

**UNIQUENESS, EXISTENCE AND REGULARITY OF
SOLUTIONS OF INTEGRO-PDE IN DOMAINS OF \mathbb{R}^N**

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The Academic Faculty

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To my parents, wife and daughter

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SUMMARY

The main goal of the thesis is to study integro-differential equations. Integro-differential equations arise naturally in the study of stochastic processes with jumps. These types of processes are of particular interest in finance, physics and ecology.

In the first part of my thesis, we study interior regularity for the regional fractional Laplacian operator. We first obtain the integer order differentiability of the regional fractional Laplacian. We further extend the integer order differentiability to the fractional order of the regional fractional Laplacian. Schauder estimates for the regional fractional Laplacian are also provided.

In the second and third parts of my thesis, we consider uniqueness and existence of viscosity solutions for a class of nonlocal equations. This class of equations includes Bellman-Isaacs equations containing operators of Lévy type with measures depending on x and control parameters, as well as elliptic nonlocal equations that are not strictly monotone in the u variable.

In the fourth part of my thesis, we obtain semiconcavity of viscosity solutions for a class of degenerate elliptic integro-differential equations in \mathbb{R}^n . This class of equations includes Bellman equations containing operators of Lévy-Itô type. Hölder and Lipschitz continuity of viscosity solutions for a more general class of degenerate elliptic integro-differential equations are also proved.

In the last part of my thesis, we study interior regularity of viscosity solutions of non-translation invariant nonlocal fully nonlinear equations with Dini continuous terms. We obtain C^σ regularity estimates for the nonlocal equations by perturbative methods and a version of a recursive Evans-Krylov theorem.

CHAPTER I

INTRODUCTION

The thesis contains several results about nonlocal equations. We begin with recalling basic notations which will be used in the manuscript.

1.1 Basic notions

We use 0 for both the origin in \mathbb{R} and \mathbb{R}^n . For a given open set Ω in \mathbb{R}^n with $\partial\Omega \neq \emptyset$, let

$$d_x = \text{dist}(x, \Omega^c) \text{ and } \Omega_\delta = \{x \in \Omega; d_x > \delta\}.$$

For each non-negative integer r and $0 < \alpha \leq 1$, we denote by $C^{r,\alpha}(\Omega)$ ($C^{r,\alpha}(\bar{\Omega})$) the subspace of $C^{r,0}(\Omega)$ ($C^{r,0}(\bar{\Omega})$) consisting functions whose r th partial derivatives are locally (uniformly) α -Hölder continuous in Ω . For each $j = (j_1, j_2 \cdots j_n) \in \mathbb{N}^n$, we denote $|j| = j_1 + j_2 + \cdots + j_n$ and $\partial^j u = \frac{\partial^{|j|} u}{(\partial x_1)^{j_1} (\partial x_2)^{j_2} \cdots (\partial x_n)^{j_n}}$. For any $u \in C^{r,\alpha}(\bar{\Omega})$, where r is a non-negative integer and $0 \leq \alpha \leq 1$, define

$$[u]_{r,\alpha;\Omega} = \begin{cases} \sup_{x \in \Omega, |j|=r} |\partial^j u(x)|, & \text{if } \alpha = 0; \\ \sup_{x,y \in \Omega, x \neq y, |j|=r} \frac{|\partial^j u(x) - \partial^j u(y)|}{|x-y|^\alpha}, & \text{if } \alpha > 0, \end{cases}$$

and

$$\|u\|_{C^{r,\alpha}(\bar{\Omega})} = \begin{cases} \sum_{j=0}^r [u]_{j,0;\Omega}, & \text{if } \alpha = 0; \\ \|u\|_{C^{r,0}(\bar{\Omega})} + [u]_{r,\alpha;\Omega}, & \text{if } \alpha > 0. \end{cases}$$

For simplicity, we use the notation $C^\alpha(\Omega)$ ($C^\alpha(\bar{\Omega})$), where $\alpha > 0$, to denote the space $C^{r,\alpha'}(\Omega)$ ($C^{r,\alpha'}(\bar{\Omega})$), where r is the largest integer smaller than α and $\alpha' = \alpha - r$. We note that if α is an integer r , then $C^\alpha(\Omega) = C^{\alpha-1,1}(\Omega) \neq C^{\alpha,0}(\Omega)$ ($C^\alpha(\bar{\Omega}) = C^{\alpha-1,1}(\bar{\Omega}) \neq C^{\alpha,0}(\bar{\Omega})$). We denote $C_c^\infty(\Omega)$ as the space of C^∞ functions with compact support in Ω , \mathcal{S} as the Schwartz space of rapidly decreasing C^∞ function in \mathbb{R}^n , and $\Lambda_*(\bar{\Omega})$ as the Zygmund space of all bounded functions on $\bar{\Omega}$ such that

$$[u]_{\Lambda_*(\bar{\Omega})} := \sup_{x, x+h, x-h \in \bar{\Omega}} \frac{|u(x+h) + u(x-h) - 2u(x)|}{|h|} < \infty.$$

We equip the space $\Lambda_*(\bar{\Omega})$ with the norm $\|u\|_{\Lambda_*(\bar{\Omega})} := \|u\|_{L^\infty(\bar{\Omega})} + [u]_{\Lambda_*(\bar{\Omega})}$. We will write $BUC(\mathbb{R}^n)$ for the space of bounded and uniformly continuous functions in \mathbb{R}^n . For any $1 < \theta \leq 2$ and any convex open set Ω' , we say a set of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ is uniformly θ -semiconvex with constant C in Ω' if, for any $x, y \in \Omega'$, $\alpha \in \mathcal{A}$,

$$2f_\alpha\left(\frac{x+y}{2}\right) - f_\alpha(x) - f_\alpha(y) \leq C|x-y|^\theta.$$

We say a set of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ is uniformly θ -semiconcave with constant C in Ω' if $\{-f_\alpha\}_{\alpha \in \mathcal{A}}$ is uniformly θ -semiconvex with constant C in Ω' . If the set \mathcal{A} is a unit set, i.e., $\mathcal{A} = \{\alpha_0\}$, then we just simply say that f_{α_0} is θ -semiconvex (θ -semiconcave) in Ω' .

1.2 Background and main results

1.2.1 Regional fractional Laplacian

Given real numbers $0 < s < 2$, $\epsilon > 0$, and an open set $\Omega \subset \mathbb{R}^n$, denote

$$\Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x) = \mathcal{A}(n, -s) \int_{\Omega \cap B_\epsilon^c(x)} \frac{u(y) - u(x)}{|x - y|^{n+s}} dy,$$

where $\mathcal{A}(n, -s) = \frac{|s|2^{s-1}\Gamma(\frac{n+s}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\frac{s}{2})}$, $B_\epsilon(x)$ is the open ϵ -ball in \mathbb{R}^n centered at x , and $u \in L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}})$, i.e., $\int_\Omega \frac{|u(x)|}{(1+|x|)^{n+s}} dx < \infty$. The *regional s -fractional Laplacian* $\Delta_\Omega^{\frac{s}{2}}$ on Ω is defined as

$$\Delta_\Omega^{\frac{s}{2}} u(x) = \lim_{\epsilon \rightarrow 0} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x), \quad u \in L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}}),$$

provided that the limit exists. The regional s -fractional Laplacian can be also defined on the closure $\bar{\Omega}$ of Ω by talking $\bar{\Omega}$ in place of Ω in the above. We note that, if $x \in \Omega$, then $\Delta_\Omega^{\frac{s}{2}} u(x) = \Delta_{\bar{\Omega}}^{\frac{s}{2}} u(x)$.

