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# Interlace Polynomial of a Special Eulerian Graph 

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# MONTCLAIR STATE UNIVERSITY <br> Interlace Polynomial of a Special Eulerian Graph <br> by <br> Christian A. Hyra <br> A Master's Thesis Submitted to the Faculty of Montclair State University <br> In Partial Fulfillment of the Requirements <br> For the Degree of Master of Pure and Applied Mathematics 

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College of Science and Mathematics
Department of Mathematics



#### Abstract

In a recent paper, Arratia, Bollobás and Sorkin discussed a graph polynomial defined recursively, which they call the interlace polynomial. There have been previous results on the interlace polynomials for special graphs, such as paths, cycles, and trees. Applications have been found in biology and other areas. In this research, I focus on the interlace polynomial of a special type of Eulerian graph, built from one cycle of size $n$ and $n$ cycle three graphs. I developed explicit formulas by implementing the toggling process to the graph. I further investigate the coefficients and special values of the interlace polynomial. Some of them can describe properties of the considered graph. Aigner and Holst also defined a new interlace polynomial, called the Q-interlace polynomial, recursively, which can tell other properties of the original graph. One immediate application of the Q -interlace polynomial is that a special value of it is the number of general induced subgraphs with an odd number of general perfect matchings. Thus by evaluating the Q -interlace polynomial at a specific value, we determine the number of general induced subrgaphs with an odd number of general perfect matchings of the considered Eulerian graph.


# INTERLACE POLYNOMIAL OF A SPECIAL EULERIAN GRAPH 

## A THESIS

Submitted in partial fulfillment of the requirements for the degree of Masters in Pure and Applied Mathematics
by

Christian A. Hyra
Montclair State University
Montclair, New Jersey
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## 2 History

The origin of studying Eulerian graphs came from Euler's solution to the Konigsberg Bridge Problem in 1735; he proved that it was not possible to cross each of the seven bridges exactly one time and return to the orginal starting point [13]. Through solving this problem, Euler established the foundation of graph theory. He defined the Eulerian graph to be a graph containing an Eulerian circuit, which is a circuit that includes each edge of a graph exactly once, starting and ending at the same vertex. His negative result on the $k$ Bridge Problem was the first theorem in graph thoery [13].

In recent years, an abundance of graph polynomials have been studied. The two most prominent graph polynomials that have been studied are the Tutte Polynomials and the interlace polynomial, which resembled the Martin Polynomial [12]. The Tutte polynomial, a two-variable graph polynomial, has the important universal property that essentially any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it [8]. These deletion/contraction operations are natural reductions for many network models arising from a wide range of problems at the hearts of computer science, engineering, optimization, physics, and biology [8]. Martin Polynomials can be considered as a specific type of a Tutte Polynomial, by making the two variables equal, which is how we can connect interlace polynomials to Tutte Polynomials.

Eulerian circuits are directly connected in DNA sequencing by hybridization; counting the number of 2-in, 2-out digraphs (Eulerian, directed graphs in which each vertex has in-degree two and out-degree two) [3]. The interlace polynomial can count the number of Eulerian circuits in a 2-in, 2-out digraph [3] and models the interlaced repeated subsequences of DNA that can interfere with the unique reconstruction of the original DNA strand [8]. The interlace polynomial can also be used to successfully count the k -component circuit partitions of a graph [3]. Interlace polynomials share
similar properties as Martin Polynomials and Kauffman polynomials, which encode information about the families of closed paths in Eulerian graphs [4]. Since Euler circuits have a lot to do with DNA sequencing and other graph polynomials, I choose to observe a certain type of Eulerian graph and investigate if they share common properties with other graph polynomials. In this paper I focus on the interlace polynomial for a specific Eulerian Graph, $\Gamma_{n}$, and investigate how the graph polynomials can be specialized and generalized, and if they can encode any information relevant to physical applications. Aigner and Holst defined two interlace polynomials [1]. Since each one holds its own specific properties, I develop the recursive and explicit formulas for both interlace polynomials for $\Gamma_{n}$.

## 3 Preliminaries

In this paper, we work with graphs, which are represented by an ordered pair of sets of vertices and edges.

Definition 1. A graph $G$ is an ordered pair $G=(V, E)$ such that:

1. $V$ is a set, called the vertex set;
2. $E$ is a set of two-element subsets of $V$, called the edge set, that is $E \subseteq\{\{u, v\}: u, v \in V\}$.

The number of edges incident to a specific vertex, $v \in V$, is called the degree of $v$, denoted $\operatorname{deg}(v)$. The maximum degree of a graph $G$ is the highest degree of all the possible vertices in the graph $G$. From existing results, we know that for any graph, the number of vertices with odd degree is always even. Furthermore, the degree sum formula tells us the sum of the degrees of all vertices in a graph $G$ is equivalent to 2 times the cardinality of the edge set of $G$.

Lemma 3.1. (Degree Formula) Given a graph $G=(V, E)$,

$$
\sum_{v \in V} \operatorname{deg}(v)=2|E|
$$

In this paper we focus only on simple graphs. A graph $G$ is called a simple graph if there are no multiple edges joining the same pair of vertices, as well as no loops, which would be an edge that starts and ends at the same vertex. In order to uderstand the structure of a graph, we state common elements a graph can have, which can be found in any graph theory text book.

Definition 2. Given a graph $G=(V, E)$, where $V=\left\{v_{0}, v_{1}, \ldots, v_{n}\right\}$ and $E=\left\{e_{1}, e_{2}, \ldots, e_{m}\right\}:$

1. A walk of length $k$ is a sequence $v_{0}, v_{1}, \ldots v_{k}$, which contains $k$ edges of the form $\left\{v_{0}, v_{1}\right\},\left\{v_{1}, v_{2}\right\}, \cdots,\left\{v_{k-1}, v_{k}\right\}$.
2. A path of length $k$ is a walk with $k+1$ distinct vertices, denoted by $P_{k}$.
3. A cycle of size $k$, is a path with $k$ vertices, with an additional edge between $v_{k}$ and $v_{1}$, making a closed path, denoted by $C_{k}$.
4. A sequence of distinct edges $e_{1} e_{2} \cdots e_{k}$ is called a trail if we can take a continuous walk in our graph, first walking through the edge $e_{1}$, then the edge $e_{2}$, and so on. In addition, if we start and end at the same vertex, we have a closed trail.
5. An Eulerian Trail is a trail that covers each edge of $G$ exactly once.
6. An Eulerian Circuit is an Eulerian Trail that starts and ends at the same vertex, covering each edge exactly once.

Note that the difference between a trail and a path is the uniqueness of edges and vertices. A trail has distinct edges while a path has distinct vertices.

Definition 3. [7] If for any two vertices $x$ and $y$ in a graph $G$, one can find a path from $x$ to $y$, then we say that $G$ is a connected graph.

When dealing with graphs, we can have directed or undirected graphs. A graph where each edge is assigned a direction is a directed graph. When the edges of a graph are not assigned a specific direction, we are dealing with an undirected graph.

A graph $G^{\prime}$ is called the subgraph of the graph $G$ if the set of vertices and edges of the graph $G^{\prime}$ form subsets of the vertices and edges of the original graph $G$. In other words, $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ where $V^{\prime} \subset V$ and $E^{\prime} \subset E$. An induced subgraph of $G$ by a subset $S \subseteq V(G)$ is the subgraph $G^{\prime}=\left(S, E^{\prime}\right)$ where for $u, v \in S, u v=e \in E^{\prime} \Longleftrightarrow$ $e \in E(G)$. We must take all and only those edges present in $G$ between the specified vertices in $S$. That is $E^{\prime}=\{u v \mid u, v \in S, u v \in E(G)\}$. Below we define special graphs that we will use further in the paper.

## Definition 4. Special Graphs

1. A Bipartite Graph is a graph whose vertex set is decomposed in two disjoint sets, called partite sets, such that no two graph vertices within the partite set are adjacent.
2. A Complete Graph, denoted $K_{n}$, is a simple undirected graph which every pair of distinct vertices is connected with a unique edge.
3. A Complete Bipartite Graph, denoted $K_{m, n}$, is a bipartite graph such that every pair of graph vertices in different partite sets are adjacent.
4. An Eulerian Graph, is a graph containing an Eulerian circuit, which is only possible if all vertices are of even degree.

When finding the interlace polynomial of a graph we need to recall a few definitions. If $x \in V(G), G \backslash\{x\}$ is the resulting graph after removing the vertex $x$ and all edges of $G$ incident to $x$. Further, we need to recall the pivot of a graph [13].

Consider an undirected graph $G$ and $a, b \in V(G)$, with $a b \in E(G)$. The edge $a b$ will divide the neighbors of $a$ or $b$ into three classes: (1) vertices adjacent to both $a$ and $b$, (2) vertices adjacent to $a$ alone, and (3) vertices adjacent to $b$ alone. Hence when we toggle an edge in between any two of these three classes, $x y$ will be an edge of the new graph if and only if $x y$ is not an edge of G.

Definition 5. [3] (Pivot) Let $G$ be any undirected simple graph and ab an edge of $G$. The pivot of $G$ with respect to $a b$, denoted $G^{a b}$, is the resulting graph after toggling all pairs $x, y$ where $x, y$ are from different classes of (1), (2), (3) described above, shown in Figure 1.


Figure 1: Neighborhoods of $a, b$ and the Toggle Operation

The formula for finding the interlace polynomial of a graph $G$ is given recursively by:

Definition 6. (Interlace Polynomial) For any undirected graph $G$ with $n$ vertices, the interlace polynomial of $G$ is defined recursively by:

1. $q(G, x)=x^{n}$ if $E(G)=\varnothing$;
2. $q(G, x)=q(G \backslash\{a\}, x)+q\left(G^{a b} \backslash\{b\}, x\right)$ where $a b \in E(G)$;
3. $q(G, x)=q\left(G_{1}, x\right) q\left(G_{2}, x\right) \cdots q\left(G_{n}, x\right)$ if $G=G_{1} \cup G_{2} \uplus \cdots \cup G_{n}$.

Note that it is shown [3] that the interlace polynomial is the same no matter what edge is toggled. Below we state the existing results of interlace polyomials of certain graphs.

Lemma 3.2. The interlace polynomials are known for the following graphs [3]:

1. (complete graph $\left.K_{n}\right) q\left(K_{n}, x\right)=2^{n-1} x$;
2. (complete bipartite graph $K_{m, n}$ )

$$
q\left(K_{m, n}, x\right)=\left(1+x+\cdots+x^{m-1}\right)\left(1+x+\cdots+x^{n-1}\right)+x^{m}+x^{n}-1 ;
$$

3. (path $P_{n}$ with $n$ edges) $q\left(P_{1}, x\right)=2 x, q\left(P_{2}, x\right)=x^{2}+2 x$, and for $n \geq 3$,

$$
q\left(P_{n}, x\right)=q\left(P_{n-1}, x\right)+x q\left(P_{n-2}, x\right) ;
$$

4. (small cycles) $q\left(C_{3}, x\right)=4 x$ and $q\left(C_{4}, x\right)=3 x^{2}+2 x$.

Note that the interlace polynomial of an isolated vertex is simply $x$. Below is an example of the pivot process for a graph $G$.

Example 1. Developing the Interlace Polynomial of $\Gamma_{3}$ using the pivot process.
Consider the following graph, called $\Gamma_{3}$. We pivot with respect to ab.

$\Gamma_{3} \backslash\{a\}:$

$\Gamma_{3}^{a b} \backslash\{b\}$ : In this step we toggle between $N_{1}=N(a, b)$ and $N_{2}=N(b) \backslash(\{a\} \cup N(a))$, then remove $b$ and its adjacent edges.


We will now use this pivot process multiple times, in order to find the interlace polynomial for $\Gamma_{3}: q\left(\Gamma_{3}, x\right)=q\left(\Gamma_{3} \backslash\{a\}, x\right)+q\left(\Gamma_{3}^{a b} \backslash\{b\}, x\right)$.

Step 1: Using the pivot process from above we obtain the two graphs $\Gamma_{3} \backslash\{a\}$ and $\Gamma_{3}^{a b} \backslash\{b\}$.


Step 2: Pivoting $\Gamma_{3} \backslash\{a\}$ with respect to cd, we obtain a graph we call $\Lambda_{1}$ and $P_{3}$.


Step 3: Pivoting $\Lambda_{1}$ with respect to oh gives us $C_{3}$ and $x P_{1}$.

$$
\widehat{\Delta}=\triangle+\cdot I+L+\mathbf{L}
$$

Step 4: Pivoting $\Gamma_{3}^{a b} \backslash\{b\}$ with respect to of gives us $C_{4}$ and $x P_{2}$.


