A CONTROL THEORETIC PERSPECTIVE ON SOCIAL NETWORKS

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SUMMARY

This thesis discusses the application of control theory to the study of complex networks, drawing inspiration from the behavior of social networks. There are three topic areas covered by the thesis. The first area considers the ability to control a dynamical system which evolves over a network. Specifically, this thesis introduces a network controllability notion known as herdability. Herdability quantifies the ability to encourage general behavioral change in a system via a set-based reachability condition, which describes a class of desirable behaviors for the application of control in a social network setting. The notion is closely related to the classical notion of controllability, however ensuring complete controllability of large complex networks is often unnecessary for certain beneficial behaviors to be achieved. The basic theory of herdability is developed in this thesis.

The second area of study, which builds directly on the first, is the application of herdability to the study of complex networks. Specifically, this thesis explores how to make a network herdable, an extension of the input selection problem which is often discussed in the context of controllability. The input selection problem in this case considers which nodes to select to ensure the maximal number of nodes in the system are herdable. When there are multiple single node sets which can be used to make a system completely herdable, a herdability centrality measure is introduced to differentiate between them. The herdability centrality measure, a measure of importance with respect to the ability to herd the network with minimum energy, is compared to existing centrality measures.

The third area explores modeling the spread of the adoption of a beneficial behavior or an idea, in which the spread is encouraged by the action of a social network. A novel model of awareness-coupled epidemic spread is introduced, where agents in a network are aware of a virus (here representing something which should be spread) moving through the network. If the agents have a high opinion of the virus, they are more likely to adopt it. The behavior of this viral model is considered both analytically and in simulation.

CHAPTER 1

INTRODUCTION

The modern world is increasingly understood as an interconnection of interacting parts, as a large complex network which itself can be built on the interplay of various networks [1, 2]. The network representation has found its way into varied fields ranging from biology and sociology to power systems and robotics [3, 4, 5].

Network representations, fueled by concepts from graph theory [6], have proven to be powerful tools. In biology, analyzing graphs has helped reveal beneficial drug-drug interactions [7] and helped drive understanding of properties of the human brain [8, 9, 10]. In the social and behavioral sciences, networks have helped expose the power of social networks; showing, for example, that dieting can be transmitted by social networks [11]. In robotics, the ability to control the behavior of a network of interacting robots is driven by the network structure of the communication between the robots [5].

The power of the network representation has lead to a field known generally as network science. Network science concerns itself with such questions as how networks are formed and how to discuss the structure of a network on two levels. A micro level, i.e. at the level of local interactions between nodes in the network, as well as on a macro level, searching for properties that hold across the network. This field has seen a surge in popularity recently, as can be seen not only by the large number of recent network science textbooks [3, 12, 13, 14] but the large number of popular sciences books describing the ”new” science of networks [15, 16, 17].

The fact that networks are a powerful representation tool to understand the behavior of a system comes as no surprise to two rather distinct fields in the literature: that of the social and behavioral sciences and of control theory. In the social sciences, the network representation was developed beginning in the late 1940’s [18, 19, 20] and has proved useful in a wide range of social science sub-disciplines such as those that study the diffusion of innovations, social influence, and group problem solving (see the introduction of [4] for a full discussion).

Control theory as a field has also long been interested in networks as a representation of the world, and in understanding how the properties of a system can be analyzed based on its network representation [21, 22, 23, 24, 25]. There has also been work on updating mainstays of the control literature to tackle the challenges faced in complex networks, such as decentralized control [26, 27] or as was done in extending Lyapunov stability analysis to consider connective stability of complex ecosystems [28, 29].

There has been a recent overlap between the field of network science and control theory. As a result, control theory has received considerable attention from the complex networks community and control theorists have expanded to new application areas in the study of complex networks. Researchers from the study of complex networks are primarily interested in the theory of controllability and observability, seeing the ability to control the behavior of a complex system as the ultimate test of the understanding of the behavior of the system [30, 31]. For control theorists, there have been many new application domains which are particularly amenable to study via the tools of control theory, among them the study of epidemic spread over complex networks [32, 33, 34] and opinion dynamics [35].

This thesis concerns itself with the interplay between the study of complex networks and control theory; particularly bringing ideas from the study of social networks to bear on the development of new theory. Social networks, and other large complex networks such as biological networks, require different modeling considerations than traditional engineered systems as they are driven by different types of behavior than engineered systems. Interacting with these systems in a meaningful way requires new models and new understanding of control authority in these networks.

CHAPTER 2
BACKGROUND

The following section takes a subset of a particularly wide literature, touching on topics related to a varied set of disciplines, which is necessitated by the fundamentally cross disciplinary nature of the work presented in this thesis. This subset will provide the necessary context for the work of the thesis by discussing control of networks, how node importance is determined, and the spread of viruses, opinions, and products over networks.

Much of the work presented as background is related to the general problem of translating a known graph structure into a dynamical system that evolves over that graph structure. Once this idea is treated more formally, and a number of basic definitions are introduced, this chapter will show two methods for making that transition, within the broader context of control of networks. The first approach assumes that the weights of the edges of the graph are known, which leads to a specific dynamical system which can be analyzed. The other approach assumes that the weights are unknown and asks about the properties of a system over a range of possible weights, leading to the consideration of a class of linear systems.

2.1 Preliminaries/Definitions

The following thesis deals with the interplay between two mathematical objects, both of which describe a large complex system. The first is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w(\cdot))$ where $\mathcal{V}$ is a set of n nodes, $\mathcal{E}$ is a set of possibly directed edges between nodes, and $w(\cdot)$ is a weighting function which accepts an edge and returns a weight in $\mathbb{R}$. The other object of study is a dynamical system which evolves over the graph $\mathcal{G}$. In most complex networks, this dynamic can be highly nonlinear, however in order to gain understanding about the properties of the network this dynamical system will be considered here to be a continuous time, linear system

$$\dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$. We will assume that each node in the network will have a single scalar state associated with it, which is why it holds that the state vector $x \in \mathbb{R}^n$. Note it is also possible to move in the other direction, i.e. a state $x \in \mathbb{R}^n$, and its accompanying linear system, gives rise to a graph structure with n nodes, each of which represents one element of the vector x.

Much of the work presented in the sections that follow considers how to move from one of these two representation of a system to the other, and how their relationship can provide information about the behavior of the system. In the controls literature, it often the case that the underlying graph structure is used to make concrete statements about the controllability properties of the linear system [5, 21]. This is a case where one moves from a dynamical system representation to a graph representation. In the complex networks and social networks literature, properties of the dynamic which evolves over the graph are inferred from the structure of the networks, though typically without a formal representation of the dynamic which is assumed to evolving over that network [3, 12, 4]. Before discussing either of these approaches, an introduction to the study of graphs is required.

2.1.1 Graph Theoretic Preliminaries

This section considers a number of basic concepts from the study of graph theory, providing a tool set to discuss the properties of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w(\cdot))$. Unless otherwise noted, the provided definitions follow [36]. A quick digression on the definition of the graph as weighted. Treating the graph $\mathcal{G}$ as weighted is the most general case of this graph representation. If the graph is to be unweighted, that implies for any edge e_i, $w(e_i) = 1$. It is also common for the weights to be assumed to be positive in many complex network settings, in such cases the weight represents a distance or an amount transfered. This thesis will also consider the case where the edge weights are assumed to be negative, which in a social networks setting represents a pair of agents that are enemies [12]. Having successfully digressed, we now present a number of basic definitions from graph theory.

An arbitrary element of $\mathcal{V}$ will be referred to by v_i for some index i. Denote the directed edge from v_i to v_j as (v_i, v_j) or e_{ij}. An arbitrary element of $\mathcal{E}$ will be referred to by e_i for some index i. In the case of an undirected graph the neighborhood $\mathcal{N}_i$ of a node v_i are all nodes v_j such that $(v_i, v_j) \in \mathcal{E}$. The degree d_i of a node is equal to the number of network neighbors, $d_i = |\mathcal{N}_i|$. In the case of a directed graph, there are two neighborhood sets: the in-neighborhood $\mathcal{N}_i^i$ and the out-neighborhood $\mathcal{N}_i^o$. The in-neighborhood of v_i is all nodes v_j such that $(v_j, v_i) \in \mathcal{E}$ and the out-neighborhood is all nodes v_j such that $(v_i, v_j) \in \mathcal{E}$. Similarly each node i has an in-degree $d_i^i = |\mathcal{N}_i^i|$ and an out-degree $d_i^o = |\mathcal{N}_i^o|$.

The graph can be represented by an adjacency matrix $\tilde{A}(\mathcal{G})$ where the element $\tilde{a}_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$. In the undirected case, the adjacency matrix $\tilde{A}(\mathcal{G})$ is symmetric. Depending on the level of information required about a network, the graph G can also be represented by a signed adjacency matrix $\tilde{A}_s(\mathcal{G})$ and a weighted adjacency matrix $\tilde{A}_s(\mathcal{G})$ where $\tilde{a}_{ij}^s = \text{sign}(w(e_{ji}))$ and $\tilde{a}_{ij}^w = w(e_{ji})$. In the case of an unweighted graph these three representations of the graph are equivalent.

In the case of an unweighted network, the graph can be represented as an incidence matrix $M(\mathcal{G}) \in \mathbb{R}^{n \times m_e}$ where $m_e = |\mathcal{E}|$. The incidence matrix has a column corresponding to each edge $e = (v_i, v_j) \in \mathcal{E}$, and each row follows

$$M(\mathcal{G})_{ze} = \begin{cases} 1 \text{ if } z = i \\ -1 \text{ if } z = j \\ 0 \text{ else.} \end{cases}$$

In the case of an undirected network, an arbitrary orientation is assigned to each edge to generate the incidence matrix.

The graph can also be represented by the graph Laplacian $L_{\mathcal{G}}$ which for undirected, unweighted graphs satisfies

$$L_{\mathcal{G}} = D - \tilde{A}(\mathcal{G})$$

where $D = \mathrm{diag}(d_i)$. In the case of directed graphs, the in-Laplacian is defined similarly as

$$L_{\mathcal{G}}^i = D^i - \tilde{A}(\mathcal{G}),$$

where $D^i = \mathrm{diag}(d_i^i)$. The Laplacian also satisfies

$$L_{\mathcal{G}} = M(\mathcal{G})M(\mathcal{G})^T,$$

which immediately shows that the Laplacian is positive semidefinite. The Laplacian can also be defined for the case of weighted networks, however such objects will not be used in this thesis.

A walk from v_0 to v_p, $\pi(v_0, v_p)$, is any alternating sequence of nodes and edges $\pi(v_0, v_p) = v_0, e_1, v_1, e_2, v_2 \ldots, v_{p-1}, e_p, v_p$ such that $v_i \in \mathcal{V} \ \ \forall i$ and $e_i = (v_{i-1}, v_i) \in \mathcal{E}$. The set of walks from v_0 to v_p is $\theta(v_0, v_p)$. A node v_j is reachable from v_i, which will be written as $v_i \to v_j$, if $\theta(v_i, v_j) \neq \emptyset$. Note that reachability is discussed within both graph theory and control theory. This thesis will use the term reachable in both senses, with clarification only if it is uncertain which notion of reachability is considered. The length of a walk, $\mathrm{len}(\pi)$, is equal to the number of edges in π. A walk is a path if all nodes are distinct. A walk is a cycle if the first and last node of the walk is the same.

A walk has an associated sign which follows

$$s(\pi) = \prod_{e_i \in \pi} s(e_i).$$

For the purpose of this thesis, a walk also has an associated weight which follows

$$w(\pi) = \prod_{e_i \in \pi} w(e_i).$$

This is distinct from the weight of a walk as it is treated in many applications, such as shortest path algorithms, which consider $w(\pi) = \sum_{e_i \in \pi} w(e_i)$ [37].

A semi-walk from v_0 to v_k, $\pi_s(v_0, v_k)$, is a collection of nodes $v_0, v_1, v_2 \ldots, v_{k-1}, v_k \in \mathcal{V}$, as well as k edges which satisfy $(v_{i-1}, v_i) \in \mathcal{E} \vee (v_i, v_{i-1}) \in \mathcal{E}$. For convenience, the semi-walk can be represented by $\pi_s = v_0, \hat{e}_1, v_1, \hat{e}_2, v_2 \ldots, v_{k-1}, \hat{e}_k, v_k$ where $\hat{e}_i$ is the element of $\{(v_{i-1}, v_i), (v_i, v_{i-1})\}$ that is contained in $\mathcal{E}$.

Like a walk, the sign of a semi-walk follows $s(\pi_s) = \prod_{\hat{e}_i \in \pi_s} s(\hat{e}_i)$ and the weight of a semi-walk follows $w(\pi_s) = \prod_{\hat{e}_i \in \pi_s} w(\hat{e}_i)$. A semi-walk is a semi-path if the nodes of the semi-walk are distinct and a semi-walk is a semi-cycle if the first and last element of the semi-walk are the same. A directed graph is weakly connected if there is a semi-walk between any two vertices in the graph and is strongly connected if there is a walk between any two vertices in the graph.

A graph is structurally balanced if all semi-cycles have a positive sign [38]. If a network is structurally balanced, the nodes can be partitioned into two clusters, where all inter-cluster edges are positive and all intra-cluster edges are negative. Structural balance is a well studied property of social networks [12, 39], which has implications in control [40, 41]. Despite the nice mathematical properties it provides, many real networks are not structurally balanced, leading to questions of how close to balanced they are [42].

2.1.2 Positive Linear Systems

This section introduces some basic definitions from the study of positive linear systems which will be used later. A system is positive if and only if for every non-negative initial state and for every nonnegative input its state is nonnegative. The study of positive systems covers subject areas ranging from epidemic spread and, more generally, compartmental systems in biology to opinion dynamics and robotics [43, 44, 5, 45, 32, 33].

In the following discussion, all operations will be considered element-wise. If a matrix satisfies $A > 0$ then it is element-wise non-negative. Similarly for $A \geq 0$. A matrix A

is said to be Metzler if $A + \alpha I \geq 0$ for some $\alpha \in \mathbb{R}$ or, equivalently, all non-diagonal elements of A are nonnegative [46].

For a continuous time linear system,

$$\dot{x} = Ax + Bu$$

if A is Metzler and $B \geq 0$ then the system is positive.

Definition 2.1.1 ([46])**.** *A positive system is said to be excitable if and only if each state variable can be made positive by applying an appropriate nonnegative input to the system initially at rest i.e. from* $[x(0) = 0]$.

Lemma 2.1.1 (Theorem 8 in [46])**.** *A positive system is excitable if and only if there exists at least one path from an input to each node in the underlying graph.*

2.2 Control of Networks with Known Dynamics

In this section, some basic results on the control of networked systems will be presented in the case that the dynamical system which evolves over $\mathcal{G}$ is known exactly, as is often the case with an engineered network system.

2.2.1 The Consensus Algorithm

A prime example of an engineered networked system is a robotic network, where each node $v_i \in \mathcal{V}$ represents a robotic agent and each $e_i \in \mathcal{E}$ represents communication between the agents [5, 47]. In the field of multi-agent robotics, the consensus or controlled agreement dynamic is widely used to move from a given graph $\mathcal{G}$ to a dynamic over that graph. This section will discuss the case where the graph is undirected and edge weights are assumed positive (in fact 1), though we note that the problem of directed [5] and signed consensus [40, 41] has also been considered.

Given a network $\mathcal{G}$ the consensus algorithm has each agent in the network update its state x_i as

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i).$$

Under the consensus dynamic, each agent considers its network neighbors (in-neighbors in the case of a directed graph) and changes its state based on the state of its neighbors. If the state is the robot's position, then the robot will attempt to move to the mean of its network neighbors position. This dynamic can be equivalently expressed as

$$\dot{x} = -L_{\mathcal{G}} x.$$

If the graph is connected, then the state will converge to the agreement subspace $x \in \text{span}\,(1_n)$, as 1_n is the only eigenvector of the Laplacian associated with the eigenvalue 0.

The consensus dynamic can be extended to the controlled agreement protocol which incorporates the ability to control the position of the robotic network. To do so requires that some nodes be designated leaders, which will be captured by the set $L \subset \{1, 2, \ldots, n\}$ which will follow

$$\dot{x}_l = u_l, \;\; \forall l \in L.$$

We are interested in studying the behavior of the remaining nodes in the network, the set of followers $F = \{1, 2, \ldots, n\} \setminus L$ which have an associated subgraph $\mathcal{G}_f$. By partitioning the graph, one can represent the incidence matrix as

$$M(\mathcal{G}) = \begin{bmatrix} M_f \\ M_l \end{bmatrix},$$

where $M_f \in \mathbb{R}^{n_f \times m_e}$, $M_l \in \mathbb{R}^{n_l \times m_e}$, $n_l = |L|$, and $n_f = |F|$. Then this partition implies

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} A_f & B_f \\ B_f^T & A_l \end{bmatrix}$$

where $A_f = M_f M_f^T$, $A_l = M_l M_l^T$, and $B_f = M_f M_l^T$. The system dynamic then becomes

$$\begin{aligned} \dot{x}_f &= -A_f x_f - B_f u \\ y &= B_f^T x_f. \end{aligned} \tag{2.1}$$

Controllability of this system admits a graph theoretic characterization. We present the results for the single input case here, though it has been extended to the case of multiple inputs [5].

Definition 2.2.1. *A permutation matrix, J, is a* $\{0, 1\}$*- matrix with a single nonzero element in each row and column.*

Definition 2.2.2. *The system in Equation* (2.1) *is input symmetric if there exists a nonidentity permutation* J *such that*

$$JA_f = A_f J.$$

Then it is possible to show that

Theorem 2.2.1 (Theorem 10.15 from [5])**.** *The system in Equation 2.1 is uncontrollable if it is input symmetric.*

As the name suggests, the permutation in Definition 2.2.2 captures symmetry with respect to the selected input node. If nodes are symmetric with respect to an input then they can not be controlled separately which leads to a decrease in controllability. In fact, the number of groups of symmetric nodes determine the dimension of the controllable subspace [48]. This has been studied in some depth for leader follower networks [49]. The effect of symmetry on controllability will be revisited in Chapter 4.

2.2.2 Input Selection

Often when interacting with a complex network, there are no existing interactions with input and the problem facing the researcher considering the network is where to place the appropriate control inputs to ensure desirable system properties are satisfied. This problem is known as the input selection problem: given an autonomous system

$$\dot{x} = Ax$$

how does one design a B matrix that ensures system controllability. In the specific case of multi-agent systems following consensus dynamics, this is known as the leader selection problem. There are a number of results which hold specifically for the case of leader selection which will be discussed first before moving on to the general case of input selection for an arbitrary but known system.

The leader selection problem was solved by determining whether a graph was input symmetric with respect to a given input [50, 51], often considering whether specific graph structures could be made controllable. Specifically the controllability of circulant networks [52], path and cycle graphs [53], and grid graphs [54] have been considered.

The input selection problem has been addressed more generally in the case of ensuring system controllability for a known network [55, 56, 57, 58]. These results where obtained under specific assumptions on the structure of the B matrix that is to be designed. In one case, the question is to create a B matrix with the minimal number of columns. This can addressed by the application of classic controllability test, known as the Popov-Belevitch-Hautus test, in the case of a linear system. In the complex networks community this method is known by the name of ”exact controllability” [59].

If, instead, the question is to create a diagonal B with the minimum number of non-zero elements, i.e. to find the minimum number of states with which to interact to ensure system controllability, then finding a solution was shown to be NP-Hard [56]. A similar problem was also considered, that of selecting which state nodes to apply input to such that reachability to a specific end point or subspace is ensured and was also found to be NP-hard [58].

In the case where it is possible to select from a number of agents to insure controllability, other considerations come into play. Selecting leader nodes while taking into account worst case control energy, as quantified by the smallest eigenvalue of the controllability grammian, was considered in [60]. A number of control energy centralities were introduced in [61], some of which were extended to include considerations of robustness to noise in [62].

2.2.3 Qualitative Systems Analysis

It is often the case when dealing with real systems that the parameters which govern the system dynamics are unknown. The paradigm of Qualitative Systems Analysis asks what can be said about a system for which only certain properties are known, specifically the interconnection structure of the network. This section will discuss two sub areas of Qualitative Systems Analysis: structural controllability, which considers what the interaction structure says about controllability of a network, and sign controllability which considers what the sign of the interaction structure says about controllability of a network. There has been a surge in interest on the qualitative controllability of complex networks following the work of Liu et. al. [63]. This work applied the notion of structural controllability to large complex networks, which lead to a characterization of both the whole network and individual nodes based on controllability properties [63, 64, 65, 66, 67, 68, 69].

Structural Controllability

Structural controllability makes statements about the controllability properties of a system based on solely on its structure, without worrying that the system parameters are perfectly known [21, 22, 23]. Structured systems [70] as typically used in the complex networks literature, provides a set of rules to move from a graph structure of a complex network to a linear dynamical system. The system matrices are said to be structured matrices, where the structure of the system matrices is defined by which elements of the matrices are zero and which are nonzero. Further these structured system matrices are related to the edge set E of the graph, i.e. a nonzero element of the system matrices corresponds to a edge in E.

Consider a system

$$\dot{x} = A_s x + B_s u,$$

where A_s, B_s are structured matrices, that is they consist of either zero or indeterminate non-zero elements corresponding to an underlying graph. Specifically A_s has a non-zero element at position ij if $e_{ji} \in \mathcal{E}$. Let $\mathbb{R}^{n_A} \times \mathbb{R}^{n_B}$ denote the parameter space associated with the structured system, where n_A (n_B) is the number of non-zero elements of $A_s(B_s)$. The matrices $\mathcal{A}_s, \mathcal{B}_s$ are a specific incarnation of the matrices A_s, B_s, which are found by fixing the non-zero parameter values.

Definition 2.2.3. *A property holds generically for the structured system in Eq.* (3.2) *if it is satisfied outside of a proper variety on the parameter space* $\mathbb{R}^{n_A} \times \mathbb{R}^{n_B}$.

Definition 2.2.4. *A system is structurally controllable if it is generically controllable.*

The benefit of structural controllability is that it admits a structural answer to the question of controllability, i.e. by inspecting the graph of the system, the structural controllability of that system can be determined. There are two sets of structural conditions which can be used to quantify structural controllability. The first set of conditions uses two network structures, a stem and a bud, which are shown in Figure 2.1.

The graph P is called a cactus if and only if one can write $P = S \cup B_1 \cup B_2 \cup \cdots \cup B_p$ where S is a stem and $B_1, B_2, \ldots B_p$ are buds such the origin e_i for bud B_i is the only node in B_i that belongs to $S \cup B_1 \cup B_2 \cup \cdots \cup B_{i-1}$. If the graph $\mathcal{G}_{A,B}$ is a cactus, then the system (A_s, B_s) is structurally controllable.

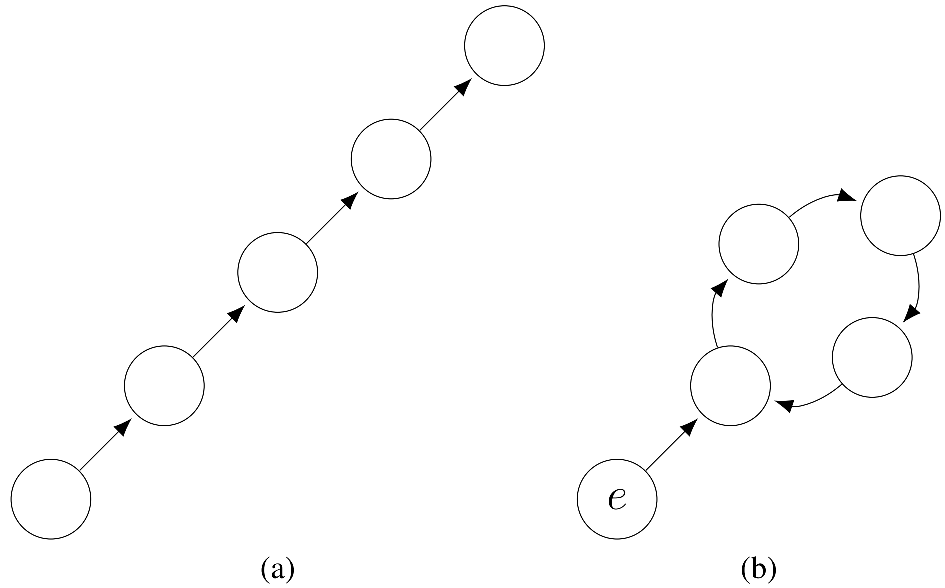


Figure 2.1: (a) A stem (b) a bud with origin e

Structural controllability can also be equivalently expressed as there being no dilations and no non-accessible nodes in the graph $\mathcal{G}_{A,B}$.

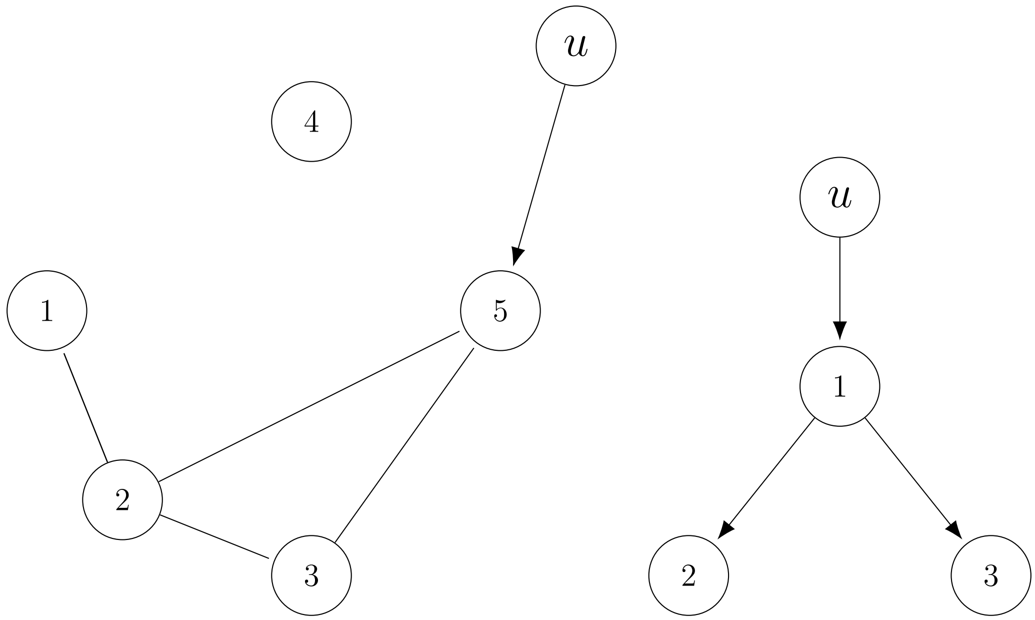
Definition 2.2.5. *Consider S any set of nodes in the graph and $T(S)$ the set of nodes that have an edge which enters a member of S. A dilation occurs when for some S, $|T(S)| < |S|$.*

Definition 2.2.6. *A node is accessible if there is a path from an input to the node. Equivalently, a set S is non-accessible if $T(S) = \emptyset$.*

Figure 2.2 shows examples of a non-accessible node and a dilation.

Structural Input Selection

The question of structural input selection is also of some interest to the study of structural controllability, similar to the case of controllability. Liu et al expanded structural controllability to the analysis of large complex networks [63]. They found that the number of unmatched nodes in a maximum matching on a bipartite representation of a directed graph determined the number of nodes needed to ensure structural controllability. The nodes selected via this method tended to have lower degree than the average degree of the network.



(a) (b)

Figure 2.2: (a) Accessibility: Node 4 is non accessible as there is no path from the input to the node (b) A dilation: here 2 nodes have in-bound edges from 1 node

This suggests that structural controllability based driver node selection avoids hubs, which matches well with the fact that dilations degrade controllability. Other work has shown that an estimate of the number of driver nodes needed can be derived from the average betweenness centrality and the average closeness centrality of a network, suggesting that degree centrality might not be appropriate to characterize network controllability [71].

This work has been expanded to consider subsets of the problem of ensuring complete controllability, for example whether the state can be controlled to specific areas of interest in the state space or if the certain nodes of the graph can be controlled [64, 65, 66]. This work has also been expanded to a control centrality measure, which ranks the importance of node i based on the generic rank of the controllable subspace while node i is the sole node which receives input [72].

Despite the popularity of the work on structural controllability there is an assumption made in the original analysis of networks that bears further examination. It has been shown that if the internal dynamics of all nodes of a network are assumed to have a finite time constant, or equivalently all nodes are assumed to have a self loop, then the system can always be controlled via one input [73]. The analysis done by Liu et al assumes an infinite time constant for most of the nodes in the networks under investigation, as these networks tend not to have self loops. As almost all systems considered for application of these control principles have a finite time constant for internal nodal dynamics, this raises serious issues about the applicability of structural controllability driven driver node selection.

As opposed to the case of controllability, where it was determined to NP-Hard, finding the minimum number of manipulated state variables to ensure structural controllability can be done in polynomial time [74].

Extensions

A number of problems similar to the structural controllability problem have been addressed in the literature. Strong structural controllability has been considered, in which the condition of controllability must hold for all values of the parameters as opposed to all parameters outside of a zero measure set [75]. Sign controllability has also been considered, which asks whether the sign pattern of the system matrices determines the controllability of the matrix [76, 77, 78, 79]. While the specific results of sign controllability will not be used in the thesis, it is interesting to note the general approach used, which is based on the study of sign solvable linear system [80].

A signing is a diagonal matrix with elements on the diagonal that are in $\{0, -1, 1\}$. A vector is called balanced if it is the zero vector or it has both positive and negative elements. A vector is unisigned if it is not balanced. The set of signings S such that the columns of SX are all balanced is denoted $\mathcal{B}(X)$. Central to the study of sign solvability is the concept of an L-matrix. A matrix X is an L-matrix if all matrices that share the same sign pattern have linearly independent rows. Equivalently X is an L-matrix if and only if $\mathcal{B}(X) = \emptyset$ [80]. In the context of sign controllability often the objective is to show that the matrix $[A\ B]$ is an L-matrix [78]. In the thesis the concept of an L-matrix is too strong for what will be considered, and as such other conditions will be studied.

2.3 Characterizing Importance in Complex Networks: Centrality Measures

There is an extensive literature on characterizing complex networks, which spans the work of varied research communities such as statistical physics and social network analysis, as well as the control community. We refer the reader to [3] for a discussion on other characterizations of complex networks.

This survey focuses on the problem of determining node importance in complex networks. This problem has been addressed in a number of different ways, though this chapter will focus on finding which node in the network maximizes some desired graph structure based objective function. The objective function for this problem is known as a centrality measure, and it is of interest to determine the highest centrality node. In the context of social network analysis, these objectives are structural properties that are determined to be important by external verification. There is great interest in determining not only the appropriate structural property but how to efficiently compute it.