When Ω is a bounded Lipschitz open set, the regional s -fractional Laplacian $\Delta_\Omega^{\frac{s}{2}}$ is in fact the generator of the so-called reflected symmetric s -stable process $(X_t)_{t \geq 0}$ on $\bar{\Omega}$, i.e., a Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\bar{\Omega}, dx)$:

$$\mathcal{E}(u, v) = \frac{1}{2} \mathcal{A}(n, -s) \int_{\bar{\Omega}} \int_{\bar{\Omega}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+s}} dx dy,$$

$$\mathcal{F} = \left\{ u \in L^2(\bar{\Omega}), \int_{\bar{\Omega}} \int_{\bar{\Omega}} \frac{(u(x) - u(y))^2}{|x - y|^{n+s}} dx dy < \infty \right\}.$$

It is first shown in [11] that if $0 < s \leq 1$, then the censored s -stable process in Ω is essentially the reflected s -stable process $(X_t)_{t \geq 0}$, and if $1 < s < 2$, then the censored s -stable process in Ω is identified as a proper subprocess of $(X_t)_{t \geq 0}$ killed upon leaving Ω . Later, it is shown in [21] that $(X_t)_{t \geq 0}$ can be refined to be a process starting from each point of $\bar{\Omega}$ which admits a Hölder continuous transition density function. In [33], not only is the generator of $(X_t)_{t \geq 0}$ on $\bar{\Omega}$ shown to be the regional s -fractional Laplacian, but also a semi-martingale decomposition of $(X_t)_{t \geq 0}$ is obtained by studying the differentiability of the regional fractional Laplacian and its integration by parts property. For other studies on regional fractional Laplacians, we refer the

reader to [31] for a more general integration by parts formula of the regional fractional and fractional-like Laplacian, and to [32] for some boundary Harnack inequalities for the regional fractional Laplacian on $C^{1,\beta-1}(\Omega)$, $s < \beta \leq 2$.

If $\Omega = \mathbb{R}^n$, the regional fractional Laplacian $\Delta_{\mathbb{R}^n}^{\frac{s}{2}}$ becomes the usual fractional Laplacian $-(-\Delta)^{\frac{s}{2}}$ defined via Fourier transform: $\mathcal{F}((-\Delta)^{\frac{s}{2}}u)(\xi) = |\xi|^s \mathcal{F}(u)(\xi)$ (see [64]). If we let s tend to 2, then the fractional Laplacian $-(-\Delta)^{\frac{s}{2}}$ becomes the classical Laplacian Δ , and it is clear that $u \in C^\alpha$ for some integer $\alpha > 2$ implies that $\Delta u \in C^{\alpha-2}$. In the case that $u \in C^\alpha$ for some $\alpha > s$ with $\alpha - s$ not being an integer, one also has $-(-\Delta)^{\frac{s}{2}}u \in C^{\alpha-s}$ ([72, Proposition 2.7]). A natural problem then is whether the regional fractional Laplacian shares similar regularity properties as that of the classical and fractional Laplacian. This problem is first investigated in [33] in which the following results are proved.

Theorem 1.2.1. *Let Ω be an open set in \mathbb{R}^n and $u \in L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}})$ for some $0 < s < 2$. Then the following holds.*

- a) ([33, Proposition 8.3]) *If $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > s$ when $0 < s < 1$ or $u \in C^{2,\alpha}(\Omega)$ for some $\alpha > s - 1$ when $1 \leq s < 2$, then $\Delta_{\Omega}^{\frac{s}{2}}u \in C^{1,0}(\Omega)$.*
- b) ([33, Theorem 8.1]) *In the case $n = 1$, if r is a non-negative integer such that $u \in C^{r,\alpha}(\Omega)$ for some $\alpha > s$ when $0 < s < 1$ or $u \in C^{r+1,\alpha}(\Omega)$ for some $\alpha > s - 1$ when $1 \leq s < 2$, then $\Delta_{\Omega}^{\frac{s}{2}}u \in C^{r,0}(\Omega)$.*

It is conjectured in [33] that part b) of the above theorem should hold for higher dimensions as well. In Section 2.1, we gave an affirmative answer to this conjecture. Unlike the fractional Laplacian, the differential operator and the regional fractional Laplacian are not exchangeable in order. To overcome this difficulty, we derive a class of integral identities (see Lemma 2.1.1) and use them to conclude that all possible singular terms of $D^r(\Delta_{\Omega,\epsilon}^{\frac{s}{2}}u)$ as $\epsilon \rightarrow 0^+$ are in fact non-singular. Making further estimates, we are able to extend the integer order differentiability result to a fractional order. Then we have the result analogous to [72, Proposition 2.7] in the case of regional fractional Laplacian.

Schauder estimate is well-known for the classical Laplacian Δ (see [28]) as well as for the fractional Laplacian (see [15, 30, 67, 72]). We refer the reader to [15, 30] for interior and boundary regularity theory for more general fractional operators. In Section 2.2, using Schauder estimates for the fractional Laplacian, we are able to show a similar Schauder estimate holds for the regional fractional Laplacian.

1.2.2 Nonlocal fully nonlinear equations

The nonlocal fully nonlinear equations we considered are of form

$$G(x, u, Du, D^2u, I[x, u]) = 0 \quad \text{in } \Omega, \quad (1)$$

where Ω is a domain in \mathbb{R}^n and $I[x, u]$ is an integro-differential operator. The function u is real-valued. The nonlinearity $G : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which is coercive, i.e., there is a non-negative constant γ such that, for any $x, p \in \mathbb{R}^n$, $r \geq s$, $X \in \mathbb{S}^n$, $l \in \mathbb{R}$,

$$\gamma(r - s) \leq G(x, r, p, X, l) - G(x, s, p, X, l), \quad (2)$$

and degenerate elliptic in a sense that, for any $x, p \in \mathbb{R}^n$, $r, l_1, l_2 \in \mathbb{R}$, $X, Y \in \mathbb{S}^n$

$$G(x, r, p, X, l_1) \leq G(x, r, p, Y, l_2) \quad \text{if } X \geq Y, l_1 \geq l_2. \quad (3)$$

Here \mathbb{S}^n is the set of symmetric $n \times n$ matrices equipped with its usual order. The nonlocal operator I is either of Lévy type, i.e.,

$$I_L[x, u] := \int_{\mathbb{R}^n} [u(x + z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] \mu_x(dz), \quad (4)$$

or of Lévy-Itô type, i.e.,

$$I_{LI}[x, u] := \int_{\mathbb{R}^n} [u(x + j(x, z)) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot j(x, z)] \mu(dz), \quad (5)$$

where $\mathbb{1}_{B_1(0)}$ denotes the indicator function of the unit ball $B_1(0)$, $j(x, z)$ is a function that determines the size of the jumps for the diffusion related to the operator I_{LI} and μ_x and μ are Lévy measures.

We will also be interested in equations of Bellman-Isaacs type

$$\begin{aligned} \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{ & - \operatorname{Tr}(\sigma_{\alpha\beta}(x) \sigma_{\alpha\beta}^T(x) D^2u(x)) - I_{\alpha\beta}[x, u] \\ & + b_{\alpha\beta}(x) \cdot Du(x) + c_{\alpha\beta}(x)u(x) + f_{\alpha\beta}(x) \} = 0, \quad \text{in } \Omega, \end{aligned} \quad (6)$$

where $\sigma_{\alpha\beta} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, $b_{\alpha\beta} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $c_{\alpha\beta} : \mathbb{R}^n \rightarrow \mathbb{R}$, $f_{\alpha\beta} : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous functions, $c_{\alpha\beta} \geq \gamma$ in \mathbb{R}^n and $I_{\alpha\beta}$ is either of Lévy type or of Lévy-Itô type.

