AN ALTERNATIVE VIEW ON THE BATEMAN-LUKE VARIATIONAL PRINCIPLE

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A new derivation of the Bernoulli equation for water waves in three-dimensional rotating and translating coordinate systems is given. An alternative view on the Bateman-Luke variational principle is presented. The variational principle recovers the boundary value problem governing the motion of *potential water waves* in a container undergoing prescribed rigid-body motion in three dimensions. A mathematical theory is presented for the problem of three-dimensional interactions between *potential surface waves* and a floating structure with interior *potential fluid sloshing*. The complete set of equations of motion for the exterior gravity-driven water waves, and the exact nonlinear hydrodynamic equations of motion for the linear momentum and angular momentum of the floating structure containing fluid, are derived from a second variational principle.

1 Introduction

The Bateman-Luke variational principle (Bateman 1932; Luke 1967) for the problem of fluid sloshing in a container undergoing prescribed rigid-body motion in three dimensions is given by Lukovsky (1990), Lukovsky (2015), Faltinsen & Timokha (2009), Timokha (2016) and Faltinsen *et al.* (2000) as

$$\delta\mathcal{L}\left(\Phi,\xi\right)=\delta\int_{t_1}^{t_2}\int_{\mathcal{Q}(t)}-\rho\left(\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi-\nabla\Phi\cdot\left(\boldsymbol{v}_0+\boldsymbol{\omega}\times\boldsymbol{r}\right)+U\right)\mathrm{d}\mathcal{Q}\,\mathrm{d}t=0\,, \tag{1}$$

where $\mathcal{Q}\left(t\right)$ is the fluid volume bounded by the free surface $\Sigma\left(t\right)$ and the wetted tank surface $S\left(t\right)$, $\Phi\left(x,y,z,t\right)$ is the velocity potential of the interior irrotational flow in a non-inertial (body) coordinate system $Oxyz$ fixed with respect to the rigid tank, the origin of the body coordinate system $Oxyz$ is in the unperturbed free surface and moves with the velocity $\boldsymbol{v}_0$ relative to an inertial (laboratory) coordinate system $O'x'y'z'$, $\xi\left(x,y,t\right)$ is the free surface height relative to the body frame O, $\boldsymbol{\omega}$ is the angular velocity of the tank relative to the laboratory coordinate system $O'x'y'z'$, and $U\left(x,y,z,t\right)$ is the gravity field potential defined as

$$U\left(x,y,z,t\right)=-\boldsymbol{g}\cdot\boldsymbol{r}'\quad\text{with}\quad\boldsymbol{r}'=\boldsymbol{r}_0'+\boldsymbol{r}\,, \tag{2}$$

where $\boldsymbol{r}'$ is the position vector of a point of the fluid-body system with respect to the laboratory frame O', $\boldsymbol{r}_0'$ is the coordinate vector of the origin of the body frame O with respect to the origin of the laboratory frame O', $\boldsymbol{r}$ is the position vector with respect to O and $\boldsymbol{g}$ is the gravity acceleration vector. Taking the variations $\delta\Phi$ and $\delta\xi$ in the variational principle (1) subject to the restrictions $\delta\Phi=0$ at the end points of the time interval, t_1 and t_2, gives the following boundary value problem (Faltinsen *et al.* 2000)

$$\left.\begin{aligned}
&\Delta\,\Phi:=\Phi_{xx}+\Phi_{yy}+\Phi_{zz}=0\quad\text{in}\quad\mathcal{Q}\left(t\right)\,,\\
&\frac{\partial\Phi}{\partial\boldsymbol{n}}=\boldsymbol{v}_0\cdot\boldsymbol{n}+\boldsymbol{\omega}\cdot\left(\boldsymbol{r}\times\boldsymbol{n}\right)\quad\text{on}\quad S\left(t\right)\,,\\
&\frac{\partial\Phi}{\partial\boldsymbol{n}}=\boldsymbol{v}_0\cdot\boldsymbol{n}+\boldsymbol{\omega}\cdot\left(\boldsymbol{r}\times\boldsymbol{n}\right)+\frac{\xi_t}{\sqrt{1+\xi_x^2+\xi_y^2}}\quad\text{on}\quad\Sigma\left(t\right)\,,\\
&\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi-\nabla\Phi\cdot\left(\boldsymbol{v}_0+\boldsymbol{\omega}\times\boldsymbol{r}\right)+U=0\quad\text{on}\quad\Sigma\left(t\right)\,,
\end{aligned}\right\} \tag{3}$$

where $\boldsymbol{n}$ is the outward-pointing normal to the boundary of $\mathcal{Q}\left(t\right)$. The second equation in (3) gives the rigid-wall boundary condition, the third equation in (3) is the kinematic free surface boundary condition, and the last equation in (3) is the dynamic free surface boundary condition deduced from Bernoulli's equation, which is the pressure field p vanishes at the free surface. The Bernoulli equation for the hydrodynamic pressure p in $\mathcal{Q}(t)$ takes the form (Faltinsen *et al.* 2000; Faltinsen & Timokha 2009)

$$\frac{\partial \Phi}{\partial t} + \frac{p}{\rho} + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi - \nabla\Phi\cdot\left(\boldsymbol{v}_0 + \boldsymbol{\omega}\times\boldsymbol{r}\right) + U = 0\,, \tag{4}$$

where ρ is the density of the fluid and $\partial\Phi/\partial t$ is calculated in the body coordinate system, i.e. for a point rigidly connected with the system $Oxyz$.

It is stated in the literature that Bernoulli's equation which is a result of integrating the Euler equations relative to the laboratory coordinate system, i.e. $O'x'y'z'$, is only valid in an inertial system and hence cannot directly be applied to an accelerated coordinate system, i.e. $Oxyz$. Hence, the Bernoulli equation (4), in the body coordinate system, is obtained by transforming the Bernoulli equation from the laboratory coordinate system O' to the body coordinate system O, by relating $\partial\Phi/\partial t$ between the inertial and non-inertial coordinate systems.

Our main goal in the first part of the current paper, §2 and §3, is to present a new derivation of the Bernoulli equation (4) by integrating the Euler equations relative to the rotating and translating coordinate system attached to the moving container, using the vorticity equation. The proposed Bernoulli equation is then used to present an alternative view on the Bateman-Luke variational principle (1) and the boundary value problem (3) for water waves in moving (rotating and translating) coordinate systems.

Variational principles for the motion of a rigid body dynamically coupled to its interior fluid motion are given by Moiseyev & Rumyantsev (1968) and Lukovsky (2015) (and references therein). In the work by Lukovsky (2015), the Bateman-Luke variational principle is used to develop a mathematical theory for interactions between potential surface waves and a floating rigid body containing cavities filled partially with a homogeneous incompressible ideal liquid. Alemi Ardakani (2019) derived a variational principle for three-dimensional interactions between gravity-driven potential water waves and a floating rigid body dynamically coupled to its interior inviscid and incompressible fluid sloshing governed by the Euler equations relative to the rotating-translating coordinate system attached to the body. The variational principle gives the complete set of equations of motion for the exterior water waves, the Euler-Poincaré equations for the angular momentum and linear momentum of the rigid-body, and the Euler equations for the motion of the interior fluid of the rigid-body relative to the body coordinate system.

