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Null geodesics in conformal gravity

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Abstract

We present an analysis of the null geodesics of the static, spherically symmetric, vacuum solution to the equations of conformal (Weyl) gravity. We classify the full range of exotic spacetimes arising from the parameter space of the metric. The nature of various notable features of these spacetimes is investigated including light spheres, horizons and physical singularities.

Keywords: null geodesics, exotic spacetimes, conformal gravity

1. Introduction

For over one hundred years Einstein's theory of general relativity has dominated the narrative in our understanding of space, time, matter and energy in fields from planetary dynamics to cosmology. Its longevity has been due to its originality, relative simplicity and tremendous success in explaining a wide range of astrophysical phenomena.

At the scale of the Solar System such phenomena include: radar echo delay from planets in conjunction [1]; deflection of starlight near the Sun [2]; redshift of light leaving the gravitational potential of the earth [3] and perihelion precession of elliptical orbits [4]. A great many more distant observations also seem to validate general relativity: changing orbital periods of binary pulsars [5]; redshift of light leaving the gravitational potential of white dwarfs [6] and gravitational waves released by merging black holes [7].

For all its success, however, general relativity on galactic and cosmological scales requires the invocation of extra matter and energy to explain observations. This extra content in the Universe has so far resisted investigation and has, therefore, been referred to as dark. Dark matter is required at galactic scales to explain the asymptotic flatness of galaxy rotation curves

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[8], the extent of gravitational lensing [9] and the hydrostatic equilibrium of x-ray gas in galaxy clusters [10]. Dark energy, meanwhile, is required to explain the accelerating expansion of the Universe [11].

Research into alternative gravitational theories is often motivated in part by the desire to do away with the need for these dark substances. Other reasons may include the potential for quantisation or unification of gravity with other fundamental forces. The consequences of various alternative theories have profound implications for both cosmology (such as the energy content, origin and fate of the Universe) and astrophysics (such as the feasibility and nature of black holes). Regardless of the theory, however, to be viable it must replicate the predictive success of general relativity and for this extensive testing is required.

One candidate theory to have been advanced in recent decades is conformal (Weyl) gravity [12, 13]. General relativity relies on the symmetry of Lorentz invariance, developed in Einstein's special relativity, and prioritises the simplicity of second-order field equations

$$G_{\mu\nu} = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu}, \quad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor. Conformal gravity, on the other hand, prioritises the symmetry of local conformal invariance (a property shared by the other fundamental forces). Local conformal invariance means invariance to local isotropic stretching of the spacetime geometry with metric tensor $g_{\mu\nu}(x)$ of the form $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$, where $\Omega(x)$ is a smooth, positive function—a Weyl transformation. An unfortunate consequence of this extra symmetry is that conformal gravity is characterised by fourth-order field equations [12] given by

$$4\alpha_g W_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

where α_g is a parameter for the strength of the gravitational field. These equations differ from those used in general relativity because both the Bach tensor $W_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ are traceless in conformal gravity and the only constant α_g is dimensionless. The lack of conformal invariance exhibited by the Einstein tensor $G_{\mu\nu}$ and the dimensions of the constants G and Λ prevent conformal scaling of general relativity.

While a number of solutions to the field equation (2) have been found [14], the focus of this paper is on the static, spherically symmetric, vacuum solution (with equivalents in general relativity such as Schwarzschild, de Sitter and anti-de Sitter). This solution to the source-free, fourth-order Poisson equation is commonly known as the Mannheim–Kazanas (MK) metric [12] and is given by

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

where

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \kappa r^2,$$

with parameters β , γ and κ picked up during integration. For the metric given (3) and all further calculations we adopt geometrised units where $c = 1$.

How then does conformal gravity hold up under the same tests applied to general relativity? At the Solar System scale, conformal gravity can match all the results of general relativity with the constraint that γ and κ are very small [15–17]. In this limit, with β proportional to the mass of a central body, the MK metric (3) reduces to the Schwarzschild metric of general relativity. At the galactic scale, rotation curves have been fitted with constraints on the magnitude of γ and κ consistent with Solar System tests and without the need for dark matter [18].

Gravitational lensing effects, however, have been thought to require $\gamma < 0$ [15, 16, 19] as opposed to $\gamma > 0$ found in the rotation curve fitting. The wrong sign for γ would result in gravitational lensing away from massive objects which is contrary to observations. Much work has gone into resolving this from a variety of perspectives [20–22].

The parameters of the MK metric (3), β , γ and κ , characterise a wide variety of spacetimes containing features such as light spheres, horizons (coordinate singularities of the metric) and physical singularities. The values of the parameters alter the position, configuration and nature of these features and any deviation from the familiar $\gamma = \kappa = 0$ of the Schwarzschild metric introduces more exotic cases. A light sphere occurs at a radius in a curved spacetime where photons follow a circular orbit. In the Schwarzschild case there exists one unstable light sphere at $r = 3\beta$. Any deviation from the circular orbit of an unstable light sphere causes the photon to either fall into the singularity or escape to infinity. Other values of γ and κ , however, can give rise to stable light spheres or ‘accumulation points’ [23]. A build up of photons in such a stable light sphere could alter the geometry further—providing a potential formation mechanism for a self-gravitating shell of photons or orbiting radiation star [24]. The other notable feature of the Schwarzschild case is the black hole event-horizon at $r = 2\beta$. This is a causal boundary in spacetime across which light can only travel in one direction (towards $r = 0$). As with the light spheres, however, different values of γ and κ can result in other horizons at different positions and acting in different directions [25].