1.2.2.1 Uniqueness

In Chapter 3, we study comparison principles and uniqueness of viscosity solutions for a simplified version of (1), i.e.,

$$G(x, u, I[x, u]) = 0 \quad \text{in } \Omega, \quad (7)$$

where Ω is a bounded domain in \mathbb{R}^n , $I[x, u]$ is of Lévy type and $\{\mu_x : x \in \Omega\}$ is a family of Lévy measures, i.e. non-negative, Borel measures on $\mathbb{R}^n \setminus \{0\}$ such that

$$\int_{\mathbb{R}^n} \min\{|z|^2, 1\} \mu_x(dz) < +\infty \quad \text{for all } x \in \Omega. \quad (8)$$

The operator $I[x, u]$ is thus well defined at least for functions $u \in C^2(B_\delta(x)) \cap BUC(\mathbb{R}^n)$ for some $\delta > 0$. We point out that the solution u has to be given in the whole space \mathbb{R}^n even if (7) is satisfied only in Ω . We will also be interested in studying comparison principles and uniqueness of viscosity solutions for equations of Bellman-Isaacs type

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-I_{\alpha\beta}[x, u] + f_{\alpha\beta}(x)\} = 0, \quad \text{in } \Omega, \quad (9)$$

where each $I_{\alpha\beta}[x, u]$ is of Lévy type.

Comparison principles and uniqueness results are well known for equations (1) and (6) when $\gamma > 0$ and the nonlocal operators I and $I_{\alpha\beta}$ are of Lévy-Itô type. In this case the Lévy measure is fixed which, in the stochastic control/differential game interpretation of the Bellman-Isaacs equations, means that we can only control the state through the diffusion coefficients $j_{\alpha\beta}$ of a stochastic differential equation driven by a fixed Lévy process or a fixed random measure. The first comparison and uniqueness results for such equations were obtained in [68, 74, 75] and many other results can be found in the literature, including results for equations with second order PDE terms, see [1, 2, 3, 4, 7, 6, 8, 9, 20, 35, 39, 40].

The case when we have a family of μ_x measures depending on x is much more difficult. Some comparison results for time dependent equation like (7) were obtained in [2] however with restrictive assumptions. In particular the measures $\mu_{t,x}$, which depend on t and x there, are bounded. In Chapter 3, we prove several comparison theorems for equations (7) and (9). In Section 3.2, we first look at the case when equations are strictly monotone in the u variable, i.e. when $\gamma > 0$ in (2) and in (9). Since standard comparison proofs do not work for these equations, the idea is to try to prove comparison assuming that either a viscosity subsolution or a supersolution is more regular. Of particular interest is the case when one of them is in $C^r(\Omega)$ for some $r > 1$. We adapt to the nonlocal case the technique from [22], Section 5.6 (see also [41]). There are many recent $C^r(\Omega)$ regularity results [6, 8, 13, 12, 14, 42, 48, 70] for equations (7) and (9) and we show in Section 3.6 that comparison theorems obtained in previous sections can be applied to various classes of problems.

Another largely open problem considered in Chapter 3 is comparison results for equations (7) and (9) when they are not strictly monotone in the u variable, i.e. when $\gamma = 0$. The only result in this direction in [12], Section 5, is for equations

corresponding to the case when the measures μ_x are independent of x . There is also a remark made in [34], Theorem 9.2, about comparison for a class of equations being a consequence of an Alexandrov-Bakelman-Pucci estimate for nonlocal equations, however it is not supported by any proof and it is probably false without additional assumptions about the nonlocal operator. Our small contribution here in Section 3.3 is in showing how comparison results of Section 3.2 can be extended to the case $\gamma = 0$ when equations are elliptic with respect to a good enough class of linear nonlocal operators. We follow a typical strategy of perturbing viscosity sub/supersolutions to strict viscosity sub/supersolutions (see [22, 38]). The reader can consult [5, 22, 38, 41] for comparison results for fully nonlinear elliptic PDE which are not strictly monotone in the u variable.

In Section 3.4 we show how viscosity sub/supersolutions of equations (7) and (9) can be regularized by special sup- and inf-convolutions that depend on a family of smooth functions. We also show how to use these special sup/inf-convolutions to prove that the difference of a viscosity subsolution and a viscosity supersolution of the same elliptic equation is a viscosity subsolution of a nonlocal Pucci extremal equation. Knowing this one can use an Alexandrov-Bakelman-Pucci estimate of [34] to prove a comparison principle but this part appears to be missing in [34].

1.2.2.2 Existence

In Chapter 4, we use Perron's method to establish existence of a viscosity solution of

$$\begin{cases} G(x, u, I[x, u]) = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega^c, \end{cases} \quad (10)$$

where Ω is a bounded domain, $I[x, u]$ is of Lévy type, g is a bounded continuous function in \mathbb{R}^n and $\{\mu_x : x \in \Omega\}$ is a family of Lévy measures. We will also be interested in existence of viscosity solutions of

$$\begin{cases} \gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-I_{\alpha\beta}[x, u] + f_{\alpha\beta}(x)\} = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega^c. \end{cases} \quad (11)$$

Existence of viscosity solutions is well known for equations (1) and (6) with nonlocal operators of Lévy-Itô type and $\gamma > 0$ or with uniformly elliptic translation-invariant nonlocal operators of Lévy type and $\gamma = 0$, see [7, 12]. In these two cases, since comparison principle holds, the existence of a viscosity solution can be proved directly by Perron's method. The case when we have a family of μ_x measures depending on x is slightly more difficult since we do not have a good comparison principle, see [63]. To our knowledge, the only available results for existence of solutions for non-translation invariant equations are the following. In Proposition 4.2 of [70], J.

Serra proved existence of a viscosity solutions of a nonlocal Bellman equation. H. Chang Lara and D. Kriventsov obtained existence of viscosity solutions of time dependent nonlocal Isaacs equations in Proposition 5.5 of [19]. In both proofs, the authors used a fixed point argument. The reader can consult [22, 36] for Perron's method for viscosity solutions of fully nonlinear partial differential equations.

In Section 4.2, we adapt to the nonlocal case the approach from [36, 44] for obtaining existence of discontinuous viscosity solutions of (74) and (75). Here we assume that there exist a viscosity subsolution and a viscosity supersolution of each equation satisfying the boundary condition. Under this assumption, we can construct a discontinuous viscosity solution by Perron's method without using a comparison principle. In Section 4.3, we obtain Hölder estimates for discontinuous viscosity solutions of (74) and (75) constructed in Section 4.2 under uniform ellipticity assumption for nonlocal terms. The main tool we use is the weak Harnack inequality proved in [12]. In Section 4.4, we construct a continuous viscosity subsolution and a continuous viscosity supersolution of (74) and (75) satisfying the boundary condition under uniform ellipticity assumption for nonlocal terms. Here we follow the idea of [65] to construct appropriate barrier functions. With all these ingredients in hand, we can finally conclude that there exists continuous viscosity solutions of (74) and (75) when both equations are uniformly elliptic.

1.2.2.3 Semiconcavity

In Chapter 5, we study semiconcavity of viscosity solutions of (1), satisfying (2) and (3) with $\gamma > 0$, where the nonlocal operator I is of Lévy-Itô type. The Lévy measure μ is a Borel measure on $\mathbb{R}^n \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^n \setminus \{0\}} \rho(\xi)^2 \mu(d\xi) < +\infty, \quad (12)$$

where $\rho : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^+$ is a Borel measurable, locally bounded function satisfying $\lim_{\xi \rightarrow 0} \rho(\xi) = 0$ and $\inf_{\xi \in B_r^c(0)} \rho(\xi) > 0$ for any $r > 0$. We will also be interested in equations of Bellman type

$$\sup_{\alpha \in \mathcal{A}} \{ -Tr(\sigma_\alpha(x)\sigma_\alpha^T(x)D^2u(x)) - I_\alpha[x, u] + b_\alpha(x) \cdot Du(x) + c_\alpha(x)u(x) + f_\alpha(x) \} = 0, \quad \text{in } \mathbb{R}^n, \quad (13)$$

where I_α is of Lévy-Itô type and $c_\alpha \geq \gamma > 0$ in \mathbb{R}^n .

