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When the Bullwhip Effect is an Increasing Function of the Lead Time

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Abstract:
We study the relationship between lead times and the bullwhip effect produced by the order-up-to policy. The *usual* conclusion in the literature is that longer lead-time increase the bullwhip effect, we show that this is not always the case. Indeed, it seems to be rather rare. We achieve this by first showing that a positive demand impulse response leads to a bullwhip effect that is *always increasing in the lead time* when the order-up-to policy is used to make supply chain inventory replenishment decisions. By using the zeros and poles of the z-transform of the demand process, we reveal when this demand impulse is positive. To make concrete our approach in a nontrivial example we study the ARMA(2,2) demand process.

Keywords: Bullwhip effect, Lead times, Order-up-to policy, Auto-regressive moving average demand process, Minimum mean squared error forecasting.

1. INTRODUCTION

The bullwhip effect, where outgoing order fluctuations are larger than incoming customer demand fluctuations at each supply chain echelon, has been extensively studied since the important contributions by Lee et al. (1997) and Lee et al. (2000). Wang and Disney (2016) provide a recent review of the literature on the bullwhip effect. The most often studied replenishment policy in bullwhip investigations is the linear order-up-to (OUT) policy. The OUT policy is often used in high volume settings where products are ordered and replenished every period. The OUT policy is incorporated into many ERP systems as a standard replenishment policy.

Often the first order auto-regressive, AR(1), demand process is considered in a bullwhip study (Zhang, 2007; Urban, 2005; Luong, 2007) as this is the simplest demand process with an auto-correlated structure. ARMA(1,1) demand processes were studied by Chen and Disney (2007); Gaalman (2006); Duc and Kim (2008). The second order ARMA(2,2) process is studied less often, although Gaalman and Disney (2009) considered the bullwhip implications of using a variety of the order-up-to (OUT) policies for this demand process.

General statements about the interaction between the bullwhip effect and the lead-time are missing in the literature. Dejonckheere et al. (2003) provide one of the only other references that explicitly considers the link between lead times and the bullwhip effect. They showed that for all demand processes, for all lead times, the OUT replenishment policy, with exponential smoothing and moving average forecasts, always generates bullwhip.

Our contribution herein is to determine the conditions under which the bullwhip effect increases in the lead time. We focus on the specific case of the ARMA(2,2) demand process. As a side contribution we also determine, for the class of second-order discrete time systems, when the system has a non-negative impulse response.

2. DEMAND AND REPLENISHMENT POLICY.

Ali et al. (2012) found that 75% of 1798 different SKU's in a European retailer belonged to, or were sub-sets of, the ARMA(2,2) demand process. Thus, we assume ARMA(2,2) demand is present, Box et al. (2008). The ARMA(2,2) process is given by

$$d_t = \mu_d + \sum_{i=1}^{2} \phi_i \left(d_{t-i} - \mu_d\right) - \sum_{j=1}^{2} \theta_j \epsilon_{t-j} + \epsilon_t. \quad (1)$$

Here, d_t is the demand in time period t, μ_d is the mean demand, ϕ_i are the auto-regressive coefficients, θ_j are the moving average coefficients, and ϵ_t is a stochastic independent and identically distributed (i.i.d.) random variable with zero mean and variance σ_ϵ^2.

The z-transform transfer function of the ARMA(2,2) demand process is given by

$$D[z] = \frac{B[z]}{A[z]} = \frac{z^2 - \theta_1 z - \theta_2}{z^2 - \phi_1 z - \phi_2}. \quad (2)$$

The order-up-to policy, Li et al. (2014), creates replenishment orders, o_t, via

$$o_t = \hat{d}_{t+k+1|t} - (i_t - \mu_i) - \sum_{j=1}^{k} \left(o_{t-j} - \hat{d}_{t+j|t}\right), \quad (3)$$

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where $\hat{d}_{t+k+1|t}$ is a forecast of the demand in period $t+k+1$ conditional upon the information available at time t. Herein we create the minimum mean squared error (MMSE) forecasts of the ARMA(2,2) demand process, Box et al. (2008). μ_i is the mean inventory, or safety stock, a constant that can be set arbitrarily. However, setting $\mu_i = F^{-1}[b/(b+h)]$ to the critical newsvendor fractile minimizes the expected per period inventory holding, h, and backlog b cost, where $F^{-1}[\cdot]$ is the inverse of the cumulative distribution function of the inventory distribution. The inventory balance equation completes the definition of the order-up-to policy,

$$i_{t+1} = i_t + o_{t-k} - d_{t+1}. \tag{4}$$

2.1 Stability and invertability of the ARMA(2,2) process

A stable system exists, if after a finite input, the system returns to a finite state in a finite amount of time. This is equivalent to the poles of the system transfer function lying within the unit circle in the complex plane. Jury (1974) provides an easy-to-use method to determine the stability conditions directly from the denominator of the demand transfer function, (2). Using his approach produces the following (triangular) set of stability conditions,