While exotic features such as stable light spheres [23] and multiple horizons [25] have been previously identified, we present a complete exploration of the spacetimes described by the 3-parameter MK metric. This paper will first introduce the features of the MK spacetime (section 2) before classifying all the possible configurations of these features allowed by the parameter space of β , γ and κ (section 3). We will then conclude by discussing the nature of these features in greater detail (section 4).

2. Features of the spacetime geometry of the MK metric

Four key features of spacetimes characterised by the MK metric are discussed in this section: light spheres, horizons, physical singularities and large-scale curvature. For the first two of these, significant insight can be gained by defining an effective potential for photons in orbit around a compact gravitational source. This was done using the null geodesic equation and Killing vectors to produce an equation of motion for photon orbits,

$$\left(\frac{dr}{d\phi}\right)^2 = r^4 \left(\frac{1}{b^2} - V_{\text{eff}}(r)\right), \quad (4)$$

where

$$V_{\text{eff}}(r) = \frac{B(r)}{r^2} = \frac{1 - 3\beta\gamma}{r^2} - \frac{\beta(2 - 3\beta\gamma)}{r^3} + \frac{\gamma}{r} - \kappa, \quad (5)$$

is the effective radial potential of photon orbits. These orbits are characterised by quantities conserved by the Killing vectors: L and E , the orbital angular momentum and energy of orbit respectively. The constants are combined into a single quantity by defining an impact parameter $b = L/E$.

The effective potential (plus constant parameter κ) as a function of the dimensionless parameter r/β is plotted in figure 1 for a range of values of the dimensionless parameter $\beta\gamma$. We focus initially on $\beta > 0$ due to its association with mass, however, $\beta \leq 0$ will be considered in section 3.

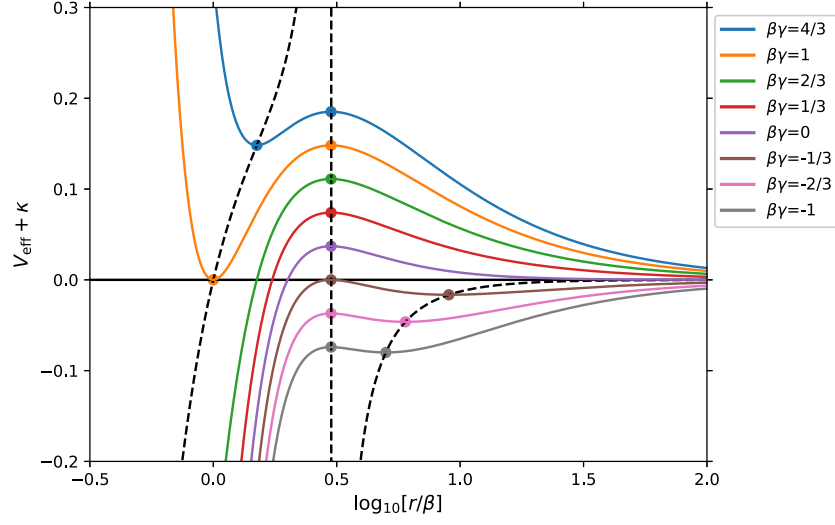


Figure 1. Effective potential plus constant parameter κ of photon orbits as a function of radius for a range of the parameter $\beta\gamma$. The positions of light spheres are marked by dots.

2.1. Light spheres

The circular photon orbits or light spheres occur at the stationary points of the effective potential.

$$\frac{dV_{\text{eff}}}{dr} = -\frac{2(1-3\beta\gamma)}{r^3} + \frac{3\beta(2-3\beta\gamma)}{r^4} - \frac{\gamma}{r^2} = 0 \quad (6)$$

can be rearranged to give the quadratic equation

$$\gamma r^2 + 2(1-3\beta\gamma)r - 3\beta(2-3\beta\gamma) = 0, \quad (7)$$

with solutions at $r = 3\beta$ and $r = 3\beta - 2/\gamma$. These then correspond to the radii of two light spheres with positions independent of κ . The effective potential informs the stability of the light spheres since the maximum in V_{eff} at $r = 3\beta$ corresponds to an unstable light sphere while the minimum at $r = 3\beta - 2/\gamma$ corresponds to a stable light sphere. The unstable light sphere is at $r = 3\beta$ independent of γ and κ . For $\gamma = 0$, the Schwarzschild case, the stable light sphere is at $r = \infty$. For $\beta\gamma < 0$ the stable light sphere at $r = 3\beta - 2/\gamma$ lies outside the unstable light sphere and for $\beta\gamma > 2/3$ there is a stable light sphere interior to the unstable one. For smaller positive values of $\beta\gamma$ only the unstable light sphere exists.

2.2. Horizons

The horizons of the metric occur at coordinate singularities where $B(r) = 0$ [25]. Rearranging this in terms of dimensionless parameters r/β , $\beta\gamma$ and $\beta^2\kappa$, the horizons are found at

$$-\beta^2\kappa\left(\frac{r}{\beta}\right)^3 + \beta\gamma\left(\frac{r}{\beta}\right)^2 + (1-3\beta\gamma)\left(\frac{r}{\beta}\right) - (2-3\beta\gamma) = 0. \quad (8)$$

Crucially, this is dependent on κ and so, unlike light spheres, the existence, nature and positions of these key features of the spacetime geometry are strongly affected by its value.

Since horizons occur where $B(r) = 0$, from the form of the effective potential (5) we have horizons where $V_{\text{eff}} = 0$. A change in κ , therefore, corresponds to adding or subtracting a constant to the effective potential seen in figure 1, with the potential to drastically change the positions of horizons. From figure 1 it is clear that for $\kappa = 0$, there are a variety of spacetime geometries characterised by the value of $\beta\gamma$. Some have no horizons ($\beta\gamma < -1/3$ and $\beta\gamma > 1$), others only the relatively familiar black hole event horizon ($0 < \beta\gamma < 2/3$) while still more have either exterior, cosmological horizons ($-1/3 < \beta\gamma < 0$) or interior, Cauchy horizons ($2/3 < \beta\gamma < 1$). Further, by visualising the variation of κ as moving the zero point on the vertical axis, we can predict even more exotic spacetimes for $\kappa \neq 0$.