The main goal in the second part of the current paper, §4, is to adapt the variational principles of Alemi Ardakani (2019) to develop a rigorous mathematical theory for three-dimensional (3–D) interactions between *potential surface waves* and a freely floating structure dynamically coupled to its interior *potential fluid sloshing* relative to the rotating and translating coordinate system attached to the moving body. The Bateman-Luke variational principle, presented in §3, recovers the Neumann boundary value problem, in terms of the velocity potential, governing the motion of the interior fluid of the floating rigid-body interacting with the exterior ocean waves. The aim in §4 is to present a second variational principle which recovers the equations of motion for the exterior potential water waves, and gives the exact hydrodynamic equations of motion for the angular momentum and linear momentum of the floating rigid-body, dynamically coupled to its interior potential fluid motion. Here we would like to highlight that the motion of the interior fluid of the floating rigid-body in Alemi Ardakani (2019) is governed by the Euler equations relative to the body coordinate system $\boldsymbol{x}$ (see below), while the motion of the interior fluid of the rigid-body in the current paper is governed by the revisited Bateman-Luke velocity potential theory. Adapting the variational principles developed by Alemi Ardakani (2019), the required variational principle takes the form

$$\delta\mathscr{L}\left(\phi,\eta,\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}}\right) = \delta\mathscr{L}_1\left(\phi,\eta\right) + \delta\mathscr{L}_2\left(\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}}\right) = 0\,, \tag{5}$$

where

$$\mathscr{L}_1\left(\phi,\eta\right) = \int_{t_1}^{t_2}\int_{V(t)} -\rho\left(\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + gZ\right)\mathrm{d}V\,\mathrm{d}t\,, \tag{6}$$

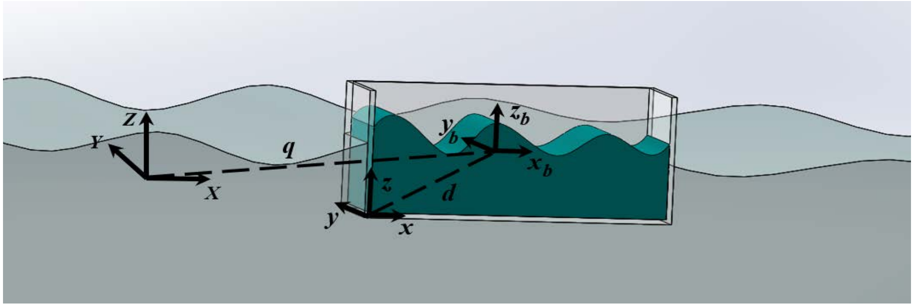

Figure 1: Schematic showing a floating structure containing fluid in hydrodynamic interaction with exterior ocean surface waves. The laboratory coordinate system is denoted by $\boldsymbol{X} = (X, Y, Z)$ and the body coordinate systems, attached to the floating rigid-body, are denoted by $\boldsymbol{x}_b = (x_b, y_b, z_b)$ and $\boldsymbol{x} = (x, y, z)$. The distance between the origin of the laboratory frame $\boldsymbol{X}$ and the point of rotation is denoted by the vector $\boldsymbol{q}\,(t)$. The distance from the point of rotation, i.e. the origin of the body frame $\boldsymbol{x}_b$, to the origin of the body frame $\boldsymbol{x}$ is denoted by the constant vector $\boldsymbol{d}$.

and

$$\begin{aligned}\mathscr{L}_2\left(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}\right) = \int_{t_1}^{t_2} \bigg(\int_{\mathscr{Q}} \Big(& \tfrac{1}{2}\left\|\boldsymbol{u}\right\|^2 + \boldsymbol{u}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})\right) \\ & + \tfrac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2 - g\left(\boldsymbol{Q}\left(\boldsymbol{x}+\boldsymbol{d}\right)+\boldsymbol{q}\right)\cdot\boldsymbol{k}\Big)\,\rho\,\mathrm{d}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_f\boldsymbol{\Omega} \\ + \tfrac{1}{2}m_v\left\|\dot{\boldsymbol{q}}\right\|^2 + \left(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v\right)\cdot\boldsymbol{Q}^T\dot{\boldsymbol{q}} + \tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_v\boldsymbol{\Omega} & - m_v g\left(\boldsymbol{Q}\overline{\boldsymbol{x}}_v + \boldsymbol{q}\right)\cdot\boldsymbol{k}\bigg)\,\mathrm{d}t\,,\end{aligned} \tag{7}$$

where in the derivation of the Lagrangian functional (5), three frames of reference are used. The laboratory frame, which is fixed in space, has coordinates denoted by $\boldsymbol{X} = (X, Y, Z)$. The first body frame, which is placed at the centre of rotation of the moving body and used for the analysis of the rigid body motion, has coordinates denoted by $\boldsymbol{x}_b = (x_b, y_b, z_b)$. The second body frame, which is attached to the moving body and used for the analysis of the fluid motion inside the tank, has coordinates denoted by $\boldsymbol{x} = (x, y, z)$. The distance between the origin of the body frame $\boldsymbol{x}_b$ (the point of rotation) and the origin of the body frame $\boldsymbol{x}$, is denoted by the position vector $\boldsymbol{d} = (d_1, d_2, d_3)$ which is a constant vector. So the position of a fluid particle relative to the body frame $\boldsymbol{x}_b$ is $\boldsymbol{x}_b = \boldsymbol{x} + \boldsymbol{d}$. The fluid-tank system has a uniform translation $\boldsymbol{q}\,(t) = (q_1(t), q_2(t), q_3(t))$ relative to the laboratory frame $\boldsymbol{X}$, which is the vector from the origin of the laboratory frame $\boldsymbol{X}$ to the origin of the body frame $\boldsymbol{x}_b$. In (5), $\boldsymbol{u}\,(x, y, z, t)$ is the Eulerian velocity of a fluid particle in the body frame $\boldsymbol{x}$, $\mathscr{Q}$ is the volume of the fluid inside the tank, $\boldsymbol{k}$ is the unit vector in the Z direction, g is the acceleration due to gravity, ρ is the water density which is assumed to be the same for the interior and exterior fluids, $\boldsymbol{Q}\,(t) \in \mathrm{SO}\,(3)$ is a proper rotation in $\mathbb{R}^3$, i.e. $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$ and $\det\left(\boldsymbol{Q}\right) = 1$, the *body angular velocity* is a time-dependent vector $\boldsymbol{\Omega}\,(t) = \left(\Omega_1\,(t), \Omega_2\,(t), \Omega_3\,(t)\right)$ relative to the body coordinate system $\boldsymbol{x}_b$ with entries determined from the rotation tensor $\boldsymbol{Q}\,(t)$ by $\widehat{\boldsymbol{\Omega}} = \boldsymbol{Q}^T\dot{\boldsymbol{Q}}$ such that the skew-symmetric matrix $\widehat{\boldsymbol{\Omega}}$ satisfies $\widehat{\boldsymbol{\Omega}}\mathbf{r} = \boldsymbol{\Omega}\times\mathbf{r}$ for any $\mathbf{r}\in\mathbb{R}^3$ (see Alemi Ardakani (2019) for more details), $\boldsymbol{I}_f$ which is defined as

$$\boldsymbol{I}_f = \int_{\mathscr{Q}} \Big(\left\|\boldsymbol{x}+\boldsymbol{d}\right\|^2\boldsymbol{I} - (\boldsymbol{x}+\boldsymbol{d})\otimes(\boldsymbol{x}+\boldsymbol{d})\Big)\,\rho\,\mathrm{d}\boldsymbol{x}\,, \tag{8}$$

is the mass moment of inertia of the interior fluid relative to the point of rotation, i.e. the origin of the body frame $\boldsymbol{x}_b$, $\otimes$ denotes the tensor product, $\boldsymbol{I}$ is the 3×3 identity matrix, $\boldsymbol{I}_v$ is the mass moment of inertia of the dry floating body relative to the point of rotation, m_v is the mass of the dry body, $\overline{\boldsymbol{x}}_v = (\overline{x}_v, \overline{y}_v, \overline{z}_v)$ is the centre of mass of the dry body relative to the body frame $\boldsymbol{x}_b$, $\phi\,(X, Y, Z, t)$ and $\eta\,(X, Y, t)$ are respectively the velocity potential and the free surface elevation of the exterior irrotational water waves, and $V\,(t)$ is the transient domain of the exterior fluid. The configuration of the fluid in a rotating and translating floating structure in hydrodynamic interaction with exterior water waves is schematically shown in Figure 1.

The paper starts with a new derivation of the Bernoulli equation and the Neumann boundary value problem for an inviscid and incompressible fluid sloshing in a container undergoing prescribed rigid-body motion in three dimensions in §2. In §3, the Bateman-Luke variational principle is revisited for the problem of fluid sloshing in rotating and translating coordinates. In §4, the variational principles of Alemi Ardakani (2019) are adapted to present a mathematical theory for 3–D interactions between ocean surface waves and a floating structure containing potential fluid motion. The paper ends with concluding remarks in §5.