$$\{\phi_1 < 1-\phi_2, \phi_1 > \phi_2 - 1, \phi_2 > -1\}. \tag{5}$$

Box et al. (2008) show that for a demand process' structure to be uniquely identified from a time series, the process must be invertable. The invertability conditions can be readily obtained by applying Jury's stability criterion to the numerator of the demand transfer function, (2):

$$\{\theta_1 < 1-\theta_2, \theta_1 > \theta_2 - 1, \theta_2 > -1\}. \tag{6}$$

2.2 The bullwhip criterion and the impulse response

The usual way to measure bullwhip effect is the ratio, BI,

$$BI = (\sigma_o^2/\sigma_d^2) > 1 \tag{7}$$

where σ_o^2 is the variance of the replenishment orders o_t and σ_d^2 is the variance of the demand, d_t, Disney and Towill (2003). These variances only exist is the demand is stationary. When demand becomes non-stationary, (7) suggests that $BI = 1$ and bullwhip is not present, but this is not true when demand is non-stationary, or near non-stationary, Gaalman and Disney (2012). In these cases, the bullwhip criterion $CB[k]$ provides a better measure,

$$CB[k] = (\sigma_o^2 - \sigma_d^2)/\sigma_\epsilon^2. \tag{8}$$

When $CB[k] > 0$, a bullwhip effect exists; when $CB[k] < 0$ the orders have less variance than the demand. To understand $CB[k]$ we need the variance of the orders, σ_o^2, and the variance of the demand σ_d^2. These variances can be readily obtained by Tsypkin's squared impulse response theorem. The impulse response of the system is the system's output when the system's input is zero for all t except at $t = 1$ when the input is unity.

Lemma 1. (Tsypkin's squared impulse response theorem). The long-run variance, σ_x^2, of the output, x_t, from a linear system reacting to an i.i.d. white noise input with variance σ_ϵ^2 is given by the sum of its squared impulse response, $\tilde{x}_t^2$.

$$\sigma_x^2 = \sigma_\epsilon^2 \sum_{t=0}^{\infty} \tilde{x}_t^2. \tag{9}$$

Proof. We refer to Tsypkin (1964, 183-192) for proof of Tsypkin's relation and Dejonckheere et al. (2003) for its link to the bullwhip effect. □

3. THE ARMA(2,2) DEMAND IMPULSE RESPONSE

A rational transfer function (2) can be represented in zero-pole form,

$$D[z] = \frac{\prod_{i=1}^{2}(z-\lambda_i^\theta)}{\prod_{i=1}^{2}(z-\lambda_i^\phi)} \tag{10}$$

where $\{\lambda_i^\theta, \lambda_i^\phi\}$ are the zeros and poles (eigenvalues) of the transfer function. The eigenvalues of the ARMA(2,2) demand process are

$$\left\{\lambda_1^\theta = \frac{1}{2}\left(\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}\right), \quad \lambda_2^\theta = \frac{1}{2}\left(\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}\right)\right\} \tag{11}$$

and

$$\left\{\lambda_1^\phi = \frac{1}{2}\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right), \quad \lambda_2^\phi = \frac{1}{2}\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)\right\}, \tag{12}$$

Gaalman et al. (2018). Note, the poles and zeros can be real, (conjugate) complex, and can have common poles or zeros. Complex zeros (poles) exist when $\theta_1^2 + 4\theta_2 < 0$, $(\phi_1^2 + 4\phi_2 < 0)$.

Lemma 2. (Impulse response of the ARMA(2,2) demand). The ARMA(2,2) demand impulse response is

$$\tilde{d}_t = \begin{cases} 1, \text{ if } t = 0, \\ r_1(\lambda_1^\phi)^{t-1} + r_2(\lambda_2^\phi)^{t-1}, \text{ if } t \geq 1, \end{cases} \tag{13}$$

where,

$$r_1 = \frac{(\lambda_1^\phi - \lambda_1^\theta)(\lambda_1^\phi - \lambda_2^\theta)}{(\lambda_1^\phi - \lambda_2^\phi)} \text{ and } r_2 = \frac{(\lambda_2^\phi - \lambda_1^\theta)(\lambda_2^\phi - \lambda_2^\theta)}{(\lambda_2^\phi - \lambda_1^\phi)}. \tag{14}$$

□

Proof. [Patterned on Moudgalaya (2007)] Using polynomial long division, we re-write (10), as

$$D[z] = 1 + \frac{\prod_{i=1}^{2}(z-\lambda_i^\theta) - \prod_{i=1}^{2}(z-\lambda_i^\phi)}{\prod_{i=1}^{2}(z-\lambda_i^\phi)}. \tag{15}$$

Partial fraction expansion then leads to

$$D[z] = 1 + \sum_{j=1}^{2} \frac{r_j}{z - \lambda_j^\phi}. \tag{16}$$

The inverse z-transform of (16) provides (13). □

Gaalman et al. (2018) derives a more general expression based on the same approach for ARMA(p,q) demands. Furthermore, common poles do not lead to fundamentally different insights.