Now we are able to substitute the existing results for the graphs left in Step 4 and determine the interlace polynomial for $\Gamma_{3}$. From our last step in the pivot process of $\Gamma_{3}$, we have $q\left(\Gamma_{3}, x\right)=q\left(C_{3}, x\right)+x q\left(P_{1}, x\right)+q\left(P_{3}, x\right)+q\left(C_{4}, x\right)+x q\left(P_{2}, x\right)$.

$$
\begin{aligned}
q\left(\Gamma_{3}, x\right) & =4 x+x(2 x)+\left(3 x^{2}+2 x\right)+\left(2 x+3 x^{2}\right)+x\left(x^{2}+2 x\right) \\
& =x^{3}+10 x^{2}+8 x
\end{aligned}
$$

This specific example is actually an example of the type of graph we work with, and is denoted by $\Gamma_{n}$. I define this type of graph further in this section, but before I do, I want to point out another graph from Step 2 of Example 1. This graph consists of a triangle and an edge from Step 2, and is called $\Lambda_{1}$. By the breakdown in Step 3 we see that $q\left(\Lambda_{1}, x\right)=2 x^{2}+4 x=2 x(x+2)$. This result will be used in section 4 .

Lemma 3.3. The interlace polynomial for $\Lambda_{1}$ is $q\left(\Lambda_{1}, x\right)=2 x(x+2)$.
In this thesis, I focus on a special type of an Eulerian graph, $\Gamma_{n}$. An Eulerian graph is a graph that contains an Eulerian circuit. Recall that an Eulerian circuit is a trail that starts and ends at the same vertex and uses each edge exactly once. A certain vertex can be repeated throughout the circuit, but an edge cannot be repeated. The graph $\Gamma_{n}$ has all vertices with degree two or four. It consists of a cycle graph $C_{n}$, where each edge contributes to a cycle graph $C_{3}$ along the perimeter of $C_{n}$, and the third vertex is of degree two. I use pivoting and other techniques to find the interlace polynomial for this graph of any size $n$. Further, I discuss any recognizable patterns noticed during the pivoting process. After finding a explicit formula for the interlace polynomial of $\Gamma_{n}$, I focus on the meaning of the coefficients, properties at specific values of $x$, properties dealing with different paraties of $x$, the relationship with matrix theory, and other applications of the interlace polynomial.

## 4 Basic Properties of $\Gamma_{n}$

Definition 7. For $n \geq 3$, define $\Gamma_{n}=(V, E)$, where $V=\left\{v_{1}, v_{2}, \cdots, v_{2 n}\right\}$ and $E=\left\{v_{1} v_{2}, v_{2} v_{3}, \cdots, v_{2 n-1} v_{2 n}, v_{2 n} v_{1}, v_{1} v_{3}, v_{3} v_{5}, \cdots, v_{2 n-3} v_{2 n-1}, v_{2 n-1} v_{1}\right\}$.

Example 2. The graph $\Gamma_{n}$ (see Figure 2): the inner part is $C_{n}$


Figure 2: $\Gamma_{n}$ with $n$ three-cycle graphs outside of the cycle graph $C_{n}$

An example of $\Gamma_{4}$ is provided below.

Example 3. The graph $\Gamma_{4}$ consists of the graph $C_{4}$, with a graph $C_{3}$ associated to each edge in $C_{4}$, around the outside perimeter: (see Figure 3).


Figure 3: $\Gamma_{4}$

Proposition 4.1. The graph $\Gamma_{n}$ has $2 n$ vertices and $3 n$ edges, with $n$ of each, degree 2 and degree 4 vertices.

Taking a look at the characteristics of the graph $\Gamma_{n}$ gives us a better perspective on how to relate the meaning of our graph at certain values of $x$. The following characteristics were observed and proved:

Theorem 4.2. Properties of $\Gamma_{n}$.

1. The independent number of $\Gamma_{n}$ is $n$.
2. Edge-connectivity and vertex-connectivity of $\Gamma_{n}$ are both 2 .
3. The circumference of $\Gamma_{n}$ is $2 n$, which is the cardinality of the vertex set, or $\left|V\left(\Gamma_{n}\right)\right|$.
4. Diameter $\left(\Gamma_{n}\right)=\left\{\begin{array}{cl}\frac{n+2}{2} & \text { if } n \text { is even } \\ \frac{n+1}{2} & \text { if } n \text { is odd }\end{array}\right.$.

## 5 The Interlace Polynomial of $\Gamma_{n}$

Throughout, for any graph $G$, we will represent the interlace polynomial $q(G, x)$ as $G(x)$. Using the toggling approach and some basic results about interlace polynomials, we first describe the interlace polynomial of $\Gamma_{n}$ for small values of $n$. The polynomials are shown below for $n$ ranging from 3 to 7 .

Lemma 5.1. The interlace polynomials for $\Gamma_{n}$, with $3 \leq n \leq 7$, are as follows:

1. $\Gamma_{3}(x)=x^{3}+10 x^{2}+8 x ;$
2. $\Gamma_{4}(x)=x^{4}+8 x^{3}+32 x^{2}+24 x$;
3. $\Gamma_{5}(x)=x^{5}+10 x^{4}+40 x^{3}+96 x^{2}+64 x$;
4. $\Gamma_{6}(x)=x^{6}+12 x^{5}+60 x^{4}+160 x^{3}+272 x^{2}+160 x$;
5. $\Gamma_{7}(x)=x^{7}+14 x^{6}+84 x^{5}+280 x^{4}+560 x^{3}+736 x^{2}+384 x$.

While working on the polynomials for the specific values of $n$, I started to notice a couple of patterns in the graphs that resulted from the toggling process. For $n$ greater than or equal to four, the toggling process produced very similar graphs, resulting in particular, three special types of graphs defined below.

Definition 8. The graphs $M_{n}, \Lambda_{n}$, and $\Delta_{n}$, with $n \geq 1$.

1. The graph $M_{n}=\left(V\left(M_{n}\right), E\left(M_{n}\right)\right)$, where $V\left(M_{n}\right)=\left\{v_{1}, v_{2}, \cdots, v_{2 n}, v_{2 n+1}\right\}$ and $E\left(M_{n}\right)=\left\{v_{1} v_{2}, v_{2} v_{3}, \cdots, v_{2 n} v_{2 n+1}, v_{1} v_{3}, v_{3} v_{5}, \cdots, v_{2 n-1} v_{2 n+1}\right\}$.
2. The graph $\Lambda_{n}=\left(V\left(M_{n}\right) \cup\left\{v_{0}\right\}, E\left(M_{n}\right) \cup\left\{v_{0} v_{1}\right\}\right)$.
3. The graph $\Delta_{n}=\left(V\left(\Lambda_{n}\right) \cup\left\{v_{2 n+2}\right\}, E\left(\Lambda_{n}\right) \cup\left\{v_{2 n+1} v_{2 n+2}\right\}\right)$.

Below we provide an example of each graph.

Example 4. The graphs $M_{5}, \Lambda_{5}$, and $\Delta_{5}$.


Figure 4: $M_{5}$ made of 5 adjacent $C_{3}$ graphs.


Figure 5: $\Lambda_{5}$ made of $M_{5}$ and an additional edge


Figure 6: $\Delta_{5}$ made of $M_{5}$ and two additional edges, one at each end.

Proposition 5.2. The recursive formula for $\Gamma_{n}(x)$, with $n \geq 4$, is:

$$
\Gamma_{n}(x)=2 \Gamma_{n-1}(x)+\Lambda_{n-2}(x)+x \Delta_{n-3}(x) .
$$

Proof. For $n \geq 4$, applying the toggling process for $\Gamma_{n}$, with respect to $a b$, creates $\Gamma_{n} \backslash\{a\}$ and $\Gamma_{n}^{a b} \backslash\{b\}$. Applying the toggling process to these resulting graphs, with respect to $c d$ and ef results in the graphs $\Lambda_{n-2}$, two $\Gamma_{n-1}$ graphs, and $\Delta_{n-3}$ with an isloted vertex. The process is shown below.


Figure 7: Toggle proccess for $\Gamma_{n}$.

We demonstrate this process for $\Gamma_{4}$, which also provides a proof for the formula for $\Gamma_{4}(x)$ shown in Lemma 5.1.

Decomposition of $\Gamma_{4}$ :
Step 1: Toggling $\Gamma_{4}$ into $\Gamma_{4} \backslash\{a\}$ and $\Gamma_{4}^{a b} \backslash\{b\}$


Step 2: Toggling $\Gamma_{4} \backslash\{a\}$ to show $\Lambda_{2}$ and $\Gamma_{3}$.

$$
\theta=\Delta+\Delta+\Delta
$$

Step 3: Further toggling $\Lambda_{2}$ to show $P_{1}$ with $P_{2}$ and a new graph.


Step 4: Further, toggling the second graph in Step 3 results in $K_{4}$ and $C_{3}$, with an isolated vertex.


Step 5: Toggling the last graph in Step 4 we achieve $\Gamma_{3}$ and a $\Delta_{1}$ with an extra vertex.


Step 6: Finally, we toggle $\Delta_{1}$ in Step 5 to achieve $\Lambda_{1}$ the graph $P_{2}$.

$$
\therefore=V 1+\boxtimes+\Delta+\Delta+\Delta+\Delta \cdot
$$

Now using the formulas from Lemma 3.2, the result for $\Gamma_{3}(x)$ in Lemma 5.1, and the result for $\Lambda_{1}(x)$ in Lemma 3.3, the interlace polynomials for the graphs from Step 6 can be expressed by:

$$
\begin{aligned}
\Gamma_{4}(x) & =2 x\left(x^{2}+2 x\right)+8 x+x(4 x)+2\left(x^{3}+10 x^{2}+8 x\right)+x\left(4 x+2 x^{2}\right)+x^{2}\left(x^{2}+2 x\right) \\
& =x^{4}+8 x^{3}+32 x^{2}+24 x
\end{aligned}
$$

If we take a look at Step 6 for the breakdown of $\Delta_{1}$, we see that $\Delta_{1}(x)=x(x+2)^{2}$. We will use this result in Section ??.

Lemma 5.3. The interlace polynomial for $\Delta_{1}$ is: $\Delta_{1}(x)=x(x+2)^{2}$.
In order to achieve an explicit formula for $\Gamma_{n}(x)$, we need to find an explicit formula for each of $\Lambda_{n}(x)$ and $\Delta_{n}(x)$.

### 5.1 Recursive Formula for $\Lambda_{n}(x)$

Applying the toggling process to the graph $\Lambda_{n}$, with respect to $a b$, gives us two graphs (see Figure 8): $M_{n}$ and $\Lambda_{n-1}$, with an additional vertex.


Figure 8: Toggling $\Lambda_{n}$ with respect to $a b$

Since toggling $\Lambda_{n}$ results in $M_{n}$ we further our process by toggling $M_{n}$ with respect to $a b$, giving us two $\Lambda_{n-1}$ graphs (See Figure 9).


Figure 9: Toggling $M_{n}$ with respect to $a b$.

From these two steps, we have a recursive formula for $\Lambda_{n}(x)$, which we use to find an explicit formula, $\Lambda_{n}(x)=M_{n}(x)+x \Lambda_{n-1}(x)=2 \Lambda_{n-1}(x)+x \Lambda_{n-1}(x)=$ $(x+2) \Lambda_{n-1}(x)$.

Lemma 5.4. The recursive formula for $\Lambda_{n}(x)$ :

$$
\Lambda_{n}(x)=(x+2) \Lambda_{n-1}(x)
$$

We can now expand on our recursive formula to achieve an explicit formula for $\Lambda_{n}(x)$ :

$$
\Lambda_{n}(x)=(x+2) \Lambda_{n-1}(x)=(x+2)^{2} \Lambda_{n-2}(x)=(x+2)^{3} \Lambda_{n-3}(x)=\cdots=(x+2)^{n-1} \Lambda_{1}(x)
$$

From Lemma 3.3, we substitute in for $\Lambda_{1}(x)=x(x+2)^{2}$ to achieve an explicit formula.

$$
\Lambda_{n}(x)=(x+2)^{n-1}\left(2 x^{2}+4 x\right)=(x+2)^{n-1}(2 x)(x+2)=2 x(x+2)^{n}
$$

This formula will be formally mentioned in the next section.

### 5.2 Recursive Formula for $\Delta_{n}(x)$

Before I introduce the recursive formula for $\Delta_{n}(x)$, I would like to share a known result for a given power series we will see. The finite sum $\sum_{k=0}^{n} r^{k}$ can be expanded as: $\sum_{k=0}^{n} r^{k}=\frac{1-r^{n+1}}{1-r}$. Applying the toggling process to the graph $\Delta_{n}$, with respect to $a b$, gives us the graph $\Lambda_{n}$, and $\Delta_{n-1}$ with an extra vertex. We demonstrate this process with $\Delta_{5}$ (see Figure 10).


