ON MV-TOPOLOGIES

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TESIS DE DOCTORADO EN CIENCIAS MATEMÁTICAS

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Rodó la piedra y otra vez como antes
la empujaré, la empujaré cuesta arriba
para verla rodar de nuevo.

Comienza la batalla que he librado mil veces
contra la piedra y Sísifo y mí mismo.

Piedra que nunca te detendrás en la cima:
te doy gracias por rodar cuesta abajo.
Sin este drama inútil sería inútil la vida.

José Emilio Pacheco, *Retorno a Sísifo.*

Contents

Introducción

La teoría de conjuntos *fuzzy* es una generalización de la teoría de conjuntos clásica. Los conjuntos fuzzy fueron definidos en 1965, por Lofti Zadeh [50], como una forma de representar la imprecisión en la vida diaria. Un conjunto fuzzy es una función $\alpha : X \longrightarrow [0,1]$ o, de forma más general, una función $\alpha : X \longrightarrow L$, donde L es un retículo.

La topología fuzzy es la rama de las matemáticas que ha resultado de una síntesis de la topología general, con ideas, nociones, y métodos de la teoría de conjuntos fuzzy. La primera definición de un espacio topológico fuzzy se debe a C. L. Chang en [7] y posteriormente varias extensiones, y definiciones alternativas, han surgido en la literatura. i Las MV-álgebras fueron introducidas por C. C. Chang, en 1958, para dar una prueba algebraica de la completitud de la lógica ∞-valuada de Łukasiewicz (ver [5, 6]). Las MV-álgebras y las correspondientes estructuras ampliadas (como las MV-álgebras producto o MV-álgebras de Riesz) han sido estudiadas extensivamente en las tres últimas décadas, sobretodo por sus conexiones naturales con estructuras algebraicas bien conocidas tales como ℓ-grupos, ℓ-anillos, semianillos, espacios de Riesz, quantales, entre otros.

En este trabajo estamos interesados en un tipo particular de topología fuzzy llamada *MV-topología*, la cual usa operaciones MV-algebraicas para generar abiertos fuzzy. Estos espacios topológicos fuzzy, introducidos por Ciro Russo en [42], permiten generalizaciones naturales de definiciones y resultados importantes de la topología clásica. En este sentido, desarrollamos algunos conceptos y resultados centrales, con el propósito de extender los correspondientes de la topología clásica, y al mismo tiempo siguiendo la ruta de la bien conocida teoría de espacios topológicos fuzzy. Mostramos que las MV-topologías son un tipo de topología fuzzy que goza de muy "buen comportamiento" matemático, en el sentido de que la mayoría de definiciones y resultados clásicos de topología general encuentran una extensión o adaptación natural en este marco. Entre otros resultados, también extendemos el concepto de haz para el caso en el que el espacio base es un espacio MV-topológico, y mostramos una respresentación por "MV-haces" para una clase de MV-álgebras.

La tesis se organiza en dos partes: la primera parte se distribuye en tres capítulos de preliminares y la segunda consta de tres capítulos en los que desarrollamos los principales tópicos de la tesis.

En el primer capítulo presentamos algunos conceptos y resultados necesarios de teoría de categorías. El segundo capítulo contiene algunos aspectos de teoría de MV-álgebras, que serán usados a lo largo de la tesis. Y finalmente, un tercer capítulo acerca de los más importantes conceptos de topologías fuzzy que se necesitarán en lo sucesivo.

En la segunda parte de la tesis, estudiamos y desarrollamos una buena parte de la teoría sobre espacios MV-topológicos. En el capítulo 4 presentamos los conceptos básicos sobre MV-topologías. Inicialmente, recopilamos algunos de los resultados presentados en el artículo pionero de MV-topologías, [42]. Después, definimos conceptos básicos adicionales como operador interior y operador clausura, espacio cociente y espacio producto, entre otros. Estudiamos también el rol de las MV-topologías dentro de las topologías fuzzy y mostramos que la extensión de la dualidad de Stone presentada en [42], restringida a las MV-topologías laminadas, es precisamente las MV-álgebras de Riesz (*limit cut complete*).

El capítulo 5 contiene algunos análogos de importantes resultados que son transversales en el estudio de la topología clásica, los cuales muestran, una vez más, el buen comportamiento de las MV-topologías y prometen un próspero desarrollo de esta teoría. Mostramos un teorema tipo-Tychonoff con varias consecuencias, y la existencia de una compactificación Stone-Čech para espacios MV-topológicos. También, caracterizamos normalidad en espacios MV-topológicos mostrando un teorema tipo-Urysohn. Finalmente, definimos MV-uniformidades y probamos que ellas inducen MV-topologías; además, cada espacio MV-topológico generado por una MV-uniformidad, es completamente regular.

En el capítulo 6, definimos haces sobre espacios MV-topológicos y mostramos una representación de una clase de MV-álgebras, mediante un MV-haz sobre el espectro maximal MV-topológico $(\operatorname{Max} A; \tau_A)$, definido en [42]. Esta representación está basada fuertemente en la representación por haces para MV-álgebras de Georgescu-Filipoiu [18], y en los resultados sobre MV-álgebras lexicográficas obtenidos por Diaconescu, Flaminio y Leustean [10]. Estamos convencidos de que los haces sobre MV-espacios abren un camino para futuros desarrollos en el estudio de propiedades globales a partir de propiedades locales en MV-topologías.

Introduction

Fuzzy Set Theory is a generalisation of classical Set Theory. Fuzzy sets were introduced by Lofti A. Zadeh [50], in 1965, as a mathematical way to represent vagueness in everyday life. A fuzzy set of X is a function $\alpha : X \longrightarrow [0,1]$ or, more generally, a function $\alpha : X \longrightarrow L$, where L has a lattice structure.

Fuzzy Topology is the branch of mathematics that arose from a synthesis of General Topology with ideas, notions, and methods of Fuzzy Set Theory. In [7], C. L. Chang gave the first definition of fuzzy topological space. Eventually, various extensions and alternative definitions appeared in the literature.

MV-algebras were introduced by C. C. Chang, in 1958, to give an algebraic proof of the completeness of Łukasiewicz ∞-valued propositional logic (see [5,6]). MV-algebras and their corresponding extended structures (such as product MV-algebras or Riesz MV-algebras) have been extensively studied in the last three decades also because of their natural connections with well-known algebraic structures, such as ℓ-groups, ℓ-rings, semirings, Riesz spaces, quantales, among others.

In this work, we are interested in a particular type of fuzzy topology, called *MV-topology*, which is based on the use of MV-algebraic operations as fuzzy sets intersections and unions. These fuzzy topological spaces, introduced by Ciro Russo in [42], allow natural generalisations of definitions and fundamental results of classical topology. In this framework, we developed some central concepts and results, with the purpose of extending the classical topological correspondents while, at the same time, following the path of the well-established theory of fuzzy topological spaces. As a matter of fact, MV-topologies are very "well-behaved" fuzzy topologies, in the sense that most of the classical definitions and results of General Topology find a pretty natural extension or adaptation in this framework. Among other results, we shall also extend the concept of sheaf to the case where the underlying space is an MV-topological one, and show a representation by *MV-sheaves* for a class of MV-algebras.

The thesis is organised in two parts: the first part consists of three chapters on preliminaries and the second one consists of three chapters in which we develop the main topics of our work.

In the first chapter, we present some necessary concepts and results of category theory. The second chapter contains some aspects of the theory of MV-algebras, which we will use throughout the thesis. Finally, the third chapter is dedicated to a review of the main fuzzy topological concepts and results which will be needed in the sequel.

In the second part of the thesis, we study and develop a big part of the theory of MV-topological spaces. In Chapter 4 we present the basic concepts on MV-topologies. The very first part comes from the first paper on MV-topologies [42], which we already mentioned. Eventually, we define additional basic concepts such as interior and closure operators, quotient space and product space, among others. We also study the role of MV-topologies within the fuzzy topologies and we show that the extension of Stone Duality, presented in [42], restricted to laminated MV-topologies, gives precisely (limit cut complete) Riesz MV-algebras.

Chapter 5 contains some analogous to important results that are ubiquitous in classical topology, which show once more the good behaviour of MV-topologies and encourage a prosperous development of this theory. We show a Tychonoff-type theorem, along with various consequences, and the existence of a Stone-Čech compactification for MV-topological spaces. Then we characterise normality in MV-topological spaces showing a Urysohn-type theorem. Last, we define MV-uniformities, and we prove that they are able to induce MV-topologies and that MV-topological spaces induced by MV-uniformities are completely regular.

In Chapter 6, we define sheaves on MV-topological spaces, and we show a representation of a class of MV-algebras by means of MV-sheaves on the maximal MV-spectrum $(\operatorname{Max} A; \tau_A)$ defined in [42]. Such a representation is strongly based on Filipoiu and Georgescu's sheaf representation for MV-algebras [18], and on the results on lexicographic MV-algebras obtained by Diaconescu, Flaminio, and Leustean [10]. We believe that MV-sheaves open a path for future developments in the study of global properties on MV-topologies from local ones.

Part I

Preliminaries

1

Category Theory

In this section we present some necessary topics about category theory. For more references and further readings, the reader can see [1, 28, 32, 38], etc.

1.1 Categories and Functors

Definition 1.1.1. A *category* $\mathcal{A}$ consist of:

- a collection of *objects* that we denote by $\mathrm{Ob}(\mathcal{A})$;
- for each $A, B \in \mathrm{Ob}(\mathcal{A})$, a collection $\mathrm{Hom}_{\mathcal{A}}(A, B)$ of *maps* or *arrows* or *morphisms* from A to B;
- for each $A, B, C \in \mathrm{Ob}(\mathcal{A})$, a function

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(A,B) \times \mathrm{Hom}_{\mathcal{A}}(B,C) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A,C) \\ (f,g) & \longmapsto & g \circ f \end{array}$$

  called *composition*;
- for each $A \in \mathrm{Ob}(\mathcal{A})$, an element $1_A \in \mathrm{Hom}_{\mathcal{A}}(A, A)$, called the *identity* on A,

satisfying the following axioms:

(i) Associativity: for each f in $\mathrm{Hom}_{\mathcal{A}}(A, B)$, g in $\mathrm{Hom}_{\mathcal{A}}(B, C)$ and h in $\mathrm{Hom}_{\mathcal{A}}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.
(ii) Identity laws: for each f in $\mathrm{Hom}_{\mathcal{A}}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.
(iii) The sets $\mathrm{Hom}_{\mathcal{A}}(A, B)$ are pairwise disjoint.

Remark 1.1.2. (a) We will often use $A \in \mathcal{A}$ to mean $A \in \mathrm{Ob}(\mathcal{A})$; $f : A \longrightarrow B$ or $A \xrightarrow{f} B$ to mean $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$; and gf to mean $g \circ f$.
(b) If $f : A \longrightarrow B$ is a morphism of $\mathcal{A}$, we call A the *domain* of f (denoted by $\mathrm{dom}(f)$) and B the *codomain* of f (denoted by $\mathrm{cod}(f)$). Observe that condition (iii) guarantees that each morphism of $\mathcal{A}$ has an unique domain and an unique codomain.

Every category $\mathcal{A}$ has an opposite or dual category $\mathcal{A}^{\mathrm{op}}$, defined by reversing the arrows. Formally, $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(B, A) = \mathrm{Hom}_{\mathcal{A}}(A, B)$ for all objects A and B. Identities in $\mathcal{A}^{\mathrm{op}}$ are the same as in $\mathcal{A}$. Composition in $\mathcal{A}^{\mathrm{op}}$ is the same as in $\mathcal{A}$, but with the arguments reversed. So, arrows $A \longrightarrow B$ in $\mathcal{A}$ correspond to arrows $B \longrightarrow A$ in $\mathcal{A}^{\mathrm{op}}$.

Example 1.1.3. The following are examples of categories:

1. the category $\mathcal{S}$et whose objects are sets and morphisms are functions;
2. the category $\mathcal{G}$rp whose objects are groups and whose morphisms are group homomorphisms;
3. in the same sense, the category $\mathcal{R}$ing whose objects are rings and whose morphisms are ring homomorphisms;
4. the category $\mathcal{MV}$ whose objects are MV-algebras and the morphisms are homomorphisms of MV-algebras (see Chapter 2 for details);
5. the category $\mathcal{T}$op whose objects are topological spaces and the morphisms are continuous maps;
6. the category $\mathcal{F}$uz whose objects are fuzzy topological spaces and the morphisms are fuzzy continuous maps (see Chapter 3);
7. Let $(P, \leq)$ a partially ordered set. We can see $(P, \leq)$ as a category whose objects are the elements of P and each $a, b \in P$ such that $a \leq b$ define an unique morphism $a \longrightarrow b$. Note that $\mathrm{Hom}(a, b)$ is empty unless $a \leq b$, in which case $\mathrm{Hom}(a, b)$ contains a single element, $a \longrightarrow b$.

Definition 1.1.4. A morphism $f : A \longrightarrow B$ in a category $\mathcal{A}$ is:

- an *isomorphism* if there exists a morphism $g : B \longrightarrow A$ with $gf = 1_A$ and $fg = 1_B$. Such a morphism g is called the *inverse* of f and is, of course, an isomorphism as well that will be denoted by f^{-1}. If there exists an isomorphism between two objects of a category, such objects are said to be *isomorphic*;
- a *monomorphism* if for all pairs $C \underset{k}{\overset{h}{\rightrightarrows}} A$ of morphisms such that $f \circ h = f \circ k$, it follows that $h = k$ (i.e., f is left-cancellable with respect to composition);
- an *epimorphism* if for all pairs $B \underset{k}{\overset{h}{\rightrightarrows}} C$ of morphisms such that $h \circ f = k \circ f$, it follows that $h = k$ (i.e., f is right-cancellable with respect to composition).

Definition 1.1.5. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

- a function
$$\begin{aligned} F : \mathcal{A} &\longrightarrow \mathcal{B} \\ A &\longmapsto F(A) \end{aligned}$$
- for each $A, A' \in \mathcal{A}$, a function
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(A, A') & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(F(A), F(A')) \\ f & \longmapsto & F(f) \end{array}$$

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{A}$;
- $F(1_A) = 1_{F(A)}$ whenever $A \in \mathcal{A}$.

Functors are sometimes called *covariant functors*. A *contravariant functor* from $\mathcal{A}$ to $\mathcal{A}'$ means a functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{A}'$.

We will often use the simplified notations FA and Ff rather than $F(A)$ and $F(f)$, and denote the action of F on both objects and morphisms by $F(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FB$.

Definition 1.1.6. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called:

- an *embedding* if it is injective on morphisms;
- *faithful* if all the hom-set restrictions $F : \mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA, FA')$ are injective and *full* if they are surjective;
- *isomorphism dense*, or *essentially surjective on objects* or, simply, *dense* provided that for any $B \in \mathcal{B}$ there exists some $A \in \mathcal{A}$ such that FA is isomorphic to B;
- an *isomorphism* if there exists a functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ such that $G \circ F$ is the identity functor of $\mathcal{A}$ and $F \circ G$ is the identity functor of B. In this case, the categories $\mathcal{A}$ and $\mathcal{B}$ are called *isomorphic*.

Even if there exists a definition of isomorphism between categories, there is a weaker concept that is much more useful in practice: the one of "categorical equivalence". Indeed, in mathematics, two object that are isomorphic can be treated as essentially the same object, and categories does not make exception; nonetheless, most of the categorical properties are preserved under equivalences and equivalences are more than isomorphisms, i.e. every isomorphism is an equivalence but not vice versa. A formal definition of this sort of "weak isomorphism" is the following:

Definition 1.1.7. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called an *equivalence* if it is full, faithful and isomorphism-dense. In this case, the categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent*.

The following properties of functors are very easy to prove.

Proposition 1.1.8. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be functors.*

(i) All functors preserve isomorphisms, i.e., whenever $f : A \longrightarrow A'$ is an isomorphism in $\mathcal{A}$, then $F(f)$ is an isomorphism in $\mathcal{B}$.

(ii) The composition $G \circ F : \mathcal{A} \longrightarrow \mathcal{C}$ defined by

$$(G \circ F)(f : A \longrightarrow A') = G(Ff) : G(FA) \longrightarrow G(FA')$$

is a functor.

(iii) A functor is an embedding if and only if it is faithful and injective on objects.

(iv) A functor is an isomorphism if and only if it is full, faithful, and bijective on objects.

(v) If F and G are both isomorphisms (respectively: embeddings, faithful, full), then so is $G \circ F$.

(vi) If $G \circ F$ is an embedding (respectively: faithful), then so is F.

(vii) If F is surjective on objects and $G \circ F$ is full, then G is full.

(viii) If F is full and faithful, then for every morphism $f : FA \longrightarrow FA'$ in $\mathcal{B}$ there exists a unique morphism $g : A \longrightarrow A'$ in $\mathcal{A}$ such that $Fg = f$. Furthermore, g is an isomorphism in $\mathcal{A}$ if and only if f is an isomorphism in $\mathcal{B}$.

(ix) If F is full and faithful, then it reflects isomorphisms; i.e., whenever $g \in \mathcal{A}$ is a morphism such that $Fg \in \mathcal{B}$ is a isomorphism, then g is an isomorphism in $\mathcal{A}$

(x) If F is an equivalence, then there exists an equivalence $H : \mathcal{B} \longrightarrow \mathcal{A}$.

(xi) If F and G are equivalences, then so is $G \circ F$.

Definition 1.1.9. Let $\mathcal{A}$ be a category. A *subcategory* $\mathcal{S}$ of $\mathcal{A}$ consists of a subclass $\mathrm{Ob}(\mathcal{S})$ of $\mathrm{Ob}(\mathcal{A})$ together with, for each $S, S' \in \mathrm{Ob}(\mathcal{S})$; a subclass $\mathrm{Hom}_{\mathcal{S}}(S, S')$ of $\mathrm{Hom}_{\mathcal{A}}(S, S')$, such that $\mathcal{S}$ is closed under composition and identities. It is a *full subcategory* if $\mathrm{Hom}_{\mathcal{S}}(S, S') = \mathrm{Hom}_{\mathcal{A}}(S, S')$ for all $S, S' \in \mathrm{Ob}(\mathcal{S})$.

For example, the category $\mathcal{AG}$ of the Abelian groups is a full subcategory of $\mathcal{G}$rp.

Definition 1.1.10. Let $\mathcal{X}$ be a category. A *concrete category* over $\mathcal{X}$ is a pair $(\mathcal{A}, U)$, where $\mathcal{A}$ is a category and $U : \mathcal{A} \longrightarrow \mathcal{X}$ is a faithful functor. Sometimes U is called the *forgetful* (or *underlying*) *functor* of the concrete category and $\mathcal{X}$ is called the *base category* for $(\mathcal{A}, U)$. A concrete category over the category $\mathcal{S}$et of sets is called a *construct*.

For example, any category whose objects are sets with some structure (namely: topological spaces, groups, lattices, etc.), is a construct. The forgetful functor is the one that sends every object into its underlying set and every morphism into itself (the latter regarded as a morphism of sets). Moreover, many constructs can also be seen as concrete categories over another category. For instance, the category of vectorial spaces is both a construct and a concrete category over $\mathcal{AG}$, the category of Abelian groups; in this case, the forgetful functor does not "forget" the whole structure but only the external multiplication.

1.2 Some limits

In this section we present definitions and examples of some limits that we will use later.

Products

Definition 1.2.1. Let $\mathcal{C}$ be a category and $X, Y \in \mathcal{C}$. A *product* of X and Y consists of an object P and maps

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

with the property that for all objects and maps

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}$$

in $\mathcal{C}$, there exists a unique map $\overline{f} : A \longrightarrow P$ such that

$$\begin{tikzcd} & A \arrow[d, dashed, "\overline{f}"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

commutes. The maps p_1 and p_2 are called the *projections*.

Equalizers

We say that a *fork* in a category consists of objects and maps

$$A \xrightarrow{f} X \underset{t}{\overset{s}{\rightrightarrows}} Y \tag{1.1}$$

such that $sf = tf$.

Definition 1.2.2. Let $\mathcal{C}$ be a category and let $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ be objects and maps in $\mathcal{C}$. An *equalizer* of s and t is an object E together with a map $E \xrightarrow{i} X$ such that

$$E \xrightarrow{i} X \underset{t}{\overset{s}{\rightrightarrows}} Y$$

is a fork, and with the property that for any fork (1.1), there exists a unique map $\overline{f} : A \longrightarrow E$ such that

$$\begin{tikzcd} A \arrow[d, dashed, "\overline{f}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"] & X \end{tikzcd}$$

commutes.

Example 1.2.3. Given the functions and sets $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ in $\mathcal{S}$et, let

$$E = \{x \in X : s(x) = t(x)\}.$$

The object E with the inclusion map $i : E \longrightarrow X$ is the equalizer of s an t.

1.3 Natural Transformations and Adjoint Functors

In some cases, it is possible to connect two functors between the same categories by means of a sort of 'map between functors', called natural transformation.

The following definitions and notations are taken from [28].

Definition 1.3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ be functors. A *natural transformation* $\alpha : F \Longrightarrow G$ is a family $\{F(A) \xrightarrow{\alpha_A} G(A)\}_{A \in \mathcal{A}}$ of morphisms in $\mathcal{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathcal{A}$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"] & G(A') \end{tikzcd}$$

commutes. The maps α_A are called the *components* of α.

If all the arrows α_A are isomorphisms, α is called a *natural isomorphism*.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $F, G, H : \mathcal{A} \longrightarrow \mathcal{B}$ be functors between $\mathcal{A}$ and $\mathcal{B}$. And let $F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H$ be natural transformations between them, then there is a composite natural transformation $F \overset{\beta\circ\alpha}{\Longrightarrow} G$ defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$, for all $A \in \mathcal{A}$. That is, given two families of morphisms of $\mathcal{B}$,

$$\{F(A) \xrightarrow{\alpha_A} G(A)\}_{A\in\mathcal{A}} \text{ and } \{G(A) \xrightarrow{\beta_A} H(A)\}_{A\in\mathcal{A}},$$

which satisfy the condition of Definition 1.3.1, we can obtain the family

$$\{F(A) \xrightarrow{\beta_A\circ\alpha_A} H(A)\}_{A\in\mathcal{A}}$$

that also satisfies the condition of Definition 1.3.1.

There is also an identity natural transformation $1_F : F \Longrightarrow F$ on any functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ defined by $(1_F)_A = 1_{F(A)}$ for each $a \in \mathcal{A}$.

From the above, we have that for any two categories $\mathcal{A}$ and $\mathcal{B}$, there is a category whose objects are the functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are the natural transformations between them. This is called the *functor category* from $\mathcal{A}$ to $\mathcal{B}$, and we written as $\mathcal{B}^{\mathcal{A}}$.

Definition 1.3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be categories, $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ be functors, and $\alpha : F \Longrightarrow G$ a natural transformation. We say that $F \overset{\alpha}{\Longrightarrow} G$ is a *natural isomorphism* if for all $A \in \mathcal{A}$, $\alpha_A : F(A) \longrightarrow G(A)$ is an isomorphism.

And we say that functors F and G are *naturally isomorphic* if there exists a natural isomorphism from F to G.

Note that a natural isomorphism between functors from $\mathcal{A}$ to $\mathcal{B}$ is actually an isomorphism in the category $\mathcal{B}^{\mathcal{A}}$.

Definition 1.3.3. Given functors $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$, we say that

$$F(A) \cong G(A)$$

naturally in A if F and G are naturally isomorphic.

Definition 1.3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ be functors between them. We say that F is *left adjoint* to G, and G is *right adjoint* to F, and write $F \dashv G$, if

$$\mathrm{Hom}_{\mathcal{B}}(F(A), B)) \cong \mathrm{Hom}_{\mathcal{A}}(A, G(B)) \tag{1.2}$$

naturally in $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

An *adjunction* between F and G is a choice of natural isomorphism (1.2).

Remark 1.3.5. We note that $\mathrm{Hom}_{\mathcal{B}}(F(-), -) : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathcal{S}\mathrm{et}$ such that

$$(A, B) \longmapsto \mathrm{Hom}_{\mathcal{B}}(F(A), B))$$

and $\mathrm{Hom}_{\mathcal{A}}(-, G(-)) : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathcal{S}\mathrm{et}$ such that

$$(A, B) \longmapsto \mathrm{Hom}_{\mathcal{A}}(A, G(B))$$

are functors from $\mathcal{A}^{\mathrm{op}} \times \mathcal{B}$ to $\mathcal{S}\mathrm{et}$. Then, it makes sense the Definition 1.3.4.

Let us better explain the naturality of the bijection (1.2): given objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the correspondence (1.2) between $F(A) \longrightarrow B$ and $A \longrightarrow G(B)$ is denoted by a horizontal bar, in both directions:

$$(F(A) \xrightarrow{g} B) \mapsto (A \xrightarrow{\overline{g}} G(B)),$$

$$(F(A) \xrightarrow{\overline{f}} B) \leftarrow\!\shortmid (A \xrightarrow{f} G(B)).$$

So $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$. We call $\overline{f}$ the *transpose* of f , and similarly for g. The naturality axiom has two parts:

$$\overline{(F(A) \xrightarrow{g} B \xrightarrow{q} B')} = (A \xrightarrow{\overline{g}} G(B) \xrightarrow{G(q)} G(B')) \tag{1.3}$$

that is, $\overline{q \circ g} = G(q) \circ \overline{g}$, for all g and q, and

$$\overline{(A' \xrightarrow{p} A \xrightarrow{f} G(B))} = (F(A') \xrightarrow{\overline{F(p)}} F(A) \xrightarrow{\overline{f}} B) \tag{1.4}$$

for all p and f.

Intuitively, naturality says that as A varies in $\mathcal{A}$ and B varies in $\mathcal{B}$, the isomorphism $\mathrm{Hom}_{\mathcal{B}}(F(A), B)) \cong \mathrm{Hom}_{\mathcal{A}}(A, G(B))$ varies in a way that is compatible with all the structure already in place. In other words, it is compatible with composition in the categories $\mathcal{A}$ and $\mathcal{B}$ and the action of the functors F and G.

For to explain this compatibility, suppose that we have maps

$$F(A) \xrightarrow{g} B \xrightarrow{q} B'$$

in $\mathcal{B}$. There are two things we can do with this data: either compose then take the transpose, which produces a map $\overline{q \circ g} : A \longrightarrow G(B')$, or take the transpose of g then compose it with $G(q)$, which produces a potentially different map $G(q) \circ \overline{g} : A \longrightarrow G(B')$. Compatibility means that they are equal; and that is the first naturality equation (1.3). The second is its dual, and can be explained in a similar way.

For each $A \in \mathcal{A}$, we have a map

$$(A \xrightarrow{\eta_A} GF(A)) = \overline{(F(A) \xrightarrow{1} F(A))}.$$

Dually, for each $B \in \mathcal{B}$, we have a map

$$(FG(B) \xrightarrow{\varepsilon_B} B) = \overline{(G(B) \xrightarrow{1} G(B))}.$$

These define natural transformations

$$\eta : 1_{\mathcal{A}} \longrightarrow G \circ F, \qquad \varepsilon : F \circ G \longrightarrow 1_{\mathcal{B}},$$

called the *unit* and *counit* of the adjunction, respectively.

The following are some necessary definitions and results for Section 5.3. The reader can find more about this in [28, 32].

A set S of objects of the category $\mathcal{C}$ is said to *generate* $\mathcal{C}$ when to any parallel pair $h, h' : c \longrightarrow d$ of arrows of $\mathcal{C}$, $h \neq h'$ implies that there is an $s \in S$ and an arrow $f : s \longrightarrow c$ with $hf \neq h'f$. This definition includes the case of a single object s generating a category $\mathcal{C}$. For example, any one-point set generates $\mathcal{S}$et, the set of integers $\mathbb{Z}$ generates $\mathcal{AG}$ and $\mathcal{G}$rp, and each ring R generates $R - \mathcal{M}od$. The set of finite cyclic groups is a generator for the category of all finite abelian groups (or, of all torsion abelian groups).

Dually, a set Q of objects is a *cogenerating set* for the category $\mathcal{C}$ when to every parallel pair $h, h' : a \longrightarrow b$ of arrows of $\mathcal{C}$ there is an object $q \in Q$ and an arrow $g : b \longrightarrow q$ with $gh \neq gh'$. A single object q is a *cogenerator* when $\{q\}$ is a cogenerating set. For example, any two-point set is a cogenerator in $\mathcal{S}$et.

A category $\mathcal{C}$ is called *small-complete* (usually just *complete*) if all small diagrams in $\mathcal{C}$ have limits in $\mathcal{C}$; that is, if every functor $F : J \longrightarrow C$, where J is a small category, has a limit. We have that $\mathcal{S}$et, $\mathcal{G}$rp, $\mathcal{AG}$, and many other categories of algebras are small-complete.

Let $\mathcal{C}$ be any category. If $f : B \longrightarrow A$ and $g : C \longrightarrow A$ are two monics with a common codomain A, write $f \leq g$ when f factors through g; that is, when $f = gh$ for some arrow h (which is then necessarily also monic). When both $f \leq g$ and $g \leq f$, write $f \equiv g$; this defines an equivalence relation $\equiv$ among the monics with codomain A, and the corresponding equivalence classes of these monics are called the *subobjects* of A. It is often convenient to say that a monic $f : C \longrightarrow A$ is a subobject of A, that is, to identify f with the equivalence class of all $g = fk$, for $k : C' \longrightarrow C$ an invertible arrow.

A category $\mathcal{C}$ is *well-powered* if for each object A in $\mathcal{C}$, the class of subobjects of A is small, that is, a set.

Now, we present a version of the *Special Adjoint Functor Theorem* which will be useful to ensure a type of Stone-Čech Compactification, for MV-topological spaces, in the Section 5.3.

Theorem 1.3.6 (Special Adjoint Functor Theorem (SAFT)). *If $\mathcal{C}$ is small-complete, well-powered, with small hom-sets, and a small cogenerating set, while $\mathcal{A}$ has small hom-sets, then a functor $G : \mathcal{C} \longrightarrow \mathcal{A}$ has a left adjoint if and only if it is continuous, i. e., preserves all small limits.*

There is a classic application of SAFT. Let $\mathcal{CHT}$op be the category of compact Hausdorff topological spaces, and $U : \mathcal{CHT}\text{op} \longrightarrow \mathcal{T}\text{op}$ the forgetful functor. By SAFT we have that U has a left adjoint β, turning any space into a compact Hausdorff space in a canonical way. To prove the existence of this left adjoint and to verify the hypotheses of SAFT, are necessary some deep theorems of topology. Given a space X, the resulting compact Hausdorff space $\beta(X)$ is called its Stone-Čech compactification.

2

MV-algebras

The MV-algebras were defined by Chang [5] as the algebraic counterpart of Łukasiewicz infinite-valued calculus. The following definitions and results about MV-algebras can be found in [8, 13, 35].

Definition 2.0.1. An *MV-algebra* is a structure $(A, \oplus, *, \mathbf{0})$ where $\oplus$ is a binary operation, $*$ is a unary operation and $\mathbf{0}$ is a constant such that the following axioms are satisfied for any $a, b \in A$:

MV1) $(A, \oplus, *, \mathbf{0})$ is an Abelian monoid,
MV2) $(a^*)^* = a$,
MV3) $\mathbf{0}^* \oplus a = \mathbf{0}^*$,
MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

We denote an MV-algebra $(A, \oplus, *, \mathbf{0})$ by its universe A. An MV-algebra is trivial if its support is a singleton.

On each MV-algebra A we define the constant $\mathbf{1}$ and two derived operations $\odot$ and $\ominus$ as follows:

- $\mathbf{1} := \mathbf{0}^*$
- $a \odot b := (a^* \oplus b^*)^*$
- $a \ominus b := a \odot b^*$

An MV-algebra is nontrivial if and only if $\mathbf{0} \neq \mathbf{1}$. The following identities are immediate consequences of the axioms in the Definition 2.0.1:

MV5) $\mathbf{1}^* = \mathbf{0}$,
MV6) $a \oplus b = (a^* \odot b^*)^*$,
MV7) $a \oplus a^* = \mathbf{1}$.

Axioms MV3) and MV4) can now be written as:

MV3') $a \oplus \mathbf{1} = \mathbf{1}$,
MV4') $(a \ominus b) \oplus b = (b \ominus a) \oplus a$

Remark 2.0.2. We consider the $*$ operation more binding than any other operation, and the $\odot$ operation more binding than $\oplus$ and $\ominus$.

The basic concepts related to MV-algebras, such as subalgebra, homomorphism, and congruence, are defined in the usual way, according to Universal Algebra. Moreover, it is clear from the definition that the class of MV-algebras is equationally defined, and therefore it forms a variety.

Let A be an MV-algebra, $a \in A$ and $n < \omega$, where ω denotes the set of all the natural numbers. We introduce the following notations:

$$0a = \mathbf{0}, na = a \oplus (n-1)a \text{ for any } n < \omega,$$

$$a^0 = \mathbf{1}, a^n = a \odot a^{n-1} \text{ for any } n < \omega.$$

We say that the element a has order n, and we write $ord(a) = n$, if n is the least natural number such that $na = 1$. We say that the element a has a *finite order*, and we write $ord(a) < \infty$, if a has order n for some $n < \omega$. If no such n exists, we say that a has *infinite order* and we write $ord(a) = \infty$.

2.1 The lattice structure of an MV-algebra

Proposition 2.1.1. *Let A an MV-algebra. For any $a, b \in A$ the following are equivalent:*

(i) $a^* \oplus b = 1$,
(ii) $a \ominus b = 0$,
(iii) $b = a \oplus b \odot a^*$,
(iv) there is $c \in A$ such that $b = a \oplus c$,
(v) there is $d \in A$ such that $a = b \odot d$.

Definition 2.1.2. Let A be an MV-algebra. We define a binary relation $\leq$ on A by $a \leq b$ iff a and b satisfy the above equivalent conditions (i)–(v). It is called the *natural order* of A.

An MV-algebra whose natural order is total is called an *MV-chain*.

Proposition 2.1.3. *The order in the Definition 2.1.2 is a partial order, moreover it is a lattice order and for each $x, y, z \in A$:*

(i) $x \vee y = (x \odot y^*) \oplus y = (x \ominus y) \oplus y$, *and*
(ii) $x \wedge y = (x^* \vee y^*)^* = x \odot (x^* \oplus y)$
(iii) $x \leq y$ *iff* $y^* \leq x^*$
(iv) $x \leq y$ *implies* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$
(v) $x \odot y \leq z$ *iff* $x \leq y^* \oplus z$
(vi) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$

Proposition 2.1.4. *For any $x \in A$ and for any family $\{y_i : i \in I\} \subseteq A$ the following properties hold whenever $\bigvee\{y_i : i \in I\}$ and $\bigwedge\{y_i : i \in I\}$ exist:*

(a) $x \odot \bigvee\{y_i : i \in I\} = \bigvee\{x \odot y_i : i \in I\}$,
(b) $x \wedge \bigvee\{y_i : i \in I\} = \bigvee\{x \wedge y_i : i \in I\}$,
(c) $x \oplus \bigvee\{y_i : i \in I\} = \bigvee\{x \oplus y_i : i \in I\}$,
(d) $x \odot \bigwedge\{y_i : i \in I\} = \bigwedge\{x \odot y_i : i \in I\}$,
(e) $x \vee \bigwedge\{y_i : i \in I\} = \bigwedge\{x \vee y_i : i \in I\}$,
(f) $x \oplus \bigwedge\{y_i : i \in I\} = \bigwedge\{x \oplus y_i : i \in I\}$.