4. THE ORDER IMPULSE RESPONSE

Having obtained the impulse response of the ARMA(2,2) demand, we now derive the corresponding impulse response for the orders.

Lemma 3. (Impulse response of the orders). The impulse response of the orders is given by

$$\tilde{o}_t = \begin{cases} \sum_{j=0}^{k+1} \tilde{d}_{t+j}, \text{ if } t = 0, \\ \tilde{d}_{t+k+1}, \text{if } t > 0. \end{cases} \tag{17}$$

Proof. Under the order-up-to policy,

$$o_t = d_t + \sum_{j=1}^{k+1} \hat{d}_{t+j|t} - \sum_{j=1}^{k+1} \hat{d}_{t+j|t-1},$$

Also, for demand as an impulse response, $\hat{d}_{t+j|t} = \tilde{d}_{t+j}$ for $t \geqslant 0$ and $\hat{d}_{t+j|t} = 0$ otherwise. □

5. DEMAND AND ORDER VARIANCES

Using Tsypkin's relation, the demand variance is

$$\sigma_d^2 = \sigma_\epsilon^2 \sum_{t=0}^{\infty} \tilde{d}_t^2, \tag{18}$$

the order variance is

$$\sigma_o^2 = \sigma_\epsilon^2 \left(\left(\sum_{j=0}^{k+1} \tilde{d}_j \right)^2 + \sum_{t=1}^{\infty} \tilde{d}_{t+k+1}^2 \right). \tag{19}$$

Using these variances, $CB[k]$ becomes

$$CB[k] = \left(\sum_{j=0}^{k+1} \tilde{d}_j \right)^2 + \sum_{t=0}^{k+1} \tilde{d}_t^2. \tag{20}$$

Theorem 4. (Necessary-sufficient condition for increasing bullwhip). Iff $\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_{k+1}\} > 0$ then $CB[k]$ is positive and increasing in the lead time.

Proof. Consider $CB[k] - CB[k-1] = 2\tilde{d}_{k+1}E[k]$ with $CB[0] = 2\tilde{d}_1 E[0] = 2\tilde{d}_1$ and $E[k] = \sum_{j=0}^{k} \tilde{d}_j$, $E[0] = \tilde{d}_0 = 1$. For $k = 0$, iff $\tilde{d}_1 > 0$ then $CB[0] > 0$. By this, $E[1] > 0$. For $k = 1$ the $CB[1] - CB[0] > 0$ iff $\tilde{d}_2 > 0$. The same reasoning can be iterated for all other k $(= 2, 3, ...)$. □

Theorem 4, the main result of this paper, reveals the question of whether bullwhip is always increasing in the lead-time is the same as the whether the demand impulse response is positive for all t; that is, if $\forall t$, $\tilde{d}_t > 0$, then $CB[k]$ is increasing for all lead-times and vice versa.

Theorem 5. (Sufficient condition for a positive impulse response). If, for each AR eigenvalue between $0 \leq \lambda_j^\phi \leq 1$, the number of MA eigenvalue smaller than λ_j^ϕ is larger than the number of AR eigenvalues smaller than λ_j^ϕ, then a positive impulse response exists.

Proof. The proof is based on the z-transform of the demand process and uses the convolution theorem. First, note the z-domain pole-zero transfer function of $\tilde{d}_t$ given in (10). Theorem 4 showed the increasing monotonicity of bullwhip is equivalent to a positive impulse response of $D[z]$. Due to the dominance of the λ_j^ϕ eigenvalues over the λ_j^θ eigenvalues in Theorem 5 we can write

$$D[z] = \prod_{j=1}^{2} D_j[z]; \text{ where } D_j[z] = \left(\frac{z - \lambda_j^\theta}{z - \lambda_j^\phi} \right), \tag{21}$$

with $\lambda_j^\phi > \lambda_j^\theta$ and $\lambda_j^\phi > 0$. By this, each $D_j[z]$ has a positive impulse response. Multiplication in the z-domain is equivalent to convolution in the time domain. Convolution involves addition and multiplication operations. Any combination of addition and multiplication of positive terms produces a positive outcome. This convolution property shows that the product $D[z]$ also has a positive impulse response. □

Note, Theorem 5 can be easily extended to all ARMA(p,q) demands, Gaalman et al. (2018). Theorem 5 shows the dominance of the λ_j^ϕ eigenvalues over the λ_j^θ eigenvalues. The largest eigenvalue must always be a pole, and the smallest one must always be a zero. Theorem 5 is insightful because it depends only on the eigenvalue ordering rather than the specific value of the eigenvalues.