The proof of semiconcavity of viscosity solutions is done in two steps. We first prove Lipschitz continuity of viscosity solutions. We then adapt to the nonlocal case the approach from [37, 38] for obtaining semiconcavity of viscosity solutions of elliptic partial differential equations. In recent years, regularity theory of viscosity solutions of

integro-differential equations has been studied by many authors under different types of ellipticity assumptions. It is impossible for us to make a complete review of all the related literature. However, the following are what we have in mind. Regularity results were initiated by assuming nondegenerate ellipticity of second order terms such as [10, 27, 29, 53, 54, 55, 56, 57, 58, 59] for both elliptic and parabolic integro-differential equations. More recently, striking regularity results were obtained under uniform ellipticity assumption for nonlocal terms. This assumption, introduced by L. A. Caffarelli and L. Silvestre, is defined using nonlocal Pucci operators. Several C^α , $C^{1,\alpha}$ and Schauder estimates for nonlocal fully nonlinear equations were obtained by various authors [12, 13, 14, 16, 17, 18, 42, 48, 69, 70, 71] under this uniform ellipticity assumption. The other notion of uniform ellipticity was defined by G. Barles, E. Chasseigne and C. Imbert. It requires either nondegeneracy of the nonlocal terms, or nondegeneracy of nonlocal terms in some directions and nondegeneracy of second order terms in the complementary directions. It was used to obtain Hölder and Lipschitz continuity for a class of mixed integro-differential equations, see [6, 8].

In Section 5.2, we study Hölder and Lipschitz continuity of viscosity solutions for (1) and (6) with nonlocal operator of Lévy-Itô type and $\gamma > 0$ in \mathbb{R}^n . Our Hölder and Lipschitz continuity results are different from these of [6, 8, 71] since we allow both the nonlocal terms and the second order terms to be degenerate. However, to compensate for degeneracy, we need to assume that the constant γ is sufficiently large. The reader can consult [39] for continuous dependence and continuity estimates for viscosity solutions of nonlinear degenerate parabolic integro-differential equations.

Having the Lipschitz continuity results, in Section 5.3 we derive semiconcavity of viscosity solutions of equations (1) and (13). To our knowledge, the only available results in this direction are about semiconcavity of viscosity solutions of time dependent integro-differential equations of Hamilton-Jacobi-Bellman (HJB) type whose proofs are based on probabilistic arguments. In [43], the author proved joint time-space semiconcavity of viscosity solutions of time dependent integro-differential equations of HJB type with terminal condition, using a representation formula based on forward and backward stochastic differential equations. However, the proof there depended on a restrictive assumption that the Lévy measure μ is finite. In another paper [24], it was shown that the value function of an abstract infinite dimensional optimal control problem is w -semiconcave, if the data in the state evolution equation are $C^{1,w}$ and the data in the cost functional are w -semiconcave. The method was then applied to the finite dimensional Euclidean space providing semiconcavity result for the value function of a stochastic optimal control problem associated with a time dependent version of (13). Later the author extended the semiconcavity result in state variables to that in time and state variables jointly in [25]. Our result for (13) extends results of [24]

to the time independent case and provide a different purely analytical approach. The result for (1) is totally new since the solution may not have an explicit probabilistic representation formula and thus the analytical proof seems to be the only available method. Finally we remark that regarding semiconcavity of viscosity solutions of PDEs of HJB type, in addition to the already mentioned analytical proofs of [37, 38], other proofs by probabilistic methods can be found in [26, 49, 72, 51, 52, 76].

1.2.2.4 C^σ regularity

In Chapter 6, we investigate interior regularity of viscosity solutions of nonlocal equations of the type

$$\inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} [u(x+y) - u(x) - \mathbb{1}_{B_1(0)}(y) Du(x) \cdot y] K_a(x, y) dy \right\} = f(x), \quad \text{in } B_1(0), \quad (14)$$

where $K_a(x, y)$ is a positive kernel. The kernels $K_a(x, y)$ are symmetric, i.e., for any $x, y \in \mathbb{R}^n$

$$K_a(x, y) = K_a(x, -y), \quad (15)$$

and are uniform elliptic, i.e., for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus \{0\}$

$$\frac{(2-\sigma)\lambda}{|y|^{n+\sigma}} \leq K_a(x, y) \leq \frac{(2-\sigma)\Lambda}{|y|^{n+\sigma}}, \quad (16)$$

where $0 < \lambda \leq \Lambda$. The symmetry assumption is essential for the regularity theory for (14), see [73]. Under the symmetry assumption, (14) can be rewritten as

$$\inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta u(x, y) K_a(x, y) dy \right\} = f(x), \quad \text{in } B_1(0),$$

where $\delta u(x, y) = u(x+y) + u(x-y) - 2u(x)$. We furthermore assume that the kernels K_a satisfy, for any $x \in \mathbb{R}^n$, any $y \in \mathbb{R}^n \setminus \{0\}$ and $i = 1, 2$

$$|D_y^i K(x, y)| \leq \frac{\Lambda(2-\sigma)}{|y|^{n+\sigma+i}}. \quad (17)$$

We will obtain C^σ regularity estimates for (14) with Dini continuous data in two steps. We first generalize the recursive Evans-Krylov theorem for translation invariant nonlocal fully nonlinear equations from the case of Hölder continuous data, see [42], to the Dini continuous case. We then use the perturbative methods to obtain C^σ regularity estimates for (14).

In Section 6.2, we establish a recursive Evans-Krylov theorem for translation invariant nonlocal fully nonlinear equations in the Dini continuous case. The sequence of equations we consider is, for $j = 0, 1, \dots, m$

$$\inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \sum_{l=0}^j \rho^{-(j-l)\sigma} w^{-1}(\rho^j) w(\rho^l) \delta v_l(\rho^{j-l}x, \rho^{j-l}y) K_a^j(y) dy + w^{-1}(\rho^j) b_a \right\} = 0, \quad \text{in } B_5(0), \quad (18)$$

where $w(t)$ is a Dini modulus of continuity, $K_a^j(x) := \rho^{j(n+\sigma)} K_a(\rho^j x)$ and $\rho \in (0, 1)$. We prove that, for any $l = 0, 1, \dots, m$, $\|v_l\|_{C^{\sigma+\bar{\beta}}(B_1(0))} \leq C$ where $0 < \bar{\beta} < 1$ and $C > 0$ are two constants independent with ρ and m . Recursive Evans-Krylov theorem was first studied by T. Jin and J. Xiong in [42]. They used it to obtain the uniform regularity estimates for the approximators at each scale. Instead of using polynomials as approximators, they used solutions for constant coefficient equations since polynomials grow too fast near infinity. We construct a slightly more general recursive Evans-Krylov theorem for our purpose. When $w(t) = t^\alpha$ for some $0 < \alpha < 1$, (18) is the case studied in [42].