And if $a, x_1, ..., x_n \in A$ for $n \geq 1$ then:

(g) $a \vee (x_1 \oplus ... \oplus x_n) \leq (a \vee x_1) \oplus ... \oplus (a \vee x_n)$,
(h) $a \wedge (x_1 \oplus ... \oplus x_n) \leq (a \wedge x_1) \oplus ... \oplus (a \wedge x_n)$,
(i) $a \vee (x_1 \odot ... \odot x_n) \geq (a \vee x_1) \odot ... \odot (a \vee x_n)$,
(j) $a \wedge (x_1 \odot ... \odot x_n) \geq (a \wedge x_1) \odot ... \odot (a \wedge x_n)$.

Some examples of MV-algebras interesting for this work, are the following:

(i) The unit interval $[0,1]$ with the operations given by: $a \oplus b = \min(1, a+b)$, $a^* = 1-a$, for each $a, b \in [0,1]$, and the constant $\mathbf{0}$ is the real number zero. Besides, we have that $a \odot b = \max(0, a+b-1)$, $a \ominus b = \max(0, a-b)$, $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$ and the order relation is the usual in $[0,1]$.
(ii) For each integer $n \geq 2$, the set $Ł_n = \{0, \frac{1}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$ is an MV-algebra with the same operations and constant of the last example. In fact, $Ł_n$ is a subalgebra of $[0,1]$.
(iii) $\mathbb{Q} \cap [0,1]$ is another subalgebra of $[0,1]$ with the induced operations.
(iv) For each nonempty set X, the set of functions $[0,1]^X$ is an MV-algebra with the operations induced by the operations in the unit interval $[0,1]$. That is, $(f \oplus g)(x) = f(x) \oplus g(x)$ and $f^*(x) = f(x)^*$ for each $f, g : X \longrightarrow [0,1]$, and the constant $\mathbf{0}$ is the constant function $\mathbf{0} : X \longrightarrow [0,1]$.

A particular inequality that holds for each MV-algebra A is the following: for all $a, b, c \in A$

$$a \odot (b \oplus c) \leq b \oplus (a \odot c). \tag{2.1}$$

2.2 Ideals and Homomorphisms

In this section we shall define ideals of an MV-algebra and we will see, among other things, that every ideal of an algebra A is the zero-class of a unique congruence on A, as in the case of rings and Boolean algebras.

First of all, we observe that, if $h : A \longrightarrow B$ is an MV-homomorphism, then for all $a, b \in A$ we have:

1. $h(\mathbf{1}) = \mathbf{1}$
2. $h(a \odot b) = h(a) \odot h(b)$
3. $h(a \wedge b) = h(a) \wedge h(b)$
4. $h(a \vee b) = h(a) \vee h(b)$

Note that h is also a lattices homomorphism from $L(A)$ to $L(B)$. In particular, h is an increasing function.

Definition 2.2.1. An *ideal* of an MV-algebra A is a nonempty subset I of A satisfying the following properties:

I1) $a \leq b$ and $b \in I$ implies $a \in I$,
I2) $a, b \in I$ implies $a \oplus b \in I$.

We denote by $\mathrm{Id}\, A$ the set of ideals of A.

An ideal is *proper* if it does not coincide with the entire algebra. The following are some natural consequences of the Definition 2.2.1:

- $\{\mathbf{0}\}$ and A are ideals,
- $\mathbf{0} \in I$ for any ideal I of A,

- An ideal I is proper if and only if $\mathbf{1} \notin I$,
- For each a, b elements of an ideal I, we have $a \odot b, a \wedge b, a \vee b, a \oplus b \in I$,
- Given a family $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals of A, the intersection $\bigcap_{\lambda \in \Lambda} I_\lambda$ is also an ideal of A.

Definition 2.2.2. In an MV-algebra A, the *distance function* $d : A \times A \longrightarrow A$ is defined by

$$d(a, b) = (a \odot b^*) \oplus (b \odot a^*).$$

Proposition 2.2.3. *For any $a, b, c, e \in A$ the following properties hold:*

d1) $d(a, b) = (a \odot b^*) \vee (b \odot a^*)$,
d2) $d(a, b) = \mathbf{0}$ *iff* $a = b$,
d3) $d(a, \mathbf{0}) = a$,
d4) $d(a, \mathbf{1}) = a^*$,
d5) $d(a^*, b^*) = d(a, b)$
d6) $d(a, b) = d(b, a)$,
d7) $d(a, c) \leq d(a, b) \oplus d(b, c)$,
d8) $d(a \oplus c, b \oplus e) \leq d(a, b) \oplus d(c, e)$,
d9) $d(a \odot c, b \odot e) \leq d(a, b) \oplus d(c, e)$.

In a Boolean algebra A the distance function is

$$d(a, b) = (a \wedge b^*) \vee (b \wedge a^*) = (a \leftrightarrow b)^*.$$

In the MV-algebras $[0, 1]$, $\mathbb{Q} \cap [0, 1]$ and $Ł_n$ the distance function is

$$d(a, b) = |a - b|$$

where $|\ |$ denotes the usual absolute value in $[0, 1]$.

Thanks to the distance function, we can establish the relationship between congruences and ideals of MV-algebras.

Lemma 2.2.4. *If I is an MV-ideal of A then the relation $\equiv_I$ defined by*

$$a \equiv_I b \text{ iff } d(a, b) \in I$$

is a congruence on A. Reciprocally, if $\equiv$ is a congruence on A then the set

$$I_\equiv = \{a \in A : a \equiv \mathbf{0}\}$$

is an MV-ideal, and it is the unique one such that $\equiv_{I_\equiv}$ is equal to $\equiv$.

Consequently, for any MV-algebra A and any ideal I of A, we shall denote by I both the ideal and the congruence $\equiv_I$.

Given an ideal I of an MV-algebra A, we shall always denote by $A/I = \{a/I : a \in A\}$ the quotient MV-algebra and by π_I the *canonical projection from* A to A/I. It is clear that $\ker(\pi_I) = I$. Therefore, in what follows, we shall call the *kernel* of an MV-homomorphism $h : A \longrightarrow B$ the set

$$\ker(h) = h^{-1}(\mathbf{0}) = \{a \in A : h(a) = \mathbf{0}\}.$$

An useful property is the following:

$$\frac{a}{I} = \{(a \oplus b) \odot c^* : b, c \in I\}. \tag{2.2}$$

For every subset $Z \subseteq A$, the intersection of all ideals $I \supseteq Z$ is said to be the ideal *generated* by Z, it will be denoted $\langle Z \rangle$ and we have the following lemma:

Lemma 2.2.5. *Let Z be a subset of an MV-algebra A. If $Z = \emptyset$ then $\langle Z \rangle = \{\mathbf{0}\}$. If $Z \neq \emptyset$, then*

$$\langle Z \rangle = \{x \in A : x \leq z_1 \oplus \cdots \oplus z_k, \text{ for some } z_1, \ldots, z_k \in Z\}.$$

In particular, for each element $z \in A$,

$$\langle z \rangle = \langle \{z\} \rangle = \{x \in A : x \leq nz \text{ for some integer } n \geq 0\}.$$

The ideal $\langle z \rangle$ is called the *principal ideal generated* by z. Note that $\langle \mathbf{0} \rangle = \{\mathbf{0}\}$ and $\langle \mathbf{1} \rangle = A$. Further, for every ideal J of an MV-algebra A and each $z \in A$ we have

$$\langle J \cup \{z\} \rangle = \{x \in A : x \leq nz \oplus a, \text{ for some } n \geq 0 \text{ and } a \in J\}.$$

In the sequel the MV-ideal generated by $I \cup J$ will be denoted by $I \oplus J$ and it is

$$I \oplus J = \{x \leq a \oplus b : \text{ for some } x \in I, y \in J\}.$$

Proposition 2.2.6. *[8] Let A be an MV-algebra and $S \subseteq A$. Then the ideal $(S]$ generated by S is proper if and only if, for any $n < \omega$ and for any $a_1, \ldots, a_n \in S$, $a_1 \oplus \cdots \oplus a_n < 1$.*

For an ideal I of an MV-algebra A, we have that the MV-subalgebra generated by I in A is $\langle I \rangle = I \cup I^*$ where $I^* = \{x^* : x \in I\}$.

Some relations between ideals and kernels of homomorphisms are summarize in the next lemma:

Lemma 2.2.7. *Let A, B be MV-algebras and $h : A \longrightarrow B$ a homomorphism. Then the following properties hold:*

(i) For each ideal J of B, the set $h^{-1}(J) = \{a \in A : h(a) \in J\}$ is an ideal of A. Thus in particular, $\ker(h)$ is an ideal of A. Besides $\ker(h) \subseteq h^{-1}(J)$.
(ii) $h(a) \leq h(b)$ iff $a \odot b^* \in \ker(h)$.
(iii) h is injective iff $\ker(h) = \mathbf{0}$.
(iv) $\ker(h) \neq A$ iff B is nontrivial.
(v) If h is surjective and $I \subseteq A$ is an ideal such that $\ker(h) \subseteq I$, then $h(I)$ is an ideal of B.

2.3 Prime ideals

The prime and the maximal spectra of MV-algebras play an important role in some results and concepts of this work. In this section we present definitions and results about them.

Proposition 2.3.1. *If P is an MV-ideal of A, then the following properties are equivalent:*

(a) For any $a, b \in A$, $a \odot b^* \in P$ or $b \odot a^* \in P$,
(b) For any $a, b \in A$, if $a \wedge b \in P$ then $a \in P$ or $b \in P$,
(c) For any $I, J \in \mathrm{Id}(A)$, if $I \cap J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

Definition 2.3.2. (i) An ideal of A is *prime* if it is proper and it satisfies one of the equivalent conditions of Proposition 2.3.1. We shall denote by $\operatorname{Spec} A$ the set of all the prime ideals of A.

(ii) An ideal M of an MV-algebra A is called *maximal* if it is proper and no proper ideal of A strictly contains M, i.e., for each ideal $J \neq M$, if $M \subseteq J$ then $J = A$. We shall denote by $\operatorname{Max} A$ the set of all the maximal ideals of A.

We have the following characterization for maximal ideals:

Proposition 2.3.3. *For any proper ideal M of an MV-algebra A the following conditions are equivalent:*

(i) M is a maximal ideal of A;
(ii) for each $a \in A$, $a \notin M$ iff $(na)^* \in M$ (or equivalently $(a^*)^n \in M$) for some integer $n \geq 1$.

Remark 2.3.4. If I and P are ideals of A such that $I \subseteq P$, then we have that

$$P \in \operatorname{Spec} A \text{ iff } \pi_I(P) \in \operatorname{Spec}(A/I).$$

And, If I and M are ideals of A such that $I \subseteq M$, we have that

$$M \in \operatorname{Max} A \text{ iff } \pi_I(M) \in \operatorname{Max}(A/I).$$

Thus, there is a bijective correspondence between the prime (maximal, resp.) ideals of A containing I and the prime (maximal, resp.) ideals of A/I.

Lemma 2.3.5. *Any maximal ideal of an MV-algebra is a prime ideal.*

Proposition 2.3.6. *Any proper ideal I of any MV-algebra A is contained in a maximal ideal M. If I is prime, M is unique.*

A special type of prime ideals which we will use are the following:

Definition 2.3.7. An ideal P of an MV-algebra A is called *primary* if P is proper and there is a unique maximal ideal containing it.

It is immediate of the Definition 2.3.7 and Proposition 2.3.6 that any prime ideal is a primary ideal.

Definition 2.3.8. The intersection of the maximal ideals of A is called the *radical* of A. It will be denoted by $\operatorname{Rad} A$.

It is easy to see that $\operatorname{Rad} A$ is an ideal, since an intersection of ideals is also an ideal.

Lemma 2.3.9. *For any $a, b \in \operatorname{Rad} A$, the following identities hold:*

(a) $a \odot b = \mathbf{0}$
(b) $a \leq b^*$

Corollary 2.3.10. $(\operatorname{Rad} A, \oplus, \mathbf{0})$ *is an ordered cancellative monoid.*

Definition 2.3.11. An element $a \in A$ is called *infinitesimal* if $a \neq \mathbf{0}$ and $na \leq a^*$ for any $n < \omega$.

Proposition 2.3.12. *For any $a \in A$, $a \neq \mathbf{0}$, the following are equivalent:*

(i) a is infinitesimal;
(ii) $a \in \operatorname{Rad} A$;

(iii) $(na)^2 = \mathbf{0}$ *for every* $n < \omega$.

Corollary 2.3.13. *If* B *is an MV-subalgebra of* A*, then* $\operatorname{Rad} B = \operatorname{Rad} A \cap B$.

Definition 2.3.14. For a nonempty subset $X \subseteq A$, the set

$$X^{\perp} = \{a \in A : a \wedge x = \mathbf{0} \text{ for any } x \in X\}$$

is called the *polar* or the *annihilator* of X. If $a \in A$ then the annihilator of $\{a\}$ will be simply denoted by $a^{\perp}$. Thus

$$a^{\perp} = \{b \in A : a \wedge b = \mathbf{0}\}.$$

Definition 2.3.15. An ideal m of A is a *minimal ideal* if it is a minimal element in $\operatorname{Spec} A$ ordered by inclusion. This means that m is a prime ideal and, whenever P is a prime ideal such that $P \subseteq m$, we get $m = P$. We shall denote by $\operatorname{Min} A$ the set of all the minimal prime ideals of A.

Lemma 2.3.16. *Let* A *be an MV-algebra and* $P \in \operatorname{Spec} A$*. Then the following are equivalent:*

(i) P *is a minimal ideal;*
(ii) For any $a \in A$*,* $a \in P$ *iff there is* $b \in A \setminus P$ *such that* $a \wedge b = \mathbf{0}$*;*
(iii) $P = \bigcup\{b^{\perp} : b \notin P\}$.

Let A be an MV-algebra and P a prime ideal of A. We set

$$O_P = \bigcap\{Q \in \operatorname{Spec} A : Q \subseteq P\}. \tag{2.3}$$

We note that O_P is an ideal of A and $O_P \subseteq P$. In the following, we show some properties of these ideals which will be useful in this work. The interested reader may refer to [17] for more information about such ideals.

Proposition 2.3.17. *[17] For each* $P \in \operatorname{Spec} A$*,* $O_P = \bigcup\{a^{\perp} : a \notin P\}$

Proposition 2.3.18. *For each* $P \in \operatorname{Spec} A$*, the ideal* O_P *is primary.*

Proof. We shall prove that there is a unique maximal ideal containing O_P. Let M_P be the unique maximal ideal such that $O_P \subseteq P \subseteq M_P$. Suppose there exists a maximal ideal $M \neq M_P$ and $O_P \subseteq M$. Then there exist $a \in M$ and $b \in M_P$ such that $a \oplus b = 1$, that is $a^* \odot b^* = \mathbf{0}$. Let $m \in \operatorname{Min} A$ with $m \subseteq P$, then $(a^*)^2 \in m$ or $(b^*)^2 \in m$. If $(b^*)^2 \in m$ then $b \oplus (b^*)^2 = b \vee b^* = \mathbf{1} \in M_P$, absurd. So $(a^*)^2 \in m$, for all $m \in \operatorname{Min} A$ with $m \subseteq P$, that is, $(a^*)^2 \in O_P \subseteq M$. Then $a \oplus (a^*)^2 = a \vee a^* = \mathbf{1} \in M$, again it is absurd. Hence O_P is primary. □

An important property of the ideals O_M when M is maximal is the following (see [18]):

Proposition 2.3.19. *For each MV-algebra* A *we have that*

$$\bigcap\{O_M : M \in \operatorname{Max} A\} = \{\mathbf{0}\} \tag{2.4}$$

So, each MV-algebra A can be seen as a subdirect product of the family $\{A/O_M\}_{M \in \operatorname{Max} A}$.

2.4 The Spectral Topology

In this section we recall a natural topology on the set of prime ideals, Spec A. We also display some properties of Spec A, Max A and Min A. Let $(A, \oplus, ^*, \mathbf{0})$ be an MV-algebra. For more about this topology, the reader may refer to [13].

For any ideal I of A we define

$$r(I) := \{P \in \operatorname{Spec} A : I \nsubseteq P\} \tag{2.5}$$

If we define $\tau := \{r(I) : I \in \operatorname{Id}(A)\}$, we have that $(\operatorname{Spec} A, \tau)$ is a topological space. Indeed,

(i) $r(\{\mathbf{0}\}) = \emptyset$,
(ii) $r(A) = \operatorname{Spec} A$,
(iii) $r(I \wedge J) = r(I) \cap r(J)$ for all $I, J \in \operatorname{Id}(A)$,
(iv) $r(\bigvee\{I_\lambda : \lambda \in \Lambda\}) = \bigcup\{r(I_\lambda) : \lambda \in \Lambda\}$ for any $\{I_\lambda : \lambda \in \Lambda\} \subseteq \operatorname{Id}(A)$.

In the sequel τ or $\mathcal{O}(\operatorname{Spec} A)$ will be referred as the *spectral topology* or the *Zariski topology*. Other properties of the sets $r(I)$ are the following:

- $I \subseteq J$ iff $r(I) \subseteq r(J)$ for any $I, J \in \operatorname{Id}(A)$,
- if $X \subseteq A$ then $\{P \in \operatorname{Spec} A : X \nsubseteq P\} = r((X])$.

For any $a \in A$ we define

$$r(a) := \{P \in \operatorname{Spec} A : a \notin P\} \tag{2.6}$$

and we have the following properties:

Lemma 2.4.1. *[13]*

(i) $r(a) = r((a])$ *for any* $a \in A$,
(ii) $r(\mathbf{0}) = \emptyset$,
(iii) $r(1) = \operatorname{Spec} A$,
(iv) $r(a \vee b) = r(a \oplus b) = r(a) \cup r(b)$ *for any* $a, b \in A$,
(v) $r(a \wedge b) = r(a) \cap r(b)$ *for any* $a, b \in A$,
(vi) $r(I) = \bigcup\{r(a) : a \in I\}$ *for any* $I \in \operatorname{Id}(A)$.

By properties (i) and (vi) in the last lemma we have that $\{r(a) : a \in A\}$ is a basis for the topology τ. We can also prove that the compact open subsets of Spec A are exactly the sets of the form $r(a)$ for some $a \in A$. In particular, Spec A is compact because $r(\mathbf{1}) = \operatorname{Spec} A$ (see [13]).

Let A be an MV-algebra. We have that, for each $a \in A$, the set

$$H(a) := \{P \in \operatorname{Spec} A : a \in O_P\}$$

is an open set of Spec A [17, Lemma 3.6].

Since $\operatorname{Max} A, \operatorname{Min} A \subseteq \operatorname{Spec} A$ we can endow Max A and Min A with the topology induced by the spectral topology τ on Spec A. This means that the open sets of Max A are

$$R(I) = r(I) \cap \operatorname{Max} A = \{M \in \operatorname{Max} A : I \nsubseteq M\}$$

So, for any $a \in A$ and $I \in \operatorname{Id} A$

$$R(a) = r(a) \cap \operatorname{Max} A = \{M \in \operatorname{Max} A : a \notin M\} \text{ and } R(I) = \bigcup\{R(a) : a \in I\}$$

Hence the family $\{R(a) : a \in A\}$ is a basis for the induced topology on $\operatorname{Max} A$. The set of opens in $\operatorname{Max} A$ will be denoted by $\mathcal{O}(\operatorname{Max} A)$.

By [13, Theorem 3.6.10], we have that for any MV-algebra A the maximal ideal space, $\operatorname{Max} A$, is a compact Hausdorff topological space with respect to the topology induced by the spectral topology on $\operatorname{Spec} A$.

Analogously, we have that the open sets of $\operatorname{Min} A$ are

$$d(I) = r(I) \cap \operatorname{Min} A = \{m \in \operatorname{Min} A : I \nsubseteq m\}.$$

We conclude the section recalling that also the *coZariski topology* on $\operatorname{Spec} A$ has been considered in the literature (see, for instance, Dubuc and Poveda [16]). Such a topology has the family $\{W_a : a \in A\}$ where $W_a = \{P \in \operatorname{Spec} A : a \in P\}$ as a basis. In particular, $W_{\mathbf{0}} = \operatorname{Spec} A$, $W_{\mathbf{1}} = \emptyset$ and $W_a \cap W_b = W_{a \oplus b}$.

2.5 Mundici's Functor

There is an interesting categorical equivalence between the category of MV-algebras $\mathcal{MV}$, and the category of Abelian ℓ-groups with strong unit $\mathcal{LAG}_u$, known as *Mundici's functor*. Let us see some preliminary definitions:

Definition 2.5.1. A *partially ordered abelian group* is an abelian group $(G, +, -, \mathbf{0})$ endowed with a partial order relation $\leq$ that is compatible with addition; i. e., $\leq$ satisfies for all $x, y, t \in G$:

$$\text{if } x \leq y \text{ then } t + x \leq t + y.$$

When the order relation is total, G is said to be a *totally ordered abelian group*, or *o-group* for short. When the order of G defines a lattice structure, G is called a *lattice-ordered abelian group*, or ℓ-group, for short. In any ℓ-group we have

$$t + (x \vee y) = (t + x) \vee (t + y) \text{ and } t + (x \wedge y) = (t + x) \wedge (t + y)$$

The positive cone G^+ of G is the set of all $x \in G$ such that $\mathbf{0} \leq x$. If $\leq$ is a total order, $G = G^+ \cup -G^+$. For each element x of an ℓ-group G, the *positive part* x^+ is defined by $x^+ := \mathbf{0} \vee x$, the *negative part* x^- by $x^- := \mathbf{0} \vee -x$, and the *absolute value* of x is defined as $|x| := x^+ + x^- = x \vee -x$.

A *strong (order) unit* u of G is an archimedean element of G, i.e., an element $\mathbf{0} \leq u \in G$ such that for each $x \in G$ there is an integer $n \geq 0$ with $|x| \leq nu$.

Let G and H be ℓ-groups. A function $h : G \longrightarrow H$ is said to be an *ℓ-group homomorphism* iff h is both a group and a lattice homomorphism; in other words, for each $x, y \in G$, $h(x - y) = h(x) - h(y)$, $h(x \vee y) = h(x) \vee h(y)$ and $h(x \wedge y) = h(x) \wedge h(y)$. Suppose that $\mathbf{0} \leq u \in G$ and $\mathbf{0} \leq v \in H$ are strong unities of G and H, respectively, and let $h : G \longrightarrow H$ be an ℓ-group homomorphism such that $h(u) = v$. Then h is said to be a *unital* ℓ-homomorphism.

For each ℓ-group G, we can construct an associated MV-algebra as follows. Given $u \in G$, $u \geq \mathbf{0}$ (no necessarily being a strong unit of G) we have the interval

$$[\mathbf{0}, u] = \{x \in G : \mathbf{0} \leq x \leq u\} \tag{2.7}$$

and define two operations on it. For each $x, y \in [\mathbf{0}, u]$,

$$x \oplus y := u \wedge (x + y) \text{ and } x^* := u - x.$$

We denote by $\Gamma(G, u)$ the structure $([\mathbf{0}, u], \oplus, ^*, \mathbf{0})$ and we have the following results:

Proposition 2.5.2. *[8]*

(1) $\Gamma(G,u)$ is an MV-algebra.
(2) Let G be an ℓ-group with strong unit u. Let $A = \Gamma(G,u)$.
(i) For all $a, b \in A$, $a + b = (a \oplus b) + (a \odot b)$;
(ii) For all $x_1, \dots x_n \in A$, $x_1 \oplus \cdots \oplus x_n = u \wedge (x_1 + \cdots + x_n)$;
(iii) The natural order of the MV-algebra A coincides with the order of $[0,u]$ inherited from G by restriction.

Let $\Gamma(h) := h \restriction [\mathbf{0}, u]$ be the restriction of h with domain $[0,u]$ and codomain $[0,v]$. Then $\Gamma(h)$ is an MV-algebra homomorphism from $\Gamma(G,u)$ into $\Gamma(H,v)$.

Let $\mathcal{LAG}_u$ denote the category whose objects are pairs (G,u) with G an ℓ-group and u a distinguished strong unit of G, and whose morphisms are unital ℓ-homomorphisms. Then Γ is a functor from $\mathcal{LAG}_u$ into the category $\mathcal{MV}$ of MV-algebras. Actually, Γ is a categorical equivalence (i.e., a full, faithful, dense functor) between $\mathcal{LAG}_u$ and $\mathcal{MV}$ (For more, see [8, 35]).

2.6 Semisimple Algebras and Belluce-Chang Representation

In this section we recall a well-known representation theorem for semisimple MV-algebras. Before that, let us see some definitions and results.

Definition 2.6.1. An MV-algebra A is called *simple* if its only proper ideal is $\{\mathbf{0}\}$. A is called *semisimple* if it is a subdirect product of simple MV-algebras.

We have that an MV-algebra A is simple if and only if it is isomorphic to a subalgebra of $[0,1]$, and that A is semisimple if and only if $\operatorname{Rad} A = \{\mathbf{0}\}$. *Archimedean* MV-algebras are MV-algebras without infinitesimals. Hence, the semisimple and Archimedean MV-algebras coincide (see [8, 13]).

We have the following representation theorem for semisimple algebras.

Theorem 2.6.2. *[3, 5, 6] For any set X, the MV-algebra $[0,1]^X$ and all of its subalgebras are semisimple. Moreover, up to isomorphisms, all the semisimple MV-algebras are of this type. More precisely, every semisimple MV-algebra can be embedded in the MV-algebra of fuzzy subsets*[1] *$[0,1]^{\operatorname{Max} A}$ of the maximal spectrum of A.*

The following is a sketch of the proof:

For any maximal ideal M the quotient algebra A/M is a simple MV-algebra and, therefore, an Archimedean MV-chain. Then A/M is isomorphic to a subalgebra of $[0,1]$ and we have this situation:

- for each $M \in \operatorname{Max} A$, there is the natural projection $\pi_M : A \longrightarrow A/M$;
- for each $M \in \operatorname{Max} A$, there exists a unique embedding $\iota_M : A/M \longrightarrow [0,1]$;
- the embedding $\iota : A \longrightarrow [0,1]^{\operatorname{Max} A}$ associates, to each $a \in A$, the fuzzy subset $\widehat{a}$ of $\operatorname{Max} A$ defined by $\widehat{a}(M) = \iota_M(\pi_M(a)) = \iota_M(a/M)$ for all $M \in \operatorname{Max} A$.

The above construction is possible for any MV-algebra A with the only difference that the homomorphism ι is not injective if A is not semisimple for the simple reason that $\ker \iota$ always coincides with $\operatorname{Rad} A$.

[1] The term *fuzzy subset* will be explained in Chapter 3.

2.7 Local MV-algebras

The theory of local MV-algebras play an important role in the sheaf representation of a particular class of MV-algebras that we will present in Chapter 6. In the following, we give some definitions and results about local MV-algebras.

Definition 2.7.1. An MV-algebra A is called *local* if it has only one maximal ideal.

In particular, it follows that, if A is local, then $\operatorname{Rad} A$ is the only maximal ideal of A. The following result is a characterisation of local MV-algebras:

Proposition 2.7.2. *For any MV-algebra A, the following are equivalent:*

(i) For any $a \in A$, $ord(a) < \infty$ or $ord(a^*) < \infty$
(ii) for any $a, b \in A$, $a \odot b = \mathbf{0}$ implies $a^n = \mathbf{0}$ or $b^n = \mathbf{0}$ for some $n < \omega$;
(iii) for any $a, b \in A$, $ord(a \oplus b) < \infty$ implies $ord(a) < \infty$ or $ord(b) < \infty$;
(iv) A has only one maximal ideal.

The following propositions show some relations between local MV-algebras and primary ideals:

Proposition 2.7.3. *For an MV-algebra A and a proper ideal $P \subseteq A$, the following are equivalent:*

(i) P is a primary ideal,
(ii) A/P is a local MV-algebra,
(iii) $a \odot b \in P$ implies $a^n \in P$ or $b^n \in P$ for some $n < \omega$,
(iv) for any $a \in A$ there exists some $n < \omega$ such that $a^n \in P$ or $(a^*)^n \in P$.

Proposition 2.7.4. *For an MV-algebra A, the following are equivalent:*

(i) A is local,
(ii) any proper ideal of A is primary,
(iii) $\{\mathbf{0}\}$ is a primary ideal,
(iv) $\operatorname{Rad} A$ contains a primary ideal.

Corollary 2.7.5. *The homomorphic image of a local MV-algebra is local.*

A type of local MV-algebra is perfect MV-algebras.

Definition 2.7.6. An MV-algebra A is called *perfect* if for any $a \in A$, $ord(a) = \infty$ iff $ord(a^*) < \infty$.

Also, we have the following characterization [13].

Proposition 2.7.7. *Let A be an MV-algebra. The following are equivalent:*

(i) A is perfect,
(ii) $A = \langle \operatorname{Rad} A \rangle = \operatorname{Rad} A \cup (\operatorname{Rad} A)^*$,
(iii) $A/\operatorname{Rad} A \simeq Ł_2$.

In the following, we display how to construct a perfect MV-algebra from an Abelian group [13].

Remark 2.7.8. Let $\mathbb{Z}$ be the Abelian ℓ-group of integers and G an Abelian ℓ-group. Note that $(1,0)$ is a strong unit of the lexicographic product $\mathbb{Z} \times_{lex} G$ and let

$$[(0,0),(1,0)] = \{(0,g) : g \in G, g \geq 0\} \cup \{(1,g) : g \in G, g \leq 0\}.$$

Set $\Delta(G) = [(0,0),(1,0)]_{\mathbb{Z}\times_{lex}G}$ the interval MV-algebra described in (2.7), then

$$(0,g)^* = (1,-g) \text{ for any } g \geq 0 \text{ and } (1,g)^* = (0,-g) \text{ for any } g \leq 0.$$

We have that $\Delta(G)$ is a perfect MV-algebra because $ord((0,g)) = \infty$ for every $g \geq 0$ and $ord((1,g)) = 2$ for every $g \leq 0$. Besides,

$$\mathrm{Rad}(\Delta(G)) = \{(0,g) : g \in G, g \geq 0\}.$$

Now, denote by $\mathcal{PERF}$ the full subcategory of $\mathcal{MV}$ whose objects are perfect MV-algebras and morphisms are MV-algebra homomorphisms. Denote by $\mathcal{LAG}$ the category whose objects are Abelian ℓ-groups and whose morphisms are ℓ-group homomorphism. Let us consider the functor

$$\Delta : \mathcal{LAG} \longrightarrow \mathcal{PERF}$$

defined as follows.

For an Abelian ℓ-group G, $\Delta(G)$ is the perfect MV-algebra from Remark 2.7.8; if $h : G \longrightarrow H$ is an ℓ-group homomorphism, $\Delta(h) : (k,g) \in \Delta(G) \longmapsto (k,h(g)) \in \Delta(H)$.

For any perfect MV-algebra A there exists an Abelian ℓ-group G such that A and $\Delta(G)$ are isomorphic MV-algebras. Moreover, Δ is a categorical equivalence whose inverse functor is $\mathcal{D} : \mathcal{PERF} \longrightarrow \mathcal{LAG}$, defined as follow.

Let A be a perfect MV-algebra. We know, by Corollary 2.3.10, that $(\mathrm{Rad}\, A, \oplus, \mathbf{0})$ is a lattice ordered cancellative monoid. We construct the generated group from $\mathrm{Rad}\, A$ in the canonical way. So, let

$$\mathcal{D}(A) := Rad(A) \times Rad(A)/\sim$$

where $\sim$ is the equivalence relation given by

$$(a,b) \sim (c,d) \text{ iff } a + d = b + c.$$

If $[a,b]$ and $[c,d]$ are elements of $\mathcal{D}(A)$, the group operations on $\mathcal{D}(A)$ are given by:

$$[a,b] + [c,d] = [a+c, b+d] \text{ and } -[a,b] = [b,a].$$

$\mathrm{Rad}\, A$ can be identified with the set $\{[a,\mathbf{0}] : a \in \mathrm{Rad}\, A\}$. The order of $\mathcal{D}(A)$ is defined by

$$[a,b] \leq [c,d] \text{ iff } [c,d] - [a,b] \in \mathrm{Rad}\, A \text{ and } -[a,b] + [c,d] \in \mathrm{Rad}\, A.$$

Hence $(\mathcal{D}(A), +, [0,0], \leq)$ is an Abelian ℓ-group and $\mathrm{Rad}\, A$ is isomorphic with the positive cone of $\mathcal{D}(A)$, $\mathcal{D}(A)^+$.

If A and B are perfect MV-algebras and $f : A \longrightarrow B$ is an MV-algebra homomorphism, we have the ℓ-group homomorphism

$$\mathcal{D}(f) : [a,b] \in \mathcal{D}(A) \longmapsto [f(a), f(b)] \in \mathcal{D}(B).$$

Lemma 2.7.9. *[13]*

1) If A is a perfect MV-algebra, then A and $\Delta(\mathcal{D}(A))$ are isomorphic MV-algebras.

2) If G is an Abelian ℓ-group, then G and $\mathcal{D}(\Delta(G))$ are isomorphic ℓ-groups.

Theorem 2.7.10. *[13] The functors Δ and $\mathcal{D}$ establish a categorical equivalence between the category $\mathcal{PERF}$ of perfect MV-algebras and the category $\mathcal{LAG}$ of Abelian ℓ-groups.*

Proof. It is a direct consequence of the Lemma 2.7.9. □

The equivalence above is due to Di Nola and Lettieri [11]. In particular, the functor $\Delta : \mathcal{LAG} \longrightarrow \mathcal{PERF}$ is usually referred to as *Di Nola–Lettieri functor*.

2.8 Lexicographic MV-algebras

Lexicographic MV-algebras were introduced by Diaconescu, Flaminio, and Leustean [10]. They proved that lexicographic MV-algebras are the counterpart of unital Abelian lattice-ordered groups defined via lexicographic products. Besides, they extended Di Nola–Lettieri equivalence to lexicographic MV-algebras.

First, we recall that an ideal I of an MV-algebra A is *retractive* if the canonical projection $\pi_I : A \longrightarrow A/I$ is retractive, i.e., there is a morphism $\delta_I : A/I \longrightarrow A$ such that $\pi_I \circ \delta_I = \mathrm{id}_{A/I}$. If an ideal I of A is retractive, then A/I is isomorphic with a subalgebra of A.

Di Nola and Lettieri also obtained the following representation in terms of lexicographic products for local MV-algebras with retractive radical.

Theorem 2.8.1. [10] *If A is an MV-algebra the following are equivalent:*

1) A is a local MV-algebra with retractive radical,
2) there exists an ℓu-subgroup $(R', 1)$ of $(\mathbb{R}, 1)$ and an ℓ-group G such that

$$A \simeq \Gamma(R' \times_{lex} G, (1, 0)).$$

The following definitions and results can be found in [10].

An ideal I of an MV-algebra A is called *strict* if

$$\frac{a}{I} = \frac{b}{I} \text{ implies } a < b \text{ for any } a, b \in A.$$

Proposition 2.8.2. *In any local MV-algebra A,* $\mathrm{Rad}\, A$ *is strict.*

An ideal I of an MV-algebra A is called *lexicographic* if the following hold:

l1) $I \neq \{\mathbf{0}\}$,
l2) I is strict
l3) I is retractive,
l4) I is prime,
l5) $\rho \leq x \leq \rho^*$, for any $\rho \in I$ and any $x \in A \setminus \langle I \rangle$.

The set of all lexicographic ideals of A is denoted by $Lex\,\mathrm{Id}(A)$.

Definition 2.8.3. An MV-algebra A is called *lexicographic* if $Lex\,\mathrm{Id}(A) \neq \emptyset$.

Theorem 2.8.4. *The following are equivalent:*

1) A is a lexicographic MV-algebra,

2) there exists an ou-group (H, u) and a non-trivial ℓ-group G such that

$$A \simeq \Gamma(H \times_{lex} G, (u, 0)).$$

Moreover $A/I \simeq \Gamma(H, u)$ and $\langle I \rangle \simeq \Delta(G)$.