In control terms, the dominance of the λ_j^ϕ eigenvalues over the λ_j^θ eigenvalues means that the demand process has the characteristics of a low pass frequency filter, Nise (2004); the high frequencies are less dominant in the demand process compared to low frequencies. The OUT policy is less able to filter low frequencies, and by this, the order variance increases (and bullwhip increases) over the lead-time. Several eigenvalue orderings have the potential to satisfy Theorem 5. With ARMA(2,2) demand there are a total number of six orderings, of which, two are low pass orderings. Many well-known demand forecasting methods, such as exponential smoothing, (Brown and Meyer, 1961), Holt-Winters linear trend model Chatfield (1978), and the general polynomial model of Harrison (1967), also have pole and zero orderings that satisfy Theorem 5.

The inverse of Theorem 5: If for each λ_j^θ, the number of AR eigenvalues smaller than λ_j^θ is larger than the number of MA eigenvalues smaller than λ_j^θ then the impulse response is not always positive and bullwhip does not increase in the lead-time. Here, the λ_j^θ eigenvalues dominate, and the demand process exhibits large high-frequency harmonics. The proof is trivial as always $\tilde{d}_1 < 0$.

From these insights, the whole set of orderings can be split into 3 subsets: one set of orderings which potentially satisfies Theorem 5, one set of orderings that satisfies the inverse of Theorem 5, and the remaining set of orderings, see Figure 1. The remaining subset contains orderings where r_2 and/or $\tilde{d}_1$ are positive or negative. Also, there are orderings that have an increasing bullwhip behaviour here. For instance, the ordering considered by Liu (2011) and Liu and Bauer (2008) has positive impulse responses that are not covered by Theorem 5.

In the next section will investigate when the ARMA(2,2) demand impulse response is positive, and by Theorem 4, when bullwhip always increases in the lead time.

6. INCREASING BULLWHIP IN THE LEAD TIME

Here, we focus on real poles only as complex poles result in a demand impulse response that oscillates and does not have an always increasing bullwhip effect in the lead time. Complex conjugate zeros are allowed. The projection of these conjugate zeros to the real axis in the complex z plane determines its eigenvalue ordering.

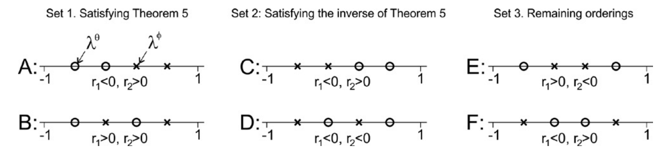

Fig. 1. The six possible real eigenvalue orderings for ARMA(2,2) demand

6.1 Potentially satisfying Theorem 5

Case A. Here the eigenvalues are in the following order: $-1 < \mathrm{Re}[\lambda_1^\theta] \le \mathrm{Re}[\lambda_2^\theta] < \lambda_1^\phi \le \lambda_2^\phi < 1$. It is easy to verify that $r_1 < 0 < r_2$, $\tilde{d}_1 > 0$, and $-r_2/r_1 > 1$. This case can exist when complex zeros are present. Depending of the sign of the poles, $\{\lambda_1^\phi, \lambda_2^\phi\}$, we need to consider the following three sub-cases:

Case A_1**:** $0 < \lambda_1^\phi \le \lambda_2^\phi$. Using $r_1 = \tilde{d}_1 - r_2$ in (13) provides

$$\tilde{d}_{t+1} = \tilde{d}_1(\lambda_1^\phi)^t + r_2((\lambda_2^\phi)^t - (\lambda_1^\phi)^t) > 0, \quad (22)$$

which is positive for all t as $\tilde{d}_1, r_2, \lambda_1^\phi, \lambda_2^\phi > 0$ and $\lambda_2^\phi > \lambda_1^\phi$. This means that the bullwhip effect is increasing in the lead time. Note, this case satisfies the requirements of Theorem 5.

Case A_2**:** $\lambda_1^\phi < 0 < \lambda_2^\phi$. The increasing in the lead time bullwhip condition, $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$, becomes

$$\left(\frac{\lambda_1^\phi}{\lambda_2^\phi}\right)^t < -\frac{r_2}{r_1}. \quad (23)$$

As $\lambda_1^\phi < 0 < \lambda_2^\phi$ and $-r_2/r_1 > 1$, two further sub-cases exist.