Using the recursive Evans-Krylov theorem in the Dini continuous case, in Section 6.3, we derive C^σ regularity estimates of viscosity solutions for (14) with Dini continuous data. To our knowledge, the only available results in this direction are the following. In Proposition 5.2 of [15], the authors proved C^σ regularity estimates for

$$u = (-\Delta)^{-\frac{\sigma}{2}} f = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{t\Delta} f(x) \frac{dt}{t^{1-\frac{\sigma}{2}}}, \quad \text{in } \mathbb{R}^n, \quad (19)$$

if $\sigma \neq 1$. For $\sigma = 1$, they obtained $\Lambda_*(\mathbb{R}^n)$ regularity estimates for (19). It can be easily deduced from Proposition 2.8 of [72] that the corresponding regularity estimates for weak solutions of $(-\Delta)^{\frac{\sigma}{2}} u = f$ in Ω hold. We notice that $C^1(\bar{\Omega}) \subsetneq \Lambda_*(\Omega)$. In Theorem 1.1(b) of [66], it was shown that C^σ regularity estimates for weak solutions hold for

$$Lu := \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} \delta u(x, \theta r) \frac{dr}{|r|^{1+\sigma}} d\mu(\theta) = f(x), \quad \text{in } B_1(0), \quad (20)$$

with a weaker ellipticity assumption

$$0 < \lambda \leq \inf_{\nu \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\nu \cdot \theta|^\sigma d\mu(\theta) \quad \text{and} \quad \mu(\mathbb{S}^{n-1}) \leq \Lambda < +\infty,$$

where $\sigma \neq 1$. If $\sigma = 1$, the authors derived $C^{\sigma-\epsilon}$ regularity estimates for (20), where ϵ can be any positive constant between 0 and σ . It was claimed in [66] that the methods there can be applied to obtain similar regularity estimates for non-translation invariant equations. In [23], H. Dong and D. Kim studied Schauder estimates for a class of nonlocal linear equations with rough kernels in both Hölder and Dini continuous case. However, in the Dini continuous case, they considered the global problem on translation invariant equations, i.e., $Lu = f$ in \mathbb{R}^n where L is defined in (128). Our results are different from the above results since we are considering the regularity theory of viscosity solutions for non-translation invariant nonlocal fully nonlinear equations. Weak solutions are not equivalent to viscosity solutions in general unless uniqueness of viscosity solutions for such equations holds. However, uniqueness of viscosity solutions for non-translation invariant nonlocal equations is still an open

question. Some recent progress has been made in [63]. Finally we refer the reader to [46, 47] for C^2 regularity estimates for viscosity solutions of classical fully nonlinear PDEs with Dini continuous terms.

CHAPTER II

INTERIOR REGULARITY FOR REGIONAL FRACTIONAL LAPLACIAN

In this chapter, we will study interior regularity for regional fractional Laplacian including both differentiability and Schauder estimate. This is a joint work with Prof. Yingfei Yi, see [62].

2.1 Differentiability

In this section, we will study the differentiability of the regional fractional Laplacian in an open set Ω . The proof will be based on some integral identities in \mathbb{R}^n .

2.1.1 Integral identities

Given $n' \in \mathbb{N}$, $z = (z_1, z_2, \dots, z_{n'}) \in \mathbb{R}^{n'}$ and $k = (k_1, k_2, \dots, k_{n'}) \in \mathbb{N}^{n'}$, we denote by z^k the monomial $\prod_{i=1}^{n'} z_i^{k_i}$. Also, for each $j = 1, 2, \dots, n$, we let e_j to denote the j th standard unit basis vector in \mathbb{R}^n .

Lemma 2.1.1. *Consider an annulus domain $R_{\delta\epsilon}(0) := \{z \in \mathbb{R}^n : \epsilon < |z| < \delta\}$, where $0 < \epsilon < \delta$. Then for any $i, j, m \in \mathbb{N}$ and $k = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, we have*

$$\frac{1}{k_i + 1} \int_{R_{\delta\epsilon}(0)} \frac{z^{k+2e_i}}{|z|^m} dz = \frac{1}{k_j + 1} \int_{R_{\delta\epsilon}(0)} \frac{z^{k+2e_j}}{|z|^m} dz.$$

Proof. The result follows from symmetry. □

Lemma 2.1.2. ([33, Lemma 8.2]) *Let Ω be an open subset of \mathbb{R}^n and $u \in L^1(\Omega)$. Suppose that u is continuous in an open neighborhood U of $x_0 = (x_1, \dots, x_n) \in \Omega$ and $\text{dist}(x_0, \partial U) > \epsilon > 0$. Then the function $f(x) = \int_{\Omega \cap B_\epsilon^c(x)} u(y) dy$ is differentiable at x_0 and*

$$\frac{\partial f}{\partial x_i}(x_0) = \int_{\partial B_\epsilon(x_0)} u(y) \frac{x_i - y_i}{|x - y|} m(dy),$$

where $m(dy)$ is the $n - 1$ dimensional surface Lebesgue measure.

2.1.2 Integer order differentiability

We first prepare some technical lemmas concerning derivatives of $\Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x)$.

Lemma 2.1.3. *Let Ω be an open set in \mathbb{R}^n and $0 < s < 2$. Suppose that $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{1,0}(\Omega)$. Then for any $x \in \Omega$, $\delta < d_x = \text{dist}(x, \partial\Omega)$, and $0 < \epsilon < \delta$, we have*

$$\begin{aligned}
& \frac{2}{\mathcal{A}(n, -s)} \frac{\partial}{\partial x_i} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x) \\
= & - \int_{R_{\delta\epsilon}(x)} \frac{\sum_{j=1}^n (\frac{\partial u}{\partial x_j}(y) - \frac{\partial u}{\partial x_j}(x))(y_j - x_j)(x_i - y_i)}{|x - y|^{n+s+2}} dy \\
& - (n-1) \int_{R_{\delta\epsilon}(x)} \frac{[u(y) - u(x) + \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)(x_j - y_j)](x_i - y_i)}{|x - y|^{n+s+2}} dy \\
& - \int_{B_{\delta}^c(x) \cap \Omega} \frac{\frac{\partial u}{\partial x_i}(x)}{|x - y|^{n+s}} dy + \int_{\partial B_{\delta}(x) \cap \Omega} \frac{(u(y) - u(x))(x_i - y_i)}{|x - y|^{n+s+1}} m(dy) \\
& - (n+s) \int_{B_{\delta}^c(x) \cap \Omega} \frac{(u(y) - u(x))(x_i - y_i)}{|y - x|^{n+s+2}} dy,
\end{aligned} \tag{21}$$

where $R_{\delta\epsilon}(x) := B_{\delta}(x) \cap B_{\epsilon}^c(x)$.

Proof. The proof follows from that of [33, Proposition 8.3]. \square

Lemma 2.1.4. *Let Ω be an open set in \mathbb{R}^n and $0 < s < 2$. Suppose that $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r+1,0}(\Omega)$ for some positive integer r . For any $x \in \Omega$, $\epsilon > 0$, $\delta < d_x = \text{dist}(x, \partial\Omega)$, and $l = (l_1, l_2, \dots, l_n)$, $k = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, if $k_i = 0$ for some $1 \leq i \leq n$, then*

$$\begin{aligned}
& \frac{\partial}{\partial x_i} \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^k dy \\
= & - (n+2p-m-1) \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+2}} (x-y)^{k+e_i} dy \\
& - \int_{R_{\delta\epsilon}(x)} \left[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j'+j} u(x)}{|x-y|^{n+s+2p+2}} (y-x)^{j'} \right] (x-y)^{k+e_i} dy,
\end{aligned}$$

where $A_j = \frac{|j|!}{j_1! j_2! \dots j_n!}$ for each $j = (j_1, j_2, \dots, j_n) \in \mathbb{N}^n$ and $m, p \in \mathbb{N}$ are such that $|l| + m = r$ and $m + |k| = 2p$.