This theorem says that the class of lexicographic MV-algebras is the widest class of MV-algebras that can be represented via the Mundici's functor, Γ, as lexicographic products between *ou*-groups and non-trivial ℓ-groups, with strong unit of the form $(u, 0)$.

Corollary 2.8.5. *If A is a lexicographic MV-algebra the following are equivalent:*

(1) $\operatorname{Rad} A \in Lex\operatorname{Id}(A)$,
(2) there exists an ℓu-subgroup $(R', 1)$ of $(\mathbb{R}, 1)$ and a non-trivial ℓ-group G such that

$$A \simeq \Gamma(R' \times_{lex} G, (1, 0)).$$

Moreover, if the above equivalent conditions are satisfied the ℓu-subgroup $(R', 1)$ of $(\mathbb{R}, 1)$ and the ℓ-group G are uniquely determined, up to isomorphism.

In [10], the authors also showed the following results which give an interesting classification of some classes of MV-algebras:

Theorem 2.8.6.
The class of lexicographic MV-algebras is strictly included in the class of local MV-algebras.

Theorem 2.8.7.
The class of local MV-algebras with retractive radical is strictly included in the class of lexicographic MV-algebras.

Proof. By Corollary 2.8.5 and Proposition 2.8.1. An example that the inclusion is strict can be found in the proof of [10, Theorem 4.7]. □

Therefore we have the following inclusions of classes of MV-algebras:

$$\text{Perfect} \subset \text{Local with retractive radical} \subset \text{Lexicographic} \subset \text{Local}.$$

2.9 Filipoiu–Georgescu sheaf representation

In this section, we recall the sheaf representation for MV-algebras obtained by Filipoiu and Georgescu in [18]. This representation is a particular case of the representation of subdirect products by sheaf spaces for universal algebras which can be found, for example, in [9]. All the results presented here are widely considered in [18].

Let A be an MV-algebra. By proposition 2.3.19 we have that $\bigcap\{O_M : M \in \operatorname{Max} A\} = \{\mathbf{0}\}$, so it is possible to represent A as a subdirect product of the family $\{A/O_M : M \in \operatorname{Max} A\}$. Since for each $M \in \operatorname{Max} A$, O_M is a primary ideal (by Proposition 2.3.18), we have that the corresponding quotient A/O_M is a local MV-algebra.

Using methods of [9], the authors Filipoiu and Georgescu define a sheaf space of MV-algebras $(E_A, \pi, \operatorname{Max} A)$ where

$$E_A = \{(\frac{a}{O_M}, M) : a \in A, M \in \operatorname{Max} A\},$$

$\pi : E_A \longrightarrow \operatorname{Max} A$ is the natural projection and $\operatorname{Max} A$ is endowed with Zariski topology. They showed that the sheaf space $(E_A, \pi, \operatorname{Max} A)$ of MV-algebras is such that for each $M \in \operatorname{Max} A$ the stalk E_M is isomorphic to the quotient A/O_M.

For each $a \in A$, the associated global section will be

$$\begin{array}{rcl} \tilde{a} : \operatorname{Max} A & \longrightarrow & E_A \\ M & \longmapsto & (\frac{a}{O_M}, M) \end{array}$$

Moreover, the following hold:

- The family $\{\tilde{a}(U) : a \in A, U \text{ is open in } \operatorname{Max} A\}$ provides a basis for the topology on E_A.
- We recall that the set of global sections[2] is denoted by $\Gamma(\operatorname{Max} A, E_A)$ and it is:

$$\Gamma(\operatorname{Max} A, E_A) = \{\sigma : \operatorname{Max} A \longrightarrow E_A \mid \sigma \text{ is continuous and } \pi \circ \sigma = \mathrm{id}_{\operatorname{Max} A}\}.$$

- The map $a \in A \longmapsto \tilde{a} \in \Gamma(\operatorname{Max} A, E_A)$ is a monomorphism of MV-algebras.
- For each $\sigma \in \Gamma(\operatorname{Max} A, E_A)$ there is an element $a \in A$ such that $\sigma = \tilde{a}$.

So, from the previous facts we have the following theorem

Theorem 2.9.1. *Every MV-algebra A is isomorphic to the MV-algebra of all global sections of a sheaf with local MV-algebras as stalks and* $\operatorname{Max} A$ *as base space.*

2.10 Riesz MV-algebras

In this section, we recall the definition of Riesz MV-algebras along with some properties of such algebras which can be consulted in [14] and [15]. Riesz MV-algebras are MV-algebras endowed with a scalar multiplication with scalars from the interval $[0, i1]$. Extending Mundici's categorical equivalence, it is possible to prove that the category of Riesz MV-algebras with MV-algebra homomorphisms is equivalent to the one of Riesz spaces with strong unit with unit preserving Riesz homomorphisms.

Definition 2.10.1. A *Riesz MV-algebra* is a structure $(R, \cdot, \oplus, ^*, \mathbf{0})$ where $(R, \oplus, ^*, \mathbf{0})$ is an MV-algebra and the operation $\cdot : [0,1] \times R \to R$ satisfies the following identities for any $x, y \in R$ and $r, q \in [0,1]$:

(RMV1) $r \cdot (x \odot y^*) = (r \cdot x) \odot (r \cdot y)^*$,
(RMV2) $(r \odot q^*) \cdot x = (r \cdot x) \odot (q \cdot x)^*$,
(RMV3) $r \cdot (q \cdot x) = (rq) \cdot x$,
(RMV4) $1 \cdot x = x$.

In [15, Theorem 2], the authors proved the following equivalence of the Definition 4.6.1:

Theorem 2.10.2. *Let $(R, \oplus, ^*, \mathbf{0})$ be an MV-algebra and $\cdot : [0,1] \times R \to R$ be an extern operation. Then $(R, \cdot, \oplus, ^*, \mathbf{0})$ is a Riesz MV-algebra if and only if the following properties are satisfied for any $a, b \in R$ and $r, q \in [0,1]$:*

RMV1') $r \cdot a \odot r \cdot b = \mathbf{0}$ and $r(a \oplus b) = (r \cdot a) \oplus (r \cdot b)$ whenever $a \odot b = \mathbf{0}$,

[2] We use the usual classical notation Γ, for the set of global sections. However, we warn the reader to pay attention in order to avoid confusion with Mundici's functor, which is also denoted by Γ.

RMV2') $r \cdot a \odot q \cdot a = \mathbf{0}$ *and* $(r \oplus q) \cdot a = (r \cdot a) \oplus (q \cdot a)$ *whenever* $r \odot q = \mathbf{0}$,
RMV3) $(r \cdot q) \cdot a = r \cdot (q \cdot a)$,
RMV4) $1 \cdot a = a$.

We write rx instead of $r \cdot x$ for $r \in [0,1]$ and $x \in R$. Note that rq is the real product for any $r, q \in [0,1]$.

Example 2.10.3. The following sets are examples of Riesz MV-algebras:

(1) The MV-algebra interval $[0,1]$ with the scalar multiplication being the usual multiplication of real numbers.
(2) Given the MV-algebra interval $[0,1]$, let

$$\mathcal{C}(X) = \{f : X \longrightarrow [0,1] \mid f \text{ is continuous}\}$$

with all the operations defined componentwise, where X is a compact Hausdorff space and the scalar multiplication is the natural.
(3) Let $R = \Gamma(\mathbb{R} \times_{lex} G, (1,0))$ where G is an Abelian ℓ-group, $\mathbb{R} \times_{lex} G$ is the lexicographic product of ℓ-groups and the scalar multiplication is defined by $r(q,x) = (rq, x)$ for any $r \in [0,1]$ and $(q,x) \in R$.

Lemma 2.10.4. *In any Riesz MV-algebra R, the following properties hold for any $r, q \in [0,1]$ and $x, y \in R$:*

(a) $0x = \mathbf{0}, r\mathbf{0} = \mathbf{0}$,
(b) $x \leq y$ *implies* $rx \leq ry$,
(c) $r \leq q$ *implies* $rx \leq qx$,
(d) $rx \leq x$.

Let $(R, \cdot, \oplus, ^*, \mathbf{0})$ be a Riesz MV-algebra. We denote by $U(R) = (R, \oplus, ^*, \mathbf{0})$ its MV-algebra reduct. If I is an ideal of $U(R)$ then, by Lemma 2.10.4 we have that $rx \in I$ whenever $r \in [0,1]$. Actually, the relation $\equiv_I$ is a congruence in R and so the quotient R/I has a canonical structure of Riesz MV-algebra. In fact, a Riesz MV-algebra R has the same theory of ideals (congruences) as its reduct $U(R)$.

3

Fuzzy Topologies

The concept of *fuzzy topology* was introduced a few years after Zadeh's famous paper on fuzzy sets [50]. It was in 1968 that C. L. Chang [7], introduced the notion of *fuzzy topological space* and made an attempt to develop basic topological notions for such spaces. The study of fuzzy topology has been pursued for many years (see, for instance, [21–23, 29, 34, 40, 41, 45, 46]). In defining a fuzzy topological space on a set X a fundamental role is played by the structure used to represent the "fuzzy powerset" of X, i.e., the fuzzy version of the Boolean algebra $\mathbf{2}^X$. According to the original definition of fuzzy set, one may find natural to consider $[0,1]^X$ as the fuzzy powerset of X. Most of the authors in this area approached fuzzy topology using either arbitrary lattice-valued fuzzy subsets or $[0,1]^X$ with its natural lattice structure. However, fuzzy topological spaces using $[0,1]^X$ equipped with a richer algebraic structure (e. g., continuous or left-continuous t-norms [20]) have been considered in the literature.

3.1 Fuzzy sets

In the following we present some basic definitions which can be found in [36, 37, 44, 48].

Definition 3.1.1. Let X be a set. A *fuzzy (sub)set* of X is a map $\alpha : X \longrightarrow [0,1]$. In this setting $\alpha(x)$ is interpreted as the degree of membership of a point $x \in X$ in a fuzzy set α, while an ordinary(or crisp) subset $A \subseteq X$ is identified with its characteristic function $\chi_A : X \longrightarrow \mathbf{2}$.

Operations on fuzzy sets

Let $\Gamma = \{\alpha_i : i \in I\}$ be a family of fuzzy sets in X. By the *union* and *intersection* of this family we mean respectively its supremum $\bigvee \Gamma := \bigvee\{\alpha_i : i \in I\}$ and infimum $\bigwedge \Gamma := \bigwedge\{\alpha_i : i \in I\}$. The *complement* of α, denoted by α^*, is defined by $\alpha^*(x) = 1 - \alpha(x)$.

3.2 Types of Fuzzy Topological Spaces

The first definition of fuzzy topology was given by Chang [7] and he gave the following definition:

Definition 3.2.1. [7] Let $\tau \subseteq [0,1]^X$. The pair (X, τ) is called a *fuzzy topological space* or *fts*, for short, iff:

(τ_1) $\mathbf{0}, \mathbf{1} \in \tau$, where $\mathbf{0}$ and $\mathbf{1}$ are the constant functions, namely, $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in X$.
(τ_2) $\alpha \wedge \beta \in \tau$ whenever $\alpha, \beta \in \tau$; and
(τ_3) $\bigvee_{i \in I} \alpha_i \in \tau$ for any family $\{\alpha_i\}_{i \in I}$ of elements of τ.

Every member of τ is called a τ-*open fuzzy set* (or simply open fuzzy set). The complement of a τ-open fuzzy set is called a τ-*closed fuzzy set* (or simply closed fuzzy set).

In 1976 Lowen gave a definition of fuzzy topological space which includes Chang's conditions and adds all the constant functions to the family of open fuzzy sets [29].

Definition 3.2.2. [29] The pair (X, τ) is called a *laminated fuzzy topological space* iff:

(τ_1) For all $a \in [0, 1]$, the a-valued constant function $\boldsymbol{a}$ is an element of τ;
(τ_2) $\alpha \wedge \beta \in \tau$ whenever $\alpha, \beta \in \tau$; and
(τ_3) $\bigvee_{i \in I} \alpha_i \in \tau$ for any family $\{\alpha_i\}_{i \in I}$ of elements of τ.

Given a fuzzy topological space (X, τ) (or a laminated fts), the set τ is also called a *fuzzy topology* on X and the elements of τ are the *open subsets* of X. The set $\tau^* = \{\alpha^* \mid \alpha \in \tau\}$ verifies the following properties:

– $\mathbf{0}, \mathbf{1} \in \tau^*$,
– for any family $\{c_i\}_{i \in I}$ of elements of τ^*, $\bigwedge_{i \in I} c_i \in \tau^*$,
– for all $\beta_1, \beta_2 \in \tau^*$, $\beta_1 \vee \beta_2 \in \tau^*$.

The elements of τ^* are called the *closed subsets* of X.

3.3 Base and Continuous Function

The concepts defined in this section hold both for fuzzy topological spaces than laminated spaces. However, we will only refer to fts. First, let us see the definitions of image and inverse image of fuzzy sets. Let X and Y be sets.

Any function $f : X \longrightarrow Y$ naturally defines a map

$$\begin{array}{rcl} f^{\leftsquigarrow} : [0,1]^Y & \longrightarrow & [0,1]^X \\ \alpha & \longmapsto & \alpha \circ f. \end{array} \tag{3.1}$$

This map $f^{\leftsquigarrow}$ is called the *preimage*, via f, of the fuzzy subsets of Y. Moreover, for any map $f : X \longrightarrow Y$ we define also a map $f^{\rightarrow} : [0,1]^X \longrightarrow [0,1]^Y$ by setting, for all $\alpha \in [0,1]^X$ and for all $y \in Y$,

$$f^{\rightarrow}(\alpha)(y) = \bigvee_{f(x)=y} \alpha(x). \tag{3.2}$$

Clearly, if $y \notin f[X]$, $f^{\rightarrow}(\alpha)(y) = \bigvee \emptyset = \mathbf{0}$ for any $\alpha \in [0,1]^X$. The map $f^{\rightarrow}$ is called the *image*, via f, of the fuzzy subsets of X.

Definition 3.3.1. [7] Let (X, τ_X) and (Y, τ_Y) be two fuzzy topological spaces. A map $f : X \longrightarrow Y$ is said to be

- *continuous* if $f^{\leftsquigarrow}[\tau_Y] \subseteq \tau_X$,
- *open* if $f^{\rightarrow}(\alpha) \in \tau_Y$ for all $\alpha \in \tau_X$,
- *closed* if $f^{\rightarrow}(\beta) \in \tau_Y^*$ for all $\beta \in \tau_X^*$
- an *homeomorphism* if it is bijective and both f and f^{-1} are continuous.

We use the same words of the classical case because it is trivial to verify that classical topological spaces are fuzzy topological spaces in the sense of Chang and a map between two classical topological spaces is continuous, open, or closed in the sense of the definition above if and only if it has the same property in the classical sense.

It is easily seen that with the definition 3.2.1, constant functions between fuzzy topological spaces are not necessarily continuous, while constant maps of laminated spaces are continuous.

Definition 3.3.2. Given a fuzzy topological space (X, τ), a subset B of τ is called

(i) a *base* for τ if every open set of τ is a join of elements of B;
(ii) a *subbase* for τ if the collection of infimum of finite subfamilies of B forms a base for (X, τ).

Fuzzy topological spaces, with continuous maps, form a category which we denote by $\mathcal{F}$uz. In addition, the category of laminated fuzzy topological spaces and their continuous maps (where Lowen continuity is understood precisely as in 3.3.1) is denoted by $\mathcal{LF}$uz.

3.4 Some relevant functors between fuzzy and classical topologies

In [29], Lowen defined the functors $\iota : \mathcal{F}\text{uz} \longrightarrow \mathcal{T}\text{op}$ and $\omega : \mathcal{T}\text{op} \longrightarrow \mathcal{LF}\text{uz}$ as follows.

1. $\iota(X, \tau) = (X, \iota(\tau))$ where $\iota(\tau)$ is the initial topology on X determined by τ and the lower limit topology on $[0, 1]$, that is, $\iota(\tau)$ is the topology generated by the subbase
$$B = \{\mu^{-1}[(r, 1]] : \mu \in \tau, r \in [0, 1)\} = \{\{x \in X : \mu(x) > r\}\}_{\mu\in\tau, r\in[0,1)}.$$
It is easy to verify that if a map $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is continuous, then the map $f : (X, \iota(\tau_X)) \longrightarrow (Y, \iota(\tau_Y))$ is continuous.
2. $\omega(X, \tau) = (X, \omega(\tau))$, with
$$\omega(\tau) = \bigcup_{r\in[0,1)} \mathcal{C}(X, I_r) = \bigcup_{r\in[0,1)} \{f : X \longrightarrow I_r \mid f \text{ is continuous}\},$$
where $I_r = (r, 1]$. Note that $\omega(\tau)$ is the set of all lower semicontinuous functions[1] from (X, τ) to the interval $[0, 1]$ equipped with the usual topology. It can be verified that the continuity of a map $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ implies the continuity of the map $f : (X, \omega(\tau_X)) \longrightarrow (Y, \omega(\tau_Y))$. Thus, ω is a functor.

A fuzzy space (X, δ) whose fuzzy topology is of the form $\delta = \omega(\tau)$ for an ordinary topology τ on X is called *topologically generated* [29] or *induced*.

It is possible to define two further functors, $e : \mathcal{T}\text{op} \longrightarrow \mathcal{F}\text{uz}$ and $j : \mathcal{F}\text{uz} \longrightarrow \mathcal{T}\text{op}$ [33], which are useful in the development of Fuzzy Topology, in the following way.

3. $e(X, \tau) = (X, e(\tau))$, where
$$e(\tau) = \{\chi_U : U \in \tau\}.$$
Since the continuity of a map $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ guarantees the continuity of the map $f : (X, e(\tau_X)) \longrightarrow (Y, e(\tau_Y))$, e is a functor.

[1] Recall that, given a topological space (X, τ), a function $f : X \longrightarrow \mathbb{R}$ is *lower-semicontinuous* $(l.s.c)$ if, for all $t \in \mathbb{R}$, the set $\{x \in X : f(x) > t\}$ is an open set.

4. $j(X,\tau) = (X, j(\tau))$, where
$$j(\tau) = \tau \cap 2^X.$$
Again, the continuity of a map $f : (X,\tau_X) \longrightarrow (Y,\tau_Y)$ guarantees the continuity of $f : (X, j(\tau_X)) \longrightarrow (Y, j(\tau_Y))$, thus j is a functor.

Note that $j(\tau)$ is the greatest topology contained in τ.

Definition 3.4.1. [33] A fuzzy space (X,τ) is said to be *weakly induced* if for each $t \in [0,1)$ and for each $\alpha \in \tau$, the characteristic function of $\alpha_t := \{x \in X : \alpha(x) > t\}$ belongs to τ.

Note that a fuzzy space (X,τ) is weakly induced if and only if $\iota(\tau) = j(\tau)$, and it is topologically generated if and only if is both laminated and weakly induced.

In the following, we present some useful properties of these functors (see [29] and [4]).

Proposition 3.4.2. *(i) For all (X,τ) in $\mathcal{T}$op, $\iota(\omega(\tau)) = \tau$.*
(ii) ι is a surjection, ω is an injection and for each $\tau_1 \subseteq \tau_2$, $\iota(\tau_1) \subseteq \iota(\tau_2)$ and $\omega(\tau_1) \subseteq \omega(\tau_2)$ (where τ_1, τ_2 are topologies or fuzzy topologies, as appropriate).
(iii) $\omega(\iota(\tau))$ is the smallest topologically generated fuzzy topology which contains τ.
(iv) δ is topologically generated iff $\omega(\iota(\delta)) = \delta$.
(v) $\iota\omega = \iota e = \mathrm{id}_{Top}$.
(vi) $(X,\tau) \in e(\mathcal{T}\mathrm{op})$ iff $e(\iota(\tau)) = \tau$ iff $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{F}\mathrm{uz}}(e(\iota(X)), X)$.
(vii) $(X,\tau) \in \omega(\mathcal{T}\mathrm{op})$ iff $\omega(\iota(\tau)) = \tau$ iff $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{F}\mathrm{uz}}(X, \omega(\iota(X)))$.
(viii) For all (X,τ) in $\mathcal{F}$uz, $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{F}\mathrm{uz}}(\omega(\iota(X)), X)$.
(ix) For all $X \in \mathcal{F}$uz and for all $Y \in \mathcal{T}$op, $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\iota(X), Y) = \mathrm{Hom}_{\mathcal{F}\mathrm{uz}}(X, e(Y))$.

Theorem 3.4.3. *[29] (X,δ) is topologically generated if and only if for each continuous function $f \in \mathcal{C}(I_r, I_r)$ and for each $\alpha \in \delta$, we have that $f \circ \alpha \in \delta$.*

Proof. ($\Rightarrow$) If (X,δ) is topologically generated, then $\delta = \omega(\tau)$ for some (X,τ) in $\mathcal{T}$op. That is, $\delta = \bigcup_{r\in[0,1)}\{f : X \longrightarrow (r,1] \mid f \text{ is continuous}\}$, so the conclusion holds.

($\Leftarrow$) We will use the item (iv) of the previous proposition. Suppose that $\alpha \in \omega(\iota(\delta))$. Since a base for $\iota(\delta)$ is provided by the finite intersections
$$\bigcap_{i=1}^{n} \beta_i^{-1}[(\varepsilon_i, 1]], \text{ where } \beta_i \in \delta, \varepsilon_i \in [0,1);$$
this is equivalent to saying
$$\forall\varepsilon, \forall x \in \alpha^{-1}([\varepsilon,1]], \exists I_{\varepsilon,x} \text{ finite such that}$$
$$x \in \bigcap_{i\in \mathrm{I}_{\varepsilon,x}} \beta_i^{-1}[(\varepsilon_1, 1]] \subseteq \alpha^{-1}([\varepsilon,1]].$$
Now fix x and let $\alpha(x) = k_x \in [0,1]$, then $\forall\varepsilon < k_x, \exists I_\varepsilon$ finite such that
$$x \in \bigcap_{i\in I_\varepsilon} \alpha_i^{-1}\beta_i^{-1}[(\varepsilon_1, 1]] \subseteq \alpha^{-1}([\varepsilon,1]].$$
Then $\forall\varepsilon < k_x$ and $\forall i \in I_\varepsilon$ put
$$\alpha_{i,\varepsilon} = \varepsilon\chi_{(\varepsilon_i,1]} \circ \beta_i;$$

then $\alpha_{i,\varepsilon} \in \delta$ and clearly

$$\alpha_{i,\varepsilon}(y) = \begin{cases} \varepsilon \text{ if } \beta_i(y) > \varepsilon_i \\ 0 \text{ if } \beta_i(y) \leq \varepsilon_i \end{cases}$$

Put $\beta_\varepsilon^x = \bigwedge_{i \in I_\varepsilon} \alpha_{i,\varepsilon} \in \delta$ then clearly too

$$\beta_\varepsilon^x(y) = \begin{cases} \varepsilon \text{ if } \beta_i(y) > \varepsilon_i, \forall i \in I_\varepsilon \\ 0 \text{ if } \exists j \in I_\varepsilon, \beta_i(y) \leq \varepsilon_i \end{cases}$$

Thus $\beta_\varepsilon^x(y) = \varepsilon$ implies $\alpha(y) > \varepsilon$, becuase, $\beta_\varepsilon^x(y) = \varepsilon$ implies $\beta_i(y) > \varepsilon_i, \forall i \in I_\varepsilon$, i. e., $y \in \beta_i^{-1}[(\varepsilon_i, 1]], \forall i \in I_\varepsilon$, so $y \in \bigcap_{i \in I_\varepsilon} \beta_i^{-1}[(\varepsilon_i, 1]] \subseteq \alpha^{-1}([\varepsilon, 1]]$; wherewith $\forall \varepsilon < k_x, \beta_\varepsilon^x \leq \alpha$.
Now, it is easily seen that

$$\alpha = \bigvee_{x \in X} \bigvee_{\varepsilon < k_x} \beta_\varepsilon^x \in \delta.$$

Then $\omega(\iota(\delta)) \subseteq \delta$ and therefore $\omega(\iota(\delta)) = \delta$.

□

Part II

MV-Topologies

4

MV-Topologies

An MV-topology is a type of fuzzy topology which is a natural generalisation of classical topology with the use of MV-algebras. This concept was introduced by Ciro Russo in [42] in order to show an extension of Stone Duality between Boolean algebras and Stone spaces to, respectively, the category of *limit cut complete MV-algebras*, namely, the full subcategory of $\mathcal{MV}$ whose objects are MV-algebras which contain the suprema of certain cuts, and a suitable category of MV-topologies, whose objects are the natural MV-version of Stone (or Boolean) spaces, called *Stone MV-spaces*.

In this chapter, we collect the basic definitions and results of the MV-topologies pioneering paper [42]. Besides, we develop some additional theory. For example, we define closure and interior operators, quotient space and product space, among others. We also study the role of MV-topologies among fuzzy topologies. In particular, we have that each MV-space is a weakly induced fuzzy space, and we study the functors defined in Section 3.4, when they are restricted to the category of MV-topological spaces, ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$. Another interesting result that we introduce here is that for each laminated MV-space (X, τ), the corresponding MV-algebra of clopen, is a Riesz MV-algebra.

4.1 Basic Concepts and Results

Most of the definitions and results of this section can be found in [42].

Definition 4.1.1. Let X be a set, A the MV-algebra $[0,1]^X$ and $\tau \subseteq A$. We say that (X, τ) is an *MV-topological space* if τ is a subuniverse both of the quantale $\langle [0,1]^X, \bigvee, \oplus \rangle$ and of the semiring $\langle [0,1]^X, \wedge, \odot, \mathbf{1} \rangle$. More explicitly, (X, τ) is an MV-topological space if

(i) $\mathbf{0}, \mathbf{1} \in \tau$,
(ii) for any family $\{\alpha_i\}_{i \in I}$ of elements of τ, $\bigvee_{i \in I} \alpha_i \in \tau$,

and, for all $\alpha_1, \alpha_2 \in \tau$,

(iii) $\alpha_1 \odot \alpha_2 \in \tau$,
(iv) $\alpha_1 \oplus \alpha_2 \in \tau$,
(v) $\alpha_1 \wedge \alpha_2 \in \tau$.

The set τ is also called an *MV-topology* on X and the elements of τ are the *open MV-subsets* of X. The set $\tau^* = \{\alpha^* \mid \alpha \in \tau\}$ is easily seen to be a subquantale of $\langle [0,1]^X, \bigwedge, \odot \rangle$ (where $\bigwedge$ has to be considered as the join w.r.t. to the dual order $\geq$ on $[0,1]^X$) and a subsemiring of $\langle [0,1]^X, \vee, \oplus, \mathbf{0} \rangle$, i.e., it verifies the following properties:

– $\mathbf{0}, \mathbf{1} \in \tau^*$,
– for any family $\{\beta_i\}_{i\in I}$ of elements of τ^*, $\bigwedge_{i\in I} \beta_i \in \tau^*$,
– for all $\beta_1, \beta_2 \in \tau^*$, $\beta_1 \odot \beta_2, \beta_1 \oplus \beta_2, \beta_1 \vee \beta_2 \in \tau^*$.

The elements of τ^* are called the *closed MV-subsets* of X.

We will usually call an MV-topological space by an *MV-space*.

Proposition 4.1.2. *Let (X, τ) be an MV-topological space. For any subset Y of X, the pair (Y, τ_Y), where $\tau_Y := \{\alpha_{\restriction Y} \mid \alpha \in \tau\}$, is an MV-topology on Y.*

Proof. Trivial. □

Definition 4.1.3. For any subset Y of X, the pair (Y, τ_Y) is called an *MV-subspace* of (X, τ).

Example 4.1.4. (a) $(X, \{\mathbf{0}, \mathbf{1}\})$ and $(X, [0, 1]^X)$ are MV-topological spaces.
(b) Any topology is an MV-topology.
(c) Let $d : X \longrightarrow [0, +\infty[$ be a distance function on X. For any fuzzy point x_t of X, and any positive real number r, we define the *open ball* of center x_t and radius r as the fuzzy set $\beta_r(x_t)$ identified by the membership function $\beta_r(x_t)(y) = \begin{cases} t \text{ if } d(x, y) < r \\ 0 \text{ if } d(x, y) \geq r \end{cases}$. Analogously, the *closed ball* $\beta_r[x_t]$ of center x_t and radius r has membership function $\beta_r[x_t](y) = \begin{cases} t \text{ if } d(x, y) \leq r \\ 0 \text{ if } d(x, y) > r \end{cases}$. It is immediate to verify that the fuzzy subsets of X that are join of a family of open balls is an MV-topology on X that is said to be *induced* by d. This example can be found also in [29].

Definition 4.1.5. If (X, τ) is an MV-topology, then $(X, \mathcal{B}(\tau))$, where $\mathcal{B}(\tau) := \tau \cap \{0, 1\}^X = \tau \cap \mathcal{B}([0, 1]^X)$, is both an MV-topology and a topology in the classical sense. The topological space $(X, \mathcal{B}(\tau))$ will be called the *skeleton space* of (X, τ).

Note that $\mathcal{B}(\tau) = j(\tau)$ where j is the functor defined in the Section 3.4, restricted to the category of the MV-topological spaces.

The skeleton space of a given MV-topological one can be equivalently defined by

$$\mathcal{B}(\tau) = \{\Delta \circ \alpha \mid \alpha \in \tau\},$$

where Δ is the so-called *Baaz delta* operator [2], i.e.,

$$\Delta : x \in [0, 1] \mapsto \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } x < 1 \end{cases} \in \{0, 1\}.$$

This operator, besides being a monotonic map, is a monoid homomorphism between $\langle [0, 1], \odot, 1\rangle$ and $\langle \{0, 1\}, \wedge, 1\rangle$. Therefore the equivalence of the two definitions follows from the fact that MV-topologies are closed under $\odot$ while classical ones are closed under $\wedge$.

4.2 Bases, Subbases, Continuous function, etc.

The following definitions extend the given definitions in 3.3, but they have their own peculiarities and some important consequences for our work.

Definition 4.2.1. Given an MV-topological space (X, τ), a subset B of $[0,1]^X$ is called a *base* for τ if $B \subseteq \tau$ and every open set of τ is a join of elements of B.

Definition 4.2.2. Given an MV-topological space (X, τ), a subset S of τ is called a *subbase* for (X, τ) if each open set of X can be obtained as a join of finite combinations of products, infima, and sums of elements of S. More precisely, S is a subbase for τ if, for all $\alpha \in \tau$, there exists a family $\{t_i\}_{i \in I}$ of terms (or polynomials) in the language $\{\oplus, \odot, \wedge\}$, such that

$$\alpha = \bigvee_{i \in I} t_i(\beta_{i1}, \ldots, \beta_{in_i}) \tag{4.1}$$

where, for all $i \in I$, $n_i < \omega$, and $\{\beta_{ij}\}_{j=1}^{n_i} \subseteq S$.

Remark 4.2.3. If S is a subbase for an MV-topology, the set B_S defined by the following conditions is obviously a base for the same MV-topology:

(B1) $S \subseteq B_S$;
(B2) if $\alpha, \beta \in B_S$ then $\alpha \star \beta \in B_S$ for $\star \in \{\oplus, \odot, \wedge\}$.

A subbase S of an MV-topology τ shall be called *large* if, for all $\alpha \in S$, $n\alpha \in S$ for all $n < \omega$.

Example 4.2.4. Let us consider the MV-topology $[0,1]$ on a singleton $\{x\}$. For any $n > 1$, all the sets of type $[0, 1/n]$, $[0, 1/n] \cap \mathbb{Q}$, and $[0, 1/n] \setminus \mathbb{Q}$ are easily seen to be (non-large) subbases for the given MV-topology. Also $[0,1] \setminus \mathbb{Q}$ is a non-large subbase — it is a base, in fact.

Example 4.2.5. It is easy to see that in the example 4.1.4 (c), the set of open balls whose center is a fuzzy point whose non-zero membership value is greater than or equal to some fixed $a < 1$ is a large subbase for the topology induced by d. On the contrary, the set of open balls whose center is a fuzzy point whose non-zero membership value is lower than or equal to some fixed $a > 0$ is a non-large subbase.

Let X and Y be sets. For a function $f : X \longrightarrow Y$ we defined the function $f^{\leftsquigarrow}$ in (3.1) as:

$$\begin{aligned} f^{\leftsquigarrow} : [0,1]^Y &\longrightarrow [0,1]^X \\ \alpha &\longmapsto \alpha \circ f. \end{aligned} \tag{4.2}$$

Seeing the sets $[0,1]^Y$ and $[0,1]^X$ as MV-algebras, we have that $f^{\leftsquigarrow}(\mathbf{0}) = \mathbf{0}$; moreover, if $\alpha, \beta \in [0,1]^Y$, for all $x \in X$ we have $f^{\leftsquigarrow}(\alpha \oplus \beta)(x) = (\alpha \oplus \beta)(f(x)) = \alpha(f(x)) \oplus \beta(f(x)) = f^{\leftsquigarrow}(\alpha)(x) \oplus f^{\leftsquigarrow}(\beta)(x)$ and, analogously, $f^{\leftsquigarrow}(\alpha^*) = f^{\leftsquigarrow}(\alpha)^*$. Then $f^{\leftsquigarrow}$ is an MV-algebra homomorphism and we shall call it the *MV-preimage* of f.

From a categorical viewpoint, once denoted by $\mathcal{S}$et, $\mathcal{B}$oole and $\mathcal{MV}$ the categories of sets, Boolean algebras, and MV-algebras respectively (with the obvious morphisms), there exist two contravariant functors $\mathscr{P} : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{B}\mathrm{oole}^{\mathrm{op}}$ and $\mathscr{F} : \mathcal{S}\mathrm{et} \longrightarrow \mathcal{MV}^{\mathrm{op}}$ sending each map $f : X \longrightarrow Y$, respectively, to the Boolean algebra homomorphism $f^{-1} : \mathscr{P}(Y) \longrightarrow \mathscr{P}(X)$ and to the MV-homomorphism $f^{\leftsquigarrow} : [0,1]^Y \longrightarrow [0,1]^X$.

The following definition is also analogous of the given in 3.3.1.

Definition 4.2.6. Let (X, τ_X) and (Y, τ_Y) be two MV-topological spaces. We say that a map $f : X \longrightarrow Y$ is:

- *MV-continuous* or *continuous* if $f^{\leftsquigarrow}[\tau_Y] \subseteq \tau_X$,
- *open* if $f^{\rightarrow}(\alpha) \in \tau_Y$ for all $\alpha \in \tau_X$,
- *closed* if $f^{\rightarrow}(\beta) \in \tau_Y^*$ for all $\beta \in \tau_X^*$,
- an *MV-homeomorphism* if it is bijective and both f and f^{-1} are continuous.

Continuity, as in Definition 4.2.6, is equivalent to $f^{\leftsquigarrow}[\tau_Y^*] \subseteq \tau_X^*$. Indeed, since $f^{\leftsquigarrow}$ is an MV-algebra homomorphism, it preserves *; therefore, for any closed set β of Y, β^* is an open set, hence $f^{\leftsquigarrow}(\beta^*) = f^{\leftsquigarrow}(\beta)^* \in \tau_X$ implies $f^{\leftsquigarrow}(\beta) \in \tau_X^*$. In a completely analogous way, it can be proved that $f^{\leftsquigarrow}[\tau_Y^*] \subseteq \tau_X^*$ implies continuity in the sense of the previous definition.

Now, we can define the category of MV-spaces, denoted by ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$, whose objects are the MV-spaces and the morphisms the continuous functions according to the Definition 4.2.6.

We note that if (X, τ_X) and (Y, τ_Y) are two MV-spaces, and $f : X \longrightarrow Y$ is a continuous function between them, then f is also a continuous map between the two skeleton spaces $(X, \mathcal{B}(\tau_X))$ and $(Y, \mathcal{B}(\tau_Y))$ and we have the restriction functor $j \restriction {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} : {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$ well defined.