2 Derivation of the Bernoulli equation for the pressure field in rotating and translating coordinate systems

The configuration of the fluid in a rotating-translating rectangular vessel is schematically shown in Figure 1. The vessel is a rigid body, which is free to rotate or translate in $\mathbb{R}^3$. The vessel is partially filled with an inviscid and incompressible fluid. The position of a fluid particle in the body frame $\boldsymbol{x}$ is related to a point in the laboratory frame $\boldsymbol{X}$ by

$$\boldsymbol{X} = \boldsymbol{Q}\left(\boldsymbol{x} + \boldsymbol{d}\right) + \boldsymbol{q}\,, \tag{9}$$

and the Eulerian velocity of a fluid particle $\boldsymbol{u}\left(\boldsymbol{x}, t\right)$ in the body frame $\boldsymbol{x}$ is related to its velocity $\boldsymbol{U}\left(\boldsymbol{X}, t\right)$ in the laboratory frame $\boldsymbol{X}$ by (Alemi Ardakani & Bridges 2011; Alemi Ardakani 2019)

$$\boldsymbol{U} = \boldsymbol{Q}\left(\boldsymbol{u} + \boldsymbol{\Omega}\times\left(\boldsymbol{x}+\boldsymbol{d}\right) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\,, \tag{10}$$

where the rotation tensor $\boldsymbol{Q}$ and the body angular velocity $\boldsymbol{\Omega}$ are defined in §1. It is shown in Alemi Ardakani & Bridges (2011) that the Euler equations for the motion of an inviscid and incompressible fluid relative to the body coordinate system $\boldsymbol{x}$ takes the form

$$\frac{D\boldsymbol{u}}{Dt} + \frac{1}{\rho}\nabla p = -2\boldsymbol{\Omega}\times\boldsymbol{u} - \dot{\boldsymbol{\Omega}}\times\left(\boldsymbol{x}+\boldsymbol{d}\right) - \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\left(\boldsymbol{x}+\boldsymbol{d}\right)\right) - \boldsymbol{Q}^T\boldsymbol{g} - \boldsymbol{Q}^T\ddot{\boldsymbol{q}}\,, \tag{11}$$

where $\boldsymbol{u} = (u, v, w)$, $D\boldsymbol{u}/Dt = \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}$ and $\boldsymbol{g} = g\boldsymbol{k}$. The term $\boldsymbol{Q}^T\boldsymbol{g}$ rotates the usual gravity vector so that its direction is viewed properly in the body frame $\boldsymbol{x}_b$. The same is true for the translational acceleration $\ddot{\boldsymbol{q}}$.

The fluid occupies the region $\mathcal{Q}\left(t\right)$ in the vessel which is bounded by the free surface $\Sigma\left(t\right)$ and the wetted tank surface $S\left(t\right)$,

$$0 \le x \le L_1\,,\quad 0 \le y \le L_2\,,\quad 0 \le z \le h\left(x, y, t\right)\,, \tag{12}$$

where the lengths L_1 and L_2 are given positive constants, and $z = h\left(x, y, t\right)$ is the position of the free surface inside the vessel, relative to the body frame $\boldsymbol{x}$.

Conservation of mass relative to the body frame $\boldsymbol{x}$ takes the form

$$\nabla\cdot\boldsymbol{u} = u_x + v_y + w_z = 0\,. \tag{13}$$

The boundary conditions are

$$\left.\begin{array}{rcllll} u &=& 0 & \text{at} & x=0 \quad \text{and} & x = L_1\,,\\ v &=& 0 & \text{at} & y=0 \quad \text{and} & y = L_2\,,\\ w &=& 0 & \text{at} & z=0\,, & \end{array}\right\} \tag{14}$$

which are the zero normal velocity component or impermeability boundary conditions on the rigid walls, and at the free surface, the kinematic and dynamic boundary conditions are respectively

$$w = h_t + uh_x + vh_y \quad \text{and} \quad p = 0 \quad \text{at} \quad z = h\left(x, y, t\right)\,, \tag{15}$$

where the surface tension is neglected in the boundary condition for the pressure p.

The vorticity vector is defined by

$$\boldsymbol{\mathcal{V}} := \nabla \times \boldsymbol{u}\,. \tag{16}$$

Differentiating this equation gives

$$\frac{D\boldsymbol{\mathcal{V}}}{Dt} = \boldsymbol{\mathcal{V}} \cdot \nabla \boldsymbol{u} + \nabla \times \left(\frac{D\boldsymbol{u}}{Dt}\right)\,. \tag{17}$$

Taking the curl of the Euler equations (11) gives

$$\nabla \times \left(\frac{D\boldsymbol{u}}{Dt}\right) = 2\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} - 2\dot{\boldsymbol{\Omega}}\,. \tag{18}$$

Substitution of (18) into (17) gives the vorticity equation

$$\frac{D\boldsymbol{\mathcal{V}}}{Dt} = (2\boldsymbol{\Omega} + \boldsymbol{\mathcal{V}}) \cdot \nabla \boldsymbol{u} - 2\dot{\boldsymbol{\Omega}}\,. \tag{19}$$

Now, if we set

$$\boldsymbol{\mathcal{V}} = -2\boldsymbol{\Omega}\,, \tag{20}$$

then

$$\frac{D\boldsymbol{\mathcal{V}}}{Dt} = -2\dot{\boldsymbol{\Omega}}\,, \tag{21}$$

and the vorticity equation (19) is satisfied. Equation (20) will be important in the derivation of Bernoulli's equation in rotating and translating coordinate systems.

Using equation (20), the vector identity

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})\,, \tag{22}$$

takes the form

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) - 2\boldsymbol{\Omega} \times \boldsymbol{u}\,. \tag{23}$$

Using the vector identity (23) the Euler equations (11) reduces to

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) + \frac{1}{\rho}\nabla p + \dot{\boldsymbol{\Omega}} \times (\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d})) + \boldsymbol{Q}^T \boldsymbol{g} + \boldsymbol{Q}^T \ddot{\boldsymbol{q}} = \boldsymbol{0}\,. \tag{24}$$

Now, if we introduce a velocity potential $\Phi\left(x, y, z, t\right)$ such that

$$\boldsymbol{u}\left(\boldsymbol{x}, t\right) = \nabla \Phi\left(\boldsymbol{x}, t\right) - \boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) - \boldsymbol{Q}^T \dot{\boldsymbol{q}}\,, \tag{25}$$

then the velocity field in (25) satisfies the vorticity equation. Noting that (Marsden & Ratiu 1999; Holm, Schmah & Stoica 2009)

$$\frac{d}{dt}\left(\boldsymbol{Q}^{-1}\right) = -\boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}}\boldsymbol{Q}^{-1} \tag{26}$$

then

$$\frac{\partial \boldsymbol{u}}{\partial t} = \nabla \Phi_t - \dot{\boldsymbol{\Omega}} \times (\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{\Omega} \times \boldsymbol{Q}^T \dot{\boldsymbol{q}} - \boldsymbol{Q}^T \ddot{\boldsymbol{q}}\,. \tag{27}$$

Substitution of (25) and (27) into the Euler equations (24) gives

$$\left.\begin{aligned} &\nabla \Phi_t + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{Q}^T \dot{\boldsymbol{q}}\right) + \frac{1}{\rho}\nabla p + \boldsymbol{Q}^T \boldsymbol{g} \\ &+\nabla\left[\tfrac{1}{2}\nabla\Phi \cdot \nabla\Phi - \nabla\Phi \cdot \left(\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{Q}^T \dot{\boldsymbol{q}}\right)\right. \\ &\qquad \left. + \tfrac{1}{2}\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) \cdot \boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) \cdot \boldsymbol{Q}^T \dot{\boldsymbol{q}}\right] = \boldsymbol{0}\,. \end{aligned}\right\} \tag{28}$$

