"One Man, One Vote" Part 2: Measurement of Malapportionment and Disproportionality and the Lorenz Curve*

Olivier de Mouzon† Thibault Laurent‡ Michel Le Breton§

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Abstract

The main objective of this paper is to explore and estimate the departure from the "One Man, One Vote" principle in the context of political representation and its consequences in distributive politics. To proceed in the measurement of the inequalities in the representation of territories (geographical under/over representation) or opinions/parties (ideological under/over representation), we import (with some important qualifications and adjustments) the Lorenz curve which is an important tool in the economics of income distribution. We consider subsequently some malapportionment and disproportionality indices. The paper contains applications of these tools to the evaluation of malapportionment and disproportionality to the 2010 Electoral College and the French parliamentary and local elections with a special attention to the electoral reform in 2015.

Classification JEL : D71, D72.

Key Words : Apportionment, Disproportionality, Electoral Justice, Lorenz Curve.

1 Introduction

This paper is in the continuation of our earlier paper dedicated to an analysis of the "one man, one vote" principle in the specific context of the U.S. Electoral College. In that paper, the focus was on the degree of violation of the "one man, one vote" principle in the context of voting. It was postulated that the variable of interest that we wanted ideally to the be the same for all voters was the probability of being decisive in an election. It was demonstrated that the "one

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†Toulouse School of Economics, INRAE, University of Toulouse Capitole, Toulouse, France.
‡Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France.
§Institut Universitaire de France and Toulouse School of Economics, University of Toulouse Capitole, France.

man, one vote" principle was violated and that the identity of the beneficiaries was dependent upon the a priori probability model which was considered. However, for the three probability models which were investigated, the ratio between the most advantaged citizens and the less advantaged ones was always around 3.

In this second paper, we want to examine the "one man, one vote" principle when, instead of a binary ideological issue, the public decision consists in the distribution of resources among several territories like districts, counties, states or countries depending upon context. It will be postulated that the actual distribution of resources among these territories is highly dependent upon how these territories are represented in the public decision body in charge of deciding the distribution of these resources. The objective is to contrast the actual distribution with the distribution of resources that would arise as the solution of a social choice or welfare optimization problem postulating an equal treatment of individuals. We are interested in comparing the positive solution (which depends upon political representation of territories) and the normative solution (which only depends upon the population of the territories) as theory (in particular bargaining theory) suggests that any deviation from equality/proportionality in representation leads to a deviation from equality/proportionality in the sharing of resources. This primary objective leads us to revisit an important issue in politics: how to measure malapportionment? Malapportionment[1] defines a situation where the allocation of seats/representatives across districts deviates from the allocation that would result from a strict application of population proportionality.[2]

This methodological issue spans a number of diverse and important situations including, in addition to legislative districting, the presidential Electoral College in the US and the European Council of Ministers. For each of these situations, we can make an instantaneous photograph of how apportionment looks like. The photograph can consist of a single measure or a set of measures or even a curve as we will see. Collecting photographs at several points in time and/or for different territories paves the way for a study the evolution of malapportionment along a time dimension (time series) or a spatial dimension (cross-section data). With such measurement tools, we will be in position to answer questions like: What has been the evolution of malapportionment in France over the last parliamentary elections? Could we say that, in the process of electing their "conseillers départementaux", malapportionment is more severe in

[1] In this paper, we will be mostly interested by malapportionment in the context of districting. Given is the partition of a territory (a union of countries, a country, a region, a "département" in France,...) into districts (countries, states, congressional districts, "cantons" in the case of the French "Départements",...). Each district is identified by its population size. The data on which measurement is based consists in the vector of seats and the vector of population sizes.

[2] Of course the concepts introduced for the measurement of malapportionment can be (and in fact are) extensively applied also in the context of party representation.

the Département "Morbihan" than it is in the Département "Creuse". We could also, using the same tools, evaluate the impact on malapportionment of a particular redistricting plan like for instance the one which has been implemented in France in March 2015 for the election of local representatives: this plan included, among other things, a division by two of the number of "cantons"

Malapportionment remains one of the key issue in political science. The one man/one vote principle is considered as a pillar of any democratic system and any violation of that principle is perceived as going against the democratic ideal. The adoption of the suffrage universal was certainly an important move towards this principle but it is well documented and recognized that in reality the voice of some citizens may count more than the voice of others. The books of Ansolabehere and Snyder (2008) and Balinski (2004) contain a lot of evidence indicating that this issue is not a secondary one. At that stage, it is important to say that this is not only a question of equality in political rights but, as we argued above, it is also a question of allocation of the resources/budgets which are under the control of the elected representatives. As demonstrated forcefully by Ansolabehere and Snyder (2008),[3] districts which are over (respectively under) represented tend to to catch a larger (respectively smaller) share of the cake. Their cross-sectional analysis shows that counties with relatively more legislative seats per person prior to redistricting receives relatively more transfers from the state per person. They calculated that population equalization significantly altered the flow of states transfers to counties, diverting approximately $7 billions annually from formerly over-represented to formerly under-represented counties. Clearly "the American experience provides clear evidence of the political consequences of unequal representation". Maaser and Stratmann (2016) reach a similar conclusion for Germany: they find that districts with a greater number of representatives receive more government funds. Kauppi and Widgrén (2004, 2007) and García-Valiñas et al. (2016) have also demonstrated that political representation within the EU council is a key driver of the distribution of the EU budgets.

In this paper, we import from economics some tools which have been developed to evaluate the intensity of inequality in income/wealth/health (or other continuous variable impacting the well being of individuals) distribution.[4] In contrast to economics, here the variable under scrutiny is seats. We argue that the tools of economists, on top of which the *Lorenz curve* and the *Gini Index* are very much appropriate to handle the measurement of malapportionment once the right variables have been introduced. In doing so, we follow the steps of Van Puyenbroeck (2006) who already suggested the fruitfulness of that connection in his pioneering must read

[3]See their documented chapter 6 as well as their (2002) paper.

[4]See e.g. Lambert (2001) for a nice presentation of the main ideas and results in that area.

paper.[5] As forcefully demonstrated by Van Puyenbroeck (2006), it is important to be careful in importing these tools as measuring departures from the equality principle in politics calls for some important adjustments. For not having paid enough attention to these points, a vein of the literature on disproportionality measurement is the subject of serious controversies.

The paper is organized as follows. In the first section, we introduce the main concepts and notations together with a general framework to evaluate the distance between ideal public decisions and real public decisions. We then move to distributive politics and define the Lorenz curve and some important indices. In the second section, we apply these tools to several situations. First, we evaluate the Lorenz curve together with the Gini and the Dauer and Kelsay (DK herefater) indices for the last national legislatures. Second, we explore the Lorenz curve of each "département" in the Metropolitan part of France as well as two indices before and after the 2015 electoral reform. Third and last, we estimate the evolution of disproportionality over the last French parliamentary elections and the 2010 US Electoral College. An appendix divided into 5 parts contains additional material on malapportionment and voting, majorization with weights and other technical developments.

2 Descriptive Statistics and Measurement of Malapportionment/Disproportionality

The framework which is developed in this paper can accommodate two different measurement issues. For both of them we want to examine and compute the "distance" to the "one man, one vote" principle. In the first subsection, we present the two settings. Then, in the second subsection, we develop a framework explaining the connection between representation and public decision. The third subsection is the key subsection. It explains how to construct the Lorenz curve in our setting and argues against some alternative constructions of this Lorenz curve. The last subsection introduces some of the main indices and in particular the two main ones which are used in section 3.

2.1 Two Settings

In the first set of applications, we consider a territory (a country, a region, a local government,...) divided into K electoral sub-territories (states, counties, electoral districts,...). The representatives of the territory are all elected at the district level. Hereafter, we will denote by N_k the population size of district k and by by R_k the number of representatives apportioned to district

[5] It is also important to point out that concepts from the theory of majorization (Marshall et al., 2011) have also been used to compare different apportionment methods (Lauwers and Puyenbroeck, 2006b,a; Marshall et al., 2002).

k for all $k = 1, 2, ..., K$. The territory can be a local/regional territory (like a "département" or a region in France or a state in the U.S.) and the assembly of representatives is a council in charge of the policies decided and implemented at the level of this territory. The inputs of the measurement issue addressed in such first case consist of two vectors: the vector of populations $\mathbf{N} = (N_1, N_2, ..., N_K)$ and the vector of representatives $\mathbf{R} = (R_1, R_2, ..., R_K)$.[6] Such a pair $(\mathbf{N}, \mathbf{R})$ will be called a *geographical pattern/situation*. In many applications, we will assume that $\mathbf{R} = (1, 1, ..., 1)$. Let us finally denote by $\mathbf{n}$ and $\mathbf{r}$ the vectors of shares $\mathbf{n} = (\frac{N_1}{N}, \frac{N_2}{N}, ..., \frac{N_K}{N})$ and $\mathbf{r} = (\frac{R_1}{R}, \frac{R_2}{R}, ..., \frac{R_K}{R})$. While our notations seem to privilege the time series analysis of a pattern, we would like to point out that the same tools allow a cross-sectional analysis (for instance we can use the tools to compare different territories,at any given point in time, as done for instance in Ansolabehere and Snyder (2008)) or a comparison between two situations describing respectively pre-reform and a post-reform patterns.

In the second type of application, the focus is on an election involving, V voters, K parties and S seats. In such case, we will denote by V_k the population of voters who voted[7] for party k and by S_k the number of seats won by party k for all $k = 1, 2, ..., K$. The inputs of the measurement issue addressed in such case consist of two vectors: the vector of votes $\mathbf{V} = (V_1, N_2, ..., V_K)$ and the vector of representatives $\mathbf{S} = (S_1, S_2, ..., S_K)$. Such a pair $(\mathbf{V}, \mathbf{S})$ will be called an *ideological pattern/situation*. We will denote by $\mathbf{v}$ and $\mathbf{s}$ the vectors of shares $\mathbf{v} = (\frac{V_1}{V}, \frac{V_2}{V}, ..., \frac{V_K}{V})$ and $\mathbf{s} = (\frac{S_1}{S}, \frac{S_2}{S}, ..., \frac{S_K}{S})$.

Given either a *geographical pattern/situation* (N, R) (or (n, r)) or an *ideological pattern/situation* (V, S) (or (v, s)), we want to measure how far we are from the "one man one vote" reference norm.

2.2 Mapping Representation into Public Decisions

To compare two different situations $(\mathbf{N}, \mathbf{R})$ and$(\mathbf{N}, \mathbf{R}')$, we introduce a set of feasible public decisions $\mathcal{D}$. We assume that each citizen $i = 1, ..., N$ has a utility function U_i on $\mathcal{D}$. Before exploring the influence of $(\mathbf{N}, \mathbf{R})$ in the positive decision making process, we define a normative reference that will be used as a benchmark in subsequent comparisons. Hereafter, we will focus on the utilitarian norm. From that perspective, the welfare attached to decision d is:

$$\sum_{i=1}^{N} U_i(d).$$

[6] Hereafter, N and R denote respectively the total number of voters and the total number of representatives.

[7] Of course the expression "who voted for party k" is possibly ambiguous if the electoral mechanism is complicated and/or if it involves several rounds. This framework only applies to elections where the ballots consist of lists of candidates (possibly one) with a party affiliation. In the case of several rounds, we retain the votes of the first round.

Let us denote by $d^*(N,U)$ the decision which maximizes utilitarian welfare. Given the decision $d(U,N,\mathbf{R})$ undertaken by the council of representatives, we may evaluate the "distance" between the two in several ways. For instance, we could consider:[8]

$$\sum_{i=1}^{N} U_i(d^*(N,U)) - \sum_{i=1}^{N} U_i(d(U,N,\mathbf{R})),$$

or we could consider the departure from the perspective of each individual i.e. :

$$(U_1(d^*(\mathbf{N},U)) - U_1(d(U,\mathbf{N},\mathbf{R})), ..., U_N(d^*(\mathbf{N},U)) - U_N(d(U,\mathbf{N},\mathbf{R}))).$$

Since these measures depend upon the particular profile U that is considered, we may prefer to consider ex ante evaluations where U is drawn randomly into a set $\mathcal{U}$ of admissible profiles according to a specific[9] probability model λ. Then for the two measures[10] above, we move to expectations with respect to λ.

$$\Delta_\lambda^1(\mathbf{N},\mathbf{R}) = \underset{\lambda}{E}\left[\sum_{i=1}^{N} U_i(d^*(\mathbf{N},U)) - \sum_{i=1}^{N} U_i(d(U,\mathbf{N},\mathbf{R}))\right],$$

$$\Delta_\lambda^2(\mathbf{N},\mathbf{R}) =$$
$$\underset{\lambda}{E}\left[(U_1(d^*(\mathbf{N},U)) - U_1(d(U,\mathbf{N},\mathbf{R})), ..., U_N(d^*(\mathbf{N},U)) - U_N(d(U,\mathbf{N},\mathbf{R})))\right] =$$
$$\underset{\lambda}{E}\left[(U_1(d^*(\mathbf{N},U)), ..., U_N(d^*(\mathbf{N},U)))\right] - \underset{\lambda}{E}\left[(U_1(d(U,\mathbf{N},\mathbf{R})), ..., U_N(d(U,\mathbf{N},\mathbf{R})))\right].$$

To proceed with these measures, we do need a detailed description of the derivation of $d(U,\mathbf{N},\mathbf{R})$. Depending upon the nature of the set $\mathcal{D}$, many alternative institutions can be considered. To study the behavior of representatives within these institutions, we will need to model the objectives of the representatives and the nature of the game that they play among themselves. We limit our attention here to two canonical cases.

The first canonical case is the classical binary framework: $\mathcal{D} = \{0,1\}$ and for each i, there are two possible utility functions: either $U_i(1) = 1$ and $U_i(0) = 0$ or $U_i(1) = 1$ and $U_i(0) = 0$. Here $d^*(\mathbf{N},U)$ is the popular majority decision. If all the representatives of territory k endorse the majority opinion in territory k, the decision $d(U,\mathbf{N},\mathbf{R})$ is the majority decision in the council. $d(U,\mathbf{N},\mathbf{R})$ does not need to be equal to $d^*(U)$: such outcome is called an election inversion. For any probability model λ, we can (in principle) compute $\Delta_\lambda^i(\mathbf{N},\mathbf{R})$ for $i = 1$ and 2. A large value indicates a large departure from the majority outcome which is here the

[8] Or in a ratio form: $\frac{\sum_{i=1}^{N} U_i(d(U,N,\mathbf{R}))}{\sum_{i=1}^{N} U_i(d^*(N,U))}$.

[9] There is no space here to discuss all details. But clearly, since the paper is about the “one man, one vote” principle, then the probability model itself must display symmetry across voters

[10] In the ratio formulation described in footnote 8, we would get instead: $\underset{\lambda}{E}\left[\frac{\sum_{i=1}^{N} U_i(d(U,\mathbf{N},\mathbf{R}))}{\sum_{i=1}^{N} U_i(d^*(\mathbf{N},U))}\right]$.

reference outcome to define at best "one man, one vote". Using Δ^1_λ informs about the distance from a decision that reflects the "one man, one vote" principle. Using Δ^2_λ gives us a more detailed information about the decomposition of the aggregate differences into its individual components. Indeed if $\Delta^1_\lambda(\mathbf{N},\mathbf{R}) > 0$, then some of the coordinates of $\Delta^2_\lambda(\mathbf{N},\mathbf{R})$ are positive. Note also that instead of measuring the distance from the reference point through utilities, we could (as in de Mouzon et al., 2020, for the US Electoral College) calculate for each state k, a number measuring the decisiveness of a voter from state k.[11] A perfect application of the "one man, one vote" principle would require the perfect equality of these K numbers. In reality, these numbers differ among themselves. Appendix 1 contains a computation of $\Delta^1_\lambda(\mathbf{N},\mathbf{R})$ and $\Delta^2_\lambda(\mathbf{N},\mathbf{R})$ and a third measure in the case where $K = 3$ and $\mathbf{R} = (1,1,1)$. These simple calculations show how the vector $\mathbf{N}$ impacts the computations.