A centrality measure characterizes the importance of a node in a network, based directly on the structure of the graph. Unless otherwise noted the presented measures are for an unweighted network, though extensions exist for many of the measures in the case of a weighted network [81, 82, 83]. In order to provide context for these centrality measures, when appropriate they will be discussed in a social networks context. In a (unweighted) social network, a node in the network represents a person and an edge represents that they are friends.

2.3.1 Degree Centrality

The degree centrality of a node i in an undirected graph is defined to be

$$c_i^d = \sum_j \tilde{a}_{ij}.$$

In a directed graph, there are two measures: the in-degree

$$c_i^{id} = \sum_j \tilde{a}_{ij}$$

and the out-degree

$$c_i^{od} = \sum_i \tilde{a}_{ij}.$$

Degree centrality captures a very basic notion of importance in a graph. In a social network, the person with the highest degree centrality has the most friends in the network and as such will likely be important.

2.3.2 Eigenvector Centrality

A extension of the degree centrality is eigenvector centrality, which accounts not only for number of neighbors (as degree centrality does) but also for the relative importance of those neighbors. It does so by describing the centrality of a node as the sum of its neighbors centralities. Specifically the centrality vector satisfies

$$\tilde{A}c^e = \lambda c^e$$

where λ is the largest eigenvalue of the adjacency matrix $\tilde{A}$. Considering this equation for an element i of c^e, shows that

$$c_i^e = \frac{1}{\lambda} \sum_j \tilde{A}_{ij} c_j^e.$$

Eigenvalue centrality has the property that c_i^e can be high if agent i has a lot of low influence neighbors or a few high influence neighbors.

2.3.3 Katz Centrality

Degree centrality can be interpreted as counting all walks of length 1 that either leave (out-degree) or enter (in-degree) a node [84]. This notion is extended to walks of all lengths by

[85]. Each walk is weighted by a constant $\alpha > 0$ and by the length of the walk giving:

$$c_i^k = \sum_{k=1}^{\infty} \sum_j \alpha^k (\tilde{a}_{ij})^k.$$

Note that in an unweighted network, $(\tilde{a}_{ij})^k = 1$ if there is a path of length k from node j to node i. In the case that $\alpha < \frac{1}{\lambda}$, where λ is the largest eigenvalue of the network then the above sum converges to

$$c_i^k = (I - \alpha A)^{-1} \mathbf{1}_n. \tag{2.2}$$

The expression in Equation (2.2) shows that katz centrality can also be seen as an extension of eigenvector centrality. Consider the equation

$$x = \alpha A x + \mathbf{1}_n$$

which is similar to the update law of eigenvector centrality however each node is assigned a centrality of 1 to start the process. This update law converges to Equation (2.2).

In this light, α is a weight that determines how similar to eigenvector centrality the calculated katz centrality is. Modifying the α parameter to be as high as possible has allowed katz centrality to be used in cases when eigenvector centrality provides illogical answers [3].

Katz centrality, of all the structure based centrality measures, has seen the most interest from the general networks community. It has been linked to an evolving centrality dynamic over the network [86], has been shown to be the best predictor of neuronal firing [9], and it has been shown that with the appropriate formulation of an infinite horizon optimal control problem, it is optimal to control the node with the highest Katz centrality [87].

2.3.4 Closeness Centrality

Like katz centrality, there are other centrality measures that are based on paths between nodes. One such is closeness centrality, which asks which node in the network is closest to the other nodes in the network. Closeness centrality follows:

$$c_i^c = \frac{1}{\sum_j d(i,j)},$$

where $d(i,j)$ is the shortest path distance between node j and node i. Closeness centrality is an intuitive notion of centrality. In the case of a social network, if the objective is to spread a message throughout the network, the person who has the least hops to everyone else in the network is the person to give the message to.

2.3.5 Betweenness Centrality

Another path based centrality is betweenness centrality which follows

$$c_i^b = \sum_{j \neq k \neq i} \frac{\sigma_{j,k}(i)}{\sigma_{j,k}},$$

where $\sigma_{j,k}$ is the number of shortest paths between node j and node k and $\sigma_{j,k}(i)$ is the number of shortest paths between node j and node k that pass through node i. A high betweenness centrality node will be part of many shortest paths between other nodes, and has a natural interpretation as a centrality measure: a person with high betweenness centrality will control information flow in a network.

2.4 Spreading Phenomena in Networks

A central issue in complex networks is understanding spreading processes over networks, whether the object that is spreading be viruses, ideas, or any host of other things. This section will discuss the spread over networks of viruses, opinions, and products/innovations.

Aspects of these three sets of spreading models will be unified in Chapter 5 to form a new model for the spread of adoptive phenomena, i.e. those spreading objects which agents should adopt (like a beneficial behavior), as opposed to viruses, which they should not.

2.4.1 Epidemic Models

One of the most fundamental spreading processes is that of a virus. Understanding viral spread has important ramifications as viral spread needs to be understood if major viral outbreaks are to be mitigated. Studying how viruses spread began in the early 20th century [88]. There are various models of epidemic spread, which cover the multitude of behavior that is possible for a virus [89, 3]. This section will present two models, the Susceptible-Infected (SI) model and the Susceptible-Infected-Susceptible (SIS) model.

Susceptible Infected Model

The SI model is the most basic model of a viral infection. This model will be considered here in the fully mixed setting, i.e. each agent has an equal chance of coming into contact with every other agent over a given time interval. This model can be extended to the case of an interaction structure which is represented by a network, however we will leave that treatment to the main object of study; the SIS model.

In the SI model, the population can either be susceptible to infection or infected with the virus which is spreading through the population. Once an agent is infected they remain infected. This infection happens with rate β. Let $s(t)$ to be the fraction of individuals in the population that are susceptible at time t and $i(t)$ to be the fraction of infected individuals. As the population can only be either susceptible or infected, $s = 1 - i$. This leads to a dynamic in i of

$$\begin{aligned} \frac{di}{dt} &= \beta s i \\ &= \beta(1-i)i \end{aligned}$$

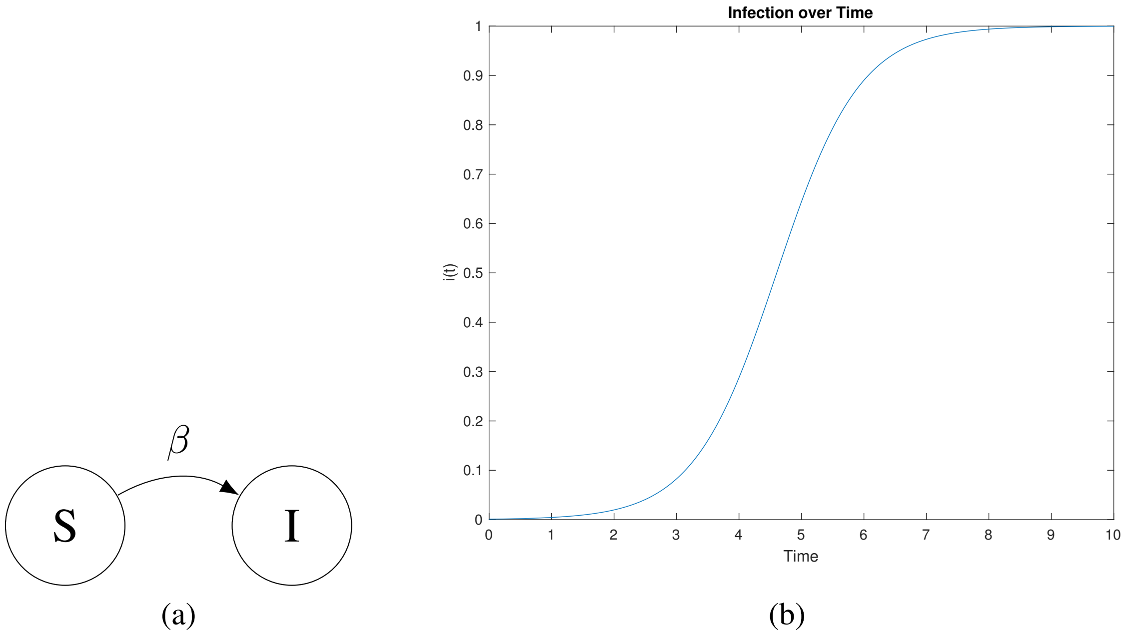


Figure 2.3: The SI Model: (a) The transition between disease states. (b) The infection over time.

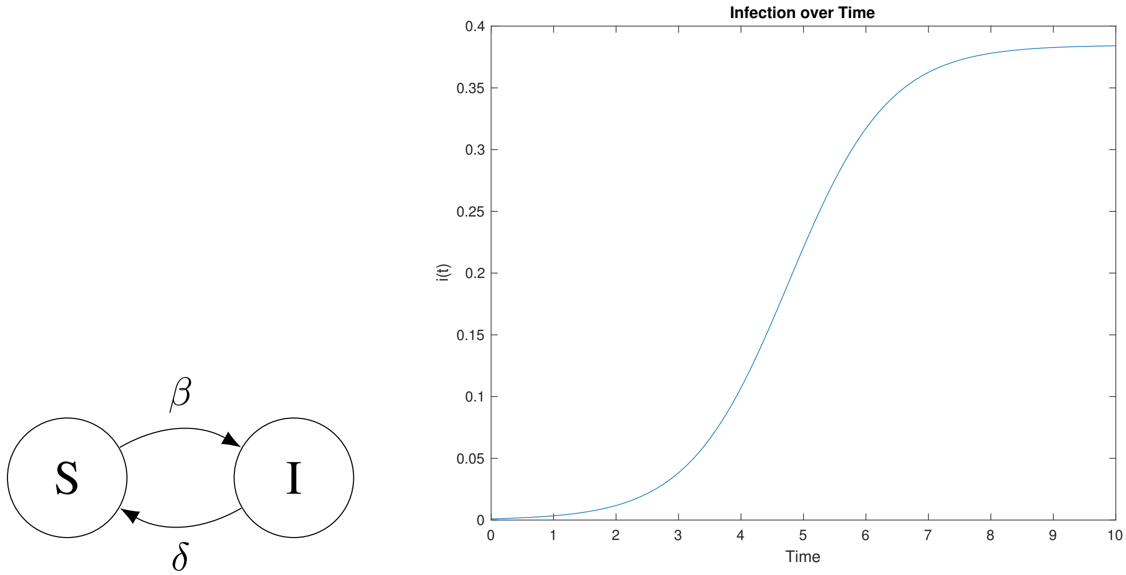
and starting from $i(0) = i_0$ gives

$$i(t) = \frac{i_0 e^{\beta t}}{1 - i_0 + i_0 e^{\beta t}}.$$

Plotting the number of infected individuals shows that the fraction of infected individuals forms an S-shaped curve, shown in Figure 2.3. This S-curve is important for the study of viral phenomena and, as will be seen, is related also to the study of diffusion of innovations.

Susceptible Infected Susceptible Model

One way to extend the SI model is to allow for the agents in the population to recover. In this case, when the agents are no longer infected it is assumed that they will once again become susceptible to infection leading to the SIS model. The agents transition from being infected to being once again susceptible with rate δ. The transitions are shown in Figure



(a) (b)

Figure 2.4: The SIS Model: (a) The transition between disease states. (b) The infection over time when $\beta > \delta$.

2.4. The state of infection in the network follows

$$i(t) = i_0 \frac{(\beta - \delta)e^{(\beta-\delta)t}}{\beta - \delta + \beta i_0 e^{(\beta-\delta)t}}.$$

The long term behavior of the system depends on the relative values of β and δ. When $\beta > \delta$, i.e. the population becomes infected at a higher rate than it heals, the SIS model also shows an S-curve behavior. However, unlike the SI model which converges to $i = 1$, the SIS model will converge to a fixed fraction of the population $i(t) = \frac{\beta-\delta}{\beta} < 1$. This steady state value is known as an endemic equilibrium.

When $\delta > \beta$, i.e. the population heals at a higher rate than it is infected, the endemic state goes to 0. This is often discussed in terms of the basic reproduction number $R_o = \frac{\beta}{\delta}$. When $R_o = 1$ there is a transition in the equilibrium behavior of the the system.

The SIS model as presented here, makes the assumption that the population is well mixed. There are a number of ways to extend the model to the case where the interactions between agents are mediated by a network structure. We consider here the Mean Field SIS model [32], in which the probability of infection of agent i follows

$$\dot{i}_i = -\delta_i i_i + (1 - i_i) \sum_{j \in \mathcal{N}_i} \beta_{ij} i_j$$

where δ_i is the healing rate of agent i and β_{ij} is the infection rate from agent j to agent i.

If δ and β are uniform then there will not be an endemic equilibrium if

$$\frac{\beta}{\delta} < \frac{1}{\lambda_{\max}(\tilde{A}_{\mathcal{G}})}$$

. In the case where δ and β can vary across nodes, the disease-free equilibrium is stable if

$$\lambda_{\max}(B\tilde{A}_{\mathcal{G}} - \text{diag}(\delta_i)) < 0_n,$$

where B is the matrix of β_{ij} (see Lemma 2 of [90] for a proof).

Network SIS Models have been studied widely studied [91, 92, 34], and have included consideration of SIS models with dynamically scaled β parameters [93, 94, 95], as well as with changing or switching graph structures [96, 97, 98, 99, 100]. The SIS model has also been used to model the spread of innovations in social networks, however the predictions of this model for innovation spread have been called into question [92].

2.4.2 Opinion Dynamics

Opinion dynamics have been of interest in sociology since the canonical models of Abelson and DeGroot [45, 101], and have since become of interest to the controls community in the form of the consensus algorithm discussed previously [5]. This section will introduce a number of continuous time models of opinion dynamics. Most of these models have a discrete time counterpart which will be mentioned briefly.

Abelson Opinion Dynamics

The Abelson opinion dynamic model is one of the oldest opinion dynamic models. If x_i is the opinion of agent i then agent i updates its opinion as

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i) \,.$$

Notice that the Abelson opinion dynamic model is exactly the consensus equation presented previously. Therefore this model predicts that the opinions of the agents converge to universal agreement. This property, while beneficial for the problem of robotic rendezvous, does not accurately model how humans update their opinions in social settings. Dissatisfaction with the universal agreement properties of this model has driven much research in the field of opinion dynamics.

The discrete time counterpart of this model is known as DeGroot opinion dynamics model [101], which has been extended to the Friedkin-Johnson model [102], in which agents update their opinions based on their neighbors opinions as well as their initial opinion.

Bounded Confidence Opinion Dynamics

One model which attempts to break the universal agreement of the Abelson opinion dynamic model is the bounded confidence model [103, 104] which follows

$$\dot{x}_i = \sum_{\mathcal{N}_i} p(x_j, x_i)(x_j - x_i).$$

where

$$p(o_j, o_i) = \begin{cases} 1 & \text{if } \|o_j - o_i\| < \epsilon \\ 0 & \text{if else.} \end{cases}$$

In the bounded confidence model, an agent will update their opinion based on whichever

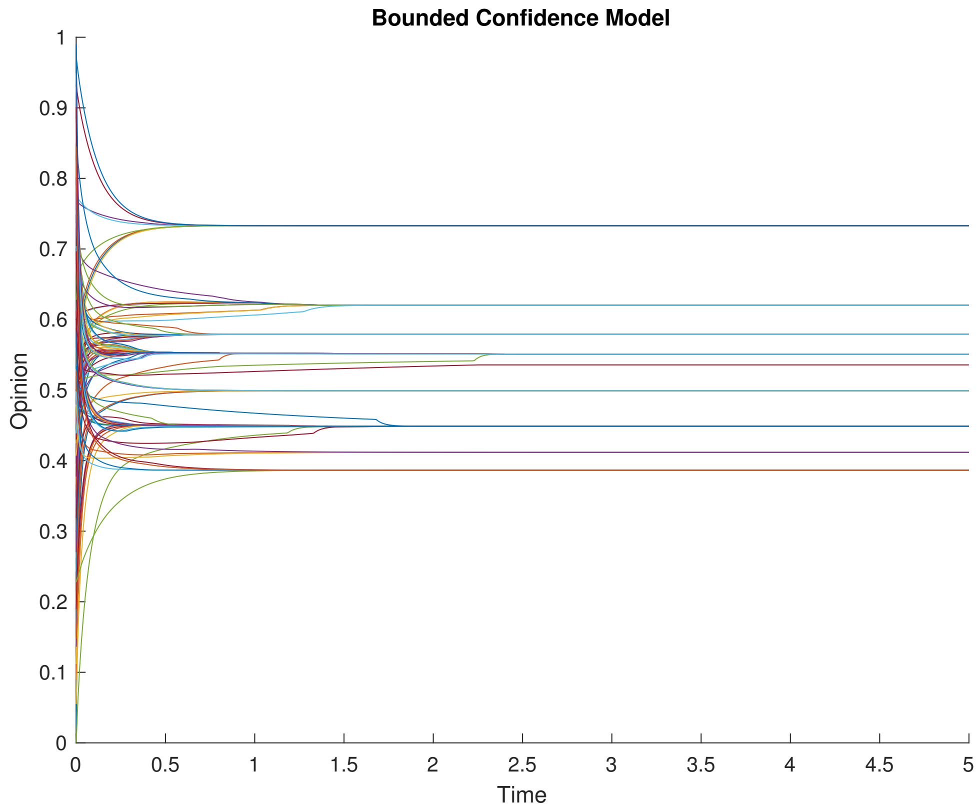


Figure 2.5: The Bounded Confidence Model: The evolution on a complete graph with 100 agents and $\epsilon = .01$.

of its network neighbors it has a similar opinion to and ignore all neighbors which have a sufficiently different opinion. Over time this model predicts that distinct clusters will form in which each agent in the cluster will have the same opinion. Figure 2.5 shows the evolution of a complete graph with random starting conditions. In the discrete time case, this model is referred to as the Hegelsman-Krause model [105].

Signed Consensus

Another extension of the Abelson opinion dynamic considers negative interactions between agents. The Signed Consensus or Altafini model [41] which is of the form

$$\dot{o}_i = \sum_{\mathcal{N}_i^{\bar{O}}} |a_{ij}|(\text{sign}(a_{ij})o_j - o_i), \tag{2.3}$$

where $\mathcal{N}_i^{\bar{O}}$ is a signed set, with negative edges for the neighbors node i distrusts.

If the opinion graph is structurally balanced, then it can give a bipartite consensus, meaning all the members of one group converge to a value and all the members of the other group converge to the negative of that value [41, 38]. Alternatively, if the graph is structurally unbalanced then the opinions converge to 0_N [41, 106].

2.4.3 Innovations Spread Models

The question of how innovations spread has been a focal point of the sociology literature for a number of years. This section will deal with the dominant paradigm within this literature, the paradigm known as diffusion of innovations. Once the diffusion of innovations paradigm has been discussed, a selection of other adoption will be reviewed, specifically models that look at the interplay between opinion and adoption behavior.

Diffusion of Innovations

The dominant paradigm for the spread of products is that of diffusion of innovations, which was introduced by [107] and was later popularized by the work of [108].[1] Diffusion of Innovations as a paradigm began with the study of the adoption of hybrid seed corn. Hybrid seed corn has a number of properties, such as better drought tolerance and higher yield, which makes it superior to the corn that was being used widely at the time. A pair of

[1] Interestingly enough the diffusion of innovations paradigm, as the paradigm is itself an innovation in the study of diffusion, has been an object of meta-study [109, 110, 111].

rural sociology researchers tracked the adoption of hybrid seed corn over the course of a 10 year period, starting around 1930. The main findings included that the communication about hybrid seed corn was crucial for the adoption of the corn, as knowledge of the hybrid seed corn preceded adoption of the corn. Those that adopted the hybrid seed corn early heard from sales people about the corn and then spread the information to their friends and neighboring farms.

This work was extended by Everett Rogers [108]. Rogers, postulated that the four main elements for the spread of an innovation are the properties of the innovation itself, time, how information about the innovation is communicated and the social system in which information about the innovation is communicated. It was also found that adoption over time of the innovation followed an S-curve, similar to the behavior of the epidemic models mentioned previously. As such, many attempts to model the diffusion of innovation are rooted in the use of epidemic models.

Opinion-Adoption Models

There have been few works that study the interplay between product adoption and opinion dynamics, i.e. allowing a consumer's opinion about the quality or value of a product affect his/her decision to purchase or adopt it. These opinions change dynamically because they are influenced by the opinions and decisions of their network neighbors.

One such model is the model of Kalish. Before diving into this model, we must discuss the model of Bass [112], as the work of Kalish is an extension of the Bass model to include awareness. This model considers the interplay between innovation and imitation. The population is split into two categories of adopters. A small fraction of the population are innovators, who will adopt early and without any of their neighbors having adopted. The rest of the population are imitators, who will become more likely to adopt as more people adopt. Bass considered the case where the probability that an agent purchase a product at time T given that no purchase has been made is

$$P(T) = p + \frac{q}{m}Y(T)$$

where p is the probability of adopting at time 0, $Y(T)$ are the total number of those who have bought until time T, m is the initial number sold and q is the coefficient of imitation, reflecting the pressure to adopt the innovation. This leads to an adoption rate of the form

$$\dot{Y}(t) = (N - Y(t))(p + \frac{q}{m}Y(T))$$

where N is the total size of the population.

Kalish proposed a coupled adoption and awareness model that includes advertising [113]. Let $I(t)$ be the fraction of the population aware at time t, $A(t)$ be an advertising effort and $f(A(t))$ be the likelihood that a random agent will be exposed to the advertising. Then if b_a is the information transfer rate of the agent that has adopted and b_i is the information transfer rate of the agent that is aware of the product but has not adopted then the information dynamic follows

$$\dot{I} = (1 - I)\left(f(A) + b_a I + (b_i - b_a)\frac{Y}{N}\right)$$

Kalish also took the price of the product P into account as well as the uncertainty of the value of the product which is parameterized by u. The model also assumes that the consumer can only adopt the product in the case where they are aware of it. Then considering a likelihood of adoption k gives

$$\dot{Y} = \left(N\left(\frac{PN}{uY}\right)I - Y\right)k$$

Similar to the first SIS model mentioned in Section 2.4.1, both Bass's and Kalish's model assumes full connectivity of the graph and models the system with only two differential equations, aggregating the population into one group. There are some other models which base the interaction dynamic on a network structure, such as the Continuous Opinion Discrete Action (CODA) model; which provides a model of discrete product adoption with Bayesian opinion updates [114]. However the Bayesian opinion update only depends on the adoption actions of network neighbors, not their opinions.

Threshold Models

The diffusion of innovations has also been modeled through the use of threshold models for a variety of innovations and behaviors [115, 116, 117, 118]. The core of these models is that an agent will consider the behavior of its network neighbors and if a sufficient number of them has adopted the innovation (i.e. the number of adopting neighbors is above a threshold), the agent will also adopt. Differences in thresholds cause the varied adoption behavior seen in social networks: agents with low thresholds will adopt early causing other agents with higher thresholds to adopt. The study of these models often seeks to understand how the distribution of thresholds affects adoption behavior.

2.5 Conclusion

In this chapter, a number of concepts from the study of complex networks were introduced. From the section on the control of networks, it was shown that the underlying graph structure of a system has important implications for the ability to control the system. For both consensus and structural dynamics, symmetry with respect to an input was shown to be a sufficient condition for a lack of controllability. It was also shown that input selection is a hard problem for known systems. Finally the spreading behavior of epidemics, opinions, and products was discussed through sociological theory and a number of models. The theory highlighted that the spreading behavior of products is similar to that of epidemics and that opinion is a vital component of product spread.

CHAPTER 3
HERDABILITY

This chapter introduces the notion of herdability, a control theoretic notion that is particularly applicable to understanding the behavior of complex networks. This section expands upon the results in [119].

A system is completely herdable if all the elements of the state can be brought above a threshold by the application of a control input. Thresholds capture an important class of behavior in biological and social systems, in which a system reaches a tipping point and as a result the behavior of a system may change dramatically. Examples of behavior driven by thresholds include the firing of a neuron [120], quorum sensing in bacteria [121], and collective social action [115, 116].

Herdability is a set reachability condition, asking if the state can be driven to the set where all elements of the state are above a given threshold. As such, herdability captures the ability to apply input to encourage a general change in behavior. This is a different approach to addressing the challenges of interacting with large, complex networks. As has been shown in Chapter 2 often these large scale systems dealt with by taking into account uncertainty in the dynamics, as exemplified by structural controllability. Herdability instead asks for a looser condition as the end goal.

The applicability of herdability while interacting with large complex networks can be best shown via an example from the context of social networks. Consider the case where the state of a dynamical system represents the percentage of a given community that will vote for a political candidate. These communities interact with each other via a network structure which is based on friendships between communities, proximity and a host of other factors. We are on the campaign staff of Candidate X and have a control input, in that we are able to advertise for our candidate. This advertising effort can be distributed in a given community to encourage them to vote for or against a certain candidate, with the goal of having Candidate X win.

If this large complex system was completely controllable, that would imply that each community could be driven to a specific desired voting percentage for Candidate X, specifically a voting percentage that is completely unrelated to the voting percentage of neighboring communities. Now while that may sound appealing (or potentially horrifying), this level of control over the voting behavior of a population would be very expensive to achieve and is unnecessary. An advertising campaign is successful if the state can be driven high enough for Candidate X to win, regardless of whether communities can be made to vote at any specific percentage as would be required by complete controllability. Instead of insisting on complete controllability, if the campaign staff selects the advertising effort such that the system is herdable to the point where the voting percentage of each community is above 50%, then Candidate X has won.

3.1 Characterizing Herdability

In this section, the basic theory of the herdability of continuous time, linear dynamical systems is presented as well as a characterization of herdability based on system matrices such as the controllability grammian W_c and controllability matrix $\mathcal{C}$. All the necessary concepts from the study of linear systems, as well as the form of the various matrices have been summarized in Appendix B. Of course before characterizing herdability, the following definitions of herdability are required.

Definition 3.1.1. *The state i of a linear system is d-herdable if $\forall \mathbf{x}(0) \in \mathbb{R}^n$, there exists a finite time t_f and an input $\mathbf{u}(t),\ t \in [0, t_f]$ such that $(\mathbf{x}(t_f))_i \geq d$ under control input $\mathbf{u}(t)$.*

If the system is d-herdable for any $d \geq 0$ it will be said to be herdable. In the case of linear systems, d-herdability for $d > 0$ and herdability are equivalent. As the following discussion concerns itself with the analysis of linear systems, we will refer only to the herdability of such systems.

Definition 3.1.2. *The state i of a linear system is herdable if $\forall \mathbf{x}(0) \in \mathbb{R}^n, h \geq 0$, there exists a finite time t_f and an input $\mathbf{u}(t),\ t \in [0, t_f]$ such that $(\mathbf{x}(t_f))_i \geq h$ under control input $\mathbf{u}(t)$.*

Definition 3.1.3. *A set of states, $\mathcal{X} \subseteq \{1, 2, \ldots, n\}$, is herdable if each individual state in $\mathcal{X}$ is herdable together, i.e. if $\forall \mathbf{x}(0) \in \mathbb{R}^n$ and $h \geq 0$, there exists a finite time t_f and an input $\mathbf{u}(t),\ t \in [0, t_f]$ such that $(\mathbf{x}(t_f))_i \geq h,\ \forall i \in \mathcal{X}$ under control input $\mathbf{u}(t)$.*

Definition 3.1.4. *A linear system is completely herdable if all states in the system are herdable together.*

To translate the definition of herdability to a necessary and sufficient condition for herdability requires some basic concepts from the study of linear systems, specifically the relation between the reachable subspace and the controllability grammian W_c and controllability matrix $\mathcal{C}$ discussed in Appendix B.

It is possible to characterize the herdability of a system based on its controllability matrix. With Lemma $B.0.1$ it is possible to prove the following Theorem, which gives a necessary and sufficient condition for the herdability of a subset of states.

Theorem 3.1.1. *A set of states $\mathcal{X} \subseteq \{1, 2, \ldots, n\}$ in a linear system is herdable if and only if there is exists a vector $\mathbf{k} \in \mathrm{range}(\mathcal{C})$ that satisfies $(\mathbf{k})_i > 0$ for all $i \in \mathcal{X}$.*

Proof. Define the set $\mathcal{K}$ to be the set that contains the positive elements of $\mathbf{k}$, $\mathcal{K} = \{p \mid p > 0\ \wedge\ \exists\, i$ such that $(\mathbf{k})_i = p\}$.

($\mathbf{k} \in \mathrm{range}(\mathcal{C}) \Rightarrow \mathcal{X}$ is herdable) Consider the problem of controlling all states in the set $\mathcal{X}$ to be greater than some lower threshold $h \geq 0$ from an initial condition $\mathbf{x}(0)$. Suppose there is a $\mathbf{k} \in \mathrm{range}(\mathcal{C})$, that satisfies $(\mathbf{k})_i > 0$ if $i \in \mathcal{X}$. As $\mathbf{k} \in \mathrm{range}(\mathcal{C})$, $\exists \boldsymbol{\alpha}$ such that

$$\mathcal{C}\boldsymbol{\alpha} = \mathbf{k}.$$

If

$$\gamma > \frac{\max_j\ (h\mathbf{1}_n - e^{At}\mathbf{x}(0))_j}{\min \mathcal{K}}$$

and $\mathbf{v} = \gamma\boldsymbol{\alpha}$ then for all $i \in \mathcal{X}$ it holds that

$$(\mathcal{C}\mathbf{v})_i > (h\mathbf{1}_n - e^{At}\mathbf{x}(0))_i.$$

As the range of $\mathcal{C}$ is the same as the reachable subspace, $\exists \mathbf{u}(\cdot)$ such that for all $i \in \mathcal{X}$

$$(e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau)_i > h$$

then all states in $\mathcal{X}$ can be made larger that h and as h is arbitrary the subset of states $\mathcal{X}$ is herdable.

($\mathcal{X}$ is herdable $\Rightarrow \mathbf{k} \in \text{range}(\mathcal{C})$) As the set of state nodes $\mathcal{X}$ is herdable, each element of $\mathcal{X}$ can be made larger than some $h^* > 0$ from any initial condition. Consider the initial condition $x(0) = \mathbf{0}_n$. Then by the herdability of the set $\mathcal{X}$ there exists a vector $\mathbf{k}^*$ that satisfies $(\mathbf{k}^*)_i > h^* \ \ \forall i \in \mathcal{X}$ and an input $\mathbf{u}(\cdot)$ such that

$$\int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau = \mathbf{k}^*$$

Then $(\mathbf{k}^*)_i > 0 \ \ \forall i \in \mathcal{X}$ by the definition of h^*. By the definition of $\mathcal{R}[0,t]$, $\mathbf{k}^* \in \mathcal{R}[0,t]$ and consequently $\mathbf{k}^* \in \text{range}(\mathcal{C})$ by Lemma $B.0.1$. □

Corollary 3.1.2. *A linear system is completely herdable if and only if there exists an element-wise positive vector* $\mathbf{k} \in \text{range}(\mathcal{C})$.

