

ON THE WELL-POSEDNESS OF REDUCED 3D PRIMITIVE GEOSTROPHIC ADJUSTMENT MODEL WITH WEAK DISSIPATION

CHONGSHENG CAO, QUYUAN LIN*, AND EDRISS S. TITI

ABSTRACT. In this paper we prove the local well-posedness and global well-posedness with small initial data of the strong solution to the reduced 3D primitive geostrophic adjustment model with weak dissipation. The term reduced model means that the relevant physical quantities depend only on two spatial variables. The weak dissipation helps us overcome the ill-posedness of the original model. We also prove the global well-posedness of the strong solution to the Voigt α -regularization of this model, and establish the convergence of the strong solution of the Voigt α -regularized model to the corresponding solution of the original model. Furthermore, we derive a criterion for existence of finite-time blow-up of the original model with weak dissipation based on Voigt α -regularization.

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1. INTRODUCTION

It is commonly believed that the dynamics of ocean and atmosphere adjusts itself toward a geostrophic balance. The following reduced 3D primitive geostrophic adjustment system is one of the main diagnostic models for studying geostrophic adjustment (cf. e.g., [28, 34, 52]):

$$u_t + u u_x + w u_z - f_0 v + p_x = 0, \quad (1)$$

$$v_t + u v_x + w v_z + f_0 u = 0, \quad (2)$$

$$p_z + T = 0, \quad (3)$$

$$u_x + w_z = 0, \quad (4)$$

$$T_t + u T_x + w T_z = 0, \quad (5)$$

where the velocity field (u, v, w) , the temperature T and the pressure p are the unknown functions of horizontal variable x , vertical variable z , and time t , and f_0 is the Coriolis parameter. System (1)–(5) is reduced from the 3D inviscid primitive equations model by assuming that the flow is independent of the third spatial variable. This system has been a standard framework for studying geostrophic adjustment of frontal anomalies in a rotating continuously stratified fluid of strictly rectilinear fronts and jets (cf. e.g., [2, 27, 28, 33, 34, 36, 52, 54] and references therein).

The first systematically mathematical studies of the viscous primitive equations (PEs) were carried out in the 1990s by Lions–Temam–Wang [45, 46, 47]. They considered the PEs with both full viscosities and full diffusivities and established the global existence of weak solutions. The uniqueness of weak solutions in the 3D viscous case is still an open problem, while the weak solutions in 2D turn out to be unique, see Bresch, Guillén-González, Masmoudi and Rodríguez-Bellido [5]. Concerning the strong solutions for the 2D case, the local existence result was established by Guillén-González, Masmoudi and Rodríguez-Bellido [30], while the global existence for 2D case was proved by Bresch, Kazhikhov and Lemoine in [6], and by Temam and Ziane

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*Corresponding author. Department of Mathematics, Texas A&M University, College Station, TX 77840, USA. E-mail address: abellyn@hotmail.com.

in [60]. The global existence of strong solutions for $3D$ case was established by Cao and Titi in [16] and later by Kobelkov in [35], see also the subsequent articles of Kukavica and Ziane [40, 41] for different boundary conditions, as well as Hieber and Kashiwabara [32] for some progress towards relaxing the smoothness on the initial data by using the semigroup method. On the other hand, it was proved that smooth solutions to the inviscid $2D$ or $3D$ PEs, with or without coupling to the temperature equation, can develop singularities in finite time, see Cao et al. [8] and Wong [61]. Motivated by physical considerations, it is of great interest to investigate the well-posedness, finite-time blow-up, or even ill-posedness of the PEs with only partial viscosities or partial diffusivities. There have been several mathematical studies of these models. The global existence and uniqueness of strong solutions for the PEs with full viscosities and with either only horizontal or only vertical diffusivity have been established by Cao–Titi [18] and Cao–Li–Titi [9, 10]. Concerning partial viscosities, global well-posedness of the PEs with only horizontal viscosity and with either only horizontal or only vertical diffusivity was established by Cao–Li–Titi in [11, 12] and [13]. On the other hand, there are no results concerning the well-posedness or finite-time blow-up for the PEs with only vertical viscosity, even in $2D$. In this paper, we are interested in this case. More specifically, we are interested in system (1)–(5) with only vertical viscosity and full diffusivity:

$$u_t - \nu u_{zz} + u u_x + w u_z - f_0 v + p_x = 0, \quad (6)$$

$$v_t - \nu v_{zz} + u v_x + w v_z + f_0 u = 0, \quad (7)$$

$$p_z + T = 0, \quad (8)$$

$$u_x + w_z = 0, \quad (9)$$

$$T_t - \kappa \Delta T + u T_x + w T_z = 0. \quad (10)$$

The main difficulty in proving well-posedness of system (6)–(10) is the lack of control over the horizontal derivatives. The case with only vertical viscosity, i.e. system (6)–(10), is more difficult than the situation with only horizontal viscosity, for which the global well-posedness was established. In fact, system (1)–(5) and system (6)–(10) turn out to be ill-posed. By assuming $f_0 = 0$, $v \equiv 0$, and $T \equiv 0$ in system (1)–(5) and system (6)–(10), we end up with the so-called $2D$ hydrostatic Euler equations:

$$u_t + u u_x + w u_z + p_x = 0, \quad (11)$$

$$p_z = 0, \quad (12)$$

$$u_x + w_z = 0, \quad (13)$$

and $2D$ hydrostatic Navier-Stokes equations:

$$u_t + u u_x + w u_z + p_x - \nu u_{zz} = 0, \quad (14)$$

$$p_z = 0, \quad (15)$$

$$u_x + w_z = 0, \quad (16)$$

correspondingly. The linear ill-posedness of system (11)–(13) near certain shear-flows was established by Renardy in [53]. The author of [53] has also indicated, without providing details, that one should be able to show the linear ill-posedness of system (14)–(16) in any Sobolev space by using matched asymptotics. The nonlinear ill-posedness of system (11)–(13) was established by Han-Kwan and Nguyen in [31], where they built an abstract framework to show that the hydrostatic Euler equations are ill-posed in any Sobolev space. One might be able to argue that the main reason for the ill-posedness in these is again the lack of control over the horizontal derivatives. From a mathematical perspective, system (14)–(16) is reminiscent of the $2D$ Prandtl system in the upper half space. Ill-posedness of Prandtl system in Sobolev spaces was established by Gérard-Varet and Dormy [24], and by Gérard-Varet and Nguyen [26]. The existence of finite-time blow-up for Prandtl system was shown in [22]. On the other hand, well-posedness results of hydrostatic Euler equations and Prandtl system have been obtained by assuming either real analyticity or some special structures of the

initial data [3, 4, 29, 37, 38, 39, 48, 49, 50]. Recently, the authors in [25] showed the local well-posedness for system (14)–(16) with convex initial data in a Gevrey class.

All the results and discussions above suggest that, in order to prove the well-posedness for system (6)–(10) in Sobolev spaces, without assuming any special structures, some additional horizontal dissipation terms are necessary. In the derivation of system (14)–(16) from the Navier-Stokes equations, only vertical viscosity has survived. This suggests instead of strong dissipation, i.e., horizontal viscosity, we should consider some weaker horizontal dissipation. Inspired by Samelson and Vallis [55], and Salmon [56, p. 150], by introducing the linear (Rayleigh-like friction) damping in both horizontal momentum equation (6) and vertical hydrostatic approximation (7), we consider the following system:

$$u_t + u u_x + w u_z + \epsilon_1 u - f_0 v + p_x - \nu u_{zz} = 0, \quad (17)$$

$$v_t + u v_x + w v_z + \epsilon_1 v + f_0 u - \nu v_{zz} = 0, \quad (18)$$

$$\epsilon_2 w + p_z + T = 0, \quad (19)$$

$$u_x + w_z = 0, \quad (20)$$

$$T_t - \kappa \Delta T + u T_x + w T_z = 0 \quad (21)$$

in the horizontal channel $\{(x, z) : 0 \leq z \leq H, x \in \mathbb{R}\}$, with the following boundary conditions:

$$(u_z, v_z, w, T)|_{z=0, H} = 0, \quad (22)$$

$$u, v, w, T \text{ are periodic in } x \text{ with period } 1.$$

Here ϵ_1 and ϵ_2 are positive constants representing the linear (Rayleigh-like friction) damping coefficients, and ν is positive constant that stands for the vertical viscosity of the horizontal momentum equations. Unlike the strong horizontal dissipation, i.e., horizontal viscosity, we regard the linear damping term $\epsilon_2 w$ as a weak dissipation. With this weak dissipation, we are able to prove the local well-posedness and global well-posedness with small initial data. Accordingly, when $\epsilon_2 > 0$, one can view the term $\epsilon_2 w$ in (19) as having a “regularizing” effect, since it annihilates the possible ill-posedness as indicated in [53] when $\epsilon_2 = 0$. This also indicates that the damping term $\epsilon_2 w$ has a nonnegligible effect on the dynamics and leads to a reliable numerical scheme. In terms of physical motivation, the damping terms $\epsilon_1 u$, $\epsilon_1 v$, and $\epsilon_2 w$ can be interpreted as the Rayleigh friction with the bottom of ocean in the original model with physical boundaries, and hence produce weak dissipation. Furthermore, in [17], the authors consider the 3D Salmon’s planetary geostrophic oceanic dynamics model, which is the three-dimensional version of system (17)–(21), and with (17)–(18) replaced by geostrophic balance equations. Indeed, it is well known that when $\epsilon_2 = 0$ the planetary geostrophic model of ocean circulation is ill-posed (see, e.g., [17] and reference therein). This in turn motivated Salmon to introduce the friction term $\epsilon_2 w$, with $\epsilon_2 > 0$, in the planetary geostrophic model to overcome this problem. Consequently, this provides an additional motivation for taking $\epsilon_2 > 0$ in our system.

In order to study the possible finite-time blow-up of system (17)–(21), and to give a more reliable numerical model/scheme, we also study the Voigt α -regularization with respect to z variable of (17)–(21). More specifically, we consider the following model:

$$(u - \alpha^2 u_{zz})_t - \nu u_{zz} + u u_x + w u_z + \epsilon_1 u - f_0 v + p_x = 0, \quad (23)$$

$$(v - \alpha^2 v_{zz})_t - \nu v_{zz} + u v_x + w v_z + \epsilon_1 v + f_0 u = 0, \quad (24)$$

$$\epsilon_2 w + p_z + T = 0, \quad (25)$$

$$u_x + w_z = 0, \quad (26)$$

$$T_t - \kappa \Delta T + u T_x + w T_z = 0 \quad (27)$$

in the horizontal channel $\{(x, z) : 0 \leq z \leq H, x \in \mathbb{R}\}$, with boundary conditions (22). We show the global well-posedness of system (23)–(27) with $\nu = 0$. The same result holds for $\nu > 0$. Based on this, we show the convergence of the strong solution of system (23)–(27) to the corresponding solution of system (17)–(21) on the interval of existence of the latter, as $\alpha \rightarrow 0$. Furthermore, we derive, based on this Voigt α -regularization,

a criterion for existence of finite-time blow-up in system (17)–(21). For more details of Voigt α -regularization of the 3D Euler equations, we refer the reader to [19, 42, 43].

The paper is organized as follows. In section 2, we introduce the main notations and collect some preliminary results. In section 3, we show the local well-posedness and global well-posedness with small initial data for the system (17)–(21). In section 3.3, by assuming $f_0 = 0$, $v \equiv 0$, and $T \equiv 0$, we reduce to the 2D hydrostatic Navier-Stokes equations (14)–(16) with damping, for which we obtain local well-posedness by requiring less on the initial conditions. In section 4, we show the global well-posedness of system (23)–(27). In section 5, by assuming $f_0 = 0$, $v \equiv 0$, and $T \equiv 0$, we show the convergence of the strong solution of system (23)–(27) to the corresponding solution of system (17)–(21) on the interval of existence of the latter, as $\alpha \rightarrow 0$. The assumptions that $f_0 = 0$, $v \equiv 0$, and $T \equiv 0$ were made for simplicity. The same convergence result holds without these assumptions. In section 6, by assuming $f_0 = 0$, $v \equiv 0$, and $T \equiv 0$, we derive, based on Voigt α -regularization, a criterion for finite-time blow-up in system (17)–(21).

2. PRELIMINARIES

In this section, we introduce some notations and collect some preliminary results which will be used in the rest of this paper. For domain $\Omega \subset \mathbb{R}^2$, we denote by $L^p(\Omega)$, for $p \geq 1$, the Lebesgue space of functions f satisfying $\int_{\Omega} |f|^p dx dz < \infty$, and denote the corresponding norm by $\|f\|_{L^p} := \|f\|_{L^p(\Omega)} = (\int_{\Omega} |f|^p dx dz)^{\frac{1}{p}}$. For the space $L^2(\Omega)$, we denote its inner product by (\cdot, \cdot) . Similarly, for $m \geq 1$ an integer, we denote by $H^m(\Omega) = W^{m,2}(\Omega)$ the Sobolev space of functions f satisfying $\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^2}^2 < \infty$, and denote the

corresponding norm by $\|f\|_{H^m} := (\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^2}^2)^{\frac{1}{2}}$. Given time $\mathcal{T} > 0$, we denote by $L^p(0, \mathcal{T}; X)$ the space

of functions $f : [0, \mathcal{T}] \rightarrow X$ satisfying $\int_0^{\mathcal{T}} \|f(t)\|_X^p dt < \infty$, where X is a Banach space and $\|\cdot\|_X$ represents its norm. Similarly, we denote by $C([0, \mathcal{T}]; X)$ the space of continuous functions $f : [0, \mathcal{T}] \rightarrow X$. We write $L^p(0, \mathcal{T}; L^2)$ and $L^p(0, \mathcal{T}; H^m)$ instead of $L^p(0, \mathcal{T}; L^2(\Omega))$ and $L^p(0, \mathcal{T}; H^m(\Omega))$, respectively, for simplicity. When $\Omega = \mathbb{T}^2$ is the unit two-dimensional flat torus, we denote by $L^2(\mathbb{T}^2)$, $H^m(\mathbb{T}^2)$ the set of all periodic functions in x, z with period 1, which have bounded $L^2(\mathbb{T}^2)$ norm or $H^m(\mathbb{T}^2)$ norm, respectively. We start by recalling the following:

Lemma 1. (cf. [14]) *Assume that $f, g, h, g_z, h_x \in L^2(\mathbb{T}^2)$. Then*

$$\int_{\mathbb{T}^2} |fgh| dx dz \leq C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} (\|g\|_{L^2}^{\frac{1}{2}} + \|g_z\|_{L^2}^{\frac{1}{2}}) \|h\|_{L^2}^{\frac{1}{2}} (\|h\|_{L^2}^{\frac{1}{2}} + \|h_x\|_{L^2}^{\frac{1}{2}}).$$

Proof. First, recall that by one-dimensional Agmon's inequality (or Gagliardo–Nirenberg interpolation inequality), for $\phi \in H^1(0, 1)$, one has

$$\|\phi\|_{L^\infty(0,1)} \leq C \left(\|\phi\|_{L^2(0,1)} + \|\phi\|_{L^2(0,1)}^{\frac{1}{2}} \|\phi_x\|_{L^2(0,1)}^{\frac{1}{2}} \right). \quad (28)$$

Therefore, by Hölder's inequality and Agmon's inequality (28),

$$\begin{aligned} \int_{\mathbb{T}^2} |fgh|(x, z) dx dz &\leq C \int_0^1 \left[\left(\int_0^1 |f(x, z)|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 |g(x, z)|^2 dx \right)^{\frac{1}{2}} \left(\sup_{0 \leq x \leq 1} |h(x, z)| \right) \right] dz \\ &\leq C \int_0^1 \left\{ \left(\int_0^1 |f(x, z)|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 |g(x, z)|^2 dx \right)^{\frac{1}{2}} \right. \\ &\quad \left. \left[\left(\int_0^1 |h(x, z)|^2 dx \right)^{\frac{1}{4}} \left(\int_0^1 |h_x(x, z)|^2 dx \right)^{\frac{1}{4}} + \left(\int_0^1 |h(x, z)|^2 dx \right)^{\frac{1}{2}} \right] \right\} dz \\ &\leq C \|f\|_{L^2} \sup_{0 \leq z \leq 1} \left(\int_0^1 |g(x, z)|^2 dx \right)^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} (\|h\|_{L^2}^{\frac{1}{2}} + \|h_x\|_{L^2}^{\frac{1}{2}}). \end{aligned} \quad (29)$$

By Minkowski's inequality, Agmon's inequality (28), and Hölder inequality,

$$\begin{aligned} \sup_{0 \leq z \leq 1} \left(\int_0^1 |g(x, z)|^2 dx \right)^{\frac{1}{2}} &\leq C \left(\int_0^1 \sup_{0 \leq z \leq 1} |g(x, z)|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^1 \left[\left(\int_0^1 |g(x, z)|^2 dz \right)^{\frac{1}{2}} \left(\int_0^1 |g_z(x, z)|^2 dz \right)^{\frac{1}{2}} + \int_0^1 |g(x, z)|^2 dz \right] dx \right)^{\frac{1}{2}} \\ &\leq C \|g\|_{L^2}^{\frac{1}{2}} (\|g\|_{L^2}^{\frac{1}{2}} + \|g_z\|_{L^2}^{\frac{1}{2}}). \end{aligned} \quad (30)$$

Inserting (30) to (29) yields the desired inequality. \square

Next we prove the following:

Lemma 2. *Assume that $f \in H^1(\mathbb{T}^2)$ and $f_{xz} \in L^2(\mathbb{T}^2)$. Then $f \in L^\infty(\mathbb{T}^2)$. Moreover,*

$$\|f\|_{L^\infty} \leq C (\|f\|_{H^1}^2 + \|f_{xz}\|_{L^2}^2)^{\frac{1}{2}}.$$

Proof. Let $\{\hat{f}_k\}_{k \in \mathbb{Z}^2}$ be the Fourier coefficients of f . By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|f\|_{L^\infty} &\leq \sum_{k \in \mathbb{Z}^2} |\hat{f}_k| = \sum_{k \in \mathbb{Z}^2} \frac{|\hat{f}_k| (1 + k_1^2 + k_2^2 + k_1^2 k_2^2)^{\frac{1}{2}}}{(1 + k_1^2 + k_2^2 + k_1^2 k_2^2)^{\frac{1}{2}}} \\ &\leq \left(\sum_{k \in \mathbb{Z}^2} |\hat{f}_k|^2 (1 + k_1^2 + k_2^2 + k_1^2 k_2^2) \right)^{\frac{1}{2}} \left(\sum_{k \in \mathbb{Z}^2} \frac{1}{(1 + k_1^2)(1 + k_2^2)} \right)^{\frac{1}{2}} \leq C (\|f\|_{H^1}^2 + \|f_{xz}\|_{L^2}^2)^{\frac{1}{2}} < \infty. \end{aligned}$$

Therefore, $f \in L^\infty(\mathbb{T}^2)$. \square

We also need the following Aubin-Lions theorem.