In the second canonical case (often recorded under the headings "Distributive Politics" or "Divide the Dollar"), the set of public policies $\mathcal{D}$ is a simplex:

$$\mathcal{D} = \mathcal{S} \equiv \left\{ X \in \mathbb{R}^K_+ : \sum_{k=1}^{K} X_k = M \right\}, \tag{1}$$

where M is a positive number.[12] The council decision consists in a distribution of the total budget M across the K territories. In such case, it is natural to assume that U_i depends only upon X_k (where k is the territory where i lives) and is strictly increasing with respect to that variable. If we assume further that the share of the budget received by territory k is divided equally among its residents, i.e. if the good which is considered is purely private (no economies of scale), the benefit of a resident of territory k is $\frac{X_k}{N_k}$ and then the utility derived by i from decision d is:

$$U_i(\frac{X_k}{N_k}).$$

Further, if we postulate symmetry in inter-comparison of utilities i.e. that U does not depend upon i, then the utilitarian welfare attached to decision d is:

$$\sum_{k=1}^{K} N_k U(\frac{X_k}{N_k}). \tag{2}$$

If U is strictly concave, maximization of (2) under constraint (1) i.e.

$$Max \sum_{k=1}^{K} N_k U(\frac{X_k}{N_k}),$$

[11] For some probability models λ the two approaches are equivalent.

[12] We will often consider the unitary simplex i.e. assume that $M = 1$.

under the constraints $X \in \mathcal{S}$ yields an unique interior solution:

$$X_k^*(\mathbf{N}, U) = \frac{N_k}{N} M \equiv n_k M \text{ for all } k = 1, ..., K.$$

The reference point is perfect proportionality. According to the utilitarian principle, each territory should receive a share of the budget proportional to its population. We note that now, in contrast to the first canonical case, the reference point does not depend upon the profile U. The are several ways to transform (N, R) into $d(U, \mathbf{N}, \mathbf{R})$. Hereafter[13], we will focus on the case where :

$$X_k(U, \mathbf{N}, \mathbf{R}) = \frac{R_k}{R} M \equiv r_k M \text{ for all } k = 1, ..., K.$$

As before, $d(U, \mathbf{N}, \mathbf{R})$ does not depend upon U in that case. As before, if the utility function[14] U is drawn randomly into a set $\mathcal{U}$ of admissible utility functions according to a specific[15] probability model λ. We obtain:[16]

$$\Delta_\lambda^1(\mathbf{N}, \mathbf{R}) = \underset{\lambda}{E}\left[\sum_{k=1}^{K} N_k \left(U(\frac{M}{N}) - U(\frac{MR_k}{RN_k})\right)\right].$$

This measures calls for several comments. For any fixed U, we want this measure to be as small as possible. These measures act as indices evaluating the "distance" between the "ideal" and the reality induced by the situation $(\mathbf{N}, \mathbf{R})$. For instance if we take $U(x) = ax - \frac{b}{2}x^2$ where a and b are positive numbers with $\frac{a}{b} > M$, then up to the multiplicative constant $\frac{bM}{2}$ and the additive constant $\sum_{k=1}^{K} N_k \left(\frac{M}{N}\right)^2$:

$$\Delta_U^1(\mathbf{N}, \mathbf{R}) = \sum_{k=1}^{K} N_k \left(\frac{R_k}{RN_k}\right)^2.$$

Up to the difference with $\left(\sum_{k=1}^{K} N_k \left(\frac{R_k}{RN_k}\right)\right)^2 = 1$, we recognize the variance of the vector $\left(\frac{R_1}{RN_1}, ..., \frac{R_k}{RN_k}\right)$. If we consider $\mathcal{U}$ to be the set of increasing and concave functions on $\mathbb{R}$, taking $-U$ instead of U provides a measure. Instead of $-ax + \frac{b}{2}x^2$, we could take an arbitrary convex function g and consider the index:

[13] An alternative to the one considered in or paper is described in appendix 4.

[14] In contrast to the first canonical case, it has been assumed that the profile if diagonal and summarized by a single increasing utility function.

[15] There is no space here to discuss all details. But clearly, since the paper is about the "one man, one vote" principle, then the probability model itself must display symmetry across voters.

[16] And similarly: $\Delta_\lambda^2(\mathbf{N}, \mathbf{R}) = \underset{\lambda}{E}\left[U(\frac{N_1}{N}M) - U(\frac{R_1}{R}M), ..., U(\frac{N_K}{N}M) - U(\frac{R_K}{R}M))\right]$.

$$\sum_{k=1}^{K} N_k g\left(\frac{R_k}{RN_k}\right).$$

2.3 The Lorenz Order

By choosing a specific concave function U (or equivalently a specific convex function g), we can order any two situations (N,R) and (N',R') after normalizing the vectors. such that the sums of the N_k,N' and of the R_k,R' are equal to 1 according to:

$$(N,R) \text{ dominates } (N',R') \text{ iff } \sum_{k=1}^{K} n_k g\left(\frac{r_k}{n_k}\right) \leq \sum_{k=1}^{K} n'_k g\left(\frac{r'_k}{n'_k}\right).$$

This dominance reads as follows: given a convex function g, the situation (N,R) is closer to the ideal "one man one vote" than the situation (N',R'). This ordering is complete i.e. any two situations can be compared. But it is also very sensitive to the choice of g. Two different convex functions could lead to two opposite statements. This suggests to consider the following *partial* ordering:

$$(N,R) \text{ unambiguously dominates } (N',R') \text{ iff } \sum_{k=1}^{K} n_k g\left(\frac{r_k}{n_k}\right) \leq \sum_{k=1}^{K} n'_k g\left(\frac{r'_k}{n'_k}\right)$$

for all convex functions g.

This partial ordering is presented in appendix 2. It is an extension of the classical majorization ordering[17] (Marshall et al., 2011) which is exclusively defined over the subclass of situations such that $n_k = n'_k = \frac{1}{K}$ for all $k = 1, ..., K$. How do we prove or disprove that (N,R) unambiguously dominates (N',R')? The main characterization theorems are also presented in appendix 2. The most important one consists in introducing the Lorenz curve which is defined here as follows.

First, for any situation (N,R) we consider the rearrangement of the coordinates of the vector $(\frac{r_1}{n_1}, \frac{r_2}{n_2}, ..., \frac{r_K}{n_K})$ from the lowest to the largest. From this vector, denoted $(\frac{\widetilde{r}_1}{\widetilde{n}_1}, \frac{\widetilde{r}_2}{\widetilde{n}_2}, ..., \frac{\widetilde{r}_K}{\widetilde{n}_K})$, we construct the following curve which contains (according to us) all the relevant statistical information on (N,R). We plot on the horizontal axis all the cumulative fractions: $0, \widetilde{n}_1, \widetilde{n}_1 + \widetilde{n}_2, \widetilde{n}_1 + \widetilde{n}_2 + \widetilde{n}_3, ..., 1$ and on the vertical axis all the cumulative ordered fractions 0, $\widetilde{r}_1, \widetilde{r}_1 + \widetilde{r}_2, \widetilde{r}_1 + \widetilde{r}_2 + \widetilde{r}_3, ..., 1$. This provides a sample of $K+1$ points in the unit square $[0,1]^1$ including $(0,0)$ and $(1,1)$. This sample is increasing and convex in the sense that:

[17] This partial ordering is also known as second order stochastic dominance.

$$\frac{\sum_{j=1}^{k}\widetilde{r}_j}{\sum_{j=1}^{k}\widetilde{n}_j} \geq \frac{\sum_{j=1}^{k-1}\widetilde{r}_j}{\sum_{j=1}^{k-1}\widetilde{n}_j} \text{ for all } k = 1, ..., K.$$

For convenience we identify this finite set of points to a curve by piece-wise linear interpolation between any pair of adjacent points. Let us denote by $L_{(N,R)}(x)$ this curve defined for all $x \in [0,1]$ and with values in $[0,1]$. For the sake of illustration, the construction of such a curve is depicted on Figure 1 when $K = 5$, $\mathbf{n}^t = (0.10, 0.32, 0.30, 0.20, 0.08)$ and $\mathbf{r}^t = (0.2, 0.2, 0.2, 0.2, 0.2)$.

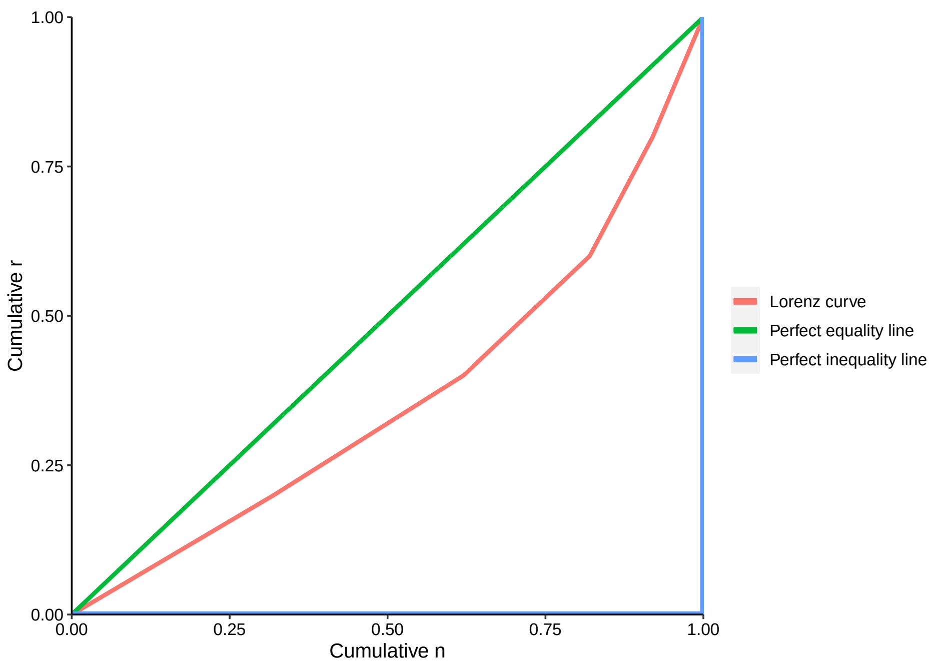

Figure 1: Illustration of the Lorenz curve

We will say that the pattern (N, R) (strictly) Lorenz dominates the pattern (N', R') iff $L_{(N,R)}(x) \geq L_{(N',R')}(x)$ for all $x \in [0,1]$ (with a strict inequality for at least one value of x).

Implicit in the above construction is the fact that the relevant units in our comparison are the individuals and what they ultimately receive into the redistribution of the resources. This should be contrasted with alternative choices as those discussed and criticized by Van Puyenbroeck (2006).[18] For instance, in many measures, the frequencies do not appear in the weighted sum and scholars[19] look at the ordering:

[18]Puyenbroek contains a lot of developments including discussions about the rearrangement. In particular, he spends time contrasting the arrangement based on the ratios $\frac{r_k}{n_k}$ with the arrangement based on the differences $r_k - n_k$. Both agree that if k and k' are such that $\frac{r_k}{n_k} > 1$ and $\frac{r_{k'}}{n_{k'}} < 1$, then k' should be on the left of k but possibly disagree on k and k' when they are on the same side.

[19]See for instance Karpov (2008); Pennisi (1998).

$$\sum_{k=1}^{K} g\left(\frac{r_k}{n_k}\right) \leq \sum_{k=1}^{K} g\left(\frac{r'_k}{n'_k}\right) \text{ for all convex functions } g.$$

In doing so, we move from the voters to the territories or the parties as relevant recipients. This choice amounts to draw a Lorenz curve where the relevant coordinates on the horizontal (vertical) axis are $0, \frac{1}{K}, \frac{2}{K}, ..., \frac{K-1}{K}, 1$ ($\left(0, \frac{\frac{\tilde{r}_1}{\tilde{n}_1}}{\sum_{k=1}^{K}\frac{\tilde{r}_k}{\tilde{n}_k}}, \frac{\frac{\tilde{r}_1}{\tilde{n}_1}+\frac{\tilde{r}_2}{\tilde{n}_2}}{\sum_{k=1}^{K}\frac{\tilde{r}_k}{\tilde{n}_k}}, ..., \frac{\sum_{k=1}^{K-1}\frac{\tilde{r}_k}{\tilde{n}_k}}{\sum_{k=1}^{K}\frac{\tilde{r}_k}{\tilde{n}_k}}, 1\right)$). Van Puyenbroeck (2006) also discusses the possibilities offered by plotting respectively the coordinates $0, \frac{1}{K}, \frac{2}{K}, ..., \frac{K-1}{K}, 1$ on the horizontal axis and the coordinates 0, $n_1, n_1 + n_2, ..., 1$ and 0, $r_1, r_1 + r_2, ..., 1$ on the vertical axis where the coordinates are rearranged according to the ordering attached to n and r (under the presumption that these two orderings are the same). This choice is debatable as the quantity $\sum_{k=1}^{K}\frac{\tilde{r}_k}{\tilde{n}_k} = \sum_{k=1}^{K}\frac{r_k}{n_k}$ is not invariant under transfers of seats between territories or parties. This leads to problematic issues when we analyse seat transfers.[20]

In addition to Van Puyenbroeck (2006), the Lorenz curve that we use in our paper has also been used by Colignatus (2017c,b,a) in a series of applications to recent electoral data. In section 3, we compute this Lorenz curve for several apportionment situations and one ideological situation. Let us conclude this section by three remarks.

First, note that when $M = 1$ and when we limit to vectors R with integer coordinates, then the Lorenz curve has a very simple shape depicted in Figure 2.

When the choice is the vector $\mathbf{R}$ where the k^{th} is equal to 1, then the curve is flat until $1-n_k$ and is linear then. This implies that if k and l are such that $n_k > n_l$, then the Lorenz curve attached to k is above the Lorenz curve attached to l. The Lorenz curve ordering is compatible with the absence of an election inversion. From the Lorenz perspective, it is always better to allocate the seat to the candidate with the highest number of votes.

Second, note that when we compare $(\mathbf{N}, \mathbf{R})$ and $(\mathbf{N}', \mathbf{R}')$ when $R = R' = (1, 1, ..., 1)$ the ordering of the units on the horizontal axis amounts to the ordering of the units from the most populated to the less populated.

Third, note that the Lorenz criterion is also useful to compare situations where the map of the districts has been reshaped. For instance, we may consider two situations $(\mathbf{N}, \mathbf{R})$ and $(\mathbf{N}', \mathbf{R}')$ where $K' = \frac{K}{2}$, N' is deduced from N through a matching i.e. according to a grouping of the old districts by pairs, $\mathbf{R} = (1, ..., 1)$ and $\mathbf{R}' = (2, ..., 2)$[21]. If we move from the first situation

[20] Van Puyenbroeck (2006) writes "Conversely, and equally unfortunately, it seems difficult to sustain that the latter construct, $\frac{1}{K}\sum_{k=1}^{K}\frac{r_k}{n_k}$, provides a reasonable benchmark of equality".

[21] In the second situation, the number of districts has been reduced by one half while the total population and the total number of representatives have remained unchanged.