A similar statement can be made about the controllability grammian W_c of a system, following directly from Lemma $B.0.1$ and Theorem $3.1.1$.

Corollary 3.1.3. *A set of states* $\mathcal{X} \subseteq \{1, 2, \ldots, n\}$ *in a linear system is herdable if and only if there is exists a vector* $\mathbf{k} \in \text{range}(W_c)$ *that satisfies* $(\mathbf{k})_i > 0$ *for all* $i \in \mathcal{X}$. *A linear system is completely herdable if and only if there exists an element-wise positive vector* $k \in \text{range}(W_c)$.

There is also a necessary condition for herdability which arises based on the characterization of Theorem 3.1.1.

Theorem 3.1.4. *If a linear system is completely herdable then there exists an element-wise positive vector* $k \in \text{range}([A\ B])$.

Proof. If a linear system is completely herdable, then by Theorem 3.1.1, there is an element-wise positive vector $k \in \text{range}(\mathcal{C})$. As such there exists a $y \in \mathbb{R}^{nm}$ such that

$$\mathcal{C}y = k$$

. Dividing y into n subcomponents, with each $y_i \in \mathbb{R}^m$:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

gives that

$$\begin{aligned} k &= \mathcal{C}y \\ &= By_1 + ABy_2 + \cdots + A^{n-1}By_n \\ &= By_1 + A(By_2 + ABy_3 + \cdots + A^{n-2}y_n). \end{aligned}$$

Then $k \in \text{range}([AB])$ as

$$k = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} By_2 + ABy_3 + \cdots + A^{n-2}y_n \\ y_1 \end{bmatrix}.$$

□

While Theorem 3.1.4 is only a necessary condition, it can still be valuable for designing the interaction with the system via the selection of a B matrix. In the case that the A matrix is such that there are no element-wise $k \in \text{range}(A)$ then B can be designed such that there is an element-wise $k \in \text{range}(A)$, with relatively little computational cost. However as Theorem $3.1.4$ is a necessary condition, other more expensive methods would be required to verify system herdability.

The final result of this section is a lemma which is useful to show that a system is not completely herdable and therefore useful for showing other necessary conditions for complete herdability.

Lemma 3.1.5. *A state i is herdable if and only if $\exists j$ such that*

$$(\mathcal{C})_{i,j} \neq 0.$$

Proof. ($(\mathcal{C})_{i,j} \neq 0 \Rightarrow i$ Herdable) If $(\mathcal{C})_{i,j} \neq 0$ then by appropriate choice of the j-th element of a vector $\mathbf{z}$ it holds for a positive constant w that:

$$(\mathcal{C}\mathbf{z})_i = w$$

Then there is a vector $\mathbf{k} \in \text{range}(\mathcal{C})$ with $(\mathbf{k})_i > 0$ and v_{xi} is herdable by Theorem $3.1.1$.

(Herdable $\Rightarrow$ $(\mathcal{C})_{i,j} \neq 0$) Suppose the contrary. Then by assumption $\forall j\ (\mathcal{C})_{i,j} = 0$. Consider making $\mathbf{x}(t) \geq h$ from an initial state $\mathbf{x}(0) = \mathbf{0}_n$. As $\forall j\ (\mathcal{C})_{i,j} = 0$, it holds that $\forall z \in \text{range}(\mathcal{C}),\ (z)_i = 0$ and by Lemma $B.0.1$ for any reachable $\mathbf{x}(t)\ \ \forall t \geq 0$, $(\mathbf{x}(t))_i = 0$ and state i is not herdable. □

As will be seen, determining the element-wise positive vector k which shows that the system is herdable is non-trivial in the case where the system is herdable but not controllable. Checking that a specific vector is in $\text{range}(\mathcal{C})$ is easy but verifying that there exists any element-wise positive vector in $\text{range}(\mathcal{C})$ can be computationally expensive. As such many of the results presented in the next section are cases when sufficient conditions for herdability can be determined relatively efficiently.

3.2 Sufficient Conditions for Herdability

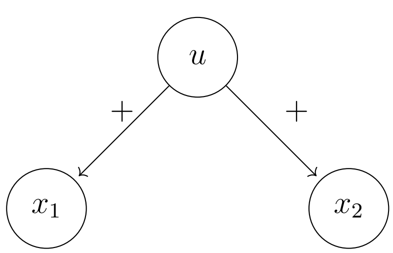
The section provides a number of sufficient conditions for herdability based on the structure of the controllability matrix $\mathcal{C}$ and controllability grammian W_c.

To do so requires the following set of definitions from the study of qualitative systems, which we recall from Chapter 2. A vector is balanced if it is the zero vector or contains both positive and negative elements. A vector is unisigned if its non-zero elements all have the same sign. A unisigned vector is positive (negative) if all non-zero elements have a positive (negative) sign.

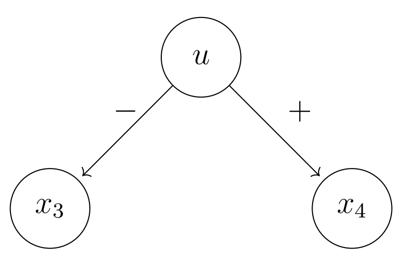
Definition 3.2.1. *A state i in the system is strictly herdable, if there $\exists k \in \mathcal{R}[0, t]$ such that k is unisigned and $k_i \neq 0$.*

Definition 3.2.2. *A state i is loosely herdable if all vectors $k \in \mathcal{R}[0, t]$ such that $k_i \neq 0$ are balanced.*

Verifying that a state is indeed loosely herdable can be difficult, as such this section focuses on verifying that a state is strictly herdable with low computation cost. As an example of loose herdability consider the signed dilation shown in Figure 3.1. If the dilation has the same sign, both nodes are strictly herdable; while if the signs are different the nodes are loosely herdable. Selecting to herd a loosely herdable node drives other nodes out of the herding set, as can be seen from the signed dilation in Figure 3.1b.



(a)



(b)

Figure 3.1: Both nodes in 3.1a are strictly herdable, while both nodes in 3.1b are loosely herdable

This section now considers the implications of the definition of strict herdability, first in the context of the controllability matrix. Let $\mathcal{S}$ be the set of nodes such that for all $i \in \mathcal{S}$ there exists a unisigned column of $\mathcal{C}$ with a non-zero element at position i.

Lemma 3.2.1. *Each element of $\mathcal{S}$ is strictly herdable.*

Proof. By the definition of $\mathcal{S}$, for node $i_s \in \mathcal{S}$ there exists a j_s such that $(\mathcal{C})_{i_s,j_s} \neq 0$ and each non-zero element of $(\mathcal{C})_{:,j_s}$, has the same sign. If $(\mathcal{C})_{i_s,j_s} > 0$, then $(\mathcal{C})_{:,j_s} \in \mathrm{range}(\mathcal{C})$ and the node i_s is strictly herdable. Alternatively if $(\mathcal{C})_{i_s,j_s} < 0$, then the positive unisigned vector $-(\mathcal{C})_{:,j_s} \in \mathrm{range}(\mathcal{C})$ and the node i_s is strictly herdable. □

Let $\mathcal{D} = \{1, 2, \ldots, n\} \setminus \mathcal{S}$. If $l \in \mathcal{D}$ there exists a j such that $(\mathcal{C})_{l,j} \neq 0$, then the column vector $(\mathcal{C})_{:,j}$ is balanced.

Definition 3.2.3. *Node z balances node l at j if it has a different sign than l in the column $(\mathcal{C})_{:,j}$ and favors node l at j if it has the same sign as l in the column $(\mathcal{C})_{:,j}$.*

Lemma 3.2.2. *If for $l \in \mathcal{D}$ there exists a j such that l is opposed only by strictly herdable nodes at j then l is strictly herdable.*

Proof. Let $\hat{\mathcal{S}}$ be the set of nodes which oppose l at j. By definition of strictly herdable nodes, for each $s \in \hat{\mathcal{S}}$ there exists a vector v^s such that $v^s_s > 0$ and each non-zero element of the vector v^s has the same sign. Consider the set of vectors $S = \{v_s, b\}$ where $b = (\mathcal{C})_{:,j}$, the vector where l is opposed by the elements of $\hat{\mathcal{S}}$. Then $\hat{s} = \sum_s v^s + sign(b_l)b$ is a vector which is positive at l, at each node that favors l and at each node $s \in \hat{\mathcal{S}}$. As $\hat{s} \in \mathrm{range}(\mathcal{C})$, l is strictly herdable. □

The following result shows why strictly herdable nodes are important.

Theorem 3.2.3. *All states $i \in \{1, 2, \ldots, n\}$ are strictly herdable if and only if the system is completely herdable.*

Proof. (Sufficiency) As each state $i \in \{1, 2, \ldots, n\}$ is strictly herdable, there exists a vector $k^i \in \text{range}(\mathcal{C})$ which is element-wise non-negative and $k_i^i > 0$. Then the element wise positive $k = \sum_i k^i \in \text{range}(\mathcal{C})$ and the system is completely herdable.

(Necessity) As the system is completely herdable, there is an element-wise positive vector $k \in \text{range}(\mathcal{C})$. Then for each state $i \in \{1, 2, \ldots, n\}$ $k_i > 0$ and the other elements are nonnegative, so state i is strictly herdable. □

These results provide a way to check for the herdability of a system efficiently from the controllability grammian, simply by inspecting the columns of $\mathcal{C}$. A similar set of results hold for the columns of the controllability grammian W_c though they are not described here. As will be seen shortly, the controllability matrix has the advantage of being related to the underlying graph structure of the network, which can present further opportunities for determining system herdability.

3.3 Characterizing Dynamical Systems via Graphs

This section presents a characterization of a dynamical system as a signed, directed graph. This characterization will allow an exploration of the relationship between the ability to control a system and the structure of the interactions between the states as well as the interaction between the inputs and the states of the system.

A continuous time, linear system can be represented by three graphs; each of which contains different levels of information about the interactions between the states and inputs. The first is an unweighted, unsigned directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex (equivalently node) set and $\mathcal{E}$ is the edge set. This graph is commonly used in the study of structural controllability to represent a class of systems which share the same structure. The second graph is a signed graph $\mathcal{G}^s = (\mathcal{V}, \mathcal{E}, s(\cdot))$ where $s(\cdot)$ accepts an edge and returns a label in $\{+1, -1\}$, which is the sign of the edge. This signed graph represents a class of systems whose edge weights have the same sign pattern. Similarly this representation was used in the study of sign controllability to represent a class of systems which share the same sign structure. The third graph is a weighted graph $\mathcal{G}^w = (\mathcal{V}, \mathcal{E}, w(\cdot))$ where $w(\cdot)$ accepts an edge and returns a weight in $\mathbb{R}$. The weighted graph is the representation of a single system.

As will be seen later, the weighted graph $\mathcal{G}^w$ can be directly related to the controllability matrix $\mathcal{C}$ and therefore the controllability properties of the system. The following sections focus on the interplay between $\mathcal{G}^s$ and $\mathcal{G}^w$, in that the presented structural results are cases where the results for the herdability of a system based on the weighted $\mathcal{G}^w$ can be extended to all signed graphs with the same sign structure $\mathcal{G}^s$ regardless of the weights of the edges in $\mathcal{G}^w$, a notion similar to strong structural controllability and sign controllability as discussed in Chapter 2. This notion is called sign herdability.

Definition 3.3.1. *A system is completely sign herdable if all systems which share the same sign structure $\mathcal{G}^s$ are completely herdable.*

The formal definition of the graphs follows. The set of vertices satisfies $\mathcal{V} = \mathcal{V}_x \cup \mathcal{V}_u,\ \mathcal{V}_x \cap \mathcal{V}_u = \emptyset$, where $\mathcal{V}_x = \{v_{x1}, v_{x2}, \ldots, v_{xn}\}$ is a set of vertices representing the states of the system and $\mathcal{V}_u = \{v_{u1}, v_{u2}, \ldots, v_{um}\}$ is a set of nodes representing the inputs to the system. An arbitrary element of $\mathcal{V}$ will be referred to by v_i for some index i, as will arbitrary elements $v_{xi} \in \mathcal{V}_x$ and $v_{ui} \in \mathcal{V}_u$. The state i will now be interchangeably referred to by the node v_{xi} as will the input j and the node v_{uj}.

The edge set satisfies $\mathcal{E} = \mathcal{E}_x \cup \mathcal{E}_u$ where the edges in $\mathcal{E}_x$ represent interactions between states of the system, while the edges in $\mathcal{E}_u$ represent interactions between the inputs and the states. Denote the directed edge from v_i to v_j as (v_i, v_j). Then $(v_{xi}, v_{xj}) \in \mathcal{E}_x \Leftrightarrow A(j,i) \neq 0$ and $(v_{ui}, v_{xj}) \in \mathcal{E}_u \Leftrightarrow B(j,i) \neq 0$. An arbitrary element of $\mathcal{E}$ will be referred to by e_i for some index i. By partitioning the node and edges sets, it is possible to define the state subgraph $\mathcal{G}_x = (\mathcal{V}_x, \mathcal{E}_x)$, which captures only interactions between states as well as the input subgraph $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u)$ which captures interactions from the inputs to the states. Note that the input nodes do not interact with each other nor is it possible to have an edge of the form (v_{xi}, v_{uj}), which would imply that the states influences the evolution of the input.

When considering the signed graph $\mathcal{G}^s$, $s((v_{xi}, v_{xj})) = \text{sgn}(A(j,i))$ and $s((v_{ui}, v_{xj})) = \text{sgn}(B(j,i))$. Similarly for $\mathcal{G}^w$, $w((v_{xi}, v_{xj})) = A(j,i)$ and $w((v_{ui}, v_{xj})) = B(j,i)$.

As an example, consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 5 & 0 & 2 \\ 4 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u} \tag{3.1}$$

which is translated into $\mathcal{G}^s$ and $\mathcal{G}^w$ in Figure 3.2.

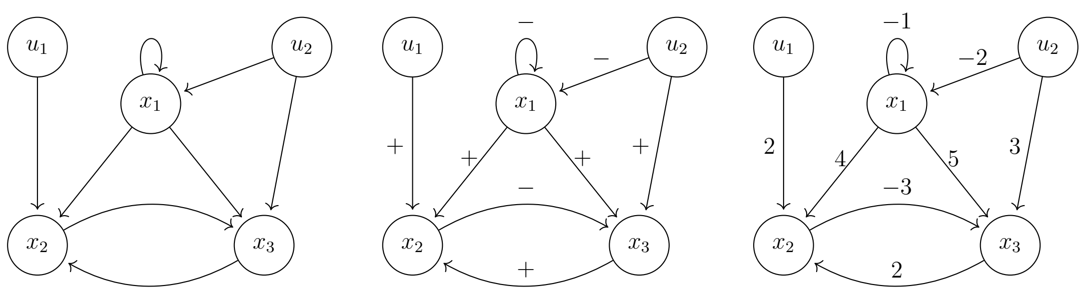


Figure 3.2: The graphs of the system in in Equation (3.1). 3.2a: $\mathcal{G}$ the unsigned, unweighted graph. 3.2b: $\mathcal{G}^s$ the signed graph. 3.2c: $\mathcal{G}^w$ the weighted graph

To begin classifying a continuous time, linear system based on the signed graph $\mathcal{G}^s$, we define two basic types of sets. Let $\mathcal{N}_d^j$ be the set of nodes reachable from v_{uj} via at least one negative walk of length d. Similarly $\mathcal{P}_d^j$ is the set of nodes reachable from v_{uj} through at least one positive walk of length d. If there is only one input to the system, the superscript will be dropped to refer to $\mathcal{N}_d$ and $\mathcal{P}_d$ instead of $\mathcal{N}_d^1$ and $\mathcal{P}_d^1$. Figure 3.3 shows an example of these sets.

As will be seen, the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ can provide sufficient information to determine the sign-herdability of a continuous time, linear system. To show this requires classifying the structure of the weighted graph $\mathcal{G}^w$. Consider the total weight of positively signed walks

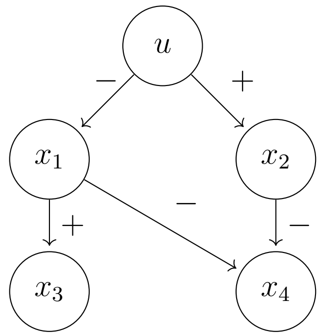


Figure 3.3: An example of $\mathcal{N}_d$ and $\mathcal{P}_d$: $\mathcal{N}_1 = \{x_1\}, \mathcal{N}_2 = \{x_3, x_4\}, \mathcal{P}_1 = \{x_2\}, \mathcal{P}_2 = \{x_4\}$

from input v_{uj} to node v_{xi} with length d,

$$\rho^+_{j\to i,d} = \sum_{\pi \in \theta^+_d(v_{uj}, v_{xi})} w(\pi),$$

where $\theta^+_d(v_{uj}, v_{xi})$ is the set of positive walks of length d from v_{uj} to v_{xi}. From the definition of $\mathcal{P}^j_d$, it holds that $\rho^+_{j\to i,d} > 0$ if $v_{xi} \in \mathcal{P}^j_d$ and 0 else. Similarly the total weight of negatively signed walks from input v_{uj} to node v_{xi} with length d is

$$\rho^-_{j\to i,d} = \sum_{\pi \in \theta^-_d(v_{uj}, v_{xi})} w(\pi),$$

where $\theta^-_d(v_{uj}, v_{xi})$ is the set of negative walks of length d from v_{uj} to v_{xi} and it follows that $\rho^-_{j\to i,d} < 0$ if $v_{xi} \in \mathcal{N}^j_d$ and 0 else. The weight of all walks from input v_{uj} of length d is

$$\rho_{j\to i,d} = \rho^+_{j\to i,d} + \rho^-_{j\to i,d}.$$

It is possible that based on the sets $\mathcal{P}^j_d$ and $\mathcal{N}^j_d$ that there be a 0 path weight even though paths exist. Consider the example shown in Figure 3.4. The signed graph represents all

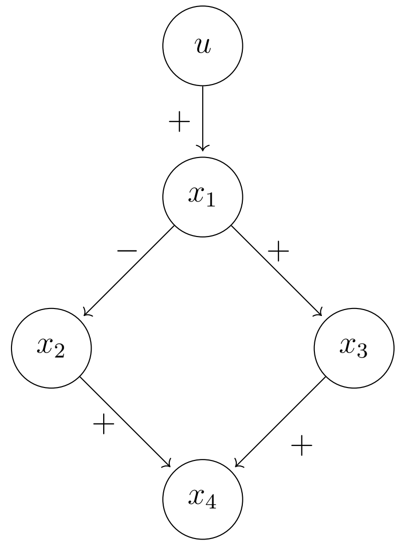


Figure 3.4: An example of a signed graph where the sets $\mathcal{N}_d$ and $\mathcal{P}_d$ do not uniquely determine the sign of $\rho_{j\to i,d}$, as there is the possibility that two paths cancel each other out

systems of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & \alpha_4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1 > 0$. Here the total walk weight to node v_{x4} at length 2 is

$$\rho_{1\to 4,2} = \beta_1 \left(\alpha_2\alpha_4 - \alpha_1\alpha_3\right)$$

which can be positive, negative or zero depending on the values of the various constants.

The case where the sign of $\rho_{j\to i,d}$ is determined by $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ is shown in the following Lemmas. These Lemmas follow directly from the definitions of the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ and as such are presented without proof.

Lemma 3.3.1. *If* $v_{xi} \in \mathcal{P}_d^j \wedge v_{xi} \notin \mathcal{N}_d^j$ *then* $\rho_{j\to i,d} > 0$.

Lemma 3.3.2. *If* $v_{xi} \in \mathcal{N}_d^j \wedge v_{xi} \notin \mathcal{P}_d^j$ *then* $\rho_{j\to i,d} < 0$.

It is possible to relate $\rho_{j\to i,d}$ with the system matrices A, B and ultimately the controllability properties of the system. Define a weighted adjacency matrix $\tilde{A}_w$ for $\mathcal{G}_x^w$, where $(\tilde{A}_w)_{i,j} = w((v_{xj}, v_{xi}))$ if $(v_{xj}, v_{xi}) \in \mathcal{E}_x$ and $(\tilde{A}_w)_{i,j} = 0$ if not. Define a weighted adjacency matrix $\tilde{B}_w$ for $\mathcal{G}_u^w$, where $(\tilde{B}_w)_{i,j} = w((v_{uj}, v_{xi}))$ if $(v_{uj}, v_{xi}) \in \mathcal{E}_u$ and $(\tilde{B}_w)_{i,j} = 0$ if not. Note that from the definition of the weight of an edge, $\tilde{A}_w = A$ and $\tilde{B}_w = B$. Then $(A^{d-1}B)_{i,j}$ is the sum of the weight of all walks of length d from v_{uj} to v_{xi}. More formally:

Lemma 3.3.3.

$$(A^{d-1}B)_{i,j} = \rho_{j \to i,d}.$$

Proof. The result will be shown via proof by induction on d. Consider the case of $d = 1$. By the definition of the weight of an edge:

$$(B)_{i,j} = \rho_{j \to i,1}.$$

Consider the weight of all walks of length d from an input v_{uj} to a state node v_{xi}. By assumption, $(A^{d-2}B)_{i,j} = \rho_{j \to i,d-1}$. As $A^{d-1}B = AA^{d-2}B$, it follows that

$$(A^{d-1}B)_{i,j} = \sum_{k=1}^{n} (A)_{i,k} \rho_{j \to k,d-1}.$$

As a walk of length d is the concatenation of a walk of length $d-1$ and a walk of length 1, it follows from the definition of the weight of a walk that

$$\sum_{k=1}^{n} (A)_{i,k} \rho_{j \to k,d-1} = \rho_{j \to i,d}.$$

□

As $\mathcal{C}$ is the concatenation of matrix products from B to $A^{n-1}B$, Lemma 3.3.3 shows that the herdability of a continuous time linear system is determined by walks on $\mathcal{G}^w$ which have lengths from 1 to n. Further:

Lemma 3.3.4. $(\mathcal{C})_{i,(m(d-1)+j)} = \rho_{j \to i,d}$.

Proof. From Lemma 3.3.3,

$$(A^{d-1}B)_{i,j} = \rho_{j\to i,d}.$$

From the definition of the controllability matrix, the sub-matrix

$$(\mathcal{C})_{:,m(d-1)+1:md} = A^{d-1}B.$$

The result follows. □

3.4 A Necessary Condition for Complete Herdability

This section shows how graph structure and system herdability are related by providing a necessary condition for complete herdability of a system known as input connectability. It also explores some examples that show why input connectability is only a necessary condition. These examples have been explored in previous sections, though in less depth.

Definition 3.4.1. *A graph is input connectable (equivalently, accessible) if*

$$\bigcup_{v_{uj}\in\mathcal{V}_u} \mathscr{R}_j = \mathcal{V}_x,$$

where $\mathscr{R}_j$ *is the set of nodes reachable from* v_{uj}*:* $\mathscr{R}_j = \{v_{xi} \in \mathcal{V}_x \mid v_{uj} \to v_{xi}\}$.

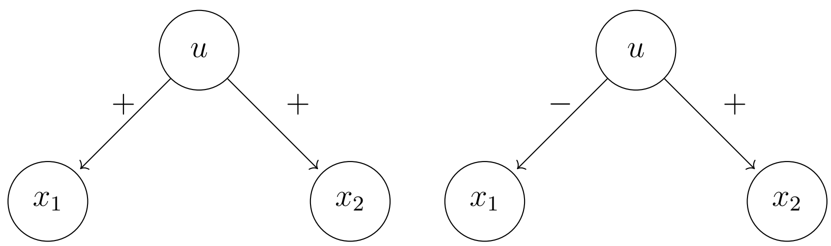
If a single node is not herdable then the system is not completely herdable. As such, Lemma 3.1.5 can be used to show the following:

Theorem 3.4.1. *If a system is completely herdable, then it is input connectable.*

Proof. Suppose not. Then by assumption, there exists a node v_{xi} such that $v_{xi} \notin \bigcup_j \mathscr{R}_j$ and as such there is no walk from an input to v_{xi}. If there is no walk to v_{xi}, then $(\mathcal{C})_{i,:} = \mathbf{0}_n$ by Lemma 3.3.4 and the node will not be herdable by Lemma 3.1.5. As such, the system is not completely herdable. □

Consider the following two examples that show why input connectability is only a necessary condition and not a sufficient condition. These examples motivate the the condition of Theorem 3.5.1 in Section 3.2, which ensures that the system is input connectable and that the cases presented in these examples do not occur.

The first example has to do with the structure of the signed graph $\mathcal{G}^s$. We return to the example of the signed dilation, which is shown again in Figure 3.5.



(a) (b)

Figure 3.5: The systems represented by the graph structure in 3.5a are completely herdable, while 3.5b shows a graph structure that is never completely herdable

Figure 3.5a represents systems of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \mathbf{u}$$

where $\beta_1, \beta_2 > 0$, which gives a controllability matrix:

$$\mathcal{C} = \begin{bmatrix} \beta_1 & 0 \\ \beta_2 & 0 \end{bmatrix}$$

And by inspection,

$$\text{range}(\mathcal{C}) = span \left(\left\{ \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right\} \right).$$

This system is always completely herdable.

On the other hand, Figure 3.5b can be translated to systems of the form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\beta_1 \\ \beta_2 \end{bmatrix} \mathbf{u}$$

where $\beta_1, \beta_2 > 0$. This gives a controllability matrix:

$$\mathcal{C} = \begin{bmatrix} -\beta_1 & 0 \\ \beta_2 & 0 \end{bmatrix}$$

And by inspection,

$$\text{range}(\mathcal{C}) = span\left(\left\{\begin{bmatrix} -\beta_1 \\ \beta_2 \end{bmatrix}\right\}\right).$$

Here either v_{x1} or v_{x2} can be made larger than any threshold $h \geq 0$ but not both. This example illustrates a fundamental trade off when herding signed digraphs, which is that at a given distance from the input either $\mathcal{N}_d$ or $\mathcal{P}_d$ can be herded but not both. In the language of social networks, it is not possible to simultaneously convince an enemy and a friend.

It turns out that Figure 3.5a is an example of a positive system. In the case of a positive system, input connectability is a necessary and sufficient condition for complete herdability.

Theorem 3.4.2. *A positive linear system is completely herdable if and only if it is input connectable.*

Proof. (Sufficiency) By Theorem 8 of [46], an input connectable, positive linear system is excitable. Then there is an element-wise positive vector in the reachable subspace, which is also the range of the controllability matrix by Lemma $B.0.1$. Then by Corollary $3.1.2$, the system is completely herdable.

(Necessity) Follows from Theorem $3.4.1$. □

The second example that shows why input connectability is only a necessary condition and not a sufficient condition can be seen based on the weighted graph $\mathcal{G}^w$, specifically the cancellation of walk weights from an input to a state node. It is possible that a node be included in both $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ which could lead to a combination of weights such that $\rho_{j\to i,d} = 0$. If the only walks to v_{xi} are of length d then the node v_{xi} is not herdable, as is the case for v_{x4} in Figure 3.4. The following lemma shows a condition which ensures this undesirable interaction does not occur.

Lemma 3.4.3. *If* $v_{xk} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j \wedge v_{xk} \notin \mathcal{N}_d^j \cap \mathcal{P}_d^j$ *then* $\rho_{j\to i,d} \neq 0$.

Proof. Suppose the contrary. Then

$$\rho_{j\to k,d} = 0$$
$$\rho^+_{j\to k,d} + \rho^-_{j\to k,d} = 0.$$

As $v_{xk} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j$ it holds that

$$\rho^+_{j\to k,d} > 0,\, \rho^-_{j\to k,d} < 0$$

which implies that

$$v_{xi} \in \mathcal{P}_d^j,\, v_{xi} \in \mathcal{N}_d^j$$
$$v_{xi} \in \mathcal{P}_d^j \cap \mathcal{N}_d^j$$

□

It is also possible to show that such a condition will not hold in a generic sense, which requires a brief digression into structural herdability. Consider a system

$$\dot{x} = A_s x + B_s u, \tag{3.2}$$

which was described in Section 2.2.3.

Definition 3.4.2. *A node i is individually structurally herdable if the node i is generically individually herdable.*

Theorem 3.4.4. *If node i is accessible, then node i is individually structurally herdable.*

Proof. By Lemma 3.1.5, the herdability of a node is equivalent to a non-zero weight path from an input. As node i is accessible, assume that there is path of length d from input node j. Then the weight of this path $\rho_{j \to i,d}$, is a polynomial in the edge weights. The set $\mathcal{O} = \{\tilde{A}_s, \tilde{B}_s \in \mathbb{R}^{n_A} \times \mathbb{R}^{n_B} | \rho_{j \to i,d} = 0\}$ is a proper variety. This can be seen by the fact that it is possible to select an edge weight combination that is results in a non-zero path weight (i.e. $\mathcal{O}$ is not the full parameter space) nor is it empty. Equivalently perturbing a single edge weight by epsilon will results in a non-zero path weight. As such the weight of the path is non-zero generically. □

Even though a node is individually structurally herdable, accessibility does not imply structural herdability of groups of nodes. The counter example is the signed dilation shown most recently in Figure 3.5.

For this graph the set of edge weights where the two nodes are not herdable at the same time is not zero measure, as the weights can both be perturbed by some ϵ and the system is still not herdable. The information about the presence of dilations in the underlying system graph is captured in the matrix $[A\ B]$, as A and B serve as weighted adjacency matrices for the graph. This hints at a connection here between the structural herdability of the system and the matrix $[A\ B]$ which has yet to be fully explored.

This section has explored a necessary condition for system herdability based on the structure of the underlying graph structure. In the next section, a number of sufficient conditions will be presented.

3.5 Using the Sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ to Determine Herdability

This section considers two variants on the theme of using the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ to determine system herdability. The first recasts the sufficient conditions of Section 3.1 in terms of the underlying graph structure of the system. The second considers whether, given the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ how one can determine whether the system is completely herdable.

3.5.1 Sufficient Graph Conditions for Herdability

This section will now consider the sufficient condition of Section 3.1 in light of the characterization of the controllability matrix given in Lemma 3.3.4. The following Theorems provide a case where the composition of the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ uniquely determines the herdability of the graph, i.e. one is able to show the sign-herdability of the system.

