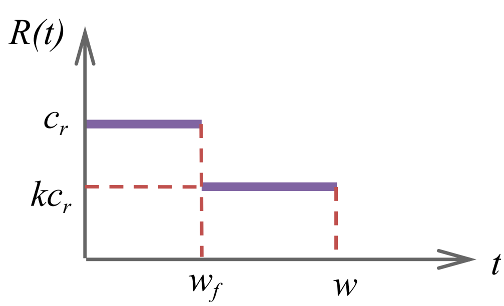

Fig. 5 The refund function for a FRW/LSW policy

$$E[c_w(x)] = E[N_w(x)] \times E[R(t)] \tag{17}$$

4.2.4 Total mean cost

For reliability improvement given by model II, total mean cost for a non-repairable second-hand product is:

$$c(x,\tau,w) = \frac{\overline{F}(x)}{\overline{F}(x+\tau)}[c_x + c_\tau(x)] + E[c_w(x)] \tag{18}$$

5 Distribution of the past age

Dealers of second-hand products deal with products of different ages in an interval [m, M]. The expected reliability improvement cost and warranty costs for older products are higher than those of younger products. If these high costs are built in to sale price, then older products become unattractive to the new buyers. To make the sale price of the older products attractive to the new buyers, the dealer can subsidize older products by charging less than the expected warranty cost and recover the subsidy from younger products by charging more than the expected warranty cost [10].

We model this by viewing X as a random variable assuming values in [m, M] and being characterized by a distribution function $H(x)$, with $H(m) = 0$ and $H(M) = 1$. Let $h(x) = \frac{d}{\mathrm{d}x}H(x)$ denote the density function associated with $H(x)$. On carrying out the expectation over X, we have the total mean cost per product, $c(\tau, w)$, given by:

$$c(\tau,w) = \int_m^M c(x,\tau,w) \times h(x)\mathrm{d}x \tag{19}$$

To stimulate consumers' purchase willingness, dealers must minimize the total mean cost per product and determine the reasonable selling prices for their second-hand products. One form of $h(x)$, which is analytically tractable, is the following truncated exponential distribution [22]:

$$h(x) = \frac{\rho e^{-\rho x}}{e^{-\rho m} - e^{-\rho M}} \tag{20}$$

where $\rho>0$ is the parameter for the truncated exponential distribution used in the life distribution of second-hand products and m and M are the lower and upper limits of past age X at statutory base. In real life, distribution of lifetime coverage might not be possible to model using a particular distribution and can be modeled using probability mass function.

Table 6 $c(x, \tau, w)$ for different combinations of τ, w for 1-year second-hand circuit boards sold under FRW, PRW, and FRW/LSW policies

		τ=0.05	τ=0.1	τ=0.15	τ=0.2
FRW	w=1.0	5.17	5.52	5.88	6.25
	w=1.5	6.80	7.24	7.68	8.14
	w=2.0	10.09	10.68	11.29	11.91
	w=2.5	16.09	16.95	17.84	18.74
FRW/LSW w_f=0.5, k=0.3	w=1.0	4.83	4.96	5.08	5.22
	w=1.5	6.07	6.26	6.46	6.66
	w=2.0	8.53	8.84	9.16	9.49
	w=2.5	12.97	13.49	14.02	14.57
PRW	w=1.0	4.65	4.75	4.86	4.98
	w=1.5	5.41	5.56	5.72	5.89
	w=2.0	7.00	7.24	7.49	7.75
	w=2.5	10.02	10.42	10.84	11.27

Example 2 Consider a dealer who sells second-hand circuit boards that are used in electrical devices such as plasma. It is non-repairable and is currently sold without any screening test under a non-renewing free replacement warranty in market. Let $F(t)$ be given by the cumulative function of the Weibull distribution $F(t) = 1 - \exp\left[-(\lambda t)^{\beta}\right]$. The dealer desires to satisfy the expectations of his customers and to improve the quality of his products by providing screening test. This test includes a series of industrial stages such as component cleaning, reconditioning, and electrical test (the test time << the life of the item). As a result, after the screening test, the intensity function for product failure is:

$$F_{\tau}(t) = \frac{\exp\left[-(\lambda(x+\tau))^{\beta}\right] - \exp\left[-(\lambda(x+\tau+t))^{\beta}\right]}{\exp\left[-(\lambda(x+\tau))^{\beta}\right]}$$

We assume the following values for the model parameters: P_0=5, L=5, k_0=0.975, λ=0.3/year and β=2.0. This implies that the mean time to first failure is 2.954 years. Let c_s=0.2 and c_u=5/unit of time and c_r=10. Table 6 shows the total mean cost $c(x, \tau, w)$ or 1-year second-hand circuit boards and different combinations of w varying from 1 to 2 years and τ varying from 0.05 to 0.2 years for FRW, FRW/LSW, and PRW policies.