Proposition 3. *(Aubin-Lions Lemma, cf. Simon [57] Corollary 4) Assume that X , B and Y are three Banach spaces, with $X \hookrightarrow B \hookrightarrow Y$. Then it holds that*

(i) *If \mathcal{F} is a bounded subset of $L^p(0, \mathcal{T}; X)$, where $1 \leq p < \infty$, and $\mathcal{F}_t := \{\frac{\partial f}{\partial t} | f \in \mathcal{F}\}$ is bounded in $L^1(0, \mathcal{T}; Y)$, then \mathcal{F} is relative compact in $L^p(0, \mathcal{T}; B)$.*

(ii) *If \mathcal{F} is a bounded subset of $L^\infty(0, \mathcal{T}; X)$ and \mathcal{F}_t is bounded in $L^q(0, \mathcal{T}; Y)$, where $q > 1$, then \mathcal{F} is relative compact in $C([0, \mathcal{T}]; B)$.*

3. WELL-POSEDNESS OF SYSTEM (17)–(21)

In this section we study the system (17)–(21) in the horizontal channel $\{(x, z) : 0 \leq z \leq H, x \in \mathbb{R}\}$. We complement this system with the boundary conditions (22) and the initial condition

$$(u, v, T)|_{t=0} = (u_0, v_0, T_0). \quad (31)$$

In particular, without loss of generality, we choose $H = \frac{1}{2}$. Instead of considering this physical problem, in this section, we consider another problem, that is, system (17)–(21) in the unit two dimensional torus \mathbb{T}^2 , subject to the following symmetric boundary conditions and initial conditions:

$$u, v, w, p \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1; \quad (32)$$

$$u, v, p \text{ are even in } z, \text{ and } w, T \text{ are odd in } z; \quad (33)$$

$$(u, v, T)|_{t=0} = (u_0, v_0, T_0). \quad (34)$$

The periodicity and symmetry are valid due to the fact that the periodic subspace \mathcal{H} , give by

$\mathcal{H} := \{(u, v, w, p, T) \mid u, v, w, p \text{ and } T \text{ are spatially periodic in both } x \text{ and } z \text{ variables with period } 1, \text{ and are even, even, odd, even, odd with respect to } z \text{ variable, respectively}\},$

is invariant under the evolution system (17)–(21). After solving this problem in the flat torus, the solution restricted on original horizontal channel $\{(x, z) : 0 \leq z \leq \frac{1}{2}, x \in \mathbb{R}\}$ will solve the original physical problem with corresponding boundary conditions (22) and initial conditions (31).

3.1. Reformulation of The Problem. First, let us reformulate the system (17)–(21) by deriving equations for w, p_x and p_z in terms of u, v and T . For the sake of simplicity, we drop the argument t in functions when there is no confusion. First, from (20) and by boundary condition (33), i.e., $w(x, 0) = 0$, we have

$$w(x, z) = - \int_0^z u_x(x, s) ds. \quad (35)$$

From (19) and (35), we have

$$p_z(x, z) = -T(x, z) - \epsilon_2 w(x, z) = -T(x, z) + \epsilon_2 \int_0^z u_x(x, s) ds. \quad (36)$$

Next, we will derive equation for p_x . Notice that since $w(x, 0) = w(x, 1) = 0$, from (35), one has the compatibility condition

$$\int_0^1 u_x(x, z) dz = 0. \quad (37)$$

Let us denote by $c(t) := \int_0^1 u(x, z, t) dz$ and $d(x, t) := \int_0^1 v(x, z, t) dz$. Integrating (17) with respect to z over $(0, 1)$, using boundary condition (32) and (33), one has:

$$\dot{c}(t) + \epsilon_1 c(t) + \int_0^1 \left(uu_x(x, z) + wu_z(x, z) + p_x(x, z) \right) dz = f_0 d(x, t).$$

By integration by parts and using (20), (32) and (33), we get

$$\dot{c}(t) + \epsilon_1 c(t) + \int_0^1 \left((u^2)_x(x, z) + p_x(x, z) \right) dz = f_0 d(x, t). \quad (38)$$

Integrating (38) with respect to x over $(0, 1)$, using compatibility condition (37), we have

$$\dot{c}(t) + \epsilon_1 c(t) + \int_0^1 \int_0^1 \left((u^2)_x(x, z) + p_x(x, z) \right) dx dz = f_0 \int_0^1 d(x, t) dx.$$

Thanks to (32), we have

$$\dot{c}(t) + \epsilon_1 c(t) = f_0 \int_0^1 d(x, t) dx. \quad (39)$$

Plugging (39) back into (38) yields

$$\int_0^1 p_x(x, z) dz = f_0 \int_0^1 v(x, z) dz - f_0 \int_0^1 \int_0^1 v(x, z) dx dz - \int_0^1 2uu_x(x, z) dz. \quad (40)$$

Next, from (35) and (36), we have

$$p(x, z) = p_s(x) + \epsilon_2 \int_0^z \int_0^s u_x(x, \xi) d\xi ds - \int_0^z T(x, s) ds, \quad (41)$$

where $p_s(x) = p(x, 0)$ is the pressure at $z = 0$. By differentiating (41) with respect to x , and integrating respect to z over $(0, 1)$, by virtue of (40), we have

$$(p_s)_x(x) = \int_0^1 \left[\int_0^{z'} T_x(x, s) ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x, \xi) d\xi ds + f_0 v(x, z') - 2uu_x(x, z') \right] dz' - f_0 \int_0^1 \int_0^1 v(x', z') dx' dz'. \quad (42)$$

Therefore, by differentiating (41) with respect to x , and using (42), we have

$$\begin{aligned} p_x(x, z) &= \epsilon_2 \int_0^z \int_0^s u_{xx}(x, \xi) d\xi ds - \int_0^z T_x(x, s) ds \\ &+ \int_0^1 \left[\int_0^{z'} T_x(x, s) ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x, \xi) d\xi ds + f_0 v(x, z') - 2uu_x(x, z') \right] dz' \\ &- f_0 \int_0^1 \int_0^1 v(x', z') dx' dz'. \end{aligned} \quad (43)$$

By virtue of (35), (36) and (43), and since p is determined up to a constant, the unknowns for system (17)–(21) are only (u, v, T) . Therefore, we reformulate system (17)–(21) to the following system:

$$u_t - \nu u_{zz} + u u_x + w u_z + \epsilon_1 u - f_0 v + p_x = 0, \quad (44)$$

$$v_t - \nu v_{zz} + u v_x + w v_z + \epsilon_1 v + f_0 u = 0, \quad (45)$$

$$T_t - \kappa \Delta T + u T_x + w T_z = 0, \quad (46)$$

with w, p_x, p_z defined by:

$$w(x, z) := - \int_0^z u_x(x, s) ds, \quad (47)$$

$$\begin{aligned} p_x(x, z) &:= \epsilon_2 \int_0^z \int_0^s u_{xx}(x, \xi) d\xi ds - \int_0^z T_x(x, s) ds \\ &+ \int_0^1 \left[\int_0^{z'} T_x(x, s) ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x, \xi) d\xi ds + f_0 v(x, z') - 2uu_x(x, z') \right] dz' \\ &- f_0 \int_0^1 \int_0^1 v(x', z') dx' dz', \end{aligned} \quad (48)$$

$$p_z(x, z) := -T(x, z) + \epsilon_2 \int_0^z u_x(x, s) ds. \quad (49)$$

In this section, we are interested in system (44)–(46) with (47)–(49) in the unit two dimensional torus $\Omega = \mathbb{T}^2$, subject to the following symmetry boundary conditions and initial conditions:

$$u, v \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1; \quad (50)$$

$$u, v \text{ are even in } z, \text{ and } T \text{ is odd in } z; \quad (51)$$

$$(u, v, T)|_{t=0} = (u_0, v_0, T_0). \quad (52)$$

It's worth mentioning again that our system (44)–(46) with (47)–(49) satisfies the compatibility condition (37). By virtue of (47)–(49) and (50), (51), we obtain that w, p also satisfy the symmetry conditions:

$$w \text{ and } p \text{ are periodic in } x \text{ and } z \text{ with period } 1; \quad (53)$$

$$p \text{ is even in } z, \text{ and } w \text{ is odd in } z. \quad (54)$$

From (47) and (49), and by differentiating (47) with respect to z , we have

$$\epsilon_2 w + p_z + T = 0, \quad (55)$$

$$u_x + w_z = 0. \quad (56)$$

Therefore, we conclude system (44)–(46) with (47)–(49) and subject to (50)–(52) is equivalent to original system (17)–(21) subject to (32)–(34).

3.2. Local Well-posedness. In this section, we will show the local regularity of strong solutions to the system (44)–(46) with (47)–(49), subject to boundary and initial conditions (50)–(52). First, we give the definition of strong solution to system (44)–(46) with (47)–(49).

Definition 4. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (50) and (51), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Given time $\mathcal{T} > 0$, we say (u, v, T) is a strong solution to system (44)–(46) with (47)–(49), subject to (50)–(52), on the time interval $[0, \mathcal{T}]$, if*

(i) u, v and T satisfy the symmetry conditions (50) and (51);

(ii) u, v and T have the regularities

$$\begin{aligned} u, v, T, u_x, v_x, T_x &\in L^\infty(0, \mathcal{T}; H^1), \quad u_z, v_z, u_{xz}, v_{xz} \in L^2(0, \mathcal{T}; H^1), \quad T, T_x \in L^2(0, \mathcal{T}; H^2), \\ u, v, T &\in L^\infty(0, \mathcal{T}; L^\infty) \cap C([0, \mathcal{T}]; L^2), \quad \nabla u, \nabla v, \nabla T \in L^2(0, \mathcal{T}; L^\infty), \quad \partial_t u, \partial_t v, \partial_t T \in L^2(0, \mathcal{T}; L^2); \end{aligned}$$

(iii) u, v and T satisfy system (44)–(46) in the following sense:

$$\begin{aligned} \partial_t u - \nu u_{zz} + uu_x + wu_z + \epsilon_1 u - f_0 v + p_x &= 0 \quad \text{in } L^2(0, \mathcal{T}; L^2); \\ \partial_t v - \nu v_{zz} + uv_x + wv_z + \epsilon_1 v + f_0 u &= 0 \quad \text{in } L^2(0, \mathcal{T}; L^2); \\ \partial_t T - \kappa \Delta T + uT_x + wT_z &= 0 \quad \text{in } L^2(0, \mathcal{T}; L^2), \end{aligned}$$

with w, p_x, p_z defined by (47)–(49), and fulfill the initial condition (52).

We have the following result concerning the existence and uniqueness of strong solutions to system (44)–(46) with (47)–(49), subject to (50)–(52), on $\mathbb{T}^2 \times (0, \mathcal{T})$, for some positive time \mathcal{T} .

Theorem 5. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (50) and (51), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Then there exists some time $\mathcal{T} > 0$ such that there exists a unique strong solution (u, v, T) of system (44)–(46) with (47)–(49), subject to (50)–(52), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.*

In section 3.2.1, we establish the existence of solutions to system (44)–(46) with (47)–(49) by employing the standard Galerkin approximation procedure. In section 3.2.2, we establish formal *a priori* estimates for the solutions of system (44)–(46) with (47)–(49). These estimates can be justified rigorously by deriving them first to the Galerkin approximation system and then passing to the limit using the Aubin-Lions compactness theorem. In section 3.2.3, we establish the uniqueness of strong solutions, and its continuous dependence on the initial data.

3.2.1. Galerkin approximating system. In this section, we employ the standard Galerkin approximation procedure. Let

$$\phi_k = \phi_{k_1, k_2} := \begin{cases} \sqrt{2} \exp(2\pi i k_1 x) \cos(2\pi k_2 z) & \text{if } k_2 \neq 0 \\ \exp(2\pi i k_1 x) & \text{if } k_2 = 0, \end{cases} \quad (57)$$

$$\psi_k = \psi_{k_1, k_2} := \sqrt{2} \exp(2\pi i k_1 x) \sin(2\pi k_2 z), \quad (58)$$

and

$$\begin{aligned}\mathcal{E} &:= \{\phi \in L^2(\mathbb{T}^2) \mid \phi = \sum_{k \in \mathbb{Z}^2} a_k \phi_k, a_{-k_1, k_2} = a_{k_1, k_2}^*, \sum_{k \in \mathbb{Z}^2} |a_k|^2 < \infty\}, \\ \mathcal{O} &:= \{\psi \in L^2(\mathbb{T}^2) \mid \psi = \sum_{k \in \mathbb{Z}^2} a_k \psi_k, a_{-k_1, k_2} = a_{k_1, k_2}^*, \sum_{k \in \mathbb{Z}^2} |a_k|^2 < \infty\}.\end{aligned}$$

Observe that functions in \mathcal{E} and \mathcal{O} are even and odd with respect to z variable, respectively. Moreover, \mathcal{E} and \mathcal{O} are closed subspace of $L^2(\mathbb{T}^2)$, orthogonal to each other and consist of real valued functions. For any $m \in \mathbb{N}$, denote by

$$\begin{aligned}\mathcal{E}_m &:= \{\phi \in L^2(\mathbb{T}^2) \mid \phi = \sum_{|k| \leq m} a_k \phi_k, a_{-k_1, k_2} = a_{k_1, k_2}^*\}, \\ \mathcal{O}_m &:= \{\psi \in L^2(\mathbb{T}^2) \mid \psi = \sum_{|k| \leq m} a_k \psi_k, a_{-k_1, k_2} = a_{k_1, k_2}^*\},\end{aligned}$$

the finite-dimensional subspaces of \mathcal{E} and \mathcal{O} , respectively. For any function $f \in L^2(\mathbb{T}^2)$, we denote by $\hat{f}_j = (f, \bar{\phi}_j)$ and $\tilde{f}_j = (f, \bar{\psi}_j)$, and we write $P_m f := \sum_{|k| \leq m} \hat{f}_k \phi_k$ and $\Pi_m f := \sum_{|k| \leq m} \tilde{f}_k \psi_k$. Then P_m and Π_m are the orthogonal projections from $L^2(\mathbb{T}^2)$ to \mathcal{E}_m and \mathcal{O}_m , respectively. Now let

$$u_m = \sum_{|k|=0}^m a_k(t) \phi_k(x, z), \quad v_m = \sum_{|k|=0}^m b_k(t) \phi_k(x, z), \quad T_m = \sum_{|k|=0}^m c_k(t) \psi_k(x, z),$$

and consider the following Galerkin approximation system for our model (44)–(46), with (47)–(49):

$$\partial_t u_m - \nu \partial_{zz} u_m + P_m [u_m \partial_x u_m + w_m \partial_z u_m] + \epsilon_1 u_m - f_0 v_m + \partial_x p_m = 0, \quad (59)$$

$$\partial_t v_m - \nu \partial_{zz} v_m + P_m [u_m \partial_x v_m + w_m \partial_z v_m] + \epsilon_1 v_m + f_0 u_m = 0, \quad (60)$$

$$\partial_t T_m - \kappa \Delta T_m + \Pi_m [u_m \partial_x T_m + w_m \partial_z T_m] = 0, \quad (61)$$

with $w_m, \partial_x p_m, \partial_z p_m$ defined by:

$$w_m(x, z) := - \int_0^z \partial_x u_m(x, s) ds, \quad (62)$$

$$\begin{aligned}\partial_x p_m(x, z) &:= \epsilon_2 \int_0^z \int_0^s \partial_{xx} u_m(x, \xi) d\xi ds - \int_0^z \partial_x T_m(x, s) ds \\ &\quad + \int_0^1 \left[\int_0^{z'} \partial_x T_m(x, s) ds - \epsilon_2 \int_0^{z'} \int_0^s \partial_{xx} u_m(x, \xi) d\xi ds + f_0 v_m(x, z') \right] dz' \\ &\quad - P_m \int_0^1 2u_m \partial_x u_m(x, z') dz' - f_0 \int_0^1 \int_0^1 v_m(x', z') dx' dz',\end{aligned} \quad (63)$$

$$\partial_z p_m(x, z) := -T_m(x, z) + \epsilon_2 \int_0^z \partial_x u_m(x, s) ds, \quad (64)$$

subject to the following initial conditions:

$$u_m(0) = P_m u_0, \quad v_m(0) = P_m v_0, \quad T_m(0) = \Pi_m T_0. \quad (65)$$

Observe that the definitions of $w_m, \partial_x p_m$ and $\partial_z p_m$ are inspired by (47)–(49). Moreover, notice that

$$(\partial_x p_m)_z(x, z) = -\partial_x T_m(x, z) + \epsilon_2 \int_0^z \partial_{xx} u_m(x, s) ds = (\partial_z p_m)_x(x, z),$$

and hence (63) and (64) are compatible. For each $m \geq 1$, the Galerkin approximation, system (59)–(61), together with (62)–(64) corresponds to a first order system of ordinary differential equations, in the coefficients a_k, b_k and c_k for $0 \leq |k| \leq m$, with quadratic nonlinearity. Therefore, by the theory of ordinary differential

equations, there exists some $t_m > 0$ such that system (59)–(61) together with (62)–(64) admit a unique solution (u_m, v_m, T_m) on the interval $[0, t_m]$. Observe that from (65), we have $a_k(0), b_k(0), c_k(0) \in \mathbb{C}$ satisfying $a_{-k_1, k_2}(0) = a_{k_1, k_2}^*(0), b_{-k_1, k_2}(0) = b_{k_1, k_2}^*(0)$, and $c_{-k_1, k_2}(0) = c_{k_1, k_2}^*(0)$. Thanks to the uniqueness of the solutions of the ODE system, we conclude that $a_{-k_1, k_2}(t) = a_{k_1, k_2}^*(t), b_{-k_1, k_2}(t) = b_{k_1, k_2}^*(t)$, and $c_{-k_1, k_2}(t) = c_{k_1, k_2}^*(t)$, for $t \in [0, t_m]$. Therefore, $u_m, v_m \in \mathcal{E}_m$, and $T_m \in \mathcal{O}_m$. In the next section, we perform formal *a priori* estimates for the original system (44)–(46) with (47)–(49). These formal *a priori* estimates can be justified rigorously by establishing them first to the Galerkin approximation system and then passing to the limit using the Aubin-Lions compactness theorem.