Proof. Since $k_i = 0$, we have by Lemma 2.1.2 that

$$\begin{aligned}
I : &= \frac{\partial}{\partial x_i} \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^k dy \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= -\frac{1}{m!} \sum_{|j|=m} A_j \partial^{l+j+e_i} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p}} dy, \\
I_2 &:= -(n+s+2p) \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+2}} (x-y)^{k+e_i} dy, \\
I_3 &:= \int_{\partial B_\epsilon(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+1}} (x-y)^{k+e_i} m(dy) \\
&\quad - \int_{\partial B_\delta(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+1}} (x-y)^{k+e_i} m(dy).
\end{aligned}$$

Using the identity $m + |k| = 2p$, integration by parts yields

$$\begin{aligned}
I_2 &= -(n+s+2p) \int_\epsilon^\delta r^{-n-s-2p-1} dr \int_{\partial B_1(x)} r^{n+2p-m-1} (x-y)^{k+e_i} \\
&\quad \left[\partial^l u(x+r(y-x)) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x) r^{|j|} \right] m(dy) \\
&= \int_\epsilon^\delta r^{n+2p-m-1} \left\{ \int_{\partial B_1(x)} \left[\partial^l u(x+r(y-x)) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x) r^{|j|} \right] \right. \\
&\quad \left. (x-y)^{k+e_i} m(dy) \right\} dr^{-(n+s+2p)} \\
&= -I_3 - (n+2p-m-1) \int_\epsilon^\delta \int_{\partial B_1(x)} r^{-m-2-s} (x-y)^{k+e_i} \\
&\quad \left[\partial^l u(x+r(y-x)) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x) r^{|j|} \right] m(dy) dr \\
&\quad - \int_\epsilon^\delta \int_{\partial B_1(x)} r^{-m-1-s} (x-y)^{k+e_i} \\
&\quad \sum_{|j'|=1} \left[\partial^{l+j'} u(x+r(y-x)) (y-x)^{j'} \right. \\
&\quad \left. - \sum_{|j|=0}^{|j|=m-1} \frac{A_j}{|j|!} (y-x)^{j'+j} \partial^{l+j'+j} u(x) r^{|j|} \right] m(dy) dr \\
&= -I_3 - (n+2p-m-1) \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+2}} (x-y)^{k+e_i} dy \\
&\quad - \int_{R_{\delta\epsilon}(x)} \left[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{|j|=0}^{|j|=m-1} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j'+j} u(x)}{|x-y|^{n+s+2p+2}} \right. \\
&\quad \left. (y-x)^{j'} \right] (x-y)^{k+e_i} dy.
\end{aligned}$$

Therefore,

$$I_2 + I_3 = I_{23}^{(1)} + I_{23}^{(2)} + I_{23}^{(3)},$$

where

$$\begin{aligned} I_{23}^{(1)} &= -(n+2p-m-1) \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+2}} (x-y)^{k+e_i} dy, \\ I_{23}^{(2)} &= -\frac{n+2p}{(m+1)!} \sum_{|j|=m+1} A_j \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy, \\ I_{23}^{(3)} &= - \int_{R_{\delta\epsilon}(x)} \left[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j'+j} u(x)}{|x-y|^{n+s+2p+2}} (y-x)^{j'} \right] (x-y)^{k+e_i} dy. \end{aligned}$$

Observe that

$$\begin{aligned} & I_1 + I_{23}^{(2)} \\ &= - \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} \frac{C_m^{2t}}{m!} \sum_{|j|=m, j_i=2t} A_{j-j_i e_i} \partial^{l+j+e_i} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\ & \quad - \frac{n+2p}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} C_{m+1}^{2t+1} \sum_{|j|=m+1, j_i=2t+1} A_{j-j_i e_i} \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy \\ &= - \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} \frac{C_{m+1}^{2t+1}}{(m+1)!} \sum_{|j|=m, j_i=2t} A_{j-j_i e_i} (2t+1) \partial^{l+j+e_i} u(x) \\ & \quad \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\ & \quad - \frac{n+2p}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} C_{m+1}^{2t+1} \sum_{|j|=m+1, j_i=2t+1} A_{j-j_i e_i} \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy \end{aligned}$$

where $j = (j_1, j_2, \dots, j_n)$. An application of Lemma 2.1.1 yields

$$\begin{aligned} & I_1 + I_{23}^{(2)} \\ &= \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} \frac{C_{m+1}^{2t+1}}{(m+1)!} \sum_{|j|=m, j_i=2t} A_{j-j_i e_i} \left[n+2p - (|k| + n + m) \right] \partial^{l+j+e_i} u(x) \\ & \quad \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}} dy = 0. \end{aligned}$$

Thus, $I = I_{23}^{(1)} + I_{23}^{(3)}$ and the lemma is proved. \square

Lemma 2.1.5. *Let Ω be an open set in \mathbb{R}^n and $0 < s < 2$. Suppose that $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r+1,0}(\Omega)$ for some positive integer r . For any $x \in \Omega$, $\epsilon > 0$,*

$\delta < d_x = \text{dist}(x, \partial\Omega)$, and $l = (l_1, l_2, \dots, l_n)$, $k = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, if $k_i \neq 0$ for some $1 \leq i \leq n$, then

$$\begin{aligned} & \frac{\partial}{\partial x_i} \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{\substack{|j|=m \\ |j|=0}} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^k dy \\ = & k_i \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{\substack{|j|=m+1 \\ |j|=0}} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^{k-e_i} dy \\ & - (n+2p-m-1) \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{\substack{|j|=m+1 \\ |j|=0}} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p+2}} (x-y)^{k+e_i} dy \\ & - \int_{R_{\delta\epsilon}(x)} \left[\sum_{|j'|=1} \frac{\partial^{l+j'} u(y) - \sum_{\substack{|j|=m \\ |j|=0}} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j'+j} u(x)}{|x-y|^{n+s+2p+2}} (y-x)^{j'} \right] (x-y)^{k+e_i} dy \end{aligned}$$

where $m, p \in \mathbb{N}$ are such that $|l| + m = r$ and $m + |k| = 2p$.

Proof. Since $k_i \neq 0$, we have by Lemma 2.1.2 that

$$I = I_1 + \bar{I}_1 + I_2 + I_3,$$

where I and I_i , $i = 1, 2, 3$, are as in the proof of Lemma 2.1.4 and

$$\bar{I}_1 = k_i \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{\substack{|j|=m \\ |j|=0}} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^{k-e_i} dy.$$

By the proof of Lemma 2.1.4, we have

$$I_2 + I_3 = I_{23}^{(1)} + I_{23}^{(2)} + I_{23}^{(3)}.$$

where $I_{23}^{(i)}$, $i = 1, 2, 3$, are defined in the proof of Lemma 2.1.4. Write

$$\bar{I}_1 = \bar{I}_1^{(1)} + \bar{I}_1^{(2)},$$

where

$$\begin{aligned} \bar{I}_1^{(1)} &= k_i \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{\substack{|j|=m+1 \\ |j|=0}} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^{k-e_i} dy, \\ \bar{I}_1^{(2)} &= \frac{k_i}{(m+1)!} \sum_{|j|=m+1} A_j \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k-e_i}}{|x-y|^{n+s+2p}} dy. \end{aligned}$$

Note that

$$\begin{aligned}
& I_1 + \bar{I}_1^{(2)} + I_{23}^{(2)} \\
= & -\frac{1}{m!} \sum_{|j|=m} A_j \partial^{l+j+e_i} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& + \frac{k_i}{(m+1)!} \sum_{|j|=m+1} A_j \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& - \frac{n+2p}{(m+1)!} \sum_{|j|=m+1} A_j \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy.
\end{aligned}$$

First let k_i be an even number. Then

$$\begin{aligned}
& I_1 + \bar{I}_1^{(2)} + I_{23}^{(2)} \\
= & - \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} \frac{C_m^{2t}}{m!} \sum_{|j|=m, j_i=2t} A_{j-j_i e_i} \partial^{l+j+e_i} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& + \frac{k_i}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} C_{m+1}^{2t+1} \sum_{|j|=m+1, j_i=2t+1} A_{j-j_i e_i} \partial^{l+j} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& - \frac{n+2p}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} C_{m+1}^{2t+1} \sum_{|j|=m+1, j_i=2t+1} A_{j-j_i e_i} \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy \\
= & - \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} \frac{C_{m+1}^{2t+1}}{(m+1)!} \sum_{|j|=m, j_i=2t} A_{j-j_i e_i} (2t+k_i+1) \partial^{l+j+e_i} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& + \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} \frac{C_{m+1}^{2t+1}}{(m+1)!} \sum_{|j|=m, j_i=2t} A_{j-j_i e_i} (n+2p) \partial^{l+j+e_i} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}} dy.
\end{aligned}$$