- A_{2i}. When $-\lambda_1^\phi < \lambda_2^\phi$ the RHS of (23) oscillates with decaying amplitude, with amplitude strictly less than one, meaning that the impulse response is positive. The $-\lambda_1^\phi < \lambda_2^\phi$ condition is equivalent to $\lambda_1^\phi + \lambda_2^\phi > 0 \implies \phi_1 > 0$, see (12). Interestingly, case A_{2i} does not meet the requirements of Theorem 5, but still an increasing bullwhip in the lead time is present.
- A_{2ii}. If $-\lambda_1^\phi > \lambda_2^\phi$, the impulse may initially be positive, but the LHS of (23) will oscillate with ever increasing amplitude and eventually the increasing bullwhip condition will not hold.

Case A_3**:** $\lambda_1^\phi \le \lambda_2^\phi < 0$. First note, that for a positive impulse response $(\lambda_1^\phi/\lambda_2^\phi)^t < (-r_2/r_1)$ is required for an even t and $(\lambda_1^\phi/\lambda_2^\phi)^t > (-r_2/r_1)$ is required for an odd t. That $\lambda_1^\phi/\lambda_2^\phi > 1$, implies $(\lambda_1^\phi/\lambda_2^\phi)^t$ is increasing in t. This means that while a positive impulse may initially exist, with a large enough t, eventually the impulse response will be negative for large odd t, while for even t the impulse response is always positive. The increasing in the lead time bullwhip criteria will be violated.

Note, case A_1 conforms to the requirements of Theorem 5, whereas the A_2 and A_3 cases do not.

Case B. The eigenvalues are ordered as $-1 < \lambda_1^\theta < \lambda_1^\phi < \lambda_2^\theta < \lambda_2^\phi < 1$, both $\{r_1, r_2\} > 0$, and $0 < \tilde{d}_1 = r_1 + r_2 < 1$. As the zeros enclose a pole, this ordering cannot exist with complex conjugate zeros.

Case B_1**:** $0 < \lambda_1^\phi < \lambda_2^\phi$. As $\{r_1, r_2, \lambda_1^\phi, \lambda_2^\phi\} > 0$, it is obvious that $\forall t$, $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$.

Case B_2**:** $-\lambda_1^\phi < 0 < \lambda_2^\phi$. The increasing in the lead time bullwhip condition, $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$ becomes

$$\left(\frac{\lambda_1^\phi}{\lambda_2^\phi}\right)^t > -\frac{r_2}{r_1}. \quad (24)$$

In case B_2, all odd t have positive impulse responses. Extra conditions lead to positive impulse responses for even t:

Sub-case B_{2i}. If $-\lambda_1^\phi < \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ oscillates between positive and negative numbers that tend towards zero as t increases. This leads to two further sub-cases:

- B_{2ia}. When $\tilde{d}_2 > 0$, the minimum $(\lambda_1^\phi/\lambda_2^\phi)^t$, which occurs at $t = 1$, means all impulses are positive, revealing bullwhip increases in the lead time.
- B_{2ib}. When $\tilde{d}_2 < 0$, the impulse response initially has a negative impulse for even t, but with a large enough t, $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$, implying that $\tilde{d}_{t+1} > 0$, and as $E[\infty] > 0$ then an increasing bullwhip in the lead-time effect will return with long lead-times.

Sub-case B_{2ii}. If $\lambda_1^\phi > \lambda_2^\phi$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ oscillates with ever-increasing amplitude which will eventually break the condition $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$. There are two further sub-cases:

- B_{2iia}. If $\tilde{d}_2 < 0$ then for all odd t, $\tilde{d}_{t+1} < 0$ and for even t, $\tilde{d}_{t+1} > 0$, indicating that bullwhip does not always increasing in the lead time.
- B_{2iib}. If $\tilde{d}_2 > 0$ then for small t, $\tilde{d}_{t+1} > 0$; for large odd t, $\tilde{d}_{t+1} < 0$ and for large even t, $\tilde{d}_{t+1} > 0$, indicating that the bullwhip effect may initially increase in the lead time for small k, but for large k is does not.

Case B_3**:** $\lambda_1^\phi < \lambda_2^\phi < 0$. From $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t$, $r_1, r_2 > 0$, and $\lambda_1^\phi, \lambda_2^\phi < 0$. Together these imply $\tilde{d}_{t+1} > 0$ for even t and $\tilde{d}_{t+1} < 0$ for odd t. This oscillating impulse response does not produce a bullwhip effect that always increasing in the lead time.

Case B_1, satisfies the conditions of Theorem 5, but cases B_2 and B_3 do not. Case B_{2ia}, with its positive impulse response, is an example that confirms Theorem 5 is a sufficient, but not a necessary, condition.

6.2 Satisfying the inverse of Theorem 5

This set of orderings do not conform to the requirements of Theorem 5. That is, it is not possible to pair up all poles and zeros such that each pair has a zero to the left of a pole. For this class of eigenvalue orderings, a positive demand impulse response does not always exist, and bullwhip is not always increasing in the lead-time.