3.2.2. *A priori estimates.* The constant C appears in the following inequalities may change from line to line. By taking the L^2 -inner product of equation (44) with $u, -\Delta u, \Delta u_{xx}$, equation (45) with $v, -\Delta v, \Delta v_{xx}$, equation (55) with $w, -\Delta w, \Delta w_{xx}$ and equation (46) with $T, -\Delta T, \Delta T_{xx}$, and by integration by parts, thanks to (50) and (53), we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|\nabla v_x\|_{L^2}^2 + \|T\|_{L^2}^2 + \|\nabla T\|_{L^2}^2 + \|\nabla T_x\|_{L^2}^2 \right) \\
& + \nu \left(\|u_z\|_{L^2}^2 + \|v_z\|_{L^2}^2 + \|\nabla u_z\|_{L^2}^2 + \|\nabla v_z\|_{L^2}^2 + \|\nabla u_{xz}\|_{L^2}^2 + \|\nabla v_{xz}\|_{L^2}^2 \right) \\
& + \epsilon_1 \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|\nabla v_x\|_{L^2}^2 \right) \\
& + \epsilon_2 \left(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla w_x\|_{L^2}^2 \right) + \kappa \left(\|\nabla T\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta T_x\|_{L^2}^2 \right) \\
& = \int_{\mathbb{T}^2} [uu_x + wu_z - f_0v + p_x](-u + \Delta u - \Delta u_{xx}) + [uv_x + wv_z + f_0u](-v + \Delta v - \Delta v_{xx}) \\
& \quad + (p_z + T)(-w + \Delta w - \Delta w_{xx}) + (uT_x + wT_z)(-T + \Delta T - \Delta T_{xx}) \, dx dz. \tag{66}
\end{aligned}$$

By integration by parts, thanks to (50), (53) and (56), we have

$$\begin{aligned}
& \int_{\mathbb{T}^2} (-f_0v + p_x)(-u + \Delta u - \Delta u_{xx}) + f_0u(-v + \Delta v - \Delta v_{xx}) + p_z(-w + \Delta w - \Delta w_{xx}) \\
& \quad + (uu_x + wu_z)(-u + u_{zz}) + (uv_x + wv_z)(-v) + (uT_x + wT_z)(-T) \, dx dz = 0.
\end{aligned}$$

Therefore, the right-hand side of (66) becomes

$$\begin{aligned}
& \int_{\mathbb{T}^2} (uu_x + wu_z)(u_{xx} - u_{xxxx} - u_{xxzz}) + (uv_x + wv_z)(\Delta v - v_{xxxx} - v_{xxzz}) \\
& \quad + T(-w + \Delta w - \Delta w_{xx}) + (uT_x + wT_z)(\Delta T - \Delta T_{xx}) \, dx dz =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned}$$

Let us denote by

$$\begin{aligned}
Y & := 1 + \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|\nabla v_x\|_{L^2}^2 + \|T\|_{L^2}^2 + \|\nabla T\|_{L^2}^2 + \|\nabla T_x\|_{L^2}^2, \\
F & := \|u_z\|_{L^2}^2 + \|v_z\|_{L^2}^2 + \|\nabla u_z\|_{L^2}^2 + \|\nabla v_z\|_{L^2}^2 + \|\nabla u_{xz}\|_{L^2}^2 + \|\nabla v_{xz}\|_{L^2}^2, \\
G & := \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla w_x\|_{L^2}^2, \quad K := \|\nabla T\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta T_x\|_{L^2}^2. \tag{67}
\end{aligned}$$

From (35), by Hölder inequality and Minkowski inequality, we have

$$\begin{aligned}
\|w\|_{L^2} & = \left(\int_0^1 \int_0^1 \left| \int_0^z u_x(x, s) ds \right|^2 dx dz \right)^{\frac{1}{2}} \leq \int_0^z \left(\int_0^1 \int_0^1 |u_x(x, s)|^2 dx dz \right)^{\frac{1}{2}} ds \\
& \leq \int_0^1 \left(\int_0^1 \int_0^1 |u_x(x, s)|^2 dx dz \right)^{\frac{1}{2}} ds \leq \left(\int_0^1 \int_0^1 \int_0^1 |u_x(x, s)|^2 dx dz ds \right)^{\frac{1}{2}} = \|u_x\|_{L^2}. \tag{68}
\end{aligned}$$

Similarly, one can get

$$\|w_x\|_{L^2} \leq \|u_{xx}\|_{L^2}. \tag{69}$$

Let us estimate terms I_1 – I_8 . By integration by parts, using Cauchy–Schwarz inequality, Young’s inequality and Lemma 1, thanks to (50), (53), (56), (68) and (69), we have

$$|I_1| = \left| \int_{\mathbb{T}^2} (uu_x + wu_z) u_{xx} dx dz \right| \leq C \|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_z\|_{L^2}^{\frac{1}{2}}) \|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xx}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2} \\ + C \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_z\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2} \leq CY^{\frac{3}{2}} \leq CY^3,$$

$$|I_2| = \left| \int_{\mathbb{T}^2} [uu_x + wu_z] u_{xxxx} dx dz \right| = \left| \int_{\mathbb{T}^2} [3u_x u_{xx} + w_{xx} u_z + 2w_x u_{xz}] u_{xx} dx dz \right| \\ \leq C \left[\|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xx}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{3}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|w_{xx}\|_{L^2} \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|w_x\|_{L^2} \|u_{xz}\|_{L^2}^{\frac{1}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \right] \\ \leq \frac{\epsilon_2}{6} \|w_{xx}\|_{L^2}^2 + \frac{\nu}{6} \|u_{xxx}\|_{L^2}^2 + CY^3 \leq \frac{\nu}{10} F + \frac{\epsilon_2}{6} G + CY^3,$$

$$|I_3| = \left| \int_{\mathbb{T}^2} [uu_x + wu_z] u_{xxzz} dx dz \right| = \left| \int_{\mathbb{T}^2} (u_x u_{xz} + w_x u_{zz}) u_{xz} dx dz \right| \\ \leq C \left[\|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xz}\|_{L^2}^{\frac{3}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|u_{xz}\|_{L^2} \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{zz}\|_{L^2}^{\frac{1}{2}} (\|u_{zz}\|_{L^2}^{\frac{1}{2}} + \|u_{xzz}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. \leq \frac{\nu}{10} (\|u_{zz}\|_{L^2}^2 + \|u_{xzz}\|_{L^2}^2 + \|u_{xxx}\|_{L^2}^2) + CY^2 \leq \frac{\nu}{10} F + CY^3, \right.$$

$$|I_4| = \left| \int_{\mathbb{T}^2} [uv_x + wv_z] \Delta v dx dz \right| \leq C (\|u\|_{L^2} + \|u_z\|_{L^2}) (\|v_x\|_{L^2} + \|v_{xx}\|_{L^2}) (\|v_{xx}\|_{L^2} + \|v_{zz}\|_{L^2}) \\ + C (\|w\|_{L^2} + \|w_z\|_{L^2}) (\|v_z\|_{L^2} + \|v_{xz}\|_{L^2}) (\|v_{xx}\|_{L^2} + \|v_{zz}\|_{L^2}) \leq \frac{\nu}{10} \|v_{zz}\|_{L^2}^2 + CY^2 \leq \frac{\nu}{10} F + CY^3,$$

$$|I_5| = \left| \int_{\mathbb{T}^2} [uv_x + wv_z] v_{xxxx} dx dz \right| = \left| \int_{\mathbb{T}^2} [u_{xx} v_x + w_{xx} v_z + 2u_x v_{xx} + 2w_x v_{xz}] v_{xx} dx dz \right| \\ \leq C \left[\|v_{xx}\|_{L^2} \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xx}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|w_{xx}\|_{L^2} \|v_z\|_{L^2}^{\frac{1}{2}} (\|v_z\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|v_{xx}\|_{L^2}^{\frac{3}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxx}\|_{L^2}^{\frac{1}{2}}) \|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xz}\|_{L^2}^{\frac{1}{2}} (\|v_{xz}\|_{L^2}^{\frac{1}{2}} + \|v_{xxx}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2} \right] \\ \leq \frac{\epsilon_2}{6} \|w_{xx}\|_{L^2}^2 + \frac{\nu}{10} (\|u_{xxx}\|_{L^2}^2 + \|v_{xxx}\|_{L^2}^2) + CY^3 \leq \frac{\nu}{10} F + \frac{\epsilon_2}{6} G + CY^3,$$

$$|I_6| = \left| \int_{\mathbb{T}^2} [uv_x + wv_z] v_{xxzz} dx dz \right| = \left| \int_{\mathbb{T}^2} [u_{xz} v_x + v_{xx} u_z - v_z u_{xx} + w_x v_{xz}] v_{xz} dx dz \right| \\ \leq C \left[\|v_{xz}\|_{L^2} \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xz}\|_{L^2}^{\frac{1}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|v_{xz}\|_{L^2} \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|v_{xz}\|_{L^2} \|v_z\|_{L^2}^{\frac{1}{2}} (\|v_z\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\ \left. + \|v_{xz}\|_{L^2} \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xz}\|_{L^2}^{\frac{1}{2}} (\|v_{xz}\|_{L^2}^{\frac{1}{2}} + \|v_{xxx}\|_{L^2}^{\frac{1}{2}}) \right] \\ \leq \frac{\nu}{10} (\|u_{xxx}\|_{L^2}^2 + \|v_{xxx}\|_{L^2}^2) + CY^2 \leq \frac{\nu}{10} F + CY^3,$$

$$\begin{aligned}
|I_7| &= \left| \int_{\mathbb{T}^2} T(-w + \Delta w - \Delta w_{xx}) dx dz \right| \\
&\leq \|T\|_{L^2} \|w\|_{L^2} + \|\nabla T\|_{L^2} \|\nabla w\|_{L^2} + \|\nabla T_x\|_{L^2} \|\nabla w_x\|_{L^2} \leq \frac{\epsilon_2}{6} G + CY, \\
|I_8| &= \left| \int_{\mathbb{T}^2} [uT_x + wT_z] (\Delta T - \Delta T_{xx}) dx dz \right| \\
&\leq \left| \int_{\mathbb{T}^2} [uT_x + wT_z] \Delta T dx dz \right| + \left| \int_{\mathbb{T}^2} [u_x T_x + uT_{xx} + wT_{xz} + w_x T_z] \Delta T_x dx dz \right| \\
&\leq C \left[\|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_x\|_{L^2}^{\frac{1}{2}}) \|T_x\|_{L^2}^{\frac{1}{2}} (\|T_x\|_{L^2}^{\frac{1}{2}} + \|T_{xz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_z\|_{L^2}^{\frac{1}{2}}) \|T_z\|_{L^2}^{\frac{1}{2}} (\|T_z\|_{L^2}^{\frac{1}{2}} + \|T_{xz}\|_{L^2}^{\frac{1}{2}}) \right] \|\Delta T\|_{L^2} \\
&\quad + C \left[\|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xx}\|_{L^2}^{\frac{1}{2}}) \|T_x\|_{L^2}^{\frac{1}{2}} (\|T_x\|_{L^2}^{\frac{1}{2}} + \|T_{xz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_z\|_{L^2}^{\frac{1}{2}}) \|T_{xx}\|_{L^2}^{\frac{1}{2}} (\|T_{xx}\|_{L^2}^{\frac{1}{2}} + \|T_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|T_{xz}\|_{L^2}^{\frac{1}{2}} (\|T_{xz}\|_{L^2}^{\frac{1}{2}} + \|T_{xzz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|T_z\|_{L^2}^{\frac{1}{2}} (\|T_z\|_{L^2}^{\frac{1}{2}} + \|T_{xz}\|_{L^2}^{\frac{1}{2}}) \right] \|\Delta T_x\|_{L^2} \\
&\leq \frac{\kappa}{2} (\|\Delta T\|_{L^2}^2 + \|\Delta T_x\|_{L^2}^2) + CY^3 \leq \frac{\kappa}{2} K + CY^3.
\end{aligned}$$

From the estimates above, (66) becomes

$$\frac{dY}{dt} + \nu F + \epsilon_2 G + \kappa K \leq CY^3. \quad (70)$$

Therefore, we have $\frac{dY}{dt} \leq CY^3$, and this implies that

$$Y(t) \leq \sqrt{\frac{Y(0)^2}{1 - Y(0)^2 C t}}.$$

Choose

$$\mathcal{T} = \frac{3}{4CY(0)^2}.$$

From above, we have $Y(t) \leq 2Y(0)$ on $[0, \mathcal{T}]$. Plugging it in (70), we have

$$\frac{dY}{dt} + \nu F + \epsilon_2 G + \kappa K \leq 8CY(0)^3, \text{ for } t \in [0, \mathcal{T}]. \quad (71)$$

Integrating above from 0 to t for any time $t \in [0, \mathcal{T}]$, we obtain

$$Y(t) + \int_0^t (\nu F(s) + \epsilon_2 G(s) + \kappa K(s)) ds \leq Y(0) + 8CtY(0)^3.$$

Therefore, we have

$$u, v, T, u_x, v_x, T_x \in L^\infty(0, \mathcal{T}; H^1), \quad u_z, v_z, u_{xz}, v_{xz} \in L^2(0, \mathcal{T}; H^1), \quad T, T_x \in L^2(0, \mathcal{T}; H^2). \quad (72)$$

By virtue of (72) and (68), we have

$$w \in L^\infty(0, \mathcal{T}; H^1). \quad (73)$$

Thanks to Lemma 2, from (72), we also have

$$u, v, T \in L^\infty(0, \mathcal{T}; L^\infty), \quad \nabla u, \nabla v, \nabla T \in L^2(0, \mathcal{T}; L^\infty). \quad (74)$$

3.2.3. *Uniqueness of solutions and its continuous dependence on the initial data.* In this section, we will show the continuous dependence on the initial data and the uniqueness of the strong solutions. Let $(u_1, v_1, w_1, p_1, T_1)$ and $(u_2, v_2, w_2, p_2, T_2)$ be two strong solutions of system (44)–(46) with (47)–(49), with the initial data $((u_0)_1, (v_0)_1, (T_0)_1)$ and $((u_0)_2, (v_0)_2, (T_0)_2)$, respectively. Denote by $u = u_1 - u_2$, $v = v_1 - v_2$, $w = w_1 - w_2$, $p = p_1 - p_2$, $T = T_1 - T_2$. It is clear that

$$\partial_t u - \nu u_{zz} + u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z + \epsilon_1 u - f_0 v + p_x = 0, \quad (75)$$

$$\partial_t v - \nu v_{zz} + u_1 v_x + w_1 v_z + u(v_2)_x + w(v_2)_z + \epsilon_1 v + f_0 u = 0, \quad (76)$$

$$\epsilon_2 w + p_z + T = 0, \quad (77)$$

$$u_x + w_z = 0, \quad (78)$$

$$\partial_t T - \kappa \Delta T + u_1 T_x + w_1 T_z + u(T_2)_x + w(T_2)_z = 0. \quad (79)$$

By taking the inner product of equation (75) with u , (76) with v , (77) with w , and (79) with T , in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (50), (56), and (78), we get

$$\begin{aligned} & \frac{1}{2} \frac{d(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2)}{dt} + \epsilon_1(\|u\|_{L^2}^2 + \|v\|_{L^2}^2) + \nu(\|u_z\|_{L^2}^2 + \|v_z\|_{L^2}^2) + \epsilon_2 \|w\|_{L^2}^2 + \kappa \|\nabla T\|_{L^2}^2 \\ & \leq \left| \int_{\mathbb{T}^2} [u(u_2)_x + w(u_2)_z] u \, dx dz \right| + \left| \int_{\mathbb{T}^2} [u(v_2)_x + w(v_2)_z] v \, dx dz \right| \\ & \quad + \left| \int_{\mathbb{T}^2} wT \, dx dz \right| + \left| \int_{\mathbb{T}^2} [u(T_2)_x + w(T_2)_z] T \, dx dz \right| =: I + II + III + IV. \end{aligned}$$

By integration by parts, using Hölder inequality and Young's inequality, thanks to (50), (56), and (78), we have

$$I = \left| \int_{\mathbb{T}^2} [u(u_2)_x + w(u_2)_z] u \, dx dz \right| \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + (\|(u_2)_x\|_{L^\infty} + \|(u_2)_z\|_{L^\infty}) \|u\|_{L^2}^2,$$

$$II = \left| \int_{\mathbb{T}^2} [u(v_2)_x + w(v_2)_z] v \, dx dz \right| \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + (\|(v_2)_x\|_{L^\infty} + \|(v_2)_z\|_{L^\infty}) (\|u\|_{L^2}^2 + \|v\|_{L^2}^2),$$

$$III = \left| \int_{\mathbb{T}^2} wT \, dx dz \right| \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + C \|T\|_{L^2}^2,$$

$$IV = \left| \int_{\mathbb{T}^2} [u(T_2)_x + w(T_2)_z] T \, dx dz \right| \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + (\|(T_2)_x\|_{L^\infty} + \|(T_2)_z\|_{L^\infty}) (\|u\|_{L^2}^2 + \|T\|_{L^2}^2).$$

From the estimates above, we obtain

$$\begin{aligned} & \frac{d(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2)}{dt} + \epsilon_1(\|u\|_{L^2}^2 + \|v\|_{L^2}^2) + \nu(\|u_z\|_{L^2}^2 + \|v_z\|_{L^2}^2) + \epsilon_2 \|w\|_{L^2}^2 + \kappa \|\nabla T\|_{L^2}^2 \\ & \leq CK (\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2), \end{aligned}$$

where

$$K = 1 + \|\nabla u_2\|_{L^\infty}^2 + \|\nabla v_2\|_{L^\infty}^2 + \|\nabla T_2\|_{L^\infty}^2.$$

Thanks to (74), we obtain $K \in L^1(0, \mathcal{T})$. Therefore, by Gronwall inequality, we obtain

$$\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|T(t)\|_{L^2}^2 \leq (\|u(t=0)\|_{L^2}^2 + \|v(t=0)\|_{L^2}^2 + \|T(t=0)\|_{L^2}^2) e^{C \int_0^t K(s) ds},$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $u(t=0) = v(t=0) = T(t=0) = 0$, we have $u(t) = v(t) = T(t) = 0$, for all $t \geq 0$. Therefore, the strong solution is unique.

3.3. The Special Case: $\mathbf{f}_0 = \mathbf{0}$, $\mathbf{v} \equiv \mathbf{0}$ and $\mathbf{T} \equiv \mathbf{0}$. In this section, we assume that $f_0 = 0$, $v \equiv 0$ and $T \equiv 0$. In this case, system (17)–(21) will be reduced to:

$$\partial_t u - \nu u_{zz} + uu_x + wu_z + \epsilon_1 u + p_x = 0, \quad (80)$$

$$\epsilon_2 w + p_z = 0, \quad (81)$$

$$u_x + w_z = 0, \quad (82)$$

Remark 1. There are two reasons why we consider this special case. Firstly, notice that when $\epsilon_1 = \epsilon_2 = 0$, system (80)–(82) is exactly the hydrostatic Navier-Stokes equations (14)–(16). So we can regard system (80)–(82) as the hydrostatic Navier-Stokes equations with damping. Secondly, as we will see later, we can show the local regularity of strong solution to system (80)–(82) for initial conditions with less regularity. The reason why we need to assume more regularity for initial data to system (44)–(46) is that we need to bound terms which contain v_{xx} . For u_{xx} , we can use incompressible condition $u_{xx} = -w_{xz}$ to avoid such an issue. Therefore, in the case when we do not have the evolution equation for v , we can require less for the initial data.