Lemma 4.2.7. *Let (X, τ) and (Y, τ') be two MV-topological spaces and let B be a base for τ'. A map $f : X \longrightarrow Y$ is continuous if and only if $f^{\leftsquigarrow}[B] \subseteq \tau$.*

Proof. One implication is trivial, since B is a family of open sets. Conversely, assuming that $f^{\leftsquigarrow}[B] \subseteq \tau$, let $\alpha = \bigvee \Gamma$, with $\Gamma \subseteq B$, be any open set of τ'. As we observed, $f^{\leftsquigarrow}$ is an MV-algebra homomorphism, hence $f^{\leftsquigarrow}(\alpha) = f^{\leftsquigarrow}(\bigvee \Gamma) = \bigvee f^{\leftsquigarrow}[\Gamma]$, i.e. $f^{\leftsquigarrow}(\alpha)$ is the join of open sets of τ and, therefore, open itself. □

Lemma 4.2.8. *Let (X, τ) and (Y, τ') be two MV-topological spaces and let S be a subbase for τ'. A map $f : X \longrightarrow Y$ is continuous if and only if $f^{\leftsquigarrow}[S] \subseteq \tau$.*

Proof. Assuming that $f^{\leftsquigarrow}[S] \subseteq \tau$, let

$$\alpha = \bigvee_{i \in I} t_i(\beta_{i1}, \ldots, \beta_{in_i}) \tag{4.3}$$

where, for all $i \in I$, $n_i < \omega$, and $\{\beta_{ij}\}_{j=1}^{n_i} \subseteq S$, be an open set of τ'. As $f^{\leftsquigarrow}$ is an MV-algebra homomorphism, we have that

$$f^{\leftsquigarrow}(\alpha) = f^{\leftsquigarrow}(\bigvee_{i \in I} t_i(\beta_{i1}, \ldots, \beta_{in_i})) = \bigvee_{i \in I} f^{\leftsquigarrow}(t_i(\beta_{i1}, \ldots, \beta_{in_i})) \tag{4.4}$$

and

$$\bigvee_{i \in I} f^{\leftsquigarrow}(t_i(\beta_{i1}, \ldots, \beta_{in_i})) = \bigvee_{i \in I} (t_i(f^{\leftsquigarrow}(\beta_{i1}), \ldots, f^{\leftsquigarrow}(\beta_{in_i}))) \in \tau$$

because $f^{\leftsquigarrow}(\beta_{ij}) \in \tau$ for all $i \in I, j \in \{1, ..., n_i\}$ $n_i < \omega$ and so

$$t_i(f^{\leftsquigarrow}(\beta_{i1}), \ldots, f^{\leftsquigarrow}(\beta_{in_i})) \in \tau$$

and therefore $f^{\leftsquigarrow}(\alpha) \in \tau$.

Since S is a family of open sets, we have the other implication. □

4.3 Coverings and Compactness

A *covering* of X is any subset Γ of $[0,1]^X$ such that $\bigvee \Gamma = \mathbf{1}$ [7], while an *additive covering* ($\oplus$-covering, for short) is a finite family $\{\alpha_i\}_{i=1}^n$ of elements of $[0,1]^X$, $n < \omega$, such that $\alpha_1 \oplus \cdots \oplus \alpha_n = \mathbf{1}$. It is worthwhile remarking that we used the expression "finite family" in order to include the possibility for such a family to have repetitions. In other words, an additive covering is a finite subset $\{\alpha_1, \ldots, \alpha_k\}$ of $[0,1]^X$, along with natural numbers $n_1, \ldots, n_k$, such that $n_1\alpha_1 \oplus \cdots \oplus n_k\alpha_k = \mathbf{1}$.

Proposition 4.3.1. *For any set X, any covering of fuzzy subsets of X which is closed under $\oplus$, $\odot$, and $\wedge$ is a base for an MV-topology on X.*

Proof. Let $\Gamma \subseteq [0,1]^X$ be a covering closed under $\oplus$, $\odot$, and $\wedge$, and let $\tau = \{\bigvee G \mid G \subseteq \Gamma\}$. We have $\mathbf{1} \in \tau$, by definition of covering, and $\mathbf{0} = \bigvee \emptyset \in \tau$. On the other hand, τ is trivially closed under arbitrary joins and $\odot$, $\oplus$, and $\wedge$ distribute over any existing join. Then, given $\alpha_1, \alpha_2 \in \tau$, $\alpha_1 = \bigvee_{i\in I} \alpha_i$ and $\alpha_2 = \bigvee_{j\in J} \beta_j$, with $\{\alpha_i\}_{i\in I}, \{\beta_j\}_{j\in J} \subseteq \Gamma$, whence

$$\alpha_1 \bullet \alpha_2 = \left(\bigvee_{i\in I} \alpha_i\right) \bullet \left(\bigvee_{j\in J} \beta_j\right) = \bigvee_{i\in I} \left(\alpha_i \bullet \bigvee_{j\in J} \beta_j\right) = \bigvee_{i\in I} \bigvee_{j\in J} (\alpha_i \bullet \beta_j),$$

for $\bullet \in \{\oplus, \odot, \wedge\}$. So τ verifies Definition 4.1.1, i.e. it is an MV-topology, and Γ is a base for it. □

The presence of strong and weak conjunctions and disjunction, in the structure of open sets of an MV-topology, naturally suggests different fuzzy versions (weaker or stronger) of most of the classical topological concepts (separation axioms, compactness etc.). Let us see:

Definition 4.3.2. An MV-topological space (X, τ) is said to be *compact* if any open covering of X contains an additive covering; it is called *strongly compact* if any open covering contains a finite covering.[1]

It is obvious that strong compactness implies compactness and, since the operations $\oplus$ and $\vee$ coincide on Boolean elements of MV-algebras, in the case of topologies of crisp subsets the two notions collapse to the classical one. For the same reason, it is evident as well that the skeleton spaces of both compact and strongly compact MV-spaces are compact. The following example shows that compactness does not imply strong compactness, i.e., they are not equivalent.

Example 4.3.3. Let X be a non-empty set and τ the set of all constant fuzzy subsets of X, which is clearly an MV-topology. For each $r \in [0,1]$, let o_r be the fuzzy set constantly equal to r. Then, for any family $\{r_i\}_{i\in I} \subseteq [0,1)$ such that $\bigvee_{i\in I} r_i = 1$, the set $\{o_{r_i} \mid i \in I\}$ is an open covering and all the coverings not containing $\mathbf{1}$ are of this form. On the other hand, all of such coverings do not contain finite coverings but do include additive ones.

Lemma 4.3.4. *A closed subspace (Y, τ_Y) of a compact (respectively: strongly compact) space (X, τ) is compact (resp.: strongly compact).*

[1] What we call strong compactness here is called simply compactness in the theory of lattice-valued fuzzy topologies [7].

Proof. Since Y is a subspace, in particular it is a crisp subset of X and, therefore, all of its open sets are of the form $\alpha_{\restriction Y}$ with $\alpha \in \tau$. So let $\{\alpha_i\}_{i\in I} \subseteq \tau$ such that $\bigvee_{i\in I} \alpha_i \geq Y$. Since Y is closed, Y^* is open and $\{\alpha_i\}_{i\in I} \cup \{Y^*\}$ is an open covering of X. By compactness of X, there exists a finite family $\{\alpha_j\}_{j=1}^n$ of elements of $\{\alpha_i\}_{i\in I}$ such that $\alpha_1 \oplus \cdots \oplus \alpha_n \oplus Y^* = X$. Then, since $Y \wedge Y^* = \mathbf{0}$, we have (with a slight abuse of notation) $Y = Y \wedge (\alpha_1 \oplus \cdots \oplus \alpha_n) = (Y \wedge \alpha_1) \oplus \cdots \oplus (Y \wedge \alpha_n)$, the latter equality easily following from the properties of Boolean elements of MV-algebras, whence Y is compact.

The case of strong compactness is completely analogous. □

Definition 4.3.5. Let (X, τ) be an MV-topological space. X is called a *Hausdorff* (or *separated*) *space* if, for all $x \neq y \in X$, there exist $\alpha_x, \alpha_y \in \tau$ such that

(i) $\alpha_x(x) = \alpha_y(y) = 1$,
(ii) $\alpha_x \wedge \alpha_y = \mathbf{0}$.

There is no interesting "weak" version of the above definition, since it is immediate to verify that Definition 4.3.5 is equivalent to the following:

for all $x \neq y \in X$, there exist $\alpha'_x, \alpha'_y \in \tau$ verifying

(i) $\alpha'_x(x) = \alpha'_y(y) = 1$,
(ii') $\alpha_x \odot \alpha_y = \mathbf{0}$.

Indeed, overlooking the trivial implication, assume there such two open sets α'_x and α'_y exist, and set $\alpha_x = \alpha'^2_x$ and $\alpha_y = \alpha'^2_y$. Then, by the quasi-equation $x \odot y = \mathbf{0} \Rightarrow x^2 \wedge y^2 = \mathbf{0}$ (which holds in every MV-algebra), α_x and α_y satisfy Definition 4.3.5.

As for compactness, Definition 4.3.5 coincide with the classical T_2 property on crisp topologies and implies that the corresponding skeleton space is Hausdorff in the classical sense.

The following result is obvious.

Lemma 4.3.6. *If (X, τ) is an Hausdorff space, then all crisp singletons of X are closed.*

4.4 MV-Topologies among Fuzzy Topologies

The MV-topologies are fuzzy topologies in the sense of Chang's definition, and they are contained in the class of weakly induced spaces (see Definition 3.4.1). Indeed, we have the following

Proposition 4.4.1. *Each MV-topological space is a weakly induced fuzzy topological space.*

Proof. Let (X, τ) be an MV-topological space. In the following, we will identify a subset of A with its characteristic map, so we will make no difference between χ_A and A when $A \subseteq X$. We have to show that for each $t \in [0, 1)$ and each $\alpha \in \tau$, the set $\{x \in X : \alpha(x) > t\}$ is an element of τ. We will proceed in three steps.

Claim 1. For $\alpha \in \tau$, we have that $\operatorname{supp} \alpha \in \tau$.

Indeed, as $\operatorname{supp} \alpha = \{x \in X : \alpha(x) > 0\}$, then for each $x \in \operatorname{supp} \alpha$, there is some natural number n such that

$$n\alpha(x) = \underbrace{\alpha(x) \oplus \cdots \oplus \alpha(x)}_{n \text{ times}} = 1$$

then

$$\chi_{\operatorname{supp}\alpha} = \bigvee_{n=1}^{\infty} n\alpha \in \tau.$$

Claim 2. For each $\alpha \in \tau$ and every irreducible fraction $t = \frac{k}{2^n} \in (0,1)$,

$$\alpha_t = \{x \in X : \alpha(x) > t\} \in \tau.$$

By induction,

- For $n = 1$ and $k = 1$. For each $x \in X$

$$\alpha(x) > \frac{1}{2} \text{ iff } \alpha(x) \odot \alpha(x) > 0.$$

  Hence, $\alpha_{1/2} = \operatorname{supp}(\alpha \odot \alpha) \in \tau$.
- Inductive step. Let's see that if $\alpha_t \in \tau$ for all t of the form $\frac{k}{2^n}$, then $\alpha_t \in \tau$ for all t of the form $\frac{k}{2^{n+1}}$.
  If $t < 1/2$, then $t \oplus t = \frac{k}{2^n}$, hence $\alpha_t = (\alpha \oplus \alpha)_{t \oplus t} \in \tau$.
  If $t \geq 1/2$, then $k \geq 2^n$ and $t \odot t = \frac{k-2^n}{2^n}$, hence $\alpha_t = (\alpha \odot \alpha)_{t \odot t} \in \tau$.

Claim 3. For all $\mu \in \tau$ and $t \in [0,1)$,

$$\mu_t = \{x \in X : \mu(x) > t\} = \bigcup \{\mu_s : s = \frac{k}{2^n}, s \geq t\} \in \tau.$$

□

We will say that an MV-space (X, τ) is *laminated* if all constant functions on X are elements of τ. It is clear that such spaces form a full subcategory of ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$, which will be denoted by ${}^{\mathrm{LMV}}\mathcal{T}\mathrm{op}$.

We will now see some properties about the functors defined in Section 3.4, when they are restricted to ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ both in the domain and in the codomain.

Proposition 4.4.2. *The functors ω, ι, e and j have the following properties with respect to the category ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$.*

1. *If (X, τ) is a topological space then $\omega(\tau)$ is an MV-topology, so the codomain of the functor ω is actually the ${}^{\mathrm{LMV}}\mathcal{T}\mathrm{op}$ category.*
2. *For all (X, τ) in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ and for all (Y, δ) in $\mathcal{T}\mathrm{op}$,*

$$\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(Y, \iota(X)) = \mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(\omega(Y), X).$$

   This implies that ω is a left adjoint of $\iota \restriction {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} : {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$.
3. *The functor e can be seen as $e : \mathcal{T}\mathrm{op} \longrightarrow {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$.*
4. *For all (X, τ) in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ and for all (Y, δ) in $\mathcal{T}\mathrm{op}$,*

$$\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\iota(X), Y) = \mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(X, e(Y)).$$

   This says that e is a right adjoint of $\iota \restriction {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} : {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$.
5. *For all X in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ and for all Y in ${}^{\mathrm{LMV}}\mathcal{T}\mathrm{op}$,*

$$\mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(X, Y) \neq \emptyset \Leftrightarrow X \in {}^{\mathrm{LMV}}\mathcal{T}\mathrm{op}\,.$$

6. ${}^{\mathrm{LMV}}\mathcal{T}\mathrm{op} \bigcap {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} = \omega(\mathcal{T}\mathrm{op})$.

Proof. 1. We recall that $\omega(\tau)$ is the following fuzzy topology

$$\omega(\tau) = \bigcup_{r\in[0,1)} \mathcal{C}(X, I_r) = \bigcup_{r\in[0,1)} \{f : X \longrightarrow I_r : f \text{ is continuous}\}$$

where $I_r = (r, 1]$. Let us see that $\omega(\tau)$ is closed for $\oplus$ and $\odot$. If $f : X \longrightarrow I_r$ and $g : X \longrightarrow I_s$ are elements of $\omega(\tau)$, then $f \oplus g : X \longrightarrow I_{\min(r,s)}$ given by

$$(f \oplus g)(x) = f(x) \oplus g(x) = \min(f(x) + g(x), 1)$$

is continuous. Analogously, $f \odot g : X \longrightarrow [0, 1]$ given by

$$(f \odot g)(x) = f(x) \odot g(x) = \max(f(x) + g(x) - 1, 0)$$

is continuous.

2. The sentence holds because a function $f : (Y, \delta) \longrightarrow (X, \iota(\tau))$ is continuous in $\mathcal{T}\mathrm{op}$ if and only if $f : (Y, \omega(\delta)) \longrightarrow (X, \tau)$ is continuous in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$. Let us see, f continuous in $\mathcal{T}\mathrm{op}$ means that for all $t \in [0, 1)$ and $\mu \in \tau$, if $U = \{x \in X : \mu(x) > t\} \in \iota(\tau)$ then

$$f^{-1}(U) = \{y \in Y : f(y) \in U\} = \{y \in Y : \mu(f(y)) > t\} \in \delta.$$

And this is equivalent to say that $\mu \circ f : Y \longrightarrow I_t$ is continuous, i. e., $\mu \circ f \in \omega(\delta)$, that is, f is continuous in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$.

3. It is clear because $e(\tau) = \{\chi_U : U \in \tau\}$ is an MV-topology whenever τ is a topology.
4. It is enough to observe that if $\alpha \in e(\delta)$ then $\alpha = \chi_U$ for some $U \in \delta$. So, f is continuous in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ if for all $U \in \delta$, $\chi_U \circ f \in \tau$, that is, $\chi_{f^{-1}(U)} \in \tau$, and it is equivalent to say that $f^{-1}(U) \in \iota(\tau)$, and so f is continuous in $\mathcal{T}\mathrm{op}$.

□

4.5 Extending Stone Duality

In [42] C. Russo proved that Stone Duality can be extended to a class of semisimple MV-algebras and compact separated MV-topologies having a base of clopens. He showed an extension of Stone Duality between Boolean algebras and Stone spaces to, respectively, the category of limit cut complete MV-algebras, namely, the full subcategory of $\mathcal{MV}$ whose objects are algebras which contain the suprema of certain cuts, and a suitable category of MV-topologies, whose objects are the natural MV-version of Stone (or Boolean) spaces — called Stone MV-spaces. Such an extension is "proper" in the sense that its restriction to, respectively, Boolean algebras and Stone spaces — which are full subcategories of the ones involved in the duality — yields the classical well-known duality, up to a trivial reformulation in terms of maximal ideals instead of ultrafilters.

In the following, we show how this duality is obtained. For that, we follow the route drawn in [42]. First, let us see some preliminary facts. All theory and results can find in [42].

In what follows, we shall always denote by $\widehat{a}$ and $\widehat{X}$, respectively, $\iota(a) \in [0, 1]^{\mathrm{Max}\, A}$ and $\iota(X) \subseteq [0, 1]^{\mathrm{Max}\, A}$, for $a \in A$ and $X \subseteq A$, according we defined in the proof of Theorem 2.6.2.

The class of semisimple MV-algebras form a full subcategory of $\mathcal{MV}$ that we shall denote by $\mathcal{MV}^{\mathrm{ss}}$. As usual, for subsets $Z \subseteq Y$ of an ordered set $\langle X \leq \rangle$ we shall denote by $l_Y Z$ (or

simply lZ when $Y = X$) the set of lower bounds of Z in Y and by $u_Y Z$ (respectively: uZ) the set of all upper bounds of Z in Y. We also recall that a subset Y of X is called a *cut* if $Y = luY$. We set the following

Definition 4.5.1. Let A be a semisimple MV-algebra. We say that a cut X of A is a *limit cut* iff

$$d(\widehat{X}, \widehat{uX}) = \bigwedge\{d(\widehat{a}, \widehat{b}) \mid b \in uX, a \in X\} = \bigwedge\{\widehat{b} \ominus \widehat{a} \mid b \in uX, a \in X\} = 0. \tag{4.5}$$

We shall say that A is *limit cut complete* (*lcc* for short) if, for any limit cut X of A, there exists in A the supremum of X or, equivalently, the supremum of $\widehat{X}$ in $[0,1]^{\mathrm{Max}\,A}$ belongs to $\widehat{A}$.

Proposition 4.5.2. *Let A be a semisimple MV-algebra. Then a cut X of A is a limit cut if and only if there exists a cut Y of A such that, in $[0,1]^{\mathrm{Max}\,A}$, $\bigvee \widehat{X} = \bigwedge \widehat{Y}^*$, where $Y^* = \{y^* \mid y \in Y\}$. Moreover, Y is a limit cut too.*

Proof. Let X be a limit cut of A and set $Y = (uX)^*$. From $x \leq y$ iff $x^* \geq y^*$ readily follows that $a \in uY$ iff $a^* \in luX = X$, whence $uY = X^*$. Analogously $a \in luY$ iff $a^* \in uX$. Therefore $luY = (uX)^* = Y$, i.e., Y is a cut. Now, since $x \ominus y = 0$ iff $x \leq y$ in any MV-algebra, from $d(\widehat{X}, \widehat{uX}) = 0$, we get $\bigvee \widehat{X} = \bigwedge \widehat{uX} = \bigwedge \widehat{Y}^*$. Moreover, from $y^* \ominus x^* = y^* \odot x = x \ominus y$, we have that $d\left(\widehat{Y}, \widehat{uY}\right) = d\left(\widehat{(uX)}^*, \widehat{X}^*\right) = d(\widehat{X}, \widehat{uX}) = 0$, and therefore Y is a limit cut.

Conversely, let X and Y be cuts such that $\bigvee \widehat{X} = \bigwedge \widehat{Y}^*$, so in particular $d(\widehat{X}, \widehat{Y}) = 0$. Then $Y^* \subseteq uX$, whence $d(\widehat{X}, \widehat{uX}) \leq d(\widehat{X}, \widehat{Y}^*) = 0$, and X is a limit cut. The fact that also Y is a limit cut is an immediate consequence of the mutual roles of X and Y in this part of the proof. □

Corollary 4.5.3. *A semisimple MV-algebra A is lcc if and only if, for all $X, Y \subseteq A$ and $\alpha \in [0,1]^{\mathrm{Max}\,A}$, $\alpha = \bigvee \widehat{X} = \bigwedge \widehat{Y}$ implies $\alpha \in \widehat{A}$.*

Proof. Follows immediately from Proposition 4.5.2 by observing that, for any subset X of A, $\bigvee \widehat{X} = \bigvee \widehat{luX}$. Then, if $\alpha = \bigvee \widehat{X} = \bigwedge \widehat{Y}$, luX and $lu(Y^*)$ form a pair of limit cuts as in Proposition 4.5.2. □

The distance $d(\widehat{X}, \widehat{uX})$ considered in (4.5) do not necessarily coincide with $\iota(d(X, uX))$, as the following example shows.

Example 4.5.4. Let B the finite-cofinite Boolean algebra on the natural numbers. Let $\mathbb{E}$ be the set of even numbers and consider the set X of all finite subsets of $\mathbb{E}$ and the set Y of all cofinite subsets of $\mathbb{N}$ which include $\mathbb{E}$. Then it is self-evident that X and Y^* are cuts in B, $Y = uX$, and $d(X, Y) = 0$ in B. However, by the Boolean Prime Ideal Theorem, we know that there exists a maximal ideal M of B which separates X and Y, i.e. such that $X \subset M$ and $Y \cap M = \emptyset$. It follows that $d(\widehat{X}, \widehat{Y}) \neq 0$.

Note that limit cut completeness is a distinctive feature of Boolean algebras among semisimple MV-algebras; in other words, all Boolean algebras are limit cut complete, while not all semisimple MV-algebras are. So, the definition of limit cut complete MV-algebras is somehow ad hoc but, on the other hand, it turns out that the class of limit cut complete MV-algebras can play an important role for the theory of MV-algebras, as shown by the

results of [42] and, in particular, by the fact that it is a reflective subcategory of $\mathcal{MV}$ and a completion subcategory of $\mathcal{MV}^{\mathrm{ss}}$.

Let us now consider an MV-algebra A. By Theorem 2.6.2 and the comments following it, up to an isomorphism, $A' = A/RadA$ is a subalgebra of $[0,1]^{\mathrm{Max}\,A}$. Therefore, A' is a covering of $\mathrm{Max}\,A$ and, since it is an MV-subalgebra of $[0,1]^{\mathrm{Max}\,A}$, it is closed under $\oplus$, $\odot$ and $\wedge$. Then, by Proposition 4.3.1, it is a base for an MV-topology τ_A on $\mathrm{Max}\,A$. Conversely, given an MV-topological space (X,τ), the set $\mathrm{Clop}\,X = \tau \cap \tau^*$ of the *clopen* subsets of X, i.e. the fuzzy subsets of X that are both open and closed, is a semisimple MV-algebra. Indeed $\mathbf{0}, \mathbf{1} \in \mathrm{Clop}\,X$ and, obviously, $\mathrm{Clop}\,X$ is closed under $\oplus$ and *; $\mathrm{Clop}\,X$ is semisimple as an obvious consequence of being a subalgebra of $[0,1]^X$.

We recall that ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ is the category whose objects are MV-topological spaces and morphisms are MV-continuous functions between them. We shall denote by ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ the full subcategory of ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ whose objects are *Stone MV-spaces*, i.e., compact, separated MV-topological spaces having a base of clopen sets (*zero-dimensional*).

In the proof of the following results we shall often identify any semisimple MV-algebra A with its isomorphic image included in $[0,1]^{\mathrm{Max}\,A}$; so any element a of a semisimple MV-algebra will be identified with the fuzzy set $\widehat{a}$. The reader may refer to [3, 5, 6, 8] for further details.

Let us now consider the following class functions:

$$\begin{array}{lcccc} \mathrm{Clop} : (X,\tau) \in {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} & \longmapsto & \mathrm{Clop}\,X & \in & \mathcal{MV} \\ \mathrm{Max} : \quad A \quad \in \mathcal{MV} & \longmapsto & (\mathrm{Max}\,A, \tau_A) & \in & {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}\,. \end{array} \tag{4.6}$$

Moreover, we set the following:

- for any two MV-topological spaces (X,τ) and (X',τ'), and for any continuous function $f : X \to X'$,
$$(\mathrm{Clop}\,f)(\alpha) = f^{\leftsquigarrow}(\alpha), \text{ for all } \alpha \in \mathrm{Clop}\,X';$$
- for any two MV-algebras A and B, and for any MV-algebra homomorphism $h : A \to B$,
$$(\mathrm{Max}\,h)(N) = h^{-1}[N], \text{ for all } N \in \mathrm{Max}\,B.$$

Lemma 4.5.5. *With the above notations,* Clop *and* Max *are two contravariant functors.*

Proof. Let (X,τ_X) and (Y,τ_Y) be two MV-topologies, and let $f : X \longrightarrow Y$ be a continuous map between them. As we already remarked $f^{\leftsquigarrow} : [0,1]^Y \longrightarrow [0,1]^X$ is a homomorphism of MV-algebras. On the other hand, by Definition 3.3.1, $f^{\leftsquigarrow}[\tau_Y] \subseteq \tau_X$ and, as we observed right after the same definition, $f^{\leftsquigarrow}[\tau_Y^*] \subseteq \tau_X^*$; therefore $f^{\leftsquigarrow}[\mathrm{Clop}\,Y] \subseteq \mathrm{Clop}\,X$. Hence, for all $f \in \mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(X,Y)$, $\mathrm{Clop}\,f$ is an MV-algebra homomorphism from $\mathrm{Clop}\,Y$ to $\mathrm{Clop}\,X$, i.e., a morphism from $\mathrm{Clop}\,X$ to $\mathrm{Clop}\,Y$ in $(\mathcal{MV}^{\mathrm{ss}})^{\mathrm{op}}$. The fact that Clop preserves composition and identities is absolutely trivial.

Let now A and B be two MV-algebras and $h : A \longrightarrow B$ an MV-algebra homomorphism. It is known that the preimage of a maximal ideal under an MV-algebra homomorphism is a maximal ideal; then it is well-defined the map

$$\mathrm{Max}\,h : N \in \mathrm{Max}\,B \longmapsto h^{-1}[N] \in \mathrm{Max}\,A.$$

The function $\mathrm{Max}\,h$, on its turn, defines an MV-algebra homomorphism

$$(\mathrm{Max}\,h)^{\leftsquigarrow} : \alpha \in [0,1]^{\mathrm{Max}\,A} \longmapsto \alpha \circ \mathrm{Max}\,h \in [0,1]^{\mathrm{Max}\,B}.$$

Let us prove that $(\operatorname{Max} h)^{\leftrightsquigarrow}[A'] \subseteq \tau_B$.

So let N be an arbitrary maximal ideal of B and $M = h^{-1}[N] = (\operatorname{Max} h)(N)$. We have

$$(\operatorname{Max} h)^{\leftrightsquigarrow}(\widehat{a})(N) = (\widehat{a} \circ \operatorname{Max} h)(N) = \widehat{a}(M), \text{ for all } a \in A.$$

The map $h' : a/M \in A/M \longrightarrow h(a)/N \in B/N$ is well-defined since

$$\begin{array}{l} a/M = a'/M \quad \Rightarrow \quad (a \odot a'^*) \oplus (a' \odot a^*) \in M \quad \Rightarrow \\ (h(a) \odot h(a')^*) \oplus (h(a') \odot h(a)^*) \in N \;\Rightarrow\; h(a)/N = h(a')/N; \end{array}$$

moreover it can be proved in a similar way that $h(a)/N = h(a')/N$ implies $a/M = a'/M$, that is, h' is injective. Now, if we look at A/M and B/N as subalgebras of $[0,1]$, we get that the fuzzy set $\widehat{h(a)}$ takes, in any given $N \in \operatorname{Max} B$, precisely the same value taken by the fuzzy set $\widehat{a}$ in $M = \operatorname{Max} h(N)$. In other words, the fuzzy set $(\operatorname{Max} h)^{\leftrightsquigarrow}(\widehat{a})$ is in B', for all $a \in A$. It follows that $(\operatorname{Max} h)^{\leftrightsquigarrow}[\tau_A] \subseteq \tau_B$ and therefore, by Lemma 4.2.7, $\operatorname{Max} h$ is an MV-continuous function from $(\operatorname{Max} B, \tau_B)$ to $(\operatorname{Max} A, \tau_A)$, i.e., it is a morphism from $(\operatorname{Max} A, \tau_A)$ to $(\operatorname{Max} B, \tau_B)$ in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}^{\mathrm{op}}$. Again, it is immediate to see that Max is well-behaved w.r.t. composition and identity morphisms. □

Theorem 4.5.6 (Duality theorem). Clop *and* Max *form a duality between* $\mathcal{MV}^{\mathrm{lcc}}$ *and* ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$.

Proof. It is immediate to verify that both the functors, restricted to ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ and $\mathcal{MV}^{\mathrm{lcc}}$ respectively, are faithful. We shall prove that

$$\operatorname{Max}\operatorname{Clop} X \cong_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}} X \quad \text{ and } \quad \operatorname{Clop}\operatorname{Max} A \cong_{\mathcal{MV}} A,$$

for all $(X, \tau) \in {}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ and for all $A \in \mathcal{MV}^{\mathrm{lcc}}$. The assertion will therefore follow from the fact that such isomorphisms, together with faithfulness, yield two natural isomorphisms between the two compositions Max Clop and Clop Max and, respectively, $\mathrm{id}_{{}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}}$ and $\mathrm{id}_{\mathcal{MV}^{\mathrm{lcc}}}$.

First, let us prove that $\operatorname{Max} A \in {}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ for any semisimple MV-algebra A and that $\operatorname{Clop}\operatorname{Max} A \cong A$ if $A \in \mathcal{MV}^{\mathrm{lcc}}$.

$(\operatorname{Max} A, \tau_A)$ is zero-dimensional by definition. $\operatorname{Clop}\operatorname{Max} A$ is obviously semisimple, and every element of $\operatorname{Clop}\operatorname{Max} A$ can be obtained as both a join and a meet of elements of $\widehat{A}$. Therefore, if $A \in \mathcal{MV}^{\mathrm{lcc}}$, by Proposition 4.5.2 and Corollary 4.5.3, $A \cong \operatorname{Clop}\operatorname{Max} A$. Now we need to prove only that $\operatorname{Max} A$ is compact and Hausdorff. Let Γ be an open covering of $\operatorname{Max} A$ and assume, by contradiction, that it does not contain any additive covering. By Proposition 2.2.6, $(\Gamma]$ is a proper ideal of A and, therefore, it is contained in some $M \in \operatorname{Max} A$; but this implies that, for any $a \in \Gamma$, $a(M) = 0$, i.e. Γ is not a covering of $\operatorname{Max} A$, which is absurd.

In order to prove separation, let us consider $M \neq N \in \operatorname{Max} A$ and let $a \in M \setminus N$; we have $\widehat{a}(M) = 0$ and $\widehat{a}(N) \neq 0$. So, since $[0,1]$ is hyper-Archimedean, there exists $k < \omega$ such that $k\widehat{a}(N) = 1$. Then we have $k\widehat{a}(N) = 1$ and $\widehat{a}^*(M) = \widehat{a}(M)^* = 1$, which implies $(\widehat{a}^*)^k(M) = 1$. Moreover, $(\widehat{a}^*)^k \odot k\widehat{a} = (k\widehat{a})^* \odot k\widehat{a} = \mathbf{0}$; then $(\operatorname{Max} A, \tau_A)$ is a Stone MV-space.

Now let us prove that X and $\operatorname{Max}\operatorname{Clop} X$ are homeomorphic for any Stone MV-space (X, τ). Let (X, τ) be a Stone MV-space and, for each $x \in X$, let

$$f(x) = \{\alpha \in \operatorname{Clop} X \mid \alpha(x) = 0\}.$$

It is self-evident that $f(x)$ is a proper ideal of the algebra $A = \operatorname{Clop} X$ for all $x \in X$. For any fixed x and for each $\alpha \in A$, $\alpha \notin f(x)$ implies $\alpha(x) > 0$ and, therefore, $\alpha^*(x) < 1$. Then

there exists $n < \omega$ such that $(\alpha^*)^n(x) = 0$, i.e. $(\alpha^*)^n \in f(x)$, and Proposition 2.3.3 ensures us that $f(x)$ is a maximal ideal.

Now we must prove that the map $f : X \longrightarrow \operatorname{Max} A$ is a homeomorphism of MV-spaces. First, let $x \neq y \in X$; since X is Hausdorff, there exist $\alpha_x, \alpha_y \in \tau$ that satisfy Definition 5.1.1, and each of these open sets is the join of a set of clopens because X is zero-dimensional. By Lemma 4.3.6, $\{x\}$ and $\{y\}$ are closed, whence, by Lemma 4.3.4, they are compact; then there exist two finite families of such sets — say $\{\alpha_{xi}\}_{i=1}^n$ and $\{\alpha_{yj}\}_{j=1}^m$ — which are additive open coverings of $\{x\}$ and $\{y\}$ respectively, and are such that $(\alpha_{x1} \oplus \cdots \oplus \alpha_{xn})(y) = 0 = (\alpha_{y1} \oplus \cdots \oplus \alpha_{ym})(x)$. Moreover, $\alpha_{x1} \oplus \cdots \oplus \alpha_{xn}$ and $\alpha_{y1} \oplus \cdots \oplus \alpha_{ym}$ are both clopen, hence the former belongs to $f(y)$ and the latter to $f(x)$. It follows $f(x) \neq f(y)$, namely, f is injective.

In order to prove that f is onto, let $M \in \operatorname{Max} A$ and assume, by contradiction, that M is not the image under f of any element of X, that is, for all $x \in X$ there exists $\alpha \in M$ such that $\alpha(x) > 0$. Then, for each $x \in X$, there exist $\alpha \in M$ and $m < \omega$ such that $m\alpha(x) = 1$, and $m\alpha \in M$ because M is an ideal. So let, for each $x \in X$, α_x be an element of M whose value in x is 1; the family $\{\alpha_x\}_{x \in X}$ is an open covering of X whence, by the compactness of X, it contains an additive covering $\{\alpha_i\}_{i=1}^n$. It follows that $\mathbf{1} = \alpha_1 \oplus \cdots \oplus \alpha_n \in M$ which contradicts the hypothesis that M is a proper ideal. Such a contradiction follows from the assumption that for all $x \in X$ there exists $\alpha \in M$ such that $\alpha(x) > 0$; hence there exists $x \in X$ such that $\alpha(x) = 0$ for all $\alpha \in M$, i.e., such that $M = f(x)$, and f is onto.

We need to prove that both f and f^{-1} are continuous. To this purpose, we first observe that, for all $x \in X$ and $\alpha \in \operatorname{Clop} X$, $\alpha/f(x)$ is a real number in $[0, 1]$ and coincide with the membership value $\alpha(x)$ of the point x to the clopen α. Indeed, by the property (2.2), $\alpha/f(x) = \{(\alpha \oplus p) \odot q^* \mid p, q \in f(x)\}$ and, on the other hand, $((\alpha \oplus p) \odot q^*)(x) = (\alpha(x) \oplus 0) \odot 1 = \alpha(x)$ for all $p, q \in f(x)$. Therefore, $(\operatorname{Clop} X)/f(x) = \{\alpha(x) \mid \alpha \in \operatorname{Clop} X\}$ and $\pi_{f(x)} : \alpha \in \operatorname{Clop} X \mapsto \alpha(x) \in (\operatorname{Clop} X)/f(x) \subseteq [0, 1]$.