But

$$\nabla\left(\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) \cdot \boldsymbol{Q}^T \dot{\boldsymbol{q}}\right) = -\boldsymbol{\Omega} \times \boldsymbol{Q}^T \dot{\boldsymbol{q}}\,, \tag{29}$$

and

$$\nabla\left(\tfrac{1}{2}\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})\cdot\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})\right)=-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))\;, \tag{30}$$

and so equation (28) simplifies to

$$\nabla\left(\Phi_t+\frac{p}{\rho}+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi-\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)+\boldsymbol{Q}^T\boldsymbol{g}\cdot(\boldsymbol{x}+\boldsymbol{d})\right)=\boldsymbol{0}\,. \tag{31}$$

Integrating this equation in space gives

$$\Phi_t+\frac{p}{\rho}+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi-\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)+\boldsymbol{Q}^T\boldsymbol{g}\cdot(\boldsymbol{x}+\boldsymbol{d})=Be\,(t)\;, \tag{32}$$

where $Be\,(t)$ is the Bernoulli function which can be absorbed into $\Phi\,(\boldsymbol{x},t)$. Therefore, Bernoulli's equation for the pressure field $p\,(x,y,z,t)$ in $\mathcal{Q}\,(t)$, in the body coordinate system $\boldsymbol{x}_b$, takes the form

$$\Phi_t+\frac{p}{\rho}+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi-\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)+\boldsymbol{Q}^T\boldsymbol{g}\cdot(\boldsymbol{x}+\boldsymbol{d})=0\,. \tag{33}$$

Now, comparing the proposed Bernoulli equation (33) with the Bernoulli equation (4) proves that $\boldsymbol{v}_0$, which is in our notation here $\dot{\boldsymbol{q}}$, should be relative to the body coordinate system $Oxyz$ or in our notation $\boldsymbol{x}_b$, and hence it should be $\boldsymbol{Q}^T\boldsymbol{v}_0$, i.e. $\boldsymbol{Q}^T\dot{\boldsymbol{q}}$ as in (33), and $\boldsymbol{\omega}$ which is the angular velocity of the tank relative to the laboratory coordinate system $O'x'y'z'$ or in our notation $\boldsymbol{X}$, should be the angular velocity of the tank in the body frame $Oxyz$ or in our notation $\boldsymbol{x}_b$, and hence it should be $\boldsymbol{Q}^T\boldsymbol{\omega}$ which is in our notation $\boldsymbol{\Omega}$ as in (33). Also note that $\boldsymbol{r}=\boldsymbol{x}+\boldsymbol{d}$ in our notation. Moreover, the gravity potential $U\,(x,y,z,t)=-\boldsymbol{g}\cdot\boldsymbol{r}'$ should be relative to the body frame $\boldsymbol{x}_b$, which is $\boldsymbol{Q}^T\boldsymbol{g}\cdot(\boldsymbol{x}+\boldsymbol{d})$.

In terms of the velocity potential $\Phi\,(x,y,z,t)$, the rigid-wall boundary conditions in (14) become

$$\nabla\Phi=\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\quad\text{on}\quad S\,(t)\;, \tag{34}$$

which can be written in the form

$$\frac{\partial\Phi}{\partial\boldsymbol{n}_b}=\boldsymbol{\Omega}\cdot((\boldsymbol{x}+\boldsymbol{d})\times\boldsymbol{n}_b)+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\cdot\boldsymbol{n}_b\quad\text{on}\quad S\,(t)\;, \tag{35}$$

where $\boldsymbol{n}_b=\boldsymbol{Q}^{-1}\boldsymbol{n}$ is the outward-pointing normal to the boundary of $\mathcal{Q}\,(t)$ relative to the body frame $\boldsymbol{x}_b$. In terms of the velocity potential $\Phi\,(x,y,z,t)$, the kinematic free surface boundary condition in (15) becomes

$$\frac{\partial\Phi}{\partial\boldsymbol{n}_b}=\boldsymbol{\Omega}\cdot((\boldsymbol{x}+\boldsymbol{d})\times\boldsymbol{n}_b)+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\cdot\boldsymbol{n}_b+\frac{h_t}{\sqrt{1+h_x^2+h_y^2}}\quad\text{on}\quad\Sigma\,(t)\;. \tag{36}$$

From (33) it can be concluded that the dynamic free surface boundary condition in (15) becomes

$$\Phi_t+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi-\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)+\boldsymbol{Q}^T\boldsymbol{g}\cdot(\boldsymbol{x}+\boldsymbol{d})=0\quad\text{on}\quad\Sigma\,(t)\;. \tag{37}$$

Finally, the substitution of the velocity field (25) into the continuity equation (13) leads to Laplace's equation for $\Phi\,(x,y,z,t)$,

$$\Delta\,\Phi=\Phi_{xx}+\Phi_{yy}+\Phi_{zz}=0\quad\text{in}\quad\mathcal{Q}\,(t)\;. \tag{38}$$

Having derived a precise mathematical expression for the Bernoulli equation (33) in the body frame $\boldsymbol{x}_b$, and the boundary-value problem (38), (35), (36) and (37) for water waves in the rotating and translating (body) coordinate systems $\boldsymbol{x}$ and $\boldsymbol{x}_b$, we can transform these equations to the laboratory coordinate system $\boldsymbol{X}$. Define the velocity potential of the interior fluid relative to the laboratory frame $\boldsymbol{X}$ by $\Phi_s\,(X,Y,Z,t)$. It is related to the velocity potential $\Phi\,(x,y,z,t)$ in the body frame $\boldsymbol{x}$ by

$$\Phi\,(\boldsymbol{x},t)=\Phi_s\,(\boldsymbol{Q}\,(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{q},t)\;, \tag{39}$$

from which it follows

$$\nabla_{\boldsymbol{x}}\Phi=\boldsymbol{Q}^T\nabla_{\boldsymbol{X}}\Phi_s\,, \tag{40}$$

where $\nabla_{\boldsymbol{X}}$ is the gradient in the laboratory frame. The body angular velocity $\boldsymbol{\Omega}$ is to be contrasted with the *spatial* angular velocity, the angular velocity viewed from the laboratory frame $\boldsymbol{X}$, which is

$$\widehat{\boldsymbol{\Omega}}_s := \dot{\boldsymbol{Q}}\boldsymbol{Q}^T \,. \tag{41}$$

As vectors, the spatial (laboratory) and body angular velocities are related by $\boldsymbol{\Omega}_s = \boldsymbol{Q}\boldsymbol{\Omega}$. Now if we denote the pressure field of the interior fluid in the laboratory frame $\boldsymbol{X}$ by $\mathrm{P}(\boldsymbol{X},t)$ and multiply the vectors in the Bernoulli equation (33) by $\boldsymbol{Q}$, we obtain the Bernoulli equation relative to the laboratory frame $\boldsymbol{X}$

$$\frac{\partial \Phi_s}{\partial t} + \frac{\mathrm{P}}{\rho} + \tfrac{1}{2}\nabla_{\boldsymbol{X}}\Phi_s \cdot \nabla_{\boldsymbol{X}}\Phi_s - \nabla_{\boldsymbol{X}}\Phi_s \cdot \left((\boldsymbol{\Omega}_s \times (\boldsymbol{X}-\boldsymbol{q})) + \dot{\boldsymbol{q}} \right) + \boldsymbol{g}\cdot(\boldsymbol{X}-\boldsymbol{q}) = 0\,, \tag{42}$$

where $\boldsymbol{X}$ is the position of a fluid particle relative to the laboratory frame, and $\boldsymbol{X}-\boldsymbol{q} = \boldsymbol{Q}\,(\boldsymbol{x}+\boldsymbol{d})$. A similar analysis shows that the Neumann boundary-value problem (38), (35), (36) and (37) in the laboratory frame $\boldsymbol{X}$ takes the form