As in section 3.2, our domain is \mathbb{T}^2 , and the boundary and initial condition are

$$u, w \text{ and } p \text{ are periodic in } x \text{ and } z \text{ with period } 1, \quad (83)$$

$$u \text{ and } p \text{ are even in } z, \text{ and } w \text{ is odd in } z, \quad (84)$$

$$u|_{t=0} = u_0. \quad (85)$$

Using an analogue argument to that in section 3.1, system (80)–(82) subject to (83)–(85) is equivalent to the following:

$$u_t - \nu u_{zz} + u u_x + w u_z + \epsilon_1 u + p_x = 0, \quad (86)$$

with w, p_x, p_z defined by:

$$w(x, z) := - \int_0^z u_x(x, s) ds, \quad (87)$$

$$p_x(x, z) := \epsilon_2 \int_0^z \int_0^s u_{xx}(x, \xi) d\xi ds + \int_0^1 \left[-\epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x, \xi) d\xi ds - 2u u_x(x, z') \right] dz', \quad (88)$$

$$p_z(x, z) := \epsilon_2 \int_0^z u_x(x, s) ds, \quad (89)$$

subject to the following symmetry boundary condition and initial condition:

$$u \text{ is periodic in } x \text{ and } z \text{ with period } 1, \text{ and is even in } z; \quad (90)$$

$$u|_{t=0} = u_0. \quad (91)$$

By virtue of (87)–(89) and (90), we obtain that w, p also satisfy the symmetry conditions:

$$w \text{ and } p \text{ are periodic in } x \text{ and } z \text{ with period } 1; \quad (92)$$

$$p \text{ is even in } z, \text{ and } w \text{ is odd in } z. \quad (93)$$

By virtue of (87) and (89), and by differentiating (87) with respect to z , we have

$$\epsilon_2 w + p_z = 0, \quad (94)$$

$$u_x + w_z = 0. \quad (95)$$

In this section, we are interested in system (86) with (87)–(89) in the unit two-dimensional flat torus \mathbb{T}^2 , subject to (90)–(91). First, we give the definition of strong solution to system (86) with (87)–(89).

Definition 6. Suppose that $u_0 \in H^1(\mathbb{T}^2)$ satisfies the symmetry conditions (90), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_{xz} u_0 \in L^2(\mathbb{T}^2)$. Given time $\mathcal{T} > 0$, we say u is a strong solution to system (86) with (87)–(89), subject to (90)–(91), on the time interval $[0, \mathcal{T}]$, if

(i) u satisfies the symmetry condition (90);

(ii) u has the regularities

$$\begin{aligned} u &\in L^\infty(0, \mathcal{T}; H^1) \cap L^2(0, \mathcal{T}; H^2) \cap C([0, \mathcal{T}], L^2) \cap L^\infty(0, \mathcal{T}; L^\infty), \quad u_z \in L^2(0, \mathcal{T}; L^\infty) \\ u_{xz} &\in L^\infty(0, \mathcal{T}; L^2), \quad u_{xzz} \in L^2(0, \mathcal{T}; L^2), \quad \partial_t u \in L^2(0, \mathcal{T}; L^2); \end{aligned}$$

(iii) u satisfies system (86) in the following sense:

$$\partial_t u - \nu u_{zz} + uu_x + wu_z + \epsilon_1 u + p_x = 0, \quad \text{in } L^2(0, \mathcal{T}; L^2),$$

with w, p_x, p_z defined by (87)–(89), and fulfill the initial condition (91).

We have the following result concerning the locally existence and uniqueness of strong solutions to system (86) with (87)–(89), subject to (90)–(91), on $\mathbb{T}^2 \times (0, \mathcal{T})$, for some positive time \mathcal{T} .

Theorem 7. Suppose that $u_0 \in H^1(\mathbb{T}^2)$ satisfies the symmetry conditions (90), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_{xz} u_0 \in L^2(\mathbb{T}^2)$. Then there exists some $\mathcal{T} > 0$ such that there is a unique strong solution u of system (86) with (87)–(89), subject to (90)–(91), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution u depends continuously on the initial data.

Proof. For sake of simplicity, we will only do *a priori* estimates formally here, and we can use Galerkin method, as remarked in section 3.2, to prove the result rigorously. We denote by $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{T}^2)}$. By taking the inner product of equation (86) with $u, -u_{zz}$, and equation (94) with $w, -w_{zz}$, in $L^2(\mathbb{T}^2)$, we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|u\|^2 + \|u_z\|^2) + \nu (\|u_z\|^2 + \|u_{zz}\|^2) + \epsilon_1 (\|u\|^2 + \|u_z\|^2) + \epsilon_2 (\|w\|^2 + \|w_z\|^2) \\ &= - \int_{\mathbb{T}^2} (uu_x + wu_z) (u - u_{zz}) dx dz - \int_{\mathbb{T}^2} \left(p_x (u - u_{zz}) + p_z (w - w_{zz}) \right) dx dz. \end{aligned}$$

By integration by parts, thanks to (90), (92) and (95), we have

$$- \int_{\mathbb{T}^2} (uu_x + wu_z) (u - u_{zz}) dx dz - \int_{\mathbb{T}^2} \left(p_x (u - u_{zz}) + p_z (w - w_{zz}) \right) dx dz = 0.$$

Thanks to Gronwall inequality, we obtain

$$\|u(t)\|^2 + \|u_z(t)\|^2 + 2 \int_0^t \left[\nu (\|u_z(s)\|^2 + \|u_{zz}(s)\|^2) + \epsilon_2 (\|w(s)\|^2 + \|w_z(s)\|^2) \right] ds \leq \|u(0)\|^2 + \|u_z(0)\|^2.$$

From the estimates above, we obtain

$$\begin{aligned} u, u_z &\text{ bounded in } L^\infty(0, \mathcal{T}; L^2), \\ w, u_{zz}, w_z = -u_x &\text{ bounded in } L^2(0, \mathcal{T}; L^2), \end{aligned} \tag{96}$$

for arbitrary $\mathcal{T} > 0$. By taking the inner product of equation (86) with $-u_{xx}, u_{xxzz}$ and equation (94) with $-w_{xx}, w_{xxzz}$ in $L^2(\mathbb{T}^2)$, integrating by parts, thanks to (90) and (92) we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|u_x\|^2 + \|u_{xz}\|^2) + \nu (\|u_{xz}\|^2 + \|u_{xzz}\|^2) + \epsilon_1 (\|u_x\|^2 + \|u_{xz}\|^2) + \epsilon_2 (\|w_x\|^2 + \|w_{xz}\|^2) \\ &= \int_{\mathbb{T}^2} (uu_x + wu_z) (u_{xx} - u_{xxzz}) dx dz + \int_{\mathbb{T}^2} \left(p_x (u_{xx} - u_{xxzz}) + p_z (w_{xx} - w_{xxzz}) \right) dx dz. \end{aligned} \tag{97}$$

By integration by parts, thanks to (90), (92) and (95), we have

$$\int_{\mathbb{T}^2} \left(p_x(u_{xx} - u_{xxxz}) + p_z(w_{xx} - w_{xxxz}) \right) dx dz = 0.$$

Therefore, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u_x\|^2 + \|u_{xz}\|^2) + \nu (\|u_{xz}\|^2 + \|u_{zzz}\|^2) + \epsilon_1 (\|u_x\|^2 + \|u_{xz}\|^2) + \epsilon_2 (\|w_x\|^2 + \|w_{xz}\|^2) \\ & \leq \left| \int_{\mathbb{T}^2} (uu_x + wu_z) (u_{xx} - u_{xxxz}) dx dz \right| =: |I_1 + I_2|. \end{aligned}$$

Let us denote by

$$\begin{aligned} Y &:= 1 + \|u_x\|^2 + \|u_{xz}\|^2, \quad F := \|u_{xz}\|^2 + \|u_{zzz}\|^2, \\ G &:= \|w_x\|^2 + \|w_{xz}\|^2, \quad K := 1 + \|u\|^2 + \|u_z\|^2 + \|u_{zz}\|^2. \end{aligned}$$

By integration by parts and Lemma 1, using Young's inequality, thanks to (68), (69), (90), (92) and (95), we have

$$\begin{aligned} |I_1| &= \left| \int_{\mathbb{T}^2} (uu_x + wu_z) u_{xx} dx dz \right| \leq C \|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_x\|_{L^2}^{\frac{1}{2}}) \|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|w_{xz}\|_{L^2} \\ &\quad + C \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_z\|_{L^2}^{\frac{1}{2}}) \|w_{xz}\|_{L^2} \leq \frac{\epsilon_2}{4} G + CKY^2, \\ |I_2| &= \left| \int_{\mathbb{T}^2} [uu_x + wu_z] u_{xxxz} dx dz \right| = \left| \int_{\mathbb{T}^2} (u_x u_{xz} + w_x u_{zz}) u_{xz} dx dz \right| \\ &\leq C \left[\|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xz}\|_{L^2}^{\frac{3}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{zzz}\|_{L^2}^{\frac{1}{2}}) \right. \\ &\quad \left. + \|u_{xz}\|_{L^2} \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{zz}\|_{L^2}^{\frac{1}{2}} (\|u_{zz}\|_{L^2}^{\frac{1}{2}} + \|u_{zzz}\|_{L^2}^{\frac{1}{2}}) \right] \leq \frac{\epsilon_2}{4} G + \frac{\nu}{2} F + CKY^2. \end{aligned}$$

From the estimates above and by (96), we have

$$\frac{dY}{dt} + \nu F + \epsilon_2 G \leq CKY^2, \quad \text{with } K \in L^1(0, \mathcal{T}) \text{ for arbitrary } \mathcal{T} > 0. \quad (98)$$

Therefore, we have $\frac{dY}{dt} \leq CKY^2$, and this implies that

$$Y(t) \leq \frac{Y(0)}{1 - Y(0)C \int_0^t K ds}.$$

Let \mathcal{T} be such that

$$\int_0^{\mathcal{T}} K ds = \frac{1}{2Y(0)C}.$$

From above, we will have $Y(t) \leq 2Y(0)$ on $[0, \mathcal{T}]$. Plugging it in (98), we have

$$\frac{dY}{dt} + \nu F + \epsilon_2 G \leq 4CKY(0)^2, \quad \text{for } t \in [0, \mathcal{T}].$$

Integrating above from 0 to t for any time $t \in [0, \mathcal{T}]$, we obtain

$$Y(t) + \int_0^t (\nu F(s) + \epsilon_2 G(s)) ds \leq Y(0) + 4CY(0)^2 \int_0^t K(s) ds.$$

From the estimates above, by virtue of (95), (96) and (68), we obtain

$$u \in L^\infty(0, \mathcal{T}; H^1) \cap L^2(0, \mathcal{T}; H^2), \quad u_{xz} \in L^\infty(0, \mathcal{T}; L^2), \quad u_{zzz} \in L^2(0, \mathcal{T}; L^2), \quad (99)$$

$$w, w_z, w_{zz} \in L^\infty(0, \mathcal{T}; L^2), \quad w_x, w_{xz} \in L^2(0, \mathcal{T}; L^2). \quad (100)$$

Using Galerkin method, as remarked in section 3.2, we can obtain local existence of strong solution to system (86) with (87)–(89), subject to (90)–(91). Next, we show the continuous dependence of solutions on the initial

data and the uniqueness of the strong solutions. Let (u_1, w_1, p_1) and (u_2, w_2, p_2) be two strong solutions of system (86) with (87)–(89), and initial data $(u_0)_1$ and $(u_0)_2$, respectively. Denote by $u = u_1 - u_2, w = w_1 - w_2, p = p_1 - p_2$. It is clear that

$$\partial_t u + u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z + \epsilon_1 u - \nu u_{zz} + p_x = 0, \quad (101)$$

$$\epsilon_2 w + p_z = 0. \quad (102)$$

By taking the inner product of equation (101) with u , (102) with w in $L^2(\mathbb{T}^2)$, we have

$$\frac{1}{2} \frac{d\|u\|^2}{dt} + \epsilon_1 \|u\|^2 + \epsilon_2 \|w\|^2 + \nu \|u_z\|^2 = \int_{\mathbb{T}^2} u(u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z) + (p_x u + p_z w) dx dz.$$

By integration by parts, thanks to (90), (92) and (95), we have

$$\int_{\mathbb{T}^2} u(u_1 u_x + w_1 u_z) + (p_x u + p_z w) dx dz = 0.$$

Therefore, we have

$$\frac{1}{2} \frac{d\|u\|^2}{dt} + \epsilon_1 \|u\|^2 + \epsilon_2 \|w\|^2 + \nu \|u_z\|^2 \leq \left| \int_{\mathbb{T}^2} u \left(u(u_2)_x + w(u_2)_z \right) dx dz \right| =: |I_1 + I_2|.$$

From (99) and (100), and by lemma 2, we obtain that $w_2, (u_2)_z \in L^2(0, \mathcal{T}; L^\infty)$. Therefore, using Young's inequality and Hölder inequality, we have

$$\begin{aligned} |I_1| &= \left| \int_{\mathbb{T}^2} u^2 (u_2)_x dx dz \right| = \left| \int_{\mathbb{T}^2} u^2 (w_2)_z dx dz \right| = \left| 2 \int_{\mathbb{T}^2} u u_z w_2 dx dz \right| \\ &\leq \int_{\mathbb{T}^2} \left(\frac{\nu}{2} |u_z|^2 + C |u w_2|^2 \right) dx dz \leq C \|w_2\|_{L^\infty}^2 \|u\|^2 + \frac{\nu}{2} \|u_z\|^2, \end{aligned}$$

$$|I_2| = \left| \int_{\mathbb{T}^2} u w (u_2)_z dx dz \right| \leq \int_{\mathbb{T}^2} \left(\frac{\epsilon_2}{2} |w|^2 + C |u (u_2)_z|^2 \right) dx dz \leq C \|(u_2)_z\|_{L^\infty}^2 \|u\|^2 + \frac{\epsilon_2}{2} \|w\|^2.$$

From the estimates above, we obtain

$$\frac{d}{dt} \|u\|^2 + \epsilon_1 \|u\|^2 + \epsilon_2 \|w\|^2 + \nu \|u_z\|^2 \leq C (\|w_2\|_{L^\infty}^2 + \|(u_2)_z\|_{L^\infty}^2) \|u\|^2.$$

Thanks to Gronwall inequality, we have

$$\|u(t)\|^2 \leq \|u(0)\|^2 \exp \left(C \int_0^t (\|w_2(s)\|_{L^\infty}^2 + \|(u_2)_z(s)\|_{L^\infty}^2) ds \right).$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $u(t=0) = 0$, we have $u(t) = 0$, for all $t \in [0, \mathcal{T}]$. Therefore, the strong solution is unique. \square

3.4. Global Well-posedness with Small Initial Data. In this section, we will show the following result concerning the global existence and uniqueness of strong solutions to system (44)–(46) with (47)–(49), subject to boundary and initial conditions (50)–(52), provided that the initial data is small enough.

Theorem 8. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (50) and (51), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that*

$$\|u_0\|_{H^1} + \|v_0\|_{H^1} + C_0 \|T_0\|_{H^1} + \|\partial_x u_0\|_{H^1} + \|\partial_x v_0\|_{H^1} + C_0 \|\partial_x T_0\|_{H^1} \ll 1$$

is small enough, for some $C_0 > 0$ determined in (105). Then for any time $\mathcal{T} > 0$, there exists a unique strong solution (u, v, T) of system (44)–(46) with (47)–(49), subject to (50)–(52), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.

Proof. From Theorem 5, we know there exists time $\mathcal{T}^* > 0$ such that there is a unique strong solution (u, v, T) of system (44)–(46) with (47)–(49), subject to (50)–(52), on the interval $[0, \mathcal{T}^*]$. Assume the maximal time \mathcal{T} for existence of solution is finite, then it is necessary to have

$$\limsup_{t \rightarrow \mathcal{T}^-} (\|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|T(t)\|_{H^1} + \|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|T_x(t)\|_{H^1}) = \infty.$$

We will prove this is not true for any finite time $\mathcal{T} > 0$, and therefore $\mathcal{T} = \infty$. First, notice that since T is an odd function with respect to z variable, we have

$$\int_{\mathbb{T}^2} T \, dx dz \equiv 0. \quad (103)$$

By taking the L^2 -inner product of equation (44) with $u, -\Delta u, \Delta u_{xx}$, equation (45) with $v, -\Delta v, \Delta v_{xx}$, equation (55) with $w, -\Delta w, \Delta w_{xx}$ and equation (46) with $C_0 T, -C_0 \Delta T, C_0 \Delta T_{xx}$, in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (50), (53) and (56), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|\nabla v_x\|_{L^2}^2 + C_0 \|T\|_{L^2}^2 + C_0 \|\nabla T\|_{L^2}^2 + C_0 \|\nabla T_x\|_{L^2}^2 \right) \\ & + \nu \left(\|u_z\|_{L^2}^2 + \|v_z\|_{L^2}^2 + \|\nabla u_z\|_{L^2}^2 + \|\nabla v_z\|_{L^2}^2 + \|\nabla u_{xz}\|_{L^2}^2 + \|\nabla v_{xz}\|_{L^2}^2 \right) \\ & + \epsilon_1 \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|\nabla v_x\|_{L^2}^2 \right) \\ & + \epsilon_2 \left(\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla w_x\|_{L^2}^2 \right) + C_0 \kappa \left(\|\nabla T\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta T_x\|_{L^2}^2 \right) \\ & = \int_{\mathbb{T}^2} [uu_x + wu_z - f_0 v + p_x] (-u + \Delta u - \Delta u_{xx}) + [uv_x + wv_z + f_0 u] (-v + \Delta v - \Delta v_{xx}) \\ & \quad + (p_z + T) (-w + \Delta w - \Delta w_{xx}) + C_0 (uT_x + wT_z) (-T + \Delta T - \Delta T_{xx}) \, dx dz \\ & = \int_{\mathbb{T}^2} (uu_x + wu_z) (u_{xx} - u_{xxxx} - u_{xxzz}) + (uv_x + wv_z) (\Delta v - v_{xxx} - v_{xxzz}) \\ & \quad + T (-w + \Delta w - \Delta w_{xx}) + C_0 (uT_x + wT_z) (\Delta T - \Delta T_{xx}) \, dx dz =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8. \end{aligned}$$