Figure 10: Toggling $\Delta_{n}$ with respect to $a b$.

From the toggling process, we create a recursive formula for $\Delta_{n}(x)$ and use it to find the explicit formula.

Lemma 5.5. The recursive formula for $\Delta_{n}(x)$ is:

$$
\Delta_{n}(x)=\Lambda_{n}(x)+x \Delta_{n-1}(x)=2 x(x+2)^{n}+x \Delta_{n-1}(x) .
$$

We can now expand the recursive formula to achieve an explicit formula for $\Delta_{n}(x)$.

$$
\Delta_{n}(x)=2 x(x+2)^{n}+x\left[2 x(x+2)^{n-1}+x\left(\Delta_{n-2}(x)\right)\right]
$$

$$
\begin{aligned}
&= 2 x(x+2)^{n}+2 x^{2}(x+2)^{n-1}+x^{2} \Delta_{n-2}(x) \\
&= 2 x(x+2)^{n}+2 x^{2}(x+2)^{n-1}+x^{2}\left[2 x(x+2)^{n-2}+x\left(\Delta_{n-3}(x)\right)\right] \\
&= 2 x(x+2)^{n}+2 x^{2}(x+2)^{n-1}+2 x^{3}(x+2)^{n-2}+x^{3}\left(\Delta_{n-3}(x)\right) \\
& \vdots \\
&= x^{n-1}\left(\Delta_{1}(x)\right)+2 x^{n-1}(x+2)^{2}+2 x^{n-2}(x+2)^{3}+\cdots+2 x(x+2)^{n} \\
&=x^{n-1}\left(\Delta_{1}(x)\right)+2 x \sum_{k=2}^{n}(x+2)^{k} x^{n-k}=x^{n-1}\left(\Delta_{1}(x)\right)+2 x \sum_{k=2}^{n} x^{n}\left(\frac{x+2}{x}\right)^{k} \\
&=x^{n-1}\left(\Delta_{1}(x)\right)+2 x^{n+1} \sum_{k=2}^{n}\left(\frac{x+2}{x}\right)^{k}
\end{aligned}
$$

Now we can expand our finite sum by the known result mentioned above, but we will have to modify our expansion since we are starting with $k=2$ instead of $k=0$ :

$$
\begin{aligned}
2 x^{n+1} \sum_{k=2}^{n}\left(\frac{x+2}{x}\right)^{k} & =2 x^{n+1}\left[\frac{1-\left(\frac{x+2}{x}\right)^{n+1}}{1-\frac{x+2}{x}}\right]-\left(2 x^{n+1}\right)\left(\frac{x+2}{x}\right)^{0}-\left(2 x^{n+1}\right)\left(\frac{x+2}{x}\right)^{1} \\
& =2 x^{n+1}\left[\frac{1-\left(\frac{x+2}{x}\right)^{n+1}}{\frac{-2}{x}}\right]-\left(2 x^{n+1}\right)\left(\frac{x+2}{x}\right)^{0}-\left(2 x^{n+1}\right)\left(\frac{x+2}{x}\right)^{1} \\
& =x^{n+1}\left[1-\frac{(x+2)^{n+1}}{x^{n+1}}\right](-x)-\left(2 x^{n+1}\right)\left(\frac{x+2}{x}\right)^{0}-\left(2 x^{n+1}\right)\left(\frac{x+2}{x}\right)^{1} \\
& =-x^{n+2}+x(x+2)^{n+1}-4 x^{n+1}-4 x^{n}
\end{aligned}
$$

Now going back to the previous formula $\Delta_{n}(x)=x^{n-1}\left(\Delta_{1}(x)\right)+2 x^{n+1} \sum_{k=2}^{n}\left(\frac{x+2}{x}\right)^{k}$ and apply the result for the finite sum and the result for $\Delta_{1}(x)$, from Lemma 5.3 , to receive:

$$
\begin{aligned}
\Delta_{n}(x) & =x^{n-1}\left(x^{3}+4 x^{2}+4 x\right)-x^{n+2}+x(x+2)^{n+1}-4 x^{n+1}-4 x^{n} \\
& =x^{n+2}+4 x^{n+1}+4 x^{n}-x^{n+2}+x(x+2)^{n+1}-4 x^{n+1}-4 x^{n} \\
& =x(x+2)^{n+1}
\end{aligned}
$$

Lemma 5.6. Respectively, the explicit formula for $\Lambda_{n}(x)$ and $\Delta_{n}(x)$ are:

1. $\Lambda_{n}(x)=2 x(x+2)^{n}, \quad n \geq 1$.
2. $\Delta_{n}(x)=x(x+2)^{n+1}, \quad n \geq 1$.

Proof. 1. $\Lambda_{n}(x)=2 x(x+2)^{n}$, for $n \geq 1$. By Induction:

Let $n=1, \Lambda_{1}(x)=2 x(x+2)^{1}=2 x(x+2)$.
From Lemma 3.3, we know this is true.
From Lemma 5.4, we know $\Lambda_{n}(x)=(x+2)\left(\Lambda_{n-1}\right)$. Assume $\Lambda_{n-1}(x)=2 x(x+$ $2)^{n-1}$.

$$
\begin{aligned}
& \Lambda_{n}(x)=(x+2)\left(\Lambda_{n-1}\right)=(x+2)\left(2 x(x+2)^{n-1}\right)=2 x(x+2)^{n} . \\
& \therefore \Lambda_{n}(x)=2 x(x+2)^{n}, \text { for all } n \geq 1 .
\end{aligned}
$$

Proof. 2. $\Delta_{n}(x)=x(x+2)^{n+1}$, for $n \geq 1$. By Induction:
Let $n=1, \Delta_{1}=x(x+2)^{2}$.
From Lemma 5.3, we know this is true.
From Lemma 5.5, we know $\Delta_{n}(x)=2 x(x+2)^{n}+x\left(\Delta_{n-1}\right)$. Assume $\Delta_{n-1}(x)=$ $x(x+2)^{n}$.

$$
\begin{aligned}
\Delta_{n}(x) & =2 x(x+2)^{n}+x\left(\Delta_{n-1}\right)=2 x(x+2)^{n}+x\left(x(x+2)^{n}\right) \\
& =x(x+2)^{n+1} . \\
\therefore \quad \Delta_{n} & (x)=x(x+2)^{n+1}, \text { for all } n \geq 1 .
\end{aligned}
$$

### 5.3 Explicit Formula for $\Gamma_{n}(x)$

We can now use the recursive formula for $\Gamma_{n}(x)$ from Lemma 5.2 to find an explicit formuala for $\Gamma_{n}(x)$.

$$
\begin{aligned}
\Gamma_{n}(x) & =2 \Gamma_{n-1}(x)+\Lambda_{n-2}(x)+x \Delta_{n-3}(x)=2 \Gamma_{n-1}(x)+2 x(x+2)^{n-2}+x^{2}(x+2)^{n-2} \\
= & 2 \Gamma_{n-1}(x)+x(x+2)^{n-1}=2^{2} \Gamma_{n-2}(x)+2 x(x+2)^{n-2}+x(x+2)^{n-1} \\
= & 2^{3} \Gamma_{n-3}(x)+2^{2} x(x+2)^{n-3}+2 x(x+2)^{n-2}+x(x+2)^{n-1} \\
& \vdots
\end{aligned}
$$

$$
\begin{aligned}
& =2^{n-4} \Gamma_{4}(x)+2^{n-5}(x+2)^{4}+2^{n-6}(x+2)^{5}+\cdots+2^{0} x(x+2)^{n-1} \\
& =2^{n-4} \Gamma_{4}(x)+\sum_{k=4}^{n-1} 2^{n-1-k} x(x+2)^{k}=2^{n-4}\left(\Gamma_{4}(x)\right)+2^{n-1} x \sum_{k=4}^{n-1}\left(\frac{x+2}{2}\right)^{k} .
\end{aligned}
$$

Note that $\sum_{k=4}^{n-1}\left(\frac{x+2}{2}\right)^{k}=\sum_{k=0}^{n-1}\left(\frac{x+2}{2}\right)^{k}-\sum_{k=0}^{3}\left(\frac{x+2}{2}\right)^{k}$.

$$
\begin{gathered}
2^{n-1} x \sum_{k=0}^{n-1}\left(\frac{x+2}{2}\right)^{k}=(x+2)^{n}-2^{n} ; \\
2^{n-1} x \sum_{k=0}^{3}\left(\frac{x+2}{2}\right)^{k}=2^{n-4}(x+2)^{4}-2^{n} .
\end{gathered}
$$

We go back to our previous equation to apply the power series formula and the result for $\Gamma_{4}(x)$, from Lemma 5.1.

$$
\begin{aligned}
\Gamma_{n}(x) & =2^{n-4}\left(x^{4}+8 x^{3}+32 x^{2}+24 x\right)+(x+2)^{n}-2^{n}-\left[2^{n-4}(x+2)^{4}-2^{n}\right] \\
& =2^{n-4}\left(8 x^{2}-8 x-16\right)+(x+2)^{n}=2^{n-1}\left(x^{2}-x-2\right)+(x+2)^{n}
\end{aligned}
$$

Theorem 5.7. The explicit formula for $\Gamma_{n}(x)$, with $n \geq 3$, is:

$$
\Gamma_{n}(x)=2^{n-1}\left(x^{2}-x-2\right)+(x+2)^{n} .
$$

Proof. By Induction:
Let $n=4, \Gamma_{4}(x)=2^{3}\left(x^{2}-x-2\right)+(x+2)^{4}=x^{4}+8 x^{3}+32 x^{2}+24 x$.
It is confirmed by Lemma 5.1.
From Proposition 5.2 and Lemma 5.6 we know $\Gamma_{n}(x)=2 \Gamma_{n-1}(x)+x(x+2)^{n-1}$.
Assume $\Gamma_{n-1}(x)=2^{n-2}\left(x^{2}-x-2\right)+(x+2)^{n-1}$, then

$$
\begin{aligned}
\Gamma_{n}(x) & =2 \Gamma_{n-1}(x)+x(x+2)^{n-1} \\
& =2\left(2^{n-2}\left(x^{2}-x-2\right)+(x+2)^{n-1}\right)+x(x+2)^{n-1} \\
& =2^{n-1}\left(x^{2}-x-2\right)+2(x+2)^{n-1}+x(x+2)^{n-1} \\
& =2^{n-1}\left(x^{2}-x-2\right)+(x+2)^{n} .
\end{aligned}
$$

$\therefore \quad \Gamma_{n}(x)=2^{n-1}\left(x^{2}-x-2\right)+(x+2)^{n}$ is true for all $n \geq 3$.

## 6 Properties of $\Gamma_{n}(x)$

The interlace polynomial of a graph is a special graph invariant that can tell us different information about the graph. We are specifically interested in the coefficients and some special values of $\Gamma_{n}(x)$. Do the coefficients give us any meaning towards the graph $\Gamma_{n}$ itself, or any of the subgraphs within $\Gamma_{n}$ ? What do special values of $\Gamma_{n}(x)$ tell us and what can that information be used for? Furthermore, what kind of relation is there with $\Gamma_{n}(x)$ to the adjacency matrix of $\Gamma_{n}$ ? Within this section, I analyze the meaning of the interlace polynomial for $\Gamma_{n}$ and correlate the information to certain applications.

### 6.1 Coefficients of $\Gamma_{n}(x)$

From the explicit formulas given in Lemma 5.6 and Theorem 5.7, we are able to relate the coefficients between $\Gamma_{n}(x), \Lambda_{n}(x)$, and $\Delta_{n}(x)$. The relation is made using generating functions to define the coefficients. Obviously the constant term of any interlace polynomial is zero. From the explicit formula of $\Gamma_{n}(x)$ (Theorem 5.7) we can see that the polynomial is of degree $n$. Using the fact that the constant term is zero and expressing $(x+2)^{n}$ by the binomial formula, we can rewrite $\Gamma_{n}(x)$ as:

$$
\begin{equation*}
\Gamma_{n}(x)=2^{n-1} x^{2}-2^{n-1} x+\left[\sum_{k=1}^{n}\binom{n}{k} 2^{n-k} x^{k}\right], \quad n \geq 3 \tag{1}
\end{equation*}
$$

Definition 9. Consider the polynomials $\Gamma_{n}(x), \Lambda_{n}(x)$, and $\Delta_{n}(x)$. We use $a_{n, k}, l_{n, k}$, and $d_{n, k}$ to represent the coefficients for each polynomial respectively. That is,

$$
\Gamma_{n}(x)=\sum_{k=1}^{n} a_{n, k} x^{k}, \quad \Lambda_{n}(x)=\sum_{k=1}^{n+1} l_{n, k} x^{k}, \quad \text { and } \quad \Delta_{n}(x)=\sum_{k=1}^{n+2} d_{n, k} x^{k} .
$$

Lemma 6.1. The coefficients of $\Gamma_{n}(x)=\sum_{k=1}^{n} a_{n, k} x^{k}$, with $n \geq 3$, are given by

$$
a_{n, k}=\left\{\begin{array}{cc}
2^{n-1}(n-1) & \text { if } k=1 \\
2^{n-3}\left(n^{2}-n+4\right) & \text { if } k=2 \\
2^{n-k}\binom{n}{k} & \text { if } 2<k \leq n
\end{array} .\right.
$$

Let us take a look at an example of the coefficients of $\Gamma_{3}(x)$. We know the degree of the polynomial is three, so we will only need to concentrate on the formula for our coefficients $a_{3, k}$ with $1 \leq k \leq 3$.