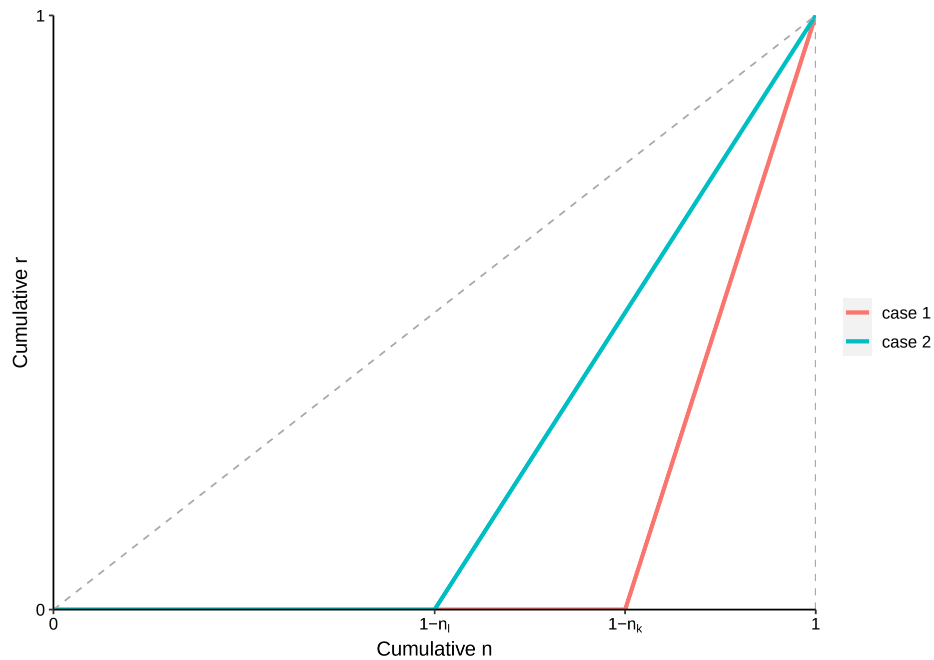

Figure 2: Illustration of the Lorenz curve

to the second one, we may wonder what are the best matchings from a Lorenz perspective.[22]

2.4 Malapportionment and Disproportionality Indices

The Lorenz ordering defined in the preceding section is partial (we cannot compare (N, R) and (N', R') when their Lorenz curves intersect). To overcome this difficulty when it arises, it is useful to complement the measurement analysis based on Lorenz by computing the value of some indices. An index is a function I which maps any situation (N, R) into a real number $I(N, R)$ and satisfies the monotonicity property:

$$\text{If } L_{(N,R)}(x) \geq L_{(N',R')}(x) \text{ for all } x \in [0,1] \text{ then } I(N,R) \leq I(N',R').$$

Among the most popular indices, are the *Gini index* defined as follows:[23]

$$G(N,R) = \frac{1}{2} \sum_{k=1}^{K} \sum_{j=1}^{K} n_k n_j \left| \frac{r_k}{n_k} - \frac{r_j}{n_j} \right|.$$

We could of course import from the inequality measurement literature other indices among which the Atkinson-Kolm's indices defined as follows:

[22]Clearly, grouping is always a good thing from a Lorenz perspective. Note that this question is formally related to the issue of aggregation. When we move to more aggregated level and average the values accordingly, we lose information and we ultimately underestimate malapportionment or disproportionality

[23]It is defined alternatively as the surface of the area between the diagonal and the Lorenz curve $L_{(N,R)}$.

$$AKT_{\alpha}\ (N,R)=\begin{cases} 1-\left(\sum_{k=1}^{K} n_k\left(\frac{r_k}{n_k}\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \text{ if } \alpha\neq 1 \\ 1-\left(\prod_{k=1}^{K}\left(\frac{r_k}{n_k}\right)^{n_k}\right) \text{ if } \alpha=1 \end{cases}.$$

The parameter α is a parameter of inequality aversion. The larger is α, the larger is the aversion to inequality. When α tends to $+\infty$, this index tends to:

$$1-\underset{1\leq k\leq K}{Min}\frac{r_k}{n_k}.$$

We could alternatively[24] consider the class of generalized entropy indices defined as follows:

$$GE_{\alpha}(N,R)=\begin{cases} \frac{1}{\alpha(\alpha-1)}\left(\sum_{k=1}^{K} n_k(\left(\frac{r_k}{n_k}\right)^{\alpha}-1\right) \text{ if } \alpha\neq 0,1 \\ \sum_{k=1}^{K} n_k\left(\frac{r_k}{n_k}\ln\frac{r_k}{n_k}\right) \text{ if } \alpha=1 \\ -\sum_{k=1}^{K} n_k\ln\frac{r_k}{n_k} \text{ if } \alpha=0 \end{cases}.$$

Up to some secondary details, the class of indices GE_{α} is part of the general class of indices I defined as follows:

$$I(N,R)=\sum_{k=1}^{K} n_k g(\frac{r_k}{n_k}) \text{ where } g \text{ is a convex function.}$$

To conclude this point,[25] let us mention the DK index (after Dauer and Kelsay, 1955) which is advocated by Ansolabehere and Snyder (2008). Let x^* be the unique value of x such that $L_{(N,R)}(x)=0.5$. From what precedes, x^* is larger than 0.5. The DK index attached to the pattern (N,R), denoted $DK(N,R)$ is the number $1-x^*$. It evaluates the smallest size of a population of citizens which control a majority of representatives in the assembly. For instance if $DK(N,R)=0.32$, it means than in the context (N,R), 32% of the electorate controls 50% of the seats/representatives. Here we prefer to have large values of DK which means that, strictly speaking, the index should be defined as being x^* itself.

All these indices are useful in the case where the Lorenz curves intersect. Drawing the Lorenz curves of situations is always important as when they do not intersect, it shows that the conclusion does not depend upon the choice of a particular index. In contrast, when they intersect, indices help to say something on the evolution of malapportionment/disproportionality. In our paper, we will focus on the Gini and DK indices.

[24] The two classes of indices are ordinally equivalent since they deduce from each other through increasing transformations. See e.g. Lambert (2001).

[25] We could also consider other measures like for instance the ratio between the largest coordinate and the smallest one but note that while popular in inequality measurement, this number is insensitive to changes in other parts of the vectors.

Let us remind that the indices which are considered in this section are those which are monotonic with respect to the Lorenz ordering which has been introduced before. This means for instance that the popular Gallagher index (Gallagher, 1991) $GA(N,R)$ defined as follows:

$$GA(N,R) = \sqrt{\frac{1}{2}\sum_{k=1}^{K}(r_k - n_k)^2},$$

is not an index as defined above since it is not always monotonic with respect to the Lorenz ordering. This difficulty with the Gallagher's index is pointed out in Goldenberg and Fisher (2019) and Renwick (2015). There is an enormous literature[26] on the measurement of disproportionality. As emphasized by Van Puyenbroeck (2006), who refers to a "zoo of no fewer than 19 proposed indices" many of them are problematic if the concern is to examine how far we are from the "one man, one vote" principle. As pointed out by Van Puyenbroeck (2006), this includes among others some versions of the Gini's index.[27]

3 Applications

This section applies to real world cases the Lorenz curve and the Gini and DK indices presented in the previous section. The four real world cases considered here are:

1. The Evolution of the geographical Lorenz curve in the "Assemblée Nationale" of the French 5^{th} Republic.

2. The Evolution of the ideological Lorenz curve in the "Assemblée Nationale" over the recent twenty five years of the French 5^{th} Republic.

3. The Evolution of the geographical Lorenz curve in the "départements" before and after the 2015 electoral reform.

4. The 2010 Electoral College in the USA.

Supplementary material including original data and code can be found at `http://www.thibault.laurent.free.fr/code/4CT`

[26] Karpov (2008) compares 18 indices. See also Chessa and Fragnelli (2012); Cox and Shugart (1991); Fry and McLean (1991); Monroe (1994); Pennisi (1998); Taagepera and Grofman (2003) out of many. There are also papers developments axiomatic analysis of some malapportionment and disproportionality indices (see e.g. Bouyssou et al., 1947; Koppel and Diskin, 2009).

[27] Many of these indices are simply defined as functions expressing (up to some normalization and/or ordinal transformation) a kind of "distance" between the population/vote shares and the seat shares which happens to be equal to 0 iff the two vectors coincide. Gallagher (1991) uses least squares but some other authors uses other absolute deviations.

3.1 Application 1: the Evolution of the geographical Lorenz curve in the "Assemblée Nationale" of the French 5^{th} Republic

There are 577 geographical zones known as "circonscriptions électorales". Each circonscription is associated to its "département" (a French administrative geographical level) and its corresponding numeric code. For example, "Département" Ain (01) had 5 circonscriptions numbered $1, 2, \ldots, 5$ in 2017.

The "Assemblée Nationale", designed to represent all citizens, must be elected on demographic grounds. In other terms, each deputy should represent the same number of citizens. The rules to determine the allocations of the deputies have changed across the time:[28]

- Before 2012, the rules of this election were based on the principle that one single "département" had at least two deputies, and one additional deputy was added every additional 108,000 inhabitants (for example, for a "département" where the population was between 216,000 and 324,000 inhabitants, there were two deputies, between 324,000 and 432,000, there were three deputies, etc.)
- After 2012, a "département" has at least one deputy and an additional deputy is allocated every additional 125,000 inhabitants (for example for a "département" with less than 125,000 inhabitants, there is one deputy, between 125,000 and 250,000 there are two deputies, etc.)

Marginal corrections can be made for a new election, but the big changes (i.e. the new rules presented above) occurred only in 1988 and 2012.

This election occurs every 5 years which means there were 6 elections between 1993 and 2017 (1993, 1997, 2002, 2007, 2012, 2017). For each election, we have the results of the votes at the two rounds.[29] Among the variables collected, we have the number of people who have the right to vote, the number of voters and the votes obtained by the different candidates.

Ii is important to point out that in the first application, only the data related to the number of people who have the right to vote is available for each election. Since the method to allocate the deputies is related to the number of inhabitants which is different from the number of people who have the right to vote, our conclusions are valid under the presumption that the ratio (voters/inhabitants) is sufficiently stable across time and space.

If this ratio was not stable in space, then our conclusions would still show that some voters have more power (because there are in a low ratio region) than others (in a higher ratio region).

[28]For more details, see for instance `https://www.liberation.fr/france/2017/06/11/pourquoi-y-a-t-il-577-circonscriptions_1575645`

[29]More information about the election process, can be found e.g. `https://fr.wikipedia.org/wiki/%C3%89lections_l%C3%A9gislatives_en_France`

Figure 3 represents the boxplot and kernel density plot of the number of people who have the right to vote per "circonscription électorale", with respect to the year of the election. On this figure, it is obvious that the distribution of the number of voters for each deputy has globally increased across the elections (meaning that the French electoral population has increased throughout the country over the years). Moreover, for the elections in 2012 and 2017, the distributions around the median seem uniform which means that the probability that a circonscription has a number of voters close to the median is higher than for the previous elections.

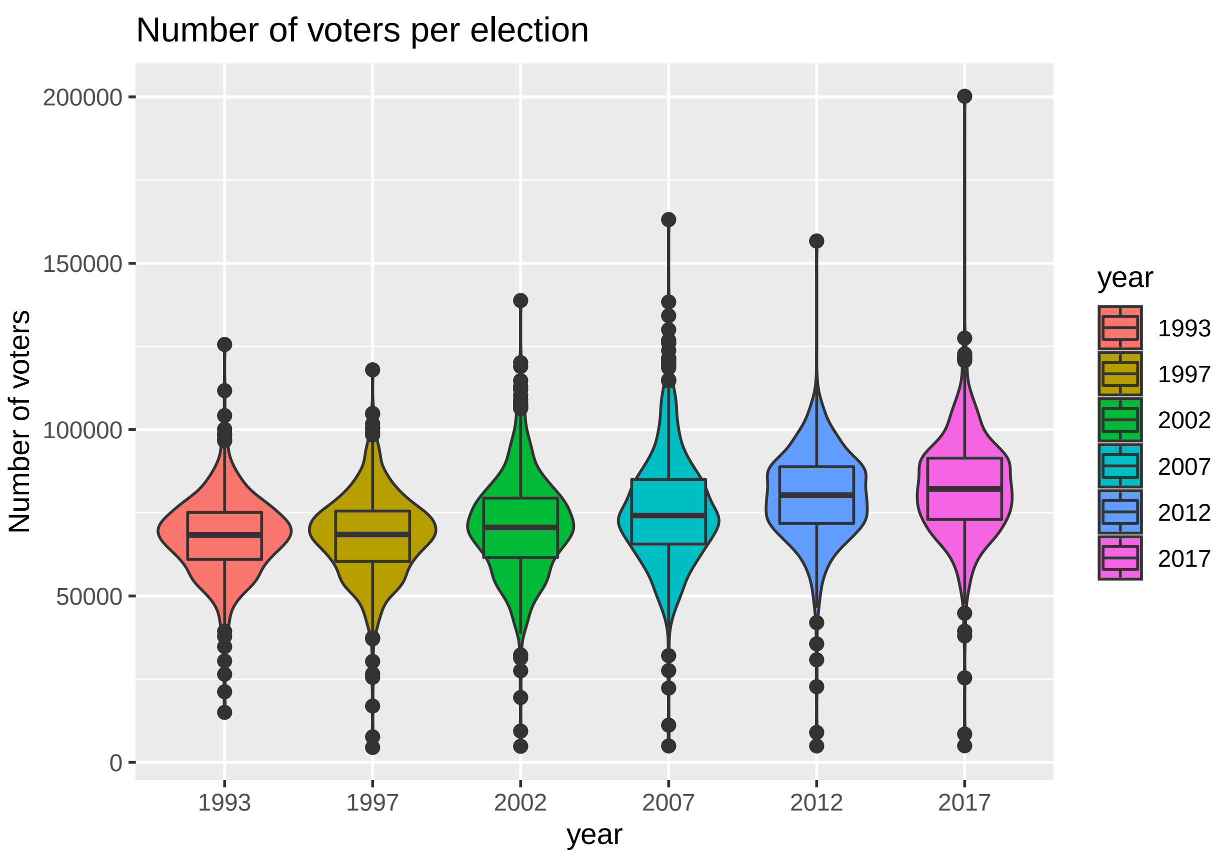

Figure 3: Year by year boxplot and kernel density plot of the number of people who have the right to vote per "circonscription électorale"

For any fixed election, we observe both outliers and a strong variance in the data, which seems to indicate that some circonscriptions are better (those with less voters) or worse (those with more voters) represented.

In the next section, we try to better understand this distribution for a fixed year.

3.1.1 Analysis of the 2017 election

We consider the population data in 2013 which is the one which is supposed to be used to settle the geographical boundaries of the circonscriptions. Those geographical boundaries were used for the 2017 election. For this election, 10 circonscriptions were allocated to the French citizens living in foreign countries. We did not include these circonscriptions hereafter.

We look at the number of deputies observed per "département" to check if the rule "a 'département' has at least one deputy and an additional deputy is allocated every additional 125,000 inhabitants" is indeed followed.

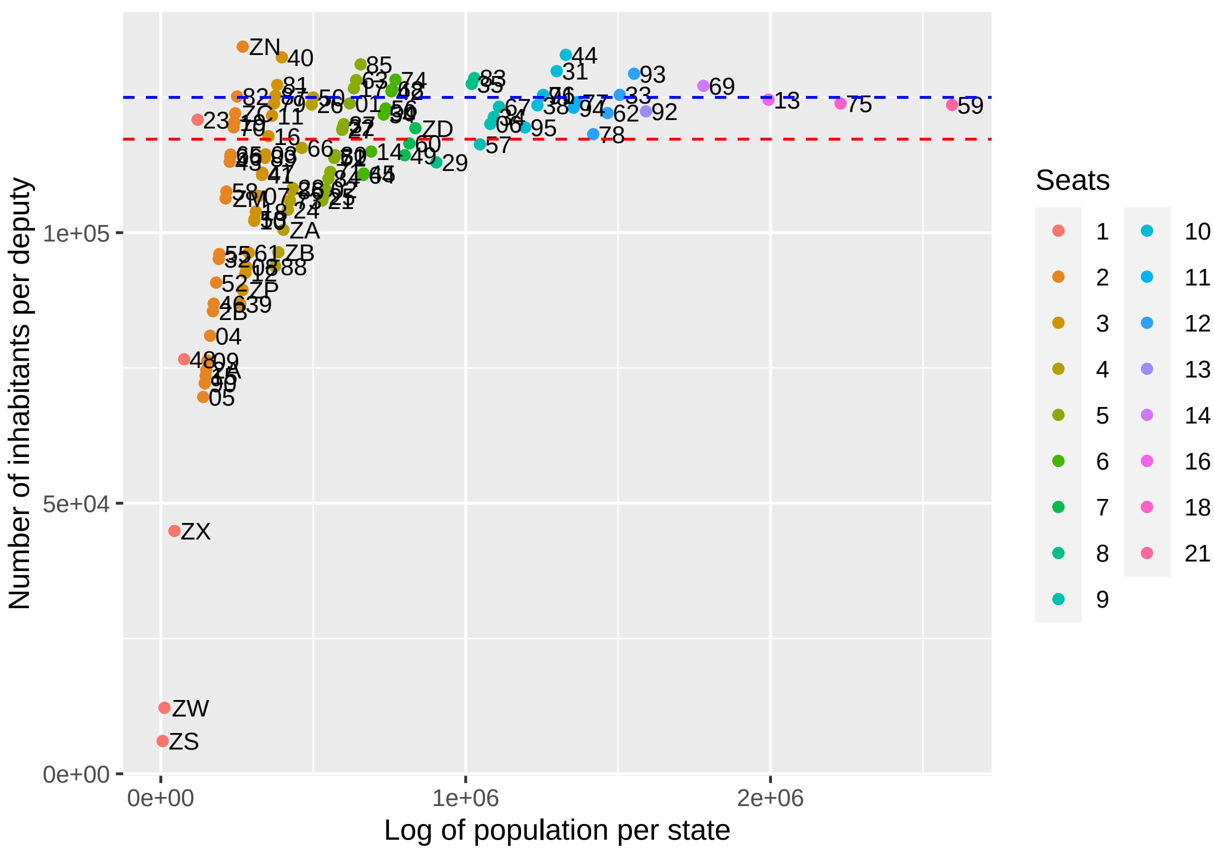

Figure 4: Number of inhabitants per deputy for each "département" with respect to its population and dotted lines representing the average level (in red) and the theoretical threshold leading to a supplementary deputy (in blue)

Figure 4 plots the number of inhabitants per representative in each "département" with respect to its population. Note that "Départements" ZS, ZW and ZX have few inhabitants and have 1 deputy each. It explains why the ratio (number of inhabitants)/(number of deputy) is very low for these circonscriptions.