Due to the diverse nature and positions of horizons in the MK metric it is necessary to define precisely what we mean by each of them and to classify them consistently. To this end we will cease to refer to horizons as Cauchy or black hole: this system changes if you view them from the other side (i.e. cosmological horizons). Instead, we will use ‘in’ and ‘out’ horizons which can be either ‘interior’ or ‘exterior’ to the unstable light sphere at $r = 3\beta$ (for $\beta > 0$). We establish the nature (in or out) of these horizons by analogy with the Schwarzschild horizon. A transition from $B(r) > 0$ to $B(r) < 0$ as radius decreases, like in the Schwarzschild case, corresponds to an in-horizon. Conversely, a transition from $B(r) > 0$ to $B(r) < 0$ as radius increases corresponds to an out-horizon.

An examination of various possible arrangements of these horizons and the positions of stable and unstable light spheres is presented in section 3.

2.3. Physical singularities and large-scale curvature

The final features of spacetime that need explaining are the behaviours of the metric at the limits of r . To accomplish this we shall analyse the scalar curvature as $r \rightarrow 0$ and as $r \rightarrow \infty$.

Our initial analysis will focus only on the Einstein frame in which the metric takes the form (3) without a Weyl transformation. This will prove helpful when interpreting the null geodesics and causal structure (both of which are invariant under Weyl transformations). We will discuss the effects of conformal scaling of the metric in section 4.2.

Beginning with the behaviour at $r = 0$ it is immediately clear that for the MK metric (3), just as for the Schwarzschild metric, there is a singularity. Importantly, this singularity is not merely a coordinate singularity as no coordinate transformation can make it disappear. A physical singularity is then a singularity that exists independent of coordinate system. We can check this is the case by finding a curvature scalar that diverges at the position of the singularity.

The most simple of these is the Ricci scalar R^μ_μ derived from contractions of the Riemann tensor,

$$R^\mu_\mu = 12\kappa - \frac{6\gamma}{r} + \frac{6\beta\gamma}{r^2}. \quad (9)$$

This clearly diverges unless $\gamma = 0$ and therefore all spacetimes with $\gamma \neq 0$ contain physical singularities. There are, however, other curvature scalars we might examine to investigate the case where $\gamma = 0$ and indeed one such scalar is required to prove the existence of a physical singularity for the Schwarzschild metric where $\gamma = 0$ and $\kappa = 0$. The Kretschmann scalar, $K = R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa}$, for the MK metric takes the form

$$K = 24\kappa^2 - \frac{24\gamma\kappa}{r} + \frac{24\beta\gamma\kappa}{r^2} + \frac{6\gamma^2}{r^2} - \frac{24\beta\gamma^2}{r^3} + \frac{36\beta^2\gamma^2}{r^4} + \frac{24\beta^2\gamma(2-3\beta\gamma)}{r^5} + \frac{12\beta^2(2-3\beta\gamma)^2}{r^6}. \quad (10)$$

Having already shown a singularity exists for $\gamma \neq 0$ all that remains is to take $\gamma = 0$ such that

$$K = 24\kappa^2 + \frac{48\beta^2}{r^6}, \quad (11)$$

where it is clear that the curvature diverges at $r = 0$ unless we also have $\beta = 0$ (this will be discussed further in section 4.1). The MK metric, therefore, has a physical singularity at the origin and none elsewhere as long as both γ and β are not equal to zero.

At the other extreme, as $r \rightarrow \infty$, the curvature scalars (9) and (10) depend only on the parameter κ . It is for this reason that κ is associated with the large scale curvature of spacetime. Indeed, from the linear dependence of the Ricci scalar R^μ_μ on κ it is clear that a negative value of κ corresponds to an hyperbolic, open Universe while a positive value of κ corresponds to a spherical, closed Universe. $\kappa = 0$, meanwhile, indicates no large scale curvature and an asymptotically flat Universe.

3. Exploration of spacetime geometries in the parameter space of the MK metric

All twenty distinct configurations of features permitted by the MK metric are shown in table 1 including the signs of β , γ and κ for which they are permitted. Figure 2 shows the locations of the light spheres and horizons as a function of the dimensionless parameter $\beta\gamma$. Each horizontal line on either side of $r/\beta = 0$ corresponds to a spacetime containing some configuration of features. Figure 2 shows clearly that varying κ only affects the horizons (blue) and not the stable or unstable light spheres (orange and red respectively). The stable light spheres tend towards the unstable light sphere at $r = 3\beta$ as $\beta\gamma \rightarrow \pm\infty$. Figure 3, meanwhile, maps the domains of the $\beta^2\kappa - \beta\gamma$ parameter space for $\beta > 0$ and $\beta < 0$. Each of the 20 domains correspond to an entry in table 1 and each point in figure 3 corresponds to a horizontal line in figure 2.

The wide variety of interesting spacetimes described by the MK metric have in common some combination of the features outlined above in section 2. They all contain a physical singularity at $r = 0$ (unless $\gamma, \beta = 0$) and the large-scale geometry may be hyperbolic, spherical or flat. With the restriction of $\beta > 0$, they also all contain an unstable light sphere at $r = 3\beta$. Beyond this they may contain up to one stable light sphere at either a larger or smaller radius than the unstable one and some combination of up to three of the four possible horizons.