Theorem 3.5.1. *If for each $v_{xi} \in \mathcal{V}_x$, there exists a distance d and an input v_{uj} such that $v_{xi} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j$ and $\mathcal{N}_d^j = \emptyset \veebar \mathcal{P}_d^j = \emptyset$, where $\veebar$ denotes exclusive OR, then the system is completely sign herdable.*

Proof. Consider the herdability of a node v_{xi} which satisfies $v_{xi} \in \mathcal{N}_{d^i}^{j^i} \cup \mathcal{P}_{d^i}^{j^i}$ and $\mathcal{N}_{d^i}^{j^i} = \emptyset \veebar \mathcal{P}_{d^i}^{j^i} = \emptyset$ for some d^i and v_{uj^i}. The fact that $\mathcal{N}_{d^i}^{j^i} = \emptyset \veebar \mathcal{P}_{d^i}^{j^i} = \emptyset$ implies that $\mathcal{N}_{d^i}^{j^i} \cap \mathcal{P}_{d^i}^{j^i} = \emptyset$, and as such it must be that $v_{xi} \in \mathcal{N}_{d^i}^{j^i} \cup \mathcal{P}_{d^i}^{j^i}$ and $v_{xi} \notin \mathcal{N}_{d^i}^{j^i} \cap \mathcal{P}_{d^i}^{j^i}$.

From Lemma 3.3.4 and Lemma 3.4.3, this implies $(\mathcal{C})_{i,m(d^i-1)+j^i} \neq 0$. Additionally, as $\mathcal{N}_{d^i}^{j^i} = \emptyset \veebar \mathcal{P}_{d^i}^{j^i} = \emptyset$, Lemma 3.3.1 and Lemma 3.3.2 show that all nonzero elements of $(\mathcal{C})_{:,m(d^i-1)+j^i}$ have the same sign and that the sign does not depend on the edge weights. Each node is strictly sign herdable by Lemma 3.2.1. As this hold for all v_{xi}, the system is completely herdable Theorem 3.2.3. As the conditions only rely on the sign of the edges, the system is completely sign herdable. $\square$

Theorem 3.5.1 is an extension of Lemma 3.2.1 to the sign structure of the network. Consider the following definition which allows the extension of Lemma 3.2.2.

Definition 3.5.1. *A node v_{xi} is said to be sign balanced if there exists a distance d and an input v_{uj} such that $v_{xi} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j$, $v_{xi} \notin \mathcal{N}_d^j \cap \mathcal{P}_d^j$, and all nodes that balance v_{xi} at distance d from an input v_{uj} are sign herdable.*

Theorem 3.5.2. *If all nodes are herding balanced then the system is completely sign herdable.*

Proof. As for each v_{xi} there exists a distance d and an input v_{uj} such that $v_{xi} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j$, $v_{xi} \notin \mathcal{N}_d^j \cap \mathcal{P}_d^j$, there is a column of $\mathcal{C}$ whose sign with respect to v_{xi} is always consistent regardless of the weight of the edges in the walks that connect the input v_{uj} and v_{xi}. As it is balanced by sign herdable nodes, node v_{xi} is strictly sign herdable. As all nodes are strictly sign herdable then the whole system is sign herdable by Theorem 3.2.3. □

Theorem 3.5.1 and Theorem 3.5.2, as well as Lemmas 3.2.1 and 3.2.2 (their counterpart based on the controllability matrix $\mathcal{C}$), provide sufficient conditions to verify that a node is strictly herdable. However as they are only sufficient there are completely herdable systems which can not be identified by verifying the conditions of Lemmas 3.2.1 and 3.2.2 and Theorems 3.5.1 and 3.5.2. Figure 3.9 shows a simple example.

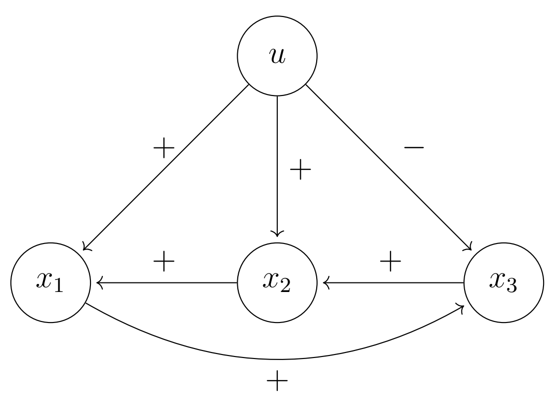


Figure 3.6: An example of a completely herdable graph which can not be identified by inspecting the columns of $\mathcal{C}$ nor the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$

3.5.2 The Subset Herdability Problem

This section will now consider another version of the herdability problem, that of determining the herdability of a system from the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$, which will be called the subset herdability problem. Admittedly this ignores any information about the weights of walks and as such only in specific cases (i.e. the system is sign herdable) is a solution to this problem a solution to the general herdability problem. Note that the sufficient conditions presented earlier in this section are instances when there is a solution to the subset herdability problem that herds all nodes and it coincides with the solution based on the controllability matrix.

The subset herdability problem is NP-hard, to show this consider the NP-hard Maximum Coverage Problem.

Definition 3.5.2. *Given a number γ and a collection of sets $S = \{S_1, S_2, \ldots, S_z\}$. The Maximum Coverage Problem asks that we find the collection of at most γ elements of S that maximizes the number of elements covered, $|\bigcup_1^\gamma S_i|$.*

Theorem 3.5.3. *The subset herdability problem is NP-hard.*

Proof. Subset herdability is an instance of the Maximum Coverage Problem. To see this consider this reformulation of the set-based herdability problem. The set-based herdability problem considers a collection of $2mn$ sets, which are the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ for distances up to n and for each of the m inputs. The task is to select at most mn of those sets, where an arbitrary selected set will be denoted with S_i, such that the most states are herded, i.e. such the number of state nodes contained in $|\bigcup_1^{mn} S_i|$ is maximized. □

Maximum Coverage is a NP-hard problem, however with subset herdability there is a restriction on which sets can be chosen to form a cover. This is because both $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ for some fixed d and j can not be selected simultaneously. Instead subset herdability involves solving an extension of the Maximum Coverage Problem known as Maximum Coverage

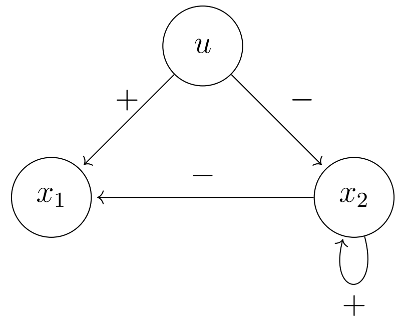


Figure 3.7: An example where the subset herdability problem returns an invalid result

with Group Budgets[122]. It has been shown that the greedy algorithm is in the best case a 2-factor approximation, i.e. that $|G| \geq \frac{1}{2}|O|$ where G is the set selected by the greedy algorithm and O is the optimal set[122].

As mentioned previously, it is possible that the solution to the subset herdability problem does not lead to a viable solution to the original herdability problem, based on the weights. To see an example as to why, consider a modification of the signed dilation shown below.

This leads to a system of the form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -\alpha_1 \\ 0 & \alpha_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_1 \\ -\beta_2 \end{bmatrix} \mathbf{u}$$

and a controllability matrix of

$$\mathcal{C} = \begin{bmatrix} \beta_1 & \beta_2\alpha_1 \\ -\beta_2 & -\beta_2\alpha_2 \end{bmatrix}$$

A greedy algorithm would add x_1 at the first distance and x_2 at the second distance (or vice versa). If $\beta_1\alpha_2 = \beta_2\alpha_1$ then the columns are linearly dependent and it is only possible to herd one node, i.e. the greedy algorithm returned an infeasible solution.

3.5.3 Subset Selection: Directed Out-branchings

If a system is not completely herdable, it is still possible to control a subset of the system nodes to enter the set $\mathcal{H}_d$. This section presents such a selection procedure in the special case of graphs that are a rooted out-branching. In such cases the structure of the system allows a greedy algorithm to return a solution that is always feasible.

A directed graph, $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ is a rooted out-branching if it has a root node $v_i \in \hat{\mathcal{V}}$ such that for every other node $v_j \in \hat{\mathcal{V}}$ there is a single directed walk from v_i to v_j. The case considered here is that of a single input, input rooted out-branching, which means that every node $v_{xi} \in \hat{\mathcal{V}}_x$ has a single in-bound walk from the single input v_u. The unique walk from v_u to v_{xi} in the input-rooted out-branching will be referred to as $\pi_t(v_u, v_{xi})$. Consider the maximum walk length between v_u and a state node, which is

$$d_{\max} = \max_{v_{xi} \in \hat{\mathcal{V}}_x} \operatorname{len}(\pi_t(v_u, v_{xi})).$$

Let $\mathbb{H}_u$ be the set of nodes made larger than some lower threshold $h \geq 0$ via a signal from the input v_u.

Theorem 3.5.4. *In an input rooted, out-branching, $\mathbb{H}_u$ follows*

$$\mathbb{H}_u = \bigcup_{d=1}^{d_{\max}} \mathcal{X}_d,$$

where $\mathcal{X}_d \in \{\mathcal{P}_d, \mathcal{N}_d, \emptyset\}$.

Proof. Consider the ability to herd a node v_{xi} and assume that $\operatorname{len}(\pi_t(v_u, v_{xi})) = d_i$. As there is only one walk from v_u to v_{xi} it holds that $(\mathcal{C})_{i,d} = 0, \ \forall d \in \mathcal{D},$ such that $d \neq d_i$ and $(\mathcal{C})_{i,d_i} \neq 0$. Further v_{xi} is either in $\mathcal{P}_d$ or in $\mathcal{N}_d$ but can not be in both as there is only one path to v_{xi}. Then if v_{xi} is in $\mathcal{P}_{d_i}$, $\rho_{u \to i,d} > 0$ by Lemma 3.3.1 and consequently $(\mathcal{C})_{i,d_i} > 0$ by Lemma 3.3.4 or if v_{xi} is in $\mathcal{N}_{d_i}$, $\rho_{u \to i,d} < 0$ by Lemma 3.3.2 and $(\mathcal{C})_{i,d_i} < 0$ by Lemma 3.3.4.

Then it follows that $(\mathcal{C})_{:,d_i}$ uniquely determines the ability to herd all nodes at distance d_i. If $\alpha_{di} = 1$ then $((\mathcal{C})_{:,d_i}\alpha_{di})_i > 0, \quad \forall i$ such that $v_{xi} \in \mathcal{P}_{d_i}$ and $\mathcal{P}_{d_i}$ is herdable by Theorem 3.1.1. If $\alpha_{di} = -1$ then $((\mathcal{C})_{:,d_i}\alpha_{di})_j > 0, \quad \forall i$ such that $v_{xi} \in \mathcal{N}_{d_i}$ and $\mathcal{N}_{d_i}$ is herdable by Theorem 3.1.1. Finally if $\alpha_{di} = 0$ then $(\mathcal{C})_{:,d_i}\alpha_{di} = \mathbf{0}_n$ and no nodes are herded. Then by the appropriate choice of α_{di} the set of nodes that can be herded at distance d_i from u, $\mathcal{X}_{di}$ must be one of $\{\mathcal{P}_d, \mathcal{N}_d, \emptyset\}$.

Construct a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ where $\forall d \in \{1, 2, \ldots, d_{\text{max}}\}$

$$(\boldsymbol{\alpha})_d = \begin{cases} 1 \quad \text{so that } \mathcal{X}_d = \mathcal{P}_d, \\ -1 \text{ so that } \mathcal{X}_d = \mathcal{N}_d, \\ 0 \quad \text{so that } \mathcal{X}_d = \emptyset, \end{cases}$$

and where the remaining $n - d_{\max}$ elements are 0. Then $\mathcal{C}\boldsymbol{\alpha}$ shows the herdability of the set of nodes $\bigcup_{d=1}^{d_{\max}} \mathcal{X}_d$. □

Corollary 3.5.5. *The maximal collection of nodes, $\mathbb{H}_u^*$, that can be herded in a input rooted out-branching satisfies*

$$|\mathbb{H}_u^*| = \sum_{l=1}^{d_{max}} \max(|\mathcal{N}_l|, |\mathcal{P}_l|).$$

In the case of an single input, input connectable, directed out-branching where $\forall d \in \{1, 2, \ldots, d_{\text{max}}\},\ \mathcal{N}_d = \emptyset \veebar \mathcal{P}_d = \emptyset$, Corollary 3.5.5 shows that $|\mathbb{H}_u^*| = n$, or equivalently that the system is completely herdable. Figure 3.8 shows an example of selecting the set of nodes that can be herded in an input rooted, out-branching.

The graph in Figure 3.8 can be translated into the following class of systems:

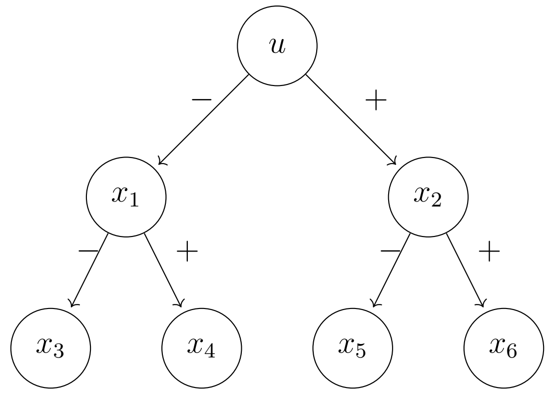


Figure 3.8: An example of an input rooted out-branching

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_3 & 0 & 0 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\beta_1 \\ \beta_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 > 0$. The system has a controllability matrix:

$$\mathcal{C} = \begin{bmatrix} -\beta_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1\beta_1 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2\beta_1 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_3\beta_2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_4\beta_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\mathrm{range}(\mathcal{C}) = span\left(\left\{\begin{bmatrix}-\beta_1\\ \beta_2\\ 0\\ 0\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 0\\ \alpha_1\beta_1\\ -\alpha_2\beta_1\\ -\alpha_3\beta_2\\ \alpha_4\beta_2\end{bmatrix}\right\}\right)$$

As such the possible sets of herded nodes are $\{1,3,6\}, \{1,4,5\}, \{2,3,6\}, \{2,4,5\}$.

The result of Theorem $3.5.4$ is similar in nature to the k-walk controllability theory[64]. The k-walk theory shows that for each $d \in \{1, 2, \ldots, d_{\text{max}}\}$ one element of either $\mathcal{N}_d$ or $\mathcal{P}_d$ can be controlled. In the graph given in Figure 3.8, the possible sets of nodes that can be controlled are $\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}\{2,5\}, \{2,6\}$. As a consequence of the k-walk theory, the maximal collection of nodes that are controlled in a directed out-branching from input v_u, $\mathbb{C}_u^*$, satisfies

$$|\mathbb{C}_u^*| = d_{\text{max}}.$$

In the case of herding a network, Corollary $3.5.5$ shows that the maximal collection of nodes, $\mathbb{H}_u^*$, will satisfy

$$d_{\text{max}} \leq |\mathbb{H}_u^*| \leq n.$$

Therefore in the worst case, the same number of nodes can be herded as can be controlled and depending on the network structure many more nodes can be herded. Note that the results of Theorem $3.5.4$ do not extend directly to the multi-input out-branching case, as in a multiple input out-branching the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ no longer uniquely determine the ability to herd a node.

3.6 Determining $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$

The previous sign herdability results all depend on the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$, which must be determined from the graph structure. To find the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ requires the use of a graph traversal algorithm. Unfortunately determining $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ via graph traversal involves considering all paths between an input node and a state node in the graph, of which there are potentially an exponential number. Let us consider the graph traversal algorithm in more depth.

The graph traversal can be done via a modified Breadth First Search, which allows nodes and edges to be revisited. Unfortunately this will increase the time complexity of the algorithm, in some cases by a lot. If A has an underlying graph which is a directed acyclic graph then each node will be visited once for each in-bound edge and each edge will be visited based on the number of times the node which is at its tail is visited. This is a linear time operation, and as such this graph traversal method is appropriate for a graph which is a directed acyclic graph.

If A is not acyclic then there is a possibility that the time complexity of the algorithm grow exponentially in the number of state nodes. The first restriction which may improve the time complexity of the algorithm is to only consider paths of up to length n from inputs as it has been previously demonstrated that these paths determine the controllability properties of the system. Unfortunately, even given this restriction the time complexity of the algorithm can be quite large. Consider the worst case graph for this algorithm, which is a complete graph with self loops, i.e. A is $\mathbf{1}_{n\times n}$. If the B matrix is $\mathbf{1}_{n\times 1}$. i.e there is a single input which interacts with each nodes, then for this graph at distance d from an input, each node will be visited n^{d-1} times. Then when the algorithm terminates at a distance n from an input, each node will have been visited n^{n-1} times, or a total time complexity of $O(n^n)$ based purely on node visits. Essentially using the graph algorithm is only feasible in the case of directed acyclic graphs.

It is possible to approximate the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ based on the controllability matrix of the system. The time complexity of this operation is dominated by calculating the last element of the controllability matrix ($A^{n-1}B$) which takes $O(n^{2.3727}log(n-1)+(n^2m))$ time. As discussed previously, when calculating the controllability matrix there is a chance that paths will cancel on the graph. This means that the approximate sets $\hat{\mathcal{N}}_d^j$ and $\hat{\mathcal{P}}_d^j$ determined via this method can be used to determine the herdability of a particular graph but not all graphs with the same sign pattern. However, if it can be shown that an additional property holds for the graph, then the controllability matrix determines the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ exactly and statements can be made about the sign-herdability of the system.

First consider the following of extension of Lemma 3.3.1 and Lemma 3.3.2, which captures the behavior that is desired to capture the sign-herdability of a network.

Theorem 3.6.1. *If* $\forall i \in \{1, 2, \ldots, n\}$ *it holds that for each* d *and* j *such that* $\mathcal{C}_{i,m*(d-1)+j} \neq 0$, i *satisfies* $v_{xi} \in \mathcal{P}_d^j \wedge v_{xi} \notin \mathcal{N}_d^j$ *or* $v_{xi} \in \mathcal{N}_d^j \wedge v_{xi} \notin \mathcal{P}_d^j$*, then the sign pattern of the controllability matrix does not depend on weights of the graph.*

Proof. If the condition of the theorem holds, then every non-zero element of $\mathcal{C}$ is associated only with paths of the same sign and as such will have the same sign as the paths no matter the weights on the graphs. □

If the underlying graph of the system satisfies Theorem 3.6.1, the system is sign-consistent. Clearly this theorem depends on knowledge of the sets $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$, which is infeasible computationally when the underlying graph structure is not a directed acyclic graph. However it is possible to show a stronger condition more easily.

Theorem 3.6.2. *If the system graph is structurally balanced, then for all* $i \in \{1, 2, \ldots, n\}$ *it holds that for each* d *and* j *such that* $\mathcal{C}_{i,m*(d-1)+j} \neq 0$, i *satisfies* $v_{xi} \in \mathcal{P}_d^j \wedge v_{xi} \notin \mathcal{N}_d^j$ *or* $v_{xi} \in \mathcal{N}_d^j \wedge v_{xi} \notin \mathcal{P}_d^j$.

Proof. Suppose not. Then there exists an i such that there is a d and j where $v_{xi} \in \mathcal{P}_d^j \wedge v_{xi} \in \mathcal{N}_d^j$. This implies there are one or more positive paths of length d from j and one or more negative paths of length d from j. Without loss of generality, consider one positive path and one negative path from input j to node i. These paths form a semi-cycle in the graph. One of the paths is negative and must have an odd number of negative edges. The other is positive and must have an even number of negative edges. As such the semi-cycle must have an odd number of negative edges, i.e. the semi-cycle must have negative weight, which implies that the graph is not structurally balanced. □

To see why structural balance is only a sufficient condition consider the graph in Figure 3.9. As can be seen from the graph, structural balance ignores the lengths of the paths which connect an input to a state node.

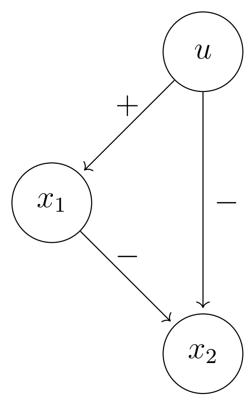


Figure 3.9: An example of a graph which is sign consistent but not structurally balanced

Structural balance can be determined in linear time[123], which implies that Theorem 3.6.2 makes it possible to characterize the sign herdability of a system from the controllability matrix with little extra computational cost.

3.7 Cardinality Herding

As determining the herdability of a system based on the sign pattern of the underlying graph is NP-Hard and potentially returns infeasible solutions, there is a need for another method to determine the herdability of a system. We propose a computational method to determine the herdability of a system based on the controllability matrix $\mathcal{C}$. The cardinality herding problem solves the following linear program:

$$\max_u \sum_{i=1}^{n} (\mathcal{C}u)_i \quad \text{subject to } \mathcal{C}u \leq 1_n. \tag{3.3}$$

Once the linear program is solved, the number of positive elements of the resultant vector $\mathcal{C}u$ is examined to determine how many states have been herded. As will be seen in Section 4.2, this relatively simple optimization problem can be used to show that a large portion of a given network is herdable from one node.

3.8 Conclusion

In this chapter, the basic theory of herdable systems was presented. The definition of herdability was shown to translate to a simple condition based on three matrices: the controllability grammian, the controllability matrix, and the matrix $[A\ B]$. As verifying this condition can be quite difficult a number of sufficient conditions where shown. Further a method for verifying this condition by inspecting the columns of the matrix under consideration was developed, which provides a computationally efficient but incomplete method to understand the herdability properties of a system.

The characterization of herdability based on the controllability matrix was extended to consider the underlying graph of the dynamical system. It was shown that a certain loss of symmetry, as shown by a balanced vector in the range of the controllability subspace, ensured that a system was no longer complete herdable. Additionally it was shown that as herdability is only dependent on the sign of an interaction, any characterization of herdability based on the controllability matrix can be extended to a class of systems with the same sign pattern if it can shown that the sign of the columns of the controllability matrix do not depend on the underlying edge weights.

CHAPTER 4

HERDABILITY INPUT SELECTION

This chapter considers the application of herdability to the study of complex networks by discussing the input selection problem: given an existing network structure, which node(s) should be selected to ensure that the system is completely herdable. Two versions of this problem will be discussed, one which focuses specifically on the context of positive systems and the other which considers the general case. This chapter considers and extends the work in [124].

4.1 Positive Systems

In Chapter 3, it was shown that an input connectable positive system is completely herdable. This section considers the following modification of the input selection problem: how to select a minimal subset, $\mathbb{H}$, consisting of N_H state nodes that ensures that the system is input connectable. Input connectability will in turn ensure that, under the assumption that the system dynamic is positive, that the system will therefore be herdable. Each element of $\mathbb{H}$ is called a herding node.

Note that herdability, as a set reachability condition, can always be achieved via one input node. The consideration then is which state nodes to communicate with. Therefore the results presented here do not explicitly depend on the structure of the B matrix and hold for the commonly used structures of B matrix, i.e. either a zero-one column vector or a diagonal matrix [56, 58].

Consider now the problem of making a given system herdable by finding an input connectable cover for a network. The solution to this problem is called a Herding Cover, as controlling the root nodes of the Herding Cover ensures herdability of a positive system. Of additional interest is insuring that the Herding Cover has the fewest possible number of root nodes. Before discussing the problem of finding a minimal Herding Cover, consider the Set Cover Problem.

Definition 4.1.1 (Set Cover Problem)**.** *Given a universe U of n elements, a collection of subsets of U say $S = \{S_1, S_2, \dots, S_k\}$ with non-negative costs specified, the minimum set cover problem asks for a minimum cost collection of sets whose union is U [125].*

The task of finding the minimum set cover is a classic example of a NP-Hard problem, see [125] for discussion and the hardness proof. Mapping the Herding Cover problem to the Set Cover problem shows that finding a minimal Herding Cover is NP-Hard.

Theorem 4.1.1. *Finding the minimum number of nodes to ensure herdability is NP-hard.*

Proof. For a network with n nodes, define $U = \{1, \dots, n\}$ as the universe set. For each node i in the network, assign the set S_i which are the nodes that can be reached through a spanning tree originating at node i. Assign each set S_i with a cost of 1. With cost of 1 for each set, the minimum cost solution selects the minimum number of sets to cover the network. Finding a minimal Herding Cover is then equivalent to this incarnation of the Set Cover Problem and is therefore NP-Hard. □

The solution to the set cover problem can be approximated via a greedy algorithm [125]. In this section, the following greedy algorithm will be implemented in order to determine the herdability properties of complex networks as follows. First find the directed spanning tree rooted at each individual node. This gives rise to the sets $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$, where the elements of $\mathcal{S}_i$ are the nodes that are in a directed spanning tree rooted at node i. The set of root nodes, $\mathbb{H}$, is increased based on the set of uncovered nodes, $\mathcal{U}$. A node is uncovered if it is not contained in $\bigcup_{k\in\mathbb{H}} \mathcal{S}_k$. At each iteration of the greedy algorithm, all unselected nodes $j \in \{1, 2, \dots, n\} \setminus \mathcal{R}$ are considered and the node that covers the most elements of $\mathcal{U}$ is added to $\mathbb{H}$. This is a H_n factor approximation algorithm, where $H_n = \sum_{z=1}^{n} \frac{1}{z}$ [125].

While in general finding the minimal Herding Cover is NP-Hard, it is possible to characterize some properties of a Herding Cover. The following Theorem provides bounds for the number of root nodes needed for a minimal Herding Cover (N_H).

Theorem 4.1.2. *The minimum number of roots needed to ensure input connectability of a directed network is bounded by*

$$N_w \leq N_H \leq \sum_{i=1}^{N_w} \max(N_i^s - 1, 1)$$

where N_w is the number of weakly connected components and N_i^s is the number of strongly connected components in weakly connected component i.

Proof. ($N_w \leq N_H$) Consider each weakly connected component in turn. If a directed spanning tree exists with covers all nodes in the weakly connected component, then the root of the directed spanning tree will be selected to make the weakly connected component input connectable. If a directed spanning tree exists for each weakly connected component, then N_w roots will form a Herding Cover.

($N_H \leq \sum_{i=1}^{N_w} \max(N_i^s - 1, 1)$) Consider each weakly component in turn. There are two cases:

- (Case 1 $N_i^s > 1$) Consider a weakly connected component with $N_s > 1$ strongly connected components. In this case, $N^s - 1 \geq 1$ and the graph requires at most $N^s - 1$ nodes to form a herding cover. Suppose not, then there is a minimal herding cover with N^s nodes. This implies that each of the N^s strongly connected components is disjoint because if there were a path between any two strongly connected components then both could be covered by one root node. Then there are in fact N^s weakly connected components, a contradiction.

- (Case 2 $N_i^s = 1$) If $N_i^s = 1$, then weakly connected component i is in fact a strongly connected component and one node is sufficient to cover this weakly connected component.

Both cases imply that for each weakly connected component, at most $\max(N^s - 1, 1)$ root nodes are needed for a minimal cover. Summing over each weakly connected component gives the upper bound. □

The upper bound holds in the case of the inverted star shown in Figure 4.1(a) which requires $N_\star - 1$ nodes to ensure herdability, where $N_\star$ is the number of nodes in the inverted star graph. In this configuration, the number of strongly connected components $N_\star^s$ within the weakly connected component is $N_\star^s = N_\star$. As links are added moving away from the center hub, $N_\star^s$ decreases along with the number of required nodes to herd the network.

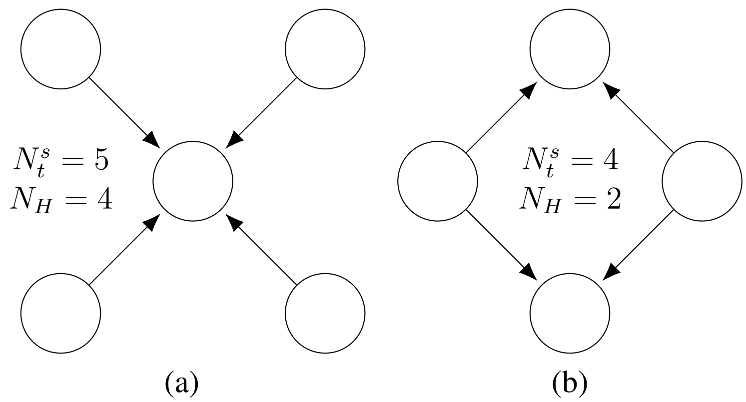


Figure 4.1: Herding Cover of a Network: (a) An inverted star graph where the upper bound is tight. (b) An example where the bound is not tight.

Theorem 4.1.2 can be extended to the following corollaries. Corollary 4.1.4 in particular will motivate the discussion in Section 4.1.1.

Corollary 4.1.3. *If the graph is undirected or consists of disjoint strongly connected components,*

$$N_H = N_w$$

.

Proof. If the graph is undirected or consists of disjoint strongly connected components, $\forall i \in \{1, 2, \ldots N_w\}, N_i^s = 1$ and so the upper bound becomes

$$\sum_{i=1}^{N_w} \max(0, 1) = N_w.$$

Then $N_w \leq N_H \leq N_w$. □

Corollary 4.1.4. *If the directed graph $\mathcal{G}$ is strongly connected and the interaction dynamic is such that the system is positive, then any one node set forms the root of a Herding Cover.*

Proof. If the system is strongly connected, then whichever node is chosen as an input node there will be a path to all nodes in the network by the definition of strong connectivity. Further if the system is a positive system then the system is an excitable, positive system and the system is herdable. □

The characterization of herdability provided in Corollary 4.1.4 allows us to consider in greater depth how herdability differs from controllability. The primary difference comes from the inherent inability of controllability analysis to deal with symmetry with respect to an input. This symmetry occurs due to dilations in the underlying graph, an example is which is shown in Figure 4.2(a). This graph represents systems of the form

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} \beta_1 \\ 0 \\ 0 \end{bmatrix} u.$$

The controllability matrix follows

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & \alpha_2 & 0 \end{bmatrix}.$$

Symmetry causes a rank deficiency in the controllability matrix, forcing the symmetric nodes to be controlled in constant relation to each other, where the constant is dependent on the relative edge weights. The inability of the symmetric nodes to be controlled separately of each other violates the controllability condition. As herdability looks only at herding the state to be larger than some threshold, the herdability condition is satisfied even when the symmetric nodes are controlled to the same point. An illustrative case of symmetric systems is the star graph, shown in Figure 4.2(c). The fact that symmetry degrades controllability explains why past analysis of controllability of complex networks has found that driver node selection avoids hubs [63].