$$\left.\begin{aligned}
&\Delta\,\Phi_s := \frac{\partial^2\Phi_s}{\partial X^2} + \frac{\partial^2\Phi_s}{\partial Y^2} + \frac{\partial^2\Phi_s}{\partial Z^2} = 0 \quad \text{in} \quad \mathcal{Q}\,(t)\ ,\\
&\frac{\partial\Phi_s}{\partial \boldsymbol{n}} = \boldsymbol{\Omega}_s\cdot\left((\boldsymbol{X}-\boldsymbol{q})\times\boldsymbol{n}\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{n} \quad \text{on} \quad S\,(t)\ ,\\
&\frac{\partial\Phi_s}{\partial \boldsymbol{n}} = \boldsymbol{\Omega}_s\cdot\left((\boldsymbol{X}-\boldsymbol{q})\times\boldsymbol{n}\right) + \dot{\boldsymbol{q}}\cdot\boldsymbol{n} + \frac{\mathcal{H}_t}{\sqrt{1+\mathcal{H}_X^2+\mathcal{H}_Y^2}} \quad \text{on} \quad \Sigma\,(t)\ ,\\
&\frac{\partial\Phi_s}{\partial t} + \underbrace{\tfrac{1}{2}\nabla_{\boldsymbol{X}}\Phi_s\cdot\nabla_{\boldsymbol{X}}\Phi_s}_{=\frac{1}{2}\nabla_{\boldsymbol{x}}\Phi\cdot\nabla_{\boldsymbol{x}}\Phi} - \nabla_{\boldsymbol{X}}\Phi_s\cdot\left((\boldsymbol{\Omega}_s\times(\boldsymbol{X}-\boldsymbol{q}))+\dot{\boldsymbol{q}}\right) + \boldsymbol{g}\cdot(\boldsymbol{X}-\boldsymbol{q}) = 0 \quad \text{on} \quad \Sigma\,(t)\ ,
\end{aligned}\right\} \tag{43}$$

where $\mathcal{H}\,(X,Y,t)$ is the free surface height relative to the laboratory frame $\boldsymbol{X}$, and $\boldsymbol{n}$ is the outward-pointing normal relative to the laboratory frame $\boldsymbol{X}$.

3 The Bateman-Luke variational principle for fluid sloshing in vessels undergoing prescribed rigid-body motion in three dimensions

Based on the new Bernoulli equation (33), the Bateman-Luke variational principle (1) is modified to

$$\begin{aligned}
\delta\mathscr{L}\,(\Phi,h) &= \delta\int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} p\,(\boldsymbol{x},t)\;\mathrm{d}\mathcal{Q}\,\mathrm{d}t\\
&= \delta\int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} -\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi - \nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right) + \boldsymbol{Q}^T\boldsymbol{g}\cdot(\boldsymbol{x}+\boldsymbol{d})\right)\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t = 0\,,
\end{aligned} \tag{44}$$

subject to the endpoint conditions $\delta\Phi\,(\boldsymbol{x},t_1) = \delta\Phi\,(\boldsymbol{x},t_2) = 0$. Note that in (44), $\boldsymbol{\Omega}$ is the *body* angular velocity relative to the body frame $\boldsymbol{x}_b$, $\boldsymbol{Q}^T\dot{\boldsymbol{q}}$ is the translational velocity of the moving rigid body relative to the body coordinate system $\boldsymbol{x}_b$, and $\boldsymbol{Q}^T\boldsymbol{g}$ is relative to the body frame $\boldsymbol{x}_b$, while in (1), $\boldsymbol{\omega}$ is the angular velocity of the rigid body relative to the laboratory coordinate system, $\boldsymbol{v}_0$ is the translational velocity of the rigid body relative to the laboratory frame, and $U = -\boldsymbol{g}\cdot\boldsymbol{r}'$ is relative to the laboratory frame.

Hence, the new derivation of the Bernoulli equation (33) relative to the rotating and translating coordinate system attached to the moving body leads to mathematically precise definitions for the angular and translational velocities of the rigid body in the Bateman-Luke variational principle, as well as a precise definition for the gravity vector relative to the body frame $\boldsymbol{x}_b$.

According to the usual procedure in the calculus of variations, the variational principle (44) becomes

$$\left.\begin{aligned}
&\delta\mathscr{L}\,(\Phi,h) = \int_{t_1}^{t_2}\int_{\Sigma(t)} p\,(x,y,h,t)\,\delta h\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} -\left(\delta\Phi_t + \nabla\Phi\cdot\nabla\delta\Phi - \nabla\delta\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\right)\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t = 0\,,
\end{aligned}\right\} \tag{45}$$

where $\ell = \sqrt{1 + h_x^2 + h_y^2}$ and

$$p\,(x,y,h,t) \;\; = \;\; -\rho\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi - \nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right) + \boldsymbol{Q}^T\boldsymbol{g}\cdot(\boldsymbol{x}+\boldsymbol{d})\right)\Bigg|^{z=h(x,y,t)} . \tag{46}$$

But

$$\int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} -\delta\Phi_t\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{\Sigma(t)} h_t\delta\Phi\ell^{-1}\,\rho\,\mathrm{d}S\,\mathrm{d}t\,, \tag{47}$$

noting that $\delta\Phi = 0$ at $t = t_1$ and $t = t_2$. Moreover, using Green's first identity we obtain

$$\left\{\begin{aligned} &-\int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} \nabla\Phi\cdot\nabla\delta\Phi\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} \Delta\,\Phi\,\delta\Phi\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t - \int_{t_1}^{t_2}\int_{\partial\mathcal{Q}} \nabla\Phi\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} \Delta\,\Phi\,\delta\Phi\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t - \int_{t_1}^{t_2}\int_{S(t)} \nabla\Phi\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t - \int_{t_1}^{t_2}\int_{\Sigma(t)} \nabla\Phi\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t\,, \end{aligned}\right. \tag{48}$$

where $\boldsymbol{n}_b$ is the unit outward normal vector to the boundary of $\mathcal{Q}\,(t)$, and also

$$\begin{aligned} \int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} \nabla\delta\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t \;\; &= \;\; \int_{t_1}^{t_2}\int_{S(t)} \left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int_{\Sigma(t)} \left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t\,. \end{aligned} \tag{49}$$

Now, the substitution of (47), (48) and (49) into the variational principle (45) gives

$$\left.\begin{aligned} &\delta\mathscr{L}\,(\Phi,h) = \int_{t_1}^{t_2}\int_{\Sigma(t)} p\,(x,y,h,t)\,\delta h\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{\mathcal{Q}(t)} \Delta\,\Phi\,\delta\Phi\,\rho\,\mathrm{d}\mathcal{Q}\,\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int_{\Sigma(t)} \left(h_t\ell^{-1} + \left(-\nabla\Phi + \boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\cdot\boldsymbol{n}_b\right)\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int_{S(t)} \left(-\nabla\Phi + \boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\cdot\boldsymbol{n}_b\,\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t = 0\,. \end{aligned}\right\} \tag{50}$$

From (50), it can be concluded that invariance of $\mathscr{L}$ with respect to a variation in the free surface height h yields the dynamic free surface boundary condition (37). Similarly, the invariance of $\mathscr{L}$ with respect to a variation in the velocity potential Φ at the free surface $\Sigma\,(t)$ yields the kinematic free surface boundary condition (36), and the invariance of $\mathscr{L}$ with respect to a variation in the velocity potential Φ along the wetted surface $S\,(t)$ recovers the rigid-wall boundary conditions (35). Moreover, the invariance of $\mathscr{L}$ with respect to a variation in the velocity potential Φ yields the field equation (38).

4 A mathematical theory for 3–D interactions between potential surface waves and a floating structure with interior potential fluid sloshing

The interest in this section is twofold. Firstly, to adapt the variational principles of Alemi Ardakani (2019) to present a mathematical theory for three-dimensional rotational and translational motion of a rigid-body containing fluid such that the interior fluid motion satisfies the velocity potential theory developed in §2 and §3. Secondly, to extend the variational principle to present a theory for the problem of interactions between potential water waves and a floating structure dynamically coupled to its interior potential fluid sloshing.