Since the seat varaible is integer valued, the rule is constant over intervals of populations and admits discontinuous jumps:

- 1 deputy if the number of inhabitants is lower than 125,000
- 2 deputies if the number of inhabitants is between 125,000 and 250,000
- etc.

The "départements" which are close to the lower bound are favored and the "départements" which are close to the upper bound are disadvantaged. For example "Départements" 05 and ZN have two deputies, but the first one has 139,279 inhabitants and the second has 268,767 inhabitants. In this case, it is interesting to notice that this last "Département" should have three deputies like "Département" 39 which has 3 deputies and 260,502 inhabitants. The departure lies in the fact that the population data considered here is not the same as the one used to design the circonscription.

For the biggest "départements", we observe that the ratio (number of inhabitants)/(number of deputy) is closed to the theoretical blue line 125,000. The red line corresponds to the total number of inhabitants divided by the the number of deputies and is equal to 117,274.

3.1.2 Lorenz curve

Figure 5 plots the Lorenz curve for each year. Election of year 1993 was not kept because the data was incomplete.

Zooming on this figure leads to the following observations:

- 2012 seems always above the other curves except in two cases, where it is just under but still very close to the maximum curve (1997 when $x < 0.0075$ and mainly 2017 when $x > 0.9625$).
- 2007 is below all the other curves when $x < 0.72$ except for $x < 0.007$ where it is just above but very close to 2017.
- 2002 is below all the other curves when $x > 0.72$ except for $x > 0.985$ where it is just under but very close to 1997.
- at the beginning of the curve (i.e. $x < 0.5$), 1997 and 2017 are very close (except for $x < 0.05$ where 2017 is below 1997), then (when $x > 0.5$) 1997 is below 2017 (except for $x > 0.999$).

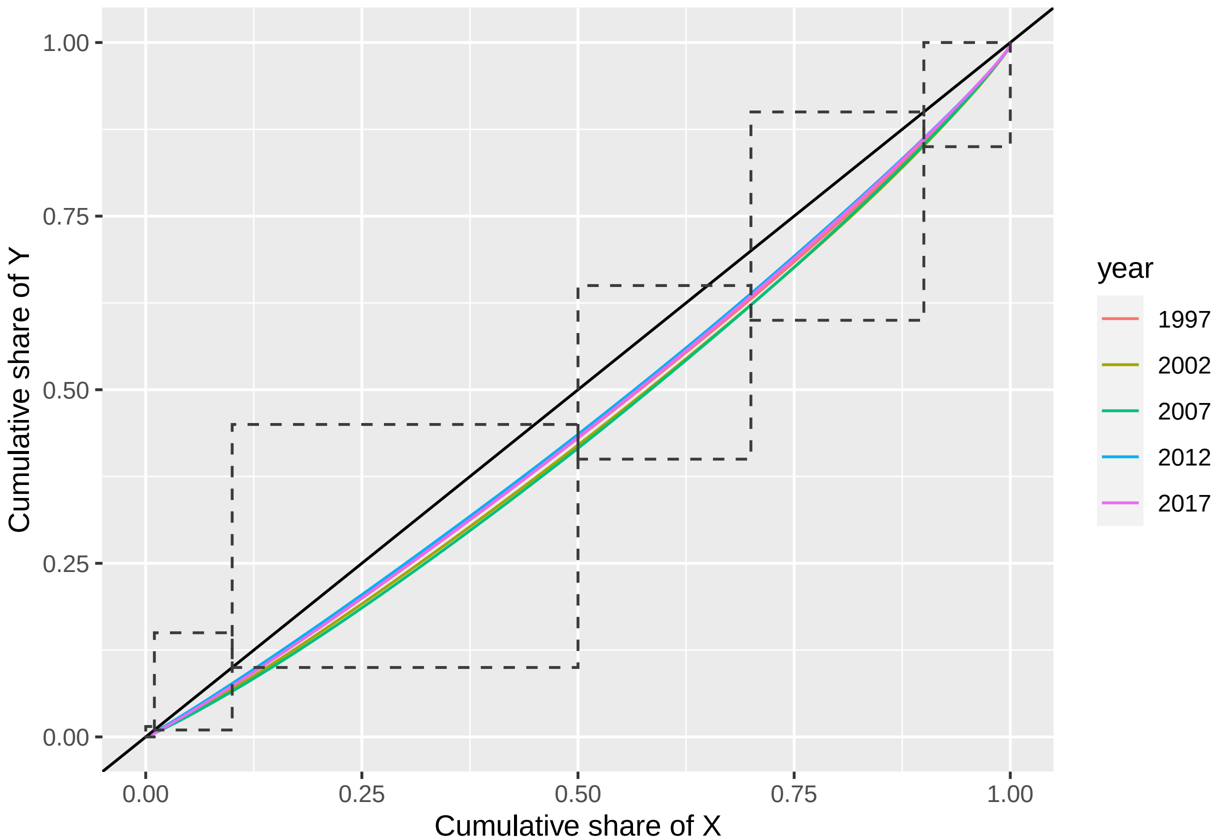

Figure 5: Lorenz curve for the last five French “Assemblée Nationale” elections (zooms for the dotted rectangles can be found in the supplementary material)

Moreover, we observe very few crossings between the curves. To check this, the rankings of the 5 studied elections were computed seat by seat (for each of the 576 seats) and are presented on Figure 6. The link with the specific Lorenz curve of this application is the following: All the curves are based on 576 dots, which share the same ordinates (cumulative share of Y, here seats). Hence, the ranking is easily obtained.

All the curves cross one another at least once. Yet, in all pairs of curves but one, there is always one curve that clearly is above the other one for most of the graph (at least around 94% of the graph). And the 6% or less of the graph where the situation is reversed is always at the very beginning or the very end of the graph.

The only pair of curves that does not match this trend is 2002 and 2007: In about 72% of the graph the 2002 curve is above the 2007 one. Then, for the 28% or so remaining part of the graph the situation is reversed (except at the very end where the curves cross three times).

In spite of these few intersections, it seems that the elections in 2007 and 2002 were the least fair ones, then 1997, 2017 and finally the election in 2012 was the most fair one (the one

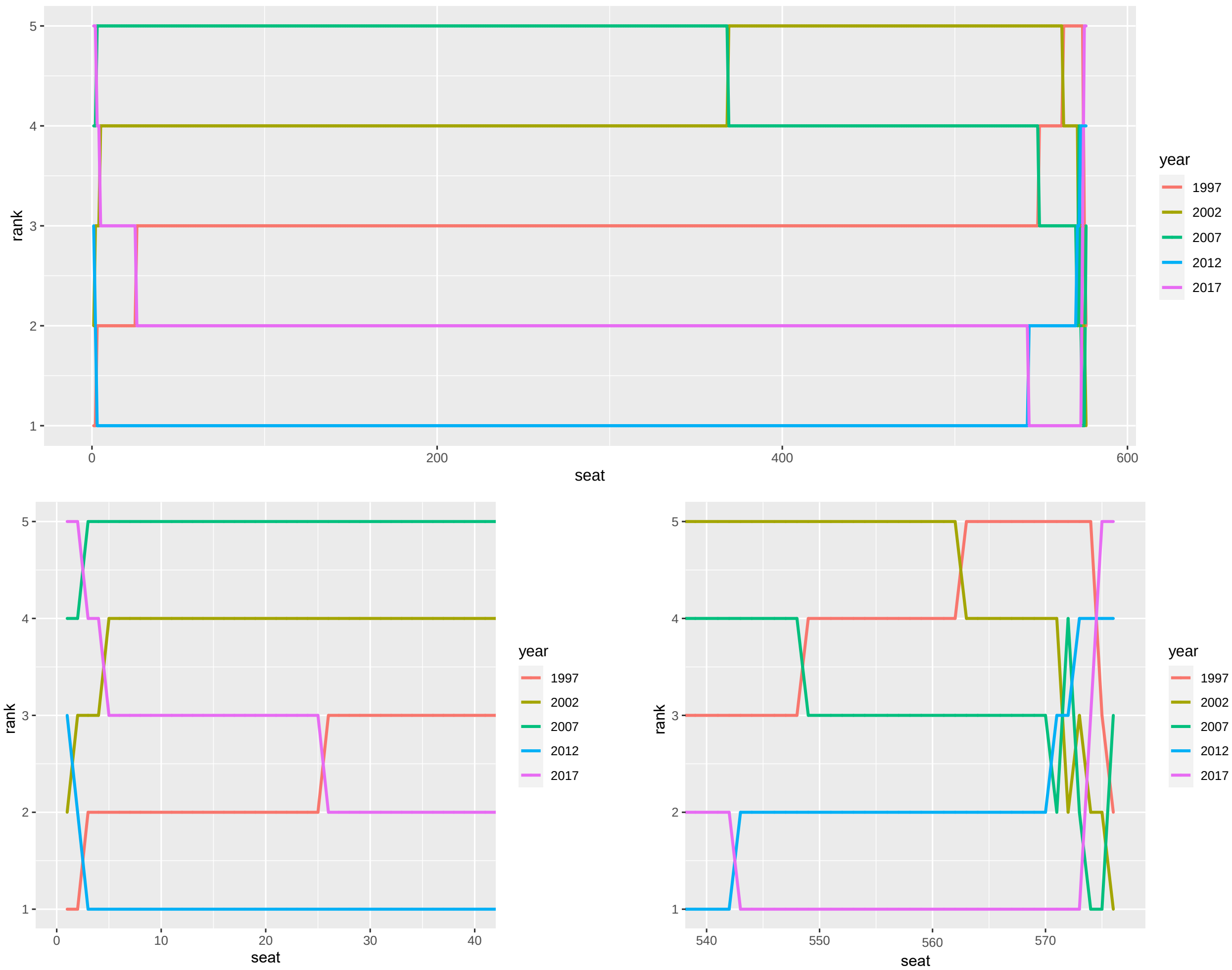

Figure 6: Ranking of the closest Lorenz curve to the diagonal for each seat in the last five French "Assemblée Nationale" elections (and zooms for the first and last seats, where most crosses are observed)

just after the application of the new rules).

3.1.3 Gini Index

The Gini index leads to the following results:

Best: 2012 ($G = 0.0464$) $<$ 2017 ($G = 0.0497$) $<$ 1997 ($G = 0.0517$) $<$ 2002 ($G = 0.0589$) $<$ Worst: 2007 ($G = 0.0613$)

Unsurprisingly, the Gini index confirms the conclusion derived from the Lorenz analysis: the elections in 2007 and 2002 were the least fair ones, then 1997, 2017 and finally the election in 2012 was the most fair one.

3.1.4 DK Index

For the DK index, both versions (discrete[30] and continuous[31]) lead to similar results.

Here are those for the continuous case:

Best: 2012 ($DK = 0.435$) $<$ 2017 ($DK = 0.431$) $<$ 1997 ($DK = 0.429$) $<$ 2002 ($DK = 0.419$) $<$ Worst: 2007 ($DK = 0.416$)

Again, the ranking is the same as with the Gini index and in line with what was conjectured from the Lorenz curve shapes.

3.1.5 Conclusion on Application 1

In this application, the Lorenz curve ordering is almost conclusive and consequently the Gini and DK indices computations are aligned with it for the fairness ranking of the studied elections. Further, the curves and indices are very close from one year to another, meaning that the fairness of the different elections seems quite stable in time. It is clear that the 2012 reform has designed circonscriptions fitting "at best" the population distribution of that year, leading to the most fair election. For the following election, the population had evolved a little, leading to a small decrease in the fairness of year 2017 election. But its fairness seems very close to year 1997. And year 1997 is two elections after the previous circonscription apportionment (which occurred in 1988). Then 2002 is one more election away as 2007. So it seems quite logical that the fairness tends to decrease when moving away from the last apportionment, because population changes tend to follow a time trend.

3.2 Application 2: the Evolution of the ideological Lorenz curve in the "Assemblée Nationale" of the French 5^{th} Republic

In this section, we consider the same data used previously. However, instead of considering the effect of apportionment (as in the previous section), we focus here on the differences between the vote shares and seat shares obtained for each competing party. For instance, Figure 7 shows these differences for each of the 17 competing parties of year 2012 election.

On this figure, the parties are ranked with respect to the Lorenz curve order: the first party is the one that was best off for this election and the last party is the one that was worst off (highest vote shares with no seat). The 4 first parties benefited from the electoral system (higher seat shares than vote shares) at the expense of the 11 others. Some parties with higher vote shares than others still get lower seat shares than the latter.

[30] In the discrete case: we search the value of $x^* = \min(x_k)$, $k = 1, ..., n$ so that $L(x_k) > 0.5$ and we get the DK with $1 - x_k^*$

[31] More computation time is needed (due to the linear interpolation) for the continuous case, but the results are even more accurate.

Hence, the correlation between the two shares is unclear. These huge differences cannot be explained by the small malapportionment studied in the previous section (especially in the case of year 2012 where the malapportionment was the lowest). In fact, these differences are mainly due to the electoral system.[32] Similar discrepancies are observed for the five other elections (years 1993, 1997, 2002, 2007 and 2017). They can be seen in the supplementary material.