Now, any clopen α of X can be identified (see the proof of Theorem 2.6.2) with a clopen $\widehat{\alpha}$ of $\operatorname{Max} \operatorname{Clop} X$ in a unique way: $\widehat{\alpha}(M) = \iota_M(\pi_M(\alpha)) = \iota_{f(x)}(\pi_{f(x)}(\alpha)) = \iota_{f(x)}(\alpha(x))$, for all $M = f(x) \in \operatorname{Max} \operatorname{Clop} X$, and $\iota_{f(x)}$ is simply the inclusion map of $(\operatorname{Clop} X)/f(x)$ in $[0, 1]$. Therefore, for any basic clopen $\widehat{\alpha}$ of $\operatorname{Max} \operatorname{Clop} X$, and for each $x \in X$, $f^{\leftsquigarrow}(\widehat{\alpha})(x) = (\widehat{\alpha} \circ f)(x) = \widehat{\alpha}(f(x)) = \alpha(x)$, with $\alpha \in \operatorname{Clop} X$. It follows that the fuzzy preimage, under f, of any basic open set of $\operatorname{Max} \operatorname{Clop} X$ is open in X, that is, f is continuous. Analogously, for each $M = f(x) \in \operatorname{Max} \operatorname{Clop} X$, $(f^{-1})^{\leftsquigarrow}(\alpha)(M) = (\alpha \circ f^{-1})(f(x)) = \alpha(x) = \widehat{\alpha}(M)$, and f^{-1} is continuous as well. We can conclude that X and $\operatorname{Max} \operatorname{Clop} X$ are homeomorphic spaces.

The proof is complete. □

4.6 MV-Topologies and Riesz MV-algebras

We write again the definition of Riesz MV-algebra given in the Section 2.10.

Definition 4.6.1. A *Riesz MV-algebra* is a structure $(R, \cdot, \oplus, ^*, \mathbf{0})$, where $(R, \oplus, ^*, \mathbf{0})$ is an MV-algebra and $\cdot : [0, 1] \times R \to R$ is a function such that for any $r, q \in [0, 1]$ and $a, b \in R$:

RMV1) $r \cdot (a \odot b^*) = (r \cdot a) \odot (r \cdot b)^*$,
RMV2) $\max(r - q, 0) \cdot a = (r \cdot a) \odot (q \cdot a)^*$,
RMV3) $(r \cdot q) \cdot a = r \cdot (q \cdot a)$,
RMV4) $1 \cdot a = a$.

We show another interesting example of a Riesz MV-algebra which is related with MV-topologies.

Proposition 4.6.2. *If (X, τ) is a laminated MV-topological space then the MV-algebra* $\operatorname{Clop} X$ *is a Riesz MV-algebra.*

Proof. We saw before, in the Claim 3 of Proposition 4.4.1, that for any $\alpha \in \tau$, the crisp set $\alpha_a = \{x \in X : \alpha(x) > a\}$ is an element of τ, for each $a \in [0, 1)$. As X is laminated, we have that the constant function $\mathbf{r}$ is an element of $\operatorname{Clop} X$, for each $r \in [0, 1]$. Then for each $r \in [0, 1]$, $r \cdot \alpha_a = \mathbf{r} \odot \alpha_a \in \tau$ and $r \cdot \alpha_a^* = \mathbf{r} \odot \alpha_a^* \in \tau^*$. Thus, we have the following:

Claim 1.: For each $r \in [0, 1]$ and $\alpha \in \operatorname{Clop} X$, $r \cdot \alpha \in \tau$. Moreover

$$\bigvee_{a \in [0,1)} ra \cdot \alpha_a = r \cdot \alpha.$$

Proof of Claim. For each $x \in X$

$$(\bigvee_{a \in [0,1)} ra \cdot \alpha_a)(x) = (\bigvee_{a \geq \alpha(x)} ra \cdot \alpha_a)(x) \vee (\bigvee_{a < \alpha(x)} ra \cdot \alpha_a)(x) = r\alpha(x)$$

because $\alpha_a(x) = 0$ if $\alpha(x) \leq a$ and $\alpha_a(x) = 1$ if $\alpha(x) > a$. This shows that $r \cdot \alpha$ is an open fuzzy set in (X, τ).

Claim 2.: For each $r \in [0, 1]$ and $\alpha \in \operatorname{Clop} X$, $r \cdot \alpha \in \tau^*$,

$$\bigwedge_{a \in [0,1)} ra \cdot \alpha_a^* = r \cdot \alpha.$$

Proof of Claim. For each $x \in X$

$$(\bigwedge_{a \in [0,1)} ra \cdot \alpha_a^*)(x) = (\bigwedge_{a \geq \alpha(x)} ra \cdot \alpha_a^*)(x) \wedge (\bigwedge_{a < \alpha(x)} ra \cdot \alpha_a^*)(x) = r\alpha(x)$$

because $\alpha_a^*(x) = 0$ if $\alpha(x) > a$ and $\alpha_a^*(x) = 1$ if $\alpha(x) \leq a$. So $r \cdot \alpha$ is a closed fuzzy set in (X, τ).

We have proved that for each $r \in [0, 1]$ and each $\alpha \in \operatorname{Clop} X$, $r\alpha \in \operatorname{Clop} X$. As $[0, 1]^X$ is a Riesz MV-algebra, and $\operatorname{Clop} X$ is closed for scalar multiplication, then we have that $\operatorname{Clop} X \subseteq [0, 1]^X$ is also a Riesz MV-algebra. □

Let A be a semisimple MV-algebra and let $\hat{A}_l$ the subalgebra of $[0, 1]^{\operatorname{Max} A}$ generated by $\hat{A} \cup \{\mathbf{r} : r \in [0, 1]\}$, where $\mathbf{r}$ is the constant function whose value is r, for each $r \in [0, 1]$. Then, $\hat{A}_l$ is a base for a laminated MV-topology on $\operatorname{Max} A$ and it is the coarsest MV-topology that contains $\hat{A} \cup \{\mathbf{r} : r \in [0, 1]\}$. We call this topology $\tau_{\hat{A}_l}$.

Corollary 4.6.3. *The MV-algebra* $\operatorname{Clop} \operatorname{Max} A$ *of the clopen fuzzy sets of* $\tau_{\hat{A}_l}$ *is a Riesz MV-algebra.*

Proof. It is immediate from last proposition. □

4.7 Closure and Interior

In the following, we define the interior and the closure of an open fuzzy set of an MV-topological space. This definition is the same for a fuzzy topological space. However, below we will define an interior operator and a closure operator for MV-spaces, which have some differences and particularities with respect to fuzzy topological spaces.

Definition 4.7.1. Let (X, τ) be an MV-topological space and let α be a fuzzy set in X.

1. The join of all open sets contained in α is called the *interior* of α, denoted by α°, i. e.,

$$\alpha^\circ = \bigvee\{\beta \in \tau : \beta \leq \alpha\}.$$

2. The meet of all closed sets containing α is called the *closure* of α, denoted by $\overline{\alpha}$, i.,e.,

$$\overline{\alpha} = \bigwedge\{\beta \in \tau^* : \alpha \leq \beta\}$$

It is clear that α° is the largest open set contained in α and $\overline{\alpha}$ is the smallest closed set containing α. The following are properties of the interior and the closure:

Proposition 4.7.2. *Let (X, τ) be an MV-topological space and let α, β be fuzzy sets in X. Then:*

1. $(\alpha^\circ)^\circ = \alpha^\circ$
2. $\alpha^\circ \leq \alpha$
3. If $\alpha \leq \beta$ *then* $\alpha^\circ \leq \beta^\circ$
4. $\overline{\overline{\alpha}} = \alpha$
5. $\alpha \leq \overline{\alpha}$
6. If $\alpha \leq \beta$ *then* $\overline{\alpha} \leq \overline{\beta}$

Definition 4.7.3. A mapping $f : [0,1]^X \longrightarrow [0,1]^X$ is called an *MV-interior operator* on X iff f satisfies the following axioms:

1. $f(\mathbf{1}) = \mathbf{1}$;
2. $f(\alpha) \leq \alpha$;
3. $f(f(\alpha)) = f(\alpha)$;
4. $f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta)$;
5. $f(\alpha) \oplus f(\beta) \leq f(\alpha \oplus \beta)$;
6. $f(\alpha) \odot f(\beta) \leq f(\alpha \odot \beta)$.

Definition 4.7.4. A mapping $f : [0,1]^X \longrightarrow [0,1]^X$ is called an *MV-closure operator* on X iff f satisfies the following axioms:

1. $f(\mathbf{0}) = \mathbf{0}$
2. $\alpha \leq f(\alpha)$
3. $f(f(\alpha)) = f(\alpha)$
4. $f(\alpha \vee \beta) = f(\alpha) \vee f(\beta)$
5. $f(\alpha \oplus \beta) = f(\alpha) \oplus f(\beta)$
6. $f(\alpha \odot \beta) = f(\alpha) \odot f(\beta)$.

Remark 4.7.5. If f is an MV-closure operator on X, then it satisfies that if $\alpha \leq \beta$ then $f(\alpha) \leq f(\beta)$ for every $\alpha, \beta \in [0,1]^X$. In fact, $\alpha \leq \beta$ implies that $\alpha \vee \beta = \beta$ then $f(\alpha) \vee f(\beta) = f(\alpha \vee \beta) = f(\beta)$ so $f(\alpha) \leq f(\beta)$. Analogously, if f is an MV-interior operator on X, if $\alpha \leq \beta$ then $f(\alpha) \leq f(\beta)$ for every $\alpha, \beta \in [0,1]^X$.

Proposition 4.7.6. *The interior of the definition 4.7.1 is an MV-interior operator.*

Proof. Let (X, τ) be an MV-space. The items 1., 2. and 3. are clear of the definition. For 4., we have that $\alpha^\circ \leq \alpha$ and $\beta^\circ \leq \beta$, then $\alpha^\circ \wedge \beta^\circ \leq \alpha \wedge \beta$ and as $\alpha^\circ \wedge \beta^\circ \in \tau$ then $\alpha^\circ \wedge \beta^\circ \leq (\alpha \wedge \beta)^\circ$. On the other hand, $\alpha \wedge \beta \leq \alpha, \beta$ then $(\alpha \wedge \beta)^\circ \leq \alpha^\circ, \beta^\circ$ and therefore $(\alpha \wedge \beta)^\circ \leq \alpha^\circ \wedge \beta^\circ$. Thus $(\alpha \wedge \beta)^\circ = \alpha^\circ \wedge \beta^\circ$. For 5., we have that $\alpha^\circ \leq \alpha$ and $\beta^\circ \leq \beta$ then $\alpha^\circ \oplus \beta^\circ \leq \alpha \oplus \beta$, so $\alpha^\circ \oplus \beta^\circ \leq (\alpha \oplus \beta)^\circ$ because $\alpha^\circ \oplus \beta^\circ \in \tau$. The proof of 6. is analogous to item 5. interchanging $\oplus$ for $\odot$. Thus we have $\alpha^\circ \odot \beta^\circ \leq (\alpha \odot \beta)^\circ$. □

Proposition 4.7.7. *The closure of the Definition 4.7.1 is an MV-closure operator.*

Proof. The items 1., 2. and 3. are clear of the definition. For the item 4., we have that $\alpha \leq \overline{\alpha}$ and $\beta \leq \overline{\beta}$ then $\alpha \vee \beta \leq \overline{\alpha} \vee \overline{\beta}$ and so $\overline{\alpha \vee \beta} \leq \overline{\alpha} \vee \overline{\beta}$ because $\overline{\alpha} \vee \overline{\beta}$ is a closed set. On the other hand, as $\alpha, \beta \leq \alpha \vee \beta$ then $\overline{\alpha}, \overline{\beta} \leq \overline{\alpha \vee \beta}$ and therefore $\overline{\alpha} \vee \overline{\beta} \leq \overline{\alpha \vee \beta}$. For the item 5. we have that $\alpha \leq \overline{\alpha}$ and $\beta \leq \overline{\beta}$ then $\alpha \oplus \beta \leq \overline{\alpha} \oplus \overline{\beta}$ so $\overline{\alpha \oplus \beta} \leq \overline{\alpha} \oplus \overline{\beta}$ because $\overline{\alpha} \oplus \overline{\beta}$ is a closed set. On the other hand,

$$\begin{aligned}
&\overline{\alpha} \oplus \overline{\beta} = \textstyle\bigwedge_{i \in I}\{\alpha_i \in \tau^* : \alpha \leq \alpha_i\} \oplus \bigwedge_{j \in J}\{\beta_j \in \tau^* : \beta \leq \beta_j\} = \\
&= \textstyle\bigwedge_{i \in I, j \in J}\{\alpha_i \oplus \beta_j : \alpha_i, \beta_j \in \tau^*; \alpha \leq \alpha_i, \beta \leq \beta_j\} \geq \\
&\geq \textstyle\bigwedge\{\gamma \in \tau^* : \alpha \oplus \beta \leq \gamma\} = \overline{\alpha \oplus \beta},
\end{aligned}$$

so $\overline{\alpha \oplus \beta} = \overline{\alpha} \oplus \overline{\beta}$. To verify 6., it is enough interchanging $\oplus$ by $\odot$ in the item 5. □

In the other direction, any MV-interior operator on X can determine some MV-topology for X and any MV-closure operator on X can determine an MV-topology for X. We have the following two theorems:

Theorem 4.7.8. *Let f be an MV-interior operator on X, let $\tau = \{\alpha \in [0,1]^X : f(\alpha) = \alpha\}$ then τ is an MV-topology for X and for every $\beta \in [0,1]^X$, $f(\beta)$ is the τ-interior of β. The topology τ thus determined will be called the MV-topology associated with an MV-interior operator.*

Proof. By the axiom 1. of the Definition 4.7.3, we have that $\mathbf{1} \in \tau$. By the axiom 2., $f(\mathbf{0}) \leq \mathbf{0}$, then $f(\mathbf{0}) = \mathbf{0}$, and so $\mathbf{0} \in \tau$. Now, let $\alpha, \beta \in \tau$, i. e., $f(\alpha) = \alpha$ and $f(\beta) = \beta$ then:

(i) $\alpha \wedge \beta \in \tau$ because, by the axiom 4., $f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta) = \alpha \wedge \beta$.
(ii) By the axiom 5., $\alpha \oplus \beta = f(\alpha) \oplus f(\beta) \leq f(\alpha \oplus \beta)$ and by the axiom 2., $f(\alpha \oplus \beta) \leq \alpha \oplus \beta$, then $f(\alpha \oplus \beta) = \alpha \oplus \beta$ and therefore $\alpha \oplus \beta \in \tau$.
(iii) By the axiom 6., $\alpha \odot \beta = f(\alpha) \odot f(\beta) \leq f(\alpha \odot \beta)$ and by the axiom 2., $f(\alpha \odot \beta) \leq \alpha \odot \beta$, then $f(\alpha \odot \beta) = \alpha \odot \beta$ and therefore $\alpha \odot \beta \in \tau$.

Let $\{\alpha_i : i \in I\}$ be a family of elements of τ, i.e, $f(\alpha_i) = \alpha_i$ for every $i \in I$. We know that for all $i \in I$, $\alpha_i \leq \bigvee_{i \in I} \alpha_i$ then by the remark 4.7.5, $f(\alpha_i) \leq f(\bigvee_{i \in I} \alpha_i)$ for each $i \in I$, so that $\bigvee_{i \in I} f(\alpha_i) \leq f(\bigvee_{i \in I} \alpha_i)$. Now, by the axiom 2., $f(\bigvee_{i \in I} \alpha_i) \leq \bigvee_{i \in I} \alpha_i = \bigvee_{i \in I} f(\alpha_i)$, then $f(\bigvee_{i \in I} \alpha_i) = \bigvee_{i \in I} \alpha_i$, and so $\bigvee_{i \in I} \alpha_i \in \tau$. We have proved that τ is an MV-topology.

It remains to show that $f(\alpha) = \alpha^\circ$. By definition, $\alpha^\circ = \bigvee\{\beta \in \tau : \beta \leq \alpha\}$ and by the axiom 3. of the Definition 4.7.3, $f(\alpha) \in \tau$ for every $\alpha \in [0,1]^X$; besides $f(\alpha) \leq \alpha$ then $f(\alpha) \leq \alpha^\circ$. On the other hand $\alpha^\circ \in \tau$ then $\alpha^\circ = f(\alpha^\circ) \leq f(\alpha)$ therefore $f(\alpha) = \alpha^\circ$. □

Theorem 4.7.9. *Let f be an MV-closure operator on X, let $F = \{\alpha \in [0,1]^X : f(\alpha) = \alpha\}$ and let $\tau = \{\alpha^* : \alpha \in F\}$ then τ is an MV-topology for X and for every $\beta \in [0,1]^X$, $f(\beta)$ is the τ-closure of β. The topology τ thus determined will be called the MV-topology associated with an MV-closure operator.*

Proof. By the axiom 1. of the Definition 4.7.4 we have that $\mathbf{0} \in F$, so that $\mathbf{1} = \mathbf{0}^* \in \tau$. By the axiom 2., $\mathbf{1} \leq f(\mathbf{1})$ so $\mathbf{1} = f(\mathbf{1})$ wherewith $\mathbf{1} \in F$ and then $\mathbf{0} = \mathbf{1}^* \in \tau$. Let $\alpha^*, \beta^* \in \tau$, i. e., $f(\alpha) = \alpha$ and $f(\beta) = \beta$ then:

(i) $\alpha^* \wedge \beta^* = (\alpha \vee \beta)^* \in \tau$ because $f(\alpha \vee \beta) = f(\alpha) \vee f(\beta) = \alpha \vee \beta$; by the axiom 4.
(ii) $\alpha^* \odot \beta^* = (\alpha \oplus \beta)^* \in \tau$ because $f(\alpha \oplus \beta) = f(\alpha) \oplus f(\beta) = \alpha \oplus \beta$; by the axiom 5.
(iii) $\alpha^* \oplus \beta^* = (\alpha \odot \beta)^* \in \tau$ because $f(\alpha \odot \beta) = f(\alpha) \odot f(\beta) = \alpha \odot \beta$ by the axiom 6.

Let $\{\alpha_i^* : i \in I\}$ be a family of elements of τ, i.e., $f(\alpha_i) = \alpha_i$ for every $i \in I$. As $\bigwedge_{i\in I} \alpha_i \leq \alpha_i$ for each $i \in I$, then by the remark 4.7.5, $f(\bigwedge_{i\in I} \alpha_i) \leq f(\alpha_i)$ for each $i \in I$, so that $f(\bigwedge_{i\in I} \alpha_i) \leq \bigwedge_{i\in I} f(\alpha_i) = \bigwedge_{i\in I} \alpha_i$. Now, by the axiom 2., $\bigwedge_{i\in I} \alpha_i \leq f(\bigwedge_{i\in I} \alpha_i)$, then $f(\bigwedge_{i\in I} \alpha_i) = \bigwedge_{i\in I} \alpha_i$, i.e., $\bigwedge_{i\in I} \alpha_i \in F$ and then $\bigvee_{i\in I} \alpha_i^* = (\bigwedge_{i\in I} \alpha_i)^* \in \tau$. We have proved that τ is an MV-topology.

Now, let us see that $f(\alpha) = \overline{\alpha}$. By definition, $\overline{\alpha} = \bigwedge\{\beta \in \tau^* : \alpha \leq \beta\} = \bigwedge\{\beta \in F : \alpha \leq \beta\}$ and by the axiom 3., $f(\alpha) \in F$ for every $\alpha \in [0,1]^X$, and as $\alpha \leq f(\alpha)$, then $\overline{\alpha} \leq f(\alpha)$. As $\overline{\alpha} \in F$ and $\alpha \leq \overline{\alpha}$ then $f(\alpha) \leq f(\overline{\alpha}) = \overline{\alpha}$; therefore $f(\alpha) = \overline{\alpha}$. □

4.8 Quotient and Product Spaces

Let (X, τ) be an MV-topological space. Let R be an equivalence relation on X. Let X/R be the quotient set, and let $\varphi : X \longrightarrow X/R$ be the projection map. Let

$$\tau' = \{\alpha \in [0,1]^{X/R} : \varphi^{\leftsquigarrow}(\alpha) \in \tau\} = \{\alpha \in [0,1]^{X/R} : \alpha \circ \varphi \in \tau\}.$$

Let us see that τ' is an MV-topology on X/R. We know that τ' is a fuzzy topology by [37], that is, $\mathbf{1}, \mathbf{0} \in \tau'$, it is closed for arbitrary joins and finite meets. Let us see that τ' is closed for $\oplus$ and $\odot$.

We know that if $\sigma \circ \varphi, \delta \circ \varphi \in \tau$ then

$$((\sigma \oplus \delta) \circ \varphi)(x) = ((\sigma \oplus \delta)(\varphi(x)) = \sigma(\varphi(x)) \oplus \delta(\varphi(x)) = ((\sigma \circ \varphi) \oplus (\delta \circ \varphi))(x)$$

and as $(\sigma \circ \varphi) \oplus (\delta \circ \varphi) \in \tau$ then $\sigma \oplus \delta \in \tau'$. The same holds for $\odot$.

Definition 4.8.1. Let R be an equivalence relation on X. The MV–space $(X/R, \tau')$ with τ' defined as above, is called the *quotient MV-space* of (X, τ).

Definition 4.8.2. Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of MV-topological spaces. According to the general definition of Category Theory, we say that an MV-topological space (X, τ), with a family $(p_i : X \to X_i)_{i\in I}$ of continuous functions, is the *product* of the spaces $(\tau_i)_{i\in I}$ if, for any MV-topological space (Y, τ_Y) and any family of continuous functions $(f_i : Y \to X_i)_{i\in I}$, there exists a unique continuous function $f : Y \to X$ such that $p_i \circ f = f_i$ for all $i \in I$.

Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of MV-topological spaces. We define the *product MV-topology* τ on the Cartesian product $X = \prod_{i\in I} X_i$ by means of the subbase

$$S = \{\pi_i^{\leftsquigarrow}(\alpha) \mid \alpha \in \tau_i, i \in I\}, \tag{4.7}$$

where $\pi_i : X \to X_i$ is the canonical projection. The name "product MV-topology" is fully justified by the following result.

Theorem 4.8.3. *The MV-topological space (X, τ), with the canonical projections π_i, is the product of $\{(X_i, \tau_i)\}_{i \in I}$.*

Proof. First, it is immediate to see that all projections π_i are continuous.

Now let Y be an MV-topological space and $(f_i : Y \to X_i)_{i \in I}$ a family of continuous functions. We set $f : y \in Y \mapsto (f_i(y))_{i \in I} \in X$. Let us show that f is continuous. Let B be the base obtained from S as in Remark 4.2.3 and consider an open set $\beta \in \tau_X$. If $\beta \in S$, namely, $\beta = \alpha \circ \pi_i$ for some $\alpha \in \tau_i$ then, for all $y \in Y$,

$$(\beta \circ f)(y) = ((\alpha \circ \pi_i) \circ f)(y) = (\alpha \circ \pi_i)(f_i(y))_{i \in I} = \alpha(f_i(y)) = (\alpha \circ f_i)(y)$$

and therefore $f^{\leftsquigarrow}(\beta) = \beta \circ f = \alpha \circ f_i \in \tau_Y$ because each f_i is continuous. Now, let us assume that $\beta = \alpha \star \gamma$, with $\star \in \{\oplus, \odot, \wedge\}$ and $\alpha, \gamma \in B$ being such that $\alpha \circ f, \gamma \circ f \in \tau_Y$. Then we have that $\beta \circ f = (\alpha \star \gamma) \circ f = (\alpha \circ f) \star (\gamma \circ f) \in \tau_Y$. Then f is continuous.

Now, in order to prove that f is the universal extension of $(f_i)_{i \in I}$, let $g : Y \to X$ be a continuous function such that $\pi_i \circ g = f_i$ for each $i \in I$. For all $y \in Y$, $g(y) = (\pi_i(g(y)))_{i \in I} = (f_i(y))_{i \in I}$, and therefore $g = f$. $\square$

5

Compactness and Separation

In this chapter we show some central results of topology in their MV-topological version. We show a Tychonoff-type Theorem, a Urysohn-type Lemma, and a Stone-Čech Compactification for MV-topologies. For Tychonoff Theorem we shall present two proofs, one of which uses the functors between fuzzy and crisp topologies previously defined, while the other uses an analogous of Alexander Subbase Lemma. Then we will show that also our Tychonoff theorem is equivalent to the Axiom of Choice in ZF.

For what concerns compactness, we will also present Stone-Čech Compactification for MV-topological spaces.

Eventually, we shall define normality and study the I-fuzzy unit interval $\mathfrak{F}(I)$ defined by Hutton in [24]. We will show that the usual fuzzy topology on $\mathfrak{F}(I)$ is also an MV-topology and then, using this fact, we will obtain Urysohn Lemma for MV-topologies. We also define MV-uniformities and we show how to induce an MV-topology from a given MV-uniformity.

Last, inspired by Hutton's paper about fuzzy uniformities [25], we define an MV-uniform structure on the fuzzy interval $\mathfrak{F}(I)$ in such a way that the MV-topology generated by this MV-uniformity is the usual MV-topology on $\mathfrak{F}(I)$. Then we will define complete regularity for MV-spaces.

This results indicate that the MV-topological spaces have a good behaviour with respect to the generalisation of the classical topological concepts and constructions and this suggests other possible derived questions and new problems.

5.1 Some results about Hausdorff MV-spaces

We recall from Definition 4.3.5 that an MV-topological space, (X, τ) is called a *Hausdorff* (or *separated*) *space* if, for all $x \neq y \in X$, there exist $\alpha_x, \alpha_y \in \tau$ such that

(i) $\alpha_x(x) = \alpha_y(y) = 1$,
(ii) $\alpha_x \wedge \alpha_y = \mathbf{0}$.

In the following we give some results about Hausdorff MV-spaces and compact MV-spaces.

Lemma 5.1.1. *The product of Hausdorff MV-topologies is Hausdorff.*

Proof. The proof proceeds analogously to the classical case with no major differences. Indeed, let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of Hausdorff MV-spaces, (X, τ) its product space, and $(x_i)_{i \in I}, (y_i)_{i \in I}$ two distinct points of X. So there exists $j \in I$ such that $x_j \neq y_j$ and, since

every X_i is Hausdorff, there exist $\alpha_x, \alpha_y \in \tau_j$ such that $\alpha_x(x_j) = \alpha_y(y_j) = 1$ and $\alpha_x \wedge \alpha_y = \mathbf{0}$. Then it is not hard to see that the open sets $\alpha_x \circ \pi_j$ and $\alpha_y \circ \pi_j$ separate the given points of X, namely, $(\alpha_x \circ \pi_j)((x_i)_{i\in I}) = (\alpha_y \circ \pi_j)((y_i)_{i\in I}) = 1$ and $(\alpha_x \circ \pi_j) \wedge (\alpha_y \circ \pi_j) = \mathbf{0}$. □

Lemma 5.1.2. *The product of zero-dimensional MV-topological spaces is zero-dimensional.*

Proof. Since sums, products, and finite infima of clopens of an MV-topology are clopens, the assertion follows immediately from (4.7) and Remark 4.2.3. □

Theorem 5.1.3. *Let Y be a crisp subspace of a Hausdorff MV-space (X, τ). If Y is compact, then Y is closed in X.*

Proof. First, we will show that, for all $x \in X \setminus Y$, there exists $\alpha \in \tau$ with $\alpha(x) = 1$ and $\alpha \leq Y^*$.

Since (X, τ) is Hausdorff, for each $y \in Y$, we can find $\alpha_y, \beta_y \in \tau$ such that $\alpha_y(x) = \beta_y(y) = 1$ and $\alpha_y \wedge \beta_y = \mathbf{0}$. Thus $\{\beta_y \restriction Y : y \in Y\}$ is an open covering of Y, so it has an additive subcovering $\{\beta_{y_1} \restriction Y, \ldots, \beta_{y_n} \restriction Y\}$. Let $\alpha = \alpha_{y_1} \wedge \cdots \wedge \alpha_{y_n}$, then $\alpha(x) = 1$ and $\alpha \wedge (\beta_{y_1} \oplus \cdots \oplus \beta_{y_n}) = \mathbf{0}$, because $\alpha \wedge (\beta_{y_1} \oplus \cdots \oplus \beta_{y_n}) \leq (\alpha \wedge \beta_{y_1}) \oplus \cdots \oplus (\alpha \wedge \beta_{y_n}) = \mathbf{0}$ in any MV-algebra. We know that for each $y \in Y$, there exists k such that $\beta_{y_k}(y) > 0$, so $\alpha(y) = 0$. Hence $\alpha \leq Y^*$. Therefore, $X \setminus Y = \bigvee_{x\in X\setminus Y} \alpha_x$ is open in X, whence the thesis follows. □

Theorem 5.1.4. *Let (X, τ_X) and (Y, τ_Y) be MV-spaces and let $f, g : X \to Y$ be continuous functions. If Y is a Hausdorff MV-space then the set*

$$Z = \{x \in X : f(x) = g(x)\}$$

is a closed crisp subset.

Proof. Let $x \in X \setminus Z$, so $f(x) \neq g(x)$. Since Y is Hausdorff, there exist $\alpha, \beta \in \tau_Y$ such that $\alpha(f(x)) = \beta(g(x)) = 1$ and $\alpha \wedge \beta = \mathbf{0}$. Moreover, f and g are continuous, so we have that $\alpha \circ f$ and $\beta \circ g$ are open sets of (X, τ_X). Set $\gamma_x = (\alpha \circ f) \wedge (\beta \circ g)$. Then $\gamma_x \in \tau_X$, $\gamma_x(x) = 1$, and

$$\gamma_x(z) = ((\alpha \circ f) \wedge (\beta \circ g))(z) = \alpha(f(z)) \wedge \beta(g(z)) = (\alpha \wedge \beta)(f(z)) = 0,$$

for each $z \in Z$

It follows that $Z^* = \bigvee_{x\in X\setminus Z} \gamma_x \in \tau_X$, whence Z is closed. □

5.2 Tychonoff-type theorem for MV-topologies

In the present section we shall prove the MV-topological correspondents of Alexander Subbase Lemma (Lemma 5.2.2) and Tychonoff Theorem (Theorem 5.2.3). As in the classical case, the latter turns out to be an immediate consequence of the former. The proof of Lemma 5.2.2 is divided in various claims with the aim of making it more readable.

Lemma 5.2.1. *Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of compact MV-topological spaces and let (X, τ_X) be their product. Then any open cover Γ of X consisting solely of elements of the form $\alpha \circ \pi_i$, $\alpha \in \tau_i$, contains an additive cover.*

Proof. Let Γ be such a cover of X, and define

$$\Gamma_i = \{\alpha \in \tau_i : \alpha \circ \pi_i \in \Gamma\}.$$

We claim that

$$\exists j \in I \ \forall x \in X_j \ \exists \alpha_x \in \Gamma_j \ (\alpha_x(x) > 0). \tag{5.1}$$

Indeed, assuming by contradiction that (5.1) does not hold, namely, that for each index $i \in I$ there exists $a_i \in X_i$ such that $\alpha(a_i) = 0$ for all $\alpha \in \Gamma_i$, then obviously $(\bigvee \Gamma_i)(a_i) = 0$ for all $i \in I$. Therefore, setting $a = (a_i)_{i \in I} \in X$, we get

$$\begin{aligned}
&(\bigvee \Gamma)(a) = \\
&= \left(\bigvee_{i \in I} \left(\bigvee_{\alpha \in \Gamma_i} (\alpha \circ \pi_i)\right)\right)(a) = \\
&= \left(\bigvee_{i \in I} ((\bigvee \Gamma_i) \circ \pi_i)\right)(a) = \bigvee_{i \in I} (\bigvee \Gamma_i(a_i)) = \\
&= 0,
\end{aligned}$$

which implies that Γ does not cover X, in contradiction with the hypothesis. Hence the statement (5.1) holds.

Now, from (5.1) it follows that, for all $x \in X_j$, there exists $n_x < \omega$ such that $n_x \alpha_x(x) = 1$. Then the family $(n_x \alpha_x)_{x \in X_j}$ is an open cover of X_j and, by the compactness of X_j, there exist $x_1, \ldots, x_m \in X_j$ such that

$$\bigoplus_{k=1}^{m} n_{x_k} \alpha_{x_k} = 1.$$

It follows that

$$\bigoplus_{k=1}^{m} (n_{x_k}(\alpha_{x_k} \circ \pi_j)) = \bigoplus_{k=1}^{m} ((n_{x_k} \alpha_{x_k}) \circ \pi_j) = \bigoplus_{k=1}^{m} n_{x_k} \alpha_{x_k} = 1,$$

whence we obtain an additive subcover of Γ by simply taking n_{x_k} copies of each $\alpha_{x_k} \circ \pi_j$, $k = 1, \ldots, m$. □

Lemma 5.2.2 (Alexander Subbase Lemma for MV-Topologies). *Let (X, τ) be an MV-topological space and S a large subbase for τ. If every collection of sets from S that cover X has an additive subcover, then X is compact.*

Proof. By contradiction, suppose that every cover of X of elements of S has an additive subcover, and X is not compact. Then the collection

$$\mathfrak{F} = \{\Gamma \subseteq \tau \mid \bigvee \Gamma = 1 \text{ and } \Gamma \text{ does not contain additive covers}\}$$

is nonempty and partially ordered by set inclusion. We use Zorn's Lemma to prove that $\mathfrak{F}$ has a maximal element. Take any chain $\{E_\alpha\}_{\alpha \in A}$ in $\mathfrak{F}$; let us see that $E = \bigcup E_\alpha$ is an upper bound of such a chain in $\mathfrak{F}$. It is clear that $E \subseteq \tau$ and $\bigvee E = 1$. To see that E contains no additive subcover, look at any finite subcollection $\{f_1, \ldots, f_n\}$ in E. Then, for each j, there exists α_j such that $f_j \in E_{\alpha_j}$. Since we have a total ordering, there is some E_{α_0} which contains all of the f_j's. Thus such a finite collection cannot be an additive cover. Now, applying Zorn's Lemma, we can assert the existence of a maximal element M in $\mathfrak{F}$.

First of all, let see some properties of M.

Claim 1. $\alpha \notin M$ iff $M \cup \{\alpha\}$ has an additive subcover.

In other words $\alpha \notin M$ iff there exist $\beta_1, \dots, \beta_n \in M$ such that $\alpha \oplus \beta_1 \oplus \cdots \oplus \beta_n = 1$, and that is obvious.

Claim 2. $\alpha_1, \dots, \alpha_n \notin M$ implies $\alpha_1 \star \cdots \star \alpha_n \notin M$, for $\star \in \{\wedge, \oplus, \odot\}$.

Proof of Claim 2. First of all note that, for each $i \in \{1, \dots, n\}$, there exists a finite family $\{\beta_{ij}\}_{j=1}^{m_i}$ of elements of M such that

$$\alpha_i \oplus \bigoplus_{j=1}^{m_i} \beta_{ij} = 1, \quad \text{and} \quad \alpha_i \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \beta_{ij} = 1.$$

Hence, if we set $\beta := \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \beta_{ij}$, we have $\alpha_i \oplus \beta = 1$ for each $i \in \{1, \dots, n\}$.

For $\star = \wedge$, for each $x \in X$, we have that $(\alpha_1 \wedge \cdots \wedge \alpha_n)(x) = \alpha_{j_x}(x)$ for some $j_x \in \{1, \dots, n\}$. So, for each $x \in X$,

$$(\alpha_1 \wedge \cdots \wedge \alpha_n)(x) \oplus \beta(x) = \alpha_{j_x}(x) \oplus \beta(x) = 1$$

namely, $\alpha_1 \wedge \cdots \wedge \alpha_n \oplus \beta = 1$, and then $\alpha_1 \wedge \cdots \wedge \alpha_n \notin M$.