Gerrits & Veldman (2003) and Veldman *et al.* (2007) studied the problem of coupled liquid-solid dynamics for a liquid-filled spacecraft in three dimensions. They presented the differential equations for the motion of the spacecraft containing fluid, describing the conservation of linear momentum and angular momentum. The equation for the conservation of angular momentum of the rigid body takes the form (see equation (6) of Gerrits and Veldman (2003) and the work of Alemi Ardakani (2019) for minor modifications of this equation)

$$m_v \overline{\boldsymbol{x}}_v \times \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} + \boldsymbol{I}_v\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \boldsymbol{I}_v\boldsymbol{\Omega} = \boldsymbol{\mathcal{T}} - m_v g \overline{\boldsymbol{x}}_v \times \boldsymbol{\chi}\,, \tag{51}$$

where m_v is the mass of the dry body, $\overline{\boldsymbol{x}}_v$ is the centre of mass of the dry body relative to the body frame $\boldsymbol{x}_b$, $\boldsymbol{I}_v$ is the mass moment of inertia of the dry body relative to the point of rotation, i.e. the origin of the body frame $\boldsymbol{x}_b$, $\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}$ is the linear acceleration of the origin of the moving coordinate frame relative to the body frame $\boldsymbol{x}_b$, and

$$\boldsymbol{\mathcal{T}} = \int_{\mathcal{Q}(t)} (\boldsymbol{x} + \boldsymbol{d}) \times \nabla p \,\mathrm{d}\boldsymbol{x}\,, \tag{52}$$

is the torque that the interior fluid exerts on the boundary of the rigid body via pressure, and

$$\boldsymbol{\chi} = \boldsymbol{Q}^{-1}\boldsymbol{k}\,, \tag{53}$$

where $\boldsymbol{k}$ is the unit vector in the Z direction. For our problem in this paper, the interior fluid is inviscid and incompressible and satisfies the velocity potential theory developed in §2 and §3. Now, after substituting for ∇p for the interior fluid of the rigid body from (31) in (52), equation (51) for the body angular velocity $\boldsymbol{\Omega}(t)$ takes the form

$$\begin{cases} \displaystyle\int_{\mathcal{Q}} -(\boldsymbol{x}+\boldsymbol{d}) \times \left[\nabla\Phi_t + \nabla\left(\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi\right) - \nabla\left(\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\right)\right]\rho\,\mathrm{d}\boldsymbol{x} \\ -mg\overline{\boldsymbol{x}}\times\boldsymbol{\chi} - m_v\overline{\boldsymbol{x}}_v\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \boldsymbol{I}_v\dot{\boldsymbol{\Omega}} + \boldsymbol{I}_v\boldsymbol{\Omega}\times\boldsymbol{\Omega} = \boldsymbol{0}\,, \end{cases} \tag{54}$$

where m is the total mass of the (interior fluid + body) system,

$$m = m_f + m_v \quad \text{with} \quad m_f = \int_{\mathcal{Q}} \rho\,\mathrm{d}\boldsymbol{x}\,, \tag{55a,b}$$

where m_f is the mass of the interior fluid, which is time independent. Note that in (54) $\overline{\boldsymbol{x}}$ is the centre of mass of the coupled (interior fluid + body) system relative to the body frame $\boldsymbol{x}_b$, which is time dependent and satisfies

$$m\overline{\boldsymbol{x}}(t) = m_f\overline{\boldsymbol{x}}_f + m_v\overline{\boldsymbol{x}}_v\,, \tag{56}$$

with

$$m_f\overline{\boldsymbol{x}}_f = \int_{\mathcal{Q}} (\boldsymbol{x}+\boldsymbol{d})\,\rho\,\mathrm{d}\boldsymbol{x}\,, \tag{57}$$

where $\overline{\boldsymbol{x}}_f(t)$ is the centre of mass of the interior fluid relative to the body frame $\boldsymbol{x}_b$.

Similarly, the equation for the conservation of linear momentum of the rigid body takes the form (see equation (5) of Gerrits and Veldman (2003))

$$m_v\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{\Omega}}\times m_v\overline{\boldsymbol{x}}_v + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v) = \boldsymbol{\mathcal{F}} - m_v g\boldsymbol{\chi}\,, \tag{58}$$

where

$$\boldsymbol{\mathcal{F}} = \int_{\mathcal{Q}(t)} \nabla p\,\mathrm{d}\boldsymbol{x}\,, \tag{59}$$

is the force that the interior fluid exerts on the boundary of the rigid body via pressure. Now, after substituting for ∇p from (31) in (59), equation (58) for the translational motion of the rigid body $\boldsymbol{q}(t)$ takes the form

$$\begin{cases} \displaystyle\int_{\mathcal{Q}} -\left[\nabla\Phi_t + \nabla\left(\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi\right) - \nabla\left(\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\right)\right]\rho\,\mathrm{d}\boldsymbol{x} \\ -mg\boldsymbol{\chi} - m_v\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{\Omega}}\times m_v\overline{\boldsymbol{x}}_v - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v) = \boldsymbol{0}\,. \end{cases} \tag{60}$$

The Lagrangian action for the motion of a rigid-body dynamically coupled to its interior fluid motion takes the form

$$\mathscr{L} = \int_{t_1}^{t_2} \left(\mathrm{KE}^{fluid} - \mathrm{PE}^{fluid} + \mathrm{KE}^{body} - \mathrm{PE}^{body} \right) \mathrm{d}t \,, \tag{61}$$

where KE^{fluid} is the kinetic energy of the fluid, KE^{body} is the kinetic energy of the rigid body, PE^{fluid} is the potential energy of the fluid and PE^{body} is the potential energy of the rigid body. It is shown in Alemi Ardakani (2019) that the action functional (61) for a rigid body which contains an inviscid and incompressible fluid and undergoes three dimensional rotational and translational motions takes the form (7). The equations of motion for the body angular velocity $\boldsymbol{\Omega}(t)$ and the translational motion $\boldsymbol{q}(t)$ of the rigid body are provided by Hamilton's variational principle:

$$\delta \mathscr{L}_2 \left(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}} \right) = 0 \,, \tag{62}$$

subject to the fixed endpoints $\delta \boldsymbol{q}(t_1) = \delta \boldsymbol{q}(t_2) = \boldsymbol{0}$, and noting that the variations $\delta \boldsymbol{Q}$ are taken among paths $\boldsymbol{Q}(t) \in \mathrm{SO}(3)$, $t \in [t_1, t_2]$, with fixed endpoints, so that $\delta \boldsymbol{Q}(t_1) = \delta \boldsymbol{Q}(t_2) = \boldsymbol{0}$. It is proved in Alemi Ardakani (2019) that taking the variations $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$ in the variational principle (62), using the Euler-Poincaré framework (Marsden & Ratiu 1999; Holm, Schmah & Stoica 2009), gives the Euler-Poincaré equation for $\boldsymbol{\Omega}(t)$ as

$$\int_{\mathcal{Q}} - (\boldsymbol{x} + \boldsymbol{d}) \times \left(\frac{D\boldsymbol{u}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u} \right) \rho \, \mathrm{d}\boldsymbol{x} - mg\overline{\boldsymbol{x}} \times \boldsymbol{\chi} = m\overline{\boldsymbol{x}} \times \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} + \boldsymbol{I}_t\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \boldsymbol{I}_t\boldsymbol{\Omega} \,, \tag{63}$$

and the Euler-Poincaré equation for $\boldsymbol{q}(t)$ as

$$\int_{\mathcal{Q}} \left(-\frac{D\boldsymbol{u}}{Dt} - 2\boldsymbol{\Omega} \times \boldsymbol{u} \right) \rho \, \mathrm{d}\boldsymbol{x} - m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{\Omega}} \times m\overline{\boldsymbol{x}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times m\overline{\boldsymbol{x}}) - mg\boldsymbol{\chi} = \boldsymbol{0} \,, \tag{64}$$

where $\boldsymbol{I}_t = \boldsymbol{I}_f + \boldsymbol{I}_v$ is the mass moment of inertia of the coupled (interior fluid + body) system.