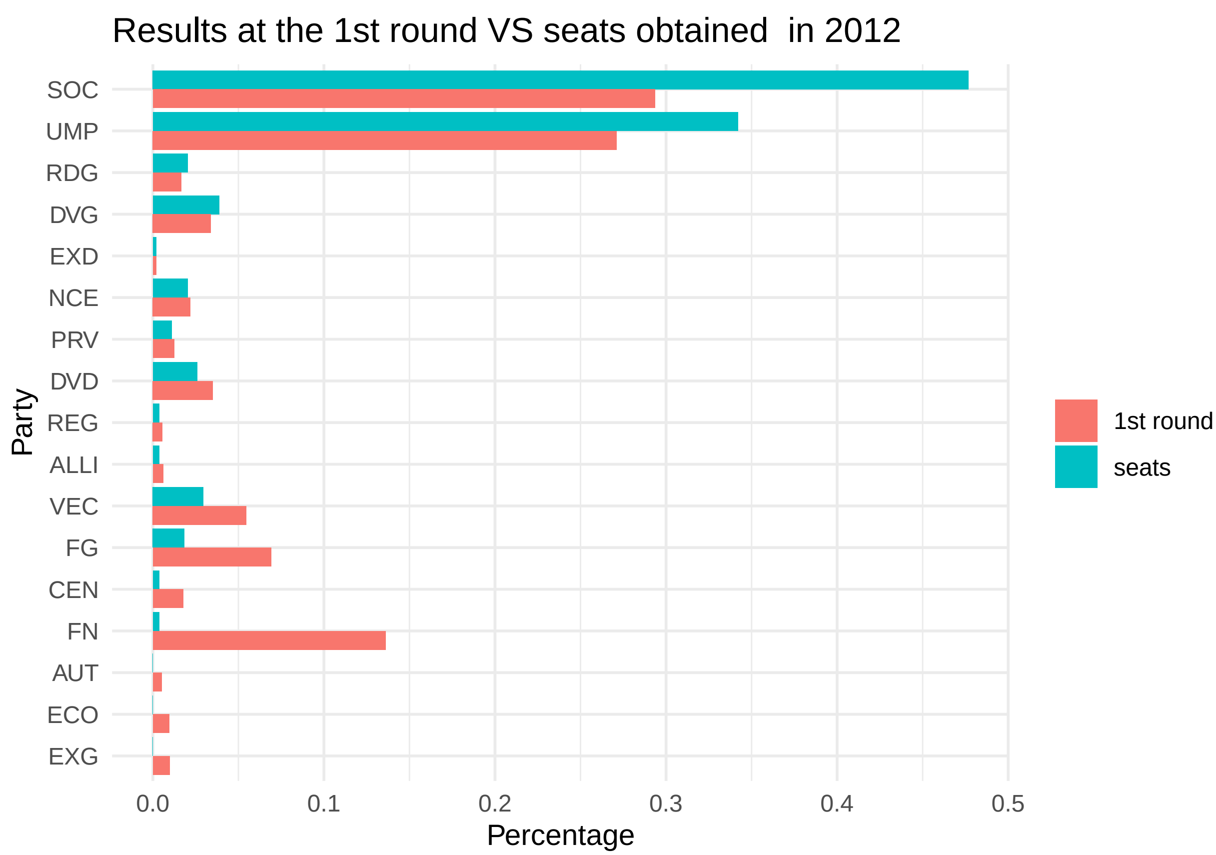

Figure 7: Differences between the vote shares and seat shares obtained for each of the 17 competing parties of year 2012 election (zooms for the dotted rectangles can be found in the supplementary material).

3.2.1 Lorenz curve

Figure 8 plots the different elections' Lorenz curves on the same graph. Yet, as the party choice set differs from one election to another (even in quantity) and also from one circonscription to another, it is difficult to explain the observed differences.

[32]Note that the vote shares are computed at the first round of the election and the seat shares are computed after the second round. The rules to be able to maintain candidacy between the two rounds, and the game of political alliances may strongly differ from the proportional rule.

If we compare to Figure 5, it appears that the Lorenz curves of Figure 8 are much further away from the diagonal and with a much higher variability from one election to another. In fact, a deputy represents more or less the same number of voters throughout the country, but the seat shares are not necessarily in-line with the vote shares.

Up to $x = 26\%$, the election in 2017 seems to be the most proportional one (closest to the diagonal). Then, after $x = 26\%$, it is 2007.

On the contrary, up to $x = 21\%$, the election in 2002 seems to be the least proportional one (farthest from the diagonal). Then, after $x = 21\%$, it is 1993. In fact, up to $x = 14\%$ the two curves overlap each other. So 2002 is a little worse than 1993 only for 5% of the vote shares.

Hence, 1993 seems the farthest from proportional rule. Then, 1997, 2002 and 2017 seem close. Finally, 2007 seems the closest to the proportional rule, followed by 2012.

There are some curves crossing. Most of the crosses are on the first third of the vote shares. Then, after $x = 54\%$, the curves do not cross.

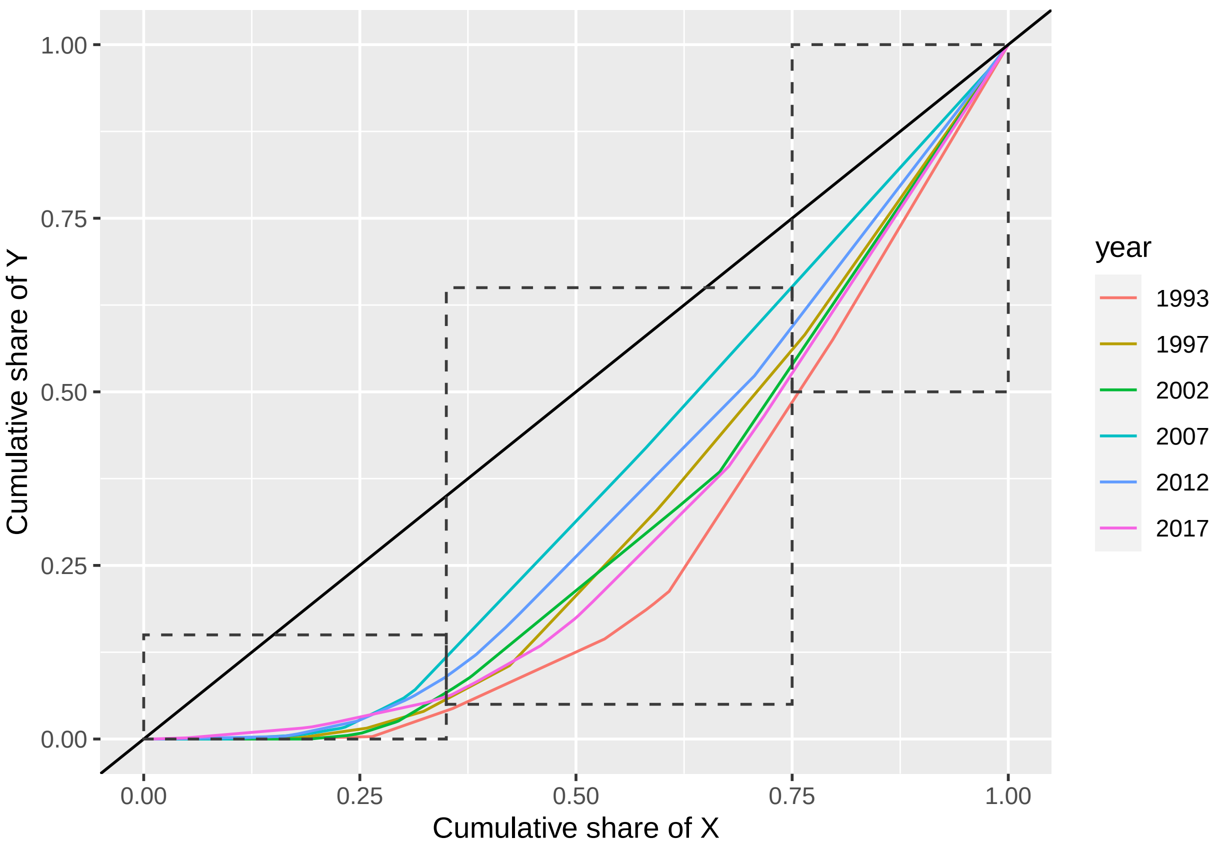

Figure 8: Lorenz curves for 1993, 1997, 2002, 2007, 2012 and 2017 elections

3.2.2 Gini Index

The Gini index leads to the following results:

Best: 2007 ($G = 0.134$) $<$ 2012 ($G = 0.162$) $<$ 1997 ($G = 0.189$) $<$ 2002 ($G = 0.195$) $<$ 2017 ($G = 0.201$) $<$ Worst: 1993 ($G = 0.233$)

As in the previous application, the Gini index is in line with the almost Lorenz ordering: the elections in 2007 and 2012 were the most proportional ones, then 1997, 2002 and 2017 and finally the election in 1993 was the least proportional one.

Specifically, the Gini index enables to break the ties i.e. to rank the three years that were close but not straightforward ranked in terms of Lorenz curves.

3.2.3 DK Index

The continuous DK index leads to the following results which coincide with those derived from Gini:

Best: 2007 ($DK = 0.360$) $<$ 2012 ($DK = 0.312$) $<$ 1997 ($DK = 0.291$) $<$ 2002 ($DK = 0.271$) $<$ 2017 ($DK = 0.264$) $<$ Worst: 1993 ($DK = 0.242$)

3.2.4 Conclusion on Application 2

Like in the first application, the Gini and DK indices are aligned and complete the almost complete ranking derived from Lorenz. However, in this application, the curves and indices are far from the principle of proportionality which seems mainly due to the electoral system. Moreover, we observe more variability from one election to another that could also be explained by the different party choice-set across time and space.

3.3 Application 3: The Evolution of the Geographical Lorenz Curve in the "Départements" before and after the 2015 Electoral Reform

The main objective of this section is to explore how the geographical Lorenz curve at the "département" level has changed as the result of an electoral reform concomitant with some redistricting.[33] Each "Départment" elects a chamber of representatives. This legislative body is in charge of a number of local policies and redistributes resources across the territories within the perimeter of the "Départment". This election proceeds from a division of the "Département" into districts called cantons. Before 2015, the district magnitude was equal to 1: There was one seat per district and ballots consisted of a single candidate. From 2015, several changes

[33] More information about the election process, can be found e.g. https://fr.wikipedia.org/wiki/%C3%89lections_d%C3%A9partementales_en_France

were implemented. First, the number of districts has been basically divided by two.[34] Second, the district magnitude was increased from 1 to 2 with a very peculiar "the winner takes all" electoral formula: Each ballot consists in a ticket (not a list) of candidates (one male, one female). The main objective of this reform was to guarantee the perfect equality of the two genders in the chamber. The electoral reform leaves unchanged the size of the chamber[35] and has two components: A new map of the districts and a new electoral formula. It must also be pointed out that the reform was been exploited as an opportunity to solve at least partially the severe malapportionment problems of the historical electoral maps. This combination of multiple changes makes the problem quite complicated to analyse.

In the rest of this section, we will proceed to an evaluation of the 2015 reform from the perspective of the Lorenz curve. But before doing so, let us call the attention of the reader on the alternative evaluation, motivated by voting among two camps, that was presented in sub-section 2.2. We could indeed apply this method here with a focus on the color (D or R) of the chamber. It may well happen that a majority of the voters of the "départment" vote D and a majority of districts vote R. This is what we have called an election inversion. We could in particular compute how the measures Δ^1_λ and Δ^2_λ have changed under the reform, for some λ. This question is explored in Le Breton et al. (2017), where another index (called an index of disproportionality) is introduced. As demonstrated there, if the principle "one man one vote" is defined from that voting perspective, it is not clear that the reform leads to an improvement.

In the context of distributive politics, things are different. Had the reform exclusively consisted in merging two old districts to create a new one, the post electoral reform Lorenz curve would have been closer to the diagonal than the pre-electoral reform one. This follows from a sequential application of the Pigou-Dalton principle. When two districts merge, the equal distribution within the new district dominates the unequal distribution prevailing in the union of the two old ones. We cannot apply without qualification this argument to the actual reform for many reasons, on top of which the fact that the redrawing of the map of districts was not as simple as a series of paring. In this section, we look carefully at this question. As in Application 1, we focus here the geographical distribution of the seats.[36]

[34]In fact, the result of the division is rounded to the closest upper even number. Moreover, this number is at least 17 for "départments" with 500,000 or more inhabitants, and 13 with 150,000 or more.

[35]In fact, the size of the chamber has slightly increased in some "départments", as explained in footnote 34.

[36]Again, we consider here the number of voters (available for the elections before and after the reform) and not the number of inhabitants (as such data was not easily available for each canton). So our conclusions are valid under the assumption that the ratio (voters/inhabitants) is sufficiently stable across time and space.

3.3.1 Lorenz curve

Figure 9 shows 100 graphs (one per "département" before the reform). Each graph shows the Lorenz curve at the last "département" election before the reform (red line) and, for 98 of them, the one just after the reform (green line).[37]

It is obvious that 96 "départements" out of 98 are better off after the reform: The green line is always closer to the diagonal than the red one. This means that the reform has enabled to take into account the population changes that had occurred with time, similarly to the case of Application 1 (2012 election, right after the reform, was more fair than the previous one). There is a clear exception in the case of one "Département" (Mayotte, last graph on the figure) which is worse off after the reform. This may occur when the actual number of voters and number of inhabitants are not so well correlated throughout that "Département".

The only questionable case is for "Département" 94 (Val-de-Marne), but both red and green lines are very close to the diagonal and almost overlapping. In fact the green line is closer to the diagonal at the beginning and at the end of the graph. The red line is only very slightly closer to the diagonal from $x = 34\%$ to $x = 74\%$. So, in the case of "Département" 94, a few cantons are worse off, but overall "Département" 94 seems better off.

It is interesting to notice that Figure 9 before the reform shows red lines that can still be not so far from the diagonal (as all studied elections of Application 1) and others that are much further away (similarly to the worst case of Application 2). The graphs with a red line far from the diagonal correspond to "départements" where the population has probably most changed, so they most benefit of the reform, even if all of them reach a more fair situation.

3.3.2 Gini index

Figure 10 on the left represents the Gini index for each "département" before and after the reform. A line represents the evolution of the same "département". The position of the boxes and the slopes of the lines indicate clearly a negative trend (except for two "Départements": Mayotte, clearly positive, and Val-de-Marne, flat but positive). We have represented the lines with different colors with respect to the absolute values of the slopes. Our idea is to represent on a map those classes of "départements" and visualize[38] if it exists a spatial autocorrelation. It appears that there exists a spatial autocorrelation and a trend North/South. The "départements" with the highest changes are mostly located in the South of France. The "départements" nearby Paris seem the ones with the lowest changes. Finally, the "départements" with medium changes are mostly located in the North. It would be interesting to use a spatial econometric

[37] Note that the green line is present only for 98 "départements": Two small "départements" (overseas) disappear after the reform, so only the red line appears for them.