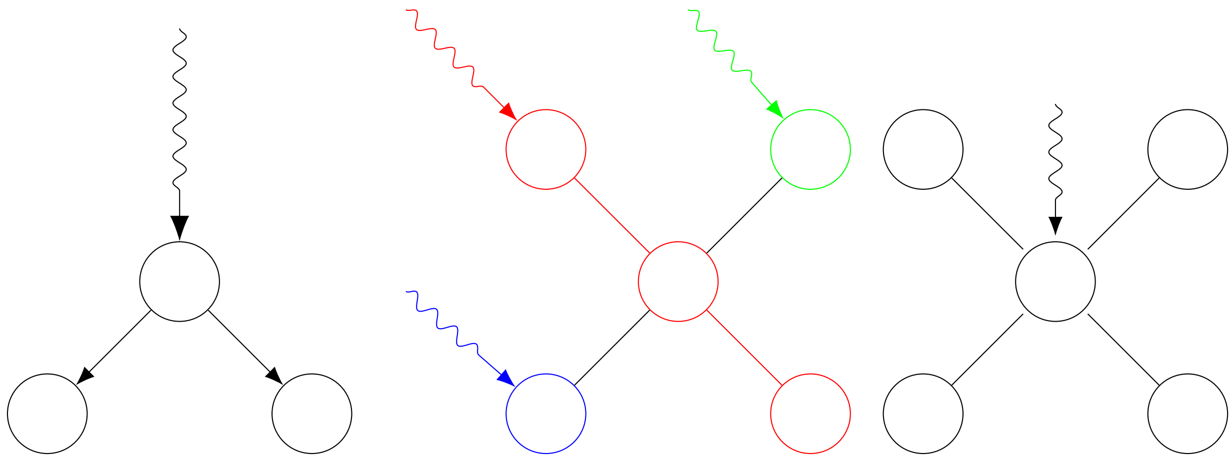


Figure 4.2: The Effect of Symmetry on Control: (a) A dilation: Two nodes (x_2 and x_3) have in-bound edges from one node. Nodes x_2 and x_3 are both symmetric with respect to the control input u making the system uncontrollable. (b) Controllability analysis will select 3 nodes as driver nodes in order to ensure controllability of the system. (c) Herdability can select the middle node as symmetry does not necessarily degrade the ability to herd the network.

Table 4.1 shows results for analysis of the fraction of herding nodes, n_H, compared with the fraction of driver nodes, n_c, from the controllability analysis of [63]. These results are approximate in the case of directed networks and exact in the case of undirected networks. Across all considered networks $n_H \leq n_c$. In 15 of the 24 networks, Table 4.1 shows that influencing complex networks requires communication with fewer nodes than controlling the network as $n_H < n_c$. There are some networks, such as the Western US Power Grid, where $n_H << n_c$. These networks consist of a single strongly connected component, which can be made herdable with one herding node as shown in Corollary 4.1.4.

Table 4.1: For each network, the table shows the number of node N, the number of edges L, whether the network is Undirected or Directed, the ratio of number of herding nodes to number of weakly connected components $n_w = \frac{N_H}{N_w}$, the fraction of herding nodes $n_H = \frac{N_H}{N}$, the fraction of driver nodes $n_c = \frac{N_c}{N}$, the fraction associated with the theoretical upper bound $n_u = \frac{\sum_{i \in N_w} \max(N_i^s - 1, 1)}{N}$.

Type	Name	N	L	Dir.	n_w	n_H	n_u	n_c
Collaboration	Astro-Physics[126]	16,706	242,502	U	1	0.062	0.062	0.080
	Condensed Matter Physics[126]	16,726	95,188	U	1	0.071	0.071	0.108
	Cond. Mat. Physics 2003[126]	31,163	240,058	U	1	0.051	0.051	0.090
	Cond. Mat. Physics 2005[126]	40,421	351,384	U	1	0.045	0.045	0.083
	High Energy Physics[126]	8,361	31,502	U	1	0.159	0.159	0.208
	Network Science[127]	1,589	5,484	U	1	0.249	0.249	0.260
	Jazz[128]	198	5,484	U	1	0.005	0.005	0.005
	General Relativity[129]	26,196	28,980	U	1	0.813	0.813	0.827
Biological	C. Elegans Neural [130]	306	2,345	D	3.7	0.121	0.212	0.190
	Protein Interaction[131]	2,114	4,480	U	1	0.197	0.197	0.462
	Dolphin Social [132]	62	318	U	1	0.016	0.016	0.032
Infrastructure	Western US Power Grid [130]	4,941	13,188	U	1	0.0002	0.0002	0.116
	Top Airports[133]	500	5960	U	1	0.002	0.002	0.250
	Football Games[134]	115	1,226	U	1	0.009	0.009	0.009
Online	UCIonline[83]	1,899	20,296	D	138	0.291	0.315	0.323
	Political Blogs[135]	1,490	19,025	D	1.89	0.340	0.460	0.471
Friendship	Third Grade[136]	22	177	D	1	0.046	0.046	0.046
	Fourth Grade[136]	24	161	D	1	0.042	0.042	0.042
	Fifth Grade[136]	22	103	D	1	0.046	0.046	0.046
	Highschool[137]	73	243	D	2	0.137	0.233	0.178
	Fraternity[138]	58	1,934	U	1	0.017	0.017	0.017
	EIES 1[139]	32	650	D	1	0.031	0.031	0.031
	EIES 2[139]	32	759	D	1	0.031	0.031	0.031
	Mine[140]	15	88	U	1	0.067	0.067	0.067

4.1.1 Herdability Centrality

If the system is herdable from any one node, a secondary issue arises of selecting which one node to use as the herding node. To select between nodes in a strongly connected component, a new herdability centrality measure is proposed which takes into account the energy required to drive the system into the set $\mathcal{H}_d$. Herdability centrality explicitly takes the dynamics of the system into account, unlike many existing centrality measures which carry implicit assumptions about the network processes they describe [141].

While many networks are not necessarily strongly connected, any directed graph can be broken down into a non-overlapping set of strongly connected components, allowing each strongly connected component to be considered individually to determine the herdability centrality. The strongly connected components of a graph can be found in linear time via Kosaraju's algorithm [142].

Consider the problem of entering the set $\mathcal{H}_d = \{x \in \mathbb{R}^n | x_i \geq d\}$ from the origin with minimal control energy:

$$\begin{aligned} J(B,d) = \min_{u(t)} \int_0^{t_f} & \|u(\tau)\|^2 d\tau \\ \text{s.t. } & \dot{x}(t) = Ax(t) + Bu(t),\ t \in [0, t_f] \\ & x(t_f) \in \mathcal{H}_d \\ & x(0) = 0_n, \end{aligned} \tag{4.1}$$

where the minimum energy, J, is parameterized by the structure of the interaction with control inputs, which is given in the matrix B, and by $d > 0$ which is assumed to be fixed.

The formulation in Equation (4.1) can be contrasted with the minimum energy optimal control problem as typically studied, i.e. in the context of completely controllable systems. Specifically, in such cases the desired end position of the system is typically a desired final point x_f instead of the set $\mathcal{H}$. In general, for systems that are not completely controllable, there is no guarantee that a desired x_f or even $\mathcal{H}$ can be reached. However if the system is herdable, then by definition the reachable subspace from 0_n, $\mathcal{R}(0)$ intersects the set $\mathcal{H}_d$. As such it is possible to characterize the form of the minimum energy to reach $\mathcal{H}_d$.

Lemma 4.1.5. *If the system is herdable, then the minimum energy to reach $\mathcal{H}_d$ is of the form*

$$x_f^T W_c^+ x_f,$$

where $x_f \in \mathcal{H}_d \cap \mathcal{R}(0)$, and W_c^+ is the Moore-Penrose pseudo-inverse of the Controllability Grammian.

Proof. If the network is herdable then $\exists x_f \in \mathcal{H}_d \cap \mathcal{R}(0)$. This reachable x_f allows the use of a number of properties of the controllability grammian. To reach $\forall x_f \in \mathcal{R}(0) \cap \mathcal{H}_d$ requires an input $u(t)$ that satisfies $\int_0^t e^{A(t-\tau)} Bu(\tau)d\tau = x_f$. This $u(t)$ will have the form $u(t) = B^T e^{At} p$ where $W_c p = x_f$. The equation $W_c p = x_f$ has at least one solution as $\mathcal{R}(0) = \text{range}(W_c)$ i.e. that $x_f \in \text{range}(W_c)$. These solutions are of the form

$$p^* = W_c^+ x_f + [I - W_c^+ W_c] x_f$$

with $p^* = W_c^+ x_f$ as the unique solution in the range of W_c, where W_c^+ can here refer to any generalized inverse [143]. If W_c^+ refers specifically to the Moore Penrose Inverse (or any generalized reflexive inverse) the form of the minimum energy to reach x_f is $x_f W_c^+ x_f$. $\square$

With the analytical expression for the minimum energy to reach x_f, it is possible to re-frame the earlier energy minimization problem as the problem of choosing the optimal x_f in the set $\mathcal{H}_d \cap \mathcal{R}(0)$. The optimization becomes the following:

$$
\begin{aligned}
&\min_{x_f} x_f^T W_c^+ x_f \\
\texttt{s.t. } & x_f \geq d \\
& x_f \in \mathcal{R}(0) \\
& x(0) = 0_n
\end{aligned}
\tag{4.2}
$$

Here the problem can once again be simplified further based on properties of the controllability grammian. As W_c is a symmetric, real matrix, the eigenvectors of W_c are mutually orthogonal and the eigenvectors with non-zero eigenvalues span the range of W_c [144]. When $\text{rank}(W_c) = r \leq n$ there are r eigenvectors $\{v_1, \ldots, v_r\}$ associated with the r non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$ which form an orthonormal basis for $\text{range}(W_c)$. Therefore if $x_f \in \text{range}(W_c)$ then x_f can represented as

$$
x_f = \sum_{i=1}^{r} \alpha_i v_i. \tag{4.3}
$$

Then using that v_i are orthonormal and also eigenvectors of W_c^+ with associated eigenvalues $\frac{1}{\lambda_i}$, substituting in Equation (4.3) gives

$$
\begin{aligned}
x_f^T W_c^+ x_f &= \left(\sum_{i=1}^{r} \alpha_i v_i\right)^T W_c^+ \left(\sum_{i=1}^{r} \alpha_i v_i\right) \\
&= \left(\sum_{i=1}^{r} \alpha_i v_i\right)^T \left(\sum_{i=1}^{r} \frac{\alpha_i}{\lambda_i} v_i\right) \\
&= \sum_{i=1}^{r} \frac{\alpha_i^2}{\lambda_i}
\end{aligned}
$$

The optimization in Equation (4.2) becomes

$$\min_{\alpha} \sum_{i=1}^{r} \frac{\alpha_i^2}{\lambda_i}$$

$$\texttt{s.t.}\ V\alpha \geq d,$$

where

$$V = [v_1 \dots v_r] .$$

4.1.2 Calculating Herdability Centrality

With a simplified version of the minimum energy optimal control problem in hand, it is possible to move to calculating the herdability centrality of each node in the network. In order to calculate herdability centrality, each state node of the herdable system is considered in turn as the sole input node allowing the calculation of $J_i = J(e_i, d)$, where $e_i \in \mathbb{R}^n$ is 1 at position i and 0 elsewhere, and $d > 0$ is fixed. The quantity J_i is the minimum energy to reach $\mathcal{H}$ using only node i as control input. In order to compare the minimum energy across nodes, the herdability centrality for node i, Hc_i, is defined as

$$Hc_i = \frac{\min_k \{J_k\}}{J_i}$$

Herdability centrality is normalized to be between 0 and 1. As reaching $\mathcal{H}$ with minimum energy is the chosen metric when interacting with these networks, the node(s) with minimum energy to reach $\mathcal{H}$ across all nodes will have the highest herdability centrality.

For the purpose of calculating herdability centrality of existing complex networks, the largest strongly connected component of each considered network is used as the underlying interaction topology. The dynamics are assumed to be consensus dynamics, though the model presented here is related to the model of Taylor, which captures the effect of an external source of information on the opinion of an agent [145]. When node i is the sole herding node, the consensus dynamics are as follows:

$$\dot{x}_j(t) = \sum_{z \in \mathcal{N}_j} (x_z(t) - x_j(t)), \;\; \forall j \neq i$$
$$\dot{x}_i(t) = \sum_{k \in \mathcal{N}_i} (x_k(t) - x_i(t)) + u(t) - x_i(t),$$

where $\mathcal{N}_i$ is the set of nodes with edges entering node i. It's important to note that this model provides a stable A matrix allowing the calculation of the controllability grammian W_c. While the simulation results presented here make use of consensus dynamics, it possible to calculate herdability centrality in this formulation for any herdable, stable linear dynamic.

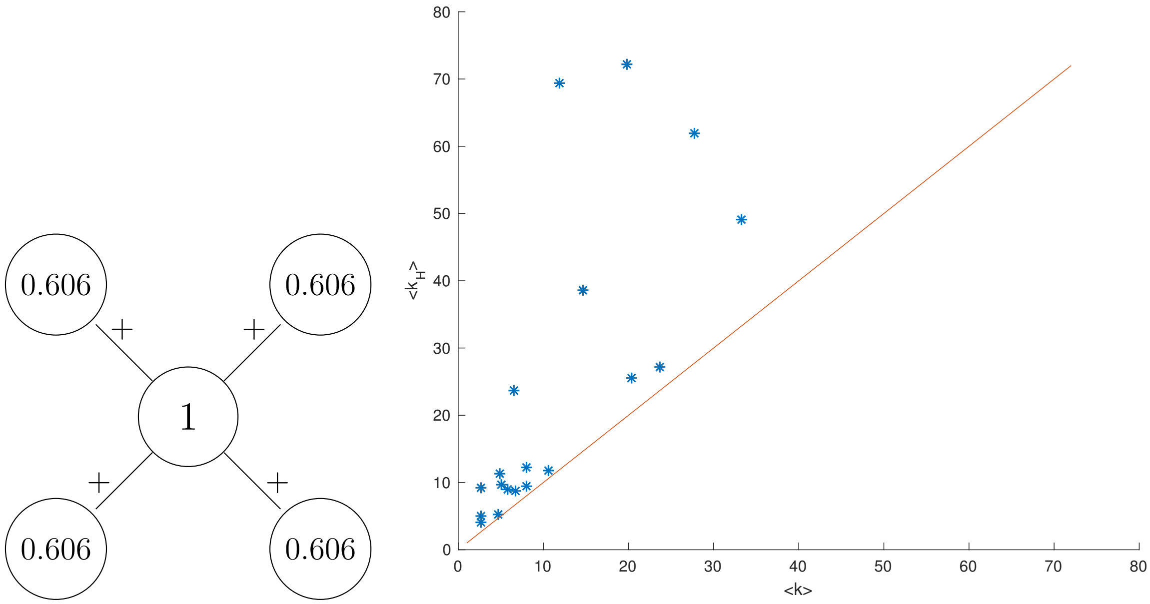
In order to improve efficiency of the calculation, the final time is taken to be $t_f = \infty$ as the infinite horizon controllability grammian can be solved for efficiently, if A is stable, as the solution to the continuous time Lyapunov equation:

$$AW_c + W_c A + BB^T = 0.$$

As mentioned previously, the more general framework of herdability allows hubs to be selected to herd complex systems, though it is not known a priori that hubs will indeed be selected. Figure 4.3(a) shows that the center node of the hub has the highest herdability centrality, and therefore requires the least energy to reach $\mathcal{H}_d$. Figure 4.3(b) shows that the introduced herdability centrality tends to select nodes that have higher than average degree, suggesting that using herdability centrality to select herding nodes targets hubs.

4.1.3 Comparison to Other Centrality Measures

Given that herdability centrality tends to select high degree nodes, the question becomes whether it is possible to forgo the computationally expensive herdability centrality calcula-



(a) (b)

Figure 4.3: Herdability Centrality and Hubs: (a) Herdability centrality of a hub. The middle node has the highest herdability centrality. (b) Plot of average degree of the complete network vs average degree of the top 10% most herdable nodes, with a line representing average network degree. The top 10% most herdable nodes of each graph all have greater than average degree.

tion in favor of an inexpensive degree centrality calculation, or some other graph structure based centrality measure.

Figure 4.4 shows that while high herdability centrality nodes tend to have high degree, the highest in-degree node does not necessarily have high herdability centrality. Further this holds for all centrality measures considered. As shown in Figure 4.4, in 8 of the 19 networks considered the traditional centrality measures overlap with the highest herdability centrality nodes. However, there is no single centrality measure which can be used reliably to select the herding node that reaches the set $\mathcal{H}$ with minimum control energy across nodes. The overlap between herdability centrality and existing measures tends to occur in undirected networks, which concurs with past results that have shown that centrality measures are often correlated in undirected networks[146]. Examining the directed networks shows that size of the network seems to have no impact on overlap with existing centrality measures. For example, in the Fifth Grade Friendship network, $N = 22$, all considered centrality

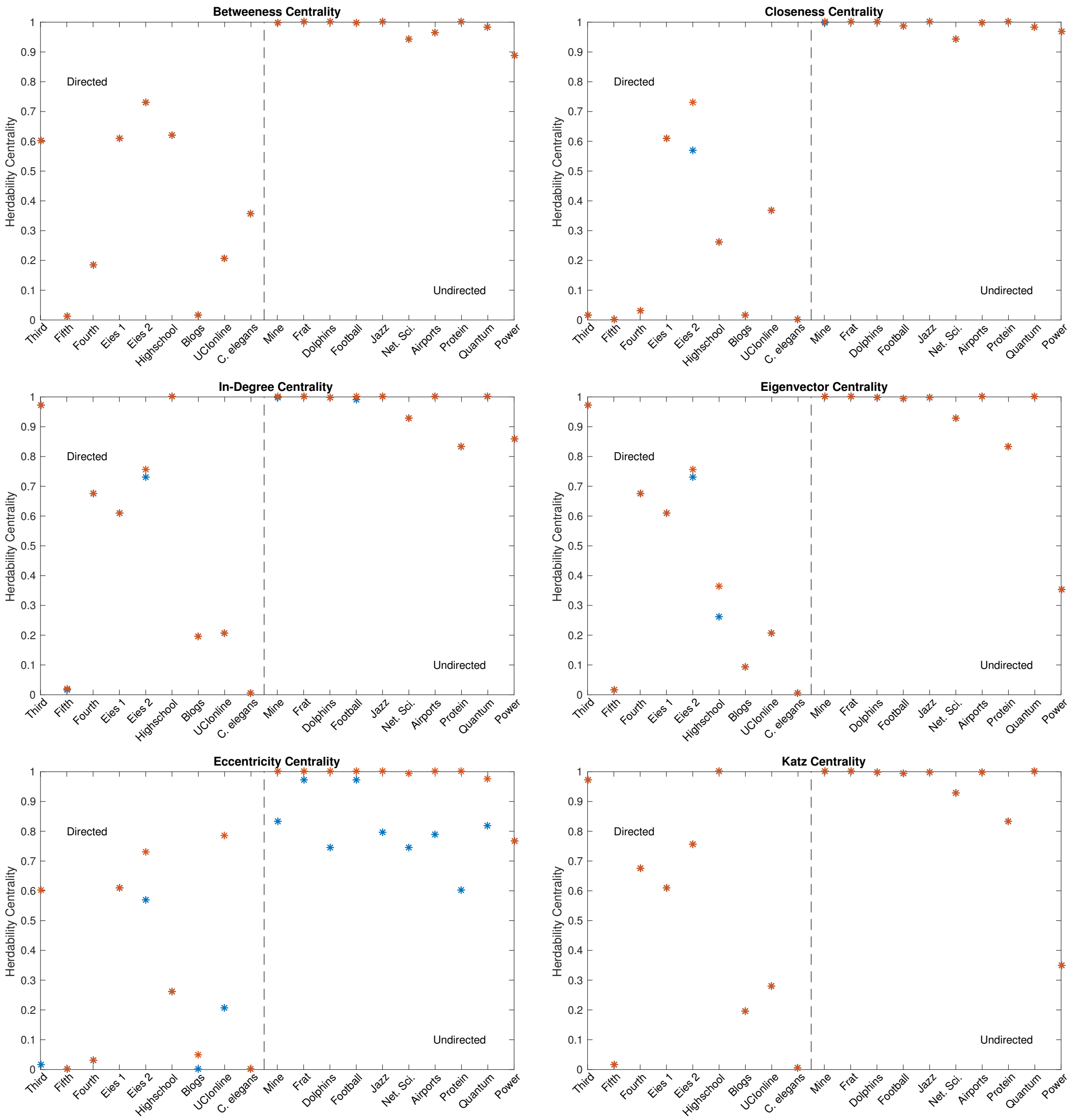


Figure 4.4: Selecting the Highest Herdability Node via Other Centrality Measures: Each subgraph considers a different centrality measure and shows the highest (in red) and lowest (in blue if present) herdability centrality of the node(s) identified as having the highest value for each respective centrality across all considered networks. Within each categorization (Direct or Undirected) the networks are ordered with the smallest networks on the left and the largest on the right. In all undirected Networks, all calculated centrality measures have high herdability centrality. In some directed networks, In-Degree, Eigenvector, and Katz centrality identify high herdability nodes.

measures select a node with low herdability centrality.

It is not entirely clear what causes this lack of overlap between existing centrality measures and herdability measures. It may be that as the dynamics being used are consensus dynamics, the overlap would be better if existing centrality measures were applied to the meta-structure determined by the graph automorphism, as the graph automorphism has been shown to determine the controllability of consensus dynamics [5].

4.2 Signed Networks

This section considers the question of which single node can be selected to herd a signed complex network, or, in the case that it is not possible to herd the complete network, which node can be used to herd the largest number of nodes.

Answering the input selection problem depends on the analysis of the herdability of a system discussed in Chapter 3. Recall, that there were three characterizations of the herdability of a system based on various system matrices: a necessary and sufficient condition based on the controllability grammian W_c or the controllability matrix $\mathcal{C}$ and a necessary condition based on the matrix $[A\ B]$.

There are some caveats for each of these methods. As mentioned previously, the test based on $[AB]$ is only a necessary condition. As such this test can only provide an upper bound for system herdability but it can be verified by considering a matrix in $\mathbb{R}^{n\times(n+m)}$.

There is a matrix test which deals with a smaller matrix than in the case of $[A\ B]$, namely the controllability grammian $W_c \in \mathbb{R}^{n\times n}$. The controllability grammian has the additional advantage that, if it can be computed, it can be computed in linear time for the case of the infinite horizon controllability grammian. However there is a major caveat for this method in that A must be stable for the controllability grammian to be computed. As will be seen, under the assumptions used here to move from a network representation to a dynamical system representation this test can not be used.

The third test is based on the controllability matrix $\mathcal{C}$, which can be quite computationally expensive. To calculate the full controllability matrix $\mathcal{C}$ requires $O(n^{2.3727}log(n-1)+(n^2m))$ time and the resulting matrix that must be analyzed $\mathcal{C} \in \mathbb{R}^{n\times nm}$. However, the controllability matrix does have the advantage that if the complex network is structurally balanced then the controllability matrix test gives information about the sign herdability of a system. Additionally if the system is unstable, the controllability matrix can be partially computed, which gives some information about the herdability of the system.

A collection of complex networks from the literature are used in the analysis. The networks are summarized in Table 4.2. Each network has been checked for structural balance, based on the linear time algorithm of [123]. None of the networks examined are structurally balanced, i.e. the controllability matrix results hold for a specific weight combination and not for all networks that share the same sign pattern.

Each network referenced in Table 4.2, has an associated signed adjacency matrix $\tilde{A}_s(\mathcal{G})$. It is assumed that the dynamics of the linear system which evolves over the network follows $A = \tilde{A}_s(\mathcal{G})$. Under these assumptions, all of the systems were shown to be unstable, and as such the matrix product $A^m \to \infty$ for some $m < n$. This implies that the controllability matrix can not be fully computed. However as mentioned previously, partial information on system herdability can be obtained.

To analyze a network, each node is considered in turn as the sole input. To consider the ability to herd from node i, it is assumed that $B = e_i$ and the herdability of the system is considered via the controllability matrix. For each network the controllability matrix is calculated and then the sets $\mathcal{P}_d$ and $\mathcal{N}_d$ are approximated by checking the sign of each column of $\mathcal{C}$. Once the approximate $\mathcal{P}_d$ and $\mathcal{N}_d$ are obtained, the sets are iterated through until the conditions of Lemmas 3.2.1 and 3.2.2 are no longer met. This is contrasted of the results of the cardinality herding linear program. Given the size of these networks, a random sample of 100 nodes was taken as potential input nodes for the three Slashdot networks. Table 4.2 shows the highest and lowest percentage of nodes that can be herded for the various methods.

Table 4.2: Signed networks used to test system herdability: Each network has its name, number of nodes N, number of edges L, $\%$ Pos the fraction of positive edges, $[AB]_h$ the highest percentage of that can be herded based on the necessary condition on $[AB]$, H_h the highest percentage of the network that can be herded based on the sign of $\mathcal{C}$, H_l the lowest percentage of the network that can be herded based on the sign of $\mathcal{C}$, C_h the highest percentage of the network that can be herded based on cardinality herding of $\mathcal{C}$, and C_h the lowest percentage of the network that can be herded based on cardinality herding of $\mathcal{C}$.

Network Name	N	L	$\%$ Pos	$[A\ B]_h$	H_h	H_l	C_h	C_l
Bitcoin Alpha[147]	5,881	35,592	93	86.175	22.522	0.026	83.109	0.026
Bitcoin OTC[147]	3,783	24,186	89	79.221	17.650	0.017	76.365	0.017
Slashdot 11/06/08[148]	77,357	516,575	77	50.385	0.019	0.001	38.517	0.001
Slashdot 02/16/09[148]	81,871	545,671	77	49.562	0.024	0.001	36.699	0.001
Slashdot 02/21/09[148]	82,144	549,202	77	50.464	0.052	0.001	35.789	0.001

Based on Table 4.2, one can see that, even in the best case, the sign pattern of the controllability matrix predicts that only a small fraction of the network can be herded. Solving the cardinality herding problem from Equation (3.3) shows that large fractions of the network can be herded, up to 83% from one node. The highest percentage of herded nodes for each network is also lower than the upper bound based on the analysis of $[A\ B]$, suggesting that the answer is reasonable. It's also interesting to note that the gap between the upper bound and the maximum percentage herded based on cardinality herding is related to the fraction of positive edges in the network. When there are more negative edges the system is harder to herd.

The results of Table 4.2 show the best and worst case for the various algorithms. The question remains whether these are outliers, special nodes which are optimal to choose for herdability. The smallest network is the Bitcoin Alpha network, and as such the percentage of nodes that can be herded was calculated for every node. The results are shown in Figure 4.5 and Figure 4.6. The sign pattern of the controllability matrix $\mathcal{C}$ suggests that certain nodes can herd significantly more nodes that others. However, as shown in Figure 4.6, based on cardinality herding problem the opposite is true, most nodes can herd $70-80\%$ of the network. This suggests that many nodes can be selected to herd the a large portion of the network.

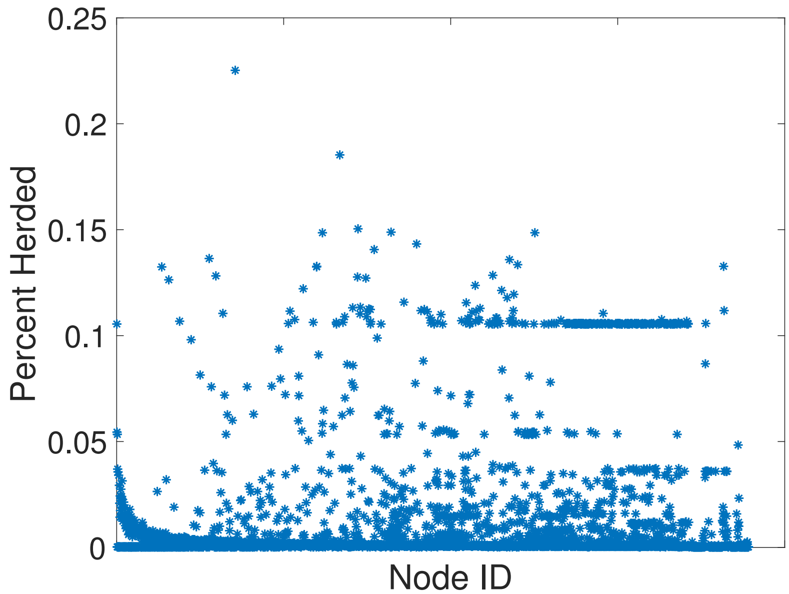


Figure 4.5: Percent of system nodes herded based on the sign pattern of the controllability matrix when taking each node as the sole input in turn.

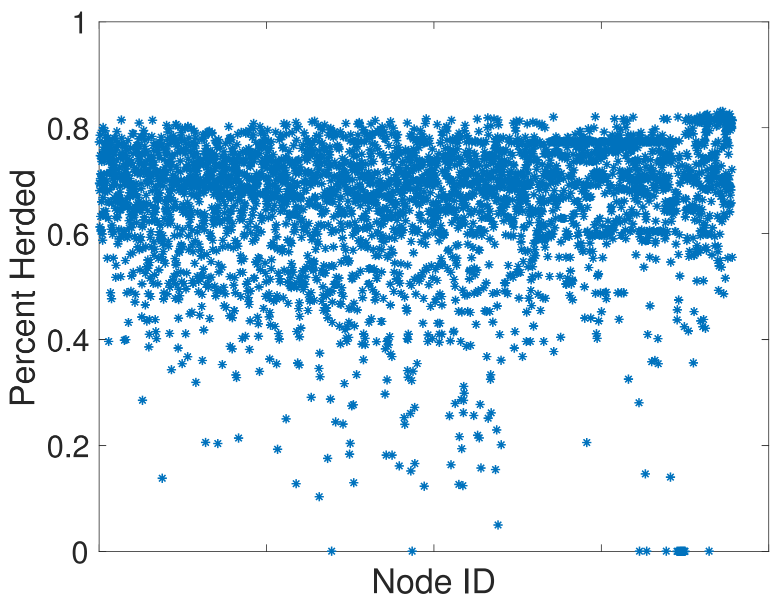


Figure 4.6: Percent of system nodes herded based on the cardinality herding analysis of the controllability matrix when taking each node as the sole input in turn.

4.3 Conclusion

In this chapter, herdability input selection was discussed in two contexts: under the assumption that the underlying system is positive and under the assumption that it is not. In the positive systems context, determining an minimal set of inputs to determine herdability was shown to be NP-Hard; similar to the results for controllability. By using approximate algorithms, the fraction of nodes required to ensure herdability was computed, and it was shown empirically that for many complex networks, herdability picks fewer nodes than controllability. Existing centrality measures, while providing an intuitive explanation of how a network dynamic should evolve, were shown to inadequately capture the behavior of the network dynamics, implying that other considerations, specifically consideration of the system dynamic, are needed in order to provide more realistic measures of importance. In the general case, the herdability of signed complex networks was considered and it was shown by simulation that a large portion of a network can be herded by one node and that in one network, many nodes can be selected to herd the network.