We denote by:

$$\begin{aligned} Y &:= \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|\nabla v_x\|_{L^2}^2 + C_0 \|T\|_{L^2}^2 + C_0 \|\nabla T\|_{L^2}^2 + C_0 \|\nabla T_x\|_{L^2}^2, \\ F &:= \|u_z\|_{L^2}^2 + \|v_z\|_{L^2}^2 + \|\nabla u_z\|_{L^2}^2 + \|\nabla v_z\|_{L^2}^2 + \|\nabla u_{xz}\|_{L^2}^2 + \|\nabla v_{xz}\|_{L^2}^2, \\ G &:= \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla w_x\|_{L^2}^2, \\ H &:= \|\nabla T\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 + \|\Delta T_x\|_{L^2}^2, \\ K &:= \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|\nabla v_x\|_{L^2}^2. \end{aligned}$$

By integration by parts, using Poincaré inequality, Young's inequality and Lemma 1, thanks to (50), (53), (56), (68), (69) and (103), we have

$$\begin{aligned} |I_1| &= \left| \int_{\mathbb{T}^2} (uu_x + wu_z) w_{xz} \, dx dz \right| \\ &\leq C \|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_z\|_{L^2}^{\frac{1}{2}}) \|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xx}\|_{L^2}^{\frac{1}{2}}) \|w_{xz}\|_{L^2} \\ &\quad + C \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_z\|_{L^2}^{\frac{1}{2}}) \|w_{xz}\|_{L^2} \\ &\leq C (\|w\|_{L^2}^2 + \|w_z\|_{L^2}^2 + \|w_{xz}\|_{L^2}^2) (\|u\|_{L^2} + \|u_z\|_{L^2} + \|u_{xz}\|_{L^2}) \leq CGY^{1/2}, \end{aligned}$$

$$\begin{aligned}
|I_2| &= \left| \int_{\mathbb{T}^2} [uu_x + wu_z] u_{xxxx} dx dz \right| = \left| \int_{\mathbb{T}^2} [3w_z w_{xz} + w_{xx} u_z + 2w_x u_{xz}] u_{xx} dx dz \right| \\
&\leq C \left[\|w_z\|_{L^2}^{\frac{1}{2}} (\|w_z\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|w_{xz}\|_{L^2} \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad + \|w_{xx}\|_{L^2} \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \\
&\quad \left. + \|w_x\|_{L^2} \|u_{xz}\|_{L^2}^{\frac{1}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \right] \\
&\leq C (\|\nabla w_x\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u_x\|_{L^2}^2 + \|u_{xxz}\|_{L^2}^2) (\|\nabla u\|_{L^2} + \|\nabla u_x\|_{L^2}) \leq C(F + G + K)Y^{1/2}, \\
|I_3| &= \left| \int_{\mathbb{T}^2} [uu_x + wu_z] u_{xxzz} dx dz \right| = \left| \int_{\mathbb{T}^2} (u_x u_{xz} + w_x u_{zz}) u_{xz} dx dz \right| \\
&\leq C \left[\|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xz}\|_{L^2}^{\frac{3}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|u_{xz}\|_{L^2} \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{zz}\|_{L^2}^{\frac{1}{2}} (\|u_{zz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \right] \leq C(F + G + K)Y^{1/2}, \\
|I_4| &= \left| \int_{\mathbb{T}^2} [uv_x + wv_z] \Delta v dx dz \right| \\
&\leq C (\|u\|_{L^2} + \|u_z\|_{L^2}) (\|v_x\|_{L^2} + \|v_{xx}\|_{L^2}) (\|v_{xx}\|_{L^2} + \|v_{zz}\|_{L^2}) \\
&\quad + C (\|w\|_{L^2} + \|w_z\|_{L^2}) (\|v_z\|_{L^2} + \|v_{xz}\|_{L^2}) (\|v_{xx}\|_{L^2} + \|v_{zz}\|_{L^2}) \leq C(K + F)Y^{1/2}, \\
|I_5| &= \left| \int_{\mathbb{T}^2} [uv_x + wv_z] v_{xxxx} dx dz \right| = \left| \int_{\mathbb{T}^2} [u_{xx} v_x + w_{xx} v_z + 2u_x v_{xx} + 2w_x v_{xz}] v_{xx} dx dz \right| \\
&\leq C \left[\|v_{xx}\|_{L^2} \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xx}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad + \|w_{xx}\|_{L^2} \|v_z\|_{L^2}^{\frac{1}{2}} (\|v_z\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \\
&\quad + \|v_{xx}\|_{L^2}^{\frac{3}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xx}\|_{L^2}^{\frac{1}{2}}) \\
&\quad \left. + \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xz}\|_{L^2}^{\frac{1}{2}} (\|v_{xz}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2} \right] \leq C(K + F + G)Y^{1/2}, \\
|I_6| &= \left| \int_{\mathbb{T}^2} [uv_x + wv_z] v_{xxzz} dx dz \right| = \left| \int_{\mathbb{T}^2} [u_{xz} v_x + v_{xx} u_z - v_z u_{xx} + w_x v_{xz}] v_{xz} dx dz \right| \\
&\leq C \left[\|v_{xz}\|_{L^2} \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xz}\|_{L^2}^{\frac{1}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad + \|v_{xz}\|_{L^2} \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \\
&\quad + \|v_{xz}\|_{L^2} \|v_z\|_{L^2}^{\frac{1}{2}} (\|v_z\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \\
&\quad \left. + \|v_{xz}\|_{L^2} \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xz}\|_{L^2}^{\frac{1}{2}} (\|v_{xz}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \right] \leq C(K + F + G)Y^{1/2}, \\
|I_7| &= \left| \int_{\mathbb{T}^2} T(-w + \Delta w - \Delta w_{xx}) dx dz \right| \\
&\leq \|T\|_{L^2} \|w\|_{L^2} + \|\nabla T\|_{L^2} \|\nabla w\|_{L^2} + \|\nabla T_x\|_{L^2} \|\nabla w_x\|_{L^2} \\
&\leq \frac{\epsilon_2}{2} G + \frac{1}{2\epsilon_2} (\|T\|_{L^2}^2 + \|\nabla T\|_{L^2}^2 + \|\nabla T_x\|_{L^2}^2) \\
&\leq \frac{\epsilon_2}{2} G + \frac{1}{2\epsilon_2} (C_p \|\nabla T\|_{L^2}^2 + \|\nabla T\|_{L^2}^2 + \|\nabla T_x\|_{L^2}^2) \leq \frac{\epsilon_2}{2} G + \frac{1}{2\epsilon_2} (C_p + 1)H,
\end{aligned}$$

where, thanks to (103), we apply Poincaré inequality to obtain the last inequality.

$$\begin{aligned}
|I_8| &= C_0 \left| \int_{\mathbb{T}^2} [uT_x + wT_z] (\Delta T - \Delta T_{xx}) dx dz \right| \\
&\leq \left| \int_{\mathbb{T}^2} [uT_x + wT_z] \Delta T dx dz \right| + \left| \int_{\mathbb{T}^2} [u_x T_x + u T_{xx} + w T_{xz} + w_x T_z] \Delta T_x dx dz \right| \\
&\leq C_0 C \left[\|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_z\|_{L^2}^{\frac{1}{2}}) \|T_x\|_{L^2}^{\frac{1}{2}} (\|T_x\|_{L^2}^{\frac{1}{2}} + \|T_{xx}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_z\|_{L^2}^{\frac{1}{2}}) \|T_z\|_{L^2}^{\frac{1}{2}} (\|T_z\|_{L^2}^{\frac{1}{2}} + \|T_{xz}\|_{L^2}^{\frac{1}{2}}) \right] \|\Delta T\|_{L^2} \\
&\quad + C_0 C \left[\|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|T_x\|_{L^2}^{\frac{1}{2}} (\|T_x\|_{L^2}^{\frac{1}{2}} + \|T_{xx}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_z\|_{L^2}^{\frac{1}{2}}) \|T_{xx}\|_{L^2}^{\frac{1}{2}} (\|T_{xx}\|_{L^2}^{\frac{1}{2}} + \|T_{xxx}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|T_{xz}\|_{L^2}^{\frac{1}{2}} (\|T_{xz}\|_{L^2}^{\frac{1}{2}} + \|T_{xzz}\|_{L^2}^{\frac{1}{2}}) \right. \\
&\quad \left. + \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xz}\|_{L^2}^{\frac{1}{2}}) \|T_z\|_{L^2}^{\frac{1}{2}} (\|T_z\|_{L^2}^{\frac{1}{2}} + \|T_{xz}\|_{L^2}^{\frac{1}{2}}) \right] \|\Delta T_x\|_{L^2} \leq C_0 C H Y^{1/2}.
\end{aligned}$$

From the estimates above, we obtain

$$|I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8| \leq C(K + G + F + C_0 H) Y^{\frac{1}{2}} + \frac{\epsilon_2}{2} G + \frac{1}{2\epsilon_2} (C_p + 1) H.$$

Therefore, we obtain

$$\frac{1}{2} \frac{dY}{dt} + \nu \left(1 - \frac{C}{\nu} Y^{\frac{1}{2}}\right) F + \epsilon_1 \left(1 - \frac{C}{\epsilon_1} Y^{\frac{1}{2}}\right) K + \epsilon_2 \left(\frac{1}{2} - \frac{C}{\epsilon_2} Y^{\frac{1}{2}}\right) G + C_0 \kappa \left(1 - \frac{C}{\kappa} Y^{\frac{1}{2}} - \frac{C_p + 1}{2\epsilon_2 \kappa C_0}\right) H \leq 0. \quad (104)$$

Choose

$$C_0 = \frac{C_p + 1}{\epsilon_2 \kappa}. \quad (105)$$

Observe that if $Y_0 < \min(\frac{\nu^2}{C^2}, \frac{\epsilon_1^2}{C^2}, \frac{\epsilon_2^2}{4C^2}, \frac{\kappa^2}{4C^2})$, there exists $t^* > 0$ such that $\frac{dY}{dt} \leq 0$ on $[0, t^*]$, and hence $Y(t) \leq Y_0$ for $t \in [0, t^*]$, and in particular, $Y(t^*) < \min(\frac{\nu^2}{C^2}, \frac{\epsilon_1^2}{C^2}, \frac{\epsilon_2^2}{4C^2}, \frac{\kappa^2}{4C^2})$. Thus we can repeat this procedure to arbitrary time $t > 0$ to get $Y(t) \leq Y_0 < \min(\frac{\nu^2}{C^2}, \frac{\epsilon_1^2}{C^2}, \frac{\epsilon_2^2}{4C^2}, \frac{\kappa^2}{4C^2})$ for all time. This implies the required bound for the global in time existence of strong solution. \square

4. GLOBAL WELL-POSEDNESS OF SYSTEM (23)–(27)

In this section we study system (23)–(27) with $\nu = 0$. Similar as in section 3.1, our domain is \mathbb{T}^2 , and the boundary conditions and initial condition are

$$u, v, w, p \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1, \quad (106)$$

$$u, v \text{ and } p \text{ are even in } z, \text{ and } w, T \text{ are odd in } z, \quad (107)$$

$$(u, v, T)|_{t=0} = (u_0, v_0, T_0). \quad (108)$$

Using analogue argument as in section 3.1, system (23)–(27) subject to (106)–(108) is equivalent to the following:

$$(u - \alpha^2 u_{zz})_t + u u_x + w u_z + \epsilon_1 u - f_0 v + p_x = 0, \quad (109)$$

$$(v - \alpha^2 v_{zz})_t + u v_x + w v_z + \epsilon_1 v + f_0 u = 0, \quad (110)$$

$$T_t - \kappa \Delta T + u T_x + w T_z = 0, \quad (111)$$

with w, p_x, p_z defined by (47)–(49), subject to the symmetry boundary conditions and initial conditions (50)–(52). We also have (55) and (56), for which we repeat here:

$$\epsilon_2 w + p_z + T = 0, \quad (112)$$

$$u_x + w_z = 0. \quad (113)$$

In this section, we are interested in system (109)–(111) with (47)–(49), subject to (50)–(52). We will show the global regularity of strong solution to system (109)–(111) with (47)–(49), subject to (50)–(52). First, we give the definition of strong solution to system (109)–(111) with (47)–(49), subject to (50)–(52).

Definition 9. *Suppose that $u_0, v_0 \in H^2(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (50) and (51), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_{xxz} u_0, \partial_{xxz} v_0 \in L^2(\mathbb{T}^2)$. Given time $\mathcal{T} > 0$, we say (u, v, T) is a strong solution to the system (109)–(111) with (47)–(49), subject to (50)–(52), on the time interval $[0, \mathcal{T}]$, if*

(i) u, v and T satisfy the symmetry conditions (50) and (51);

(ii) u, v and T have the regularities

$$\begin{aligned} u &\in L^\infty(0, \mathcal{T}; H^2) \cap C([0, \mathcal{T}]; H^1), \quad u_{xxz} \in L^\infty(0, \mathcal{T}; L^2), \quad \partial_t u \in L^\infty(0, \mathcal{T}; L^2) \cap L^2(0, \mathcal{T}; H^1), \\ v &\in L^\infty(0, \mathcal{T}; H^2) \cap C([0, \mathcal{T}]; H^1), \quad v_{xxz} \in L^\infty(0, \mathcal{T}; L^2), \quad \partial_t v \in L^\infty(0, \mathcal{T}; H^1), \\ T &\in L^2(0, \mathcal{T}; H^2) \cap L^\infty(0, \mathcal{T}; H^1) \cap C([0, \mathcal{T}]; L^2), \quad \partial_t T \in L^2(0, \mathcal{T}; L^2); \end{aligned}$$

(iii) u, v and T satisfy system (44)–(46) in the following sense:

$$\begin{aligned} \partial_t(u - \alpha^2 u_{zz}) + uu_x + wu_z + \epsilon_1 u - f_0 v + p_x &= 0 \quad \text{in } L^\infty(0, \mathcal{T}; L^2) \cap L^2(0, \mathcal{T}; H^1); \\ \partial_t(v - \alpha^2 v_{zz}) + uv_x + wv_z + \epsilon_1 v + f_0 u &= 0 \quad \text{in } L^\infty(0, \mathcal{T}; H^1); \\ \partial_t T - \kappa \Delta T + uT_x + wT_z &= 0 \quad \text{in } L^2(0, \mathcal{T}; L^2), \end{aligned}$$

with w, p_x, p_z defined by (47)–(49), and fulfill the initial condition (52).

We have the following result concerning the existence and uniqueness of strong solutions to system (109)–(111) with (47)–(49), subject to (50)–(52), on $\mathbb{T}^2 \times (0, \mathcal{T})$, for any positive time \mathcal{T} .

Theorem 10. *Suppose that $u_0, v_0 \in H^2(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (50) and (51), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_{xxz} u_0, \partial_{xxz} v_0 \in L^2(\mathbb{T}^2)$. Let time $\mathcal{T} > 0$. Then there exists a unique strong solution (u, v, T) of system (109)–(111) with (47)–(49), subject to (50)–(52), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.*

Proof. We will only do *a priori* estimates formally here, and we can use Galerkin method as in section 3.2 to prove the result rigorously. By taking the L^2 -inner product of equation (109) with u , equation (110) with v , equation (112) with w and equation (111) with T , in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (50), we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2 \right) + \epsilon_1 \|u\|_{L^2}^2 + \epsilon_1 \|v\|_{L^2}^2 + \epsilon_2 \|w\|_{L^2}^2 + \kappa \|\nabla T\|_{L^2}^2 \\ &= - \int_{\mathbb{T}^2} [uu_x + wu_z] u \, dx dz - \int_{\mathbb{T}^2} [wv_x + wv_z] v \, dx dz \\ &\quad - \int_{\mathbb{T}^2} [up_x + wp_z + wT] \, dx dz - \int_{\mathbb{T}^2} [uT_x + wT_z] T \, dx dz. \end{aligned}$$

By integration by parts, thanks to (50), (53) and (113), we have

$$\begin{aligned} & - \int_{\mathbb{T}^2} [uu_x + wu_z] u \, dx dz - \int_{\mathbb{T}^2} [uv_x + wv_z] v \, dx dz \\ & - \int_{\mathbb{T}^2} [up_x + wp_z] \, dx dz - \int_{\mathbb{T}^2} [uT_x + wT_z] T \, dx dz = 0. \end{aligned}$$

By Cauchy–Schwarz inequality and Young’s inequality, we have

$$- \int_{\mathbb{T}^2} wT \, dx dz \leq \|w\|_{L^2} \|T\|_{L^2} \leq \frac{\epsilon_2}{2} \|w\|_{L^2}^2 + C \|T\|_{L^2}^2.$$

As a result of the above, we have

$$\begin{aligned} & \frac{d}{dt} \left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2 \right) + 2\epsilon_1 \|u\|_{L^2}^2 + 2\epsilon_1 \|v\|_{L^2}^2 + \epsilon_2 \|w\|_{L^2}^2 + 2\kappa \|\nabla T\|_{L^2}^2 \\ & \leq C \|T\|_{L^2}^2 \leq C (\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2). \end{aligned}$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned} & (\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2)(t) \\ & + \int_0^t (2\epsilon_1 \|u(s)\|_{L^2}^2 + 2\epsilon_1 \|v(s)\|_{L^2}^2 + \epsilon_2 \|w(s)\|_{L^2}^2 + 2\kappa \|\nabla T(s)\|_{L^2}^2) \, ds \\ & \leq (\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|T_0\|_{L^2}^2 + \alpha^2 \|\partial_z u_0\|_{L^2}^2 + \alpha^2 \|\partial_z v_0\|_{L^2}^2) e^{Ct}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} u, v, u_z, v_z & \in L^\infty(0, \mathcal{T}; L^2), \\ w & \in L^2(0, \mathcal{T}; L^2), \\ T & \in L^\infty(0, \mathcal{T}; L^2) \cap L^2(0, \mathcal{T}; H^1), \end{aligned} \tag{114}$$

for arbitrary $\mathcal{T} > 0$. By taking the L^2 -inner product of equation (109) with $-u_{zz}$ and equation (112) with $-w_{zz}$ in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (50) and (53), we get

$$\begin{aligned} & \frac{1}{2} \frac{d(\|u_z\|_{L^2}^2 + \alpha^2 \|u_{zz}\|_{L^2}^2)}{dt} + \epsilon_1 \|u_z\|_{L^2}^2 + \epsilon_2 \|w_z\|_{L^2}^2 \\ & = \int_{\mathbb{T}^2} [uu_x + wu_z - f_0 v] u_{zz} + [u_{zz} p_x + w_{zz} p_z + w_{zz} T] \, dx dz. \end{aligned}$$

By integration by parts, thanks to (50), (53) and (113), we have

$$\begin{aligned} & \int_{\mathbb{T}^2} [uu_x + wu_z] u_{zz} \, dx dz + \int_{\mathbb{T}^2} [u_{zz} p_x + w_{zz} p_z] \, dx dz \\ & = - \int_{\mathbb{T}^2} [u_z u_x + u u_{xz} + w_z u_z + w u_{zz}] u_z \, dx dz - \int_{\mathbb{T}^2} p(u_x + w_z)_{zz} \, dx dz \\ & = - \frac{1}{2} \int_{\mathbb{T}^2} u_z^2 (u_x + w_z) \, dx dz - \int_{\mathbb{T}^2} p(u_x + w_z)_{zz} \, dx dz = 0. \end{aligned}$$