[38] This kind of representation has been studied by Laurent et al. (2012)

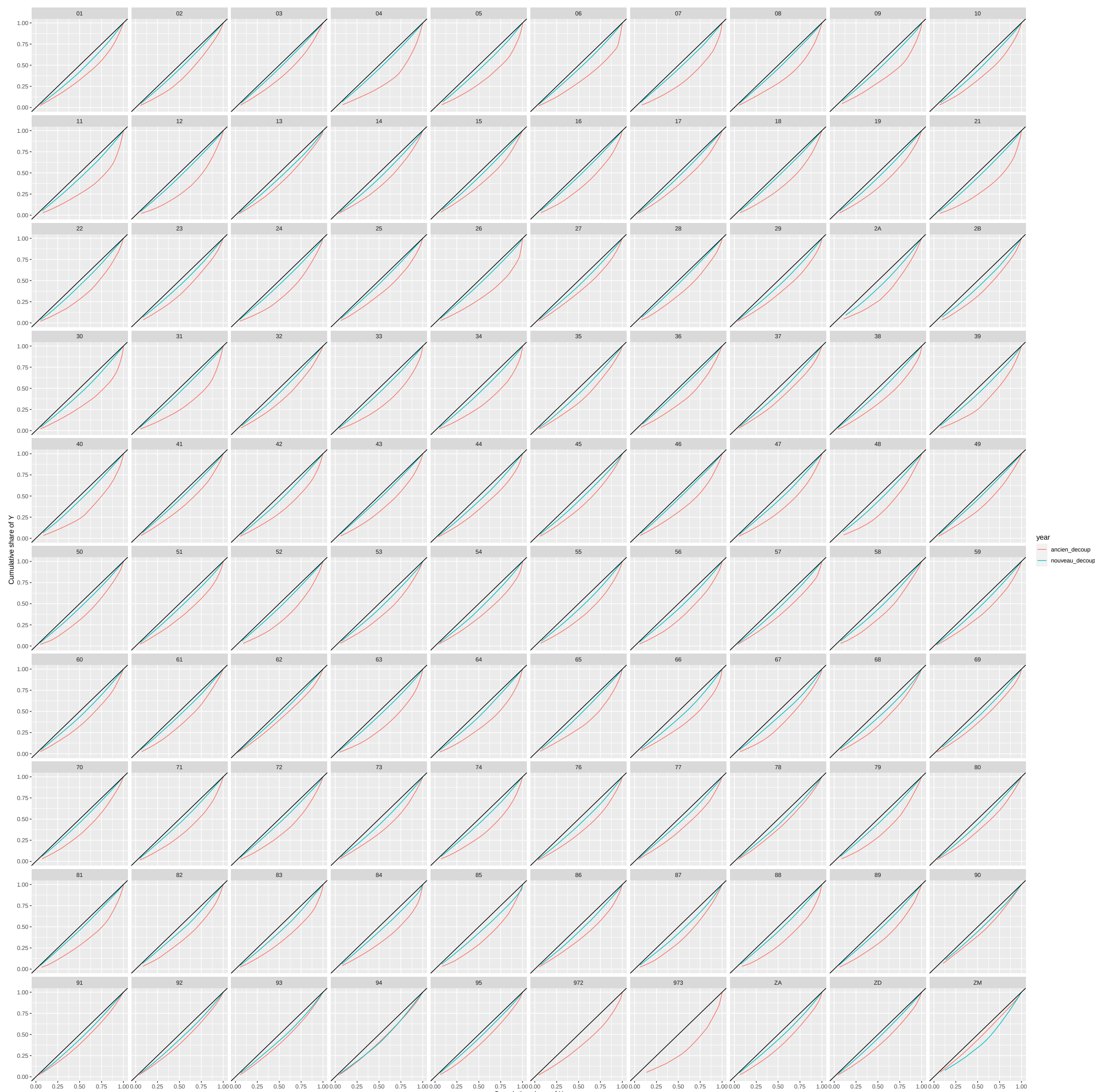

Figure 9: Lorenz curves for each French “Département” in the case of the 2015 election (green lines), after the reform, compared to the previous one (red lines, just before the reform)

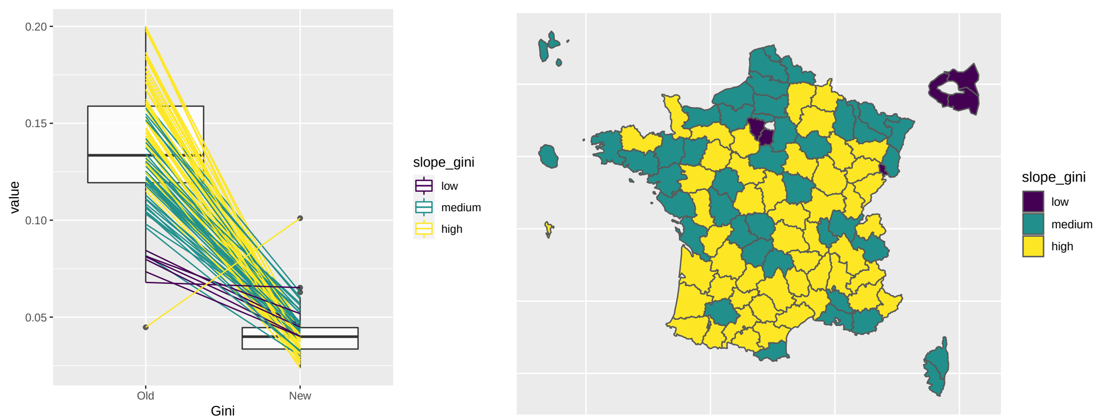

Figure 10: Gini index before and after the reform. The 3 classes "high", "medium" and "low", are defined with respect to the slopes and are represented on the map.

approach and model the Gini index by some socio-economic factors, to explain these differences of behaviour.

As before, and unsurprisingly, the Gini index corroborates the judgments based on the Lorenz curves.

3.3.3 DK index

Figure 11 on the left represents the DK index for each "département" before and after the reform. A line represents the evolution of the same "département". We have represented the lines with different colors with respect to the absolute values of the slopes, as in the previous section (cf. Gini index). The positions of the boxes and the slopes of the lines seem reversed compared to those obtained with the Gini index, so the conclusions are very similar (with Gini index, 0 is associated to fair and 0.5 to unfair, while with DK index it is the reverse). The only noticeable change is for Val-de-Marne ("Département" 94) where the more or less flat curve shows a small reduction of fairness, which is not in line with what the Gini index shows (we focus more on the area under the diagonal, which coincides with the Gini index).

3.3.4 Conclusion on Application 3

We have shown that, except in the case of "Département" Mayotte, all the new Lorenz curves are closer to the Diagonal. As explained, the case of Mayotte could be explained by a change in the percentage of people who have the right to vote. In 96 out of 98 "départements", the Lorenz Curves do not cross, so all indices (G or DK) confirm an improvement also. The interesting part of this application is for "Département" Val de Marne (94): The Lorenz curves cross twice, meaning that the cantons that have less seats per inhabitants and those that have most seats per

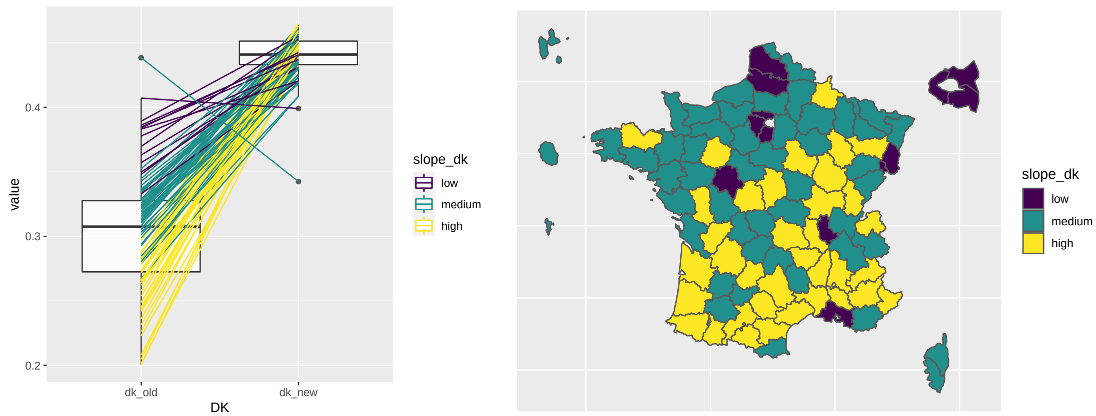

Figure 11: DK index before and after the reform. The 3 classes "high", "medium" and "low", are defined with respect to the slopes and are represented on the map.

inhabitants are better off, whereas the intermediary ones are worse off. This part is interesting because it shows that indices might not be aligned when the curves cross. Here, Gini index concludes that the situation, overall is better off: The population update benefit to a majority of inhabitants. On the contrary, DK index concludes that the situation is worse off because a smaller number of people may have half of the seats. To avoid misinterpretation, note that the inhabitants are not necessarily at the same abscissa before and after the reform, especially as the cantons are not the same in the two cases. So it might be the case that cantons that have the most population are better off after the reform, but they may concern totally different people. A spatial analysis is required if we want to look at which areas are better/worse off. In fact, in this example, the situation might even not have occurred at all, had we access to the population of each canton (and not only to the number of voters): The two curves with population might not cross. Yet, we would need to have access to the number of inhabitants in each canton to test this. Also, we have looked at the curves as if they were continuous, but in reality it might be difficult to obtain a set of cantons corresponding to at least half of the cantons, where the total population is under half of the total population. So the unfair situation captured by DK index might in fact not ever occur. Finally, both curves are very close to the diagonal, so the situation is not so different before and after the reform. For all these reasons, no strong conclusion should be based on this result, but it is interesting to stress that Gini and DK indices might not always be aligned, as in this case.

3.4 Application 4: Electoral College

In this section, we consider the presidential US elections during the 2010-2019 time period (based on year 2010 census). The number of electoral votes (called hereafter 'seats') of a state is the sum of its number of representatives and number of senators (which is 2 for all states). The District of Columbia is allocated 3 seats. This data is fully presented in Table 1 of de Mouzon et al. (2020). The aim of this section is to compare the malapportionment when considering the allocation of the seats or the allocation of the representatives.

3.4.1 Lorenz curve

Figure 12 presents the Lorenz curve when considering the number of seats (red curve) and the number of representatives (green curve), based on year 2010 census. The green curve is very close to the diagonal which shows that the representatives are allocated proportionally to the population of the State. However the red curve is always further away from the diagonal which indicates that the fact to allocate automatically 2 senators per state, creates malapportionment.

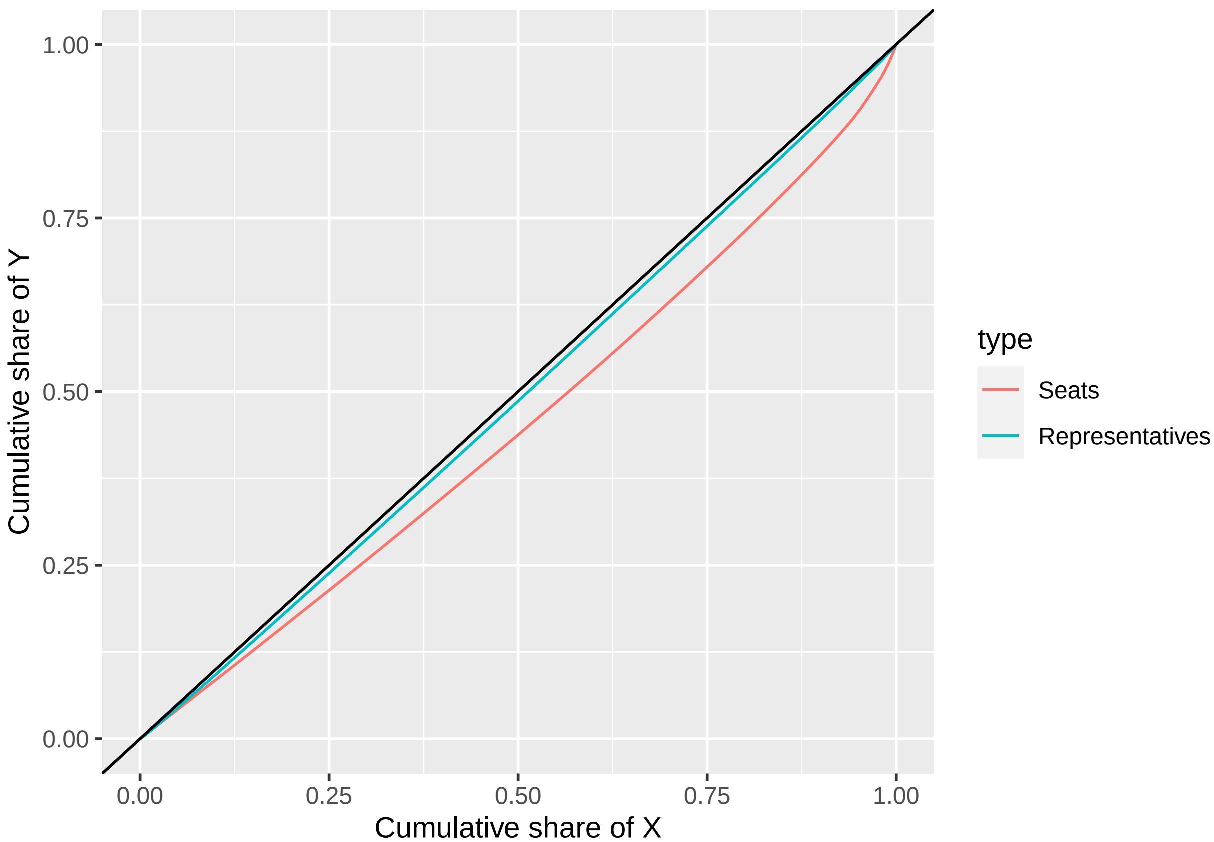

Figure 12: Lorenz curve for the Electoral College in the US elections based on year 2010 census

3.4.2 Gini Index

The Gini index is equal to $G = 0.0484$ for the number of seats and $G = 0.0106$ when considering the number of representatives.

As the Lorenz curves do not cross, the Gini index is of course in line with what was observed in the previous section.

The ideal situation corresponding to $G = 0$ (Lorenz curve aligned with the diagonal), it is interesting to observe that the number of seats has a Gini index 4.6 times higher than the number of representatives.

3.4.3 DK Index

The DK index is equal to $DK = 0.433$ for the number of seats and $DK = 0.486$ when considering the number of representatives.

Again, as expected, the DK index is in line with what was observed in the two previous sections.

It is interesting to observe that the distance to the ideal situation ($DK = 0.5$ when the Lorenz curve is aligned with the diagonal) is 4.8 times higher for the number of seats than the number of representatives.

Hence, both indices, Gini and DK, give very similar relative difference to the ideal situation between the number of seats and the number of representatives.

3.4.4 Conclusion on Application 4

This fourth application, enables to show, in a simple setting, that both indices, Gini and DK, can sometimes be aligned even up to the relative difference to the ideal situation between two settings. This is of course not a general rule (e.g. in the previous application we even had totally opposite outcomes in "Départment" 94: Gini index finding an improvement after the reform, whereas DK index finding a worse off situation).

Of course, this result is straightforward: The representatives are allocated on a proportional basis (and only suffer from the curse of rounding to integers their numbers). Adding two senators, whatever the population of the state, necessarily moves the curve even further from proportionality of seats to populations of the states. And obviously, this result is not dependant on the census year: Equivalent result is obtained for year 2000 census and any other.

Depending on the population distribution throughout the states in the different census years, it could be the case that some green (resp. red) curves are closer to the diagonal than others. But the green curves are always very close to the diagonal and the red ones always a little further away (although they are still close to the diagonal, as the curves of Application 1).

A more pragmatic question is to know whether the two "senatorial" seats really give a bonus to the small states in the presidential elections or whether they more or less correct some other unfairness (due to the fact that the biggest states have more representatives and thus more power in deciding who will be president). This question has been studied in the light of the three main voting probability models in de Mouzon et al. (2020).

A Appendices

A.1 Apportionment and Elections

This appendix focuses on the first canonical model described in section 2.2 i.e. on a single-winner election (e.g. a presidential election) involving two candidates D and R. When the country is divided into districts/territories/states, we may elect the winner either directly or indirectly. By direct election, we mean an election where each citizen votes either D or R and the winner is the candidate who has received the more votes countrywide. By indirect election, we mean an election where in each state k the N_k citizens of that state vote for a list of R_k D representatives or a list of R_k R representatives. Then each member of the house/college of representatives elected in such way votes either D or R and the winner is the candidate who has received the more votes house-wide. An election inversion[39] occurs when the winners of the two procedure differ. It is straightforward to confirm that $\Delta_\lambda^1(N, R) \geq 0$ for any "reasonable" impartial probability model λ on utility profiles. The positiveness of this measure results from the existence of election inversions for some profiles of utilities. The measure takes into account the severity of the election inversion. Instead, we can compute simply the probability $\Gamma_\lambda(N, R)$ of an election inversion.

The purpose of this first appendix is to evaluate $\Gamma_\lambda(N, R)$, $\Delta_\lambda^1(N, R)$ and $\Delta_\lambda^3(N, R)$ for a popular probability model λ known under the heading IC (Impartial Culture) in the voting literature. More specifically, hereafter, we will evaluate the probability of an election inversion when the country is divided into three states 1, 2 and 3. We assume without loss of generality that $N_1 \leq N_2 \leq N_3$ and denote respectively by x and y the fractions n_1 and n_2. We assume that each state has a single representative. Therefore the situation (N, R) is described by $N = (N_1, N_2, N_3)$ and $R = (1, 1, 1)$. Finally, concerning λ, we assume that the utilities of each citizen are independent draws from the Bernoulli distribution over $\{D, R\}$ with parameter $\frac{1}{2}$. We will denote by X_i the random variable defining if citizen i voted D or not i.e.

$$X_i = \begin{cases} 1 \text{ if } i \text{ votes } D \\ 0 \text{ if } i \text{ votes } R \end{cases}.$$

For all $k = 1, 2$ and 3, let $S_N^k = \sum_{i=1}^{N_k} X_i$ denotes the number of citizens who vote D in state k and $S = \sum_{i=1}^{3} S^k$ the number of citizens in the country who vote D. An election inversion occurs iff two states vote D and the country votes R or if two states vote R and the country

[39] There is a voluminous empirical and theoretical literature on this topic that we will not review here.

vote D. Given the symmetry of the random draw, the probabilities of these two events are the same. The first event occurs either when state 1 and state 2 vote D while the country votes R or when state 1 and state 3 vote D while the country votes R, or when the state 2 and state 3 vote D while the country votes R.