We consider a case where the past age of circuit boards is distributed as Eq. 20. Data indicate that the parameters for the truncated exponential distribution are ρ=0.2/year, m =1.0, and M=2.5 (the mean value of the past age distribution is 1.712). Table 7 shows the total mean cost to the dealer for different combinations of w varying from 1 to 2 years and τ varying from 0.05 to 0.2 years for FRW, FRW/LSW, and PRW policies.

Based on the results of the analysis, the following observations may be summarized:

- The reliability improvement level depends on many factors such as product failure distribution, past age of the second-hand product, warranty length, and cost parameters.
- The improvement is beneficial if the initial failure rate of the second-hand product is large.

Table 7 $c(\tau, w)$ for different combinations of τ, w for circuit boards with past ages distributed as Eq. 20

		τ=0.05	τ=0.1	τ=0.15	τ=0.2
FRW	w=1.0	5.50	5.90	6.30	6.73
	w=1.5	8.83	9.36	9.89	10.45
	w=2.0	15.23	15.99	16.77	17.57
	w=2.5	26.73	27.91	29.13	30.37
FRW/LSW w_f=0.5, k=0.3	w=1.0	5.01	5.16	5.32	5.49
	w=1.5	7.60	7.86	8.12	8.39
	w=2.0	12.50	12.94	13.39	13.86
	w=2.5	21.23	22.00	22.80	23.61
PRW	w=1.0	4.45	4.58	4.72	4.86
	w=1.5	6.18	6.39	6.60	6.81
	w=2.0	9.65	10.00	10.36	10.73
	w=2.5	16.20	16.83	17.48	18.14

- Since the buyer pays nothing for repairs during the warranty period, there is no incentive for him/her to invest any effort into reliability improvement. It is worthwhile for the dealer to carry out improvement action only if the reduction in the warranty servicing cost is greater than the extra cost incurred with reliability improvement.
- Failures during the warranty period are costly. For example, repair costs for computer problems associated with processors, hard drives, monitors, memory components, and DVD drives are between $150 and $900 [23] and providing upgrade actions for second-hand products can reduce these costs.
- It can be shown that the reliability improvement level increases as the expected cost of each rectification increases.
- We assumed k=0.3 for FRW/LSW policy. Smaller values of k indicate that the dealer will shoulder a lower warranty cost and larger values of k lead to a higher warranty cost for the dealer.
- For a specified improvement level and warranty length, the failure-free warranty policy leads to a higher total mean cost than the other warranty policies.

6 Conclusions and recommendations for future research

In this paper, we develop two effective ways to decide on the reliability improvement strategies for second-hand products sold under failure-free warranty, rebate warranty, and a combination of free replacement and lump sum policies. The first one is based on Kijima's virtual age model and the second one is based on screening test approach. Determination of the optimal upgrade action strategy is an optimization process in which both costs and benefits should be considered from the dealer's perspective. On one hand, providing upgrade action is usually costly and adds directly to the sale price of the second-hand product, and on the other hand, upgrade action can reduce the warranty cost.

There is huge scope for future research in the area of upgrade action cost modeling for second-hand products. Some extensions are as follows:

a. Modeling the reliability improvement cost for a general repairable second-hand product (when the second-hand product fails, the action is overhaul with probability p and perfect with probability $1-p$, $0 \le p \le 1$);
b. Determining the optimal improvement level for second-hand products to minimize the dealer's total mean cost;
c. Considering the setup cost and the upgrade action cost per unit of product per unit time as a random variable that depends on the past age and usage of the second-hand product;
d. To build models for reliability improvement action cost which incorporate past usage and maintenance strategy.
e. We can consider FRW policy for non-repairable items in which the failed item is replaced with a younger second-hand item. It can be used for managerial decisions in purchasing second-hand high-tech goods for some companies in developing countries.

The authors have obtained some results for (a)–(e) and these will be reported elsewhere.

Acknowledgments The authors wish to thank the Associate Editor and the anonymous referees. Their comments have resulted in significant improvements to the content and presentation of the manuscript.

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