It follows from Lemma 2.1.1 that

$$\begin{aligned}
& I_1 + \bar{I}_1^{(2)} + I_{23}^{(2)} \\
= & \sum_{t \in \mathbb{N}, t \leq \frac{m}{2}} \frac{C_{m+1}^{2t+1}}{(m+1)!} \sum_{|j|=m, j_i=2t} A_{j-j_i e_i} \left[(n+2p) - (|k| + n + m) \right] \partial^{l+j+e_i} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}} dy = 0.
\end{aligned}$$

Now let k_i be an odd number. Then

$$\begin{aligned}
& I_1 + \bar{I}_1^{(2)} + I_{23}^{(2)} \\
= & - \sum_{t \in \mathbb{N}, t \leq \frac{m-1}{2}} \frac{C_m^{2t+1}}{m!} \sum_{|j|=m, j_i=2t+1} A_{j-j_i e_i} \partial^{l+j+e_i} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& + \frac{k_i}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m+1}{2}} C_{m+1}^{2t} \sum_{|j|=m+1, j_i=2t} A_{j-j_i e_i} \partial^{l+j} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& - \frac{n+2p}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m+1}{2}} C_{m+1}^{2t} \sum_{|j|=m+1, j_i=2t} A_{j-j_i e_i} \partial^{l+j} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy \\
= & \frac{k_i}{(m+1)!} \sum_{|j|=m+1, j_i=0} A_j \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& - \frac{(n+2p)}{(m+1)!} \sum_{|j|=m+1, j_i=0} A_j \partial^{l+j} u(x) \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy \\
& - \sum_{t \in \mathbb{N}, t \leq \frac{m-1}{2}} \frac{C_m^{2t+1}}{m!} \sum_{|j|=m, j_i=2t+1} A_{j-j_i e_i} \partial^{l+j+e_i} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& + \frac{k_i}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m-1}{2}} C_{m+1}^{2t+2} \sum_{|j|=m+1, j_i=2t+2} A_{j-j_i e_i} \partial^{l+j} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k-e_i}}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& - \frac{n+2p}{(m+1)!} \sum_{t \in \mathbb{N}, t \leq \frac{m-1}{2}} C_{m+1}^{2t+2} \sum_{|j|=m+1, j_i=2t+2} A_{j-j_i e_i} \partial^{l+j} u(x) \\
& \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy.
\end{aligned}$$

It again follows from Lemma 2.1.1 that

$$\begin{aligned}
& I_1 + \bar{I}_1^{(2)} + I_{23}^{(2)} \\
&= \frac{1}{(m+1)!} \sum_{|j|=m+1, j_i=0} A_j \left[(m+n+|k| - (n+2p)) \right] \partial^{l+j} u(x) \\
& \quad \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+e_i}}{|x-y|^{n+s+2p+2}} dy \\
& \quad - \sum_{t \in \mathbb{N}, t \leq \frac{m-1}{2}} \frac{C_{m+1}^{2t+2}}{(m+1)!} \sum_{|j|=m, j_i=2t+1} A_{j-j_i e_i} (2t+2+k_i) \partial^{l+j+e_i} u(x) \\
& \quad \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p+2}} \sum_{|j'|=1} (x-y)^{2j'} dy \\
& \quad + \sum_{t \in \mathbb{N}, t \leq \frac{m-1}{2}} \frac{C_{m+1}^{2t+2}}{(m+1)!} \sum_{|j|=m, j_i=2t+1} A_{j-j_i e_i} (n+2p) \partial^{l+j+e_i} u(x) \\
& \quad \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}} dy \\
&= \sum_{t \in \mathbb{N}, t \leq \frac{m-1}{2}} \frac{C_{m+1}^{2t+2}}{(m+1)!} \sum_{|j|=m, j_i=2t+1} A_{j-j_i e_i} \left[(n+2p) - (|k| + n + m) \right] \partial^{l+j+e_i} u(x) \\
& \quad \int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^{k+2e_i}}{|x-y|^{n+s+2p+2}} dy = 0.
\end{aligned}$$

Thus, for any $k_i \neq 0$, $I = \bar{I}_1^{(1)} + I_{23}^{(1)} + I_{23}^{(3)}$ and the lemma is proved. \square

Lemma 2.1.6. *Let Ω be an open set in \mathbb{R}^n and $0 < s < 2$. Suppose that $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r,0}(\Omega)$ for some positive integer r . Then for any $x \in \Omega$, $\epsilon > 0$, $\hat{r} = (r_1, r_2, \dots, r_n) \in \mathbb{N}^n$ with $|\hat{r}| = r$, $\delta < d_x = \text{dist}(x, \partial\Omega)$, all ϵ -dependent terms of $\partial^{\hat{r}} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x)$ have the form*

$$I_{l,k,m,p}^{\epsilon, \delta}(x) = \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{\substack{|j|=m \\ |j|=0}} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^k dy, \quad (22)$$

where $l = (l_1, l_2, \dots, l_n)$, $k = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, $m, p \in \mathbb{N}$ are such that $|l| + m = r$ and $m + |k| = 2p$.

Proof. We will prove the lemma by induction. In the case of $r = 1$, we observe that the only ϵ -dependent terms on the right hand side of (21) are its first two terms. They clearly have the form (22) with the first term corresponding to $|l| = 1$, $m = 0$, $|k| = 2$ and $p = 1$, and the second term corresponding to $|l| = 0$, $m = 1$, $|k| = 1$ and $p = 1$.

Now suppose that (22) is satisfied when $r = q$, where q is a fixed positive integer. We want to show that it is also satisfied when $r = q + 1$, i.e., for any $l, k \in \mathbb{N}^n$ with

$|l| + m = q$ and $m + |k| = 2p$, all ϵ -dependent terms of

$$I = \frac{\partial}{\partial x_i} \int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^k dy$$

have the form

$$\int_{R_{\delta\epsilon}(x)} \frac{\partial^{l'} u(y) - \sum_{|j|=0}^{|j|=m'} \frac{A_j}{|j|!} (y-x)^j \partial^{l'+j} u(x)}{|x-y|^{n+s+2p'}} (x-y)^{k'} dy, \quad (23)$$

where $|l'| + m' = q + 1$ and $m' + |k'| = 2p'$.

If $k_i = 0$, then, by Lemma 2.1.4, we have $I = I_{23}^{(1)} + I_{23}^{(3)}$, where $I_{23}^{(1)}, I_{23}^{(3)}$ are as in the proof of Lemma 2.1.4 which clearly have the form (23) with $|l'| + m' = |l| + m + 1 = q + 1$ and $m' + |k'| = m + |k| + 2 = 2(p + 1)$.

If $k_i \neq 0$, then, by Lemma 2.1.5, we have $I = \bar{I}_1^{(1)} + I_{23}^{(1)} + I_{23}^{(3)}$, where $\bar{I}_1^{(1)}$ is as in the proof of Lemma 2.1.5 which is clearly of the form (23) with $l' + m' = |l| + m + 1 = q + 1$ and $m' + |k'| = m + k = 2p$. \square

Theorem 2.1.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}})$ for some $0 < s < 2$. If r is a non-negative integer such that $u \in C^{r,\alpha}(\Omega)$ for some $1 \geq \alpha > s$ or $u \in C^{r+1,\alpha}(\Omega)$ for some α with $2 \geq 1 + \alpha > s \geq \alpha$, then $\Delta_{\Omega}^{\frac{s}{2}} u \in C^{r,0}(\Omega)$.*

Proof. For any $\epsilon > 0$, $x \in \Omega$, $\hat{r} = (r_1, r_2, \dots, r_n) \in \mathbb{N}^n$ with $|\hat{r}| = r$, and $\delta < d_x = \text{dist}(x, \partial\Omega)$, we have by Lemma 2.1.6 that all ϵ -dependent terms of $\partial^{\hat{r}} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x)$ have the form (22).