CHAPTER 5

ADOPTIVE SPREAD MODELING

This chapter marks the transition from the discussion of threshold driven social network phenomena to spread modeling; from exploring how to drive a system above a threshold to examining the behavior of a specific epidemic spread model with information. This model describes behavior when the idea, habit or product that is spreading over a network is encouraged to spread by the agent's information. In doing so, a link has been formed between the fields of epidemic modeling, opinion dynamics and the diffusion of innovations. This section expands upon the results in [149].

5.1 The Coupled Adoptive Spread Model

In this chapter, the standard SIS epidemic ODE dynamics are modified to incorporate the coupling between the "epidemic-like" spread of adoption behavior and the opinion exchange dynamics. For ease of narration, we will refer to the object being adopted as a product, though there are other objects whose adoption can be modeled by the coupled model present here. The adoption dynamics occur over a weighted, directed network $\mathcal{G}_A$ of n agents, or nodes. The opinion dynamics occur over a weighted digraph $\mathcal{G}_O$ with the same node set as $\mathcal{G}_P$, but whose edges may or may not coincide with $\mathcal{G}_P$. We denote the neighborhood set of agent i as $\mathcal{N}_i^X$ for $X = A, O$.

Each node i has an adoption probability $x_i \in [0, 1]$ for the product, which represents how likely the consumer is to adopt the product ($x_i = 0$ means the consumer has not adopted, $x_i = 1$ means the consumer has). The consumer represented by node i also has an opinion $o_i \in [0, 1]$, modeling how much the consumer values the product ($o_i = 0$ means very averse to the product, $o_i = 1$ means very receptive to the product). The adoption dynamics for each node evolve as a function of time:

$$\begin{aligned}\dot{x}_i &= f_i(x, o) \\ &\equiv -\delta_i x_i(1 - o_i) + (1 - x_i)o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right)\end{aligned} \tag{5.1}$$

where $\delta_i > 0$ is the drop rate for agent i, $\beta_{ij} \geq 0$ is the exogenous adoption rate, and $\beta_{ii} \geq 0$ is the endogenous adoption rate. The parameters β_{ij} are the weights on the adoption graph.

The opinion dynamic model that will be considered in conjunction with the adoptive spread model in Equation (5.1) is the canonical Abelson model, which is discussed in Chapter 2. The modified Abelson dynamics follow

$$\dot{o}_i = g_i(x, o) \equiv \sum_{j \in \mathcal{N}_i^O} w_{ij}^o (o_j - o_i) + w_i^x (x_i - o_i), \tag{5.2}$$

where $w_{ij}^o \geq 0$ is the weight on the opinion network and $w_i^x > 0$ is a weight that represents the quality of the product. In the following discussion, it is assumed that $w_{ij}^o = 1, \ \forall i, j$. The last term of Equation (5.2) moves an agent's opinion toward its adoption state. Hence, an agent's opinion is affected by its neighbor's opinions and its own adoption level.

Translating the opinion into vector form, shows that the opinion dynamic satisfies

$$\dot{o} = Wx - (\mathcal{L}_o + W)o$$

where $\mathcal{L}_O$ is the (in) graph Laplacian of the opinion network and $W = \mathrm{diag}\,(w_i^x)$.

By combining Equations (5.1) and (5.2), the adoption-opinion dynamic follows

$$\begin{aligned} \dot{x}_i &= -\delta_i x_i (1 - o_i) + (1 - x_i) o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\ \dot{o}_i &= \sum_{j \in \mathcal{N}_i^O} w_{ij}^o (o_j - o_i) + w_i^x (x_i - o_i) \end{aligned} \tag{5.3}$$

It is assumed the initial conditions $x_i(0), o_i(0) \in [0,1] \ \ \forall i$ are known. As will be shown later, $x_i(0), o_i(0) \in [0,1] \ \ \forall i$ implies $x_i(t), o_i(t) \in [0,1] \ \ \forall i, t \geq 0$. Hence, $x_i(t)$ and $o_i(t)$ are functions from $[0,\infty)$ to $[0,1]$. When convenient, we denote the aggregate $2n$-state vector by $z = [x^T, o^T]^T$.

5.2 Analysis of the Coupled Dynamic

For the coupled adoption opion model in Equation (5.3), each x_i represents a probability of adoption, or the proportion of a subpopulation that has adopted, and each o_i is a scaled opinion. As such the proposed model is only meaningful for $x_i, o_i \in [0,1]$. As such the model must be show to be well-posed with respect to $x_i(t)$.

Lemma 5.2.1. *For the model in Equation* (5.1)*, if* $x(0) \in [0,1]^n$ *and* $o(t) \in [0,1]^n$ *for all* $t \geq 0$*, then* $x_i(t) \in [0,1]$ *for all* $t \geq 0$*.*

Proof. Assume $o(t) \in [0,1]$ for all $t \geq 0$.
If $x_i(0) = 0$ and $x_j(0) \in [0,1]$ for all $j \neq i$, then by Equation (5.1), $\dot{x}_i(0) \geq 0$, driving $x_i(t) \geq 0$ for $t > 0$, since $\beta_{ij} \geq 0$.
If $x_i(0) = 1$ and $x_j(0) \in [0,1]$ for all $j \neq i$, then by Equation (5.1), $\dot{x}_i(0) = -\delta_i x_i (1 - o_i) \leq 0$, driving $x_i(t) \leq 1$ for $t > 0$, since $\delta_i \geq 0$.

Since there exists a derivative by Equation (5.1), $x_i(t)$ is continuous. Therefore since $x_i(0) \in [0,1]$ for all i, and the above has shown that for t such that $x_i(t) = 1$, $\dot{x}_i(t) \leq 0$ and for t such that $x_i(t) = 0$, $\dot{x}_i(t) \geq 0$, it holds that $x_i(t) \in [0,1]$ for all $t \geq 0$. □

As the proper behavior of the adoptive spread model is dependent on the behavior of the opinion model, the combined model is now shown to be well-posed in the opinion.

Proposition 5.2.2. *For the model in Equation* (5.2)*, if* $x(0) \in [0,1]^n$ *and* $o(0) \in [0,1]^n$ *, then* $x_i(t), o_i(t) \in [0,1]$ *for all* $t \geq 0$*.*

Proof. If $o_i(0) = 0$, $x(0) \in [0,1]^n$, and $o_j(0) \in [0,1]$ for all $j \neq i$, then by Equation (5.2), $\dot{o}_i(0) \geq 0$, driving $o_i(t) \geq 0$ for $t > 0$. If $o_i(0) = 1$, $x(0) \in [0,1]^n$, and $o_j(0) \in [0,1]$ for all $j \neq i$, then by Equation (5.2), $\dot{o}_i(0) \leq 0$, driving $o_i(t) \leq 1$ for $t > 0$.

Since there exists a derivative by Equation (5.2), $o_i(t)$ is continuous. Therefore since $o_i(0) \in [0,1]$ for all i, and the above has shown that for t such that $o_i(t) = 1$, $\dot{o}_i(t) \leq 0$ and for t such that $o_i(t) = 0$, $\dot{o}_i(t) \geq 0$, it holds that $o_i(t) \in [0,1]$ for all $t \geq 0$.

As the preceding argument holds for all $i \in \{1, 2, \ldots, N\}$ this implies $o(t) \in [0,1]^n$. By applying Lemma 5.2.1, it also holds that $x(t) \in [0,1]^n$. □

Having shown the well-posedness of the adoption model, the properties of the adoptive spread model will now be discussed by considering the partial derivatives of the function in Equation (5.1). Note

$$\frac{\partial f_i}{\partial x_i} = -\delta_i(1 - o_i) - o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right), \tag{5.4}$$

which is always negative under the assumptions of Lemma 5.2.1 since $\beta_{ij}, \delta_i \geq 0$. The other set of partial derivatives with respect to x is

$$\frac{\partial f_i}{\partial x_j} = \begin{cases} (1 - x_i) o_i \beta_{ij} & \text{if } j \in \mathcal{N}_i^A, j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^A \cup \{i\}, \end{cases}$$

which is always non-negative under the assumptions of Lemma 5.2.1 and since $\beta_{ij} \geq 0$.

Considering the derivatives with respect to opinion shows that

$$\frac{\partial f_i}{\partial o_i} = \delta_i x_i + (1 - x_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \tag{5.5}$$

which is always non-negative under the assumptions of Lemma 5.2.1 and since $\beta_{ij}, \delta_i \geq 0$. Finally,

$$\frac{\partial f_i}{\partial o_j} = 0 \ \ \forall j \neq i. \tag{5.6}$$

As in the classic SIS epidemic model, the adoption of network neighbors encourages the consumer to adopt. In the new coupled model, the opinion of the consumer modifies the impact of adoption in Equation (5.4) and encourages adoption via Equation (5.5).

Consider the behavior of the opinion dynamic model via the partial derivatives of the function in Equation (5.2).

$$\frac{\partial g_i}{\partial x_i} = w_i^x \tag{5.7}$$

$$\frac{\partial g_i}{\partial x_j} = 0 \ \ \forall j \neq i$$

$$\frac{\partial g_i}{\partial o_i} = -d_i^O - w_i^x$$

$$\frac{\partial g_i}{\partial o_j} = \begin{cases} 1 & \text{if } j \in \mathcal{N}_i^O \ \ \forall j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^O \cup \{i\}, \end{cases} \tag{5.8}$$

where d_i^O is the (in)degree of node i in the opinion network. Here the agent's adoption state and the opinion of their network neighbors affects the opinion of the agent.

5.3 The Stability of 1_{2n} and 0_{2n}

This system has at least two equilibrium points, $z^* \in \{0_{2n}, 1_{2n}\}$, i.e. the equilibrium is either no one adopts the product and everyone has an opinion equal to zero, or everyone adopts the product and has an opinion equal to one. These equilibria have a mirrored condition for stability, though to see this requires discussing both local and asymptotic stability results. The discussion of the stability of these two equilibria requires some concepts from matrix analysis which are summarized in Appendix A.

Lemma 5.3.1. *The equilibrium point $z^* = 0_{2n}$ is locally stable if $\forall i,\ \ \delta_i > \beta_{ii}$.*

Proof. The Jacobian of the dynamics can be written in block form as:

$$J(z) = \left[\begin{array}{c|c} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial o} \\ \hline \frac{\partial g}{\partial x} & \frac{\partial g}{\partial o} \end{array}\right].$$

The first n rows of the Jacobian are governed by Equations (5.4) - (5.6) while the second n rows are governed by Equations (5.7)-(5.8).

Consider the Jacobian matrix at the equilibrium point

$z^* = 0_{2n}$:

$$J(z^*) = \left[\begin{array}{c|c} \text{diag}\,(-\delta_i) & \text{diag}\,(\beta_{ii}) \\ \hline W & -(\mathcal{L}_O + W) \end{array}\right].$$

As $\delta_i > \beta_{ii}\ \ \forall i$ and the graph Laplacian is diagonally dominant, $J(z^*)$ is diagonally dominant. The first n rows of $J(z^*)$ are strictly diagonally dominant while the second n rows of $J(z^*)$ are merely diagonally dominant. However for all $j \in \{n+1, n+2, \ldots, 2n\}$ the element of the Jacobian $a_{(j-n)j} = w_j^x$. Then as $w_i^x > 0\ \ \forall i$ the Jacobian is weakly chained diagonally dominant and therefore nonsingular.

As the diagonal elements of $J(z^*)$ are negative, the Gershgorin disc theorem shows that the Jacobian is Hurwitz, and thus $z^* = 0_{2n}$ is a locally stable equilibrium. □

Having shown local stability of the equilibrium $z^* = 0_{2n}$, the asymptotic stability of the equilibrium $z^* = 0_{2n}$ is now shown.

Consider the matrix

$$P = \begin{bmatrix} -\bar{B} & \bar{B} \\ W & -(\mathcal{L}_o + W) \end{bmatrix}$$

where $\bar{B} = \operatorname{diag}\left(\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}\right)$.

Lemma 5.3.2. *If $\delta_i > \sum \beta_{ij} + \beta_{ii}$, then the coupled dynamic satisfies $\dot{z} \leq Pz$.*

Proof. Consider the adoption dynamic for the case where the adoption parameters satisfy $\delta_i > \sum \beta_{ij} + \beta_{ii}$ and when the state satisfies $x_i = o_i \notin \{0, 1\}$ or $x_i \neq o_i$:

$$\begin{aligned} \dot{x}_i &= -\delta_i x_i(1 - o_i) + (1 - x_i)o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\ &\leq -\delta_i x_i(1 - o_i) + (1 - x_i)o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} \right) \\ &< -\left(\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} \right) x_i(1 - o_i) \\ &\quad + (1 - x_i)o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} \right) \\ &= (o_i - x_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} \right). \end{aligned}$$

When $x_i = o_i \in \{0, 1\}$ then $\dot{x}_i = 0$ and $(o_i - x_i)\left(\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}\right) = 0$ so

$$\dot{x}_i = (o_i - x_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} \right).$$

Translating this to matrix form gives that

$$\dot{z} \leq \begin{bmatrix} -\bar{B} & \bar{B} \\ W & -(\mathcal{L}_o + W) \end{bmatrix} z$$

$$\dot{z} \leq Pz.$$

□

Lemma 5.3.3. *The eigenvalues of P are non-positive.*

Proof. P is diagonally dominant and has negative diagonal entries. By the Gershgorin disc theorem the eigenvalues are non-positive. □

It is interesting to note that the eigenvalues of P do not depend explicitly on the structure of the opinion graph $\mathcal{G}_O$, i.e. Lemma 5.3.3 can be shown without enforcing a connectivity constraint on $\mathcal{G}_O$. This is related to the fact that the graph Laplacian is always diagonally dominant independent of the structure of the underlying graph. This has implications for the behavior under a different opinion dynamic which will be introduced in Section 5.5.

Under the assumptions that the opinion graph is undirected and $\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} = w_i^x, \ \forall i$ then the above matrix P is symmetric which implies that it is negative semi-definite.

If the opinion graph is directed, the graph Laplacian $\mathcal{L}_o$ is no longer symmetric, however the symmetric part $P_{sym} = \frac{1}{2}(P + P^T)$ of the matrix P can be considered instead and the same condition on the parameters follows. Therefore in the following discussion, the opinion graph will be treated as undirected for notational convenience.

The case where P is symmetric gives the following characterization of the asymptotic stability of 0_{2n}.

Theorem 5.3.4. *If $\delta_i > \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}, \ \forall i$ and $\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} = w_i^x, \ \forall i$ then 0_{2n} is asymptotically stable on $[0, 1]^{2n} \setminus 1_{2n}$.*

Proof. Consider the Lyapunov function $V(z) = \frac{1}{2}z^T z$. Then when $x_i \neq 1$ and $o_i \neq 1, \ \forall i$

$$\begin{aligned} \dot{V}(z) &= z^T \dot{z} \\ &\leq z^T P z \\ &\leq 0. \end{aligned} \tag{5.9}$$

As P is negative semi-definite, it has an eigenvalue at 0 which has a corresponding eigenvector $z \in \text{span}(1_{2n})$. In order to show that V can be used to show stability via Lyapunov's direct method, consider the behavior of $\dot{V}$ in the case of $z^T P z = 0$.

If $x_i = o_i$ and $x_i = x_j, \ \forall i, j$ then $Pz = 0_{2n}$ and Equation (5.9) shows that $\dot{V} \leq 0$. Consider the dynamics for x_i when $x_i = o_i \neq 0$ and $x_i = x_j, \ \forall i, j$ and

$$\begin{aligned} \dot{x}_i^* &= -\delta_i x_i^*(1 - o_i^*) + (1 - x_i^*)o_i^* \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii} \right) \\ &= -\delta_i x_i^*(1 - x_i^*) + (1 - x_i^*)x_i^* \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii} \right) \\ &= x_i^*(1 - x_i^*) \left(-\delta_i + \sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii} \right) \\ &< x_i^*(1 - x_i^*) \left(-\delta_i + \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} \right) \\ &< 0 \end{aligned}$$

Considering the opinion dynamic

$$\begin{aligned} \dot{o}_i &= \sum_{j \in \mathcal{N}_i^O} (o_j - o_i) + w_i^x (x_i - o_i) \\ &= \sum_{j \in \mathcal{N}_i^O} (x_j - x_i) + w_i^x (x_i - x_i) \\ &= 0 \end{aligned}$$

Then

$$\begin{aligned}\dot{V} &= x^T \dot{x} + o^T \dot{o} \\ &= x^T \dot{x} \\ &< 0.\end{aligned}$$

Note that it is possible that if the opinion graph $\mathcal{G}_O$ is disconnected, there will be multiple eigenvectors corresponding to the eigenvalue 0, however the analysis still holds when considering each such eigenvector in turn.

When $x_i = o_i = 0, \;\; \forall i$, the Lyapunov function satisfies $V(0_{2n}) = 0$ and $\dot{V}(0_{2n}) = 0$. Outside of the cases considered above, $z^T P z < 0$ as P is negative semi-definite and z is not the eigenvector corresponding to 0 which shows $\dot{V} < 0$.

Then $V(z) > 0 \; z \neq 0_{2n}$ by the form of the Lyapunov function and $\dot{V}(z) < 0 \; z \neq 0_{2n}$ which shows the stability of 0_{2n}. □

The results for the equilibrium point $z^* = 1_{2n}$ follow similarly.

Lemma 5.3.5. *The equilibrium point $z^* = 1_{2n}$ is locally stable if* $\forall i, \;\; \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} > \delta_i$.

Proof. Consider the Jacobian matrix at the equilibrium point $z^* = 1_{2n}$

$$J(z^*) = \left[\begin{array}{c|c} \text{diag}\left(-\sum_{\mathcal{N}_i^A} \beta_{ij} - \beta_{ii}\right) & \text{diag}\,(\delta_i) \\ \hline W & -(\mathcal{L}_O + W) \end{array}\right].$$

Similarly to the case of the equilibrium at 0_{2n} the condition that $\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} > \delta_i$ and the fact that $w_i^x > 0$ shows that $J(z^*)$ is weakly chained diagonally dominant and the fact that the diagonal elements are negative shows the Jacobian is Hurwitz by the Gershgorin disc theorem. As the Jacobian is Hurwitz, $z = 1_{2n}$ is a locally stable equilibrium. □

Theorem 5.3.6. *If $\beta_{ii} > \delta_i, \;\; \forall i$ and if $\delta_i = w_i^x, \;\; \forall i$, then 1_{2n} is asymptotically stable on* $[0, 1]^{2n} \setminus 0_{2n}$.

Proof. To show asymptotic stability of 1_{2n}, consider the change of variables $\hat{x}_i = 1 - x_i$ and $\hat{o}_i = 1 - o_i$.

Then $\dot{\hat{x}}_i = -\dot{x}_i$ and $\dot{\hat{o}}_i = -\dot{o}_i$. It follows that:

$$
\begin{aligned}
\dot{\hat{x}}_i &= \delta_i x_i(1-o_i) - (1-x_i)o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\
&= \delta_i (1-\hat{x}_i)\hat{o}_i - \hat{x}_i(1-\hat{o}_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij}(1-\hat{x}_j) + \beta_{ii} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\hat{o}}_i &= \sum_{\mathcal{N}_i^O} (o_i - o_j) + w_i^x \left(o_i - x_i \right) \\
&= \sum_{\mathcal{N}_i^O} (\hat{o}_j - \hat{o}_i) + w_i^x \left(\hat{x}_i - \hat{o}_i \right)
\end{aligned}
$$

Consider the dynamic in $\hat{x}_i$

$$
\begin{aligned}
\dot{\hat{x}}_i &= \delta_i (1-\hat{x}_i)\hat{o}_i - \hat{x}_i(1-\hat{o}_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij}(1-\hat{x}_j) + \beta_{ii} \right) \\
&\le \delta_i (1-\hat{x}_i)\hat{o}_i - \hat{x}_i(1-\hat{o}_i) \left(\beta_{ii} \right) \\
&< \delta_i (1-\hat{x}_i)\hat{o}_i - \hat{x}_i(1-\hat{o}_i) \left(\delta_i \right) \\
&= \delta_i (\hat{o}_i - \hat{x}_i)
\end{aligned}
$$

Then the matrix

$$
\hat{P} = \begin{bmatrix} -D & D \\ W & -(\mathcal{L}_o + W) \end{bmatrix},
$$

where $D = \mathrm{diag}\left(\delta_i\right)$, satisfies

$$
\dot{\hat{z}} \le \hat{P}\hat{z}.
$$

By similar logic to that in Lemma 5.3.3, $\hat{P}$ is negative semidefinite if $\delta_i = w_i^x, \ \forall i$. Then as in Theorem 5.3.4, one can use $\hat{z}^T\hat{z}$ as a Lyapunov function to show stability of 1_{2n}. □

The conditions for stability of 1_{2n} and 0_{2n} are summarized in Table 5.1. It is interesting to note that the conditions for local stability of 0_{2n} ($\delta_i > \beta_{ii}$) is the complement of the

Table 5.1: Summary of Stability Conditions

Equilibrium	Local	Asymptotic
0_{2N}	$\delta_i > \beta_{ii}$	$\delta_i > \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}$
1_{2N}	$\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} > \delta_i$	$\beta_{ii} > \delta_i$

condition for asymptotic stability of 1_{2n} ($\delta_i < \beta_{ii}$) and that the same holds for the local stability of 1_{2n} when compared to the asymptotic stability of 0_{2n}. Additionally these conditions are similar in form to the condition for the unmodified SIS epidemic model, where $\lambda_{\max}(B\tilde{A}(\mathcal{G}^A) - \text{diag}(\delta_i)) < 0_n$ implies the stability of 0_n. However the conditions presented here do not depend on the graph structure of $\mathcal{G}^A$ which suggests there may be other, tighter bounds on the parameters to show stability of these equilibrium points.

5.4 An Unstable Equilibrium

If the stability conditions presented previously for the stability of $z^* = 1_{2n}$ or $z^* = 0_{2n}$ are not satisfied, there is the possibility that a third equilibrium point exists for the system. One class of these equilibria is studied and is shown to be unstable.

Lemma 5.4.1. *If $\exists z^*$ such that for all i it holds that $\delta_i = \sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii}$, $x_i^* = o_i^*$ and $\sum_{j \in \mathcal{N}_i^O} x_j - x_i = 0$ then z^* is an equilibrium point.*

Proof. Consider the dynamic in x_i at the point z^* under the assumption that $\delta_i = \sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii}$:

$$\begin{aligned}
\dot{x}_i &= -\delta_i(1 - o_i^*)x_i^* + (1 - x_i^*)o_i^* \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii} \right) \\
&= -\delta_i(1 - o_i^*)x_i^* + (1 - x_i^*)o_i^* \delta_i \\
&= \delta_i(o_i^* - x_i^*)
\end{aligned}$$

then $\dot{x}_i = 0$ if $o_i^* = x_i^*$. Substituting the condition into the dynamic in o_i gives

$$\begin{aligned}\dot{o}_i &= \sum_{j \in \mathcal{N}_i^O} (o_j^* - o_i^*) + x_i^* - o_i^* \\ &= \sum_{j \in \mathcal{N}_i^O} (x_j^* - x_i^*) + x_i^* - x_i^* \\ &= \sum_{j \in \mathcal{N}_i^O} (x_j^* - x_i^*)\end{aligned}$$

Then if $\sum_{j \in \mathcal{N}_i^O} (x_j^* - x_i^*) = 0$, $\dot{o}_i = 0$. If these conditions hold for all i then z^* is an equilibrium point. □

Theorem 5.4.2. *If the equilibrium described in Lemma 5.4.1 exists and the graph* $G = (V, \mathcal{E}_A \cup \mathcal{E}_O)$ *is connected, it is unstable.*

Proof. Consider the Jacobian at the equilibrium point z^*, the properties of which are described in Lemma 5.4.1. The derivatives of the adoption dynamic at z^* follow

$$\begin{aligned}\frac{\partial f_i}{\partial x_i} &= -\delta_i \\ \frac{\partial f_i}{\partial x_j} &= \begin{cases} (1 - x_i^*) x_i^* \beta_{ij} & \text{if } j \in \mathcal{N}_i^A, j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^A \cup \{i\} \end{cases} \\ \frac{\partial f_i}{\partial o_i} &= \delta_i \\ \frac{\partial f_i}{\partial o_j} &= 0, \ \ \forall j \neq i.\end{aligned}$$

The Jacobian can be written as

$$J(z^*) = \left[\begin{array}{c|c} \frac{\partial f}{\partial x} & D \\ \hline W & -(\mathcal{L}_o + W) \end{array}\right]$$

where $D = \mathrm{diag}(\delta_i)$. If the graph $G = (V, \mathcal{E}_P \cup \mathcal{E}_O)$ is connected, the Jacobian is irreducible, which allows the application of Lemma A.0.3. Consider a vector y of the form

$$y = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ \mathbf{1}_n \end{bmatrix}$$

where $\alpha_i = 1 + \epsilon_i$ and

$$0 < \epsilon_i < \frac{\sum_{\mathcal{N}_i^A} \beta_{ij}(1 - x_i^*)x_i^*}{\delta_i}.$$

Then consider the matrix product $J(z^*)y$. The first n rows follow:

$$\begin{aligned}
&-\delta_i\alpha_i + \alpha_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij}(1 - x_i^*)x_i^* \right) + \delta_i \\
&> -\delta_i\alpha_i + \left(\sum_{\mathcal{N}_i^A} \beta_{ij}(1 - x_i^*)x_i^* \right) + \delta_i \\
&> -\delta_i\epsilon_i + \left(\sum_{\mathcal{N}_i^A} \beta_{ij}(1 - x_i^*)x_i^* \right) \\
&> -\left(\sum_{\mathcal{N}_i^A} \beta_{ij}(1 - x_i^*)x_i^* \right) + \left(\sum_{\mathcal{N}_i^A} \beta_{ij}(1 - x_i^*)x_i^* \right) \\
&= 0.
\end{aligned}$$

The last n rows follow

$$\begin{aligned}
&\alpha_i w_i^x - d_i^O - w_i^x + \sum_{\mathcal{N}_i^O} 1 \\
&= (\alpha_i - 1)w_i^x \\
&= \epsilon_i w_i^x \\
&> 0.
\end{aligned}$$

As y is element-wise positive and the resulting vector $J(z^*)y$ is element-wise positive, by Lemma A.0.3, $\alpha(J(z^*)) > 0$ and the equilibrium point is unstable. $\square$

For such systems, it holds that in the long term they will converge to a state of universal agreement, i.e. $z^* = 1_{2n}$ or $z^* = 0_{2n}$. As both equilibria are locally stable, which of the two equilibrium points is reached will depend on the initial condition, as will be explored further in the simulation section.

5.5 Varying Opinion Networks

As noted previously the non-positivity of the eigenvalues of the P and $\hat{P}$ matrices, which are used to characterize the stability of the equilibria of the coupled adoption behavior, do not depend on the structure of the opinion graph. Therefore the results for the stability of the equilibria of the coupled adoption model can be extended to two cases: the bounded confidence opinion dynamic model and the time-varying Abelson opinion dynamic model.

5.5.1 Bounded Confidence

The first varying opinion network is an extension of the Abelson opinion dynamic model; the bounded confidence model, discussed in Chapter 2, which when coupled with the adoption dynamic follows:

$$\dot{o}_i = g_i(x, o) = \sum_{j \in \mathcal{N}_i^O} p(o_j, o_i)(o_j - o_i) + w_i^x(x_i - o_i). \tag{5.10}$$

where

$$p(o_j, o_i) = \begin{cases} w_{ij}^o & \text{if } \|o_j - o_i\| < \tau \\ 0 & \text{if else.} \end{cases}$$

Under the bounded confidence model, agents will sever a link in the opinion graph if the agents have sufficiently different opinions and maintain or reintroduce the link if the respective opinions are closer than τ. This behavior is essentially a state dependent switch between opinion graph topologies. These opinion graphs may not be connected, to the point where each agent has no neighbors in the opinion graph.

However the structure of the coupling with the adoption dynamic ensures that conditions for asymptotic equilibria $z^* \in \{0_{2n}, 1_{2n}\}$ are the same. Proving this requires the definitions from the study of switched systems, which are summarized in Appendix C.

Theorem 5.5.1. *If* $\delta_i > \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}, \quad \forall i$ *and* $\sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii} = w_i^x, \quad \forall i$ *then* 0_{2n} *is uniformly asymptotically stable on* $[0,1]^{2n} \setminus 1_{2n}$ *under the bounded confidence opinion dynamic.*

Proof. Consider the finite collection of opinion graph topologies $\hat{\mathcal{G}}_o = \{\mathcal{G}_o^1, \mathcal{G}_o^2, \dots, \mathcal{G}_o^s\}$ which the bounded confidence model can switch between. The original opinion graph $\mathcal{G}_o \in \hat{\mathcal{G}}_o$, and also the empty opinion graph $\mathcal{G}_o^\emptyset \in \hat{\mathcal{G}}_o$. Consider the graph $\mathcal{G}_o^i$ which consists of k connected subgraphs. Then under the opinion dynamic on $\mathcal{G}_o^i$ and if $\delta_i > \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}, \ \forall i$ the dynamics follow

$$\dot{z} \leq P_i z$$

where

$$P_i = \begin{bmatrix} -\bar{B} & \bar{B} \\ W & -\left(\begin{bmatrix} \mathcal{L}_o^1 & 0 & \dots & 0 \\ 0 & \mathcal{L}_o^2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \mathcal{L}_o^k \end{bmatrix} + W \right) \end{bmatrix},$$

though a permutation may be required to the opinion dynamic into this form. The matrix P_i is negative semidefinite as it is diagonally dominant with negative diagonal elements and symmetric by the assumption on the parameters.

Under $\mathcal{G}_o^{\emptyset}$ and if $\delta_i > \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}, \ \forall i$ the dynamics follow

$$\dot{z} \leq \begin{bmatrix} -\bar{B} & \bar{B} \\ W & -W \end{bmatrix} z = P_{\emptyset} z.$$

The matrix $P_{\emptyset}$ is also negative semidefinite as it is diagonally dominant with negative diagonal elements and symmetric by the assumption on the parameters.

Then under any opinion graph $\mathcal{G}^j \in \hat{\mathcal{G}}_o$ the function $V(z) = \frac{1}{2} z^T z$ satisfies

$$\begin{aligned} \dot{V} &\leq z^t P_j z \\ &\leq 0. \end{aligned}$$

By similar logic to that used in Theorem 5.3.4 and Theorem 5.3.6 one can show that

$$\dot{V}(z) < 0, \ \forall z \neq 0_{2n}.$$

As the set of possible graph topologies is finite, $V(x) = \frac{1}{2} z^T z$ serves as a common Lyapunov function by Definition C.0.2 and can be used to show that the system is uniformly asymptotically stable by Theorem C.0.1. □

The theorem for the stability of the equilibrium point $z^* = 1_{2n}$ is presented without proof as the proof follows from the proof of Theorem 5.3.6 with similar logic to that in the proof of Theorem 5.5.1.