Example 5. Coefficients of $\Gamma_{3}(x)$ :

$$
a_{3,1}=2^{2}(2)=8, \quad a_{3,2}=2^{0}(9-3+4)=10, \quad a_{3,3}=2^{0}\binom{3}{3}=1
$$

This gives $\Gamma_{3}(x)=x^{3}+10 x^{2}+8 x$. It is confirmed by Lemma 5.1.

We can express the coefficients for $\Lambda_{n}(x)$ and $\Delta_{n}(x)$ in a similar manner. Recall, taking a look at Lemma 5.6 , the degree of $\Lambda_{n}(x)$ is $n+1$ and the degree of $\Delta_{n}(x)$ is $n+2$.

Lemma 6.2. Coefficients of $\Lambda_{n}(x)$ and $\Delta_{n}(x)$ are, respectively:

$$
\begin{align*}
& l_{n, k}=2^{n+2-k}\binom{n}{k-1} \quad 1 \leq k \leq n+1  \tag{2}\\
& d_{n, k}=2^{n+2-k}\binom{n+1}{k-1} \quad 1 \leq k \leq n+2 . \tag{3}
\end{align*}
$$

Proof. (2). Using the binomial formula:

$$
\Lambda_{n}(x)=2 x(x+2)^{n}=2 x \sum_{k=0}^{n}\binom{n}{k} 2^{n-k} x^{k}=\sum_{k=0}^{n}\binom{n}{k} 2^{n+1-k} x^{k+1}=\sum_{k=1}^{n}\binom{n}{k-1} 2^{n+2-k} x^{k} .
$$

Proof. (3). Using the binomial formula:

$$
\Delta_{n}(x)=x(x+2)^{n+1}=\sum_{k=0}^{n}\binom{n+1}{k} 2^{n+1-k} x^{k+1}=\sum_{k=1}^{n}\binom{n+1}{k-1} 2^{n+2-k} x^{k}
$$

Let us look at an example for the coefficients of the polynomials $\Lambda_{2}(x)$ and $\Delta_{1}(x)$.

Example 6. Coefficients of $\Lambda_{2}(x)$ :

$$
l_{2, k}=\left\{\begin{array}{ll}
2^{2+2-1}\binom{2}{0}=8 & \text { for } k=1 \\
2^{2+2-2}\binom{2}{1}=8 & \text { for } k=2 \\
2^{2+2-3}\binom{2}{2}=2 & \text { for } k=3
\end{array} .\right.
$$

This gives $\Lambda_{2}(x)=2 x^{3}+8 x^{2}+8 x$, which can be confirmed by Lemma 5.6: $\Lambda_{2}(x)=2 x(x+2)^{2}=2 x^{3}+8 x^{2}+8 x$.

Example 7. Coefficients of $\Delta_{1}(x)$ :

$$
d_{, k}=\left\{\begin{array}{ll}
2^{1+2-1}\binom{2}{0}=4 & \text { for } k=1 \\
2^{1+2-2}\binom{2}{1}=4 & \text { for } k=2 \\
2^{1+2-3}\binom{2}{2}=1 & \text { for } k=3
\end{array} .\right.
$$

This gives $\Delta_{1}(x)=x^{3}+4 x^{2}+4 x$, confirmed by Lemma 5.6: $\Delta_{1}(x)=x(x+2)^{2}=$ $x^{3}+4 x^{2}+4 x$.

Recall from Proposition 5.2, the formula $\Gamma_{4}(x)=2 \Gamma_{3}(x)+\Lambda_{2}(x)+x \Delta_{1}(x)$. From our examples, 5,6 , and 7 , we have the following expression for the polynomial $\Gamma_{4}(x)$, proving the result from Lemma 5.1.
$\Gamma_{4}(x)=2\left(x^{3}+10 x^{2}+8 x\right)+\left(2 x^{3}+8 x^{2}+8 x\right)+x\left(x^{3}+4 x^{2}+4 x\right)=x^{4}+8 x^{3}+32 x^{2}+24 x$.

Obviously, since $\Gamma_{n}(x)$ is made from $\Lambda_{n}(x)$ and $\Delta_{n}(x)$, the direct relationship can be shown among the coefficients of each polynomial.

Corollary 6.3. The direct relationship of coefficients between $\Gamma_{n}(x), \Lambda_{n}(x)$, and $\Delta_{n}(x)$ is:

$$
\begin{gather*}
l_{n, k+1}=2 a_{n, k}, \quad \text { for } n \geq k>2 ;  \tag{4}\\
d_{n, k}-l_{n, k}=a_{n, k-2} \quad \text { for } n \geq 4 \text { and } k \geq 5 . \tag{5}
\end{gather*}
$$

Proof. To prove equation (4), we proceed as follows:
From equation (2), we have

$$
l_{n, k+1}=2^{n+1-k}\binom{n}{k}=2\left(2^{n-k}\binom{n}{k}\right)=2 a_{n, k}
$$

To prove equation (5), we proceed as follows:
From equations (2) and (3), we have:

$$
d_{n, k}-l_{n, k}=2^{n+2-k}\left[\binom{n+1}{k-1}-\binom{n}{k-1}\right] ;
$$

Using Pascals Recurrence Relation, which states $\binom{n}{k-1}+\binom{n}{k-2}=\binom{n+1}{k-1}$,

$$
d_{n, k}-l_{n, k}=2^{n+2-k}\left[\binom{n+1}{k-1}-\binom{n}{k-1}\right]=2^{n-(k-2)}\binom{n}{k-2}=a_{n, k-2} .
$$

Concentrating on the coefficients for $\Gamma_{n}(x)$, we notice a few properties. The first observation made is that our leading coefficient, $a_{n, n}$, is always one. Another observation we can directly see is the coefficient of $x^{n-1}$, or $a_{n, n-1}$, is $2 n$, which represents the cardinality of the vertext set, or $\left|V\left(\Gamma_{n}\right)\right|$. This value is also the circumference of $\Gamma_{n}$.

### 6.2 Properties of $\Gamma_{n}(x)$ at Different Values of $x$

Research has been done on the values of interlace polynomials at $x=1$ and -1 [1]. We discuss the importance of these values for $\Gamma_{n}(x)$ in this section. Using the known results from previous papers, we determine some characteristics for $\Gamma_{n}(x)$.

Corollary 6.4. (Known results from [1] and [6] respectively.) Let $G$ be a graph with $n$ vertices.

1. $G(1)=$ number of induced subgraphs of $G$ with an odd number of perfect matchings (including the empty set).
2. $G(3)$ is divisble by $G(-1)$ and the quotient is an odd integer.
3. Let $A$ be the adjacency matrix of $G, n=|V(G)|$ and let $r$ be the rank of the matrix $I+A$ over the field $\mathbb{Z}_{2}$ of two elements. Then

$$
G(-1)=(-1)^{r} 2^{n-r} .
$$

We eveluate $\Gamma_{n}(1)$ and $\Gamma_{n}(-1)$ then correlate the meaning to these results.

Corollary 6.5. The number of induced subgraphs of $\Gamma_{n}$ with an odd number of perfect matchings is $3^{n}-2^{n}$.

The number of induced subgraphs for the complete graph $K_{2 n}$ is $2^{2 n}$. It is obvious that the number of induced subgraphs of $\Gamma_{n}$ is less than the number of induced subgraphs of the complete graph $K_{2 n}$.

Theorem 6.6. Let $A_{n}$ be the adjacency matrix of $\Gamma_{n}$. The matrix $B_{n}=I+A_{n}$ has a full rank, or $\operatorname{rank}\left(B_{n}\right)=2 n$, over $\mathbb{Z}_{2}$.

Proof. Note that $A_{n}, B_{n}$, and $I$ are $2 n \times 2 n$ matrices. Let $r=\operatorname{rank}\left(B_{n}\right)$ over $\mathbb{Z}_{2}$ with $0 \leq r \leq 2 n$.

From Theorem 5.7,

$$
\Gamma_{n}(-1)=2^{n-1}\left(1^{2}-(-1)-2\right)+(-1+2)^{n}=1 \quad \forall n \geq 3
$$

From Corollary 6.4,

$$
1=(-1)^{r}(2)^{2 n-r}
$$

This leaves us with only one solution, $r=2 n$, because 2 cannot divide one. Thus $B_{n}$ is of full rank.

Let us look at the $2 n \times 2 n$ adjacency matrix $A_{n}$ of $\Gamma_{n}$ :

$$
A_{n}=\left[\begin{array}{ccccccccc}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ddots & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & \cdots & 1 & 1 & 0
\end{array}\right]_{2 n \times 2 n}
$$

Since $B_{n}$ has a full rank, we know the determinant of $B_{n} \neq 0$ and the matrix $B_{n}$ is invertible.

### 6.3 Parity of $\Gamma_{n}(x)$

From Corollary 6.4, we know $\Gamma_{n}(3)$ must be an odd integer since $\Gamma_{n}(-1)$ is 1 . Now we determine parity of $x$. Recall $\Gamma_{n}(x)=2^{n-1}\left(x^{2}-x-2\right)+(x+2)^{n}$. For $n \geq 3$, the first term in $\Gamma_{n}(x)$ is a multiple of 2 , resulting in an even number. The parity of the second term depends on $x$, since $x$ is adding to an even number. Any power of an even number stays even and any power of an odd number stays odd. Therefore, the parity of $\Gamma_{n}(x)$ is the same as that of $x$.

Proposition 6.7. $\Gamma_{n}(x)$ is odd if $x$ is odd and $\Gamma_{n}(x)$ is even if $x$ is even.

Lets take a look at a couple values of $x \geq 3$ for $\Gamma_{n}(x)$.

$$
\begin{gathered}
\Gamma_{n}(3)=2^{n}(2)+5^{n} ; \\
\Gamma_{n}(4)=2^{n}(5)+6^{n}=2^{n}(2)+5^{n}+2^{n}(3)+6^{n}-5^{n}=\Gamma_{n}(3)+2^{n}(3)+6^{n}-5^{n} ; \\
\Gamma_{n}(5)=2^{n}(9)+7^{n}=\Gamma_{4}+2^{n}(4)+7^{n}-6^{n} .
\end{gathered}
$$

We can visibily see the pattern for the parity of $\Gamma_{n}(x)$ depends on $x$. Also, from the pattern of $x$ values and values inside the first term, I create a formula for finding the next value of $\Gamma_{n}(x)$ :

$$
\Gamma_{n}(x+1)=\Gamma_{n}(x)+2^{n}(x)+(x+3)^{n}-(x+2)^{n}
$$

It can be proved easily by applying Theorem 5.7.

## 7 Applications

In this section, we show applications of the interlace polynomial towards linear algebra and a related application in biology.

### 7.1 Linear Algebra

If someone was given the matrix $B_{n}=A_{n}+I$, where $A_{n}$ is the adjacency matrix for $\Gamma_{n}$ and $I$ is the identity matrix, and was asked to find the determinant of the matrix, there would be multiple steps to find the solution. By Theorem 6.6, the rank of $B_{n}$ is $2 n$, which is a full rank. One way to find this result using Linear Algebra is by showing the $2 n \times 2 n$ matrix has a nonzero determinant, or $\operatorname{det}\left(B_{n}\right) \neq 0$. Let us examine at the process to show the matrix $B_{5}$, for $\Gamma_{5}$, has a full rank.

The adjacency matrix $A_{5}$ and $I+A_{5}$, respectively, for $\Gamma_{5}$ is:

$$
A_{5}=\left[\begin{array}{llllllllll}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}\right]_{10 \times 10}
$$

$$
B_{5}=\left[\begin{array}{llllllllll}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]_{10 \times 10}
$$

One popular way to find the determinant is to use cofactor expansion multiple times, in order to reduce the matrix into a $3 \times 3$ matrix, or a triangular matrix, to easily compute the determinant.