Case C. The eigenvalue order is $\lambda_1^\phi \le \lambda_2^\phi < \mathrm{Re}[\lambda_1^\theta] \le \mathrm{Re}[\lambda_2^\theta]$ and $r_1 < 0 < r_2$. It is clear from the order of the eigenvalues that $-r_1 > r_2$. As a consequence the demand impulse at $t = 1$ is $\tilde{d}_1 = r_1 + r_2 < 0$, from which we can immediately conclude that the bullwhip is not always increasing in the lead-time for case C [1].

Case D. The eigenvalue ordering is $\lambda_1^\phi < \lambda_1^\theta < \lambda_2^\phi < \lambda_2^\theta$. As there is a pole between the two zeros, this ordering does not exist when there are complex zeros. Using the ordering it is easy to show that $\{r_1, r_2\} < 0$ and $\tilde{d}_1 = r_1 + r_2 < 0$. So immediately, we know that bullwhip is not always increasing in the lead-time[1].

6.3 Other eigenvalue orderings

This set of ordering is not covered by Theorem 5 or its inverse, so must be considered separately. This can be done by building upon Theorem 4 in the same manner that we used to verify the previous two sets.

Case E. The eigenvalue ordering in this case is $\lambda_1^\theta < \lambda_1^\phi \leq \lambda_2^\phi < \lambda_2^\theta$ and $r_1 > 0 > r_2$. As the two zeros are separated by the two poles, this ordering does not exist when complex poles are present.

Case E_1**:** $0 < \lambda_1^\phi \leq \lambda_2^\phi$. Using $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t$ and $\lambda_2^\phi > \lambda_1^\phi$, as t becomes large $-r_2(\lambda_2^\phi)^t > r_1(\lambda_1^\phi)^t$ and $\tilde{d}_{t+1}$ turns negative after one change of sign, indicating that bullwhip does not always increase in the lead-time.

Case E_2**:** $\lambda_1^\phi < 0 < \lambda_2^\phi$. As $\lambda_1^\phi < 0$, $r_1(\lambda_1^\phi)^t$ oscillates between positive and negative numbers. $r_2(\lambda_2^\phi)^t$ is always negative. Depending on the relative sizes of $\{r_1, r_2, \lambda_1^\phi, \lambda_2^\phi\}$, $\tilde{d}_{t+1}$ is either always negative or oscillating between positive and negative:

- *Sub-case* E_{2i}. If $-\lambda_1^\phi > \lambda_2^\phi$, after $t = 0$, $\tilde{d}_t$ is initially negative, but after some time, falls into an oscillation, where for odd t, $\tilde{d}_t > 0$, and for even t, $\tilde{d}_t < 0$.
- *Sub-case* E_{2ii}. If $-\lambda_1^\phi < \lambda_2^\phi$, after $t = 0$, $\tilde{d}_t$ is initially oscillating between a positive and negative number, but after some time will become forever after negative, $\tilde{d}_t < 0$.

Case E_3**:** $\lambda_1^\phi \leq \lambda_2^\phi < 0$. Using $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t > 0$, as $\lambda_1^\phi/\lambda_2^\phi > 1$, $(\lambda_1^\phi/\lambda_2^\phi)^t$ quickly tends to ∞ as t increases. The finite $-r_2/r_1 > 0$. For even t, we require $(\lambda_1^\phi/\lambda_2^\phi)^t > -r_2/r_1$; for odd t, we require $(\lambda_1^\phi/\lambda_2^\phi)^t < -r_2/r_1$. For even t, the increasing bullwhip criterion is satisfied, for odd t, it is not. This leads to an odd/even lead time effect in the bullwhip behaviour.

Case F. In case F, the eigenvalue order is $\lambda_1^\phi < \mathrm{Re}[\lambda_1^\theta] \leq \mathrm{Re}[\lambda_2^\theta] < \lambda_2^\phi$ and $r_1 < 0 < r_2$. Nothing is known about the relative size of r_1 and r_2. Case F can exist with complex conjugate poles.

Case F_1**:** $0 < \lambda_1^\phi \leq \lambda_2^\phi$. Using the time domain impulse response, $\tilde{d}_{t+1} = r_1(\lambda_1^\phi)^t + r_2(\lambda_2^\phi)^t$, as $\lambda_2^\phi \geq \lambda_1^\phi$ over time $r_2(\lambda_2^\phi)^t > -r_1(\lambda_1^\phi)^t$ implying that $\lim_{t\to\infty} \tilde{d}_{t+1} = 0^+$. Consider now the increasing bullwhip criterion (23), as $0 < (\lambda_1^\phi/\lambda_2^\phi) < 1$ then $(\lambda_1^\phi/\lambda_2^\phi)^t$ is decreasing t. We know $-r_2/r_1 > 0$, thus $\tilde{d}_1$ is sufficient to reveal the long term behaviour of F_1 solutions.