By integration by parts, using Cauchy–Schwarz inequality and Young’s inequality, thanks to (50) and (53), we have

$$\begin{aligned} \int_{\mathbb{T}^2} (-f_0 v u_{zz} + w_{zz} T) \, dx dz & = \int_{\mathbb{T}^2} (f_0 v_z u_z - w_z T_z) \, dx dz \leq f_0 \|v_z\|_{L^2} \|u_z\|_{L^2} + \|w_z\|_{L^2} \|T_z\|_{L^2} \\ & \leq \frac{\epsilon_2}{2} \|w_z\|_{L^2}^2 + C \|T_z\|_{L^2}^2 + C \|v_z\|_{L^2} (1 + \|u_z\|_{L^2}^2 + \alpha^2 \|u_{zz}\|_{L^2}^2). \end{aligned}$$

As a result of the above, we have

$$\frac{d(1 + \|u_z\|_{L^2}^2 + \alpha^2 \|u_{zz}\|_{L^2}^2)}{dt} + \epsilon_1 \|u_z\|_{L^2}^2 + \epsilon_2 \|w_z\|_{L^2}^2 \leq C \|v_z\|_{L^2} (1 + \|u_z\|_{L^2}^2 + \alpha^2 \|u_{zz}\|_{L^2}^2) + C \|T_z\|_{L^2}^2.$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned} & \|u_z(t)\|_{L^2}^2 + \alpha^2 \|u_{zz}(t)\|_{L^2}^2 + \int_0^t (2\epsilon_1 \|u_z(s)\|_{L^2}^2 + \epsilon_2 \|w_z(s)\|_{L^2}^2) ds \\ & \leq C \left(1 + \int_0^t \|T_z(s)\|_{L^2}^2 ds + \|\partial_z u_0\|_{L^2}^2 + \alpha^2 \|\partial_{zz} u_0\|_{L^2}^2 \right) \exp \left(C \int_0^t \|v_z(s)\|_{L^2} ds \right). \end{aligned}$$

By virtue of (114) and the above, we have

$$u_{zz} \in L^\infty(0, \mathcal{T}; L^2), \quad w_z = -u_x \in L^2(0, \mathcal{T}; L^2), \quad (115)$$

for arbitrary $\mathcal{T} > 0$. By taking the L^2 -inner product of equation (109) with $-u_{xx}$ and equation (112) with $-w_{xx}$, in $L^2(\mathbb{T}^2)$, by integration by parts, thanks to (50) and (53), we get

$$\begin{aligned} & \frac{1}{2} \frac{d(\|u_x\|_{L^2}^2 + \alpha^2 \|u_{xz}\|_{L^2}^2)}{dt} + \epsilon_1 \|u_x\|_{L^2}^2 + \epsilon_2 \|w_x\|_{L^2}^2 \\ & = \int_{\mathbb{T}^2} [uu_x + wu_z - f_0 v] u_{xx} + [u_{xx} p_x + w_{xx} p_z + w_{xx} T] dx dz. \end{aligned}$$

By integration by parts, thanks to (50), (53) and (113), we have

$$\int_{\mathbb{T}^2} [u_{xx} p_x + w_{xx} p_z] dx dz = 0.$$

By integration by parts, using Cauchy-Schwarz inequality and Young's inequality, thanks to (50), (53) and (113), we have

$$\begin{aligned} - \int_{\mathbb{T}^2} f_0 v u_{xx} dx dz & = \int_{\mathbb{T}^2} f_0 v w_{xz} dx dz = - \int_{\mathbb{T}^2} f_0 v_z w_x dx dz \leq C \|v_z\|_{L^2}^2 + \frac{\epsilon_2}{6} \|w_x\|_{L^2}^2, \\ \int_{\mathbb{T}^2} T w_{xx} dx dz & = - \int_{\mathbb{T}^2} T_x w_x dx dz \leq C \|T_x\|_{L^2}^2 + \frac{\epsilon_2}{6} \|w_x\|_{L^2}^2. \end{aligned}$$

By integration by parts, using Young's inequality and Lemma 1, thanks to (50), (53) and (113), we have

$$\begin{aligned} & \int_{\mathbb{T}^2} [uu_x + wu_z] u_{xx} dx dz = - \int_{\mathbb{T}^2} [(u_x)^3 + w_x u_z u_x] dx dz \\ & = - \int_{\mathbb{T}^2} [-w_z (u_x)^2 + w_x u_z u_x] dx dz = - \int_{\mathbb{T}^2} [2w u_x u_{xz} + w_x u_z u_x] dx dz \\ & \leq C [\|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xz}\|_{L^2} \\ & \quad + \|w_x\|_{L^2} \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_x\|_{L^2}^{\frac{1}{2}} (\|u_x\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}})] \\ & \leq C (1 + \|w\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_z\|_{L^2}^2) (1 + \|u_x\|_{L^2}^2 + \alpha^2 \|u_{xz}\|_{L^2}^2) + \frac{\epsilon_2}{6} \|w_x\|_{L^2}^2. \end{aligned}$$

From the estimates above, we have

$$\begin{aligned} & \frac{d(1 + \|u_x\|_{L^2}^2 + \alpha^2 \|u_{xz}\|_{L^2}^2)}{dt} + \epsilon_1 \|u_x\|_{L^2}^2 + \epsilon_2 \|w_x\|_{L^2}^2 \\ & \leq C (1 + \|w\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_z\|_{L^2}^2) (1 + \|u_x\|_{L^2}^2 + \alpha^2 \|u_{xz}\|_{L^2}^2) + C (\|v_z\|_{L^2}^2 + \|T_x\|_{L^2}^2). \end{aligned}$$

By Gronwall inequality, we obtain

$$\begin{aligned} & \|u_x(t)\|_{L^2}^2 + \alpha^2 \|u_{xz}(t)\|_{L^2}^2 + \int_0^t (2\epsilon_1 \|u_x(s)\|_{L^2}^2 + \epsilon_2 \|w_x(s)\|_{L^2}^2) ds \\ & \leq C \left(1 + \int_0^t (\|v_z(s)\|_{L^2}^2 + \|T_x(s)\|_{L^2}^2) ds + \|\partial_x u_0\|_{L^2}^2 + \alpha^2 \|\partial_{xz} u_0\|_{L^2}^2 \right) \\ & \quad \times \exp \left(C \int_0^t (1 + \|w(s)\|_{L^2}^2 + \|u_x(s)\|_{L^2}^2 + \|u_z(s)\|_{L^2}^2) ds \right). \end{aligned}$$

By virtue of (114), (115) and the above, we have

$$u, u_z \in L^\infty(0, \mathcal{T}; H^1), \quad w \in L^2(0, \mathcal{T}; H^1), \quad (116)$$

for arbitrary $\mathcal{T} > 0$. By virtue of (116), (68) and (69), we have

$$w \in L^\infty(0, \mathcal{T}; L^2) \cap L^2(0, \mathcal{T}; H^1), \quad (117)$$

for arbitrary $\mathcal{T} > 0$. By taking the L^2 -inner product of equation (110) with $-\Delta v$ in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (50), we have

$$\begin{aligned} & \frac{1}{2} \frac{d(\|\nabla v\|_{L^2}^2 + \alpha^2 \|\nabla v_z\|_{L^2}^2)}{dt} + \epsilon_1 \|\nabla v\|_{L^2}^2 \\ & = \int_{\mathbb{T}^2} [(uv_x + wv_z)(v_{xx} + v_{zz}) + f_0 u \Delta v] dx dz =: I + II + III + IV + V. \end{aligned}$$

By integration by parts, using Cauchy–Schwarz inequality and Lemma 1, thanks to (50), (53) and (113), we have

$$\begin{aligned} |I| & = \left| \int_{\mathbb{T}^2} uv_x v_{xx} dx dz \right| = \left| \int_{\mathbb{T}^2} \frac{1}{2} u_x v_x^2 dx dz \right| = \left| \int_{\mathbb{T}^2} \frac{1}{2} w_z v_x^2 dx dz \right| = \left| \int_{\mathbb{T}^2} w v_x v_{xz} dx dz \right| \\ & \leq C \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xz}\|_{L^2}, \\ |II| & = \left| \int_{\mathbb{T}^2} w v_x v_{zz} dx dz \right| \leq C \|u\|_{L^2}^{\frac{1}{2}} (\|u\|_{L^2}^{\frac{1}{2}} + \|u_x\|_{L^2}^{\frac{1}{2}}) \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{zz}\|_{L^2}, \\ |III| & = \left| \int_{\mathbb{T}^2} w v_z v_{xx} dx dz \right| = \left| \int_{\mathbb{T}^2} v_x (w_x v_z + w v_{xz}) dx dz \right| \\ & \leq C \|w_x\|_{L^2} \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_z\|_{L^2}^{\frac{1}{2}} (\|v_z\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \\ & \quad + C \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xz}\|_{L^2}, \\ |IV| & = \left| \int_{\mathbb{T}^2} w v_z v_{zz} dx dz \right| \leq C \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|v_z\|_{L^2}^{\frac{1}{2}} (\|v_z\|_{L^2}^{\frac{1}{2}} + \|v_{zz}\|_{L^2}^{\frac{1}{2}}) \|v_{zz}\|_{L^2}, \\ |V| & = \left| \int_{\mathbb{T}^2} f_0 u \Delta v dx dz \right| = \left| \int_{\mathbb{T}^2} f_0 \nabla u \nabla v dx dz \right| \leq C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}. \end{aligned}$$

As a result of the above and by Young's inequality, we conclude

$$\begin{aligned} & \frac{d(1 + \|\nabla v\|_{L^2}^2 + \alpha^2 \|\nabla v_z\|_{L^2}^2)}{dt} + 2\epsilon_1 \|\nabla v\|_{L^2}^2 \\ & \leq C (1 + \|u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|w_x\|_{L^2}^2) (1 + \|\nabla v\|_{L^2}^2 + \alpha^2 \|\nabla v_z\|_{L^2}^2). \end{aligned}$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned} & \|\nabla v(t)\|_{L^2}^2 + \alpha^2 \|\nabla v_z(t)\|_{L^2}^2 + \int_0^t 2\epsilon_1 \|\nabla v(s)\|_{L^2}^2 ds \\ & \leq (1 + \|\nabla v_0\|_{L^2}^2 + \alpha^2 \|\nabla \partial_z v_0\|_{L^2}^2) \exp\left(C \int_0^t (1 + \|u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|w_x\|_{L^2}^2)(s) ds\right). \end{aligned}$$

By virtue of (114), (115), (116) and the above, we have

$$v, v_z \in L^\infty(0, \mathcal{T}; H^1), \quad (118)$$

for arbitrary $\mathcal{T} > 0$. By taking the L^2 -inner product of equation (111) with $-\Delta T$ in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (50), we have

$$\frac{1}{2} \frac{d\|\nabla T\|_{L^2}^2}{dt} + \kappa \|\Delta T\|_{L^2}^2 = \int_{\mathbb{T}^2} (uT_x + wT_z)\Delta T.$$

By Lemma 1 and Young's inequality, thanks to (113), we have

$$\begin{aligned} & \int_{\mathbb{T}^2} (uT_x + wT_z)\Delta T \\ & \leq C\left(\|u\|_{L^2}^{\frac{1}{2}}(\|u\|_{L^2}^{\frac{1}{2}} + \|u_z\|_{L^2}^{\frac{1}{2}})\|T_x\|_{L^2}^{\frac{1}{2}}(\|T_x\|_{L^2}^{\frac{1}{2}} + \|T_{xx}\|_{L^2}^{\frac{1}{2}})\|\Delta T\|_{L^2}\right. \\ & \quad \left. + \|w\|_{L^2}^{\frac{1}{2}}(\|w\|_{L^2}^{\frac{1}{2}} + \|w_z\|_{L^2}^{\frac{1}{2}})\|T_z\|_{L^2}^{\frac{1}{2}}(\|T_z\|_{L^2}^{\frac{1}{2}} + \|T_{xz}\|_{L^2}^{\frac{1}{2}})\|\Delta T\|_{L^2}\right) \\ & \leq \frac{\kappa}{2} \|\Delta T\|_{L^2}^2 + C(1 + \|u\|_{L^2}^4 + \|u_z\|_{L^2}^4 + \|w\|_{L^2}^4 + \|w_z\|_{L^2}^4) \|\nabla T\|_{L^2}^2, \\ & = \frac{\kappa}{2} \|\Delta T\|_{L^2}^2 + C(1 + \|u\|_{L^2}^4 + \|u_z\|_{L^2}^4 + \|w\|_{L^2}^4 + \|u_x\|_{L^2}^4) \|\nabla T\|_{L^2}^2. \end{aligned}$$

As a result of the above we conclude

$$\frac{d\|\nabla T\|_{L^2}^2}{dt} + \kappa \|\Delta T\|_{L^2}^2 \leq C(1 + \|u\|_{L^2}^4 + \|u_z\|_{L^2}^4 + \|w\|_{L^2}^4 + \|u_x\|_{L^2}^4) \|\nabla T\|_{L^2}^2.$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned} & \|\nabla T(t)\|_{L^2}^2 + \kappa \int_0^t \|\Delta T(s)\|_{L^2}^2 ds \\ & \leq C\|\nabla T_0\|_{L^2}^2 \exp\left(C \int_0^t (1 + \|u\|_{L^2}^4 + \|u_z\|_{L^2}^4 + \|w\|_{L^2}^4 + \|u_x\|_{L^2}^4)(s) ds\right). \end{aligned}$$

By virtue of (114), (115), (116), (117) and the above, we have

$$T \in L^\infty(0, \mathcal{T}; H^1) \cap L^2(0, \mathcal{T}; H^2). \quad (119)$$

By taking the L^2 -inner product of equation (109) with u_{xxxx} , equation (110) with v_{xxxx} , and equation (112) with w_{xxxx} in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (50) and (53), we get

$$\begin{aligned} & \frac{1}{2} \frac{d(\|u_{xx}\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 + \alpha^2 \|u_{xxz}\|_{L^2}^2 + \alpha^2 \|v_{xxz}\|_{L^2}^2)}{dt} + \epsilon_1 \|u_{xx}\|_{L^2}^2 + \epsilon_1 \|v_{xx}\|_{L^2}^2 + \epsilon_2 \|w_{xx}\|_{L^2}^2 \\ & = - \int_{\mathbb{T}^2} [uu_x + wu_z - f_0v] u_{xxxx} dx dz - \int_{\mathbb{T}^2} [uv_x + wv_z + f_0u] v_{xxxx} dx dz \\ & \quad - \int_{\mathbb{T}^2} [u_{xxxx}p_x + w_{xxxx}p_z + w_{xxxx}T] dx dz. \end{aligned}$$

By integration by parts, thanks to (50), (53) and (113), , we have

$$\int_{\mathbb{T}^2} (-f_0vu_{xxxx} + f_0wv_{xxxx}) dx dz - \int_{\mathbb{T}^2} (u_{xxxx}p_x + w_{xxxx}p_z) dx dz = 0.$$

By integration by parts, using Young's inequality and Lemma 1, thanks to (50), (53) and (113), we have

$$\begin{aligned}
& \left| \int_{\mathbb{T}^2} [uu_x + wu_z] u_{xxxx} dx dz \right| = \left| \int_{\mathbb{T}^2} \left[5wu_{xxz} - \frac{1}{2}w_{xx}u_z - 2w_xu_{xz} \right] u_{xx} dx dz \right| \\
& \leq C \left[\|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \|u_{xxz}\|_{L^2} \right. \\
& \quad + \|w_{xx}\|_{L^2} \|u_z\|_{L^2}^{\frac{1}{2}} (\|u_z\|_{L^2}^{\frac{1}{2}} + \|u_{xz}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \\
& \quad \left. + \|w_x\|_{L^2} \|u_{xz}\|_{L^2}^{\frac{1}{2}} (\|u_{xz}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \|u_{xx}\|_{L^2}^{\frac{1}{2}} (\|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u_{xxz}\|_{L^2}^{\frac{1}{2}}) \right] \\
& \leq \frac{\epsilon_2}{6} \|w_{xx}\|_{L^2}^2 + C (1 + \|w\|_{L^2}^2 + \|w_x\|_{L^2}^2 + \|u_z\|_{L^2}^2 + \|u_{xz}\|_{L^2}^2) (1 + \|u_{xx}\|_{L^2}^2 + \alpha^2 \|u_{xxz}\|_{L^2}^2),
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_{\mathbb{T}^2} [uv_x + wv_z] v_{xxxx} dx dz \right| = \left| \int_{\mathbb{T}^2} [u_{xx}v_x + w_{xx}v_z - 4wv_{xxz} + 2w_xv_{xz}] v_{xx} dx dz \right| \\
& \leq \left| \int_{\mathbb{T}^2} [w_xv_{xz} + w_{xx}v_z - 4wv_{xxz} + 2w_xv_{xz}] v_{xx} dx dz \right| + \left| \int_{\mathbb{T}^2} w_xv_xv_{xxz} dx dz \right| \\
& \leq C \left[\|w_x\|_{L^2} \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \|v_{xz}\|_{L^2}^{\frac{1}{2}} (\|v_{xz}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \right. \\
& \quad + \|w_{xx}\|_{L^2} \|v_z\|_{L^2}^{\frac{1}{2}} (\|v_z\|_{L^2}^{\frac{1}{2}} + \|v_{xz}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \\
& \quad + \|w\|_{L^2}^{\frac{1}{2}} (\|w\|_{L^2}^{\frac{1}{2}} + \|w_x\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \|v_{xxz}\|_{L^2} \\
& \quad + \|w_x\|_{L^2} \|v_{xz}\|_{L^2}^{\frac{1}{2}} (\|v_{xz}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \|v_{xx}\|_{L^2}^{\frac{1}{2}} (\|v_{xx}\|_{L^2}^{\frac{1}{2}} + \|v_{xxz}\|_{L^2}^{\frac{1}{2}}) \\
& \quad \left. + \|w_x\|_{L^2}^{\frac{1}{2}} (\|w_x\|_{L^2}^{\frac{1}{2}} + \|w_{xx}\|_{L^2}^{\frac{1}{2}}) \|v_x\|_{L^2}^{\frac{1}{2}} (\|v_x\|_{L^2}^{\frac{1}{2}} + \|v_{xx}\|_{L^2}^{\frac{1}{2}}) \|v_{xxz}\|_{L^2} \right] \\
& \leq \frac{\epsilon_2}{6} \|w_{xx}\|_{L^2}^2 + C (1 + \|w\|_{L^2}^2 + \|w_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + \|v_z\|_{L^2}^2 + \|v_{xz}\|_{L^2}^2) (1 + \|v_{xx}\|_{L^2}^2 + \alpha^2 \|v_{xxz}\|_{L^2}^2).
\end{aligned}$$