Definition 10. [11] Let $A \in M_{n \times n}(F)$. For $n \geq 2$, we $\operatorname{define} \operatorname{det}(A)$, or $|A|$, recursively as

$$
|A|=\sum_{j=1}^{n}(-1)^{1+j} A_{1 j} \cdot\left|A_{i j}\right| .
$$

By the cofactor expansion on the first column of $B_{5}$, we obtain:

$$
\left|B_{5}\right|=\operatorname{det}\left(\left[\begin{array}{lllllllll}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]\right)-\operatorname{det}\left(\left[\begin{array}{lllllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]\right)
$$

$$
-\operatorname{det}\left(\left[\begin{array}{lllllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]\right)
$$

In order to have all the matrices reduced to $3 \times 3$ or trianglar matrices, this process would need to be continued multiple times. It is tedious and time consuming. This is where the interlace polynomial for this graph comes extremely useful. For $\Gamma_{5}(-1)$, we use the formula from 6.4 and show that the matrix has a full rank.

$$
\Gamma_{5}(-1)=1=(-1)^{r} 2^{10-r} \Rightarrow r=10, \text { where } r=\operatorname{rank}\left(B_{5}\right) \text { over } \mathbb{Z}_{2} .
$$

Hence we know the $\left|B_{5}\right| \neq 0$ and $B_{5}$ has a full rank. Using Maple Software, we can show $\left|B_{5}\right|=1$, confirming the result.

### 7.2 Biology

As mentioned in Section 2, the study of interlace polynomials grew from trying to reconstruct DNA strings. String reconstruction is the process of reassembling a long string of symbols from a set of its subsequences together with some sequencing information [9]. For example, fragmenting and reassembling messages is a common network protocol, and reconstruction techniques might be applied when the network protocol has been disrupted, yet the original message must be reassembled from the fragments [9]. Sequencing by hybridization is a method of reconstructing a long DNA string from knowledge of its short substrings. Unique reconstruction is not always
possible, and the goal is to study the number of reconstructions of a random string [4]. This is where the interlace polynomial for 4-regular Eulerian digraphs come to play. The different types of reconstructions can be looked at through the different Eulerian circuits in that graph, which the interlace polynomial can tell about. The probability of correctly sequencing the original strand is thus the reciprocal of the total number of Euler circuits in the graph [9].

A de Bruijn graph is a directed graph representing overlaps between sequences of symbols in string reconstruction [5]. A DNA string is represented by a 2 -in 2 -out de Bruijn Graph. Tracing the original DNA sequence of this graph can be represented by a Eulerian Circuit that starts at the vertex representing the beginning and end of the strand [9]. Therefore, once we have the de Bruijn Graph of the DNA sequence, we look at one of the Eulerian Circuits that begins with the orginating vertex in the strand. Then we represent it by a chord diagram, and construct the circle graph from the chord diagram. Given this circle graph, we calculate the interlace polynomial of the graph, and relate it to the Circuit Parition Polynomial to calculate the number of Eulerian Circuits in the orignial graph.

The coefficient of $x$ in a Circuit Partition Polynomial counts the number of Eulerian circuits for the graph. The Circuit Partition Polynomial, represented by $f(G, x)$, where $G$ represents the de Bruijn 2-in 2-out digraph, can be represented by the interlace polynomial of the circle graph, created from the Eulerian Circuit of the 2-in 2-out digraph. We modify the relation to deal with only one component graphs Proposition 7.1. [5] If $G$ is a 2 -in 2-out Eulerian digraph, $C$ is any Eulerian circuit of $G$, and $H$ is the circle grah of the chord diagram determined by $C$, then $f(G, x)=$ $x H_{n}(x+1)$.

This is how the interlace polynomial is applied to DNA sequencing, but unfortunately the graph $\Gamma_{n}$ does not represent a circle graph so it does not represent any type of DNA string. Fortunately, research has shown a modification of the interlace
polynomial can show more properties on induced subgraphs and specifically induced Eulerian subgraphs. This modified interlace polynomial is defined in the next chapter, and also defined explicitly for $\Gamma_{n}$.

## 8 A New Interlace Polynomial

A related interlace polynomial is introduced by Aigner and Holst [1], which tells us a few more distinct properties about the graph. We represent this new interlace polynomial by $Q(G, x)$. The difference between the previous interlace polynomial and $Q(G, x)$ is an additional term in the formula. The graph $G * a$ is obtained from $G$ by interchanging edges $\leftrightarrow$ non-edges in the neighborhood of $a$ [1]. The modified definition for $Q(G, x)$ is defined below.

Definition 11. [1] Let $G$ be a simple graph, where $G=\{V, E\}$. The $Q$-interlace polynomial, $Q(G, x)$ is given by:

$$
\begin{gathered}
Q(G, x)=Q(G \backslash\{a\}, x)+Q(G * a \backslash\{a\}, x)+Q\left(G^{(a b)} \backslash\{b\}, x\right) \text { where } a, b \in V(G) \\
\text { and } a b \in E(G) .
\end{gathered}
$$

As we see, respectively the first and last term for $Q(G, x)$ follow the same process as our previous interlace polynomial. The additional term is what makes the new interlace polynomial different. An example is shown below.

Example 8. Consider a graph $G$ and a vertex a of $G$ shown below.


The neighborhood of $a$, or $N(a)$, is $\{b, d\}$, and $b d \in E(G)$. For $G * a$, we must interchange the edges $\leftrightarrow$ non-edges within the neighborhood of $a$. Therefore, the edge $b d$ does not exist in $G * a$, shown below.
$G * a$ :


In order to find $Q\left(\Gamma_{n}, x\right)$, we needed to work with more graphs, and the method was a lot more tedious. The formula is more complicated than the previous polynomial, but can still be used to calculate an explicit $Q\left(\Gamma_{n}, x\right)$. Below we discuss the breakdown of $\Gamma_{n}$ using definition 11.

### 8.1 The Q-Interlace Polynomial $Q\left(\Gamma_{n}, x\right)$

In order to avoid confusion, in this section, we write $Q\left(G_{n}\right)$ for any graph $G$, to represent the Q-interlace polynomial $Q\left(G_{n}, x\right)$.

Theorem 8.1. The Q-Interlace Polynomials for $\Gamma_{3}, \Gamma_{4}$, and $\Gamma_{5}$ are:

1. $Q\left(\Gamma_{3}, x\right)=2 x^{3}+47 x^{2}+84 x$;
2. $Q\left(\Gamma_{4}, x\right)=2 x^{4}+64 x^{3}+363 x^{2}+468 x$;
3. $Q\left(\Gamma_{5}, x\right)=2 x^{5}+100 x^{4}+800 x^{3}+2421 x^{2}+2388 x$.

Before introducing the recursive formula for $Q\left(\Gamma_{n}\right)$, I define a new graph we will see in our breakdown. If we take the graph $\Gamma_{n}$ and eliminate any two edges, not in $C_{n}$, that form the same $C_{3}$ graph around the perimeter of $C_{n}$, we are left with $\Omega_{n-1}$.

Definition 12. The graph $\Omega_{n}=\left(V\left(\Gamma_{n}\right) \backslash\left\{v_{2}\right\}, E\left(\Gamma_{n}\right) \backslash\left\{v_{1} v_{2}, v_{2} v_{3}\right\}\right)$.


Figure 11: $\Omega_{n}$ with $n C_{3}$ graphs.

We begin by giving the initial recursive formula for $Q\left(\Gamma_{n}\right)$.
Lemma 8.2. The recursive formula for $Q\left(\Gamma_{n}\right)$, for $n \geq 4$, is:

$$
Q\left(\Gamma_{n}\right)=2 Q\left(\Gamma_{n-1}\right)+Q\left(\Omega_{n-1}\right)+Q\left(M_{n-1}\right)+x Q\left(\Delta_{n-3}\right)
$$

Proof. Breaking down $\Gamma_{n}$ with respect to definition 11:


Figure 12: Respectively from left to right, the graph $\Omega_{n-1}, M_{n-1}, \Gamma_{n-1}, \Gamma_{n-1}$, and $x \Delta_{n-3}$.

In the recursive formula for $Q\left(\Gamma_{n}\right)$, we deal with two graphs that were defined in the previous sections, $M_{n}$ and $\Delta_{n}$. Also, when using the toggling process for these two graphs, we achieve another familiar graph, $\Lambda_{n}$. During the devlopment for an explicit formula for $Q\left(\Gamma_{n}\right)$, I use the $Q$-interlace polynomials at specific values of $n$, for $\Lambda_{n}, M_{n}, \Delta_{n}$, and $\Omega_{n}$.

Lemma 8.3. $Q$-interlace polynomials at specific $n$ :

1. $Q\left(\Lambda_{0}\right)=3 x$.
2. $Q\left(\Lambda_{1}\right)=5 x^{2}+12 x$.
3. $Q\left(\Lambda_{2}\right)=7 x^{3}+44 x^{2}+48 x$.
4. $Q\left(M_{0}\right)=x$
5. $Q\left(M_{1}\right)=x^{2}+6 x$.
6. $Q\left(M_{2}\right)=x^{3}+16 x^{2}+24 x$.
7. $Q\left(\Delta_{1}\right)=x^{3}+16 x^{2}+24 x$.
8. $Q\left(\Delta_{2}\right)=x^{4}+30 x^{3}+112 x^{2}+96 x$.
9. $Q\left(\Omega_{2}\right)=15 x^{2}+36 x$.
10. $Q\left(\Omega_{3}\right)=14 x^{3}+133 x^{2}+204 x$.

We start by working for an explicit formula for $Q\left(\Lambda_{n}\right)$.
Lemma 8.4. The recursive formula for $Q\left(\Lambda_{n}\right)$, with $n \geq 1$, is:

$$
Q\left(\Lambda_{n}\right)=x Q\left(\Lambda_{n-1}\right)+2 Q\left(M_{n}\right) .
$$

Proof. Looking at $Q\left(\Lambda_{n}\right)$ :


Obviously, the statement is true.

Since we have the graph $M_{n}$ inside our recursive formula, let us take a look at the recursive formula for $Q\left(M_{n}\right)$ and try to relate them.

Lemma 8.5. The recursive formula for $Q\left(M_{n}\right)$, with $n \geq 1$, is:

$$
\begin{equation*}
Q\left(M_{n}\right)=x Q\left(M_{n-1}\right)+2 Q\left(\Lambda_{n-1}\right) . \tag{6}
\end{equation*}
$$

Proof. Looking at $Q\left(M_{n}\right)$ :


Obviously, the statement is true.

We substitute equation 6 into $Q\left(\Lambda_{n}\right)$ and solve for $Q\left(M_{n-1}\right)$.

$$
\begin{gathered}
Q\left(\Lambda_{n}\right)=x Q\left(\Lambda_{n-1}\right)+4 Q\left(\Lambda_{n-1}\right)+2 x Q\left(M_{n-1}\right) ; \\
Q\left(M_{n-1}\right)=\frac{Q\left(\Lambda_{n}\right)-(x+4) Q\left(\Lambda_{n-1}\right)}{2 x}
\end{gathered}
$$

Now plug this into the recursive formula for $Q\left(\Lambda_{n}\right)$.

$$
\begin{gathered}
Q\left(\Lambda_{n}=x Q\left(\Lambda_{n-1}\right)+2\left(\frac{Q\left(\Lambda_{n+1}\right)-(x+4) Q\left(\Lambda_{n}\right)}{2 x}\right)\right. \\
x Q\left(\Lambda_{n}\right)=x^{2} Q\left(\Lambda_{n-1}\right)+Q\left(\Lambda_{n+1}\right)-(x+4) Q\left(\Lambda_{n}\right)
\end{gathered}
$$

$$
\begin{equation*}
Q\left(\Lambda_{n+1}\right)=(2 x+4) Q\left(\Lambda_{n}\right)-x^{2} Q\left(\Lambda_{n-1}\right) . \tag{7}
\end{equation*}
$$

We represent this recurrence relation by its characteristic equation and solve for the roots, shown below.

$$
y^{2}-(2 x+4) y+x^{2}=0 \quad \Rightarrow \quad y=\frac{2 x+4 \pm \sqrt{(2 x+4)^{2}-4 x^{2}}}{2} .
$$

The solutions to the characteristic equation are:

$$
y=x+2 \pm 2 \sqrt{x+1}
$$

Definition 13. The roots of the characteristic equation for $Q\left(\Lambda_{n}\right)$ are defined as:

$$
\begin{equation*}
y_{1}(x)=x+2+2 \sqrt{x+1}, \quad y_{2}(x)=x+2-2 \sqrt{x+1} . \tag{8}
\end{equation*}
$$

We use these roots to express $\Lambda_{n}$ explicitly, shown below.

$$
Q\left(\Lambda_{n}\right)=c_{1}(x)\left(y_{1}(x)\right)^{n}+c_{2}(x)\left(y_{2}(x)\right)^{n} .
$$

We know the values for $Q\left(\Lambda_{0}\right)$ and $Q\left(\Lambda_{1}\right)$, so we use them to find our values for the two coefficient functions $c_{1}(x)$ and $c_{2}(x)$.

$$
\begin{gathered}
3 x=c_{1}+c_{2} \\
5 x^{2}+12 x=c_{1}(x+2+2 \sqrt{x+1})+c_{2}(x+2-2 \sqrt{x+1})
\end{gathered}
$$

Solving this set of linear equations, I find the values for our coefficient functions of $x$.

$$
\begin{equation*}
c_{1}(x)=\frac{x^{2}+3 x+3 x \sqrt{x+1}}{2 \sqrt{x+1}} \tag{9}
\end{equation*}
$$

$$
\begin{equation*}
c_{2}(x)=\frac{-x^{2}-3 x+3 x \sqrt{x+1}}{2 \sqrt{x+1}} . \tag{10}
\end{equation*}
$$

In order to simplify the explicit formula for $Q\left(\Lambda_{n}\right)$, I determine relations between the coefficient functions and the roots to the characteristic eqution.