$\beta < 0$ can be associated with some gravitational source of either negative density or pressure as required by many dark energy theories. With $\beta < 0$ there are no unstable light spheres although there may be up to one stable light sphere and two horizons. Without an unstable light sphere to provide a reference point our naming system breaks down, however, as these features can exist arbitrarily close to the singularity, we record them in table 1 as interior features.

For $\kappa = 0$ in figure 2, the horizon and light sphere curves tend to zero as $r/\beta \rightarrow \infty$ but the horizon curve tends faster. The blue curve indicating the horizons does not flatten off in the large- r limit for non-zero κ , it either continues to rise or fall depending on the sign of κ and β . It should also be noted that, with the exception of $r = 0$, the light sphere curves intersect the horizon curve at its stationary points: stable light spheres at maxima and unstable light spheres at minima for $r/\beta > 0$ and the reverse for $r/\beta < 0$. A brief calculation reveals the horizons cross the unstable light sphere at $r/\beta = 3$ when

Table 1. Possible radial configurations of light spheres (LS) and horizons (H) in the $\beta\text{-}\gamma\text{-}\kappa$ parameter space of the MK metric. The four leftmost columns contain the signs of parameters and domain labels to identify each case in figure 3. I/O = in/out horizons. S/U = stable/unstable light spheres. E = empty. The singularities at $r = 0$ are timelike (T) or spacelike (S), and naked (N) or concealed (C).

β	γ	κ	Domain	$r = 0$	Interior			Unstable LS	Exterior		
					Out-H	Stable LS	In-H		Out-H	Stable LS	In-H
+	+	+	SUO	T/N		x		x	x		
+	+	+	OSIUO	T/C	x	x	x	x	x		
+	+	+	OSU	T/N	x	x		x			
+	+	+	IUO	S/C			x	x		x	
+	+	+	U	S/N				x			
<hr/>											
+	+	-	SU	T/N		x		x			
+	+	-	OSIU	T/C	x	x	x	x			
+	+	-	IU	S/C			x	x			
<hr/>											
+	-	+	IUOS	S/C			x	x	x	x	
+	-	+	US	S/N				x		x	
<hr/>											
+	-	-	IUS	S/C			x	x		x	
+	-	-	IUOSI	S/C			x	x	x	x	x
+	-	-	USI	S/N				x		x	x
<hr/>											
-	+	+	O	T/N	x						
<hr/>											
-	+	-	E	T/N							
<hr/>											
-	-	+	E	S/N							
-	-	+	OS	T/N	x	x					
<hr/>											
-	-	-	I	S/C			x				
-	-	-	OSI	T/C	x	x	x				
-	-	-	S	T/N		x					

$$\beta^2\kappa = \frac{1 + 3\beta\gamma}{27}. \tag{12}$$

This has important implications for which configuration of features occurs in a particular space-time. Indeed, (12) gives the domain boundary $IUO \rightarrow U$ in figure 3. Other domain boundaries ($OSI \rightarrow S$) in figure 3 can be derived similarly where the horizon curves intersect the stable light spheres in figure 2 giving

$$\beta^2\kappa = \frac{\beta\gamma - 1}{\left(3 - \frac{2}{\beta\gamma}\right)^2}. \tag{13}$$

The final domain boundaries in figure 3 occur at $\beta\gamma = 0$, $\beta\gamma = 2/3$ and $\beta^2\kappa = 0$ for reasons that are clear from figure 2.

To better understand figures 2 and 3 we investigate the six domains and five boundaries which occur along $\beta^2\kappa = -0.05$. These are found in the top panel of figure 3 ($\beta > 0$) and the horizons are given by the darkest blue curve in figure 2. Starting with $\beta\gamma \gg 1$, the SU case containing a stable and an unstable light sphere, we decrease $\beta\gamma$ until at $\beta\gamma = 0.958$

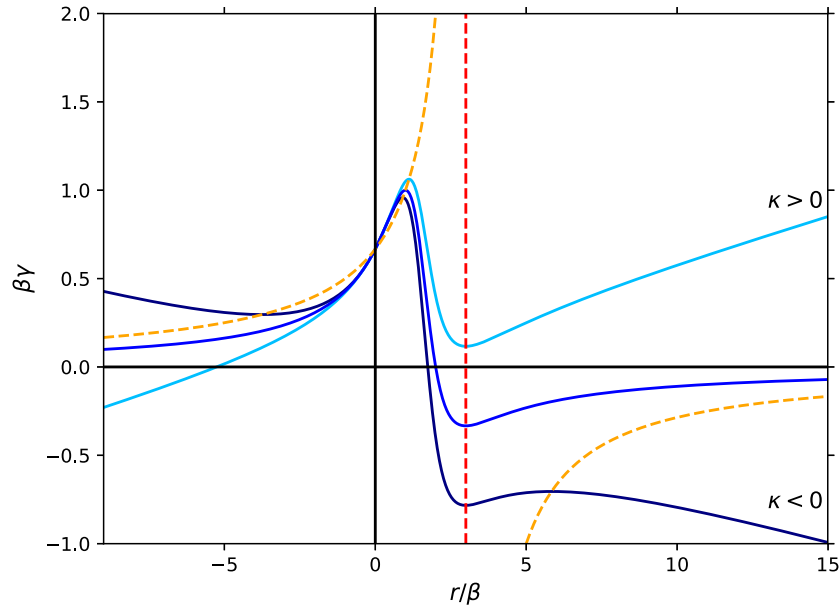


Figure 2. Dimensionless plot of the parameter space of $\beta\gamma$ against the radius r/β showing features of the spacetime geometry. The horizons are in blue, stable light spheres in orange and unstable light spheres in red. The lightest blue corresponds to $\beta^2\kappa = 0.05$, the darkest to $\beta^2\kappa = -0.05$ and the third to $\kappa = 0$.