Concerning $\odot$, using (2.1), we have that

$$\bigodot_{i=1}^{n} \alpha_i \oplus \beta \geq \bigodot_{i=1}^{n-1} \alpha_i \odot (\alpha_n \oplus \beta) = \bigodot_{i=1}^{n-1} \alpha_i \odot 1 = \bigodot_{i=1}^{n-1} \alpha_i$$

then

$$\bigodot_{i=1}^{n} \alpha_i \oplus \beta \oplus \beta \geq \bigodot_{i=1}^{n-1} \alpha_i \oplus \beta \geq \bigodot_{i=1}^{n-2} \alpha_i$$

whereby

$$\bigodot_{i=1}^{n} \alpha_i \oplus \underbrace{\beta \oplus \cdots \oplus \beta}_{n-1 \text{ times}} \geq \alpha_1$$

and therefore

$$\bigodot_{i=1}^{n} \alpha_i \oplus \underbrace{\beta \oplus \cdots \oplus \beta}_{n \text{ times}} \geq \alpha_1 \oplus \beta = 1.$$

It follows that $(\alpha_1 \odot \cdots \odot \alpha_n) \oplus \underbrace{\beta \oplus \cdots \oplus \beta}_{n \text{ times}} = 1$ where $\beta \in M$ and then $\alpha_1 \odot \cdots \odot \alpha_n \notin M$.

Last, for $\star = \oplus$, if $\alpha_1, \dots, \alpha_n \notin M$ then, in particular, $\alpha_1 \oplus \beta = 1$. It follows that $\alpha_1 \oplus \cdots \oplus \alpha_n \oplus \beta = 1$ and, therefore, $\alpha_1 \oplus \cdots \oplus \alpha_n \notin M$.

Claim 3. If $\alpha \notin M$ and $\alpha \leq \beta$ then $\beta \notin M$.

Proof of Claim 3. Indeed, if $\alpha \notin M$ there exist $\beta_1, \dots, \beta_n \in M$ such that $\alpha \oplus \beta_1 \oplus \cdots \oplus \beta_n = 1$ and then $1 = \alpha \oplus \beta_1 \oplus \cdots \oplus \beta_n \leq \beta \oplus \beta_1 \oplus \cdots \oplus \beta_n$, so $\beta \notin M$.

Claim 4. M is an ideal of the MV-algebra $[0,1]^X$.

Proof of Claim 4. M is non-empty and, if $\alpha \in M$ and $\beta \leq \alpha$, then $\beta \in M$ by Claim 3. Moreover, if $\alpha, \beta \in M$ then $\alpha \oplus \beta \in M$ because, otherwise, if $\alpha \oplus \beta \notin M$ there exist $\beta_1, \ldots, \beta_n \in M$ such that $\alpha \oplus \beta \oplus \beta_1 \oplus \cdots \oplus \beta_n = 1$. But this is impossible because M does not contain additive subcovers.

Observe that, as a consequence of Claims 2 and 3, the set $F = \{\beta \in \tau : \beta \notin M\}$ is a filter of the MV-algebra $[0,1]^X$.

Let us now consider the set $T = M \cap S$, and let us prove that T is a cover of X. Since M is a covering of X, for each $a \in X$ there exists $\alpha_a \in M$ such that $\alpha_a(a) > 0$. On the other hand, since S is a subbase, there exists a family $\{t_i\}_{i \in I}$ of terms (or polynomials) in the language $\{\oplus, \odot, \wedge\}$, such that

$$\alpha_a = \bigvee_{i \in I} t_i(\beta_{i1}, \ldots, \beta_{in_i}) \tag{5.2}$$

where, for all $i \in I$, $n_i < \omega$, and $\{\beta_{ij}\}_{j=1}^{n_i} \subseteq S$.

Claim 5. Let t be a term in the language $\{\oplus, \odot, \wedge\}$ and let $t(\beta_1, \ldots, \beta_n) \in M$ with $\{\beta_1, \ldots, \beta_n\} \subseteq S$. If $t(\beta_1, \ldots, \beta_n)(a) > 0$ for some $a \in X$, then there exists $j \in \{1, \ldots, n\}$ such that $\beta_j \in M$ and $\beta_j(a) > 0$.

Proof of Claim 5. Let us proceed by induction on the length of the term t. If t has length 1, then $t(\beta) = \beta$ with $\beta \in S$, and the condition clearly holds.

Suppose for inductive hypothesis that the assertion holds for all term of length $< m$, and let $t(\beta_1, \ldots, \beta_n) \in M$ be a term of length m such that $t(\beta_1, \ldots, \beta_n)(a) > 0$ for some $a \in X$. Since t has length m then $t = r \star s$, where r and s are terms of length $< m$ and $\star \in \{\wedge, \odot, \oplus\}$. Then we have to distinguish three cases.

If $t = r \wedge s$ then $t(\beta_1, \ldots, \beta_n)(a) = r(\beta_1, \ldots, \beta_n)(a) \wedge s(\beta_1, \ldots, \beta_n)(a)$, so $r(\beta_1, \ldots, \beta_n)(a) > 0$ and $s(\beta_1, \ldots, \beta_n)(a) > 0$ because $t(\beta_1, \ldots, \beta_n)(a) > 0$. Furthermore, since $(r \wedge s)(\beta_1, \ldots, \beta_n) \in M$, by Claim 2, $r(\beta_1, \ldots, \beta_n) \in M$ or $s(\beta_1, \ldots, \beta_n) \in M$. Without loss of generality we can assume that $r(\beta_1, \ldots, \beta_n) \in M$, then for inductive hypothesis we have that there exists $j \in \{1, \ldots, n\}$ such that $\beta_j \in M$ and $\beta_j(a) > 0$. So the assertion holds for $t = r \wedge s$.

If $t = r \odot s$ then $t(\beta_1, \ldots, \beta_n)(a) = r(\beta_1, \ldots, \beta_n)(a) \odot s(\beta_1, \ldots, \beta_n)(a)$, so $r(\beta_1, \ldots, \beta_n)(a) > 0$ and $s(\beta_1, \ldots, \beta_n)(a) > 0$ because $t(\beta_1, \ldots, \beta_n)(a) > 0$. As in the previous case, $r(\beta_1, \ldots, \beta_n) \in M$ or $s(\beta_1, \ldots, \beta_n) \in M$ for Claim 2. Without loss of generality we can assume that $r(\beta_1, \ldots, \beta_n) \in M$, then for inductive hypothesis we have that there exists $j \in \{1, \ldots, n\}$ such that $\beta_j \in M$ and $\beta_j(a) > 0$. So Claim 5 holds if $t = r \odot s$.

Last, if $t = r \oplus s$ then $t(\beta_1, \ldots, \beta_n)(a) = r(\beta_1, \ldots, \beta_n)(a) \oplus s(\beta_1, \ldots, \beta_n)(a)$, so $r(\beta_1, \ldots, \beta_n)(a) > 0$ or $s(\beta_1, \ldots, \beta_n)(a) > 0$ because $t(\beta_1, \ldots, \beta_n)(a) > 0$. Furthermore $r(\beta_1, \ldots, \beta_n) \in M$ and $s(\beta_1, \ldots, \beta_n) \in M$, because $t(\beta_1, \ldots, \beta_n) \in M$, $r(\beta_1, \ldots, \beta_n), s(\beta_1, \ldots, \beta_n) \leq t(\beta_1, \ldots, \beta_n)$, and M is an ideal. Without loss of generality we can assume that $r(\beta_1, \ldots, \beta_n)(a) > 0$, then for inductive hypothesis we have that there exists $j \in \{1, \ldots, n\}$ such that $\beta_j \in M$ and $\beta_j(a) > 0$. So the assertion holds also for $t = r \oplus s$, and this completes the proof of Claim 5.

Now, from the representation of α_a in (5.2) we have that for each $i \in I$, $t_i(\beta_{i1}, \ldots, \beta_{in_i}) \in M$ because $t_i(\beta_{i1}, \ldots, \beta_{in_i}) \leq \alpha_a$ for each $i \in I$ and α_a is an element of the ideal M.

Moreover, there exists $j \in I$ such that $t_j(\beta_{j1}, \ldots, \beta_{jn_j})(a) > 0$ because $\alpha_a(a) > 0$. Then, by Claim 5, we have that there exists $\beta_a = \beta_{jk} \in M$ with $k \in \{1, \ldots, n_j\}$ such that $\beta_a(a) > 0$. Therefore we get $n_a \beta_a(a) = 1$ for some $n_a < \omega$.

It means that the family $\{n_a\beta_a\}_{a\in X}$ is a covering of X which is contained in $T = M \cap S$. From the hypothesis about S we have that T has an additive subcover, so there exists a finite subset $\{n_{a_1}\beta_{a_1}, \ldots, n_{a_t}\beta_{a_t}\}$ of T such that $n_{a_1}\beta_{a_1} \oplus \cdots \oplus n_{a_t}\beta_{a_t} = 1$. But this means that M has an additive subcover too, which is a contradiction.

Therefore, our original collection $\mathfrak{F}$ must be empty, whence X is compact. □

Theorem 5.2.3 (Tychonoff-type Theorem for MV-Topologies). *If $\{(X_i, \tau_i)\}_{i\in I}$ is a family of compact MV-topological spaces, then so is their product space (X, τ_X).*

Proof. Let us consider as a subbase for the product MV-topology on X the collection

$$S = \{\pi_i^{\leftsquigarrow}(\beta) : \beta \in \tau_i, i \in I\}.$$

Note that S is a large subbase; indeed, for each $n < \omega$, $n(\beta \circ \pi_i) = n\beta \circ \pi_i$, and $n\beta \in \tau_i$ whenever $\beta \in \tau_i$. By Lemma 5.2.1, any subcollection of S that covers X has an additive subcover. Then the compactness of X follows from Lemma 5.2.2. □

We remark that Theorem 5.2.3 can be also obtained as a corollary of the following two results.

Theorem 5.2.4. *Every MV-topological space (X, τ) is compact if, and only if it is ultra-fuzzy compact in the sense of Lowen [31], i.e., the topological space $(X, \iota(\tau))$ is compact.*

Proof. The "only if" part is trivial. For what concerns the converse implication, suppose that $(X, \iota(\tau))$ is a compact topological space and $\{\alpha_i : i \in I\}$ is an open cover of X. For each $\beta \in \tau$ and $t \in [0, 1)$, we remember that $\beta_t = \{x \in X : \beta(x) > t\}$. Since the family $\{(\alpha_i)_{\frac{1}{2}} : i \in I\}$, is an open cover of the topological space $(X, \iota(\tau))$, there exists a finite subfamily $\{(\alpha_{i_1})_{\frac{1}{2}}, \ldots, (\alpha_{i_m})_{\frac{1}{2}}\}$ that covers X. This means $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ is an additive open cover of (X, τ). □

Theorem 5.2.5. *[31, Theorem 3.3] Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of fuzzy topological spaces. The product space $(\prod_{i\in I} X_i, \tau)$ is ultra-fuzzy compact if and only if for all $i \in I$, (X, τ_i) is ultra-fuzzy compact.*

In the following we shall briefly discuss some immediate consequences of Theorem 5.2.3.

Corollary 5.2.6. *The product of Stone MV-spaces is a Stone MV-space.*

Proof. It is a consequence of Lemmas 5.1.1, 5.1.2 and the Theorem 5.2.3. □

Corollary 5.2.7. *The category $\mathcal{MV}^{\mathrm{lcc}}$, of limit cut complete MV-algebras and MV-algebra homomorphisms, has coproducts.*

Proof. It is an immediate consequence of Theorem 5.2.3, Lemmas 5.1.1 and 5.1.2, and the duality between $\mathcal{MV}^{\mathrm{lcc}}$ and ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ [42, Theorem 4.9]. □

It is important to observe that Corollary 5.2.7 does not guarantee that the coproduct, in $\mathcal{MV}$, of lcc MV-algebras is lcc too. Moreover, as Mundici observed in [35, Corollary 7.4], the classes of totally ordered, hyperarchimedean, simple, and semisimple MV-algebras are not preserved under coproducts in the category of MV-algebras.

In order to better understand coproducts of lcc MV-algebra we prove the following result.

Proposition 5.2.8. *Let $(A_i)_{i\in I}$ be a family of lcc MV-algebras, and let A, A', and A'' be the coproducts of such a family in $\mathcal{MV}^{\mathrm{lcc}}$, $\mathcal{MV}^{\mathrm{ss}}$, and $\mathcal{MV}$, respectively. Then we have $A \cong A' \cong A''/\operatorname{Rad} A''$.*

Proof. Let $(\mu_i)_{i\in I}$, $(\nu_i)_{i\in I}$, and $(\eta_i)_{i\in I}$ be, respectively, the embeddings of the given family in A, A', and A''. For any semisimple MV-algebra B and morphisms $(f_i : A_i \to B)_{i\in I}$, there exists a morphism $f : A'' \to B$ such that $f\eta_i = f_i$ for all $i \in I$. The semisimplicity of B guarantees that $\ker f \subseteq \operatorname{Rad} A''$ and, therefore, there exists a morphism $g : A''/\operatorname{Rad} A'' \to B$ such that $g\pi = f$, where π is the canonical projection of A'' over $A''/\operatorname{Rad} A''$. So, for all $i \in I$, $g\pi\eta_i = f\eta_i = f_i$. Therefore $A''/\operatorname{Rad} A''$ is the coproduct in $\mathcal{MV}^{\mathrm{ss}}$ of $(A_i)_{i\in I}$, with embeddings $(\pi\eta_i)_{i\in I}$, whence $A' \cong A''/\operatorname{Rad} A''$.

Now, by [42, Corollary 5.8], the lcc completion $(A')^{\mathrm{lcc}}$ of A' is also a coproduct of the family $(A_i)_{i\in I}$ in $\mathcal{MV}^{\mathrm{lcc}}$. Therefore, if we denote by $\iota : A' \to (A')^{\mathrm{lcc}}$ the inclusion morphism, by $\mu : A' \to A$ the morphism such that $(\mu\nu_i)_{i\in I} = (\mu_i)_{i\in I}$, and by $\overline{\mu}$ the unique extension of μ to $(A')^{\mathrm{lcc}}$ as in [42, Corollary 5.8], we get that $\overline{\mu}$ is an isomorphism for the essential uniqueness of coproducts in any given category, and μ is onto because it is surjective on a generating set of A. On the other hand, the families $(\mu_i)_{i\in I}$, $(\nu_i)_{i\in I}$, and $(\iota\nu_i)_{i\in I}$ are right-cancellable, for being epi-sinks. It follows that $\overline{\mu}\iota = \mu$, i.e., μ is injective too. Then μ is an isomorphism, and we get $A' = (A')^{\mathrm{lcc}}$, $\iota = \mathrm{id}_{A'}$, $\mu = \overline{\mu}$, and $A \cong A'$. The diagram below will better illustrate the last part of the proof.

$$\begin{tikzcd}
 & A \arrow[dr, shift left, "\overline{\mu}^{-1}"] & \\
A_i \arrow[ur, "\mu_i"] \arrow[dr, "\nu_i"'] & & \\
 & A' \arrow[uu, "\mu"] \arrow[r, "\iota"'] & (A')^{\mathrm{lcc}} \arrow[uul, shift left, "\overline{\mu}"]
\end{tikzcd}$$

□

In [30] the author proved Tychonoff theorem for lattice-valued fuzzy topology. Theorem 5.2.3 obviously imply classical Tychonoff theorem because every classical topological space is an MV-topological space too. On the other hand, it is known that the same holds – although less obviously – for Lowen's result, as we show in the next proposition which can be easily deduced from the results in [31].

We recall the categorical full embedding $\omega : \mathcal{T}\mathrm{op} \to \mathcal{F}\mathrm{uz}$, of the category of topological spaces and continuous functions into the one of laminated fuzzy topologies, with fuzzy continuous functions, which associates, to each topological space, the so-called topologically generated fuzzy topological space (see Section 3.4 and [29]).

Proposition 5.2.9. *Lowen's Tychonoff theorem implies Tychonoff theorem.*

Proof. We need to prove that if the product of every family of fuzzy compact topological spaces is fuzzy compact, then the product of every family of compact topological spaces is compact. In order to do that, we recall the following facts.

- Fuzzy compactness is a *good* fuzzy topological property, namely, a topologically generated fuzzy topological space is compact iff the underlying topological space is compact [29, Theorem 4.1] and [31, Theorem 2.1].

- The ω functor commutes with products [43, Corollary 3.7].

Let $\{(X_i, \tau_i)\}_{i\in I}$ a family of compact topological spaces. Since compactness is a good property, the topogically generated fuzzy spaces of the family $\{(X_i, \omega(\tau_i))\}_{i\in I}$ are compact. On the other hand, the product of such fuzzy spaces is topologically generated by the product of the spaces X_i, because ω commutes with products. By Lowen's theorem, such a product is fuzzy compact. Then the product of the X_i is compact, again, because compactness is a good property. □

Actually, they both need the Axiom of Choice, which is known to be equivalent to Tychonoff theorem in ZF [27]. Therefore, the following holds.

Theorem 5.2.10. *The following statements are equivalent in ZF:*

(a) the Cartesian product of a non-empty family of non-empty sets is non-empty (AC);
(b) the product space of compact topological spaces is compact [47];
(c) the product space of fuzzy compact topological spaces (in the sense of [29]) is fuzzy compact [30];
(d) the product space of compact MV-topological spaces is compact (Theorem 5.2.3).

5.3 Compactification

In 1981, Cerruti [4] studied some concepts of fuzzy topological spaces from the categorical point of view. In particular, he studied compactness and developed a compactification theory. For that, he showed the existence of a left adjoint functor to the embedding $e : \mathcal{HCA}\mathcal{F}\text{uz} \longrightarrow \mathcal{F}\text{uz}$ where $\mathcal{HCA}\mathcal{F}\text{uz}$ is the category of Hausdorff Compact weakly induced spaces. We do an analogous categorical proof on the MV-topological spaces.

Recall that we are denoting by ${}^{\text{CMV}}\mathcal{T}\text{op}$ the full subcategory of ${}^{\text{MV}}\mathcal{T}\text{op}$ consisting of compact MV-spaces, and by ${}^{\text{HCMV}}\mathcal{T}\text{op}$ the full subcategory formed by compact and Hausdorff MV-spaces.

The Stone-Cech Compactification

Let i be the embedding $i : {}^{\text{HCMV}}\mathcal{T}\text{op} \hookrightarrow {}^{\text{MV}}\mathcal{T}\text{op}$.

Proposition 5.3.1. *The category ${}^{\text{HCMV}}\mathcal{T}\text{op}$ satisfies the following properties:*

(a) ${}^{\text{HCMV}}\mathcal{T}\text{op}$ has all products.
(b) ${}^{\text{HCMV}}\mathcal{T}\text{op}$ has equalizer.
(c) ${}^{\text{HCMV}}\mathcal{T}\text{op}$ has a small cogenerator.

Proof. (a) Follows from Theorem 5.2.3 and Lemma 5.1.1.
(b) Let $f, g : X \longrightarrow Y$ be morphisms in ${}^{\text{HCMV}}\mathcal{T}\text{op}$. Seeing these morphisms in $\mathcal{S}\text{et}$, we know that $Z = \{x \in X : f(x) = g(x)\}$ is the equalizer of them. Now, since Y is a Hausdorff space, Z is closed in X by Theorem 5.1.4. So Z is a compact MV-space (Lemma 4.3.4) and the canonical injection $m : Z \longrightarrow X$ is the equalizer. Note that Z is an element of ${}^{\text{HCMV}}\mathcal{T}\text{op}$.

(c) Let us consider the interval I in $\mathcal{T}\mathrm{op}$ with the usual topology and show that the cogenerator in ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ is $e(I)$. Indeed, let X be an element of ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$, $x, y \in X$, $x \neq y$. As $\iota(X)$ is a compact Hausdorff space, there exists a morphism $f : \iota(X) \longrightarrow I$ in $\mathcal{T}\mathrm{op}$ such that $f(x) \neq f(y)$. By the Proposition 4.4.2, $f : X \longrightarrow e(I)$ is a morphism in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$.

□

From (a) and (b) of the last proposition, we have that ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ is small-complete and we obtain the following:

Theorem 5.3.2. *The functor $i : {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op} \longrightarrow {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ has a left adjoint.*

Proof. As ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ is small-complete and it has a small cogenerator, we have the result by the Special Adjoint Functor Theorem 1.3.6. □

We denote by $\widehat{\beta} : {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \longrightarrow {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ the left adjoint functor of i. Then we write $\widehat{\beta} \dashv i$. Note that ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ is a reflective subcategory of ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ because ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ is a full subcategory of ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$, then we have that each object X of ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ is isomorphic to its reflection, that is, $X \simeq \widehat{\beta}(X)$.

We will show now that $\widehat{\beta}$ is the Stone-Čech Compactification functor in the category ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$. In what follows, we denote by β the compactification functor in $\mathcal{T}\mathrm{op}$.

Theorem 5.3.3. *The functors β and $\iota\widehat{\beta}\omega$ are naturally isomorphic.*

Proof. Let us consider the following adjunctions:

(i) ω is a left adjoint of $\iota \restriction {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} : {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$ (see Proposition 4.4.2 (2.)),
(ii) $\widehat{\beta}$ is a left adjoint of i, and
(iii) $\iota \restriction {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$ is a left adjoint of e (see Proposition 4.4.2 (4.)).

Now, from the following compositions:

$$\mathcal{T}\mathrm{op} \xrightarrow{\omega} {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \xrightarrow{\widehat{\beta}} {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op} \xrightarrow{\iota} \mathcal{CH}\mathcal{T}\mathrm{op}$$

and

$$\mathcal{CH}\mathcal{T}\mathrm{op} \xrightarrow{e} {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op} \xrightarrow{i} {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \xrightarrow{\iota} \mathcal{T}\mathrm{op},$$

we obtain the adjunction $\iota\widehat{\beta}\omega \dashv \iota i e$.

But $\iota i e = i : \mathcal{CH}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$. So $\iota\widehat{\beta}\omega$ is a left adjoint of the embedding of $\mathcal{CH}\mathcal{T}\mathrm{op}$ in $\mathcal{T}\mathrm{op}$, and then it is naturally equivalent to β. □

Now, we show that for an MV-space X, the initial topology of X determines the initial topology of the *MV-compactification* of X.

Theorem 5.3.4. *For each X in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$, $\iota\widehat{\beta}(X) \cong \beta\iota(X)$.*

Proof. If we consider the adjunctions $\widehat{\beta} \dashv i$ and $\iota \dashv e$ used in the Theorem 5.3.3, and we compose them:

$${}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \xrightarrow{\widehat{\beta}} {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op} \xrightarrow{\iota} \mathcal{CH}\mathcal{T}\mathrm{op},$$

$$\mathcal{CH}\mathcal{T}\mathrm{op} \xrightarrow{e} {}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op} \xrightarrow{i} {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$$

then we obtain the following adjunction:

$$\iota\widehat{\beta} \dashv ie.$$

It is enough to show that the restriction of $\beta\iota$ on ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ is a left adjoint of $ie = e$.

From the Proposition 4.4.2 (4.), we have that $\iota \restriction {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} : {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$ is a left adjoint of e, that is, for all (X, τ) in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ and for all (Y, δ) in $\mathcal{T}\mathrm{op}$,

$$\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\iota(X), Y) = \mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(X, e(Y)).$$

Since $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\iota(X), Y) \cong \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\beta\iota(X), Y)$, then we have that

$$\mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(X, e(Y)) \cong \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\beta\iota(X), Y)$$

and so the thesis follows. □

As a consequence of this theorem we have that for each MV-space X, the canonical morphism $\eta_X : X \longrightarrow \widehat{\beta}(X)$ is equal to the canonical morphism $\iota(X) \longrightarrow \beta\iota(X)$.

Finally, we have the following result:

Theorem 5.3.5. *(i)* $\widehat{\beta}e = e\beta$.
(ii) If X is topologically generated then $\widehat{\beta}(X) = \omega\beta\iota(X)$.

Proof. (i) It is clear.

(ii) Let X be a topologically generated space, then $X = \omega\iota(X)$ by Proposition 3.4.2 (iv). Now, for a morphism $\varepsilon_X : \iota(X) \longrightarrow \beta\iota(X)$, then we have that

$$\varepsilon_X \in \mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(\omega\iota(X), \omega\beta\iota(X)) = \mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(X, \omega\beta\iota(X)).$$

Since $\omega\beta\iota(X)$ is an object of ${}^{\mathrm{HCMV}}\mathcal{T}\mathrm{op}$, there exists a unique f that makes the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\varepsilon_X"'] & \widehat{\beta}(X) \arrow[d, dashed, "f"] \\
& \omega\beta\iota(X)
\end{tikzcd}$$

That is, $f \in \mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(\widehat{\beta}(X), \omega\beta\iota(X))$ and from the properties 5. and 6. of the Proposition 4.4.2 we have that $\widehat{\beta}(X) \in \omega(\mathcal{T}\mathrm{op})$. So, $\widehat{\beta}(X) = \omega\iota\widehat{\beta}(X)$ and from $\omega(\mathcal{T}\mathrm{op}) \subseteq {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ and by Theorem 5.3.4, we have that $\widehat{\beta}(X) = \omega\iota\widehat{\beta}(X) = \omega\beta\iota(X)$.

□

As a consequence of the item (ii) of the last theorem, we have that $\widehat{\beta}\omega = \omega\beta$. In fact, for each $X \in {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$, $\widehat{\beta}\omega(X) = \omega\beta\iota\omega(X) = \omega\beta(X)$.

On the other hand, we have that if X is an MV-space and $\widehat{\beta}(X)$ is topologically generated, then X is topologically generated. This follows because η_X is an element of $\mathrm{Hom}_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(X, \widehat{\beta}(X))$ and using the properties 5. and 6. of Proposition 4.4.2.

With the previous results, we showed that $\widehat{\beta}$ has similar properties to those of the Stone-Čech Compactification β and is an extension of it. Besides, we proved that for each topologically generated MV-space X, the compactification $\widehat{\beta}(X)$ is completely determined.

5.4 Normality and Urysohn's Lemma

The first definition of normality for fuzzy topological spaces was given by Hutton in [24]. In this section we adopt that definition and its respective consequences for MV-topological spaces.

Definition 5.4.1. An MV-topological space (X, τ) is *normal* if for every closed set $\alpha \in \tau^*$ and open set $\beta \in \tau$ such that $\alpha \leq \beta$, there exists $\gamma \in \tau$ such that

$$\alpha \leq \gamma \leq \overline{\gamma} \leq \beta.$$

Proposition 5.4.2. *An MV-topological space (X, τ) is normal if and only if for each pair of closed fuzzy sets α and β such that $\alpha \odot \beta = \mathbf{0}$ there are $\gamma, \delta \in \tau$ such that $\alpha \leq \gamma$, $\beta \leq \delta$ and $\gamma \odot \delta = \mathbf{0}$.*

Proof. $\Rightarrow$) Let $\alpha, \beta \in \tau^*$ such that $\alpha \odot \beta = \mathbf{0}$. This means that $\alpha \in \tau^*$, $\beta^* \in \tau$ and $\alpha \leq \beta^*$. As (X, τ) is normal, there exists a set $\gamma \in \tau$ such that $\alpha \leq \gamma \leq \overline{\gamma} \leq \beta^*$. So $\alpha \leq \gamma$, $\beta \leq \overline{\gamma}^*$ where $\gamma, \overline{\gamma}^* \in \tau$ satisfy $\gamma \odot \overline{\gamma}^* = \mathbf{0}$, and we have the conclusion.

$\Leftarrow$) Let $\alpha \in \tau^*$, $\beta \in \tau$ and $\alpha \leq \beta$, i.e., $\alpha, \beta^* \in \tau^*$ such that $\alpha \odot \beta^* = \mathbf{0}$, then there exist $\gamma, \delta \in \tau$ such that $\alpha \leq \gamma$, $\beta^* \leq \delta$ and $\gamma \odot \delta = \mathbf{0}$, that is $\gamma \leq \delta^* \in \tau^*$, then $\gamma \leq \overline{\gamma} \leq \delta^*$ and $\alpha \leq \gamma \leq \overline{\gamma} \leq \delta^* \leq \beta$, then (X, τ) is normal.

□

In [24], Hutton also introduced and defined the L-fuzzy unit interval and three years later Gantner et al., [19], generalized this idea to the L-fuzzy real line. Hutton used this fuzzy concept to give a fuzzy version of the Urysohn's Lemma. In the following, we present the I-fuzzy real line (when $L = I := [0, 1]$). This construction can be found in [44], although with different notations from the ones used here. Finally, we show a type of Urysohn's Lemma for MV-spaces.

Let $Z_I(\mathbb{R})$ be the set of monotonic decreasing functions $f \in [0, 1]^{\mathbb{R}}$ such that:

(i) $f(x) = 1$ for all $x \in (-\infty, 0)$ and
(ii) $f(x) = 0$ for all $x \in (1, \infty)$.

That is,

$$Z_I(\mathbb{R}) = \{f \in [0, 1]^{\mathbb{R}} : \forall x < 0, f(x) = 1; \forall x > 1, f(x) = 0; f \text{ is decreasing}\}.$$

Definition 5.4.3. Let $f \in Z_I(\mathbb{R})$, we define for each $x \in \mathbb{R}$

$$f(x+) := \bigvee_{t>x} f(t) \text{ and } f(x-) := \bigwedge_{t<x} f(t)$$

Note that if $f, g \in Z_I(\mathbb{R})$ then:

(i) If $f \leq g$ then $f(x+) \leq g(x+)$ and $f(x-) \leq g(x-)$ for each $x \in \mathbb{R}$.
(ii) For all $x \in \mathbb{R}$, $f(x+) \leq f(x) \leq f(x-)$.

On $Z_I(\mathbb{R})$ we introduce the following equivalence relation:

$$f_1 \sim f_2 \text{ iff } f_1(x+) = f_2(x+) \text{ and } f_1(x-) = f_2(x-) \text{ for all } x \in \mathbb{R}.$$

We actually have that

$$f_1(x+) = f_2(x+) \text{ iff } f_1(x-) = f_2(x-) \text{ for all } x \in \mathbb{R}.$$

Definition 5.4.4. The *fuzzy unit interval* $\mathfrak{F}(I)$ is the set of all monotonic decreasing maps $f \in Z_I(\mathbb{R})$ after the identification by the relation $\sim$. That is, $\mathfrak{F}(I) = Z_I(\mathbb{R})/\sim$.

Note that for each equivalence class $[f] \in \mathfrak{F}(I)$, there is only one left semicontinuous function in it. We introduce a partial order in $\mathfrak{F}(I)$ by

$$[f_1] \leq [f_2] \text{ iff } f_1(x+) \leq f_2(x+) \text{ for every } x \in \mathbb{R}.$$

We define an MV-topology σ on $\mathfrak{F}(I)$ by taking as a subbase the family of fuzzy sets $\{L_t, R_t : t \in \mathbb{R}\}$ where $L_t, R_t : \mathfrak{F}(I) \longrightarrow [0,1]$ are such that

$$L_t(f) = f(t-)^* \text{ and } R_t(f) = f(t+) \text{ for all } t \in \mathbb{R}$$

Proposition 5.4.5. *We have that the functions L_t, R_t with $t \in \mathbb{R}$ satisfy the following properties:*

1. $R_t \wedge R_s = R_{t \vee s}$;
2. $\bigvee_{j \in J} R_{t_j} = R_{\bigwedge_{j \in J} t_j}$;
3. $L_t \wedge L_s = L_{t \vee s}$;
4. $\bigwedge_{j \in J} L_{t_j} = L_{\bigvee_{j \in J} t_j}$;
5. $R_t \oplus L_s = \mathbf{1}$ *if* $t \leq s$. *In particular,* $R_t \oplus L_t = \mathbf{1}$;
6. $R_t \odot L_s = \mathbf{0}$ *if* $t \geq s$. *In particular,* $R_t \odot L_t = \mathbf{0}$;
7. $\underline{L_t} \leq \overline{L_t} \leq R_t^* \leq L_s$ *if* $t \leq s$;
8. $\overline{R_s} \leq L_s^* \leq R_t$ *if* $t \leq s$;
9. $L_t \leq L_s$ *if* $t \leq s$;
10. $R_s \leq R_t$ *if* $t \leq s$;
11. $L_t \oplus L_s \leq 2L_{t \oplus s}$;
12. $L_{t \odot s} \leq L_t \oplus L_s$;
13. $L_t \odot L_s \leq L_{t \oplus s}$;
14. $R_t \odot R_s \leq R_{t \odot s}$;
15. $R_{t \oplus s} \leq R_t \oplus R_s$.

Remark 5.4.6. The I-fuzzy unit interval can be extended to the *fuzzy real line* replacing $Z_I(\mathbb{R})$ for the following set denoted by $Z(\mathbb{R})$:

$$Z(\mathbb{R}) := \{f \in [0,1]^{\mathbb{R}} : \bigvee_{x \in \mathbb{R}} z(x) = 1, \bigwedge_{x \in \mathbb{R}} z(x) = 0, f \text{ is monotonic decreasing}\}.$$

In this case, the fuzzy real line is $\mathfrak{F}(\mathbb{R}) := Z(\mathbb{R})/\sim$.

Theorem 5.4.7 (Urysohn-type Lemma). *An MV-topological space (X, τ) is normal if and only if for every closed $\beta \in \tau^*$ and open set $\alpha \in \tau$, with $\beta \leq \alpha$, there exists a continuous function $f : X \longrightarrow \mathfrak{F}(I)$ such that for every $x \in X$,*

$$\beta(x) \leq f(x)(1-) \leq f(x)(0+) \leq \alpha(x).$$

Proof. $\Leftarrow$) Let $\alpha, \beta \in \tau^*$, such that $\beta \odot \alpha = \mathbf{0}$, i. e., $\beta \leq \alpha^*$. By hypothesis, there exists a continuous function $f : X \longrightarrow \mathfrak{F}(I)$ such that for every $x \in X$,

$$\beta(x) \leq f(x)(1-) \leq f(x)(0+) \leq \alpha^*(x).$$

Then we have that, for each $t \in (0,1)$, and for each $x \in X$,

$$\beta(x) \leq f(x)(t+) \leq f(x)(t-) \leq \alpha^*(x).$$

On the other hand, note that for each $x \in X, t \in \mathbb{R}$

$$f^{\leftsquigarrow}(L_t^*)(x) = (L_t^* \circ f)(x) = L_t^*(f(x)) = f(x)(t-)$$

and

$$f^{\leftsquigarrow}(R_t)(x) = (R_t \circ f)(x) = R_t(f(x)) = f(x)(t+),$$

then

$$\beta(x) \leq f^{\leftsquigarrow}(R_t)(x) \leq f^{\leftsquigarrow}(L_t^*)(x) \leq \alpha^*(x)$$

and so

$$\beta \leq f^{\leftsquigarrow}(R_t) \leq f^{\leftsquigarrow}(L_t^*) \leq \alpha^*.$$

Since $\{L_t, R_t : t \in \mathbb{R}\}$ is a subbase for the MV-topology of $\mathfrak{F}(I)$ and f is continuous, we have that $f^{\leftsquigarrow}(L_t)$ and $f^{\leftsquigarrow}(R_t)$ are open sets of (X, τ), moreover as $f^{\leftsquigarrow}$ is an MV-homomorphism, $(f^{\leftsquigarrow}(L_t^*))^* = f^{\leftsquigarrow}(L_t)$ and we have that $\beta \leq f^{\leftsquigarrow}(R_t)$, $\alpha \leq (f^{\leftsquigarrow}(L_t^*))^* = f^{\leftsquigarrow}(L_t)$ and $f^{\leftsquigarrow}(R_t) \odot f^{\leftsquigarrow}(L_t) = \mathbf{0}$ because $f^{\leftsquigarrow}(R_t) \leq (f^{\leftsquigarrow}(L_t))^*$. Then (X, τ) is normal by the characterization given by the Proposition 5.4.2.