[1] Further insights for case C and D can be gained but as they have no positive impulse responses we have omitted the details for brevity.

- *Sub-case* F_{1i}. Here $\tilde{d}_1 = \phi_1 - \theta_1 < 0$, and thus bullwhip is not always increasing in the lead-time. However, when t gets sufficiently large, $\tilde{d}_t$ becomes, and remains, positive.
- *Sub-case* F_{1ii}. If $\tilde{d}_1 = \phi_1 - \theta_1 > 0$ then $-r_2/r_1 > 1$ and $\forall t$ $\tilde{d}_{t+1} > 0$. As the requirements of Theorem 5 are not satisfied here, then is another illustration that Theorem 5 is a sufficient, but not a necessary condition for increasing bullwhip in the lead-time.

Case F_2**:** $\lambda_1^\phi < 0 < \lambda_2^\phi$. Consider the increasing bullwhip criterion, (23); $\lambda_1^\phi/\lambda_2^\phi < 0$ and $-r_2/r_1 > 0$.

- *Sub-case* F_{2i}. If $\tilde{d}_1 = \phi_1 - \theta_1 > 0$ and $-\lambda_1^\phi < \lambda_2^\phi$, then $-r_2/r_1 > 1$ and $\forall t$ $\tilde{d}_{t+1} > 0$. That is, bullwhip is increasing in the lead-time.
- *Sub-case* F_{2ii}. If $\tilde{d}_1 = \phi_1 - \theta_1 < 0$ and $-\lambda_1^\phi < \lambda_2^\phi$ then $(\lambda_1^\phi/\lambda_2^\phi)^t$ oscillates, but over time the oscillations dampen out and $\forall t$ $(\lambda_1^\phi/\lambda_2^\phi)^t \leq 1$. As $\tilde{d} < 0$ then initially for small odd values of t, $\tilde{d}_t < 0$; however, as t becomes large $\tilde{d}_t > 0$ for both odd and even t.
- *Sub-case* F_{2iii}. If $-\lambda_1^\phi > \lambda_2^\phi$, then $(\lambda_1^\phi/\lambda_2^\phi)^t$ oscillates with ever increasing amplitude as t increases and the bullwhip criterion does not hold.

Case F_3**:** $\lambda_1^\phi \leq \lambda_2^\phi < 0$. For exactly the same reasons as case E_3, case F_3 does not have an increasing in the lead time bullwhip behaviour.

7. CONCLUSIONS

We have introduced a new bullwhip metric, $CB[k]$, useful when large order and demand variances are present. Theorem 4 showed the positivity of the order impulse response determines the essential character of $CB[k]$ over the lead-time. We confirmed this by studying the eigenvalues, $\{\lambda_i^\phi, \lambda_j^\theta\}$ of the demand process rather than AR and MA parameters, $\{\phi_i, \theta_j\}$, directly. This proved to be efficient as only the order of the eigenvalues determines a lead-time/bullwhip relationship, not the specific value of the eigenvalues or the demand parameters. We found three different sets of eigenvalue orderings exist:

- a set where increasing bullwhip increases over the lead-time is possible,
- an inverse set in which bullwhip is not increasing in the lead-time,
- and a third set which includes decreasing bullwhip over the lead-time, and a bullwhip/lead-time relationship that depends on specific values of eigenvalues.

Theorem 5 identified a class of easy-to-identify orderings for which the general demand processes behaves as a low pass filter that is sufficient to describe when the bullwhip is an increasing function of the lead-time. This class is important because well-known forecasting methods are part of this class, Li and Disney (2018). Within this class, the strength of the low pass filter directly influences the strength of $CB[k]$.

We illustrated our results by studying all the possible eigenvalue orderings of the ARMA(2,2) demand process. We were able to fully characterize bullwhip over the lead-time, for all possible stable and invertible ARMA(2,2) demand processes. This is a unique and important contribution to the study of the bullwhip problem. Our analysis is also important beyond the bullwhip application we have studied here. Having obtained a complete understanding of stable and invertible ARMA(2,2) we have actually obtained a complete (necessary and sufficient) understanding of all second order discrete time control systems. We believe this also is a unique and important contribution to the field of automatic control.

The impulse response and bullwhip properties for higher order ARMA demand processes that do not belong to the low pass subset is generally complex and we do not yet understand its behaviour completely. The characteristics of both the demand process and the OUT policy are important factors that determines whether a bullwhip effect is increasing in the lead-time or not. Studying how the lead-time influences the bullwhip behaviour in other inventory replenishment policies is an interesting topic for future research.