we reach the domain boundary corresponding to the maximum of the horizon curve, coincident with the interior stable light sphere. Crossing the boundary changes $SU \rightarrow OSIU$ as we introduce interior in and out-horizons on either side of the stable light sphere. The next boundary, $OSIU \rightarrow IU$, occurs at $\beta\gamma = 2/3$, where the interior out-horizon and the stable light sphere reach $r = 0$ ($OS \rightarrow 0$) leaving IU (an interior in-horizon and unstable light sphere). Crossing the $IU \rightarrow IUS$ boundary at $\beta\gamma = 0$ then introduces an exterior stable light sphere from $r/\beta = \infty$ ($S \rightarrow \infty$ in reverse). At $\beta\gamma = -0.705$ the exterior maximum of the horizon curve in figure 2, coincident with the exterior stable light sphere, again introduces in and out-horizons via the domain boundary. This time, however, they are exterior to the unstable light sphere giving $IUS \rightarrow IUOSI$. Finally, at $\beta\gamma = -47/60$, we encounter the minimum of the horizon curve corresponding to the domain boundary $IUOSI \rightarrow USI$ where the interior in-horizon and exterior out-horizon meet at the unstable light sphere. To summarise, for $\beta > 0$ and $\beta^2\kappa = -0.05$, as we lower $\beta\gamma$ from $+\infty$ to $-\infty$ we encounter five domain boundaries effecting space-time transitions in the sequence $SU \rightarrow OSIU \rightarrow IU \rightarrow IUS \rightarrow IUOSI \rightarrow USI$. This sequence can be traced in figure 3 in the upper panel ($\beta > 0$) as we move from right to left with $\beta^2\kappa = -0.05$.

A second example can explain what occurs when $\beta^2\kappa$ is changed and some of the cases which occur on the boundaries. We start with $\beta^2\kappa = 0.05$ such that the horizons are given by the lightest blue curve in figure 2 and $\beta\gamma = 0.1$: small enough that we are in the U domain of figure 3. Decreasing $\beta^2\kappa$ then brings the minimum of the horizon curve through $\beta\gamma = 0.1$ crossing the domain boundary $U \rightarrow IUO$. A further decrease flattens the large-scale curvature until as $\kappa \rightarrow 0$, we arrive at the domain boundary $IUO \rightarrow IU$ where the exterior out-horizon tends to infinite radius ($O \rightarrow \infty$). Staying on the boundary we now consider the medium blue horizon curve in figure 2 for which $\kappa = 0$. Moving along the domain boundary by decreasing

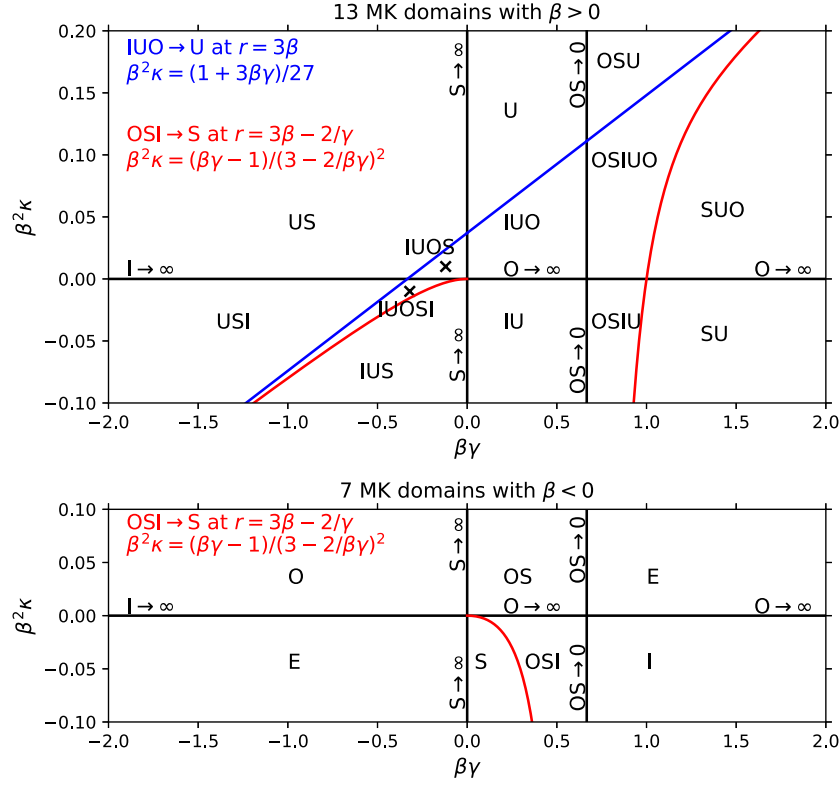


Figure 3. Dimensionless plot of the parameter space $\beta^2\kappa$ against $\beta\gamma$ showing domains and their boundaries; each corresponding to spacetimes in table 1. I/O = in/out horizons. S/U = stable/unstable light spheres. E = empty. The top plot is for $\beta > 0$ while the bottom is for $\beta < 0$.

$\beta\gamma$ then gradually increases the radius of the interior in-horizon until, where $\gamma = \kappa = 0$, we have the Schwarzschild case. This occurs at the intersection of domain boundaries in the top panel of figure 2 and, consistent with GR, the interior in-horizon occurs at $r = 2\beta$ and the unstable light sphere at $r = 3\beta$.