By integration by parts, using Cauchy-Schwarz inequality and Young's inequality, thanks to (50) and (53), we have

$$\left| \int_{\mathbb{T}^2} T w_{xxxx} dx dz \right| = \left| \int_{\mathbb{T}^2} T_{xx} w_{xx} dx dz \right| \leq \|T_{xx}\|_{L^2} \|w_{xx}\|_{L^2} \leq \frac{\epsilon_2}{6} \|w_{xx}\|_{L^2}^2 + C \|T_{xx}\|_{L^2}^2.$$

As a result of the above, we conclude

$$\begin{aligned}
& \frac{d(1 + \|u_{xx}\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 + \alpha^2 \|u_{xxz}\|_{L^2}^2 + \alpha^2 \|v_{xxz}\|_{L^2}^2)}{dt} + 2\epsilon_1 \|u_{xx}\|_{L^2}^2 + 2\epsilon_1 \|v_{xx}\|_{L^2}^2 + \epsilon_2 \|w_{xx}\|_{L^2}^2 \\
& \leq C (1 + \|w\|_{L^2}^2 + \|w_x\|_{L^2}^2 + \|u_z\|_{L^2}^2 + \|u_{xz}\|_{L^2}^2 + \|v_x\|_{L^2}^2 + \|v_z\|_{L^2}^2 + \|v_{xz}\|_{L^2}^2 + \|T_{xx}\|_{L^2}^2) \\
& \quad \times (1 + \|u_{xx}\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 + \alpha^2 \|u_{xxz}\|_{L^2}^2 + \alpha^2 \|v_{xxz}\|_{L^2}^2).
\end{aligned}$$

By Gronwall inequality, we obtain

$$\begin{aligned}
& \|u_{xx}(t)\|_{L^2}^2 + \|v_{xx}(t)\|_{L^2}^2 + \alpha^2 \|u_{xxz}(t)\|_{L^2}^2 + \alpha^2 \|v_{xxz}(t)\|_{L^2}^2 \\
& \quad + \int_0^t (2\epsilon_1 \|u_{xx}(s)\|_{L^2}^2 + 2\epsilon_1 \|v_{xx}(s)\|_{L^2}^2 + \epsilon_2 \|w_{xx}(s)\|_{L^2}^2) ds \\
& \leq C (1 + \|\partial_{xx}u_0\|_{L^2}^2 + \|\partial_{xx}v_0\|_{L^2}^2 + \alpha^2 \|\partial_{xxz}u_0\|_{L^2}^2 + \alpha^2 \|\partial_{xxz}v_0\|_{L^2}^2) \exp \left(C \int_0^t (1 + \|w(s)\|_{L^2}^2 \right. \\
& \quad \left. + \|w_x(s)\|_{L^2}^2 + \|u_z(s)\|_{L^2}^2 + \|u_{xz}(s)\|_{L^2}^2 + \|\nabla v(s)\|_{L^2}^2 + \|v_{xz}(s)\|_{L^2}^2 + \|T_{xx}(s)\|_{L^2}^2) ds \right)
\end{aligned}$$

From (114), (115), (116), (117), (118), (119) and the above, we have

$$u, v \in L^\infty(0, \mathcal{T}; H^2), \quad u_{xxz}, v_{xxz} \in L^\infty(0, \mathcal{T}; L^2), \quad w \in L^2(0, \mathcal{T}; H^2), \quad (120)$$

for arbitrary $\mathcal{T} > 0$. By virtue of (120), thanks (68) and (69), we have

$$w \in L^\infty(0, \mathcal{T}; H^1) \cap L^2(0, \mathcal{T}; H^2), \quad (121)$$

for arbitrary $\mathcal{T} > 0$. Using standard Galerkin method as in section 3.2, we can establish the existence result, and we omit the details here. Next, we show the continuous dependence on the initial data and the uniqueness of the strong solutions. Let $(u_1, v_1, w_1, p_1, T_1)$ and $(u_2, v_2, w_2, p_2, T_2)$ be two strong solutions of the system (109)–(111) with (47)–(49), and initial data $((u_0)_1, (v_0)_1, (T_0)_1)$ and $((u_0)_2, (v_0)_2, (T_0)_2)$, respectively. Denote by $u = u_1 - u_2, v = v_1 - v_2, w = w_1 - w_2, p = p_1 - p_2, T = T_1 - T_2$. Thanks to (112) and (113), it is clear that

$$\partial_t (u - \alpha^2 u_{zz}) + u_1 u_x + w_1 u_z + u(u_2)_x + w(u_2)_z + \epsilon_1 u - f_0 v + p_x = 0, \quad (122)$$

$$\partial_t (v - \alpha^2 v_{zz}) + u_1 v_x + w_1 v_z + u(v_2)_x + w(v_2)_z + \epsilon_1 v + f_0 u = 0, \quad (123)$$

$$\epsilon_2 w + p_z + T = 0, \quad (124)$$

$$u_x + w_z = 0, \quad (125)$$

$$\partial_t T - \kappa \Delta T + u_1 T_x + w_1 T_z + u(T_2)_x + w(T_2)_z = 0. \quad (126)$$

By taking the inner product of equation (122) with u , (123) with v , (124) with w , and (126) with T in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (50), (113) and (125), we get

$$\begin{aligned} & \frac{1}{2} \frac{d(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2)}{dt} + \epsilon_1 \|u\|_{L^2}^2 + \epsilon_1 \|v\|_{L^2}^2 + \epsilon_2 \|w\|_{L^2}^2 + \kappa \|\nabla T\|_{L^2}^2 \\ & \leq \left| \int_{\mathbb{T}^2} [u(u_2)_x + w(u_2)_z] u \, dx dz \right| + \left| \int_{\mathbb{T}^2} [u(v_2)_x + w(v_2)_z] v \, dx dz \right| \\ & \quad + \left| \int_{\mathbb{T}^2} w T \, dx dz \right| + \left| \int_{\mathbb{T}^2} [u(T_2)_x + w(T_2)_z] T \, dx dz \right| =: I + II + III + IV. \end{aligned}$$

By integration by parts, using Hölder inequality and Young's inequality, thanks to (50), (113) and (125) and Lemma 1,

$$\begin{aligned} I & = \left| \int_{\mathbb{T}^2} [u(u_2)_x + w(u_2)_z] u \, dx dz \right| \leq C \int_{\mathbb{T}^2} |w u_z u_2| + |w(u_2)_z u| \, dx dz, \\ & \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + C \|u_2\|_{L^\infty} \|u_z\|_{L^2}^2 + C \|(u_2)_z\|_{L^2} (\|(u_2)_z\|_{L^2} + \|(u_2)_{xz}\|_{L^2}) (\|u\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2), \\ II & = \left| \int_{\mathbb{T}^2} [u(v_2)_x + w(v_2)_z] v \, dx dz \right| \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + C \|(v_2)_x\|_{L^\infty} (\|u\|_{L^2}^2 + \|v\|_{L^2}^2) \\ & \quad + C \|(v_2)_z\|_{L^2} (\|(v_2)_z\|_{L^2} + \|(v_2)_{xz}\|_{L^2}) (\|v\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2), \end{aligned} \quad (127)$$

$$III = \left| \int_{\mathbb{T}^2} w T \, dx dz \right| \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + C \|T\|_{L^2}^2,$$

$$\begin{aligned} IV & = \left| \int_{\mathbb{T}^2} [u(T_2)_x + w(T_2)_z] T \, dx dz \right| \\ & \leq C \|T\|_{L^2} \|(T_2)_x\|_{L^2}^{\frac{1}{2}} (\|(T_2)_x\|_{L^2}^{\frac{1}{2}} + \|(T_2)_{xx}\|_{L^2}^{\frac{1}{2}}) (\|u\|_{L^2} + \|u_z\|_{L^2}) \\ & \quad + \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + \frac{\kappa}{2} \|T_z\|_{L^2}^2 + C(1 + \|(T_2)_z\|_{L^2}^2) (\|(T_2)_z\|_{L^2}^2 + \|(T_2)_{xz}\|_{L^2}^2) \|T\|_{L^2}^2 \\ & \leq \frac{\epsilon_2}{8} \|w\|_{L^2}^2 + \frac{\kappa}{2} \|\nabla T\|_{L^2}^2 + C(1 + \|(T_2)_x\|_{L^2} + \|(T_2)_{xx}\|_{L^2} + \|(T_2)_z\|_{L^2}^4 + \|(T_2)_z\|_{L^2}^2 \|(T_2)_{xz}\|_{L^2}^2) \\ & \quad \times (\|u\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2). \end{aligned} \quad (128)$$

From the estimates above, we obtain

$$\begin{aligned} & \frac{d(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2)}{dt} + \epsilon_1 \|u\|_{L^2}^2 + \epsilon_1 \|v\|_{L^2}^2 + \epsilon_2 \|w\|_{L^2}^2 + \kappa \|\nabla T\|_{L^2}^2 \\ & \leq CK (\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|T\|_{L^2}^2 + \alpha^2 \|u_z\|_{L^2}^2 + \alpha^2 \|v_z\|_{L^2}^2), \end{aligned}$$

where

$$\begin{aligned} K = 1 & + \|u_2\|_{L^\infty}^2 + \|(v_2)_x\|_{L^\infty} + \|(u_2)_z\|_{L^2}^2 + \|(u_2)_{xz}\|_{L^2}^2 + \|(v_2)_z\|_{L^2}^2 + \|(v_2)_{xz}\|_{L^2}^2 \\ & + \|(T_2)_x\|_{L^2} + \|(T_2)_{xx}\|_{L^2} + \|(T_2)_z\|_{L^2}^4 + \|(T_2)_z\|_{L^2}^2 \|(T_2)_{xz}\|_{L^2}^2. \end{aligned}$$

Using lemma 2, thanks to (119) and (120), we obtain $K \in L^1(0, \mathcal{T})$. Therefore, by Gronwall inequality, we obtain

$$\begin{aligned} & \|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|T(t)\|_{L^2}^2 + \alpha^2 \|u_z(t)\|_{L^2}^2 + \alpha^2 \|v_z(t)\|_{L^2}^2 \\ & \leq (\|u(t=0)\|_{L^2}^2 + \alpha^2 \|u_z(t=0)\|_{L^2}^2 + \|v(t=0)\|_{L^2}^2 + \alpha^2 \|v_z(t=0)\|_{L^2}^2 + \|T(t=0)\|_{L^2}^2) e^{C \int_0^t K(s) ds}, \end{aligned}$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when $u(t=0) = v(t=0) = T(t=0) = 0$, we have $u(t) = v(t) = T(t) = 0$, for all $t \geq 0$. Therefore, the strong solution is unique. \square

In the case when $f_0 = 0$ and $v \equiv 0$, our system (23)–(27) will be reduced to the following system:

$$(u - \alpha^2 u_{zz})_t + u u_x + w u_z + p_x = 0, \quad (129)$$

$$\epsilon_2 w + p_z + T = 0, \quad (130)$$

$$u_x + w_z = 0, \quad (131)$$

$$T_t - \kappa \Delta T + u T_x + w T_z = 0, \quad (132)$$

in \mathbb{T}^2 . We impose similar boundary and initial conditions for this system:

$$u, w, p \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1, \quad (133)$$

$$u, p \text{ are even in } z, \text{ and } w, T \text{ are odd in } z, \quad (134)$$

$$(u, T)|_{t=0} = (u_0, T_0). \quad (135)$$

Using analogue argument as in section 3.1, the system (129)–(132) subject to (133)–(135) is equivalent to the following:

$$(u - \alpha^2 u_{zz})_t + u u_x + w u_z + \epsilon_1 u + p_x = 0, \quad (136)$$

$$T_t - \kappa \Delta T + u T_x + w T_z = 0, \quad (137)$$

with w, p_x, p_z defined by:

$$w(x, z) := - \int_0^z u_x(x, s) ds, \quad (138)$$

$$\begin{aligned} p_x(x, z) & := \epsilon_2 \int_0^z \int_0^s u_{xx}(x, \xi) d\xi ds - \int_0^z T_x(x, s) ds \\ & + \int_0^1 \left[\int_0^{z'} T_x(x, s) ds - \epsilon_2 \int_0^{z'} \int_0^s u_{xx}(x, \xi) d\xi ds - 2wu_x(x, z') \right] dz', \end{aligned} \quad (139)$$

$$p_z(x, z) := -T(x, z) + \epsilon_2 \int_0^z u_x(x, s) ds. \quad (140)$$

We are interested in the system (136)–(137) with (138)–(140) in the unit two dimensional torus \mathbb{T}^2 , subject to the following symmetry boundary conditions and initial conditions:

$$u \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1; \quad (141)$$

$$u \text{ is even in } z, \text{ and } T \text{ is odd in } z; \quad (142)$$

$$(u, T)|_{t=0} = (u_0, T_0). \quad (143)$$

We have the global well-posedness for system (136)–(137) with (138)–(140), for initial condition with less regularity. i.e., for $u_0, \partial_z u_0 \in H^1$ and $T_0 \in H^1$. Let us give the definition of strong solution first.

Definition 11. *Suppose that $u_0 \in H^1(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (141) and (142), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_z u_0 \in H^1$. Given time $\mathcal{T} > 0$, we say (u, T) is a strong solution to the system (136)–(137) with (138)–(140), subject to (141)–(143), on the time interval $[0, \mathcal{T}]$, if*

(i) u and T satisfy the symmetry conditions (141) and (142);

(ii) u and T have the regularities

$$\begin{aligned} u &\in L^\infty(0, \mathcal{T}; H^1) \cap C([0, \mathcal{T}]; L^2), \quad u_z \in L^\infty(0, \mathcal{T}; H^1), \quad \partial_t u \in L^2(0, \mathcal{T}; L^2), \\ T &\in L^2(0, \mathcal{T}; H^2) \cap L^\infty(0, \mathcal{T}; H^1) \cap C([0, \mathcal{T}]; L^2), \quad \partial_t T \in L^2(0, \mathcal{T}; L^2); \end{aligned}$$

(iii) u, T satisfy system (136)–(137) in the following sense:

$$\begin{aligned} (u - \alpha^2 u_{zz})_t + u u_x + w u_z + \epsilon_1 u + p_x &= 0 \text{ in } L^2(0, \mathcal{T}; L^2); \\ T_t - \kappa \Delta T + u T_x + w T_z &= 0 \text{ in } L^2(0, \mathcal{T}; L^2), \end{aligned}$$

with w, p_x, p_z defined by (138)–(140), and fulfill the initial condition (143).

Based on theorem 10, we have the following theorem on the existence and uniqueness of strong solutions to system (136)–(137) with (138)–(140), subject to (141)–(143), on $\mathbb{T}^2 \times (0, \mathcal{T})$, for any positive time \mathcal{T} . The proof is similar as theorem 10, and we omit the details here.

Theorem 12. *Suppose that $u_0 \in H^1(\mathbb{T}^2)$ and $T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (141) and (142), with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that $\partial_z u_0 \in H^1$. Given time $\mathcal{T} > 0$. Then there exists a unique strong solution (u, T) of the system (136)–(137) with (138)–(140), subject to (141)–(143), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, T) depends continuously on the initial data. Same result holds when $T \equiv 0$.*

Remark 2. The reason why we need to assume more regularity for the initial data to system (109)–(111) is that we need a bound for $\|(v_2)_x\|_{L^\infty}$ appears in (127). If we do not have the evolution equation in v , we can require less for the initial data.

5. CONVERGENCE

In this section, we will prove the convergence of the strong solution of the following system

$$(u^\alpha - \alpha^2 u_{zz}^\alpha)_t - \nu u_{zz}^\alpha + u^\alpha u_x^\alpha + w^\alpha u_z^\alpha + \epsilon_1 u^\alpha + p_x^\alpha = 0, \quad (144)$$

$$\epsilon_2 w^\alpha + p_z^\alpha = 0, \quad (145)$$

$$u_x^\alpha + w_z^\alpha = 0, \quad (146)$$

subject to the following symmetric boundary conditions and initial condition

$$u^\alpha, w^\alpha \text{ and } p^\alpha \text{ are periodic in } x \text{ and } z \text{ with period } 1; \quad (147)$$

$$u^\alpha, p^\alpha \text{ are even in } z, \text{ and } w^\alpha \text{ is odd in } z; \quad (148)$$

$$u^\alpha|_{t=0} = u_0^\alpha, \quad (149)$$

to the strong solution of system (80)–(82) subject to (83)–(85), as $\alpha \rightarrow 0$.

Remark 3. The global well-posedness of system (144)–(146) subject to (147)–(149) can be easily obtained as in section 4. Moreover, as indicated in the last part of section 4, we only need to assume that u_0^α and $u_{0z}^\alpha \in H^1(\mathbb{T}^2)$ since we do not have the evolution equation in v^α .

Theorem 13. *Suppose that $u_0, \{u_0^\alpha\}_{0 < \alpha \leq 1} \subset H^1(\mathbb{T}^2)$ satisfy the symmetry conditions (83)–(84) and (147)–(148), with the compatibility conditions $\int_0^1 \partial_x u_0 dz = 0$ and $\int_0^1 \partial_x u_0^\alpha dz = 0$, for $\forall 0 < \alpha \leq 1$, and suppose that $\partial_{xz} u_0 \in L^2(\mathbb{T}^2), \{\partial_z u_0^\alpha\}_{0 < \alpha \leq 1} \subset H^1(\mathbb{T}^2)$. Moreover, suppose there exists some constant $M > 0$ such that the following uniform bound for initial data holds:*

$$\sup_{0 < \alpha \leq 1} \left(\|u_0^\alpha\|_{L^2} + \|\partial_z u_0^\alpha\|_{L^2} + \alpha \|\partial_{zz} u_0^\alpha\|_{L^2} \right) \leq M. \quad (150)$$

Let $\mathcal{T} > 0$ be such that u is the strong solution of system (80)–(82) on $[0, \mathcal{T}]$ with initial data u_0 . Let u^α be the strong solution to system (144)–(146) on $[0, \mathcal{T}]$ with initial data u_0^α . If $u_0^\alpha \rightarrow u_0$ in L^2 , as $\alpha \rightarrow 0$, then $u^\alpha \rightarrow u$ in $L^\infty(0, \mathcal{T}; L^2)$, and $u_z^\alpha \rightarrow u_z$ in $L^2(0, \mathcal{T}; L^2)$, as $\alpha \rightarrow 0$.