$\Rightarrow$) Conversely, since (X, τ) is normal, we have that, for every $\beta \in \tau^*$ and $\alpha \in \tau$ such that $\beta \leq \alpha$, there exists a family of fuzzy sets $\{\gamma_t : t \in (0,1)\}$ such that $\beta \leq \gamma_t \leq \alpha$, for all $t \in (0,1)$. Indeed, we have that $0 < r < s < 1$ implies $\gamma_s \leq \overline{\gamma_s} \leq \gamma_r^\circ \leq \gamma_r$.
We define $f : X \longrightarrow \mathfrak{F}(I)$ such that $f(x)(t) = \gamma_t(x)$. Note that for each $x \in X$, $f(x)$ is decreasing and

$$\beta(x) \leq f(x)(1-) \leq f(x)(0+) \leq \alpha(x),$$

because $f(x)(1-) = \bigwedge_{s<1} \gamma_s(x)$ and $f(x)(0+) = \bigvee_{s>0} \gamma_s(x)$. Thus, f is well defined.
In order to prove that f is continuous, we use Lemma 4.2.8. So it is enough to show that $f^{\leftsquigarrow}(L_t)$ and $f^{\leftsquigarrow}(R_t)$ are open sets of (X, τ), for each $t \in (0,1)$, which is equivalent to show that $f^{\leftsquigarrow}(R_t) \in \tau$ and $f^{\leftsquigarrow}(L_t^*) \in \tau^*$, for each t. We have:

$$f^{\leftsquigarrow}(R_t)(x) = R_t(f(x)) = f(x)(t+) = \bigvee_{s>t} f(x)(s) = \bigvee_{s>t} \gamma_s(x) = \bigvee_{s>t} \gamma_s^\circ(x)$$

where the last equality holds because for each $s \in (0,1)$, there exists $0 < r < s$ such that $\gamma_s \leq \overline{\gamma_s} \leq \gamma_r^\circ \leq \gamma_r$, and therefore $f^{\leftsquigarrow}(R_t) = \bigvee_{s>t} \gamma_s^\circ \in \tau$.
On the other hand,

$$f^{\leftsquigarrow}(L_t^*)(x) = L_t^*(f(x)) = f(x)(t-) = \bigwedge_{s<t} f(x)(s) = \bigwedge_{s<t} \gamma_s(x) = \bigwedge_{s<t} \overline{\gamma_s}(x)$$

then $f^{\leftsquigarrow}(L_t^*) = \bigwedge_{s<t} \overline{\gamma_s} \in \tau^*$.
Thus f is continuous.

□

Another equivalent way to present the Urysohn-type Lemma is the following:

Theorem 5.4.8 (Urysohn-type Lemma – 2nd version). *An MV-topological space (X, τ) is normal if and only if for every closed $\beta, \alpha \in \tau^*$ such that $\beta \odot \alpha = \mathbf{0}$ ("disjoint"), there exists a continuous function $f : X \longrightarrow \mathfrak{F}(I)$ such that for every $x \in X$,*

$$\beta \leq f^{\leftsquigarrow}(L_1^*) \leq f^{\leftsquigarrow}(R_0) \leq \alpha.$$

5.5 MV-uniformities and Complete Regularity

In this section we define MV-uniformities and we show that each MV-uniformity generates an MV-topology. We define an MV-uniformity for the fuzzy unite interval $\mathfrak{F}(I)$. Besides, we define complete regularity for MV-spaces and we show that each MV-topological space that is generated from an MV-uniformity is completely regular. Definitions and results here are inspired by [25, 26, 39].

Let Ω_X denote the family of all the functions $f : [0,1]^X \longrightarrow [0,1]^X$ with the following properties:

1. $\alpha \leq f(\alpha)$ for each $\alpha \in [0,1]^X$
2. $f(\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I} f(\alpha_i)$ for each family $\{\alpha_i\} \subseteq [0,1]^X$,
3. $f(\alpha \oplus \beta) \leq f(\alpha) \oplus f(\beta)$ for each $\alpha, \beta \in [0,1]^X$.

For $f \in \Omega_X$, the function $f^{-1} \in \Omega_X$ is defined by:

$$f^{-1}(\alpha) = \bigwedge\{\beta : f(\beta^*) \leq \alpha^*\}.$$

Some useful properties of f^{-1} are the following [25, Proposition 10]:

Proposition 5.5.1. *Let $f, g \in \Omega_X$. Then:*

(a) $f(\alpha) \leq \beta$ if and only if $f^{-1}(\beta^*) \leq \alpha^*$;
(b) $(f^{-1})^{-1} = f$;
(c) $f \leq g$ if and only if $f^{-1} \leq g^{-1}$;
(d) $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Definition 5.5.2. An *MV-quasi-uniformity* on a set X is a subset $\mathfrak{D}$ of Ω_X such that:

(MV-QU1) $\mathfrak{D} \neq \emptyset$,
(MV-QU2) $f \in \mathfrak{D}$ and $f \leq g$, with $g \in \Omega_X$, implies $g \in \mathfrak{D}$
(MV-QU3) $f \in \mathfrak{D}$ and $g \in \mathfrak{D}$ implies $f \wedge g \in \mathfrak{D}$,
(MV-QU4) $f \in \mathfrak{D}$ and $g \in \mathfrak{D}$ implies $f \odot g \in \mathfrak{D}$,
(MV-QU5) $f \in \mathfrak{D}$ implies there exists $g \in \mathfrak{D}$ such that $g \circ g \leq f$.

The pair $(X, \mathfrak{D})$ is called an *MV-quasi-uniform space*.

This definition agrees with the usual definition when $[0,1]$ is replaced by $\{0,1\}$.

Remark 5.5.3. The following are direct consequences of the previous definition.

- By the third property of Ω_X, each function $f \in \mathfrak{D}$ is increasing.
- Condition (MV-QU3) may be replaced by
  (MV-QU3') If $f, g \in \mathfrak{D}$ then there exists $h \in \mathfrak{D}$ such that $h \leq f$ and $h \leq g$.

Definition 5.5.4. An *MV-uniformity* on X is an MV-quasi-uniformity $\mathfrak{D}$ that also satisfies:

$$f \in \mathfrak{D} \text{ implies } f^{-1} \in \mathfrak{D}.$$

In this case, the pair $(X, \mathfrak{D})$ is said an *MV-uniform space*.

A *sub-basis for an MV-quasi-uniformity* $\mathfrak{D}$ is a nonempty subset B of Ω_X which satisfies (MV-QU5). If B also satisfies (MV-QU3') then B is called a *basis*. A *base for an MV-uniformity* on X is a base B for an MV-quasi-uniformity which also has the property that for each $f \in B$, there exists $g \in B$ such that $g \leq f^{-1}$.

In what follows we shall define the MV-topology induced by an MV-uniformity.

Definition 5.5.5. Let $(X, \mathfrak{D})$ be an MV-uniform space. Define $\mathrm{Int} : [0,1]^X \to [0,1]^X$ by

$$\mathrm{Int}(\beta) = \bigvee\{\alpha \in [0,1]^X : f(\alpha) \leq \beta \text{ for some } f \in \mathfrak{D}\}.$$

Proposition 5.5.6. *The operator* Int *is an MV-interior operator.*

Proof. 1. $\mathrm{Int}(\mathbf{1}) = \mathbf{1}$ because $\mathbf{1} \leq f(\mathbf{1})$ for all $f \in \mathfrak{D}$.

2. $\mathrm{Int}(\beta) \leq \beta$ because $\mathrm{Int}(\beta) = \bigvee\{\alpha \in [0,1]^X : f(\alpha) \leq \beta \text{ for some } f \in \mathfrak{D}\}$ and $\alpha \leq f(\alpha)$, for each $\alpha \in [0,1]^X$ and each $f \in \mathfrak{D}$.
3. Let $\alpha, \beta \in [0,1]^X$ and $f \in \mathfrak{D}$ such that $f(\alpha) \leq \beta$, that is, $\alpha \leq \mathrm{Int}(\beta)$. Then, by (MV-QU5) there exists $g \in \mathfrak{D}$ such that $g \circ g \leq f$. So $g(g(\alpha)) \leq f(\alpha) \leq \beta$. Thus $g(\alpha) \leq \mathrm{Int}(\beta)$ which implies $\alpha \leq \mathrm{Int}(\mathrm{Int}(\beta))$. Hence $\mathrm{Int}(\beta) \leq \mathrm{Int}(\mathrm{Int}(\beta))$. The other inequality holds by the item 2. of this proposition, so $\mathrm{Int}(\beta) = \mathrm{Int}(\mathrm{Int}(\beta))$.
4. Let us see that $\mathrm{Int}(\alpha \wedge \beta) = \mathrm{Int}(\alpha) \wedge \mathrm{Int}(\beta)$. Let $\gamma \in [0,1]^X$, $f \in \mathfrak{D}$ such that $f(\gamma) \leq \alpha \wedge \beta$ then $f(\gamma) \leq \alpha, \beta$ and therefore $\mathrm{Int}(\alpha \wedge \beta) \leq \mathrm{Int}(\alpha), \mathrm{Int}(\beta)$. Thus $\mathrm{Int}(\alpha \wedge \beta) \leq \mathrm{Int}(\alpha) \wedge \mathrm{Int}(\beta)$.
   In order to prove $\mathrm{Int}(\alpha) \wedge \mathrm{Int}(\beta) \leq \mathrm{Int}(\alpha \wedge \beta)$, we note that for arbitrary $f, g \in \mathfrak{D}$ and for arbitrary $\alpha, \beta, \gamma, \delta \in [0,1]^X$ such that $f(\gamma) \leq \alpha$ and $g(\delta) \leq \beta$, we have
   $$(f \wedge g)(\gamma \wedge \delta) \leq f(\gamma) \wedge g(\delta) \leq \alpha \wedge \beta$$
   This is because $f \wedge g$ is increasing and $(f \wedge g)(\gamma \wedge \delta) \leq (f \wedge g)(\gamma) = f(\gamma) \wedge g(\gamma) \leq f(\gamma)$, and analogously, $(f \wedge g)(\gamma \wedge \delta) \leq (f \wedge g)(\delta) \leq g(\delta)$. Thus
   $$\begin{aligned}&\mathrm{Int}(\alpha) \wedge \mathrm{Int}(\beta) = \bigvee\{\gamma \wedge \delta : \gamma, \delta \in [0,1]^X, \exists f, g \in \mathfrak{D}, f(\gamma) \leq \alpha, g(\delta) \leq \beta\}\\ &\leq \bigvee\{\gamma \wedge \delta : \gamma, \delta \in [0,1]^X, \exists f, g \in \mathfrak{D}, (f \wedge g)(\gamma \wedge \delta) \leq \alpha \wedge \beta\}\\ &\leq \bigvee\{\gamma \in [0,1]^X : \exists f \in \mathfrak{D}, f(\gamma) \leq \alpha \wedge \beta\}\\ &= \mathrm{Int}(\alpha \wedge \beta).\end{aligned}$$
5. For $\mathrm{Int}(\alpha) \odot \mathrm{Int}(\beta) \leq \mathrm{Int}(\alpha \odot \beta)$, we note that for each $f, g \in \mathfrak{D}$ and for each $\alpha, \beta, \gamma, \delta \in [0,1]^X$ such that $f(\gamma) \leq \alpha$ and $g(\delta) \leq \beta$,
   $$(f \odot g)^2(\gamma \odot \delta) \leq f(\gamma) \odot g(\delta) \leq \alpha \odot \beta.$$
   This is because $f \odot g \in \mathfrak{D}$ and then $f \odot g$ is increasing. Therefore,
   $$(f \odot g)(\gamma \odot \delta) \leq (f \odot g)(\gamma) \leq f(\gamma) \text{ and } (f \odot g)(\gamma \odot \delta) \leq (f \odot g)(\delta) \leq g(\delta).$$
   Thus
   $$\begin{aligned}&\mathrm{Int}(\alpha) \odot \mathrm{Int}(\beta) = \bigvee\{\gamma \odot \delta : \gamma, \delta \in [0,1]^X, \exists f, g \in \mathfrak{D}, f(\gamma) \leq \alpha, g(\delta) \leq \beta\}\\ &\leq \bigvee\{\gamma \odot \delta : \gamma, \delta \in [0,1]^X, \exists f, g \in \mathfrak{D}, (f \odot g)^2(\gamma \odot \delta) \leq \alpha \odot \beta\}\\ &\leq \bigvee\{\gamma \in [0,1]^X : \exists f \in \mathfrak{D}, f(\gamma) \leq \alpha \odot \beta\}\\ &= \mathrm{Int}(\alpha \odot \beta).\end{aligned}$$
6. Let see that $\mathrm{Int}(\alpha) \oplus \mathrm{Int}(\beta) \leq \mathrm{Int}(\alpha \oplus \beta)$. Note that for arbitrary $f, g \in \mathfrak{D}$, we have that $f \odot g \in \mathfrak{D}$. Then for each $\alpha, \beta, \gamma, \delta \in [0,1]^X$ such that $f(\gamma) \leq \alpha$ and $g(\delta) \leq \beta$ we have $\gamma \leq (f \odot g)(\gamma) \leq f(\gamma) \leq \alpha$ and $\delta \leq (f \odot g)(\delta) \leq f(\delta) \leq \beta$. Using the property 4. of Ω_X, we obtain the inequalities:
   $$\gamma \oplus \delta \leq (f \odot g)(\gamma \oplus \delta) \leq (f \odot g)(\gamma) \oplus (f \odot g)(\delta) \leq f(\gamma) \oplus g(\delta) \leq \alpha \oplus \beta.$$

Thus

$$\begin{aligned}
&\mathrm{Int}(\alpha) \oplus \mathrm{Int}(\beta) = \bigvee\{\gamma \oplus \delta : \gamma, \delta \in [0,1]^X, \exists f, g \in \mathfrak{D}, f(\gamma) \leq \alpha, g(\delta) \leq \beta\} \\
&\leq \bigvee\{\gamma \oplus \delta \in [0,1]^X : \exists h \in \mathfrak{D}, h(\gamma \oplus \delta) \leq \alpha \oplus \beta\} \\
&\leq \bigvee\{\rho \in [0,1]^X : \exists h \in \mathfrak{D}, h(\rho) \leq \alpha \oplus \beta\} \\
&= \mathrm{Int}(\alpha \oplus \beta).
\end{aligned}$$

This completes the proof. □

Definition 5.5.7. The MV-topology generated by an MV-uniformity $\mathfrak{D}$ is the MV-topology generated by the MV-interior operator Int of the previous proposition.

Note that for each $f \in \mathfrak{D}$ we have that $f(\alpha) \geq \alpha$ for each open α in the MV-topology generated by the MV-uniformity $\mathfrak{D}$.

Definition 5.5.8. Let $(X, \mathfrak{D})$ and $(Y, \mathfrak{E})$ be MV-quasi-uniform spaces. A map $\varphi : X \longrightarrow Y$ is said to be *MV-quasi-uniformly continuous* if for every $g \in \mathfrak{E}$, there exists an $f \in \mathfrak{D}$ such that $f \leq \varphi^{-1}(g)$, that is, for $\alpha \in [0,1]^X$, $f(\alpha) \leq \varphi^{-1}(g(\varphi(\alpha)))$.

Proposition 5.5.9. *Every MV-quasi-uniformly continuous function is continuous in the induced MV-topologies.*

Proof. Let $(X, \mathfrak{D})$ and $(Y, \mathfrak{E})$ be MV-quasi-uniform spaces and let $\varphi : X \longrightarrow Y$ be MV-quasi-uniformly continuous. Let α be an open set in the MV-topology generated by $\mathfrak{E}$. So, $\alpha = \bigvee\{\beta : g(\beta) \leq \alpha \text{ for some } g \in \mathfrak{E}\}$. If $g(\beta) \leq \alpha$ then there exists $f \in \mathfrak{D}$ such that

$$f(\varphi^{-1}(\beta)) \leq \varphi^{-1}(g(\varphi(\varphi^{-1}(\beta)))) \leq \varphi^{-1}(g(\beta)) \leq \varphi^{-1}(\alpha).$$

So, $\varphi^{-1}(\beta) \leq \mathrm{Int}(\varphi^{-1}(\alpha))$, and hence

$$\bigvee\{\varphi^{-1}(\beta) : g(\beta) \leq \alpha \text{ for some } g \in \mathfrak{E}\} \leq \mathrm{Int}(\varphi^{-1}(\alpha)).$$

But $\varphi^{-1}(\bigvee_{i \in I} \beta_i) = \bigvee_{i \in I} \varphi^{-1}(\beta_i)$ and hence $\varphi^{-1}(\alpha) \leq \mathrm{Int}(\varphi^{-1}(\alpha))$. That is, $\varphi^{-1}(\alpha)$ is an open set in the MV-topology generated by $\mathfrak{D}$. Therefore, φ is continuous. □

Now, we will construct an MV-uniform structure on the fuzzy interval $\mathfrak{F}(I)$ defined in Section 5.4 in such a way that the MV-topology generated by this MV-uniformity is the usual MV-topology on $\mathfrak{F}(I)$.

Definition 5.5.10. For $\epsilon > 0$, we define

$$B_\epsilon : [0,1]^{\mathfrak{F}(I)} \longrightarrow [0,1]^{\mathfrak{F}(I)}$$

by $B_\epsilon(\alpha) = R_{t-\epsilon}$ where $t = \sup\{s \in \mathbb{R} : \alpha \leq L_s^*\}$. That is, $B_\epsilon(\alpha) = \bigwedge\{R_{s-\epsilon} : \alpha \leq L_s^*\}$.

Lemma 5.5.11. *The family $\{B_\epsilon : \epsilon > 0\}$ satisfies the following properties:*

1. *For each $\epsilon > 0$, $B_\epsilon \in \Omega_X$*
2. *$B_\epsilon^{-1}(\alpha) = \bigwedge\{L_{s+\epsilon} : \alpha \leq R_s^*\}$.*
3. *$B_\epsilon \circ B_\delta \leq B_{\epsilon+\delta}$. In particular, $B_\epsilon \circ B_\epsilon \leq B_{2\epsilon}$.*

Proof. In this proof we will use some of the properties given in Proposition 5.4.5.

1. Let us verify that $\{B_\epsilon : \epsilon > 0\}$ satisfies the three conditions of Ω_X:

(i) $\alpha \leq B_\epsilon(\alpha)$ because $L_s^* \leq R_{s-\epsilon}$ for s.
(ii) If $\alpha_j \leq L_{s_j}^*$ then for each $\delta > 0$, we have the following implications:
$\alpha_i \leq R_{s_j-\delta}$
$\Rightarrow \bigvee_{j\in J} \alpha_j \leq \bigvee_{j\in J} R_{s_j-\delta}$
$\Rightarrow \bigvee_{j\in J} \alpha_j \leq R_{\bigwedge_{j\in J} s_j-\delta}$.
Then $\bigvee_{j\in J} \alpha_j \leq L^*_{\bigwedge_{j\in J} s_j} \leq R_{\bigwedge_{j\in J} s_j-\epsilon}$ implies

$$B_\epsilon(\bigvee_{j\in J} \alpha_j) \leq R_{\bigwedge_{j\in J} s_j-\epsilon} = \bigvee_{j\in J} B_\epsilon(\alpha_j).$$

The other inequality is trivial.
(iii) We have that $B_\epsilon(\alpha \oplus \beta) = R_{t-\epsilon}$ with $t = \sup\{s \in \mathbb{R} : \alpha \oplus \beta \leq L_s^*\}$. On the other hand, $B_\epsilon(\alpha) = R_{t_\alpha-\epsilon}$ with $t_\alpha = \sup\{s \in \mathbb{R} : \alpha \leq L_s^*\}$ and $B_\epsilon(\beta) = R_{t_\beta-\epsilon}$ with $t_\beta = \sup\{s \in \mathbb{R} : \beta \leq L_s^*\}$. Then $B_\epsilon(\alpha) \oplus B_\epsilon(\beta) = R_{t_\alpha-\epsilon} \oplus R_{t_\beta-\epsilon}$. Now, as $t_\alpha, t_\beta \geq t$ then $t_\alpha - \epsilon, t_\beta - \epsilon \geq t - \epsilon$. Therefore, $R_{t_\alpha-\epsilon}, R_{t_\beta-\epsilon} \geq R_{t-\epsilon}$ and then $R_{t_\alpha-\epsilon} \oplus R_{t_\beta-\epsilon} \geq 2R_{t-\epsilon} \geq R_{t-\epsilon}$. That is

$$B_\epsilon(\alpha \oplus \beta) \leq B_\epsilon(\alpha) \oplus B_\epsilon(\beta).$$

2. $B_\epsilon^{-1}(\alpha) = \bigwedge\{\beta : B_\epsilon(\beta^*) \leq \alpha^*\}$
$= \bigwedge\{L_t : B_\epsilon(L_t^*) \leq \alpha^*\}$
$= \bigwedge\{L_t : R_{t-\epsilon} \leq \alpha^*\}$
$= \bigwedge\{L_{t+\epsilon} : \alpha \leq R_t^*\}$
$= L_{t+\epsilon}$ where $t = \inf\{s : \alpha \leq R_s^*\}$
3. $B_\epsilon(B_\delta(\alpha)) = B_\epsilon(R_{t-\delta})$ where $t = \sup\{s : \alpha \leq L_s^*\}$
$= R_{t-\delta-\epsilon}$
$= B_{\epsilon+\delta} = R_{t-\delta-\epsilon}$.

□

Theorem 5.5.12. *The set $\{B_\epsilon, B_\epsilon^{-1} : \varepsilon > 0\}$ is a sub-basis for an MV-uniformity on $\mathfrak{F}(I)$. The MV-topology generated by the MV-uniformity is the usual MV-topology. This MV-uniformity is called the usual MV-uniformity for the usual MV-topology on $\mathfrak{F}(I)$.*

Proof. By Lemma 5.5.11, $\{B_\epsilon : \epsilon > 0\}$ is a subbase for an MV-quasi-uniformity. So, $\{B_\epsilon, B_\epsilon^{-1} : \epsilon > 0\}$ is a subbase for an MV-uniformity $\mathfrak{D}$. Besides, as $B_\epsilon(\alpha) = R_t$ and $B_\epsilon^{-1}(\alpha) = L_s$, for some t, s, then the MV-topology generated by $\mathfrak{D}$ is the usual MV-topology on $\mathfrak{F}(I)$. □

Theorem 5.5.13. *Let $(X, \mathfrak{D})$ be an MV-uniform space and let $f \in \mathfrak{D}$. If $f(\alpha) \leq \beta$, then there exists a uniformly continuous function $\varphi : X \longrightarrow \mathfrak{F}(I)$ such that for each $x \in X$,*

$$\alpha(x) \leq \varphi(x)(1-) \leq \varphi(x)(0+) \leq \beta(x).$$

Proof. We construct fuzzy sets $\{\gamma_t : t \in \mathbb{R}\} \subseteq [0,1]^X$ such that

(i) $\gamma_t = \mathbf{1}$ for $t < 0$,
(ii) $\gamma_t = \mathbf{0}$ for $t > 1$,
(iii) $\gamma_0 = \beta$,
(iv) $\gamma_1 = \alpha$,

and symmetric elements $\{f_\epsilon : \epsilon > 0\}$ of the MV-uniformity such that $f_\epsilon(\gamma_t) \leq \gamma_{t-\epsilon}$, for $t \in \mathbb{R}$. Since f_ϵ is symmetric we have that $f_\epsilon(\gamma_t^*) = \gamma_{r+\epsilon}^*$. Now, we define $\varphi : X \longrightarrow \mathfrak{F}(I)$ by $\varphi(x)(t) = \gamma_t(x)$. We observe that φ is well defined and for each $x \in X$,

$$\alpha(x) \leq \varphi(x)(1-) \leq \varphi(x)(0+) \leq \beta(x).$$

We show that φ is uniformly continuous. First, note that $\varphi^{-1}(R_t) = \bigvee_{s>t} \gamma_s$ and $\varphi^{-1}(L_t^*) = \bigwedge_{s<t} \gamma_s$. Hence
$f_\epsilon(\varphi^{-1}(L_t^*)) \leq f_\epsilon(\gamma_{t-\delta})$ for any $\delta > 0$
$\leq \gamma_{t-\delta-\epsilon}$
$\leq \bigvee_{s>t} \gamma_{s-2\delta-\epsilon}$
$\leq \varphi^{-1}(B_{\epsilon+2\delta}(L_t^*))$. Letting $\delta = \frac{1}{2}\epsilon$, we have that $f_\epsilon \leq \varphi^{-1}(B_{2\epsilon})$. Similarly, $f_\epsilon \leq \varphi^{-1}(B_{2\epsilon}^*)$ and so φ is uniformly continuous. □

Definition 5.5.14. An MV-topological space (X, τ) is *completely regular* if for each $\alpha \in \tau$ there are a family of fuzzy sets $\{\gamma_i : i \in I\}$ and a family of maps $\{f_i : X \longrightarrow \mathfrak{F}(I) \mid i \in I\}$ such that $\bigvee_{i \in I} \gamma_i = \alpha$ and

$$\gamma_i(x) \leq f_i(x)(1-) \leq f_i(x)(0+) \leq \alpha(x)$$

for all $i \in I$ and $x \in X$.

Corollary 5.5.15. *Let $(X, \mathfrak{D})$ be an MV-uniform space and (X, τ) be the MV-topological space such that τ is the MV-topology generated by $\mathfrak{D}$. Then (X, τ) is completely regular.*

Proof. It is enough to apply the previous theorem and to observe that for each $\alpha \in \tau$, $\alpha = \bigvee\{\gamma : f(\gamma) \leq \alpha$ for some $f \in \mathfrak{D}\}$. □

6

Sheaf Representation

In this chapter, we study a generalisation of the concept of sheaf in the context of MV-topological spaces. The main definitions, indeed, are suitable adaptations from the classical theory of sheaves.

Besides the basic notions, we use MV-sheaves in order to represent a particular class of MV-algebras in the wake of Filipoiu and Georgescu's representation for MV-algebras. In that case, any MV-algebra is represented as an algebra of global section of a sheaf over the maximal spectrum of the algebra and whose stalks are local MV-algebras [18]. Our aim is to represent the whole semisimple skeleton of the given algebra only by means of the MV-topology, while the stalks should carry just the "non-semisimple (or infinitesimal) information" of the elements of the algebra. Unfortunately, this is not possible for any MV-algebra, and therefore we must restrict to the class of MV-algebras for which certain quotients have retractive radical [12]. We shall make use also of results presented by Diaconescu, Flaminio, and Leustean in [10].

6.1 MV-presheaves

Let (X, τ) be an MV-topological space. The poset of open fuzzy subsets $\tau \subseteq [0,1]^X$, with the fuzzy inclusion $\leq$, can be viewed as a category in the usual manner, namely, τ is the object class and, for all $\alpha, \beta \in \tau$, there is exactly one morphism $\alpha \longrightarrow \beta$ if $\alpha \leq \beta$, there are none otherwise.

Definition 6.1.1. Let (X, τ) be an MV-topological space and let $\mathcal{C}$ be a category (of algebras). An *MV-presheaf* of $\mathrm{Ob}(\mathcal{C})$ on X is a contravariant functor $F : \tau \longrightarrow \mathcal{C}$, that is:

(i) for each fuzzy open set α in τ, $F(\alpha)$ is an object of $\mathcal{C}$, called the set of sections of F over α;
(ii) for each pair of fuzzy open sets $\beta \leq \alpha$ in τ, the image of the morphism $\beta \longrightarrow \alpha$ is the so-called *restriction map* $\rho^\alpha_\beta : F(\alpha) \longrightarrow F(\beta)$ with the following properties:
a) $\rho^\alpha_\alpha = \mathrm{id}_\alpha$, for all α;
b) $\rho^\alpha_\gamma = \rho^\beta_\gamma \circ \rho^\alpha_\beta$, whenever $\gamma \leq \beta \leq \alpha$ in τ.

Definition 6.1.2. Let F and G be MV-presheaves of $\mathrm{Ob}(\mathcal{C})$ over (X, τ). A *morphism of MV-presheaves* from F to G is a natural transformation $f : F \Longrightarrow G$, that is, a family

$\{f(\alpha) : F(\alpha) \longrightarrow G(\alpha)\}_{\alpha\in\tau}$ such that, whenever $\beta \leq \alpha$ are open fuzzy sets in τ, the diagram

$$\begin{array}{ccc} F(\alpha) & \xrightarrow{f(\alpha)} & G(\alpha) \\ {\scriptstyle \rho^{\alpha}_{\beta}}\downarrow & & \downarrow{\scriptstyle \rho'^{\alpha}_{\beta}} \\ F(\beta) & \xrightarrow{f(\beta)} & G(\beta) \end{array}$$

commutes.

Example 6.1.3. 1. Let A be a fixed object in the category $\mathcal{C}$ and (X, τ_X) be an MV-space. We define the constant MV-presheaf $A_X : \tau_X \longrightarrow \mathcal{C}$ on (X, τ_X), by setting:
$A_X(\alpha) = A$ for all α in τ_X, and
$\rho^{\alpha}_{\beta} = \mathrm{id}_A : A_X(\alpha) \longrightarrow A_X(\beta)$ for $\beta \leq \alpha$ in τ_X.

2. Let (X, τ_X) and (Y, τ_Y) be MV-topological spaces. Let us consider the following MV-presheaf $C^Y : \tau_X \to \mathcal{S}\mathrm{et}$, defined by

$$C^Y(\alpha) = \{f : \mathrm{supp}(\alpha) \longrightarrow Y \mid f \text{ is continuous}\},$$

with $\rho^{\alpha}_{\beta} : C^Y(\alpha) \longrightarrow C^Y(\beta)$ such that $\rho^{\alpha}_{\beta}(f) = f \restriction \mathrm{supp}(\beta)$ for $\beta \leq \alpha$ in τ_X. Note that $\mathrm{supp}(\beta) \subseteq \mathrm{supp}(\alpha)$ if $\beta \leq \alpha$.

Definition 6.1.4. A *directed set* I is a set with a pre-order $\leq$ which satisfies the following:

(a) for all $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

A *direct system* of sets indexed by a directed set I is a family $\{\alpha_i\}i \in I$ of sets together with, for each $i \leq j$, a map of sets $\rho_{ij} : \alpha_i \longrightarrow \alpha_j$, satisfying

(b) For all $i \in I$, $\rho_{ij} = \mathrm{id}_{\alpha_i}$;
(c) For all $i, j, k \in I$, $i \leq j \leq k$ implies $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$.

Let F be an MV-presheaf of $\mathrm{Ob}(\mathcal{C})$ over an MV-topological space (X, τ) and fix $x \in X$. Then $F(\alpha)$, with α running through all the fuzzy open sets such that $x \in \mathrm{supp}(\alpha)$, form a direct system with maps $\rho^{\alpha}_{\beta} : F(\alpha) \longrightarrow F(\beta)$, whenever $\beta \leq \alpha$, and $x \in \mathrm{supp}(\beta) \subseteq \mathrm{supp}(\alpha)$.

Definition 6.1.5. The *MV-stalk* F_x of F at x is

$$\lim_{x\in\mathrm{supp}(\alpha)} F(\alpha).$$

This comes equipped with maps $F(\alpha) \longrightarrow F_x$ such that $s \longmapsto s_x$ whenever $x \in \mathrm{supp}(\alpha)$ for $\alpha \in \tau$. The members of F_x are also called *germs* (of sections of F).

6.2 MV-sheaves

Definition 6.2.1. An MV-presheaf of sets over the MV-topological space (X, τ_X) satisfying the following two conditions is called an *MV-sheaf* of $\mathrm{Ob}(\mathcal{C})$.

1. If α is a fuzzy open set of X and the family $\{\alpha_i\}_{i\in I} \subseteq [0,1]^X$ is an open covering of α, i.e., $\alpha = \bigvee_{i\in I} \alpha_i$, and $s, s' \in F(\alpha)$ are two sections of F such that for all $i \in I$

$$\rho^{\alpha}_{\alpha_i}(s) = \rho^{\alpha}_{\alpha_i}(s')$$

then $s = s'$.

2. If α is a fuzzy open set of X and the family $\{\alpha_i\}_{i\in I} \subseteq [0,1]^X$ is an open covering of α; and if there is a family $\{s_i\}_{i\in I}$ of sections of F with $s_i \in F(\alpha_i)$ for all $i \in I$, such that for all $i, j \in I$
$$\rho^{\alpha_i}_{\alpha_i\wedge\alpha_j}(s_i) = \rho^{\alpha_j}_{\alpha_i\wedge\alpha_j}(s_j)$$
then there is $s \in F(\alpha)$ such that for all $i \in I$
$$\rho^{\alpha}_{\alpha_i}(s) = s_i.$$

In other words, if the system $(s_i)_{i\in I}$ is given on a covering and is consistent on all of the overlaps, then it comes from a section over all of the α's.

Definition 6.2.2. If F, G are MV-sheaves of $\mathrm{Ob}(\mathcal{C})$ and $f : F \Longrightarrow G$ is an MV-presheaf morphism, we also call f a *morphism of MV-sheaves*.

Example 6.2.3. The MV-presheaf C^Y, described in the Example 6.1.3, is an MV-sheaf. Let us see that C^Y satisfies the two conditions of the definition of MV-sheaf.

Let α be a fuzzy open set of X and let $\{\alpha_i\}_{i\in I} \subseteq [0,1]^X$ be an open covering of α, i.e., $\alpha = \bigvee_{i\in I} \alpha_i$,

1. let $f, f' \in C^Y(\alpha)$ be two sections of C^Y such that for all $i \in I$,
$$\rho^{\alpha}_{\alpha_i}(f) = \rho^{\alpha}_{\alpha_i}(f'),$$
that is,
$$f \restriction \mathrm{supp}(\alpha_i) = f' \restriction \mathrm{supp}(\alpha_i)$$
where $f, f' : \mathrm{supp}(\alpha) \to Y$.
Note that $\bigcup_{i\in I} \mathrm{supp}(\alpha_i) = \mathrm{supp}(\alpha)$ because $\alpha = \bigvee_{i\in I} \alpha_i$. Let us see that $f = f'$.
If $x \in \mathrm{supp}(\alpha)$, then there exists $i \in I$ such that $x \in \mathrm{supp}(\alpha_i)$, so
$$f(x) = f \restriction \mathrm{supp}(\alpha_i)(x) = f'|\,\mathrm{supp}(\alpha_i)(x) = f'(x)$$
then $f = f'$.
2. For the second condition, suppose that there is a family $\{f_i\}_{i\in I}$ of sections of C^Y with $f_i \in C^Y(\alpha_i)$ for all $i \in I$, such that for all $i, j \in I$
$$\rho^{\alpha_i}_{\alpha_i\wedge\alpha_j}(f_i) = \rho^{\alpha_j}_{\alpha_i\wedge\alpha_j}(f_j)$$
We define $f := \bigcup_{i\in I} f_i : \mathrm{supp}(\alpha) \longrightarrow Y$ by $f(x) = f_i(x)$ if $x \in \mathrm{supp}(\alpha_i) = \mathrm{dom}(f_i)$. We know that $\mathrm{supp}(\alpha) = \bigcup_{i\in I} \mathrm{supp}(\alpha_i)$, then f is well defined because for all $i, j \in I$, $x \in \mathrm{supp}(\alpha_i) \cap \mathrm{supp}(\alpha_j)$ iff $x \in \mathrm{supp}(\alpha_i \wedge \alpha_j)$, and by hypothesis
$$f_i \restriction \mathrm{supp}(\alpha_i \wedge \alpha_j)(x) = f_j \restriction \mathrm{supp}(\alpha_i \wedge \alpha_j)(x)$$
where $f_i : \mathrm{supp}(\alpha_i) \longrightarrow Y$ and $f_j : \mathrm{supp}(\alpha_j) \longrightarrow Y$. It is clear that $f \restriction \mathrm{supp}(\alpha_i) = f_i$, for each $i \in I$.
Now, let us prove that f is continuous.
Let $\gamma \in \tau_Y$, and let us prove that $\gamma \circ f \in \tau_{\mathrm{supp}(\alpha)}$. For each $i \in I$, $\gamma \circ f_i \in \tau_{\mathrm{supp}(\alpha_i)}$, i.e., $\gamma \circ f_i = \beta \wedge \mathrm{supp}(\alpha_i)$ with $\beta \in \tau_X$. As $\mathrm{supp}(\alpha_i) = \mathrm{supp}(\alpha_i) \wedge \mathrm{supp}(\alpha)$, then $\gamma \circ f_i = \beta \wedge \mathrm{supp}(\alpha_i) \wedge \mathrm{supp}(\alpha)$. Thus, for each $i \in I$, $\gamma \circ f_i \in \tau_{\mathrm{supp}(\alpha)}$ because $\beta \wedge \mathrm{supp}(\alpha_i) \in \tau_X$. Therefore, $\gamma \circ f = \bigvee_{i\in I}(\gamma \circ f_i) \in \tau_{\mathrm{supp}(\alpha)}$.