We have so far neglected to mention the cases where $\beta = 0$, however, using the tools developed above this becomes relatively simple. In the limit $\beta \rightarrow 0$ there is no unstable light sphere, the stable light sphere occurs at $r = -2/\gamma$ and the horizons occur where

$$B(r) = 1 + \gamma r - \kappa r^2 = 0. \quad (14)$$

Plots of γ against r and κ against γ can then be constructed and interpreted the same way as the dimensionless plots (figures 2 and 3 respectively). In effect the κ - γ parameter space looks very similar to the bottom panel of figure 3. The OS \rightarrow 0 domain boundary does not exist for $\beta = 0$ and neither do the I and E domains on the large $\beta\gamma$ side of it. The absence of $\beta < 0$ also inverts the horizontal axis and the curved domain boundary OSI \rightarrow S is given by $\kappa = -\gamma^2/4$. The five domains occurring in the limit $\beta \rightarrow 0$ also all occur for $\beta < 0$.

While all these spacetimes are found at the $\beta = 0$ intersection of the domain boundaries in the bottom panel of figure 3, we can also consider the domain boundaries in the κ - γ parameter space. Specifically, the S \rightarrow ∞ boundary at $\gamma = 0$ hosts de Sitter and anti-de Sitter spacetimes

depending on the sign of κ and if we then take $\kappa = 0$ as well ($\beta = \gamma = \kappa = 0$) we discover Minkowski spacetime at the intersection of domain boundaries. It should be noted, however, that care must be taken when taking the limits of parameters describing a family of spacetimes and an accurate analysis of the MK metric in the limits of β, γ and κ may require a coordinate-free approach [26].

The full $\beta\text{-}\gamma\text{-}\kappa$ parameter space which spans the MK metric contains twenty unique arrangements of features (including repetition of the empty case E due to large-scale curvature). Many of these have strict restrictions on the parameter ranges which produce them while others occur more freely. The five astrophysically significant cases are tightly constrained to the region near $\gamma = \kappa = 0$ in the top panel of figure 3 while cosmologically significant cases may even include $\beta \leq 0$. Conformal invariance also allows this theory to scale to much smaller length scales where even more exotic regions of the parameter space might become important.

4. Discussion

In this section we investigate in more depth some of the features of MK spacetimes using specific examples drawn from figures 2 and 3.

4.1. Physical singularities

If we return to consider the curvature scalars in section 2.3, we find (in the Einstein frame) they are finite everywhere in spacetime only if $\beta = \gamma = 0$. As mentioned above, these conditions reduce the MK metric to the de Sitter, anti-de Sitter or Minkowski metrics depending on κ . These are the only conditions under which the MK spacetime does not contain a physical singularity at $r = 0$.

It should be obvious that any spacetime with $\beta\gamma \neq 0$ not containing an interior in-horizon contains a naked singularity. Whether the singularity is naked or concealed behind a horizon is not, however, the only way in which the various singularities at $r = 0$ can differ. Depending on the sign of $B(r)$ in the region containing $r = 0$, the singularity could be timelike or spacelike [27]. If the singularity is in a region where $B(r) < 0$ it is spacelike—like the Schwarzschild black hole—but if $B(r) > 0$ the singularity is timelike. For $\beta > 0$ there are timelike singularities where $\beta\gamma > 2/3$ and spacelike singularities where $\beta\gamma < 2/3$. The reverse is true for $\beta < 0$ with timelike singularities where $\beta\gamma < 2/3$ and spacelike singularities where $\beta\gamma > 2/3$. These conditions coincide with the OS \rightarrow 0 transition in figure 3.

Examples of these singularities can be found in figure 4 where the OSI plot (on the left) contains a timelike singularity and the IUS plot (on the right) contains a spacelike singularity. Anything in the $B(r) < 0$ region around a spacelike singularity will reach $r = 0$ in finite time. The timelike singularities in $B(r) > 0$ regions, on the other hand, can be avoided and indeed all non-radial, null geodesics are eventually deflected away from the singularity. Not all MK spacetimes with spacelike singularities have interior in-horizons like the Schwarzschild black hole; some have only an exterior in-horizon or even no horizons at all. Similarly, timelike singularities can exist with or without an interior out-horizon (a Cauchy horizon).

4.2. Conformal scaling

Having used the presence of spacelike and timelike singularities in the Einstein frame of the MK metric (3) to explain the behaviour of null geodesics, we now consider what happens when we use a Weyl transformation to conformally scale the metric. Under

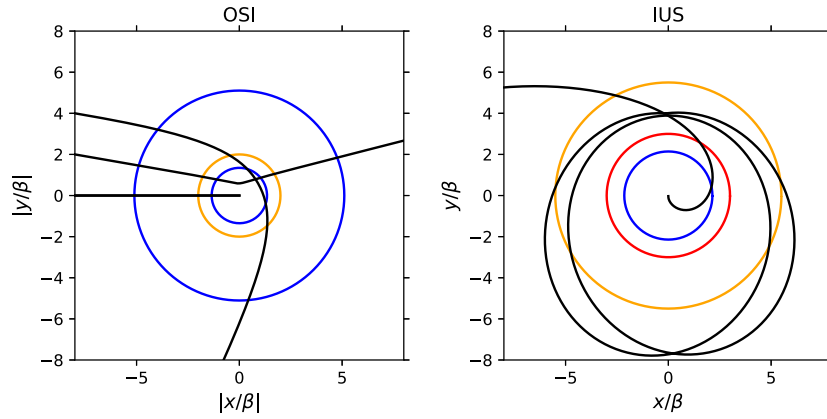


Figure 4. Polar plots of null geodesics. The horizons are in blue, stable light spheres in orange and unstable light spheres in red. Left: the domain OSI with parameters $\beta^2\kappa = -0.08$ and $\beta\gamma = 0.4$. Three null geodesics are shown including the unique, radial trajectory which reaches the timelike singularity. All null geodesics terminating at infinity will cross both horizons exactly twice while photons bound by the stable light sphere may cross repeatedly. Right: the domain IUS with parameters $\beta^2\kappa = -0.08$ and $\beta\gamma = -0.8$. Two photon trajectories are shown: one bound by the stable light sphere and the other captured by the spacelike singularity.

transformations of the form $g_{\mu\nu} \rightarrow \Omega^2(r)g_{\mu\nu}$ much of the geometry changes (i.e. massive geodesics and curvature scalars). Crucially, however, the null geodesics, light cones and causal structure remain invariant. This means the radial order of light spheres and horizons is unaffected and the configurations of features presented in table 1 are preserved under conformal scaling.