Proposition 8.6. Relationship between $y_{1}(x), y_{2}(x), c_{1}(x)$, and $c_{2}(x)$.

$$
\begin{aligned}
c_{1}(x) & =\frac{3 x \sqrt{x+1}+x^{2}+3 x}{2 \sqrt{x+1}}, \quad c_{2}(x)=\frac{3 x \sqrt{x+1}-x^{2}-3 x}{2 \sqrt{x+1}} ; \\
y_{1}(x)+y_{2}(x) & =2 x+4, \quad y_{1}(x) y_{2}(x)=x^{2} ; \\
y_{1}(x)-y_{2}(x) & =4 \sqrt{x+1} \quad \text { and } \quad\left(\left(y_{1}(x)-x\right)\left(y_{2}(x)-x\right)=-4 x ;\right. \\
c_{1}(x)+c_{2}(x) & =3 x \quad \text { and } \quad c_{1}(x) y_{2}(x)+c_{2}(x) y_{1}(x)=x^{2} ; \\
\Lambda_{n}(x) & =c_{1}(x)\left(y_{1}(x)\right)^{n}+c_{2}(x)\left(y_{1}(x)\right)^{n} .
\end{aligned}
$$

Notice that since $c_{1}(x)$ and $c_{2}(x)$ are fractions, we must make sure the denominator is never zero. It is easy to see that when $x=-1$, the denominators are zero. $Q\left(\Lambda_{n},-1\right)$ can be obtained seperately:

$$
\begin{gathered}
y_{1}(-1)=(-1)+2+0=1 \\
y_{2}(-1)=(-1)+2-0=1 \\
y_{1}=1=y_{2}
\end{gathered}
$$

Since the roots are the same, the solution looks like:

$$
\begin{gathered}
Q\left(\Lambda_{n},(-1)\right)=c_{3}(-1)(1)^{n}+c_{4}(-1) n(1)^{n} \\
Q\left(\Lambda_{n},(-1)\right)=c_{3}(-1)+c_{4}(-1) n
\end{gathered}
$$

Applying values at $Q\left(\Lambda_{1},-1\right)$ and $Q\left(\Lambda_{2},-1\right)$ :

$$
\begin{gathered}
Q\left(\Lambda_{1},(-1)\right)=-7=c_{3}+c_{4} \\
Q\left(\Lambda_{2},(-1)\right)=-11=c_{3}+2 c_{4}
\end{gathered}
$$

Solving this system of linear equations, we obtain:

$$
c_{3}(-1)=-3 \text { and } c_{4}(-1)=-4
$$

Theorem 8.7. The interlace polynomial $Q\left(\Lambda_{n}, x\right)$ is:

1. $Q\left(\Lambda_{n}, x\right)=\sum_{m=0}^{\lfloor n / 2\rfloor} 4^{m}(x+1)^{m}(x+2)^{n-2 m-1}\left[\left(3 x^{2}+6 x\right)\binom{n}{2 m}+\left(2 x^{2}+6 x\right)\binom{n}{2 m+1}\right]$ for $x \neq-1$;
2. $Q\left(\Lambda_{n},(-1)\right)=-3-4 n$.

Proof. Note that

$$
\begin{aligned}
& Q\left(\Lambda_{n}\right)=c_{1}(x) y_{1}^{n}(x)+c_{2}(x) y_{2}^{n}(x) \\
&=\frac{3 x \sqrt{x+1}+x^{2}+3 x}{2 \sqrt{x+1}} \cdot y_{1}^{n}(x)+\frac{3 x \sqrt{x+1}-x^{2}-3 x}{2 \sqrt{x+1}} \cdot y_{2}^{n}(x) \\
&=\frac{3 x}{2}\left(y_{1}^{n}(x)+y_{2}^{n}(x)\right)+\frac{x^{2}+3 x}{2 \sqrt{x+1}}\left(y_{1}^{n}(x)-y_{2}^{n}(x)\right) \\
& y_{1}^{n}(x)=(x+2+2 \sqrt{x+1})^{n}=\sum_{k=0}^{n}\binom{n}{k}(x+2)^{n-k} 2^{k}(x+1)^{k / 2}, \\
& y_{2}^{n}(x)=(x+2-2 \sqrt{x+1})^{n}=\sum_{k=0}^{n}\binom{n}{k}(x+2)^{n-k} 2^{k}(-1)^{k}(x+1)^{k / 2},
\end{aligned}
$$

Thus set $k=2 m$.

$$
\begin{aligned}
y_{1}^{n}(x)+y_{2}^{n}(x) & =\sum_{k=0}^{n}\binom{n}{k}\left(1+(-1)^{k}\right)(x+2)^{n-k} 2^{k}(x+1)^{k / 2} \\
& =2 \sum_{m=0}^{\lfloor n / 2\rfloor}\binom{n}{2 m} 4^{m}(x+1)^{m}(x+2)^{n-2 m} .
\end{aligned}
$$

Similarly, let $k=2 m+1$.

$$
\begin{aligned}
y_{1}^{n}(x)-y_{2}^{n}(x) & =\sum_{k=0}^{n}\binom{n}{k}\left(1-(-1)^{k}\right)(x+2)^{n-k} 2^{k}(x+1)^{k / 2} \\
& =4(x+1)^{(1 / 2)} \sum_{m=0}^{\lfloor n / 2\rfloor}\binom{n}{2 m+1} 4^{m}(x+1)^{m}(x+2)^{n-2 m-1}
\end{aligned}
$$

From here you can combine the above and obtain the formula in Theorem 8.7. Note that equation (2) in Theorem 8.7 can be proven by looking at the development prior to the theorem.

We now use the explicit formula for $Q\left(\Lambda_{n}\right)$ to find the explicit formulas for $Q\left(M_{n}\right)$ and $Q\left(\Delta_{n}\right)$. Recall, from Lemma 8.4, we have

$$
\begin{equation*}
Q\left(M_{n}\right)=\frac{Q\left(\Lambda_{n}\right)-x Q\left(\Lambda_{n-1}\right)}{2} \tag{11}
\end{equation*}
$$

Theorem 8.8. The interlace polynomial $Q\left(M_{n}, x\right)$, for $n \geq 1$, with $y_{1}(x)$ and $y_{2}(x)$ given by equation 8 is:

1. $Q\left(M_{n}\right)=\left[\left(x^{2}+6 x\right)+\frac{4 x^{2}+6 x}{\sqrt{x+1}}\right] \cdot y_{1}^{n-1}(x)+\left[\left(x^{2}+6 x\right)-\frac{4 x^{2}+6 x}{\sqrt{x+1}}\right] \cdot y_{2}^{n-1}(x)$ for $x \neq-1 ;$
2. $Q\left(M_{n},(-1)\right)=-1-4 n$.

Proof. 1.
Note that

$$
y_{1}-x=2(1+\sqrt{x+1}) \text { and } y_{2}-x=2(1-\sqrt{x+1}) .
$$

Then

$$
\begin{aligned}
Q\left(M_{n}\right)= & \frac{Q\left(\Lambda_{n}\right)-x Q\left(\Lambda_{n-1}\right)}{2} \\
= & \frac{c_{1}(x) y_{1}^{n}(x)+c_{2}(x) y_{2}^{n}(x)-x c_{1}(x) y_{1}^{n-1}(x)-x c_{2}(x) y_{2}^{n-1}}{2} \\
= & \frac{c_{1}(x) y_{1}^{n-1}(x)\left(y_{1}(x)-x\right)}{2}+\frac{c_{2}(x) y_{2}^{n-1}(x)\left(y_{2}(x)-x\right)}{2} \\
= & \frac{\left(3 x \sqrt{x+1}+x^{2}+3 x\right)(1+\sqrt{x+1})}{2 \sqrt{x+1}} \cdot y_{1}^{n-1}(x) \\
& +\frac{\left(3 x \sqrt{x+1}-x^{2}-3 x\right)(1-\sqrt{x+1})}{2 \sqrt{x+1}} \cdot y_{2}^{n-1}(x) \\
= & \frac{\left(x^{2}+6 x\right) \sqrt{x+1}+4 x^{2}+6 x}{\sqrt{x+1}} \cdot y_{1}^{n-1}(x) \\
& +\frac{\left(x^{2}+6 x\right) \sqrt{x+1}-4 x^{2}-6 x}{\sqrt{1+x}} \cdot y_{2}^{n-1}(x) .
\end{aligned}
$$

Proof. 2. By mathematical induction. Assume $x=-1$.

Our inital case for $n=1$ holds true.

Since $Q\left(M_{n}, x\right)=x^{2}+6 x$,

$$
Q\left(M_{1},-1\right)=(-1)^{2}+6(-1)=-5=-1-4(1)
$$

Now assume $Q\left(M_{n-1},-1\right)=-1-4(n-1)$ is true. Recall that $Q\left(\Lambda_{n}, x\right)=-3-4 n$.

$$
\begin{aligned}
Q\left(M_{n},-1\right) & =(-1) Q\left(M_{n-1},-1\right)+2 Q\left(\Lambda_{n-1}\right) \\
& =-1(-1-4(n-1))+2(-3-4(n-1))=-1-4 n
\end{aligned}
$$

Thus $Q\left(M_{n},-1\right)=-1-4 n$ for all $n \geq 1$.

We now develop a formula for $\sum_{i=0}^{n} x^{k} Q\left(\Lambda_{n-k}\right)$.
Lemma 8.9. For all $n \geq 0$,

$$
\sum_{i=0}^{n} x^{n-k} Q\left(\Lambda_{k}\right)=\frac{1}{4}\left[Q\left(\Lambda_{n+1}\right)-x Q\left(\Lambda_{n}\right)-2 x^{n+1}\right]
$$

Proof. By the recursive formula $Q\left(M_{n}\right)-x Q\left(M_{n-1}\right)=2 Q\left(\Lambda_{n-1}\right)$, for $n \geq 0$,

$$
\begin{aligned}
Q\left(M_{n+1}\right)-x Q\left(M_{n}\right) & =2 Q\left(\Lambda_{n}\right) \\
x Q\left(M_{n}\right)-x^{2} Q\left(M_{n-1}\right) & =2 x Q\left(\Lambda_{n-1}\right) \\
x^{2} Q\left(M_{n-1}\right)-x^{3} Q\left(M_{n-2}\right) & =2 x^{2} Q\left(\Lambda_{n-2}\right) \\
& \vdots \\
x^{n} Q\left(M_{1}\right)-x^{n+1} Q\left(M_{0}\right) & =2 x^{n} Q\left(\Lambda_{0}\right)
\end{aligned}
$$

Add the above equations we obtain:

$$
Q\left(M_{n+1}\right)-x^{n+1} Q\left(M_{0}\right)=2 \sum_{i=0}^{n} x^{n-k} Q\left(\Lambda_{k}\right) \Longrightarrow Q\left(M_{n+1}\right)=x^{n+2}+2 \sum_{i=0}^{n} x^{k} Q\left(\Lambda_{n-k}\right) .
$$

But from equation 11, $Q\left(M_{n+1}\right)=\frac{Q\left(\Lambda_{n+1}\right)-x Q\left(\Lambda_{n}\right)}{2} \Longrightarrow$

$$
\frac{Q\left(\Lambda_{n+1}\right)-x Q\left(\Lambda_{n}\right)}{2}=x^{n+2}+2 \sum_{i=0}^{n} x^{n-k} Q\left(\Lambda_{k}\right)
$$

Which derives the result.