The interior fluid of the rigid body has a velocity field of the form (25). Now, it can be proved that the substitution of the velocity field (25) into the Euler-Poincaré equation (63) recovers the $\boldsymbol{\Omega}$-equation (54) obtained from the balance of angular momentum of the rigid body. Similarly, the substitution of the velocity field (25) into the Euler-Poincaré equation (64) recovers the $\boldsymbol{q}$-equation (60) obtained from the balance of linear momentum of the rigid body.

The variational principle (62), with the Lagrangian action defined in (7), can be extended to the problem of 3–D water waves in hydrodynamic interaction with a freely floating rigid body containing fluid by the addition of Luke's variational principle (Luke 1967; Van Daalen, Van Groesen & Zandbergen 1993; Alemi Ardakani 2019) to Hamilton's variational principle (62). The variational principle for the motion of the exterior water waves and the motion of the rigid body containing fluid takes the form (5). See the work of Alemi Ardakani (2019) for more details. In (5), $\phi(\boldsymbol{X}, t)$ is the velocity potential of the exterior irrotational fluid lying between $Z = -H(X,Y)$ and $Z = \eta(X,Y,t)$ with the gravity acceleration g acting in the negative Z direction. In the horizontal directions X and Y, the fluid domain is cut off by a cylindrical vertical surface $\mathcal{S}$ of infinite radius which extends from the bottom to the free surface, and the transient fluid domain $V(t)$ consists of a fluid bounded by the impermeable bottom S_b defined by the equation $Z = -H(X,Y)$, the free surface S_η defined by the equation $Z = \eta(X,Y,t)$, the vertical surface $\mathcal{S}$ and the wetted surface S_w of the rigid body interacting with exterior water waves. It can be proved that taking the variations $\delta\eta$ and $\delta\phi$ of the first component of the variational principle (5) subject to the restrictions $\delta\phi = 0$ at the end points of the time interval, t_1 and t_2, recovers the complete set of equations of motion for the classical water-wave problem in three dimensions as (Luke 1967; Miles 1977; Lukovsky 2015; Van Daalen, Van Groesen & Zandbergen 1993)

$$\left.\begin{aligned}
&\Delta\phi := \phi_{XX} + \phi_{YY} + \phi_{ZZ} = 0 \quad \text{for} \quad -H(X,Y) < Z < \eta(X,Y,t) \,, \\
&\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + gZ = 0 \quad \text{on} \quad Z = \eta(X,Y,t) \,, \\
&\phi_Z = \eta_t + \phi_X\eta_X + \phi_Y\eta_Y \quad \text{on} \quad Z = \eta(X,Y,t) \,, \\
&\phi_Z + \phi_X H_X + \phi_Y H_Y = 0 \quad \text{on} \quad Z = -H(X,Y) \,.
\end{aligned}\right\} \tag{65}$$

However, for the wave-structure interaction problem, the first component of the variational principle (5) is coupled to the second component of (5), and hence the variational Reynold's transport theorem (Flanders 1973; Daniliuk 1976; Gagarina, Van der Vegt & Bokhove 2013) should be used for the variations $\delta\eta$ and $\delta\phi$, since the domain of integration $V(t)$ is time-dependent. Then, according to the usual procedure in the calculus of variations, and following the Euler-Poincaré variational framework introduced in Alemi Ardakani (2019), it can be proved that the variational principle (5) for the variations $\delta\phi$, $\delta\eta$, $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$ subject to the restrictions that they vanish at the end points of the time interval, becomes

$$\left.\begin{aligned}
&\delta\mathscr{L}\left(\phi,\eta,\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}}\right)=\int_{t_1}^{t_2}\int_{S_\eta}-\left(\phi_t+\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+gZ\right)\Big|^{Z=\eta}\rho\,\delta\eta\,l^{-1}\mathrm{d}S\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{S_w}\left(P\left(X,Y,Z,t\right)\left\langle\boldsymbol{\Gamma},\boldsymbol{x}_w\times\boldsymbol{n}_b\right\rangle+P\left(X,Y,Z,t\right)\left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q},\boldsymbol{n}_b\right\rangle\right)\mathrm{d}S\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{S_\eta}\left(\eta_t+\eta_X\phi_X+\eta_Y\phi_Y-\phi_Z\right)\delta\phi\Big|^{Z=\eta}\rho\,l^{-1}\mathrm{d}S\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{S_w}\left(\dot{\boldsymbol{X}}_w\cdot\boldsymbol{n}-\frac{\partial\phi}{\partial\boldsymbol{n}}\right)\delta\phi\rho\,\mathrm{d}S\,\mathrm{d}t+\int_{t_1}^{t_2}\int_{V(t)}\Delta\,\phi\,\delta\phi\,\rho\,\mathrm{d}V\,\mathrm{d}t\\
&-\int_{t_1}^{t_2}\int_{S_b}\left(\phi_XH_X+\phi_YH_Y+\phi_Z\right)\delta\phi\Big|_{Z=-H}\rho\,\mathrm{d}S\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{\mathcal{Q}}\left\langle\boldsymbol{\Gamma},-\left(\boldsymbol{x}+\boldsymbol{d}\right)\times\left(\frac{D\boldsymbol{u}}{Dt}+2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\right\rangle\rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\left\langle\boldsymbol{\Gamma},-m\overline{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}-\boldsymbol{\Omega}\times\boldsymbol{I}_t\boldsymbol{\Omega}-mg\overline{\boldsymbol{x}}\times\boldsymbol{\chi}\right\rangle\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{\mathcal{Q}}\left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q},-\frac{D\boldsymbol{u}}{Dt}-2\boldsymbol{\Omega}\times\boldsymbol{u}\right\rangle\rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q},-m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\dot{\boldsymbol{\Omega}}\times m\overline{\boldsymbol{x}}-\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times m\overline{\boldsymbol{x}}\right)-mg\boldsymbol{\chi}\right\rangle\mathrm{d}t=0\,,
\end{aligned}\right\}\tag{66}$$

where $\widehat{\boldsymbol{\Gamma}}=\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}$ satisfies the so-called hat map, i.e. $\widehat{\boldsymbol{\Gamma}}\,\mathbf{r}=\boldsymbol{\Gamma}\times\mathbf{r}$ for any $\mathbf{r}\in\mathbb{R}^3$ (Alemi Ardakani 2019), $\boldsymbol{X}_w$ denotes the position of a point on the wetted body surface S_w relative to the laboratory frame $\boldsymbol{X}$, $\boldsymbol{n}$ is the unit normal vector along $\partial V\supset S_w$ in the laboratory frame $\boldsymbol{X}$, $l=\left(1+\eta_X^2+\eta_Y^2\right)^{1/2}$ giving $\mathrm{d}S=l\,\mathrm{d}X\mathrm{d}Y$, P is the pressure field of the exterior water waves defined by

$$P\left(X,Y,Z,t\right)=-\rho\left(\phi_t+\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+gZ\right)\quad\text{on}\quad S_w\,,\tag{67}$$

$\boldsymbol{x}_w$ is the position of a point on the wetted rigid body surface S_w relative to the body frame $\boldsymbol{x}_b$, and $\boldsymbol{n}_b=\boldsymbol{Q}^{-1}\boldsymbol{n}$ is the unit normal vector along S_w in the body frame $\boldsymbol{x}_b$. The derivation of the variational principle (66) can be deduced from the variational derivations presented in Alemi Ardakani (2019). From (66), we conclude that invariance of $\mathscr{L}$ with respect to a variation in the free-surface elevation η yields the dynamic free-surface boundary condition in (65), invariance of $\mathscr{L}$ with respect to a variation in the velocity potential ϕ yields the field equation in (65) in the domain $V(t)$, invariance of $\mathscr{L}$ with respect to a variation in the velocity potential ϕ at $Z=-H\left(X,Y\right)$ gives the bottom boundary condition in (65), invariance of $\mathscr{L}$ with respect to a variation in the velocity potential ϕ at $Z=\eta\left(X,Y,t\right)$ gives the kinematic free-surface boundary condition in (65) and invariance of $\mathscr{L}$ with respect to a variation in the velocity potential ϕ on S_w gives the contact condition on the wetted surface of the rigid body,

$$\frac{\partial\phi}{\partial\boldsymbol{n}}=\dot{\boldsymbol{X}}_w\cdot\boldsymbol{n}\quad\text{on}\quad S_w\,.\tag{68}$$