On the other hand because the curvature scalars are affected, the physical singularities present in the Einstein frame are not necessarily present in a spacetime which is related to the MK metric (3) by a conformal factor $\Omega^2(r)$. Indeed, studies of other conformally invariant theories have demonstrated that singularities can be removed from various spacetimes by the use of appropriate Weyl transformations [28–30].

By way of example we pick a previously used conformal factor [29] which should remove singularities at $r = 0$ while retaining the dimensions of the metric,

$$\Omega^2(r) = \left(1 + \frac{L^4}{r^4}\right) = \frac{1}{r^4} (r^4 + L^4), \quad (15)$$

where L is a length scale parameter. With a conformal factor of this form, the curvature scalars can be factorised while taking care to keep track of powers of r . This results in,

$$R = \frac{-r}{(r^4 + L^4)^3} [\text{polynomial in } r \text{ of degree } 11] \quad (16)$$

and

$$K = \frac{r^2}{(r^4 + L^4)^6} [\text{polynomial in } r \text{ of degree } 22], \quad (17)$$

neither of which are singular at $r = 0$. This demonstrates that a suitable conformal transformation can remove curvature singularities in the same way a suitable coordinate transformation

can remove coordinate singularities. Spacetimes with singularities removed in this way have been shown to be geodesically complete, with massive and null particles approaching $r = 0$ taking infinite proper time (or affine parameter value) to get there [29]. If L is sufficiently small, a conformal factor of this form could provide a singularity-free metric which is indistinguishable from the MK metric in the Einstein frame. Constraining L , as has been done for other theories [31], could be an avenue for further research.

4.3. Horizons and causal structure

The horizons occurring at $B(r) = 0$ also have an effect on the causal structure of the spacetime. The interior in-horizon in the IU case for example has an event horizon which light can only cross in one direction—it is causally equivalent to the Schwarzschild spacetime. Similarly, the addition of an interior out-horizon and an interior stable light sphere in the OSIU case makes it causally equivalent to the Reissner–Nordström spacetime with an outer event horizon and an inner Cauchy horizon. This is also causally equivalent to the Reissner–Nordström anti-de Sitter spacetime for which a study of the null geodesic effective potential [32] proves instructive for understanding the possible photon trajectories of our effective potential in figure 1. From here the domain boundary $OSIU \rightarrow SU$ is equivalent to the extremal Reissner–Nordström spacetime beyond which the timelike singularity is naked.

The exterior out-horizon can be considered equivalent to the de Sitter cosmological horizon. In the coordinates we have used, as in the static patch of de Sitter spacetime, the horizon is stationary. A more cosmologically appropriate metric on a time-evolving manifold S^3 equivalent to the Friedmann–Robertson–Walker metric could reveal further similarities or differences leading to cosmological tests of conformal gravity. In our analysis, however, the exterior out-horizon appears as an event horizon; like the Schwarzschild horizon turned inside-out. By extension we might expect the exterior in-horizon to be equivalent to an inside-out Cauchy horizon.

The OSI case in figure 4 shows how the existence of a pair of in/out-horizons on either side of a stable light sphere permits null geodesics to cross multiple times. This behaviour occurs in a total of four cases with either interior or exterior in/out-horizon pairs: IUOSI, OSIUO, OSIU and OSI. Photons in precessing, bound orbits around the stable light sphere may repeatedly cross both horizons.

To understand this behaviour in figure 4 we consider a photon falling towards the singularity from the region where $B(r) > 0$ outside the in-horizon. In the $B(r) > 0$ region the time coordinate always increases and the radial coordinate can increase or decrease. Crossing a horizon into a region with $B(r) < 0$, however, changes the sign of the temporal and radial terms in the metric (3). In the $B(r) < 0$ region the time coordinate can increase or decrease and the radial coordinate always decreases. An in-falling photon would then cross the in-horizon and, having crossed, it must by necessity cross the out-horizon on its approach to the singularity (being in a spacetime where time travels forward). Once in the $B(r) > 0$ region containing the singularity the radial and time terms resume their familiar configuration and the photon is turned back towards the interior out-horizon. Crossing this for the second time it finds itself in the $B(r) < 0$ region but this time it has emerged into a new spacetime where time flows in the opposite direction and thus it must continue to travel outwards through the interior in-horizon. If the photon is bound around the stable light sphere it will turn back and repeat this many times.

Further analysis of all horizons in MK spacetimes using other coordinates or Penrose diagrams such as that carried out for the $\beta = 0$ metric (14) [33] is warranted.

4.4. Stable light spheres and photon accumulation

While any stationary point of the effective potential (5) can have circular photon orbits; only at a minimum are these stable. Away from a circular orbit, these stable light spheres can contain precessing null geodesics similar to the massive geodesics followed by planets in the Solar System. Any photons emitted on a trajectory with sufficiently large impact parameter b from an object in the vicinity of a stable light sphere will enter a stable orbit. As more photons enter bound orbits, they could eventually contribute a non-negligible amount to the energy-momentum in the spacetime. The solution to the Einstein field equations for a radially distributed spherical shell of photons has been found [24]. Perturbatively adding this form of the energy-momentum tensor to the field equation (2) should provide a starting point for understanding how photon accumulation might distort spacetime.