The first event occurs when the following inequalities hold true:

$$\begin{aligned} S_N^1 &\geq \frac{N_1+1}{2} \\ S_N^2 &\geq \frac{N_2+1}{2} \\ S_N^1 + S_N^2 + S_N^3 &\leq \frac{N-1}{2} \end{aligned} \tag{3}$$

Consider instead the event:

$$\begin{aligned} S_N^1 &\geq \frac{N_1+1}{2} \\ S_N^2 &\geq \frac{N_2+1}{2} \\ S_N^1 + S_N^2 + S_N^3 &\geq \frac{N+1}{2} \end{aligned} \tag{4}$$

Since the probability of the joint event $\left\{S_N^1 \geq \frac{N_1+1}{2}, S_N^2 \geq \frac{N_2+1}{2}\right\}$ is equal to $\frac{1}{4}$, the probability that (3) holds true is equal to $\frac{1}{4}$ minus the probability that (4) holds true. (4) is equivalent to:

$$\begin{aligned} \frac{S_N^1 - \frac{N_1}{2}}{\sqrt{N_1}} &\geq \frac{1}{2\sqrt{N_1}} \\ \frac{S_N^2 - \frac{N_2}{2}}{\sqrt{N_2}} &\geq \frac{1}{2\sqrt{N_2}} \\ \frac{\left(S_N^1 - \frac{N_1}{2}\right) + \left(S_N^2 - \frac{N_2}{2}\right) + \left(S_N^3 - \frac{N_3}{2}\right)}{\sqrt{N}} &\leq \frac{1}{2\sqrt{N}} \end{aligned} \tag{5}$$

or in a matrix form:

$$\begin{pmatrix} Z_N^1 \\ Z_N^2 \\ Z_N^3 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sqrt{x} & -\sqrt{y} & -\sqrt{1-x-y} \end{pmatrix} \times \begin{pmatrix} Y_N^1 \\ Y_N^2 \\ Y_N^3 \end{pmatrix} \geq \begin{pmatrix} \frac{1}{2\sqrt{N_1}} \\ \frac{1}{2\sqrt{N_2}} \\ \frac{1}{2\sqrt{N}} \end{pmatrix}$$

where $Y^k \equiv \frac{S^k - \frac{N_k}{2}}{\sqrt{N_k}}$ for $k = 1, 2, 3$. From the the central limit theorem, when $N \to \infty$, the vector $\begin{pmatrix} Y_N^1 \\ Y_N^2 \\ Y_N^3 \end{pmatrix}$ converges to the Gaussian vector:

$$N(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix})).$$

Therefore, when $N \to \infty$, the vector $\begin{pmatrix} Z_N^1 \\ Z_N^2 \\ Z_N^3 \end{pmatrix}$ converges to the Gaussian vector:

$$Z \equiv N(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4}\sqrt{x} \\ 0 & \frac{1}{4} & \frac{1}{4}\sqrt{y} \\ \frac{1}{4}\sqrt{x} & \frac{1}{4}\sqrt{y} & \frac{1}{4} \end{pmatrix})).$$

The correlation matrix R of Z is then:

$$\begin{pmatrix} 1 & 0 & \sqrt{x} \\ 0 & 1 & \sqrt{y} \\ \sqrt{x} & \sqrt{y} & 1 \end{pmatrix}.$$

When $N \to \infty$, since the vector $\begin{pmatrix} \frac{1}{2\sqrt{N_1}} \\ \frac{1}{2\sqrt{N_2}} \\ \frac{1}{2\sqrt{N}} \end{pmatrix}$ tends to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ we deduce from above that the probability of the event defined by inequalities (4) tends to the probability of the event:

$$\left\{ Z \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

According to Sheppard's theorem (Sheppard, 1900) of Median Dichotomy,[40] we deduce that this probability is equal to:

$$\frac{1}{8} + \frac{1}{4\pi} \left[\arcsin(0) + \arcsin\left(\sqrt{x}\right) + \arcsin\left(\sqrt{y}\right) \right].$$

Therefore, the probability that inequalities (3) hold true is equal to

$$\frac{1}{8} - \frac{1}{4\pi} \left[\arcsin\left(\sqrt{x}\right) + \arcsin\left(\sqrt{y}\right) \right].$$

Similarly, the second event occurs when the following inequalities hold true:

$$\begin{aligned} S_N^1 &\geq \frac{N_1+1}{2} \\ S_N^3 &\geq \frac{N_3+1}{2} \\ S_N^1 + S_N^2 + S_N^3 &\leq \frac{N-1}{2} \end{aligned} \tag{6}$$

[40] See also Kendall and Stuart (1963).

Proceedings along the same lines as above we obtain that the limit value of the probability of this event is equal to:

$$\frac{1}{8}-\frac{1}{4\pi}\left[\arcsin\left(\sqrt{x}\right)+\arcsin\left(\sqrt{1-x-y}\right)\right].$$

Finally, the third event occurs when the following inequalities hold true:

$$\begin{aligned} S_N^2 &\geq \frac{N_2+1}{2} \\ S_N^3 &\geq \frac{N_3+1}{2} \\ S_N^1+S_N^2+S_N^3 &\leq \frac{N-1}{2} \end{aligned} \tag{7}$$

Proceedings along the same lines as above we obtain that the limit value of the probability of this event is equal to:

$$\frac{1}{8}-\frac{1}{4\pi}\left[\arcsin\left(\sqrt{y}\right)+\arcsin\left(\sqrt{1-x-y}\right)\right].$$

Given the symmetry between D and R, the probability of an election inversion is simply the sum of the three numbers multiplied by 2. After collecting all terms we obtain:

$$\Gamma_\lambda\left(N,R\right)=\frac{3}{4}-\frac{1}{2\pi}\left(2\arcsin\left(\sqrt{x}\right)+2\arcsin\left(\sqrt{y}\right)+2\arcsin\left(\sqrt{1-x-y}\right)\right).$$

When $x=y$, the formula simplifies to:

$$\Gamma_\lambda\left(N,R\right)=\frac{3}{4}-\frac{2}{\pi}\arcsin\sqrt{x}-\frac{1}{\pi}\arcsin\sqrt{1-2x}$$

Figure 13 represents $\Gamma_\lambda\left(N,R\right)$.

We conclude this appendix by the computation of $\Delta_\lambda^1\left(N,R\right)$ and $\Delta_\lambda^2\left(N,R\right)$. Since λ is the IC probability model, it is well known[41] (the so-called Penrose's formula) that for any voting mechanism, the expected utility of any voter is equal to one half plus one half times the probability that this voter is pivotal for that voting mechanism. It is also well know that the probability for a voter in state k to be pivotal in the indirect election mechanism is equal to:

$$\sqrt{\frac{2}{\pi N_k}}\times\frac{1}{2}=\sqrt{\frac{1}{2\pi N_k}}.$$

In contrast, the probability for any voter to be pivotal in the direct election mechanism is equal to:

[41] See e.g. Le Breton and Van Der Straeten (2015).

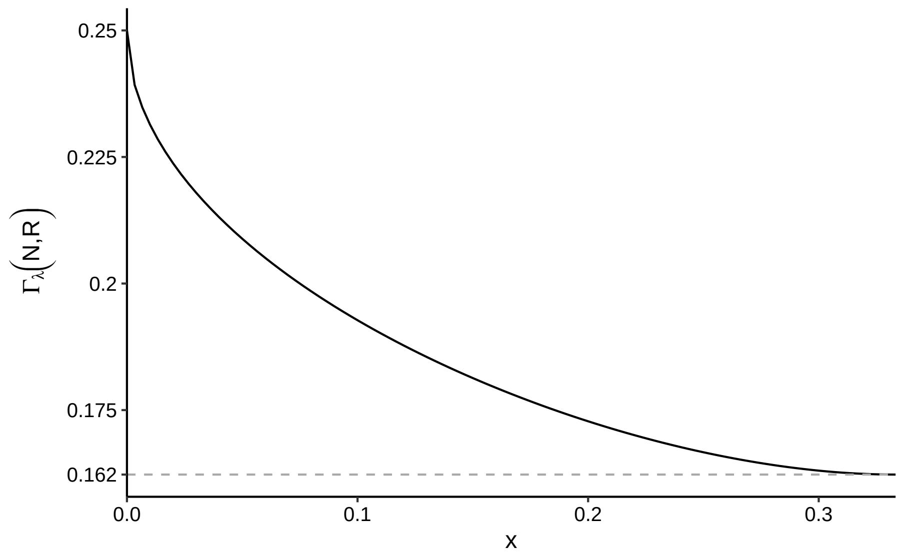

Figure 13: Graph of $\Gamma_\lambda(N,R)$

$$\sqrt{\frac{2}{\pi N}}.$$

We deduce from the Penrose's formula that:

$$\begin{aligned}
\Delta_\lambda^2(N,R) &= \frac{1}{2}\left(\sqrt{\frac{2}{\pi N}} - \sqrt{\frac{1}{2\pi N_1}}, \sqrt{\frac{2}{\pi N}} - \sqrt{\frac{1}{2\pi N_2}}, \sqrt{\frac{2}{\pi N}} - \sqrt{\frac{1}{2\pi N_3}}\right) \\
&= \sqrt{\frac{1}{4\pi N}}(\sqrt{2} - \sqrt{\frac{1}{2x}}, \sqrt{2} - \sqrt{\frac{1}{2y}}, \sqrt{2} - \sqrt{\frac{1}{2(1-x-y)}})
\end{aligned}$$

and

$$\begin{aligned}
\Delta_\lambda^1(N,R) &= N\left[\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2}{\pi N}}\right] - \sum_{k=1}^{3} N_k\left[\frac{1}{2} + \frac{1}{4}\sqrt{\frac{2}{\pi N_k}}\right] \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\sqrt{N} - \frac{1}{2}\left(\sqrt{N_1} + \sqrt{N_2} + \sqrt{N_3}\right)\right] \\
&= \sqrt{\frac{N}{2\pi}}\left[1 - \frac{1}{2}\left(\sqrt{x} + \sqrt{y} + \sqrt{1-x-y}\right)\right].
\end{aligned}$$

It is interesting to examine the graph of $\Delta_\lambda^1(N,R)$ in the area of the two-dimensional space (x,y) described by the inequalities $0 \le x \le y$ and $y \le \frac{1-x}{2}$. This area is depicted in Figure 14.

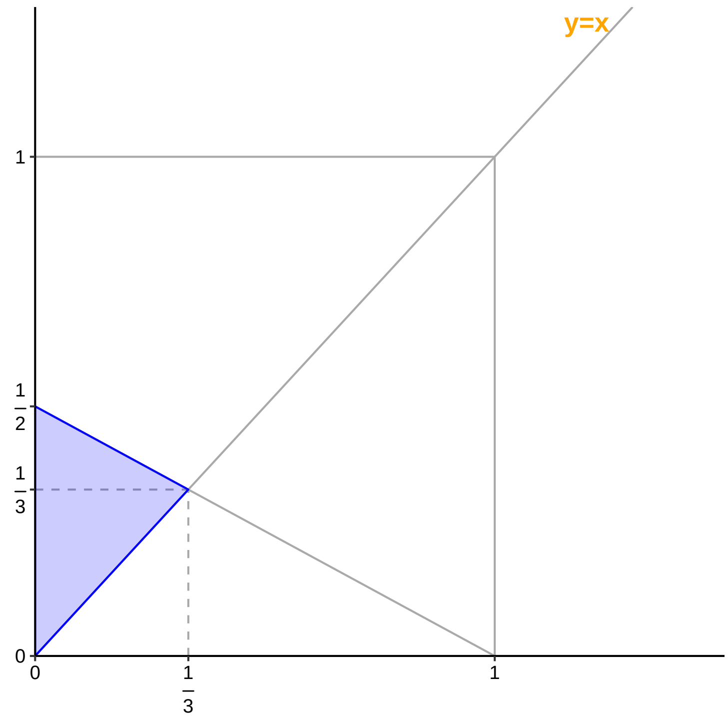

Figure 14: Area of the two-dimensional space (x, y) described by the inequalities $0 \leq x \leq y$ and $y \leq \frac{1-x}{2}$

When $x = y \leq \frac{1}{3}$, we obtain $\Delta_{\lambda}^{1}(N, R) = 1 - \frac{1}{2}(2\sqrt{x} + \sqrt{1-2x})$ whose graph is depicted in Figure 15.[42] Note that here $DK(N, R) = \frac{3x}{2}$.

A.2 Majorization with Weights[43]

Our paper is closely related to an extension of the classical theory of majorization pioneered by Hardy et al. (1952), HLP hereafter. Given two vectors x and y in $\mathbb{R}^n$, x is said to be majorized by y if:

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$$

for all convex functions $g : \mathbb{R} \rightarrow \mathbb{R}$. It is well known that this condition is equivalent to the Lorenz ordering:

$$\sum_{i=1}^{k} x_{\sigma(x,i)} \geq \sum_{i=1}^{k} y_{\sigma(y,i)} \text{ for all } k = 1, ..., n \text{ with an equality for } k = n$$

[42] Unsurprisingly, we observe that $\Delta_{\lambda}^{1}(N, R)$ is a decreasing function of x.

[43] This section borrows extensively from chapter 14 in Marshall, Olkin and Arnold (2011), MOA hereafter.

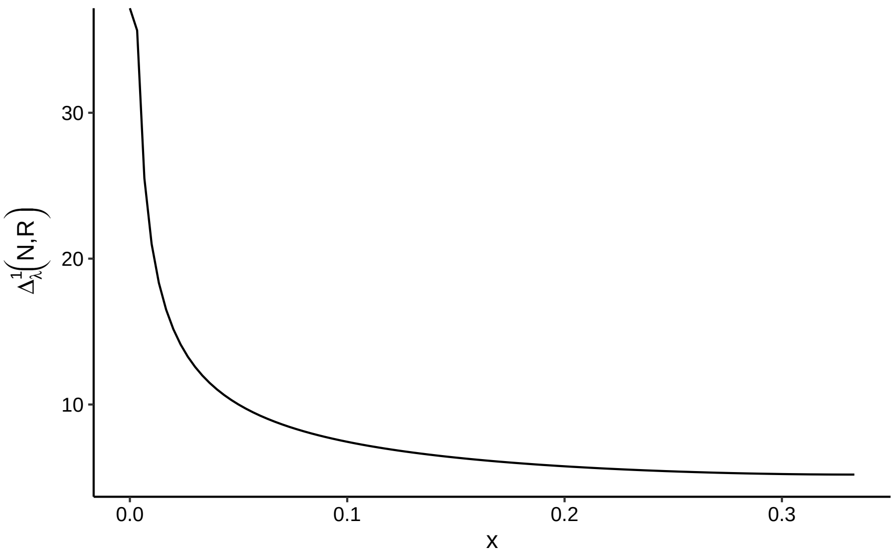

Figure 15: Graph of $\Delta^1_\lambda(N,R)$ when $x = y \leq \frac{1}{3}$

where for any vector $z \in \mathbb{R}^n$, $\sigma(z,i)$ is the index of the coordinate of the vector z with rank i (from the smallest to the largest). A rather general extension of this ordering is:

$$\sum_{i=1}^{n} p_i g(\frac{x_i}{u_i}) \leq \sum_{i=1}^{n} q_i g(\frac{y_i}{v_i}) \tag{8}$$

for all convex functions $g : \mathbb{R} \to \mathbb{R}$. In this extension, the p, q, u and v are arbitrary vectors of $\mathbb{R}^n$. Majorization corresponds to the case where $p = q = u = v = (1, 1, ..., 1)$. The general case where $u = v = (1, 1, ..., 1)$ has been studied by Blackwell (1951). The subcase where, in addition to the preceding restriction, $p = q$, is called $p-$majorization. Blackwell's characterization as well as its implications for $p-$majorization are reported in MOA. Further explorations of $p-$majorization due to Fuchs (1947) and Cheng (1977) are presented in MOA. In particular Cheng (1977) introduces a weighted version of the Lorenz order and obtains characterizations of $p-$majorization involving that order but, unfortunately, this partial ordering is only defined on vectors which are similarly ordered. This limits severely the scope of application of their theorems.