Proof. Let us first derive the uniform bounds of some norms of the strong solution u^α . By taking the L^2 -inner product of equation (144) with $u^\alpha, -u_{zz}^\alpha$, and equation (145) with $w^\alpha, -w_{zz}^\alpha$, in $L^2(\mathbb{T}^2)$, and by integration by parts, thanks to (147), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|u^\alpha\|_{L^2}^2 + (\alpha^2 + 1) \|u_z^\alpha\|_{L^2}^2 + \alpha^2 \|u_{zz}^\alpha\|_{L^2}^2 \right) + \epsilon_1 \left(\|u^\alpha\|_{L^2}^2 + \|u_z^\alpha\|_{L^2}^2 \right) \\ & + \epsilon_2 \left(\|w^\alpha\|_{L^2}^2 + \|w_z^\alpha\|_{L^2}^2 \right) + \nu \left(\|u_z^\alpha\|_{L^2}^2 + \|u_{zz}^\alpha\|_{L^2}^2 \right) \\ & = - \int_{\mathbb{T}^2} (u^\alpha u_x^\alpha + w^\alpha u_z^\alpha) (u^\alpha - u_{zz}^\alpha) dx dz - \int_{\mathbb{T}^2} \left(p_x^\alpha (u^\alpha - u_{zz}^\alpha) + p_z^\alpha (w^\alpha - w_{zz}^\alpha) \right) dx dz. \end{aligned}$$

By integration by parts, thanks to (146) and (147), we have

$$- \int_{\mathbb{T}^2} (u^\alpha u_x^\alpha + w^\alpha u_z^\alpha) (u^\alpha - u_{zz}^\alpha) dx dz - \int_{\mathbb{T}^2} \left(p_x^\alpha (u^\alpha - u_{zz}^\alpha) + p_z^\alpha (w^\alpha - w_{zz}^\alpha) \right) dx dz = 0.$$

As a result of the above, we have

$$\frac{d}{dt} \left(\|u^\alpha\|_{L^2}^2 + (\alpha^2 + 1) \|u_z^\alpha\|_{L^2}^2 + \alpha^2 \|u_{zz}^\alpha\|_{L^2}^2 \right) + \nu \left(\|u_z^\alpha\|_{L^2}^2 + \|u_{zz}^\alpha\|_{L^2}^2 \right) + \epsilon_2 \left(\|w^\alpha\|_{L^2}^2 + \|w_z^\alpha\|_{L^2}^2 \right) \leq 0.$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned} & \|u^\alpha(t)\|_{L^2}^2 + \|u_z^\alpha(t)\|_{L^2}^2 + \int_0^t \left[\nu \left(\|u_z^\alpha(s)\|_{L^2}^2 + \|u_{zz}^\alpha(s)\|_{L^2}^2 \right) + \epsilon_2 \left(\|w^\alpha(s)\|_{L^2}^2 + \|w_z^\alpha(s)\|_{L^2}^2 \right) \right] ds \\ & \leq \|u_0^\alpha\|_{L^2}^2 + (1 + \alpha^2) \|\partial_z u_0^\alpha\|_{L^2}^2 + \alpha^2 \|\partial_{zz} u_0^\alpha\|_{L^2}^2, \end{aligned}$$

for $t \in [0, \mathcal{T}]$. Thanks to the uniform bound for initial data (150), we have

$$\begin{aligned} & \sup_{0 < \alpha \leq 1} \left(\|u^\alpha\|_{L^\infty(0, \mathcal{T}; L^2)} + \|u_z^\alpha\|_{L^\infty(0, \mathcal{T}; L^2)} + \nu \|u_{zz}^\alpha\|_{L^2(0, \mathcal{T}; L^2)} \right. \\ & \left. + \epsilon_2 \|w^\alpha\|_{L^2(0, \mathcal{T}; L^2)} + \epsilon_2 \|w_z^\alpha\|_{L^2(0, \mathcal{T}; L^2)} \right) \leq C(M), \end{aligned} \quad (151)$$

where $C(M)$ is a constant depending on M , but not on α . Now subtracting (80)–(81) from (144)–(145), we obtain

$$\begin{aligned} & \partial_t[(u^\alpha - u) - \alpha^2(u_{zz}^\alpha - u_{zz})] - \nu(u_{zz}^\alpha - u_{zz}) + \epsilon_1(u^\alpha - u) + (p_x^\alpha - p_x) \\ & = (u - u^\alpha)u_x + (u_x - u_x^\alpha)u^\alpha + (w - w^\alpha)u_z + (u_z - u_z^\alpha)w^\alpha - \alpha^2\partial_t u_{zz}, \end{aligned} \quad (152)$$

$$\epsilon_2(w^\alpha - w) + (p_z^\alpha - p_z) = 0. \quad (153)$$

For simplicity, we denote $\|\cdot\| := \|\cdot\|_{L^2}$ from now on. By taking the inner product of equation (152) with $u^\alpha - u$ and equation (153) with $w^\alpha - w$, by integration by parts, and using (82) and (146), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u^\alpha - u\|^2 + \alpha^2 \|u_z^\alpha - u_z\|^2) + \nu \|u_z^\alpha - u_z\|^2 + \epsilon_1 \|u^\alpha - u\|^2 + \epsilon_2 \|w^\alpha - w\|^2 \\ & = \int_{\mathbb{T}^2} \left[(u - u^\alpha)^2 w_z + (u^\alpha - u)(u_x - u_x^\alpha)u^\alpha + (w - w^\alpha)(u^\alpha - u)u_z + (u_z - u_z^\alpha)(u^\alpha - u)w^\alpha \right. \\ & \quad \left. + (p_x - p_x^\alpha)(u^\alpha - u) + (p_z - p_z^\alpha)(w^\alpha - w) + \alpha^2 \partial_t u_{zz}(u - u^\alpha) \right] dx dz \\ & =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned}$$

By integration by parts, using Hölder inequality and Young's inequality, thanks to (82), (83), (146) and (147), we have

$$\begin{aligned} I_1 & = \int_{\mathbb{T}^2} (u - u^\alpha)^2 w_z dx dz = -2 \int_{\mathbb{T}^2} (u^\alpha - u)_z (u^\alpha - u) w dx dz \leq \frac{\nu}{2} \|u_z^\alpha - u_z\|^2 + C \|w\|_\infty^2 \|u^\alpha - u\|^2, \\ I_2 + I_4 & = \int_{\mathbb{T}^2} \left[(u^\alpha - u)(u_x - u_x^\alpha)u^\alpha + (u_z - u_z^\alpha)(u^\alpha - u)w^\alpha \right] dx dz \\ & = \frac{1}{2} \int_{\mathbb{T}^2} (u^\alpha - u)^2 (u_x^\alpha + w_z^\alpha) dx dz = 0, \\ I_3 & = \int_{\mathbb{T}^2} (w - w^\alpha)(u^\alpha - u)u_z dx dz \leq \frac{\epsilon_2}{2} \|w^\alpha - w\|^2 + C \|u_z\|_\infty^2 \|u^\alpha - u\|^2, \\ I_5 + I_6 & = \int_{\mathbb{T}^2} \left[(p_x - p_x^\alpha)(u^\alpha - u) + (p_z - p_z^\alpha)(w^\alpha - w) \right] dx dz \\ & = \int_{\mathbb{T}^2} (p - p^\alpha) [(u - u^\alpha)_x + (w - w^\alpha)_z] dx dz = 0, \\ I_7 & = \alpha^2 \int_{\mathbb{T}^2} \partial_t u_{zz} (u - u^\alpha) dx dz = \alpha^2 \int_{\mathbb{T}^2} u_t (u - u^\alpha)_{zz} dx dz \leq C \alpha^2 \|u_t\| (\|u_{zz}^\alpha\| + \|u_{zz}\|). \end{aligned}$$

From all the estimates above, we obtain

$$\begin{aligned} & \frac{d}{dt} (\|u^\alpha - u\|^2 + \alpha^2 \|u_z^\alpha - u_z\|^2) + \nu \|u_z^\alpha - u_z\|^2 + \epsilon_1 \|u^\alpha - u\|^2 + \epsilon_2 \|w^\alpha - w\|^2 \\ & \leq C (\|w\|_\infty^2 + \|u_z\|_\infty^2) (\|u^\alpha - u\|^2 + \alpha^2 \|u_z^\alpha - u_z\|^2) + C \alpha^2 \|u_t\| (\|u_{zz}^\alpha\| + \|u_{zz}\|). \end{aligned}$$

Let us denote by

$$F := \|w\|_\infty^2 + \|u_z\|_\infty^2, \quad G := \|u_t\| (\|u_{zz}^\alpha\| + \|u_{zz}\|).$$

Therefore, we obtain

$$\begin{aligned} & \frac{d}{dt} (\|u^\alpha - u\|^2 + \alpha^2 \|u_z^\alpha - u_z\|^2) + \nu \|u_z^\alpha - u_z\|^2 + \epsilon_1 \|u^\alpha - u\|^2 + \epsilon_2 \|w^\alpha - w\|^2 \\ & \leq CF (\|u^\alpha - u\|^2 + \alpha^2 \|u_z^\alpha - u_z\|^2) + C \alpha^2 G. \end{aligned}$$

Notice that the constant C appears above may change from line to line, and may depend on ν , ϵ_2 , and \mathbb{T}^2 , but not on α . Thanks to Gronwall inequality, we obtain

$$\begin{aligned} & \|u^\alpha - u\|^2(t) + \alpha^2 \|u_z^\alpha - u_z\|^2(t) + \int_0^t \left(\nu \|u_z^\alpha - u_z\|^2(s) + \epsilon_1 \|u^\alpha - u\|^2(s) + \epsilon_2 \|w^\alpha - w\|^2(s) \right) ds \\ & \leq (\|u_0^\alpha - u_0\|^2 + \alpha^2 \|\partial_z u_0^\alpha - \partial_z u_0\|^2) \exp \left(C \int_0^t F(s) ds \right) + C \alpha^2 \exp \left(C \int_0^t F(s) ds \right) \int_0^t G(s) ds \\ & =: (\|u_0^\alpha - u_0\|^2 + \alpha^2 \|\partial_z u_0^\alpha - \partial_z u_0\|^2) \exp \left(C \int_0^t F(s) ds \right) + C \alpha^2 H(t). \end{aligned} \quad (154)$$

By virtue of the regularity of strong solution to system (80)–(82) as stated in Definition 6, and the uniform bound (151), using Lemma 2, we have $F, G \in L^1(0, \mathcal{T})$. By virtue of uniform bound (151), we have $\alpha^2 H(t) \rightarrow 0$, as $\alpha \rightarrow 0$. Since $u_0^\alpha \rightarrow u_0$ in L^2 , and thanks to (150), we have $u^\alpha \rightarrow u$ in $L^\infty(0, \mathcal{T}; L^2)$, $u_z^\alpha \rightarrow u_z$ in $L^2(0, \mathcal{T}; L^2)$, and $w^\alpha \rightarrow w$ in $L^2(0, \mathcal{T}; L^2)$, as $\alpha \rightarrow 0$. \square

6. BLOW-UP CRITERION

In this section we give a blow-up criterion for system (80)–(82) subjects to (83)–(85). The following result follows the idea in [42].

Theorem 14. *With the same assumptions in Theorem 13, and take $u_0^\alpha = u_0$ for all α . Suppose there exists some time $\mathcal{T}^* < \infty$ such that*

$$\limsup_{\alpha \rightarrow 0^+} \left(\alpha^2 \sup_{t \in [0, \mathcal{T}^*]} \|u_z^\alpha(t)\|^2 \right) > 0, \quad (155)$$

then the solution for system (80)–(82) blows up on $[0, \mathcal{T}^*]$.

Proof. Assume the solution for system (80)–(82) will not blow up on $[0, \mathcal{T}^*]$, then $u \in L^\infty(0, \mathcal{T}^*; H^1)$ and $\partial_t u \in L^2(0, \mathcal{T}^*; L^2)$. By taking the inner product of equation (80) with u and equation (81) with w in $L^2(\mathbb{T}^2)$, by integration by parts and thanks to (82) and (83), we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|u_z\|^2 + \epsilon_1 \|u\|^2 + \epsilon_2 \|w\|^2 = 0. \quad (156)$$

Integrating (156) from 0 to t for $t \in [0, \mathcal{T}^*]$, we have

$$\|u(t)\|^2 + 2 \int_0^t \left(\nu \|u_z(s)\|^2 + \epsilon_1 \|u(s)\|^2 + \epsilon_2 \|w(s)\|^2 \right) ds = \|u_0\|^2. \quad (157)$$

On the other hand, using analogue argument for system (144)–(146), we have

$$\begin{aligned} & \alpha^2 \|u_z^\alpha(t)\|^2 + \|u^\alpha(t)\|^2 + 2 \int_0^t \left(\nu \|u_z^\alpha(s)\|^2 + \epsilon_1 \|u^\alpha(s)\|^2 + \epsilon_2 \|w^\alpha(s)\|^2 \right) ds \\ & = \|u_0^\alpha\|^2 + \alpha^2 \|\partial_z u_0^\alpha\|^2 = \|u_0\|^2 + \alpha^2 \|\partial_z u_0\|^2. \end{aligned} \quad (158)$$

From (154) and thanks to the fact that $u_0^\alpha = u_0$, for any $t \in [0, \mathcal{T}^*]$, we have

$$\|u^\alpha(t)\| \geq \|u(t)\| - C \alpha H^{1/2}(t) \geq \|u(t)\| - C \alpha H^{1/2}(\mathcal{T}^*), \quad (159)$$

since $H^{1/2}(t)$ is monotonically increasing. By virtue of (157), we know $\|u_0\| \geq \|u(t)\| \geq \|u(\mathcal{T}^*)\|$ for any $t \in [0, \mathcal{T}^*]$. Therefore, we can take $\alpha < \frac{\|u(\mathcal{T}^*)\|}{C H^{1/2}(\mathcal{T}^*)}$ to guarantee the right hand side of (159) is positive. Take square on (159), we obtain

$$\begin{aligned} \|u^\alpha(t)\|^2 & \geq \|u(t)\|^2 - 2\alpha C H^{1/2}(\mathcal{T}^*) \|u(t)\| + C^2 \alpha^2 H(\mathcal{T}^*) \\ & \geq \|u(t)\|^2 - 2\alpha C H^{1/2}(\mathcal{T}^*) \|u_0\| + C^2 \alpha^2 H(\mathcal{T}^*). \end{aligned} \quad (160)$$

Subtracting (158) from (157), we have

$$\begin{aligned} \|u(t)\|^2 - \|u^\alpha(t)\|^2 &= \alpha^2 \|u_z^\alpha(t)\|^2 - \alpha^2 \|\partial_z u_0\|^2 \\ &+ 2 \int_0^t \left(\nu \|u_z^\alpha(s)\|^2 + \epsilon_1 \|u^\alpha(s)\|^2 + \epsilon_2 \|w^\alpha(s)\|^2 \right) - \left(\nu \|u_z(s)\|^2 + \epsilon_1 \|u(s)\|^2 + \epsilon_2 \|w(s)\|^2 \right) ds. \end{aligned} \quad (161)$$

Combining (161) with (160), we obtain

$$\begin{aligned} \alpha^2 \|u_z^\alpha(t)\|^2 &\leq \alpha^2 \|\partial_z u_0\|^2 + 2\alpha C H^{1/2}(\mathcal{T}^*) \|u_0\| - C^2 \alpha^2 H(\mathcal{T}^*) \\ &+ 2 \int_0^t \left(\nu \|u_z(s)\|^2 + \epsilon_1 \|u(s)\|^2 + \epsilon_2 \|w(s)\|^2 \right) - \left(\nu \|u_z^\alpha(s)\|^2 + \epsilon_1 \|u^\alpha(s)\|^2 + \epsilon_2 \|w^\alpha(s)\|^2 \right) ds. \end{aligned} \quad (162)$$

By Cauchy–Schwarz inequality and Hölder inequality, thanks to (99)–(100) and the uniform bound (151), we have the estimate for the last term in (162):

$$\begin{aligned} &2 \int_0^t \left(\nu \|u_z(s)\|^2 + \epsilon_1 \|u(s)\|^2 + \epsilon_2 \|w(s)\|^2 \right) - \left(\nu \|u_z^\alpha(s)\|^2 + \epsilon_1 \|u^\alpha(s)\|^2 + \epsilon_2 \|w^\alpha(s)\|^2 \right) ds \\ &= 2 \int_0^t \left[\nu (u_z - u_z^\alpha, u_z + u_z^\alpha) + \epsilon_1 (u - u^\alpha, u + u^\alpha) + \epsilon_2 (w - w^\alpha, w + w^\alpha) \right] ds \\ &\leq 2 \int_0^t \left[\nu \|u_z - u_z^\alpha\| \|u_z + u_z^\alpha\| + \epsilon_1 \|u - u^\alpha\| \|u + u^\alpha\| + \epsilon_2 \|w - w^\alpha\| \|w + w^\alpha\| \right] ds \\ &\leq C \left(\|u_z - u_z^\alpha\|_{L^2(0, \mathcal{T}^*; L^2)} + \|u - u^\alpha\|_{L^2(0, \mathcal{T}^*; L^2)} + \|w - w^\alpha\|_{L^2(0, \mathcal{T}^*; L^2)} \right). \end{aligned}$$

Plugging this back into (162), we have

$$\begin{aligned} \alpha^2 \|u_z^\alpha(t)\|^2 &\leq \alpha^2 \|\partial_z u_0\|^2 + 2\alpha C H^{1/2}(\mathcal{T}^*) \|u_0\| - C^2 \alpha^2 H(\mathcal{T}^*) \\ &+ C \left(\|u_z - u_z^\alpha\|_{L^2(0, \mathcal{T}^*; L^2)} + \|u - u^\alpha\|_{L^2(0, \mathcal{T}^*; L^2)} + \|w - w^\alpha\|_{L^2(0, \mathcal{T}^*; L^2)} \right). \end{aligned} \quad (163)$$

By virtue of Theorem 13, the right hand side of (163) is independent of t , and it converges to 0 as $\alpha \rightarrow 0$. Therefore, by taking $\limsup_{\alpha \rightarrow 0^+} \sup_{t \in [0, \mathcal{T}^*]}$ on both hand sides of (163), we obtain

$$\limsup_{\alpha \rightarrow 0^+} \left(\alpha^2 \sup_{t \in [0, \mathcal{T}^*]} \|u_z^\alpha(t)\|^2 \right) = 0,$$

which contradicts to (155). \square

Remark 4. By considering the convergence for the whole system, i.e., the convergence of the strong solution of system (23)–(27) to the corresponding solution of system (17)–(21), we can establish similar blow-up criterion for system (17)–(21).

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(C. Cao) DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, UNIVERSITY PARK, MIAMI, FL 33199, USA.
E-mail address: caoc@fiu.edu

(Q. Lin) DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS, TX 77840, USA.
E-mail address: abellyn@math.tamu.edu

(E.S. Titi) DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS, TX 77840, USA. DEPARTMENT OF APPLIED MATHEMATICS AND THEORETICAL PHYSICS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, UK. DEPARTMENT OF COMPUTER SCIENCE AND APPLIED MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL.

E-mail address: titi@math.tamu.edu

E-mail address: Edriss.Titi@damtp.cam.ac.uk

E-mail address: edriss.titi@weizmann.ac.il