Definition 6.2.4. Let (X, τ_X) be an MV-topological space. An *MV-sheaf space* over X is a triple (E, p, X) where (E, τ_E) is an MV-topological space and $p : E \longrightarrow X$ is a *local MV-homeomorphism*, that is, p is continuous and, for all $x \in E$, there exists an open fuzzy set $\alpha \in \tau_E$ such that $\alpha(x) > 0$ and an open fuzzy set $\beta \in \tau_X$ such that $p \restriction \mathrm{supp}(\alpha) : \mathrm{supp}(\alpha) \longrightarrow \mathrm{supp}(\beta)$ is an MV-homeomorphism.

A morphism of MV-sheaf spaces over X, $f : (E, p, X) \longrightarrow (E', p', X)$, is a continuous map $f : E \longrightarrow E'$ such that $p = p' \circ f$.

In what follows we will see how to construct an MV-sheaf of sets from an MV-sheaf space.

Construction: For each MV-sheaf space E over X we can construct an MV-sheaf of sets ΓE (the sheaf of sections of E) in such a way that a morphism $f : E \longrightarrow E'$ of MV-sheaf spaces over X gives rise to a morphism $\Gamma f : \Gamma E \longrightarrow \Gamma E'$ of MV-sheaves.

The MV-sheaf of sections of (E, p, X) is constructed as following: We let, for α in τ_X,

$$\Gamma(\alpha, E) = \{f : \mathrm{supp}(\alpha) \to E | f \text{ is continuous and } p \circ f = \mathrm{id}_{\mathrm{supp}(\alpha)}\}$$

and then the MV-presheaf $\Gamma E : \tau_X \longrightarrow \mathrm{SET}$ such that $\alpha \mapsto \Gamma(\alpha, E)$ is an MV-sheaf.

Given a morphism $f : E \longrightarrow E'$ of sheaf spaces, we have the morphism of MV-sheaves $\Gamma f : \Gamma E \longrightarrow \Gamma E'$ such that for each α in τ_X, the map $\Gamma(\alpha, E) \longrightarrow \Gamma(\alpha, E')$ is such that $\sigma \longmapsto f \circ \sigma$.

Lemma 6.2.5. *Let (E, p, X) be an MV-sheaf space over (X, τ_X). Then*

(a) p is an open map;
(b) if α is a fuzzy open in X, and $\sigma \in \Gamma(\alpha, E)$, then $\sigma^{\rightarrow}(\alpha)$ is open in E; furthermore fuzzy sets of this form give a basis for the MV-topology of E;
(c) if

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & X \end{tikzcd}$$

is a commutative diagram of maps, and p, p', are local MV-homeomorphisms, then φ is continuous iff φ is open iff φ is local MV-homeomorphism.

6.3 MV-sheaf Representation

In this section, we are going to represent a particular class of MV-algebras by an MV-sheaf. Before that, we shall recall some necessary results.

Let $(\mathrm{Max}\, A, \tau_A)$ be the MV-topological space defined in [42] and recalled in Section 4.5 of this work. Let us see some of its properties and its relation with the topological space $\mathrm{Max}\, A$ with the Zariski topology. The basic opens of $\mathrm{Max}\, A$ denoted by $R(a)$ with $a \in A$, were defined in Section 2.4.

Proposition 6.3.1. *Let A be an MV-algebra and $(\mathrm{Max}\, A, \tau_A)$ be the associated MV-topological space. For each basic fuzzy open $\widehat{b} \in \tau_A$, we have that $R(b) = \mathrm{supp}(\widehat{b})$. So, each Zariski basic open $R(b)$ on $\mathrm{Max}\, A$ is a fuzzy open of τ_A, i.e., $R(b) \in \tau_A$ for each $b \in A$.*

Proof. In fact, for each $M \in \operatorname{Max} A$, $\widehat{b}(M) = \frac{b}{M} = 0$ if and only if $b \in M$. That is, $M \in \operatorname{supp} \widehat{b}$ iff $\widehat{b}(M) = \frac{b}{M} > 0$ iff $b \notin M$ iff $M \in R(b)$. □

Proposition 6.3.2. *For each $a \in A$, the set $H(a) = \{M \in \operatorname{Max} A : a \in O_M\}$ is an element of τ_A.*

Proof. We will prove that $H(a)$ is the support of a fuzzy open of τ_A. If $M \in H(a)$ then $a \in O_M$, so by Proposition 2.3.17 there exists $b_M \notin M$ such that $a \wedge b_M = \mathbf{0}$. That is, $M \in R(b_M) = \operatorname{supp}(\widehat{b_M})$. Let us see that

$$H(a) = \operatorname{supp}(\bigvee_{M \in H(a)} \widehat{b_M}).$$

In fact, if $N \in H(a)$ then there exists b_N such that $b_N \notin N$ such that $a \wedge b_N = \mathbf{0}$, then $\widehat{b_N}(N) > 0$, and therefore $(\bigvee_{M \in H(a)} \widehat{b_M})(N) = \bigvee_{M \in H(a)} \widehat{b_M}(N) > 0$, i.e, $N \in \operatorname{supp}(\bigvee_{M \in H(a)} \widehat{b_M})$. For the other inclusion, if $(\bigvee_{M \in H(a)} \widehat{b_M})(N) > 0$ then there exists $\widehat{b_M}$ with $M \in H(a)$ such that $\widehat{b_M}(N) > 0$, i.e., $b_M \notin N$ and $a \wedge b_M = \mathbf{0}$, then $a \in O_N$ and therefore $N \in H(a)$. □

In the following we will to represent an MV-algebra A through an MV-sheaf.

Let A be an MV-algebra. Let M be a maximal ideal of A. Suppose that A/O_M has retractive radical for every $M \in \operatorname{Max} A$. Then, for each $M \in \operatorname{Max} A$:

(i) A/O_M is a lexicographic MV-algebra (by Theorem 2.8.7).
(ii) $A/M \simeq (A/O_M)/(M/O_M)$.
(iii) $(M/O_M, \oplus, \mathbf{0})$ is a lattice ordered cancellative monoid.

We construct, in the usual manner, the generated lattice ordered group from $(M/O_M, \oplus, \mathbf{0})$ which we denote by $G(M/O_M)$.

Let's consider the following functors.

1. Let (X, τ) be an MV-topological space and $(X, \mathcal{B}(\tau))$ its corresponding skeleton topological space defined in [42], where $\mathcal{B}(\tau) = \tau \cap \{0,1\}^X$. As usual, we consider the posets τ and $\mathcal{B}(\tau)$ with their natural order as categories, that is, the objects are the elements of τ and $\mathcal{B}(\tau)$ respectively, and the morphisms are given by $\alpha \leq \beta$ in τ and $U \subseteq V$ in $\mathcal{B}(\tau)$, respectively. The following maps obviously define a covariant functor:

$$\begin{aligned} \mathrm{Sk} : \tau &\longrightarrow \mathcal{B}(\tau) \\ \alpha &\longmapsto \operatorname{supp}(\alpha) \end{aligned}$$

   For $\alpha \leq \beta$, we have the unique morphism $\alpha \xrightarrow{f} \beta$ in τ, and its corresponding morphism $\operatorname{supp}(\alpha) \xrightarrow{\mathrm{Sk}(f)} \operatorname{supp}(\beta)$ in $\mathcal{B}(\tau)$ is also uniquely determined, because $\alpha \leq \beta$ implies $\operatorname{supp}(\alpha) \subseteq \operatorname{supp}(\beta)$.
2. Recalling Filipoiu and Georgescu's representation [18], which we recalled in Section 2.9, we have that each MV-algebra A is representable as the MV-algebra of global sections of a sheaf whose stalks are local MV-algebras and the base space is the space of maximal ideals of A with the Zariski topology, $\mathcal{O}(\operatorname{Max} A)$. The associated sheaf in that representation is the following contravariant functor:

$$\begin{aligned} \mathfrak{F} : \mathcal{O}(\operatorname{Max} A) &\longrightarrow \mathcal{MV} \\ U &\longmapsto A/O_U \end{aligned},$$

where $O_U = \bigcap_{M \in U} O_M$, and the unique morphism between two open sets (if it exists) is sent to the natural projection between the corresponding quotient algebras.

3. We recall the category $\mathcal{ALG}$ whose objects are Abelian ℓ-groups and whose morphisms are ℓ-group homomorphisms. The following mapping defines a functor from the category of MV-algebras to the category $\mathcal{ALG}$:

$$\begin{array}{rcl} \mathfrak{G} : \mathcal{MV} & \longrightarrow & \mathcal{ALG} \\ A & \longmapsto & G(\operatorname{Rad} A) \end{array},$$

where $G(\operatorname{Rad} A)$ is the Abelian ℓ-group generated by the ordered cancellative monoid $(\operatorname{Rad}(A), \oplus, \mathbf{0})$. Actually, $G(\operatorname{Rad} A) = \mathcal{D}(A)$ where $\mathcal{D}$ is the functor described in Section 2.7 (note that the group $\mathcal{D}(A)$ can be constructed for any MV-algebra A, not necessarily perfect), and the action of the functor on morphisms is exactly the same as for $\mathcal{D}$.

Now, for each $\alpha \in \tau$, set $A_\alpha := \mathfrak{F}(\operatorname{supp}(\alpha))$. We obtain the following presheaf:

$$\begin{array}{rcl} \mathfrak{H} : \tau & \longrightarrow & \mathcal{ALG} \\ \alpha & \longmapsto & G(\operatorname{Rad}(A_\alpha)) \end{array}.$$

Note that in the construction performed by Filipoiu and Georgescu, the stalks are the local algebras A/O_M, and we have that, for each $M \in \operatorname{Max} A$,

$$\lim_{M \in \operatorname{supp}(\alpha)} \mathfrak{F}(\operatorname{supp}(\alpha)) = A/O_M$$

Such a limit can be extended to the presheaf $\mathfrak{H}$ on the category $\mathcal{ALG}$, thus obtaining the following two limits:

$$\lim_{M \in \operatorname{supp}(\alpha)} \operatorname{Rad}(\mathfrak{F}(\operatorname{supp}(\alpha))) = \operatorname{Rad}(A/O_M)$$

and

$$\lim_{M \in \operatorname{supp}(\alpha)} \mathfrak{H}(\alpha) = G(\operatorname{Rad}(A/O_M)).$$

Since $\operatorname{Rad}(A/O_M) = M/O_M$ for each $M \in \operatorname{Max} A$, we have an MV-sheaf on $\mathcal{ALG}$ where the stalks are the ℓ-groups $G(M/O_M)$ for each $M \in \operatorname{Max} A$.

If A/O_M has retractive radical, then A/O_M is a local MV-algebra with retractive radical and therefore it is lexicographic. By [10, Theorem 4.1] we have that

$$A/O_M \simeq \Gamma(H \times_{lex} G, (u, 0))$$

where

$$(H, u) \simeq \Gamma^{-1}(A/M) \text{ and } G \simeq \Delta^{-1}(\langle M/O_M \rangle) = G(M/O_M).$$

Equivalently,

$$\Gamma(H, u) \simeq A/M \text{ and } \langle M/O_M \rangle \simeq \Delta(G).$$

Now, since A/M is an MV-subalgebra of the standard MV-algebra $[0, 1]$, for each $M \in \operatorname{Max} A$, we have:

$$A/O_M \simeq \Gamma(A/M \times_{lex} G(M/O_M), (1, 0)).$$

So, according to the proof of [10, Theorem 4.1], and with an abuse of notation, we can see each element a/O_M of A/O_M in the following way:

$$\frac{a}{O_M} = \left(\frac{a}{M}, g_{aM}\right) = (\widehat{a}(M), g_{aM})$$

where $g_{aM} = (\frac{a}{O_M} \ominus \frac{a}{M}) - (\frac{a}{M} \ominus \frac{a}{O_M}) \in G(M/O_M)$. Note that

$$G(M/O_M) = \{g_{aM} : a \in A, M \in \operatorname{Max} A\}.$$

As a consequence of all that we have discussed above, we can define an MV-sheaf space $(H_A, \pi, \operatorname{Max} A)$ whose total MV-space, H_A, will be the disjoint union of stalks $\mathfrak{H}_M = G(M/O_M)$ and π will be the trivial projection:

$$H_A = \{(g_{aM}, M) : a \in A, M \in \operatorname{Max} A\},$$

and

$$\begin{array}{rccc} \pi : & H_A & \longrightarrow & \operatorname{Max} A \\ & (g_{aM}, M) & \longmapsto & M \end{array}.$$

Now for each $a \in A$ we define:

$$\begin{array}{rccc} \widetilde{a} : & \operatorname{Max} A & \longrightarrow & H_A \\ & M & \longmapsto & (g_{aM}, M) \end{array}$$

It is clear that $(\pi \circ \widetilde{a})(M) = \pi(g_{aM}, M) = M$ for all $M \in \operatorname{Max} A$.

As usual in classical sheaf representations, we set $\{\widetilde{a}^{\rightarrow}(\widehat{b})\}_{a,b \in A}$ being a subbase for an MV-topology on H_A, where

$$\widetilde{a}^{\rightarrow}(\widehat{b})(g_{cM}, M) = \bigvee_{\widetilde{a}(N) = (g_{cM}, M)} \widehat{b}(N) = \begin{cases} \widehat{b}(M) & \text{if } g_{aM} = g_{cM} \\ 0 & \text{otherwise} \end{cases}.$$

Let us see that $\alpha_{a,b} := \{M \in \operatorname{Max} A : g_{aM} = g_{bM}\}$ is an element of $\mathcal{O}(\operatorname{Max} A)$. If $g_{aM} = g_{bM}$ we have the following cases:

1. If $\frac{a}{O_M} = \frac{b}{O_M}$ then $\left(\frac{a}{M}, g_{aM}\right) = \left(\frac{b}{M}, g_{bM}\right)$, and therefore $\frac{a}{M} = \frac{b}{M}$ and therefore

$$\alpha_{a,b} = H(d(a,b)) \in \mathcal{O}(\operatorname{Max} A).$$

2. If $\frac{a}{O_M} \neq \frac{b}{O_M}$ then necessarily $\frac{a}{M} \neq \frac{b}{M}$. That is $\frac{a}{M} < \frac{b}{M}$ or $\frac{b}{M} > \frac{a}{M}$. Since A/O_M has a lexicographic order, this implies that $\frac{a}{O_M} < \frac{b}{O_M}$ or $\frac{b}{O_M} > \frac{a}{O_M}$. Therefore there exists $c \in A$ such that $\frac{a}{O_M} = \frac{b \oplus c}{O_M}$ or $\frac{a \oplus c}{O_M} = \frac{b}{O_M}$. Hence

$$\begin{array}{l} \alpha_{a,b} = \bigcup_{c \in A} \{M \in \operatorname{Max} A : \frac{a}{O_M} = \frac{b \oplus c}{O_M}\} \cup \bigcup_{c \in A} \{M \in \operatorname{Max} A : \frac{a \oplus c}{O_M} = \frac{b}{O_M}\} = \\ = \bigcup_{c \in A} H(d(a, b \oplus c)) \cup \bigcup_{c \in A} H(d(a \oplus c, b)) \in \mathcal{O}(\operatorname{Max} A) \end{array}.$$

In fact, we have proved that each $\alpha_{a,b}$ is an element of τ_A. This property guarantees that $(H_A, \pi, \operatorname{Max} A)$ is indeed an MV-sheaf space.

The MV-sheaf defined above is an MV-sheaf of lattice-ordered Abelian groups. Next, we want to obtain a representation of the MV-algebra A through this MV-sheaf.

First, let us consider for each $a \in A$, the function $\widetilde{a}$ restricting the codomain H_A to its image $\operatorname{Im}(\widetilde{a}) = \{(g_{aM}, M) : M \in \operatorname{Max} A\}$. Actually, the new $\widetilde{a}$ acts exactly like the previous one on the elements of the domain, so we shall use the same notation for them. Then, we have the bijective maps:

$$\begin{array}{rccc} \widetilde{a} : & \operatorname{Max} A & \longrightarrow & \operatorname{Im}(\widetilde{a}) \\ & M & \longmapsto & (g_{aM}, M) \end{array}$$

and for each basic open set $\widehat{a}$ in $\operatorname{Max} A$ we have the open fuzzy set $\widetilde{a}^{\rightarrow}(\widehat{a})$ in H_A satisfying

$$\widetilde{a}^{\rightarrow}(\widehat{a})(g_{aM}, M) = \frac{a}{M}, \text{ for each } (g_{aM}, M) \in \operatorname{Im}(\widetilde{a}).$$

Now, let us consider the inverse of the graphic of $\widetilde{a}^{\rightarrow}(\widehat{a})$ given by

$$\mathfrak{a} := G^{-1}(\widetilde{a}^{\rightarrow}(\widehat{a})) = \left\{\left(\frac{a}{M}, g_{aM}\right)\right\}_{M \in \operatorname{Max} A}$$

Definition 6.3.3. Let $\mathfrak{A} = \{\mathfrak{a} : a \in A\}$. We define the structure $(\mathfrak{A}, \oplus, ^*, \mathfrak{o})$ with the operations and the constant defined as follow:

for each $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$,

1. $\mathfrak{o} := G^{-1}(\widetilde{\mathbf{0}}^{\rightarrow}(\widehat{\mathbf{0}}))$
2. $\mathfrak{a} \oplus \mathfrak{b} := G^{-1}(\widetilde{a \oplus b}^{\rightarrow}(\widehat{a \oplus b}))$
3. $\mathfrak{a}^* := G^{-1}(\widetilde{a^*}^{\rightarrow}(\widehat{a^*}))$.

Theorem 6.3.4. $(\mathfrak{A}, \oplus, ^*, \mathfrak{o})$ *is an MV-algebra.*

Proof. Let us see that $\mathfrak{A}$ satisfies all properties of the Definition 2.0.1.

MV1) $(\mathfrak{A}, \oplus, \mathfrak{o})$ is a commutative monoid:

(i)

$$(\mathfrak{a} \oplus \mathfrak{b}) \oplus \mathfrak{c} = \left\{\left(\frac{(a \oplus b) \oplus c}{M}, g_{((a \oplus b) \oplus c)M}\right)\right\}_{M \in \operatorname{Max} A} =$$
$$= \left\{\left(\frac{a \oplus (b \oplus c)}{M}, g_{(a \oplus (b \oplus c))M}\right)\right\}_{M \in \operatorname{Max} A} = \mathfrak{a} \oplus (\mathfrak{b} \oplus \mathfrak{c})$$

(ii)

$$\mathfrak{a} \oplus \mathfrak{b} = \left\{\left(\frac{a \oplus b}{M}, g_{(a \oplus b)M}\right)\right\}_{M \in \operatorname{Max} A} =$$
$$= \left\{\left(\frac{b \oplus a}{M}, g_{(b \oplus a)M}\right)\right\}_{M \in \operatorname{Max} A} = \mathfrak{b} \oplus \mathfrak{a}$$

(iii)

$$\mathfrak{a} \oplus \mathfrak{o} = \left\{\left(\frac{a \oplus \mathbf{0}}{M}, g_{(a \oplus \mathbf{0})M}\right)\right\}_{M \in \operatorname{Max} A} = \left\{\left(\frac{a}{M}, g_{aM}\right)\right\}_{M \in \operatorname{Max} A} = \mathfrak{a}$$

MV2)

$$(\mathfrak{a}^*)^* = \left(\left\{\left(\frac{a^*}{M}, g_{(a^*)M}\right)\right\}_{M \in \operatorname{Max} A}\right)^* = \left\{\left(\frac{(a^*)^*}{M}, g_{(a^*)^*M}\right)\right\}_{M \in \operatorname{Max} A} =$$
$$= \left\{\left(\frac{a}{M}, g_{aM}\right)\right\}_{M \in \operatorname{Max} A} = \mathfrak{a}$$

MV3)

$$\mathfrak{a} \oplus \mathfrak{o}^* = \left\{\left(\frac{a}{M}, g_{aM}\right)\right\}_{M \in \operatorname{Max} A} \oplus \left\{\left(\frac{\mathbf{0}^*}{M}, g_{0^*M}\right)\right\}_{M \in \operatorname{Max} A}$$
$$= \left\{\left(\frac{a \oplus \mathbf{0}^*}{M}, g_{(a \oplus \mathbf{0}^*)M}\right)\right\}_{M \in \operatorname{Max} A} = \left\{\left(\frac{\mathbf{0}^*}{M}, g_{\mathbf{0}^*M}\right)\right\}_{M \in \operatorname{Max} A}$$
$$= \mathfrak{o}^*.$$

MV4)

$$
\begin{aligned}
(\mathfrak{a}^* \oplus \mathfrak{b})^* \oplus \mathfrak{b} = & \\
&= \left\{ \left(\frac{(a^* \oplus b)^*}{M}, g_{(a^* \oplus b)^* M} \right) \right\}_{M \in \operatorname{Max} A} \oplus \left\{ \left(\frac{b}{M}, g_{bM} \right) \right\}_{M \in \operatorname{Max} A} \\
&= \left\{ \left(\frac{(a^* \oplus b)^* \oplus b}{M}, g_{((a^* \oplus b)^* \oplus b)M} \right) \right\}_{M \in \operatorname{Max} A} \\
&= \left\{ \left(\frac{(b^* \oplus a)^* \oplus a}{M}, g_{((b^* \oplus a)^* \oplus a)M} \right) \right\}_{M \in \operatorname{Max} A} \\
&= \left\{ \left(\frac{(b^* \oplus a)^*}{M}, g_{(b^* \oplus a)^* M} \right) \right\}_{M \in \operatorname{Max} A} \oplus \left\{ \left(\frac{a}{M}, g_{aM} \right) \right\}_{M \in \operatorname{Max} A} \\
&= (\mathfrak{b}^* \oplus \mathfrak{a})^* \oplus \mathfrak{a}.
\end{aligned}
$$

□

Theorem 6.3.5. *The MV-algebras A and $\mathfrak{A}$ are isomorphic.*

Proof. We have the natural map $\Psi : A \longrightarrow \mathfrak{A}$ such that $a \longmapsto \mathfrak{a}$, which preserves the operations $\oplus, ^*$, and the constant $\mathbf{0}$. Indeed, by Definition 6.3.3, for each $a, b \in A$, $\Psi(a \oplus b) = G^{-1}(\widetilde{a \oplus b}^{\rightarrow}(\widehat{a \oplus b})) = \mathfrak{a} \oplus \mathfrak{b} = \Psi(a) \oplus \Psi(b)$, and $\Psi(a^*) = \mathfrak{a}^* = (\Psi(a))^*$. Analogously, we have that $\Psi(\mathbf{0}) = \mathfrak{o}$. It is clear that Ψ is a surjection, and Ψ is injective as a consequence of isomorphism constructed in order to prove the representation theorem [10, theorem 4.1] for lexicographic MV-algebras. □

Conclusion

In this thesis, we have developed a big part of the fundamental properties of MV-topological spaces, and we have shown that these spaces have good mathematical behaviour. In Chapter 4, we have defined basic concepts as interior and closure of a fuzzy set, as well as quotient and product spaces. In the development of these concepts, the influence of the MV-algebra structure is self-evident, and we can see how it contributes to characterise these spaces within the fuzzy topological spaces. As an important fact, we showed that the set of clopen of a laminated MV-topological space is a Riesz MV-algebra.

In Chapter 5 we have shown the most essential analogous theorems of the classical topology. For MV-topological spaces, we have obtained a Tychonoff-type theorem; for what concerns compactness, a Stone-Čech Compactification and an Urysohn-type Lemma. In these three relevant results we used tools from fuzzy topologies and category theory. In this chapter, we also showed that the fuzzy unit interval can see as an MV-space and this fact allows to define normality and complete regularity for MV-topological spaces. In this direction, we defined a kind of fuzzy uniformity for our spaces, MV-uniformity. All these concepts and results show that MV-topologies is a good generalisation of classical topologies and they suggest new possible questions and problems about this topic.

Finally, in Chapter 6 we presented a new kind of sheaves. We generalised the sheaves over a topological space defining sheaves over an MV-topological space: the MV-sheaves. We showed that each MV-algebra A such that A/O_M is retractive radical, can be represented by an MV-sheaf.

Without a doubt, the concepts and results shown in this work open new paths to future investigations in fuzzy topology, MV-sheaves, convergence theory, algebraic MV-topology, among others. We have only taken a first step in a theory that will develop further in the coming times.

Conclusiones

En este trabajo hemos desarrollado gran parte de las propiedades fundamentales de los espacios MV-topológicos y hemos mostrado que estos espacios tienen un buen comportamiento matemático. En el Capítulo 4, hemos definido y discutido conceptos básicos como interior y clausura de un conjunto fuzzy, así como espacio producto y espacio cociente. Definimos también operadores MV-interior y MV-clausura, y mostramos que satisfacen las propiedades topológicas naturales. En el desarrollo de estos conceptos, la influencia de la estructura de MV-álgebra es autoevidente, y es claro cómo esta contribuye a caracterizar los MV-espacios dentro de los espacios topológicos fuzzy. Como un hecho relevante, mostramos que el conjunto de clopens de MV-espacios laminados es una MV-álgebra de Riesz.

En el Capítulo 5 mostramos los más importantes teoremas análogos de la topología clásica. Para los espacios MV-topológicos, hemos obtenido un teorema tipo-Tychonoff, una compactificación Stone-Čech y un Lemma tipo-Urysohn. Destacamos que en estos tres resultados, usamos herramientas de topologías fuzzy, así como herramientas algebraicas y teoría de categorías. En este capítulo también mostramos que el intervalo unidad fuzzy puede verse como un MV-espacio y este hecho permite definir normalidad y regularidad completa para espacios MV-topológicos. En esta dirección, definimos también un tipo de uniformidad fuzzy, para nuestros espacios, que hemos llamado MV-uniformidad. Todos estos conceptos y resultados indican que las MV-topologías son una buena generalización de las topologías clásicas y esto sugiere nuevas preguntas y problemas acerca de este tópico.

Finalmente, en el Capítulo 6 generalizamos el concepto de haz sobre espacio topológico, definiendo haces sobre espacios MV-topológicos, los MV-haces. Con este concepto, mostramos que cada MV-álgebra A tal que A/O_M tiene radical retractivo puede ser representada por un MV-haz. Esta representación separa la parte real y la parte infinitesimal de la MV-álgebra A, y la parte real está codificada por los elementos del MV-espacio base.

Sin duda alguna, los conceptos y resultados mostrados en este trabajo abren nuevos caminos a futuras investigaciones en topología fuzzy, MV-haces, teoría de convergencia, MV-topología algebraica, entre otras. Sólo hemos dado un primer paso en una teoría que se desarollará aún más en los próximos tiempos.

References

[1] Adámek J., Herrlich H., Strecker G.E., *Abstract and Concrete Categories – The Joy of Cats*, John Wiley & Sons, Inc., 1990. (available on line at [51])

[2] Baaz, M.; Infinite-valued Gödel logic with 0-1-projections and relativisations. In P. Hájek (ed.), *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics*. Lecture Notes in Logic, Springer, **6** (1996), 23–33.

[3] Belluce, L. P.; Semisimple algebras of infinite valued logic and bold fuzzy set theory. *Can. J. Math*, **38**/6 (1986), 1356–1379.

[4] Cerruti, U.; The Stone-Cech compactification in the category of fuzzy topological spaces, *Fuzzy Sets and Systems*, **6** (1981), 197–204.

[5] Chang, C. C.; Algebraic analysis of many valued logic. *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

[6] Chang, C. C.; A new proof of the completeness of the Łukasiewicz axioms. *Trans. Amer. Math. Soc.* **93** (1959), 74–90.

[7] Chang, C. L.; Fuzzy topological spaces. *J. Math. Anal. App.* **24**/1 (1968), 182–190.

[8] Cignoli R. L. O., D'Ottaviano I. M. L., Mundici D., *Algebraic Foundations of Many-valued Reasoning*, Trends in Logic, **7**, Kluwer Academic Publishers, Dordrecht, 2000.

[9] Davey B., *Sheaf spaces and sheaves of universal algebras*, Math. Z. 134/4 (1973), 275–290.

[10] Diaconescu D., Flaminio T., Leuştean I., Lexicographic MV-algebras and lexicographic states. *Fuzzy Sets and Systems*, **244** (2014), 63—85.

[11] Di Nola A., Lettieri A., Perfect MV-algebras are categorically equivalent to Abelian ℓ-groups, *Studia Logica*, **53** (1994), 417–432.

[12] Di Nola A., Lettieri A., Coproduct MV-Algebras, Nonstandard reals, and Riesz Spaces, *Journal of Algebra*, **185** (1996), 605–620.

[13] Di Nola A., Leuştean I., Łukasiewicz logic and MV-algebras, *Handbook of Mathematical Fuzzy Logic 2*, (2011), 469–583.

[14] Di Nola A., Leuştean I., Riesz MV-algebras and their logic, *Proceedings of EUSFLAT-LFA*, (2011), 140–145.

[15] Di Nola A., Leuştean, I., Łukasiewicz logic and Riesz spaces, *Soft. Comput.* **18** (2014), 2349—2363.

[16] Dubuc, E.J., Poveda, Y.A., Representation theory of MV-algebras, *Annals of Pure and Applied Logic*, **161**/8 (2010), 1024–1046.

[17] Ferraioli A. R., Lettieri A., Representations of MV-algebras by sheaves, *Math. Log. Quart*, **57**/1 (2011), 27–43.

[18] Filipoiu A., Georgescu G., Compact and Pierce representations of MV-algebras, *Rev. Roum. Math. Pures Appl*, **40** (1995), No.7-8 , 599–618.

[19] Gantner, T. E. and Steinlage, R. C. ; Compactness in Fuzzy Topological Spaces. *J. Math. Anal. Appl.* **62** (1978), 547–562.

[20] Hájek, P.; *Metamathematics of fuzzy logic*. Kluwer, 1998.

[21] Höhle, U.; Fuzzy topologies and topological space objects in a topos. *Fuzzy Sets and Systems* **19** (1986), 299–304.
[22] Höhle, U., and Stout, L. N.; Foundations of fuzzy sets. *Fuzzy Sets and Systems* **40** (1991), 257–296.
[23] Höhle, U., and Šostak, A. P.; A general theory of fuzzy topological spaces. *Fuzzy Sets and Systems* **73** (1995), 131–149.
[24] Hutton, B.; Normality in fuzzy topological spaces, *J. Math. Anal. Appl.* **50** (1975), 74–79.
[25] Hutton, B.; Uniformities on fuzzy topological spaces, *J. Math. Anal. Appl.* **58** (1977), 559–571.
[26] Katsaras, A. K.; On Fuzzy Uniform Spaces, *J. Math. Anal. Appl.* **101** (1984), 97–113.
[27] Kelley, J. L.; The Tychonoff product theorem implies the axiom of choice, *Fundamenta Mathematica* **37** (1950), 75–76.
[28] Leinster T., *Basic Category Theory*, Cambridge University Press, Cambridge, united Kingdom, 2014.
[29] Lowen, R.; Fuzzy topological spaces and fuzzy compactness. *J. Math. Anal. Appl.* **56** (1976), 621–633.
[30] Lowen, R.; Initial and Final Fuzzy Topologies and the Fuzzy Tychonoff Theorem. *J. Math. Anal. Appl.* **58** (1977), 11–21.
[31] Lowen, R.; A comparison of different compactness notions in fuzzy topological spaces. *J. Math. Anal. Appl.* **64** (1978), 446–454.
[32] Mac Lane S., *Categories for the working mathematician* - 2nd Edition, Springer, 1998
[33] Martin, H.; Weakly Induced Fuzzy Topological Spaces. *J. Math. Anal. Appl.* **78** (1980) 634–639.
[34] Maruyama, Y.; Fuzzy Topology and Łukasiewicz Logics from the Viewpoint of Duality Theory. *Studia Logica* **94** (2010), 245–269.
[35] Mundici, D.; *Advanced Łukasiewicz Calculus and MV-algebras.* Trends in Logic **35** (2011), Springer.
[36] Pao-Ming, P., and Ying-Ming, L.; Fuzzy Topology. I. Neighborhood Structure of a Fuzzy Point and Moore–Smith Convergence. *Journal of Mathematical Analysis and Applications* **76** (1980), 571–599.
[37] Pao-Ming, P., and Ying-Ming, L.; Fuzzy Topology. II. Product and Quotient Spaces. *Journal of Mathematical Analysis and Applications* **77** (1980), 20–37.
[38] Roman S., *An Introduction to the Language of Category Theory*, Birkhäuser, 2017.
[39] Rodabaugh, S. E.; A Theory of Fuzzy Uniformities with Applications to the Fuzzy Real Lineis. *J. Math. Anal. Appl.* **129** (1988) 37–70.
[40] Rodabaugh, S. E.; Point-set lattice-theoretic topology. *Fuzzy Sets and Systems* **40**, 297–345, 1991.
[41] Rodabaugh, S. E.; Categorical framework for Stone representation theories. In: S. E. Rodabaugh, E. P. Klement and U. Höhle, Eds., *Applications of Category Theory to Fuzzy Subsets*, Kluwer, 177–232, 1992.
[42] Russo, C., An extension of Stone duality to fuzzy topologies and MV-algebras. *Fuzzy Sets and Systems*, **303** (2016), 80–96.
[43] Šostak, A. P.; On some modifications of fuzzy topology. *Matematički Vesnik* **41** (1989), 51–64.
[44] Šostak, A. P.; Two decades of fuzzy topology: basic ideas, notions, and results. *Uspekhi Mat. Nauk.* **44**/6 (1989), 99–147.
[45] Stout, L. N.; Topoi and categories of fuzzy sets. *Fuzzy Sets and Systems* **12** (1984), 169–184.
[46] Stout, L. N.; A survey of fuzzy set and topos theory. *Fuzzy Sets and Systems* **42** (1991), 3–14.
[47] Tychonoff, A. N.; Über die topologische Erweiterung von Räumen, *Mathematische Annalen* **102**(1) (1930), 544–561.
[48] Wong, C. K.; Fuzzy Topology: Product and Quotient Theorems. *Journal of Mathematical Analysis and Applications* **45** (1974), 512–521.
[49] Wong, C. K.; Fuzzy points and local properties of fuzzy topology. *Journal of Mathematical Analysis and Applications* **46** (1974), 316–328.
[50] Zadeh L. A., Fuzzy sets, *Inform. and Control*, **8**, 338–353, 1965.
[51] http://katmat.math.uni-bremen.de/acc

Index