Most MK metric spacetimes containing stable light spheres exist away from the five domains permitted by the astrophysical constraints on γ and κ . Further, for the spacetimes IUOS and IUOSI, the stable light sphere lies beyond an exterior out-horizon and thus cannot be seen by observers living between the interior in and exterior out-horizons, though it is accessible to observers outside the exterior out-horizon. This leaves IUS as the only domain that could contain an astrophysically observable, stable light sphere. A bound, null geodesic in an IUS spacetime is plotted in the right panel of figure 4 alongside another terminating at $r = 0$. It should be noted, however, that given the astrophysical constraints on γ any such stable light sphere could form only at galactic scales and due to the photons being bound, any observation would not be direct but via the lensing of light through and around it [24]. A lack of evidence for stable light spheres could provide further constraints on the parameters of the MK metric.

5. Conclusions

Having investigated the null geodesics of the MK metric using an effective potential, we identified several distinct features arising from the full β - γ - κ parameter space of the metric. These include stable and unstable light spheres, four varieties of horizons, and spacelike and time-like singularities. Twenty domains in the MK metric parameter space were identified, each of which corresponds to a unique arrangement of features. We linked a few further extremal cases occurring on the domain boundaries to more widely studied metrics in general relativity. Interesting properties of some of the features were also highlighted along with directions for future theoretical and observational work.

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References

- [1] Bertotti B, Iess L and Tortora P 2003 *Nature* **425** 374–6
- [2] Dyson F W, Eddington A S and Davidson C 1920 *Phil. Trans. R. Soc. A* **220** 291–333
- [3] Pound R V and Snider J L 1964 *Phys. Rev. Lett.* **13** 539–40
- [4] Clemence G M 1947 *Rev. Mod. Phys.* **19** 361–4
- [5] Weisberg J and Taylor J 2005 *Astron. Soc. Pac. Conf. Ser.* **328** 25 (<https://ui.adsabs.harvard.edu/abs/2005ASPC..328...25W>)
- [6] Popper D M 1954 *Astrophys. J.* **120** 316

- [7] Abbott B P *et al* (LIGO Scientific Collaboration and Virgo Collaboration) 2016 *Phys. Rev. Lett.* **116** 061102
- [8] Corbelli E and Salucci P 2000 *Mon. Not. R. Astron. Soc.* **311** 441–7
- [9] van den Bosch F *et al* 2017 *Mon. Not. R. Astron. Soc.* **468** 1962–80
- [10] Schmidt R W and Allen S W 2007 *Mon. Not. R. Astron. Soc.* **379** 209–21
- [11] Rapetti D A, Morris R G, Allen S W, Schmidt R W, Ebeling H and Fabian A C 2008 *Mon. Not. R. Astron. Soc.* **383** 879–96
- [12] Mannheim P D and Kazanas D 1989 *Astrophys. J.* **342** 635–8
- [13] Mannheim P D 2006 *Prog. Part. Nucl. Phys.* **56** 340–445
- [14] Mannheim P D and Kazanas D 1991 *Phys. Rev. D* **44** 417–423
- [15] Edery A and Paranjape M B 1998 *Phys. Rev. D* **58** 024011
- [16] Pireaux S 2004 *Class. Quantum Grav.* **21** 4317
- [17] Sultana J, Kazanas D and Said J L 2012 *Phys. Rev. D* **86** 084008
- [18] Mannheim P D and O’Brien J G 2012 *Phys. Rev. D* **85** 124020
- [19] Edery A, Méthot A A and Paranjape M B 2001 *Gen. Relativ. Gravit.* **33** 2075–9
- [20] Sultana J and Kazanas D 2012 *Phys. Rev. D* **85** 081502
- [21] Cattani C, Scalia M, Laserra E, Bochicchio I and Nandi K K 2013 *Phys. Rev. D* **87** 047503
- [22] Horne K 2016 *Mon. Not. R. Astron. Soc.* **458** 4122–8
- [23] Pireaux S 2004 *Class. Quantum Grav.* **21** 1897
- [24] Foster D P, Langford J and Perez-Giz G 2016 *Class. Quantum Grav.* **33** 055002
- [25] Villanueva J and Olivares M 2013 *J. Cosmol. Astropart. Phys.* **2013** 040
- [26] Paiva F, Rebouças M and McCullaum M 1993 *Class. Quantum Grav.* **10** 1165
- [27] Faraoni V 2015 *Cosmological and Black Hole Apparent Horizons* vol 907 (Berlin: Springer) (<https://doi.org/10.1007/978-3-319-19240-6>)
- [28] Naelikar J V and Kembhavi A K 1977 *Lett. Nuovo Cimento* **19** 517–20
- [29] Bambi C, Modesto L and Rachwa L 2017 *J. Cosmol. Astropart. Phys.* **JCAP09(2017)033**
- [30] Chakrabarty H, Benavides-Gallego C A, Bambi C and Modesto L 2018 *J. High Energy Phys.* **JHEP03(2018)013**
- [31] Zhou M, Cao Z, Abdikamalov A, Ayzenberg D, Bambi C, Modesto L and Nampalliwar S 2018 *Phys. Rev. D* **98** 024007
- [32] Cruz N, Olivares M, Saavedra J and Villanueva J 2011 arXiv:1111.0924
- [33] Edery A and Paranjape M B 1999 *Gen. Relativ. Gravit.* **31** 1031–47