The extension of majorization which is the more relevant for our paper is not $p-$majorization but $d-$majorization as we focus primarily on the case of private resources. In such case, if we keep interpreting p_i as the size of territory i, then our interest is in the case where $p = u$ and $q = v$. In this general formulation, we can compare vectors where not only resources but populations are different across territories. In the above formulation, we obtain the ordering:

$$\sum_{i=1}^{n} p_i g(\frac{x_i}{p_i}) \leq \sum_{i=1}^{n} q_i g(\frac{y_i}{q_i})$$

for all convex functions $g : \mathbb{R} \to \mathbb{R}$. This ordering has been studied by Ruch et al. (1978). The particular case where $p = q = u = v \equiv d$ has been investigated by Veinott (1971) and subsequently by Joe (1990) under the name $d-$majorization. It corresponds to:

$$\sum_{i=1}^{n} d_i g(\frac{x_i}{d_i}) \leq \sum_{i=1}^{n} d_i g(\frac{y_i}{d_i})$$

for all convex functions $g : \mathbb{R} \to \mathbb{R}$. Veinott (1971) defines the ordering on the basis of a matrix transformation and shows that functions of the form $\sum_{i=1}^{n} d_i g(\frac{x_i}{d_i})$ where g is convex preserve that ordering. The most important and useful characterization result due to Ruch et al. (1980) and Joe (1990) states several conditions equivalent to d-majorization. It is reported as proposition B.4 in MOA. Among these equivalent conditions, one is very useful as, like in the conventional Lorenz condition, it amounts to a finite number of checks. It asserts that x $d-$majorized y iff:

$$\sum_{i=1}^{n} \max(x_i - d_i t, 0) \leq \sum_{i=1}^{n} \max(y_i - d_i t, 0) \text{ for all } t \text{ in the set } \left\{ \frac{x_1}{d_1}, ..., \frac{x_n}{d_n}, \frac{y_1}{d_1}, ..., \frac{y_n}{d_n} \right\}.$$

This means that we have to proceed to the check of at most $2n$ inequalities. When $d = (1, ..., 1)$ the above condition already appears in HLP. For all t, the convex function $g_t(x) = \max(x - t, 0)$ is called an angle. Given x the function $\sum_{i=1}^{n} \max(x_i - t, 0)$ is piecewise linear function of t where the angularities occurring at $t = x_1, ..., x_n$. As in the case of $p-$majorization, the comparison of x and y amounts to the comparison of two piecewise linear functions with angularities for the same n values. It is enough to compare the function for these values. It is not complicated (see e.g. Berge, 1966) to verify that these n comparisons are equivalent to the Lorenz comparisons.

For an arbitrary vector d, things are a bit more tedious since the points of angularities may differ across x and y. But again, if we draw the Lorenz curves with $\sum_{i=1}^{n} d_i$ coordinates as we did before in the case of p-majorization, the ordering of the two Lorenz curves, which are piecewise linear, needs only (at most) $2n$ comparisons.

In the above formulations, we have always assumed that $n = m$ and we have seen that a necessary condition for two vectors x and y to be compared according to these majorizations is:

$$\sum_{i=1}^{n} p_i \frac{x_i}{u_i} = \sum_{i=1}^{n} q_i \frac{y_i}{v_i}.$$

For practical matters, we may want to extend the above majorization. In the political setting considered in this paper, n is the number of political districts, p_i is the number of citizens in district i and x_i is the number of representatives (seats) of district i. In applications, we may want to compare situations where n has changed, or/and p_i has changed or/and $\sum_{i=1}^n x_i$ has changed. The first and third cases may occur as the result of a reform. The second case occurs as the result of demographic and/or migration changes. In the second case if we suppose that there is one seat per district i.e. $x = y = (1, 1, ..., 1)$, d-majorization amounts to:

$$\sum_{i=1}^{n} p_i g(\frac{1}{p_i}) \leq \sum_{i=1}^{n} q_i g(\frac{1}{q_i})$$

for all convex functions $g : \mathbb{R} \to \mathbb{R}$. We see immediately[44] that we cannot compare the two situations unless $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i$. In the case where $m \neq n$, the non comparison extends to the case where $\sum_{i=1}^n p_i$ is different from $\sum_{i=1}^m q_i$. One way to exit from this dead-end, consists in the following normalization:

$$\frac{1}{\sum_{i=1}^n p_i} \sum_{i=1}^{n} p_i g(\frac{x_i}{p_i}) \leq \frac{1}{\sum_{i=1}^m q_i} \sum_{i=1}^{m} q_i g(\frac{y_i}{q_i})$$

for all convex functions $g : \mathbb{R} \to \mathbb{R}$. In the case where $p = (1, ..., 1)$ and $q = (1, ..., 1)$ this amounts to the extension considered by MOA in their section on majorization for vectors of unequal length with equal mean. They show that majorization is equivalent to Lorenz comparisons where we report on the horizontal axis the different ordered cumulative shares in the two populations and on the vertical axis the cumulative shares of seats (on this axis, we could equivalently report the average number of seats are the total are assumed to be the same).

If the total number of seats differs across the two situations, then we cannot make comparisons.[45] One possibility could consist in asking that the above comparisons hold for all for all convex and decreasing functions $g : \mathbb{R} \to \mathbb{R}$. This leads to what MOA call weak supermajorization. An adjustment in the proof shows that is equivalent to a generalized Lorenz criterion which differs from Lorenz as the values reported on the vertical axes are the cumulative number of seats and not their share.

In this paper, we will adopt the relative point of view i.e. if needed, we will replace the vectors x and y by their corresponding vectors of shares of seats/resources and the vectors p and q by their corresponding vectors of shares of populations. This axiom of scale invariance is common in inequality measurement and lead to the class of indices of relative inequality (Lambert, 2001).[46]

[44] Consider $g(x) = c$ where c is a positive or negative constant.
[45] Consider indeed $g(x) = x$ and $g(x) = -x$.
[46] In this literature, they examine the trade-off between the size of the cake and its distribution. Inequality

A.3 Integer Constraints

In the paper we have focused on the canonical problem:

$$\max_X \sum_{k=1}^{K} N_k U(\frac{X_k}{N_k}),$$

under the constraints $\sum_{k=1}^{K} X_k = M$ and $X_k \geq 0$ for all $k = 1, ..., K$.

Therefore, the variables X_k were assumed to be unrestricted continuous variables. If the variables are seats, then they are integer-valued and the optimization problem now becomes:[47]

$$\max_X \sum_{k=1}^{K} N_k U(\frac{X_k}{N_k}),$$

under the constraints $X \in \mathcal{S}$ and $X_k \in \mathbb{N}$ for all $k = 1, ..., K$.

It follows from these new constraints that the perfect proportionality solution $X_k = \frac{N_k}{\sum_{j=1}^{K} N_j} M$ for all $k = 1, ..., K$ (which does not depend on the specific U that we are considering) may not be feasible. In contrast, under integer constraints, the solution(s) of the maximization problem will depend upon U.

There is an enormous literature on proportionality in electoral science. There is no unanimity among scholars on what should be the more appropriate definition of proportionality in the face of integer constraints. Choice among these methods depend upon the axioms/properties that are expected from these methods. Among the most popular proportional methods which are used all over the world let us mention Adam, Dean, D'Hondt/Jefferson, Hill, Hamilton (largest remainders), Hill and Sainte-Lagüe/Webster.[48] We refer the reader to the excellent books of Balinski and Young (2001) and Pukelsheim (2014) for a deep and detailed exposition of the state of the art. Since we have discussed majorization in A.2, let us mention that there are a number of important contributions (Lauwers and Puyenbroeck, 2006b,a; Marshall et al., 2002) which compare these methods from a majorization perspective. Note however that they consider classical majorization where the parties/territories are ordered decreasingly from left to right on the basis of their population/vote shares i.e. from the smallest parties/territories to

measurement takes care only of the second component.

[47]In such case, M is of course, assumed to be itself an integer. M is the total number of seats.

[48]Interestingly, Sainte-Lagüe/Webster is the solution of the maximization problem when we take $U(x) = -x^2$. Other methods are solutions of maximisation problems sharing the same constraint as the maximization above but an objective quite different from the utilitarian objective. For instance, the D'Hondt/Jefferson method minimizes the Gallagher's index. We refer to Pennisi (1998); Karpov (2008) for more information on this issue. Note also that if we depart from the utilitarian principle and consider the general class of anonymous social welfare functions, we may find support for more proportional methods. For instance, the D'Hondt/Jefferson's method minimizes the objective $\underset{1 \leq k \leq K}{Max} \frac{X_k}{N_k}$.

the largest ones. Therefore classical majorization of the vector of seats amounts to look how small parties are well treated by the method. If a vector of seats Y majorizes a vector of seats X, then vector X is unambiguously better than Y from the perspective of the smallest parties. Marshall et al. (2002) prove that:

$$\text{Adams} \prec \text{ Dean} \prec \text{ Hill} \prec \text{ Webster/Sainte-Lagüe} \prec \text{ D'Hondt/Jefferson}$$

And Lauwers and Puyenbroeck (2006b,a) prove that:

$$\text{Adams} \prec \text{ Hamilton} \prec \text{ D'Hondt/Jefferson}$$

Let us insist on the fact that this literature has a different motivation from the one of this paper. These authors are mostly interested in comparing vectors from the perspective of the parties/territories rather than from the perspective of the voters/individuals.

Integer constraints raise a number of new interesting problems but this does not have any impact on the way to define the Lorenz curves attached to seat distributions and the definition of malapportionment and disproportionality indices developed in subsections 2.3 and 2.4.

A.4 Pure Local Public Goods

In this paper, we have assumed that the resources collected by a territory are distributed within the territory without economies of scale. If instead, the good which is considered is a pure local public good, the benefit of a resident of territory k is d_k and the utility derived by i from decision d is:

$$U_i(d_k)$$

and then the utilitarian welfare attached to decision d is:

$$\sum_{k=1}^{K} n_k U(d_k). \tag{9}$$

Maximization (2) under constraint (1) yields a unique interior solution:

$$d_k^*(U) = \varphi\left(\frac{\lambda}{n_k}\right) \text{ for all } k = 1, ..., K$$

where φ is the inverse of $U^{'}$ and the Lagrange multiplier λ is solution of:

$$\sum_{k=1}^{K} \varphi\left(\frac{\lambda}{n_k}\right) = M.$$

Since U' is decreasing, φ is decreasing too: $d_k^*(U)$ now depends upon U. When for instance $U(x) = 2\sqrt{x}$, we obtain:

$$d_k^*(U) = \frac{n_k^2}{\sum_{j=1}^K n_j^2} M.$$

Note also that in such case, majorization cannot be used in a straightforward manner. Take two arbitrary vectors X and Y in $\mathbb{R}_+^K$ such that $\sum_{k=1}^K X_k = \sum_{k=1}^K Y_k = M$. As pointed out in A.2, to compare x and y according to the utilitarian principle amonts to compare $\sum_{k=1}^K N_k g(X_k)$ and $\sum_{k=1}^K N_k g(Y_k)$ for all convex functions g. This looks like $p-$majorization. But unfortunately, the vectors are not comparable since $\sum_{k=1}^K N_k X_k \neq \sum_{k=1}^K N_k Y_k$.

A.5 Distributive Politics and Bargaining

In the paper, we have assumed a perfect proportionality between the number of representatives of territory k and the proportion of "private" resources allocated to territory k. This is indeed an assumption and it has to be discussed. To be as general as possible, let us denote by $X(R)$[49] the vector of resources resulting from the vector of representation R. There are of course many positive models of politics to construct equilibrium predictions pointing out in direction of a specific $X(R)$. A lobbying story a la Tullock would suggest for instance contest functions like:

$$X_k(R) = \frac{R_k^\alpha}{\sum_{j=1}^K R_j^\alpha}$$

where α is a positive parameter. This reduced form is used in the public choice literature to describe how the respective influences of groups competing against each other. When $\alpha = 1$, we are back in the specific case considered in this paper.

We may alternatively consider a legislative bargaining game à la Baron-Ferejohn (Baron and Ferejohn, 1989). Given a vector R, let $(N, \mathcal{W})$ be the weighted majority game attached to R. In this game, as long as the bargaining is not over, any player/representative has an equal chance to be chosen to be the "new" proposer i.e. to make a proposal replacing the last one which was on the table. Representatives vote in favor or against this new proposal. The yes wins if the group of supporters constitutes a winning coalition. If the yes wins, the proposal is implemented. Otherwise, the bargaining continues along the lines that we have described. This game has a unique subgame perfect Nash equilibrium depending upon R. Therefore, for each R, we have an expected distribution of resources $d(R)$. Some authors (Ansolabehere et al., 2002,

[49] We delete the dependence to U as the ordinal profile attached to U is fixed in the redistributive politics setting.

2005) have argued that the BJ solution satisfies a property of proportionality with respect to R i.e. $\frac{d_k(R)}{d_l(R)} = \frac{R_k}{R_l}$. The scope of validity of this assertion has been explored by Montero (2017).

Finally and alternatively, we could consider the TU cooperative bargaining game attached to the weighted majority game $(N, \mathcal{W})$ and explore solutions like the Shapley value $Sh(R)$ or the nucleolus $Nu(R)$. The simple game $(N, \mathcal{W})$ is a constant sum game. García-Valiñas et al. (2016); Kauppi and Widgrén (2004, 2007); Le Breton et al. (2012); Montero (2006, 2017) have investigated this direction of research. For instance, Kauppi and Widgrén (2004) uses that methodology to examine the determinants of power in the Council of the European Union. They contrast the bargaining explanation (they consider the Shapley value) with the "needs" view. Formulated with our notations, their work contrasts $X(R)$ and X^* contrast as two potential explanations of the actual distribution of power among EU members. Their empirical analysis is based on 1976–2001 data on the patterns of the EU budget shares and on measures of each member state's needs and political power. Their results indicate that at least 60% of the budget expenditures can be attributed to selfish power politics and the remaining 40% to the declared benevolent EU budget policies. a similar empirical analysis is conducted by García-Valiñas et al. (2016) with the nucleolus instead of the Shapley value.

To conclude this appendix let us mention an important point. In A.3, we have discussed some of the implications of integer constraints. Integer constraints imply that the set of feasible solutions is a finite set. Integer constraints arise naturally if the resource to be distributed is integer valued like for instance seats. But interestingly, the finiteness of the feasible set also arises when the resource is divisible like in the case of of a budget sharing. To illustrate that point, consider the cooperative game positive explanation that has been described above. True, the cooperative game depends upon the vector R. But there is a finite number of weighted majority games and therefore, even if continuous variations of R are possible, these continuous variations will not be translated into continuous variations of the solutions. What matters is truly the set of winning coalitions $\mathcal{W}$, not the vector of representation R itself. Given a solution concept (Shapley, Nucleolus,...), let us denote by $\mathcal{F}$ the finite subset of the simplex S describing the values of the solution for the all class of weighted majority games. Given U the original maximization problem can then be written as:

$$Max \sum_{k=1}^{K} N_k U(\frac{X_k}{N_k})$$

under the constraints $X \in \mathcal{F}$.

Note that the characterization of $\mathcal{F}$ is not a straightforward problem. It is known in the literature as an inverse problem (see for instance, Alon and Edelman, 2010; Kurz, 2012; Kurz and Napel, 2014). If the mapping from weighted majority games is known, the maximization problem is a mechanism design problem where the unknown is the game itself and therefore the vector R instead of the allocation X. Le Breton et al. (2012) solve that problem for low values of K[50] when the solution is the nucleolus and $U(x) = -x^2$.

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[50]When $K = 6$, the choice is among 21 weighted majority games. When $K = 7$, the size of the choice set moves to 135 and to 2470 when $K = 8$. For more information about this issue, see Le Breton et al. (2012).

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