Theorem 5.5.2. *If* $\beta_{ii} > \delta_i, \ \forall i$ *and if* $\delta_i = w_i^x, \ \forall i$*, then* 1_{2n} *is asymptotically stable on* $[0,1]^{2n} \setminus 0_{2n}$ *under the bounded confidence opinion dynamic.*

5.5.2 Time-Varying Networks

In this section, an extension of the coupled adoption dynamic in Equations (5.1) and (5.2) is considered, which allows for time-varying network effects. Consider the following model

$$\dot{x}_i = -\delta_i x_i(1 - o_i) + (1 - x_i)o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij}(t) x_j + \beta_{ii} \right)$$
$$\dot{o}_i = \sum_{\mathcal{N}_i^O} w_{ij}^o(t)(o_j - o_i) + w_i^x (x_i - o_i),$$

where the weight associated with the opinion graph $\mathcal{G}_O$ and the weight associated with the adoption graph $\mathcal{G}_A$ are now allowed to vary with time. Let $\sup_t \beta_{ij}(t) = \hat{\beta}_{ij}$ and $\sup_t w_{ij}^o(t) = \hat{w}_{ij}^o$.

Theorem 5.5.3. *If* $\delta_i > \sum_{\mathcal{N}_i^A} \hat{\beta}_{ij} + \beta_{ii}, \;\; \forall i$ *and* $\sum_{\mathcal{N}_i^A} \hat{\beta}_{ij} + \beta_{ii} = w_i^x, \;\; \forall i$ *then* 0_{2N} *is asymptotically stable on* $[0, 1]^{2n} \setminus 1_{2n}$.

Proof. Consider the adoption dynamic under time-varying network effects for the case $\delta_i > \sum \hat{\beta}_{ij} + \beta_{ii}$

$$
\begin{aligned}
\dot{x}_i &= -\delta_i x_i(1-o_i) + (1-x_i)o_i\left(\sum_{\mathcal{N}_i^A}\beta_{ij}(t)x_j + \beta_{ii}\right)\\
&\leq -\delta_i x_i(1-o_i) + (1-x_i)o_i\left(\sum_{\mathcal{N}_i^A}\beta_{ij}(t) + \beta_{ii}\right)\\
&\leq -\delta_i x_i(1-o_i) + (1-x_i)o_i\left(\sum_{\mathcal{N}_i^A}\hat{\beta}_{ij} + \beta_{ii}\right)\\
&< -\left(\sum_{\mathcal{N}_i^A}\hat{\beta}_{ij} + \beta_{ii}\right)x_i(1-o_i)\\
&\quad + (1-x_i)o_i\left(\sum_{\mathcal{N}_i^A}\hat{\beta}_{ij} + \beta_{ii}\right)\\
&= (o_i - x_i)\left(\sum_{\mathcal{N}_i^A}\hat{\beta}_{ij} + \beta_{ii}\right)
\end{aligned}
$$

Similarly for the opinion dynamic,

$$
\begin{aligned}
\dot{o}_i &= \sum_{\mathcal{N}_i^O} w_{ij}^o(t)(o_j - o_i) + w_i^x\left(x_i - o_i\right)\\
&\leq \sum_{\mathcal{N}_i^O} \hat{w}_{ij}^o(o_j - o_i) + w_i^x\left(x_i - o_i\right),
\end{aligned}
$$

Then the dynamic follows

$$
\dot{z} \leq \begin{bmatrix} -\hat{B} & \hat{B} \\ W & -\left(\hat{\mathcal{L}}_o + W\right) \end{bmatrix} z = P_{\text{sup}} z
$$

where $\hat{\mathcal{L}}_o$ is the graph Laplacian of the weighted network with weights $\hat{w}_{ij}^o$.

P_{sup} is negative semi-definite and the function $V(z) = \frac{1}{2}z^T z$ shows asymptotic stability of 0_{2n}. □

Theorem 5.5.4. *If $\beta_{ii} > \delta_i, \ \forall i$ and if $\delta_i = w_i^x, \ \forall i$, then 1_{2n} is asymptotically stable on $[0,1]^{2n} \setminus 0_{2n}$.*

Proof. Consider the dynamic in $\hat{x}_i = 1 - x_i$

$$\begin{aligned}
\dot{\hat{x}}_i &= \delta_i(1-\hat{x}_i)\hat{o}_i - \hat{x}_i(1-\hat{o}_i)\left(\sum_{\mathcal{N}_i^A} \beta_{ij}(t)(1-\hat{x}_j) + \beta_{ii}\right) \\
&\leq \delta_i(1-\hat{x}_i)\hat{o}_i - \hat{x}_i(1-\hat{o}_i)\left(\beta_{ii}\right) \\
&< \delta_i(1-\hat{x}_i)\hat{o}_i - \hat{x}_i(1-\hat{o}_i)\left(\delta_i\right) \\
&= \delta_i(\hat{o}_i - \hat{x}_i)
\end{aligned}$$

The dynamic in $\hat{o}_i = 1 - o_i$ follows

$$\begin{aligned}
\dot{\hat{o}}_i &= \sum_{\mathcal{N}_i^O} w_{ij}^o(t)(\hat{o}_j - \hat{o}_i) + w_i^x\left(\hat{x}_i - \hat{o}_i\right) \\
&\leq \sum_{\mathcal{N}_i^O} \hat{w}_{ij}^o(\hat{o}_j - \hat{o}_i) + w_i^x\left(\hat{x}_i - \hat{o}_i\right)
\end{aligned}$$

Then the coupled dynamic in $\hat{z}$ satisfies

$$\dot{\hat{z}} \leq \begin{bmatrix} -D & D \\ W & -(\hat{\mathcal{L}}_o + W) \end{bmatrix} = \hat{P}_{\text{sup}} z.$$

The matrix $\hat{P}_{\text{sup}}$ is negative semi-definite and the function $V(z) = \frac{1}{2} z^T z$ shows the asymptotic stability of 1_{2n}. □

5.6 Simulation

Having analyzed the behavior of the proposed model, the behavior of these models will now be examined via simulation. Figure 5.1, 5.2, and 5.3 show the long term behavior of the opinion dynamic and the adoption dynamic under a number of conditions on the underlying system. The simulation is run on an undirected, unweighted geometric random network with thirty nodes, serving as both the opinion and product network, $\mathcal{G}_O$ and $\mathcal{G}_A$. The initial conditions are chosen uniformly at random from $[0,1]^{2n}$ and are the same for all figures. The parameters of the adoption model are chosen randomly and it is verified that they satisfy the various stability conditions.

Figure 5.1 and 5.2 confirm the stability as shown in Theorems 5.3.4 and 5.3.6. Figure 5.3 shows that without coupling with the opinion dynamics the adoption model will converge to an endemic equilibrium. Figure 5.4 shows the behavior of the coupled model, in the case where the opinion dynamic is the bounded confidence model. Figure 5.4 shows that outside of the nonlinearities induced by the bounded confidence modification, the Abelson and bounded confidence opinion dynamics have qualitatively similar behaviors and as noted in Theorems 5.5.1 and 5.5.2 share stability conditions.

The opinion dynamics presented here induce outcomes of all adopt or all not-adopt when coupled with the adoption dynamic. This reflects scenarios where a new technology or idea either becomes the new standard or completely fails to get adopted. Examples of a new innovation being widely adopted are the invention of the steam engine and the administration of antibiotics. The practice of boiling water is an example of an innovation that failed to spread in the Peruvian village of Los Molinas, due to the inhabitants viewing it as incompatible with cultural beliefs. [108]. It is possible to introduce other opinion models and produce very different behavior in the model. The rest of this section considers in simulation the behavior of coupling with two new opinion dynamic models which are described below.

Other Opinion Dynamic Models

The first opinion dynamic model is the Signed Consensus or Altafini Model. Due to the possibility of negative opinions, the assumption of Lemma 5.2.1, that $o(t) \in [0, 1]$ for all $t \geq 0$, is difficult to meet. Therefore the adoption model requires a small change when

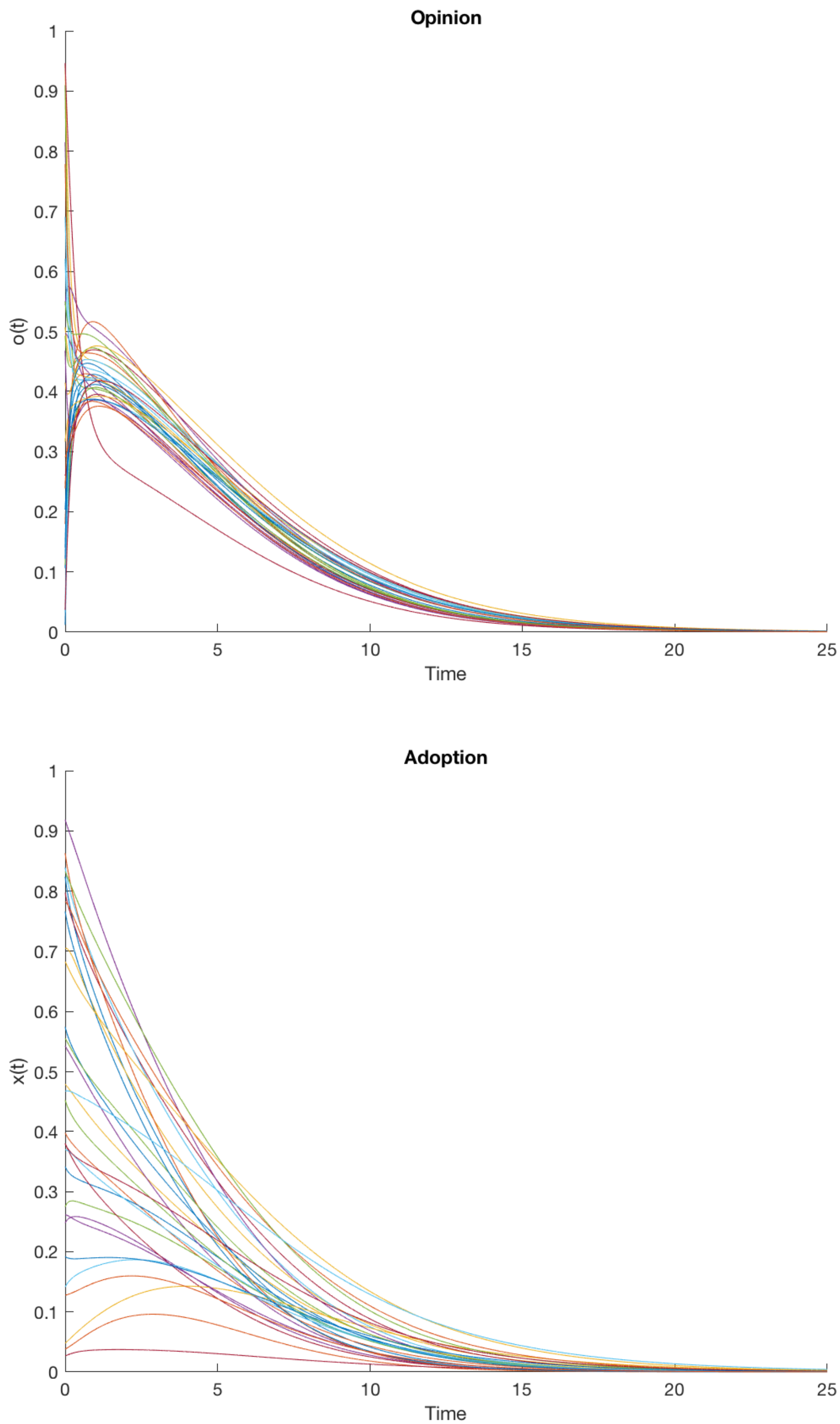


Figure 5.1: Evolution of the coupled model in the case where $\forall i \ \ \delta_i > \sum_{\mathcal{N}_i^A} \beta_{ij} + \beta_{ii}$

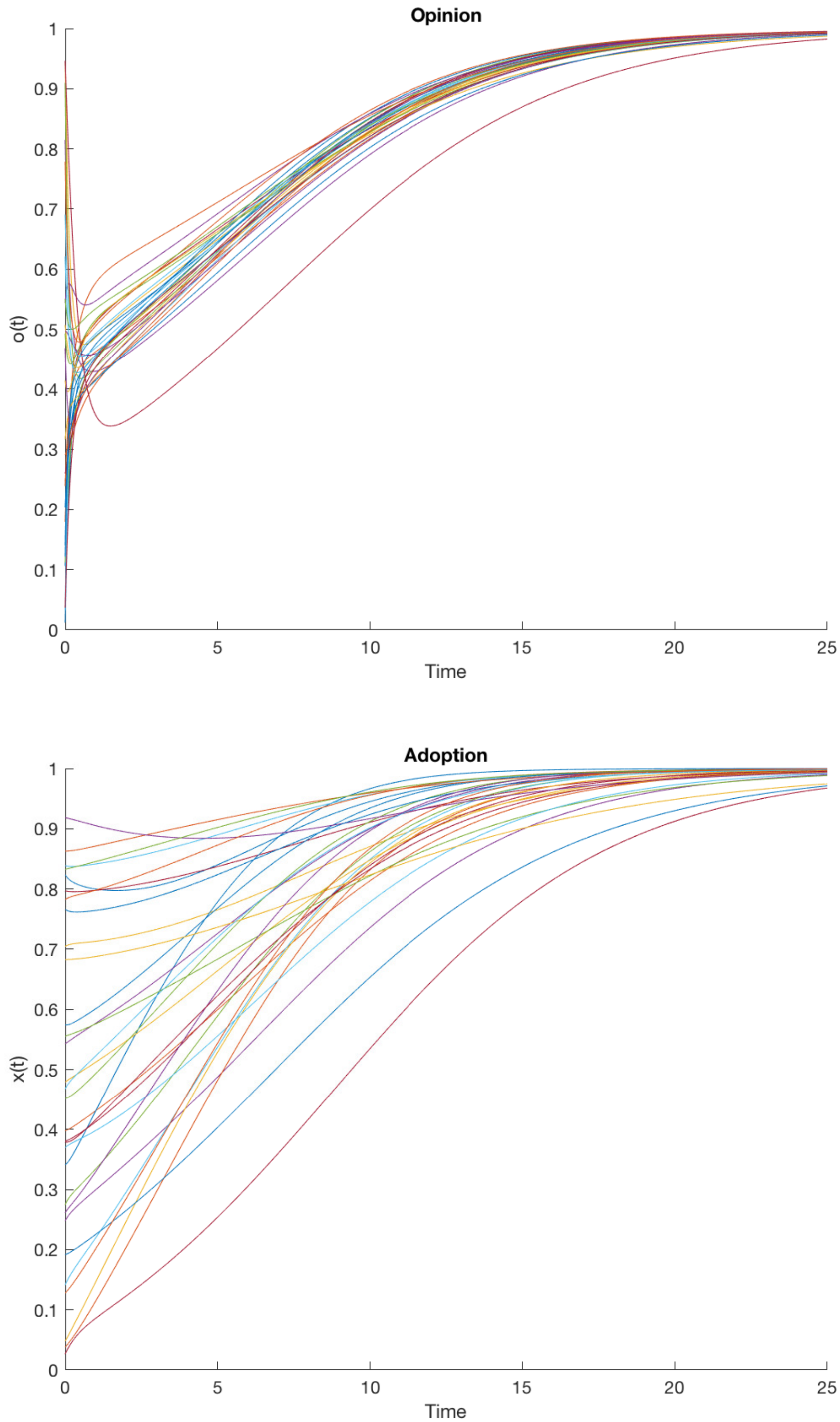
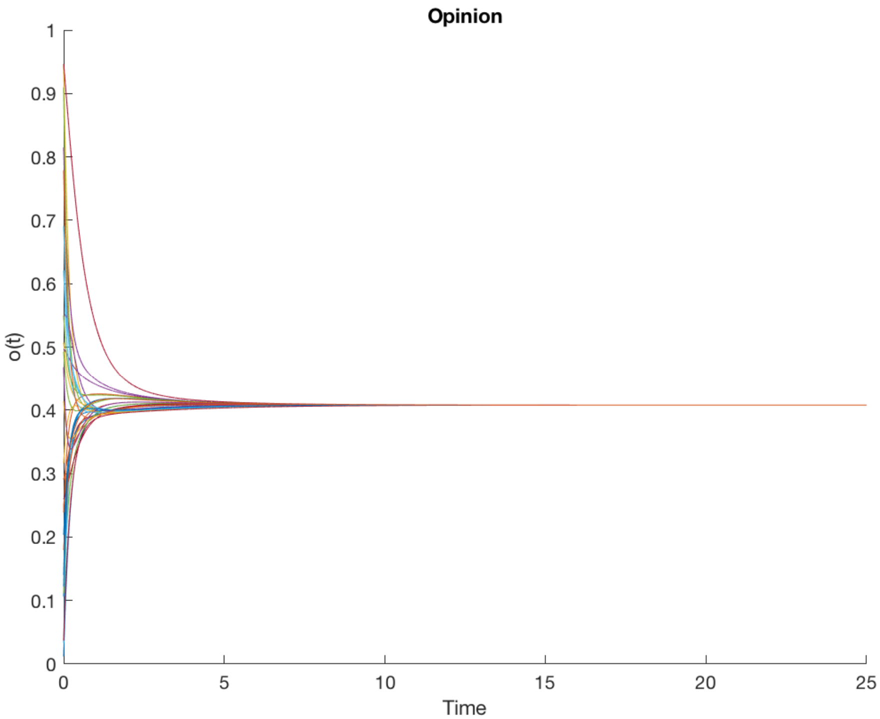


Figure 5.2: Evolution of the coupled model in the case that $\forall i \;\; \beta_{ii} > \delta_i$



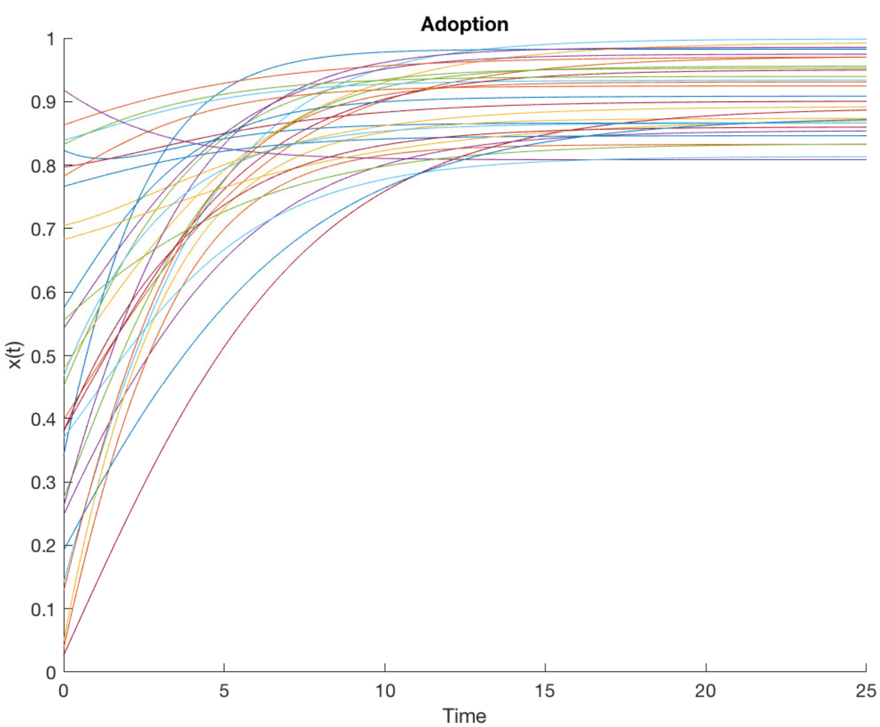


Figure 5.3: Evolution of the model when there is no coupling between the opinion and the adoption dynamics. With no coupling, the condition of Figure 5.2 produces an endemic equilibrium in the adoption dynamic

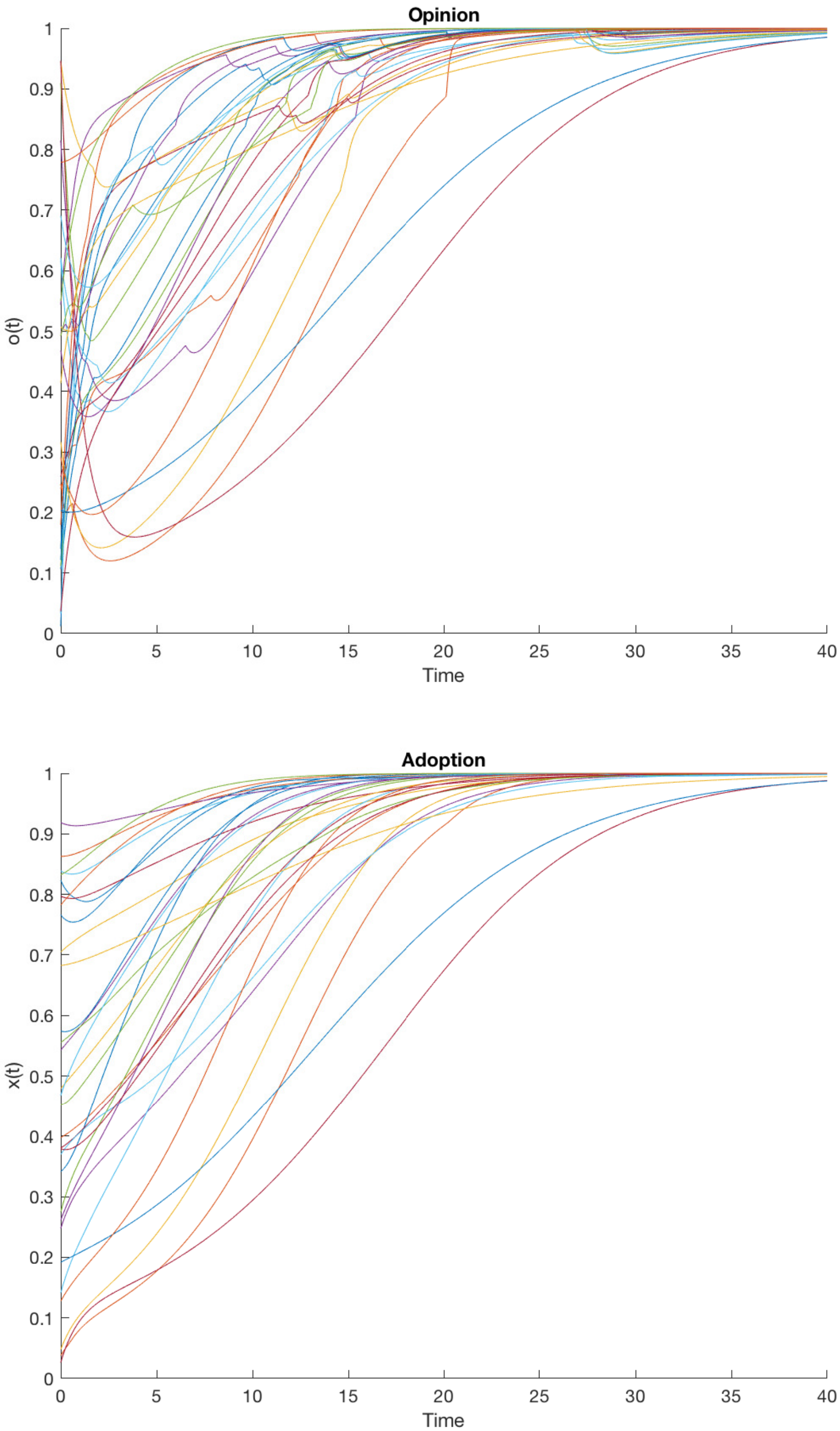


Figure 5.4: Evolution of the bounded confidence model coupled with the adoption model in the case that $\forall i \;\; \beta_{ii} > \delta_i$

employing Altafini-type dynamics. The complete model is

$$\begin{aligned}\dot{x}_i &= f_i(x, \bar{o})\\ \dot{o}_i &= \sum_{\mathcal{N}_i^{\bar{O}}} |a_{ij}|(\text{sign}(a_{ij})o_j - o_i) + w_i^x(x_i - \bar{o}_i),\end{aligned} \tag{5.11}$$

where $\bar{o}_i = o_i + .5$, it is assumed that $o_i(0) \in [-.5, .5], \ \forall i$, and the notation in Equation (5.11) is the same as in Equation (2.3). Note that when there are no negative edges this reduces to the Abelson model in Equation (5.2). When negative edges are present and the graph is structurally balanced the system can converge to a split equilibrium, that is, where some nodes are completely infected and some nodes are completely healthy.

Introducing the Altafini opinion dynamic produces rich behavior in the couple adoption opinion model. For example, if negative edges are introduced then the point 0_{2n} is no longer an equilibrium point. Figure 5.5 shows the adoption behavior under identical conditions for a 6 node network with $\delta_i > \sum_{j \in \mathcal{N}_i^O} \beta_{ij} + \beta_{ii}, \forall i$ but where the first graph has no negatives edges in the opinion dynamic and the second graph has negative edges. This is discussed more fully in [150].

The second model is a threshold driven model of opinion dynamics where individuals update their opinions using a weighted average of the opinions and product adoptions of friends, combined with a threshold. The threshold represents how stubborn or receptive one is to the influence of neighbors. As will be seen in the following simulations, this allows for polarization in opinions, resulting in coexistence of adopters and non-adopters.

Consider opinion dynamics defined by

$$\dot{o}_i = g_i(x, o) = o_i(1 - o_i)\left(h_i(x, o) - \tau_i\right), \tag{5.12}$$

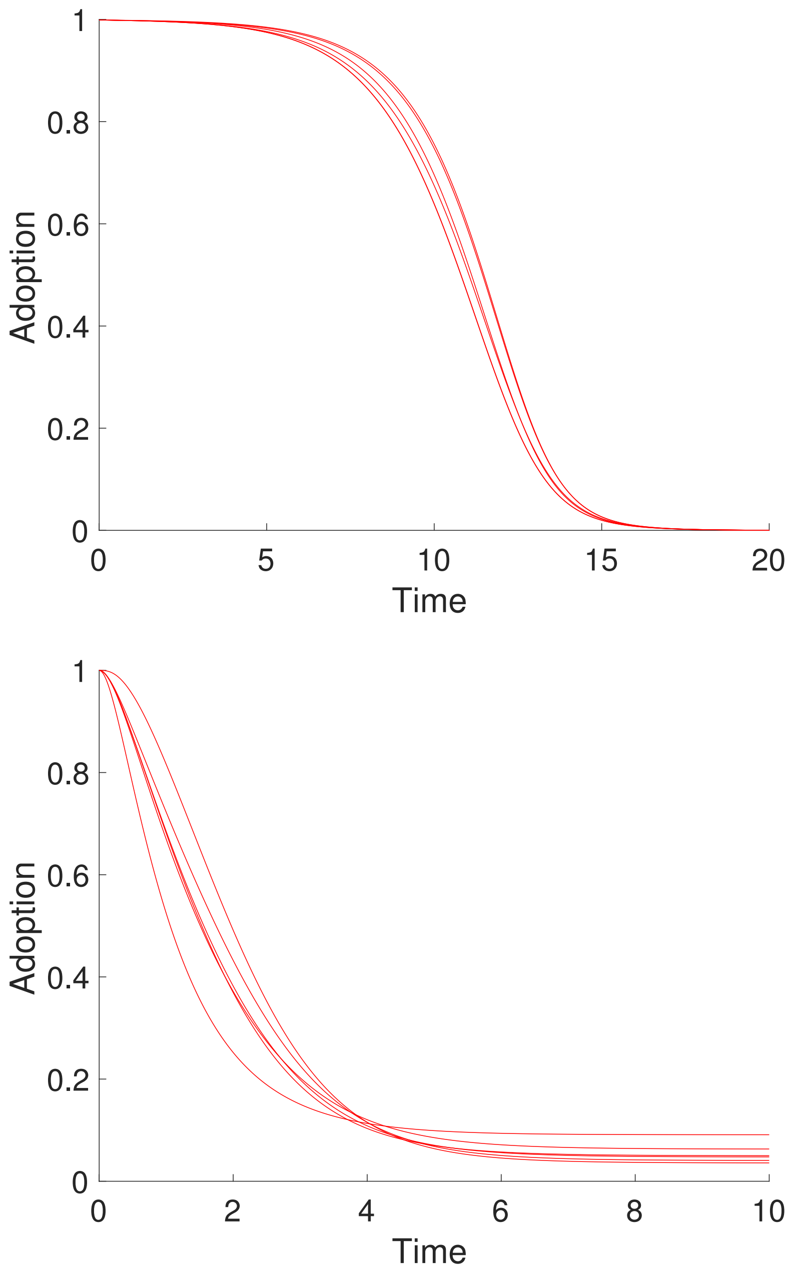


Figure 5.5: Adoption without and with Negative Edges

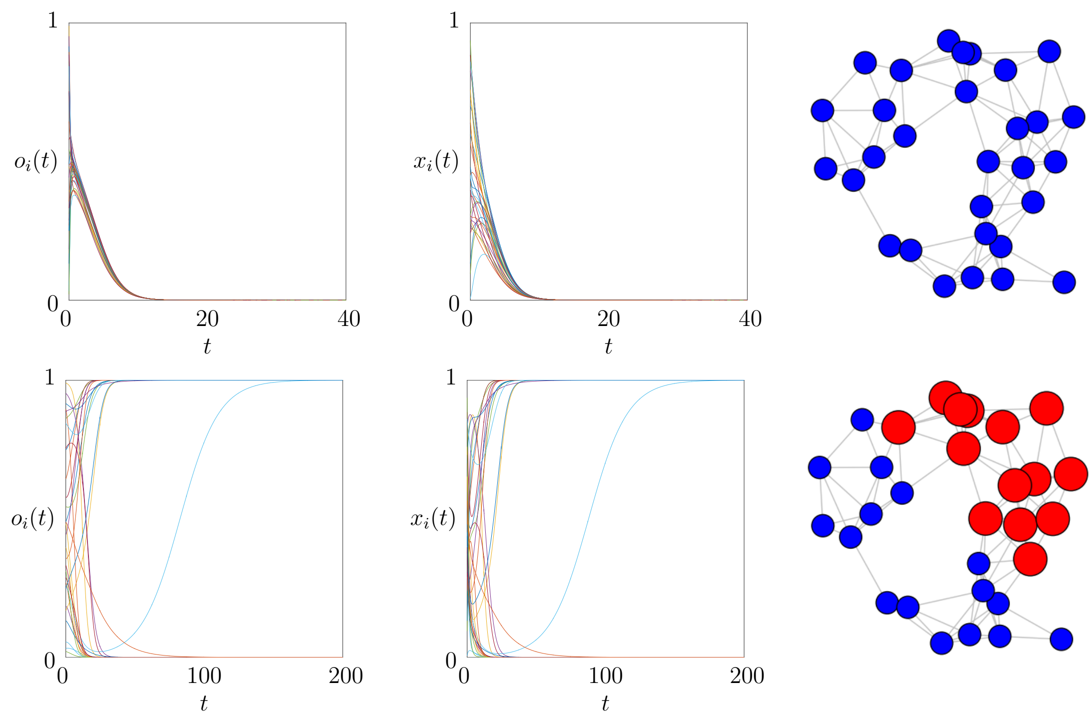


Figure 5.6: Dynamics of the Abelson coupled model (top row) and threshold-based model (bottom row). In each model, all individual opinions $o_i(t)$ (left panels) and adoptions $x_i(t)$ (center panels) are shown converging to their equilibrium values o_i^*, $x_i^* = 0$ or 1. The right panels indicate the final equilibrium layout over the 30 node geometric network. Red nodes indicate $x_i^* = 1$ and blue nodes denote $x_i^* = 0$. The largest diameters indicate $o_i(0) = 1$ and the smallest diameters indicate $o_i(0) = 0$. The network for the SIS dynamics is depicted by the gray (positive) edges. For a video of these simulations please see youtu.be/U0bWaXCeayY.

where

$$h_i(x, o) = \frac{\sum_{\mathcal{N}_i^O} w_{ij}^o o_j + \sum_{\mathcal{N}_i^P} w_{ij}^x x_j}{\sum_{\mathcal{N}_i^O} w_{ij}^o + \sum_{\mathcal{N}_i^P} w_{ij}^x}.$$

The $w_{ij}^o \in [0, 1]$ represents node i's valuation of node j's opinion, and the $w_{ij}^x \in [0, 1]$ represents the influence j's adoption decision has over i's opinion. The opinion threshold, $\tau_i \in [0, 1]$, is a measure of stubbornness to opinion change. If $\tau_i = 1$, no amount of influence will force an increase in o_i. However, if $\tau_i = 0$, any amount of influence increases o_i.