Invariance of $\mathscr{L}$ with respect to $\boldsymbol{\Gamma}$ gives the hydrodynamic equation of motion for the rotational motion $\boldsymbol{\Omega}(t)$ of the floating rigid body interacting with the exterior water waves and dynamically coupled to its interior fluid motion

$$\left.\begin{aligned}&\int_Q -\left(\boldsymbol{x}+\boldsymbol{d}\right)\times\left(\frac{D\boldsymbol{u}}{Dt}+2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}-mg\overline{\boldsymbol{x}}\times\boldsymbol{\chi}-m\overline{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}\\&-\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}-\boldsymbol{\Omega}\times\boldsymbol{I}_t\boldsymbol{\Omega}+\int_{S_w}P\left(X,Y,Z,t\right)\left(\boldsymbol{x}_w\times\boldsymbol{n}_b\right)\mathrm{d}S=\boldsymbol{0}\,.\end{aligned}\right\}\tag{69}$$

Invariance of $\mathscr{L}$ with respect to $\boldsymbol{Q}^{-1}\delta\boldsymbol{q}$ gives the hydrodynamic equation of motion for the translational motion $\boldsymbol{q}\left(t\right)$ of the floating rigid body containing fluid and in hydrodynamic interaction with the exterior water waves

$$\left.\begin{aligned}&\int_Q\left(\frac{D\boldsymbol{u}}{Dt}+2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}+m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}+\dot{\boldsymbol{\Omega}}\times m\overline{\boldsymbol{x}}+\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times m\overline{\boldsymbol{x}}\right)\\&+mg\boldsymbol{\chi}-\int_{S_w}P\left(X,Y,Z,t\right)\boldsymbol{n}_b\,\mathrm{d}S=\boldsymbol{0}\,.\end{aligned}\right\}\tag{70}$$

The terms including the pressure field $P\left(X,Y,Z,t\right)$ in the hydrodynamic equations of motion (69) and (70) are the moments and forces respectively acting on the rigid body due to interactions with the exterior water waves.

The interior fluid of the floating rigid body satisfies the velocity potential theory of §2 and §3. Hence, after substituting for the velocity field $\boldsymbol{u}\left(\boldsymbol{x},t\right)$ from (25), the governing equations for the angular momentum (69) and linear momentum (70) of the floating rigid body dynamically coupled to its interior potential fluid sloshing while interacting with the exterior ocean waves become respectively

$$\left.\begin{aligned}&\int_Q -\left(\boldsymbol{x}+\boldsymbol{d}\right)\times\left[\nabla\Phi_t+\nabla\left(\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)-\nabla\left(\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times\left(\boldsymbol{x}+\boldsymbol{d}\right)+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\right)\right]\rho\,\mathrm{d}\boldsymbol{x}\\&-mg\overline{\boldsymbol{x}}\times\boldsymbol{\chi}-m_v\overline{\boldsymbol{x}}_v\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\boldsymbol{I}_v\dot{\boldsymbol{\Omega}}+\boldsymbol{I}_v\boldsymbol{\Omega}\times\boldsymbol{\Omega}+\int_{S_w}P\left(X,Y,Z,t\right)\left(\boldsymbol{x}_w\times\boldsymbol{n}_b\right)\mathrm{d}S=\boldsymbol{0}\,,\end{aligned}\right\}\tag{71}$$

and

$$\left.\begin{aligned}&\int_Q\left[\nabla\Phi_t+\nabla\left(\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)-\nabla\left(\nabla\Phi\cdot\left(\boldsymbol{\Omega}\times\left(\boldsymbol{x}+\boldsymbol{d}\right)+\boldsymbol{Q}^T\dot{\boldsymbol{q}}\right)\right)\right]\rho\,\mathrm{d}\boldsymbol{x}\\&+mg\boldsymbol{\chi}+m_v\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}+\dot{\boldsymbol{\Omega}}\times m_v\overline{\boldsymbol{x}}_v+\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v\right)-\int_{S_w}P\left(X,Y,Z,t\right)\boldsymbol{n}_b\,\mathrm{d}S=\boldsymbol{0}\,.\end{aligned}\right\}\tag{72}$$

In summary, the equations of motion for the exterior water waves in $V\left(t\right)$ are (65) with the contact boundary condition (68). The equations of motion for the interior fluid of the rigid body are the field equation (38) and the boundary conditions (35), (36) and (37), which are dynamically coupled to the hydrodynamic equations of motion for the floating rigid body (71) and (72).

The tangent vectors $\dot{\boldsymbol{Q}}\left(t\right)\in\mathrm{TSO}\left(3\right)$ along the integral curve in the rotation group $\boldsymbol{Q}\left(t\right)\in\mathrm{SO}\left(3\right)$ may be retrieved via the reconstruction formula (Holm, Schmah & Stoica 2009)

$$\dot{\boldsymbol{Q}}=\boldsymbol{Q}\widehat{\boldsymbol{\Omega}}\,.\tag{73}$$

The solution of (73) yields the integral curve $\boldsymbol{Q}\left(t\right)\in\mathrm{SO}\left(3\right)$ for the orientation of the rigid body. Finally, differentiating the constraint equation $\boldsymbol{\chi}\left(t\right)=\boldsymbol{Q}^{-1}\left(t\right)\boldsymbol{k}$ gives

$$\dot{\boldsymbol{\chi}}\left(t\right)=\boldsymbol{\chi}\left(t\right)\times\boldsymbol{\Omega}\left(t\right)\quad\text{with}\quad\boldsymbol{\chi}\left(0\right)=\boldsymbol{Q}^{-1}\left(0\right)\boldsymbol{k}\,.\tag{74}$$

So the evolutionary system for the rigid body motion (71) and (72) is completed by (73) and (74).

5 Concluding remarks

The paper is devoted to a new derivation of the Bernoulli equation for an inviscid and incompressible fluid sloshing in a container undergoing prescribed rigid-body motion in three dimensions. The Bernoulli equation is derived by integrating the Euler equations relative to the rotating and translating coordinate system attached to the moving container and using the vorticity equation. An alternative view on the Bateman-Luke variational principle is presented. It is shown that the Neumann boundary-value problem for the problem of potential fluid sloshing in a container undergoing 3–D rigid-body motion can be derived from the Bateman-Luke variational principle (44) with mathematically precise definitions of dependent and independent variables with respect to the laboratory and body coordinate systems. A second variational principle is presented for the problem of 3–D interactions between potential water waves and a floating rigid body dynamically coupled to its interior potential fluid sloshing. The variational principle (5) recovers the complete set of equations of motion for the exterior *potential water waves* and the exact hydrodynamic equations of motion for the floating rigid body. The variational principle (5) is coupled to the variational principle (44) which gives the full set of equations of motion for the interior *potential fluid motion* of the rigid body.

The presented variational principles (5) and (44) and the corresponding partial differential equations for wave-structure-slosh interactions can be a starting point for further analytical and numerical analysis of dynamics of a liquid-filled spacecraft with interior potential fluid motion, potential fluid sloshing dynamics in moving tanks, a freely floating ship with fluid-filled tanks in hydrodynamic interaction with exterior water waves, and dynamics of floating structures such as ducted wave energy converters (Leybourne *et al.* 2014). Gagarina *et al.* (2014) developed a variational finite element method based on Luke's and Miles' variational principle (Luke 1967; Miles 1977) for nonlinear free surface gravity water waves. A direction of great interest is to extend the variational symplectic methods of Gagarina *et al.* (2014, 2016) and Kalogirou & Bokhove (2016) to develop hybrid numerical discretisations for the proposed Bateman-Luke variational principle (44) for the problem of potential water waves in rotating and translating coordinates, and the proposed variational principle (5) for 3–D interactions between exterior surface waves and a floating structure with interior potential fluid sloshing.

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