The practicing manager, having observed an ARMA(2,2) process structure in demand, may what to consider lead time reduction. Depending on the demand process observed, there may or may not be a bullwhip benefit from reducing the lead time. If there is a benefit, the cost of reducing the lead time may be offset against the reduced capacity costs, (Hosoda and Disney, 2012); if bullwhip does not increase in the lead time, perhaps different (cheaper, slower, more ecologically friendly) transport modes or production technology can be used.

REFERENCES

Ali, M., Boylan, J., and Syntetos, A. (2012). Forecast errors and inventory performance under forecast information sharing. *International Journal of Forecasting*, 28, 830–841.

Box, G., Jenkins, G., and Reinsel, G. (2008). *Time Series Analysis, Forecasting, and Control*. Holden-Day, San Francisco.

Brown, R. and Meyer, R. (1961). Fundamental theorem of exponential smoothing. *Operational Research*, 9, 673–685.

Chatfield, C. (1978). The Holt-Winters forecasting procedure. *Applied Statistics*, 27, 264–279.

Chen, Y. and Disney, S. (2007). The myopic order-up-to policy with a proportional feedback controller. *International Journal of Production Research*, 45(2), 351–368.

Dejonckheere, J., Disney, S., Lambrecht, M., and Towill, D. (2003). Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research*, 147, 590–597.

Disney, S. and Towill, D. (2003). Vendor-managed inventory and bullwhip reduction in a two-level supply chain. *International Journal of Operations & Production Management*, 23(6), 625–651.

Duc, T. Luong, H. and Kim, Y. (2008). A measure of bullwhip effect in supply chains with a mixed autoregressive-moving average demand process. *European Journal of Operational Research*, 187, 243256.

Gaalman, G. (2006). Bullwhip reduction for the ARMA demand model: The proportional out policy, versus the full-state feedback policy. *Automatica*, 42, 1283–1290.

Gaalman, G. and Disney, S. (2009). On bullwhip in a family of order-up-to policies with ARMA(2,2) demand and arbitrary lead-times. *International Journal of Production Economics*, 121(2), 454 – 463.

Gaalman, G. and Disney, S. (2012). Bullwhip behaviour in the order-up-to policy with ARIMA demand. *4th World Production and Operations Management Conference, Amsterdam, 1-5 July.*

Gaalman, G., Disney, S., and Wang, X. (2018). Bullwhip behaviour as a function of the lead-time for the order-up-to policy under ARMA demand. In *Pre-prints of the 20th International Working Seminar of Production Economics*, volume 2, 249–260. Innsbruck, Austria.

Harrison, D. (1967). Exponential smoothing and short-term sales forecasting. *Management Science*, 18, 821–851.

Hosoda, T. and Disney, S.M. (2012). On the replenishment policy when the market demand information is lagged. *International Journal of Production Economics*, 135(1), 458–467.

Jury, E. (1974). *Inners and the Stability of Dynamic Systems*. John Wiley, New York.

Lee, H., Padmanabhan, P., and Whang, S. (1997). Information distortion in a supply chain: the bullwhip effect. *Management Science*, 43, 543–558.

Lee, H., So, K., and Tang, C. (2000). The value of information sharing in a two-level supply chain. *Management Science*, 46, 626–643.

Li, Q. and Disney, S. (2018). Inventory performance of the damped trend forecasting method. In *Pre-prints of the 20th International Working Seminar of Production Economics*, volume 1, 249–260. Innsbruck, Austria.

Li, Q., Disney, S., and Gaalman, G. (2014). Avoiding the bullwhip effect using damped trend forecasting and the order-up-to replenishment policy. *International Journal of Production Economics*, 149, 3–16.

Liu, Y. (2011). *Analysis and design of systems with a nonnegative impulse response*. PhD Dissertation, University of Notre Dame.

Liu, Y. and Bauer, P. (2008). Sufficient conditions for non-negativity impulse response of arbitrary-order systems. In *IEEE Asia Pacific Conference on Circuits and Systems*, 14101413. Available at http://ieeexplore.ieee.org/abstract/document/4746294/.

Luong, H. (2007). Measure of bullwhip effect in supply chains with auto-regressive demand process. *European Journal of Operational Research*, 180, 1086–1097.

Moudgalaya, K. (2007). *Digital Control*. John Wiley & Sons, Chichester, UK.

Nise, N. (2004). *Control Systems Engineering*. John Wiley & Sons, New Jersey.

Tsypkin, Y. (1964). *Sampling Systems Theory and its Applications*, volume 1. Pergamon Press, Oxford.

Urban, T. (2005). A periodic-review model with serially-correlated, inventory-level-dependent demand. *International Journal of Production Economics*, 95, 287–295.

Wang, X. and Disney, S. (2016). The bullwhip effect: progress, trends and directions. *European Journal of Operational Research*, 250, 691–701.

Zhang, X. (2007). The impact of forecasting methods on the bullwhip effect. *European Journal of Operational Research*, 88, 15–27.