The recursive formula for $Q\left(\Delta_{n}\right)$ is given by the following lemma.

Lemma 8.10. The recursive formula for $Q\left(\Delta_{n}, x\right)$ with $n \geq 1$ is:

$$
Q\left(\Delta_{n}, x\right)=x Q\left(\Delta_{n-1}, x\right)+2 Q\left(\Lambda_{n}\right)
$$

Define $\Delta_{0}=x^{2}+6 x$. Recall the recursive formulas: $Q\left(M_{n}\right)=x Q\left(M_{n-1}\right)+$ $2 Q\left(\Lambda_{n-1}\right)$ for $n \geq 1$, and $2 Q\left(\Lambda_{n}\right)=Q\left(M_{n+1}\right)-x Q\left(M_{n}\right)$. We claim:

Theorem 8.11. For $n \geq 0$,

1. $Q\left(\Delta_{n}\right)=Q\left(M_{n+1}\right)=\frac{Q\left(\Lambda_{n+1}\right)-x Q\left(\Lambda_{n}\right)}{2}$ for $x \neq-1$;
2. $Q\left(\Delta_{n},(-1)\right)=-5-4 n$.

Proof. 1. By mathematical induction.

Obviously $Q\left(\Delta_{0}\right)=Q\left(M_{1}\right)$. Assume $Q\left(\Delta_{n-1}\right)=Q\left(M_{n}\right)$ for $n>1$. Then

$$
Q\left(\Delta_{n}\right)=x Q\left(\Delta_{n-1}\right)+2 Q\left(\Lambda_{n}\right)=x Q\left(M_{n}\right)+Q\left(M_{n+1}\right)-x Q\left(M_{n}\right)=Q\left(M_{n+1}\right) .
$$

Proof. 2. By mathematical induction. Assume $x=-1$.

Our initial value for $n=1$ holds true.

Since $Q\left(\Delta_{n}, x\right)=x^{3}+16 x^{2}+24 x$,

$$
Q\left(\Delta_{1},-1\right)=(-1)^{3}+16(-1)^{2}+24(-1)=-9=-5-4(1) .
$$

Now assume $Q\left(\Delta_{n},-1\right)$ is true for $n-1$.

$$
\begin{aligned}
& Q\left(\Delta_{n},-1\right)=(-1) Q\left(\Delta_{n-1},-1\right)+2 Q\left(\Lambda_{n},-1\right) \\
& \quad=-1(-5-4(n-1))+2(-1-4(n))=-5-4 n . \\
& \therefore \quad Q\left(\Delta_{n},-1\right)=-5-4 n \text { for all } n \geq 1 .
\end{aligned}
$$

Now we look at our last recursive relation in order to form an explicit formula for $Q\left(\Gamma_{n}, x\right)$.

Lemma 8.12. The recursive formula for $Q\left(\Omega_{n}, x\right)$, with $n \geq 2$, is:

$$
\begin{equation*}
Q\left(\Omega_{n}\right)=Q\left(\Gamma_{n}\right)+Q\left(\Lambda_{n-1}\right)+2 Q\left(\Omega_{n-1}\right)+x Q\left(\Lambda_{n-2}\right) \tag{12}
\end{equation*}
$$

The idea of the proof can be expressed by looking at $Q\left(\Omega_{4}\right)$ shown below.

Example 9. Breaking down $Q\left(\Omega_{4}\right)$.


After one step, from left to right, we have $\Lambda_{3}, \Omega_{3}, \Omega_{3}, x \Lambda_{2}$, and $\Gamma_{4}$, respectively.
Similar to the recursive relation between $Q\left(\Lambda_{n}\right)$ and $Q\left(M_{n}\right)$, we use the same technique for $Q\left(\Omega_{n}\right)$ and $Q\left(\Gamma_{n}\right)$.

Define

$$
H_{n}(x)=Q\left(\Lambda_{n}\right)+(x-2) Q\left(\Lambda_{n-1}\right)-2 x Q\left(\Lambda_{n-2}\right)+Q\left(M_{n}\right)+x Q\left(\Delta_{n-2}\right) .
$$

We simplify $H_{n}(x)$ using the previous formulas.

$$
\begin{equation*}
H_{n}=2(x+1)\left[Q\left(\Lambda_{n-1}\right)-x Q\left(\Lambda_{n-2}\right)\right], \quad n \geq 3 \tag{13}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
Q\left(\Omega_{n+1}\right)-Q\left(\Omega_{n}\right)=4\left(Q\left(\Omega_{n}\right)-Q\left(\Omega_{n-1}\right)\right)+H_{n}(x) \tag{14}
\end{equation*}
$$

In order to achieve an explicit formula for $Q\left(\Omega_{n}\right)$, we define the following:

$$
\begin{equation*}
V_{n}(x)=Q\left(\Omega_{n+1}\right)-Q\left(\Omega_{n}\right), \quad n \geq 2 \tag{15}
\end{equation*}
$$

We substitute equation (15) into equation (14) to result in the following.

$$
\begin{equation*}
V_{n}(x)=4 V_{n-1}(x)+H_{n}(x), \quad n \geq 2 \tag{16}
\end{equation*}
$$

We develop an explicit formula for $Y_{n}(x)$ using similar techniques from the previous polynomials. Note that the smallest value of $n$ for $Y_{n}(x)$ is 2 since $\Omega_{2}$ is the smallest graph of its kind.

$$
\begin{gather*}
V_{2}(x)=14 x^{3}+118 x^{2}+168 x  \tag{17}\\
V_{n}(x)=4^{n-2} V_{2}(x)+\sum_{j=3}^{n} 4^{n-j} H_{j}(x), \quad n \geq 3 . \tag{18}
\end{gather*}
$$

Now we use equation (15) to develop an explicit function for $Q\left(\Omega_{n}\right)$. Again we use similar techniques and we also use equation (18) to substitute into the function to simplify further and achieve the final function. Some steps are shown below.

$$
\begin{aligned}
Q\left(\Omega_{n}\right) & =Q\left(\Omega_{2}\right)+V_{2}(x)+\sum_{i=3}^{n-1} V_{i}(x) ; \\
& =14 x^{3}+133 x^{2}+204 x+\sum_{i=3}^{n-1}\left(4^{i-2} V_{2}(x)+\sum_{j=3}^{i} 4^{i-j} H_{j}(x)\right) ; \\
& =14 x^{3}+133 x^{2}+204 x+\frac{4^{n-2}-4}{3}\left(14 x^{3}+118 x^{2}+168 x\right)+\sum_{i=3}^{n-1} \sum_{j=3}^{i} 4^{i-j} H_{j}(x) ; \\
& =14 x^{3}+133 x^{2}+204 x+\frac{4^{n-2}-4}{3}\left(14 x^{3}+118 x^{2}+168 x\right)+\frac{1}{3} \sum_{i=3}^{n-1}\left(4^{n-i}-1\right) H_{i}(x) .
\end{aligned}
$$

Theorem 8.13. The explicit function $Q\left(\Omega_{n}, x\right)$, where $H_{n}(x)$ is given by equation (13) is:

1. $Q\left(\Omega_{n}, x\right)$
$=14 x^{3}+133 x^{2}+204 x+\frac{4^{n-2}-4}{3}\left(14 x^{3}+118 x^{2}+168 x\right)+\frac{1}{3} \sum_{i=3}^{n-1}\left(4^{n-i}-1\right) H_{i}(x)$, for $n \geq 4$ and $x \neq-1$;
2. $Q\left(\Omega_{n},-1\right)=\frac{1}{3}\left(1-(4)^{n+1}\right)$ for $n \geq 2$.

Proof. 1. By mathematical induction. Assume $x \neq-1$.

Our initial conditions for $n=4$ and $n=5$ can be confirmed by Mathematica.
$Q\left(\Omega_{4}\right)=18 x^{4}+240 x^{3}+949 x^{2}+1068 x ;$
$Q\left(\Omega_{5}\right)=22 x^{5}+440 x^{4}+2464 x^{3}+5983 x^{2}+5292 x$.

Now assume $Q\left(\Omega_{n}\right)$ and $Q\left(\Omega_{n-1}\right)$ are true.

$$
\begin{aligned}
Q\left(\Omega_{n+1}\right)= & 5 Q\left(\Omega_{n}\right)-4 Q\left(\Omega_{n-1}\right)+H_{n}(x) \\
= & (5-4)\left(14 x^{3}+133 x^{2}+204 x\right) \\
& +\left(5\left(\frac{4^{n-2}-4}{3}\right)-4\left(\frac{4^{n-3}-4}{3}\right)\right)\left(14 x^{3}+118 x^{2}+168 x\right) \\
& +\frac{5}{3}\left(\sum_{i=3}^{n-1}\left(4^{n-i}-1\right) H_{i}(x)\right)-\frac{4}{3}\left(\sum_{i=3}^{n-2}\left(4^{n-1-i}-1\right) H_{i}(x)\right)+H_{n}(x) \\
= & 14 x^{3}+133 x^{2}+204 x+\frac{4^{n-1}-4}{3}\left(14 x^{3}+118 x^{2}+168 x\right) \\
& +\frac{1}{3}\left(\sum_{i=3}^{n-2}\left[5\left(4^{n-i}-1\right)-4\left(4^{n-1-i}-1\right)\right] H_{i}(x)\right)+\frac{5}{3}(4-1) H_{n-1}(x) \\
& +H_{n}(x) \\
= & 14 x^{3}+133 x^{2}+204 x+\frac{4^{n-1}-4}{3}\left(14 x^{3}+118 x^{2}+168 x\right) \\
& +\frac{1}{3}\left(\sum_{i=3}^{n-2}\left(4^{n+1-i}-1\right) H_{i}(x)\right)+5 H_{n-1}(x)+H_{n}(x) .
\end{aligned}
$$

For this to be true for $Q\left(\Omega_{n+1}\right)$, we need $5 H_{n-1}+H_{n}$ to be equivalent to $\frac{1}{3} \sum_{i=n-1}^{n}\left(4^{n+1-i}-1\right) H_{i}$.

$$
\begin{aligned}
\frac{1}{3} \sum_{i=n-1}^{n}\left(4^{n+1-i}-1\right) H_{i}(x) & =\frac{1}{3}\left(4^{n+1-(n-1)}-1\right) H_{n-1}+\frac{1}{3}\left(4^{n+1-n}-1\right) H_{n} \\
& =\frac{1}{3}\left(4^{2}-1\right) H_{n-1}(x)+\frac{1}{3}(4-1) H_{n}(x) \\
& =5 H_{n-1}(x)+H_{n}(x) .
\end{aligned}
$$

$\therefore \quad Q\left(\Omega_{n}, x\right)$ is true for all $n \geq 4$.

Proof. 2. By mathematical induction.
Our initial case for $n=2$ holds true.
$Q\left(\Omega_{2},-1\right)=\frac{1}{3}\left(1-(4)^{3}\right)=-21$.
Now assume $Q\left(\Omega_{n-1},-1\right)$ is true, and from equations (15) and (18) we have,
$Q\left(\Omega_{n},-1\right)=Q\left(\Omega_{n-1},-1\right)+V_{n-1}(-1)$ and $V_{n}(x)=4^{n-2} V_{2}(x)+\sum_{j=3}^{n} 4^{n-j} H_{j}(x)$.
Note that for $H_{j}(x)=2(x+1)\left[2 Q\left(\Lambda_{j-1}-x Q\left(\Lambda_{j-2}\right)\right]\right.$. Clearly $H_{j}(-1)=0$.
$Q\left(\Omega_{n},-1\right)=\frac{1}{3}\left(1-(4)^{n}\right)+4^{n-3}(-64)=\frac{1}{3}\left(1-(4)^{n+1}\right)$.
$\therefore \quad Q\left(\Omega_{n},-1\right)$ holds true for all $n \geq 2$.