Figure 5.6 shows a representative simulation of the Abelson and threshold-based dynamics. In this run, the Abelson dynamics quickly converge to the all not-adopt consensus.

The threshold-based dynamic takes longer to converge to a stable equilibrium. Note that the final equilibrium outcome is heavily dependent on the initial opinions, as there are many possible stable fixed points the dynamics could converge to. As such there are individuals whose opinion $o_i(t)$ changes directions before finally settling at either $o_i^* = 0$ or 1. The simulation of the two models is run on the same undirected, unweighted geometric random network with thirty nodes as before. The parameters are $\delta_i = 1, \beta_{ij} = .15, \tau_i = .5$ for all nodes i, j. The initial condition of the simulations, chosen uniformly at random on $[0, 1]^{2n}$, are the same for both models.

The behavior of this coupled system leads the question: does influence from the adoption or opinion network drive the dynamics, or do they drive each other? Uncoupled from the opinion dynamics, the adoption state $x(t)$ would converge to its endemic equilibrium $x^* \succ 0$, as shown in Figure 5.3. Without opinions, each node reaches an intermediate value of adoption x_i^* whose value depends on its position in the network. When coupled with opinions for both Abelson and threshold-based dynamics, the $x_i(t)$ are driven to either $x_i^* = 0$ or 1, with their final opinions agreeing with their final adoption decisions. Given the difference in possible equilibria outcomes between the two models, the coupled opinion-adoption model is sensitive to the choice of opinion dynamic. Thus, opinions have a significant role in determining the final adoption state.

The adoptions $x_i(t)$ in the coupled systems follow closely the trajectories of the opinions for both Abelson and threshold-based models. In the case of the Abelson model it is difficult to determine which dynamic drives the state as the two processes evolve on similar time scales. The threshold opinion model depends on the neighborhood structure of the nodes as can be seen in Figure 5.6, which shows the final state of the time series data.

The final state of simulations of the bounded confidence, shown in Equation (5.10), and Altafini models, Equation (2.3), are shown in Figure 5.8. The coupled bounded confidence dynamics converge to the all adopt equilibrium, exhibiting the same behavior as the Abelson model. The coupled Altafini dynamics can exhibit final behavior similar to the

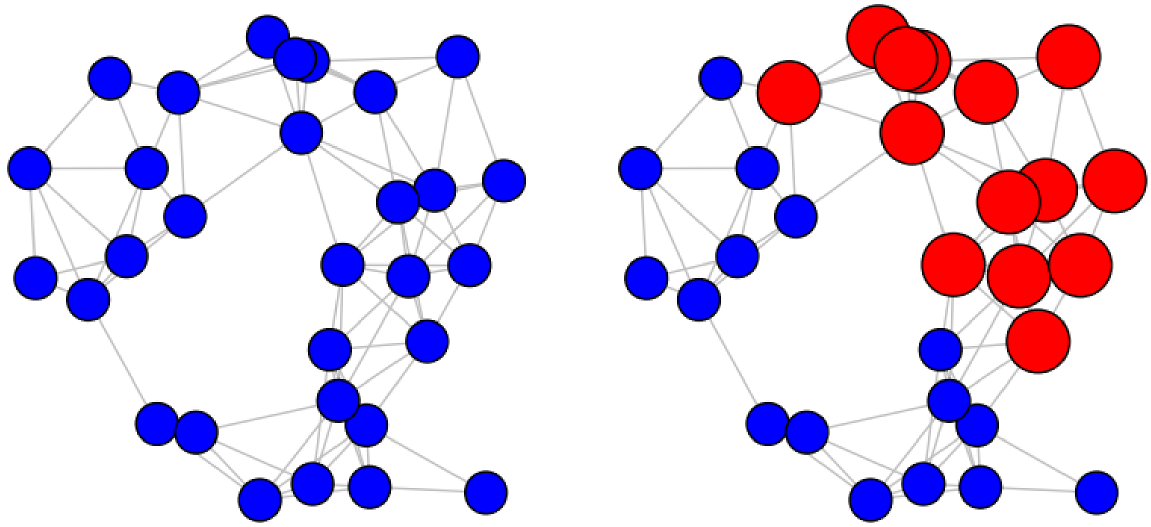


Figure 5.7: The equilibria of a simulation of the two models: (Left) the equilibrium of Equations (5.1), (5.2) and (Right) the equilibrium of Equations (5.1), (5.12). The red nodes indicate $x_i^* = 1$ and blue nodes denote $x_i^* = 0$. The largest diameters indicate $o_i(0) = 1$ and the smallest diameters indicate $o_i(0) = 0$.

threshold-based opinion model. However, static negative edges must be specified to attain such polarization. Hence while the threshold model has the possibility to reveal structure in a network, the Altafini model requires structure to be explicitly defined.

Together Figure 5.6 and Figure 5.8 show that the Abelson and Threshold opinion models considered in this section are sufficient to influence the outcome of the product spread away from the endemic state and to capture diverse equilibrium outcomes, even over a simple graph. In supplementary videos (see figure captions for URLs), $x_i(t)$ is plotted as a function of color, where $x_i = 1$ is indicated by red (r) and $x_i = 0$ is indicated by blue (b) and the color interpolates between the two. So at time t the color for agent i is given by $rx_i(t) + b(1 - x_i(t))$. The opinion of agent i $o_i(t)$ is indicated by the diameter of the node. The largest diameter indicates $o_i = 1$ and the smallest diameter indicates $o_i = 0$. The graph structure is binary and indicated by gray edges.

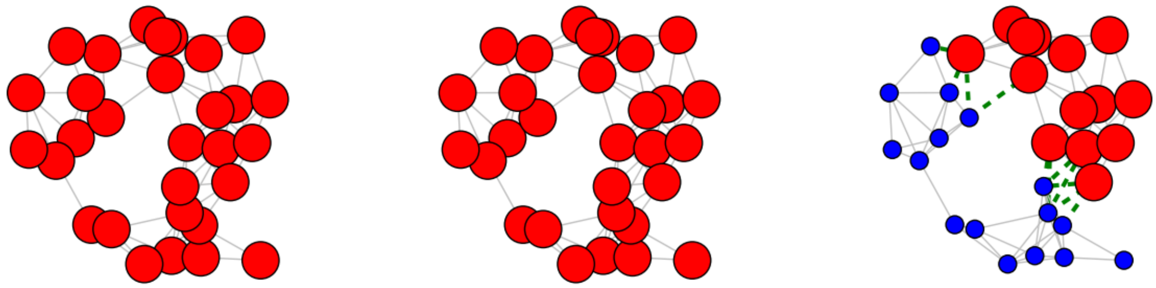


Figure 5.8: The equilibria of simulations employing the Abelson model (Left), the Bounded Confidence model (Middle), and the Altafini model (Right) with the same conditions as the simulations in Figure 5.6 except dotted lines indicate negative edges, $\beta = .5$, and the confidence parameter $\epsilon = .1$: large nodes correspond to $x_i^* = 1$ and small nodes correspond to $x_i^* = 0$. The largest diameters indicate $o_i(0) = 1$ and the smallest diameters indicate $o_i(0) = 0$. The network for the SIS dynamics is depicted by the gray (positive) edges. For a video of this simulation please see youtu.be/BXVidqntYtA

Unstable Equilibria

As mentioned in Section 5.4, there exists an unstable equilibrium if 1_{2n} and 0_{2n} are both locally stable, i.e. $\delta_i < \sum_{j \in \mathcal{N}_i^A} \beta_{ij} + \beta_{ii}$ and $\delta_i > \beta_{ii}$. This section considers the behavior of such an equilibrium in simulation. To facilitate analysis, the graph structure considered here is a completely connected 4 node graph for both $\mathcal{G}_O$ and $\mathcal{G}_A$. When $D = diag(.5, .4, .6, .3)$ and

$$B = \begin{bmatrix} 0 & .3 & .35 & .35 \\ .15 & 0 & .3 & .35 \\ .6 & .3 & 0 & .3 \\ .2 & .15 & .25 & 0 \end{bmatrix}$$

the point $z = 0.5_{2n}$ is an equilibrium point, however it is unstable. When the initial condition for this system is not $z = 0.5_{2n}$, then the long term behavior of the system depends on $x(0)$ as shown in Figure 5.10 and Figure 5.9. The mean of $x(0)$ and $o(0)$ is plotted with a blue asterisk. If the mean of both $x(0)$ and $o(0)$ is below the equilibrium is below 0.5 the system converges to 0_{2n} and if the means are above 0.5 the system converges to 1_{2n}. This makes the equilibrium at 0.5_{2n} a threshold that could be studied in the context of herdability.

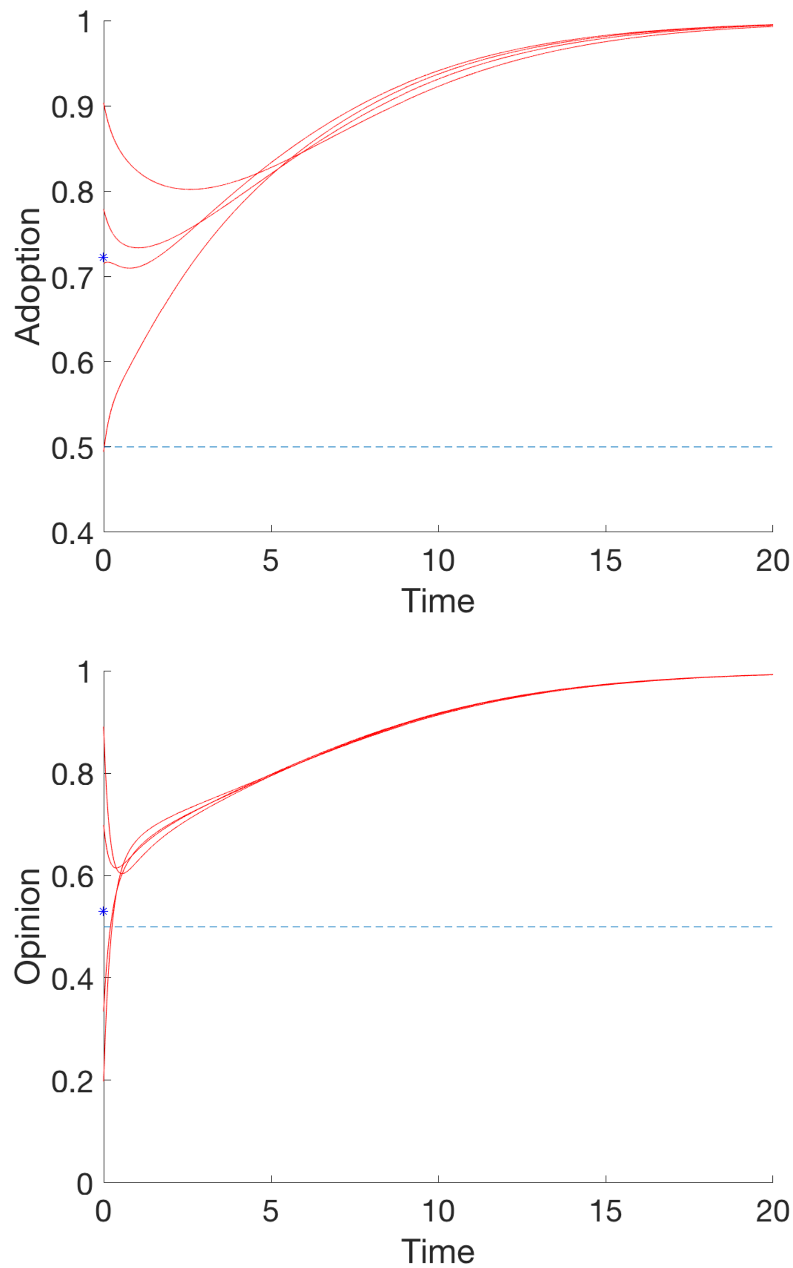


Figure 5.9: Adoption and Opinion for “High” Initial Condition. The mean of the initial condition is shown with a blue asterisk.

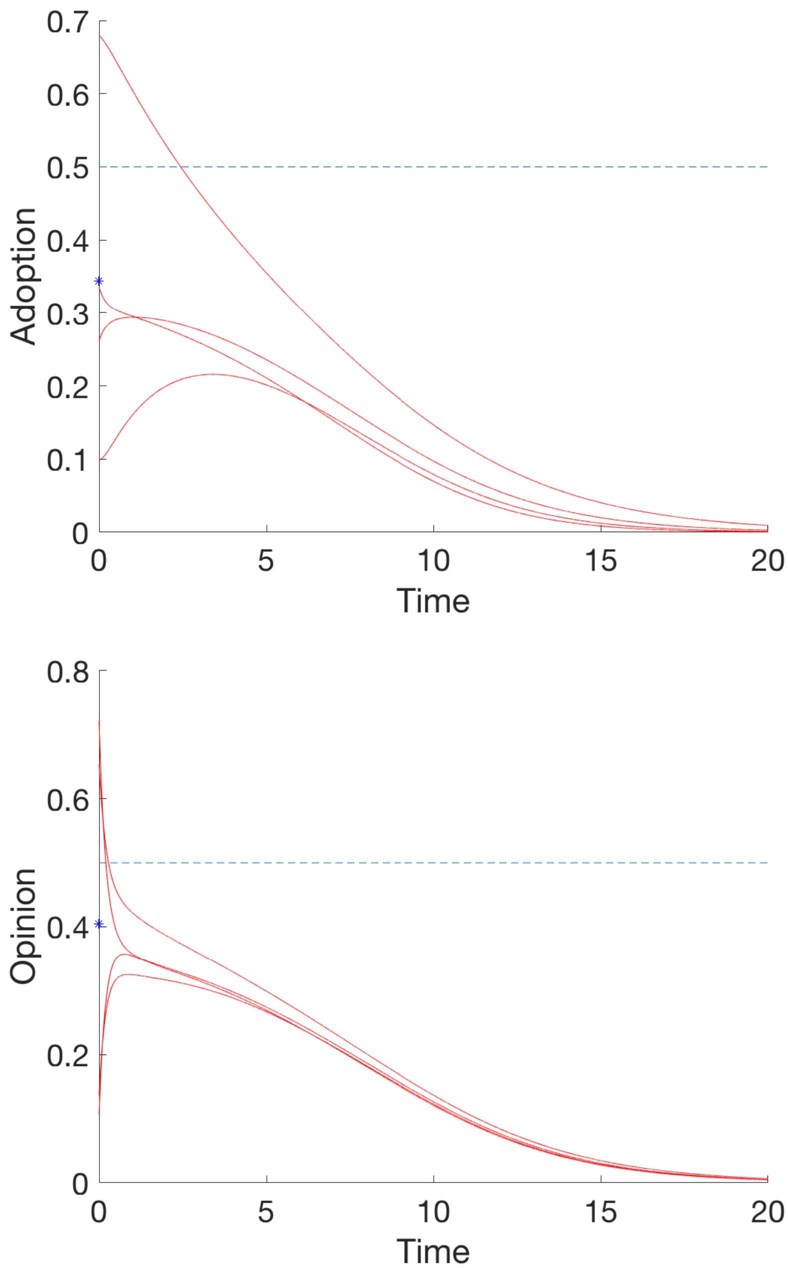


Figure 5.10: Adoption and Opinion for "Low" Initial Condition. The mean of the initial condition is shown with a blue asterisk.

5.7 Conclusion

In this chapter, the behavior of an epidemic spread model with beneficial interactions from an opinion dynamic was considered. The Abelson opinion dynamic was considered in-depth, and the stability of three equilbirum points was studied. It was found that under the assumption that there is always interaction between an agent's opinion and their adoption behavior (i.e. that $w_i^x > 0$) that there are ranges of the parameter values for which the specifics of the network structure don't impact the stability of the system. Specifically these parameters capture products that are either break away hits ($\beta_{ii} > \delta_i$) or terrible flops ($\delta_i > \sum_{j \in \mathcal{N}_i^O} \beta_{ij} + \beta_{ii}$. These results allowed the asymptotic stability results to be extended to time-varying opinion networks, however there may be tighter bounds for asymptotic stability which rely on network structure.

The behavior of the coupled model under a variety of opinion dynamics was considered in simulation and it was shown that the opinion dynamic chosen has a large impact on the behavior of the model. The presence of an intermediate unstable equilibrium was also studied in simulation. In such a case, both 1_{2n} and 0_{2n} are locally stable and if the state can be driven above the unstable equilibrium, a threshold, then the system will eventually converge to 1_{2n}. This suggests that this is a system which should be studied in the context of herdability.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

This thesis discussed the application of control theory in complex networks, drawing inspiration from two sets of phenomena seen in social networks. The first phenomenon from social networks was the threshold behavior of social action. This led to the study of herdability, which expands the discussion of control of networks beyond controllability. In doing so, herdability encourages a more thorough examination of the behaviors that can be achieved when complete controllability is not satisfied.

It was shown that if the system dynamic was positive, then input connectability was a necessary and sufficient condition for complete herdability. In the language of social networks, if everyone in the network is friends then as long as a message relayed to the system will eventually reach everyone, the system is completely herdable. The assumption that the underlying system is positive is equivalent to asking that the weight of edges between nodes is positive. It's interesting to note that this holds for most complex network structures that have been extracted from data, for a number of reasons. As an example, the Stanford Large Network Dataset Collection [1] has 61 networks from a variety of sources and 8 of them have negative edges between nodes. Essentially, most complex networks are treated as easy to herd.

It was also shown that when selecting between nodes via herdability centrality, high degree nodes are chosen. When the controllability of complex networks was first considered, the fact that selecting nodes for controllability avoids high degree nodes was touted as '‘unintuitive” [63]. From a complex networks perspective, degree is an important indicator

[1]The Stanford Large Network Dataset Collection is hosted at `snap.stanford.edu/data/`.

of the ability to influence behavior. By showing the case for which degree does matter, herdability centrality allows the unspoken assumptions of the complex networks field to be examined more clearly.

The work on herdability of complex systems has provided perspective on the general complex networks field, by providing a mathematical theory that makes plain the complex networks '"intuition" about a system. Herdability also considers the primary assumption of the work on controllability of networks: that symmetry with respect to input, a sufficient condition for loss of controllability in consensus and structured systems, produces un-desirable behavior. Herdability allows symmetry, which then encourages the exploration of the potential benefit of control under symmetry.

The second phenomenon from social networks expanded upon in this thesis was the diffusion of innovations, a process which involves the interaction of opinions about an innovation and the adoption of the innovation by others. Modeling this behavior lead to the third and final area of study: a novel model of adoptive behavior that takes opinions into account when describing spread over a complex network. By examining the parameters of this model, a set of conditions where shown which described whether an innovation was a viral hit or a major flop. It was shown that if $\delta_i < \beta_{ii},\ \forall i$ then 1_{2n}, the viral hit equilibrium, was asymptotically stable. The self infection parameter β_{ii} is an indicator of innovativeness and product quality, where an agent with a high β_{ii} is going to be an early adopter. Essentially if the population is innovative or the product is good, then the product is going to be a hit.

The case where both the hit and the flop equilibrium points were locally stable was considered in simulation and it appears that there is emergent threshold behavior for this model. This suggests that under Abelson opinion dynamics, a company attempting to spread the innovation can advertise until this threshold is reached and then let the adoption dynamics take over. A number of other models where shown in simulation to markedly change the behavior of the model, suggesting that choosing the right opinion dynamic is important to using this model to make predictions about the adoption of an innovation.

6.2 Future Work

This thesis presents a foundation for understanding certain classes of behavior in complex networks. Of particular importance is the study of herdability which hints at a fundamental limitation in interacting with complex systems. Much of the future work that can be built off this thesis is to see how the notion of herdability can be used to better understand complex network systems. Specifically, there are areas of great interest to theory of complex networks, specifically time varying, nonlinear, and multiplex networks, which are as yet not well understood in the context of herdability theory. There is also a need to examine online social network behavior in the light of herdability.

Additionally, many applications in complex networks deal with interacting with large systems. Another possible line of research is to understand how to make the tools to verify properties of herdability scale better, to increase applicability to the study of complex networks.

Finally the adoption model shown in Chapter 5 has yet to be fully explored. The simulations show that as the opinion dynamic changes, so does the behavior of the adoptive spread. Fully characterizing how varying models of opinion spread affect the proposed model are important not only in the context of the model but could lead to better understanding of the underlying process of adoption. This also points to the broader need to understand the coupling of opinion dynamic models with other types of models; leading to socio-technological or socio-ecological models, such as in [151].

At the core of this thesis lies a pair of questions: how can control theory lead to better understanding in the field of network science? How can networks science lead to new theoretical considerations for control theory? These questions form an ever evolving cycle. Take for example, herdability which takes an idea of how social and biological networks function, translates it into a mathematical theory and then returns to consider what that theory tells us about complex networks. Given the wide range of research areas which are driven by an understanding of a system as a network, this cycle continues; leading to a more detailed understanding of the modern, network-driven world.

Appendices

APPENDIX A
LINEAR ALGEBRA

This section presents a number of definitions from the study of Linear Algebra which are used to show the stability properties of the adoption dynamic considered in Chapter 5. Unless otherwise stated the discussion follows [144].

Theorem A.0.1. *The Gershgorin Disc Theorem*

For $A \in \mathbb{R}^{n\times n}$ let

$$R_i = \sum_{j\neq i} \|a_{ij}\| \text{ for } i = 1, 2, \ldots, n$$

and consider the n Gershgorin discs

$$\{z \in \mathbb{C} : \|z - a_{ii}\| \leq R_i\} \text{ for } i = 1, 2, \ldots, n$$

Then the eigenvalues of A are in the union of the Gershgorin discs

$$\bigcup_{i=1}^{n} \{z \in \mathbb{C} : \|z - a_{ii}\| \leq R_i\}$$

The Gershgorin Disc Theorem together with the following definitions will provide the tools used to characterize equilibria.

Definition A.0.1. *A matrix A is diagonally dominant if*

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}| \ \forall i$$

Definition A.0.2. *The matrix is strictly diagonally dominant if*

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}| \ \forall i$$

Consider a diagonally dominant matrix A and let

$$J = \{i \in \{1, 2, \ldots, n\} : |a_{ii}| > \sum_{j \neq i} |a_{ij}|\}.$$

Any row j such that $j \in J$ is said to be a strictly diagonally dominant row.

Definition A.0.3. *A matrix A is weakly chained diagonally dominant if it is*

- *diagonally dominant*
- *for all $i \notin J$ there is a sequence of nonzero elements of A of the form $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_r j}$ with $j \in J$.*

The second condition can be equivalently expressed as the existence of a walk from i to j on the directed graph of A. Weakly chained diagonally dominant matrices have the following characterization given in [152]:

Lemma A.0.2. *A weakly chained diagonally dominant matrix is nonsingular.*

A diagonally dominant matrix with negative diagonal entries has eigenvalues with non-positive real part by the Gershgorin disc theorem and cannot have eigenvalues on the imaginary axis; a strictly diagonally dominant matrix with negative diagonal entries has eigenvalues with negative real part by the Gershgorin disc theorem. As seen in Chapter 5 this characterization is quite powerful.

Recall also the following condition for Metzler matrices from [153]:

Lemma A.0.3. *Let A be an irreducible Metzler Matrix*

- *If there exists $x > 0$ such that $Ax > \lambda x$ then $\alpha(A) > \lambda$.*
- *If there exists $x > 0$ such that $\mu x > Ax$ then $\mu > \alpha(A)$.*

For a matrix A, $\alpha(A) = \max_{\lambda \in \mathrm{eig}(A)} \mathrm{Re}(\lambda)$, where $\mathrm{Re}()$ denotes the real part of a complex number and $\mathrm{eig}(A)$ is the set of eigenvalues of A.

APPENDIX B
CONTROL OF LINEAR SYSTEMS

In this section, basic definitions from the theory of linear systems are presented which underlie the study of network controllability, which occur in Chapters 2, 3, and 4. This section follows [154] but any text on linear systems theory will do.

When studying the response of a linear system to an input, there are two paired concepts which form the basis for the understanding of the system behavior under input. A state $x \in \mathbb{R}^n$ is reachable if there exists an input that can drive the system from 0_n to x in finite time. A state $x \in \mathbb{R}^n$ is controllable if there an input that can drive the system from x to 0_n in finite time. For a continuous time, linear system these concepts are equivalent, that is if a state is reachable then it is controllable and vice versa. As such the terms are used interchangeably to describe the behavior of a system. This section begins with a number of definitions.

Definition B.0.1. *The reachable subspace $\mathcal{R}[0,t]$ of a continuous time, linear system is*

$$\mathcal{R}[0,t] = \left\{ \mathbf{x}_1 \in \mathbb{R}^n : \exists \mathbf{u}(\cdot), \mathbf{x}_1 = \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau \right\}.$$

Definition B.0.2. *A continuous time, linear system is completely controllable if $\forall\, x(0), x_f \in \mathbb{R}^n$ there exists a finite time T and an input $u(t),\ t \in [0,T]$ s.t. $x(T) = x_f$ under control input $u(t)$. Equivalently, a continuous time, linear system is completely controllable if $\mathcal{R}[0,t] = \mathbb{R}^n$.*

Instead of calculating the reachable subspace directly, there are two matrices which are studied instead. As will be seen, these matrices given information about the reachable subspace but can computed efficiently.

Definition B.0.3. *The controllability matrix* $\mathcal{C}$ *of a linear system is*

$$\mathcal{C} = \left[B, AB, A^2B, \ldots, A^{n-1}B\right]$$

Definition B.0.4. *The Controllability Grammian on the time interval* $[0, t]$*,* $W_c[0, t]$*, of a continuous time, linear system is*

$$W_c[0, t] = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau.$$

The infinite horizon controllability grammian ($t = \infty$) can be solved for efficiently, if A is stable, as the solution to the continuous time Lyapunov equation:

$$AW_c + W_c A + BB^T = 0.$$

Lemma B.0.1. *Theorem 11.5 from [154]*

$$\mathcal{R}[0, t] = \mathrm{range}(\mathcal{C}) = \mathrm{range}(W_c[0, t]).$$

Then the next two Lemmas follow directly from Lemma $B.0.1$ and Definition $B.0.2$:

Lemma B.0.2. *A continuous time, linear system is completely controllable if and only if*

$$\mathrm{rank}(\mathcal{C}) = n.$$

Lemma B.0.3. *A continuous time, linear system is completely controllable if and only if*

$$\mathrm{rank}(W_c[0, t]) = n.$$

Note that in continuous time linear systems, the reachable subspace does not depend explicitly on the time interval used and as such the time interval will be omitted for notational convenience.

Another problem from the control of linear systems, which will be expanded upon in Chapter 4, is how to drive a system between two states with minimum energy. Consider the problem of driving a system from the origin to the state x_f with minimal control energy:

$$
\begin{aligned}
J = \min_{u(t)} \int_0^{t_f} & \|u(\tau)\|^2 d\tau \\
\text{s.t. } & \dot{x}(t) = Ax(t) + Bu(t),\ t \in [0, t_f] \\
& x(t_f) = x_f \\
& x(0) = 0_n,
\end{aligned}
$$

If the system is completely controllable, this optimization problem has an optimal solution of

$$u^*(\tau) = B^T e^{A^T(t-\tau)} (W_c)^{-1} x_f \ \ [0 \leq \tau \leq t_f].$$

The resulting minimum energy is

$$\int_0^{t_f} \|u^*(\tau)\|^2 \, d\tau = x_f^T \, (W_c)^{-1} \, x_f.$$

APPENDIX C

SWITCHING SYSTEMS

In this appendix a few definitions from the study of switched systems will be presented, which follow [155]. Consider a family of systems

$$\dot{x} = f_p(x), \;\; p \in \mathcal{P} \tag{C.1}$$

which has a switching signal $\sigma(t) : [0, \infty) \to \mathcal{P}$ which determines the switches between systems. This gives rise to a switched system,

$$\dot{x} = f_\sigma(x). \tag{C.2}$$

Definition C.0.1. *A switched system is uniformly asymptotically stable if it is asymptotically stable for all switching signals.*

Definition C.0.2. *A positive definite $\mathcal{C}^1$ function V is a common Lyapunov function for the family of systems in Eq.* (C.1) *if there is a positive definite continuous function W such that*

$$\frac{\partial V}{\partial t} f_p(x) \leq -W(x) \;\; \forall x \neq 0, \;\; \forall p \in \mathcal{P}$$

or equivalently if $\mathcal{P}$ is compact and

$$\frac{\partial V}{\partial t} f_p(x) < 0 \;\; \forall x \neq 0, \;\; \forall p \in \mathcal{P}.$$

Theorem C.0.1 (Theorem 2.1 from [155])**.** *If all systems in the family in Eq.* (C.1) *share a common Lyapunov function, then the switched system in Eq.* (C.2) *is uniformly asymptotically stable.*

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