Finally, we go back to the recursive function from Lemma 8.2 and define one more function in order to finalize an explicit function for $Q\left(\Gamma_{n}, x\right)$.

$$
R_{n}(x)=Q\left(\Omega_{n-1}\right)+Q\left(M_{n-1}\right)+x Q\left(\Delta_{n-3}\right)
$$

We use our previous formulas to simplify $R_{n}$.

$$
\begin{equation*}
R_{n}(x)=Q\left(\Omega_{n-2}\right)+\frac{Q\left(\Lambda_{n-1}\right)-x^{2} Q\left(\Lambda_{n-3}\right)}{2} \tag{19}
\end{equation*}
$$

Using the function from Lemma 8.2 and equation (19), we express $Q\left(\Gamma_{n}\right)$ as shown below.

$$
\begin{equation*}
Q\left(\Gamma_{n}\right)=2 Q\left(\Gamma_{n-1}\right)+R_{n}(x), \quad \text { for } n \geq 4 \tag{20}
\end{equation*}
$$

We apply similar techniques used previously to achieve the explicit function.

$$
Q\left(\Gamma_{n}\right)=2^{n-4} Q\left(\Gamma_{4}\right)+\sum_{i=5}^{n} 2^{n-i} R_{i}(x), \quad n \geq 5
$$

Theorem 8.14. The interlace polynomial $Q\left(\Gamma_{n}, x\right)$, with $R_{n}(x)$ given by equation (19), for $n \geq 5$ is:

1. $Q\left(\Gamma_{n}, x\right)=2^{n-4}\left(2 x^{4}+64 x^{3}+363 x^{2}+468 x\right)+\sum_{i=5}^{n} 2^{n-i} R_{i}(x)$;
2. $Q\left(\Gamma_{n},-1\right)=\frac{1}{3}\left(11-2^{2 n+1}\right)$

Proof. 1. By mathematical induction. When $x \neq-1$.

A confirmation with Mathematica shows our initial condition for $n=5$ holds true.

$$
Q\left(\Gamma_{5}, x\right)=2 x^{5}+100 x^{4}+800 x^{3}+2411 x^{2}+2388 x
$$

Now assume $Q\left(\Gamma_{n-1}, x\right)$ is true.

$$
\begin{aligned}
Q\left(\Gamma_{n}, x\right) & =2 Q\left(\Gamma_{n-1}\right)+R_{n}(x) \\
& =2\left[2^{n-5}\left(2 x^{4}+64 x^{3}+363 x^{2}+468 x\right)+\sum_{i=5}^{n-1} 2^{n-1-i} R_{i}(x)\right]+R_{n}(x) \\
& =2^{n-4}\left(2 x^{4}+64 x^{3}+363 x^{2}+468 x\right)+\sum_{i=5}^{n-1} 2^{n-i} R_{i}(x)+R_{n}(x) \\
& =2^{n-4}\left(2 x^{4}+64 x^{3}+363 x^{2}+468 x\right)+\sum_{i=5}^{n} 2^{n-i} R_{i}(x)
\end{aligned}
$$

Therefore, $Q\left(\Gamma_{n}, x\right)$ is true for all $n \geq 5$.

Proof. 2. By mathematical induction.

The initial case for $n=5$ is confirmed below.
$Q\left(\Gamma_{5},-1\right)=\frac{1}{3}\left(11-2^{11}\right)=-679$.
Now assume $Q\left(\Gamma_{n-1},-1\right)$ is true, and from equation (20) we have,
$Q\left(\Gamma_{n},-1\right)=2 Q\left(\Gamma_{n-1},-1\right)+R_{n}(-1)$.
We use equation (19) to determine $R_{n}(-1)$.

$$
\begin{aligned}
& \begin{aligned}
& Q\left(\Gamma_{n},-1\right)=2\left(\frac{1}{3}\left(11-2^{2 n-1}\right)+\frac{1}{3}\left(1-4^{n}\right)-4 ;\right. \\
&=\frac{22-2^{2 n}+1-2^{2 n}-12}{3} ; \\
&=\frac{1}{3}\left(11-2^{2 n+1}\right) . \\
& \therefore \quad Q\left(\Gamma_{n},-1\right)=\frac{1}{3}\left(11-2^{2 n+1}\right) \text { for all } n \geq 5 .
\end{aligned} .
\end{aligned}
$$

### 8.2 Applications of $Q\left(\Gamma_{n}, x\right)$

Study has been done specifically on $x$ values of 2 and 4 for the interlace polynomial $Q(G, x) . Q(G, 2)$ equals the number of general induced subgraphs of G (with possible loops attached to the vertices) with an odd number of general perfect matchings [1]. $Q(G, 4)$ equals $2 n$ times the number of induced Eulerian subgraphs of $G$ [1]. Below we will relate these specific values to our graph $\Gamma_{n}$.

Corollary 8.15. The number of general induced subgraphs for a few $\Gamma_{n}$ (with possible loops attached to the vertices), with an odd number of general perfect matchings are defined below:

1. For $\Gamma_{3}$, we have 372 general induced subgraphs with an odd number of general perfect matchings;
2. For $\Gamma_{4}$, we have 2932 general induced subgraphs with an odd number of general perfect matchings;
3. For $\Gamma_{5}$ we have 22484 general induced subgraphs with an odd number of general perfect matchings. general induced subgraphs.

Corollary 8.16. The number of induced Eulerian subgraphs of $\Gamma_{4}$ is 1536.

## 9 Future Directions

Below I discuss the direction I intend to continue with the interlace polynomial of $\Gamma_{n}(x)$ as well as $Q\left(\Gamma_{n}, x\right)$. I wish to show more direct relationships within the coefficients of $\Gamma_{n}(x)$ and $Q\left(\Gamma_{n}, x\right)$.

### 9.1 More Properties on $\Gamma_{n}(x)$

The adjacency matrix can give us specific information about our graph. Let us look at one theorem that can tell us how many walks of length $k$ are bewteen two specific vertices.

Theorem 9.1. (see, eg., [7].) Let $G$ be a graph on labeled vertices, let $A$ be its adjacency matrix, and let $k$ be a positive integer. Then $A_{i, j}^{k}$ is equal to the number of walks from $i$ to $j$ that are of lenth $k$.

Let us use 9.1 and take a look at the adjacency matrix for $\Gamma_{5}$ and concentrate on walks of size 4 and 5 .

$$
A=\left[\begin{array}{llllllllll}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}\right]_{10 \times 10}
$$

$$
\begin{aligned}
& A^{4}=\left[\begin{array}{cccccccccc}
10 & 11 & 7 & 10 & 4 & 6 & 4 & 10 & 7 & 11 \\
11 & 24 & 11 & 14 & 10 & 14 & 6 & 14 & 10 & 14 \\
7 & 11 & 10 & 11 & 7 & 10 & 4 & 6 & 4 & 10 \\
10 & 14 & 11 & 24 & 11 & 14 & 10 & 14 & 6 & 14 \\
4 & 10 & 7 & 11 & 10 & 11 & 7 & 10 & 4 & 6 \\
6 & 14 & 10 & 14 & 11 & 24 & 11 & 14 & 10 & 14 \\
4 & 6 & 4 & 10 & 7 & 11 & 10 & 11 & 7 & 10 \\
10 & 14 & 6 & 14 & 10 & 14 & 11 & 24 & 11 & 14 \\
7 & 10 & 4 & 6 & 4 & 10 & 7 & 11 & 10 & 11 \\
11 & 14 & 10 & 14 & 6 & 14 & 10 & 14 & 11 & 24
\end{array}\right]_{10 \times 10},\left[\begin{array}{llllllllll}
22 & 38 & 21 & 28 & 16 & 28 & 16 & 28 & 21 & 38 \\
38 & 50 & 38 & 59 & 28 & 44 & 28 & 44 & 28 & 59 \\
21 & 38 & 22 & 38 & 21 & 28 & 16 & 28 & 16 & 28 \\
28 & 59 & 38 & 50 & 38 & 59 & 28 & 44 & 28 & 44 \\
16 & 28 & 21 & 38 & 22 & 38 & 21 & 28 & 16 & 28 \\
28 & 44 & 28 & 59 & 38 & 50 & 38 & 59 & 28 & 44 \\
16 & 28 & 16 & 28 & 21 & 38 & 22 & 38 & 21 & 28 \\
28 & 44 & 28 & 44 & 28 & 59 & 38 & 50 & 38 & 59 \\
21 & 28 & 16 & 28 & 16 & 28 & 21 & 38 & 22 & 38 \\
38 & 59 & 28 & 44 & 28 & 44 & 28 & 59 & 38 & 50
\end{array}\right]_{10 \times 10}^{5} \\
& A^{5}=
\end{aligned}
$$

Each of the entries, $A_{i j}$, in the matrices for $A^{4}$ and $A^{5}$ represent the number of walks, respectively, of size 4 and 5 , between the two vertices $i$ and $j$. My goal here is to be able to show the number of walks of size $k$, correlate to the coefficients in my interlace polynomial for $\Gamma_{n}$ at a specific $n$. I will concentrate on specific vertices and the patterns associated in the matrix to try and see if in fact there is a correlation between a walk from two specific vertices and my coefficients in the polynomial.

Another interesting theorem that can be used for further investigation deals with
the number of spanning trees within a graph.
Theorem 9.2. (see, eg., [7].) Let $U$ be a simple undirected graph. Let $\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$ be the vertices of $U$. Define the $(n-1) \times(n-1)$ matrix $L_{0}$ by

$$
l_{i, j}=\left\{\begin{array}{l}
\text { the degree of } v_{i} \text { if } i=j \\
-1 \text { if } i \neq j \text { and } v_{i} \text { and } v_{j} \text { are connected, and } \\
0 \text { otherwise. }
\end{array}\right.
$$

where $1 \leq i, j \leq n-1$. Then $U$ has exactly $\operatorname{det}\left(L_{0}\right)$ spanning trees.

Research has been done on the interlace polynomial for arbitrary trees [2]. I will use Theorem 9.2 to show how many spanning trees $\Gamma_{n}$ has and will try to determine any significance for the interlace polynomial of arbitrary trees and the graph polynomial $\Gamma_{n}(x)$. Let us take a look at $L_{0}$ for $\Gamma_{5}$.

$$
L_{0}=\left[\begin{array}{ccccccccc}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 4 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{array}\right]_{9 \times 9}
$$

Using Maple Software, I was able to determine the determinant.

$$
\operatorname{det}\left(L_{0}\right)=810
$$

This tells us, from Theorem 9.2 that $\Gamma_{5}$ has exactly 810 spanning trees. Now the question is, how can we relate the interlace polynomial from [2], for arbitrary trees
for the spanning trees in my graph, and do they relate to my graph polynomial for $\Gamma_{5}(x) ?$

### 9.2 More Properties on $Q\left(\Gamma_{n}, x\right)$

The interlace polynomial $Q\left(\Gamma_{n}, x\right)$ is a lot more complicated from the original graph polynomial we created. I would like to further study the change in coefficients for this graph polynomial and try to relate them as I did for the coefficients of $\Gamma_{n}(x)$. Hopefully I can lead myself into a better understanding of the graph polynomial itself and in turn more properties about the graph itself.

## References

[1] Aigner, M., and Holst, H. (2004). Interlace polynomials. Linear Algebra and Its Applications, 377(0), 11-30.
[2] Anderson, C., Cutler, J., Radcliffe, A.J., and Traldi, L. On the interlace polynomials of forests, Discrete Mathematics (2009).
[3] Arratia, Richard, Bela Bollobas, and Gregory B. Sorkin. "The Interlace Polynomial Of A Graph." (2002): arXiv.
[4] Arratia, R., Bollobas, B., Coppersmith, D., and Sorkin, B. G. "Euler Circuits And DNA Sequencing By Hybridization." Discrete Applied Mathematics 104.(2000): 63-96. ScienceDirect.
[5] Arratia, R., Bollobas, B., and Sorkin, G. The interlace polnomial: a new graph polynomial, manuscript.
[6] Austin, Andrea. The Circuit Partition Polynomial with Applications and Relation to the Tutte and Interlace Polynomials (2007). Rose-Hulman Undergraduate Mathematics Journal.
[7] Balister P.N., Bollobas B., Cutler J. and Pebody L., (2002). The Interlace Polynomial of Graphs at -1, Europ. J. Combinatorics, 23, 761-767.
[8] Bona, Miklos. A Walk through Combinatorics: An Introduction to Enumeration and Graph Theory. Third ed. Hackensack, NJ: World Scientific Pub., 2006. Print.
[9] Ellis-Monaghan, Joanna, and Criel Merino. "Graph Polynomials And Their Applications I: The Tutte Polynomial." (2008): arXiv.
[10] Ellis-Monaghan, J. A. and Merino, C. "Graph Polynomials and Their Applications II: Interrelations and Interpretations." 28 Jun 2008.
[11] Ellis-Monaghan, Joanna. Properties of the Interlace Polynomial via Isotropic Systems.
[12] Friedberg, Stephen H., Arnold J. Insel, and Lawrence E. Spence. Linear Algebra. Fourth ed. Englewood Cliffs, NJ: Prentice-Hall, 1979. Print.
[13] Godlin, Benny, Emilia Katz, and Johann A. Makowsky. "Graph Polynomials: From Recursive Definitions To Subset Expansion Formulas." (2008): arXiv. Web. 9 Oct. 2014.
[14] Paoletti, Teo. "Leonard Euler's Solution to the Konigsberg Bridge Problem The Fate of Konigsberg," Loci (May 2011).