In the case $1 \geq \alpha > s$, we note that any integral of the form (22) is bounded above in absolute value by a constant times $\int_{\epsilon}^{\delta} \rho^{\alpha-s-1} d\rho$ which is convergent as $\epsilon \rightarrow 0$. It follows that $\partial^{\hat{r}} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u$ converges uniformly on any compact subset of Ω as $\epsilon \rightarrow 0$. Thus, $\partial^{\hat{r}} \Delta_{\Omega}^{\frac{s}{2}} u \in C(\Omega)$, i.e., $\Delta_{\Omega}^{\frac{s}{2}} u \in C^{r,0}(\Omega)$.

In the case $2 \geq 1 + \alpha > s \geq \alpha$, we again consider an integral $I_{l,k,m,p}^{\epsilon,\delta}(x)$ of the form (22) for some $l, k \in \mathbb{N}^n$, $m, p \in \mathbb{N}$ satisfying $|l| + m = r$ and $m + |k| = 2p$. Since $m + 1 + |k| = 2p + 1$ is an odd number, we have, for any $j \in \mathbb{N}^n$ with $|j| = m + 1$, that

$$\int_{R_{\delta\epsilon}(x)} \frac{(y-x)^j (x-y)^k}{|x-y|^{n+s+2p}} dy = 0.$$

Hence $I_{l,k,m,p}^{\epsilon,\delta}(x)$ can be re-written as

$$\int_{R_{\delta\epsilon}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^k dy$$

which is bounded above in absolute value by a constant times $\int_{\epsilon}^{\delta} \rho^{\alpha-s} d\rho$ that is convergent as $\epsilon \rightarrow 0$. It follows again that $\partial^{\hat{r}} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u$ converges uniformly on any compact subset of Ω as $\epsilon \rightarrow 0$. Thus, $\partial^{\hat{r}} \Delta_{\Omega}^{\frac{s}{2}} u(x) \in C(\Omega)$, i.e., $\Delta_{\Omega}^{\frac{s}{2}} u \in C^{r,0}(\Omega)$. \square

2.1.3 Fractional order differentiability

Let $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}}) \cap C^{r,\alpha}(\Omega)$ for some positive integer r and some real numbers $0 < s < 2$, $0 < \alpha \leq 1$. For each $\epsilon > 0$ sufficiently small, $x \in \Omega$, and any $\hat{r} \in \mathbb{N}^n$ with $|\hat{r}| \leq r$, we write

$$\partial^{\hat{r}} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x) = I_{\epsilon}(x) + I_*(x),$$

where $I_{\epsilon}(x)$ denotes the ϵ -dependent term of $\partial^{\hat{r}} \Delta_{\Omega, \epsilon}^{\frac{s}{2}} u(x)$ and $I_*(x)$ denotes the remaining term.

Lemma 2.1.8. *Let u, s, r, α be as in the above. If either $1 \geq \alpha > s$ or $2 \geq 1 + \alpha > s \geq \alpha$, then*

$$\partial^{\hat{r}} \Delta_{\Omega}^{\frac{s}{2}} u(x) = I_0(x) + I_*(x), \quad \text{if } |\hat{r}| \leq r,$$

where $I_0(x) = \lim_{\epsilon \rightarrow 0} I_{\epsilon}(x)$ which consists of terms of the form

$$I_{l,k,m,p}^{\delta}(x) = \int_{B_{\delta}(x)} \frac{\partial^l u(y) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (y-x)^j \partial^{l+j} u(x)}{|x-y|^{n+s+2p}} (x-y)^k dy, \quad (24)$$

for any $\delta < d_x = \text{dist}(x, \partial\Omega)$ and some $l = (l_1, l_2, \dots, l_n)$, $k = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, $m, p \in \mathbb{N}$ with $|l| + m = |\hat{r}|$ and $m + |k| = 2p$.

Proof. It follows immediately from Lemma 2.1.6 and the proof of Theorem 2.1.7. \square

Lemma 2.1.9. *Let u, s, r, α be as in the above. Then the following holds.*

a) *If $1 \geq \alpha > s$ and $|\hat{r}| = r$, then there exists a constant $C > 0$ such that*

$$|I_*(x) - I_*(y)| \leq C[u]_{r,\alpha;\Omega} |x-y|^{\alpha-s}, \quad x, y \in \Omega, |x-y| \ll 1.$$

b) *If $2 \geq 1 + \alpha > s \geq \alpha$ and $|\hat{r}| = r - 1$, then there exists a constant $C > 0$ such that*

$$|I_*(x) - I_*(y)| \leq C[u]_{r,\alpha;\Omega} |x-y|^{1+\alpha-s}, \quad x, y \in \Omega, |x-y| \ll 1.$$

Proof. The function I_* can be derived simply by taking higher order derivatives of the right hand side of (21) and identifying all ϵ -independent terms of the derivatives. As these terms involves only regular integrals, the lemma follows from straightforward estimates. \square

Theorem 2.1.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1(\Omega, \frac{1}{(1+|x|)^{n+s}})$ for some $0 < s < 2$. Then the following holds.*

- (i) If $u \in C^{r,\alpha}(\Omega)$ for some positive number α with $s < \alpha \leq 1$ and a non-negative integer r , then $\Delta_{\Omega}^{\frac{s}{2}}u(x) \in C^{r,\alpha-s}(\Omega)$, and moreover,

$$[\Delta_{\Omega}^{\frac{s}{2}}u]_{r,\alpha-s;\Omega} \leq C[u]_{r,\alpha;\Omega}.$$

- (ii) If $u \in C^{r,\alpha}(\Omega)$ for some positive number α with $\alpha < s < 1 + \alpha \leq 2$ and a positive integer r , then $\Delta_{\Omega}^{\frac{s}{2}}u(x) \in C^{r-1,1+\alpha-s}(\Omega)$, and moreover,

$$[\Delta_{\Omega}^{\frac{s}{2}}u]_{r-1,1+\alpha-s;\Omega} \leq C[u]_{r,\alpha;\Omega}.$$

Proof. Let $x, y \in \Omega$ and take $\delta < d_{x,y} = \min\{d_x, d_y\}$. For given $l = (l_1, l_2, \dots, l_n)$, $k = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, $m, p \in \mathbb{N}$, consider

$$J = I_{l,k,m,p}^{\delta}(x) - I_{l,k,m,p}^{\delta}(y),$$

where $I_{l,k,m,p}^{\delta}$ is as in (24). It is clear that

$$J = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &:= \int_{B_{\eta}(0)} \left[\frac{\partial^l u(x+z) - \sum_{|j|=0}^m \frac{A_j}{|j|!} z^j \partial^{l+j} u(x)}{|z|^{n+s+2p}} \right. \\ &\quad \left. - \frac{\partial^l u(y+z) - \sum_{|j|=0}^m \frac{A_j}{|j|!} z^j \partial^{l+j} u(y)}{|z|^{n+s+2p}} \right] z^k dz, \\ J_2 &:= \int_{R_{\delta\eta}(0)} \left[\frac{\partial^l u(x+z) - \sum_{|j|=0}^m \frac{A_j}{|j|!} z^j \partial^{l+j} u(x)}{|z|^{n+s+2p}} \right. \\ &\quad \left. - \frac{\partial^l u(y+z) - \sum_{|j|=0}^m \frac{A_j}{|j|!} z^j \partial^{l+j} u(y)}{|z|^{n+s+2p}} \right] z^k dz, \end{aligned}$$

and $\eta = |x - y| < \delta$.

- (i) In this case, we let $|l| + m = r$ and $m + |k| = 2p$ in J_1, J_2 . On one hand, since $|l| + m = r$ and $u \in C^{r,\alpha}(\Omega)$, there exists a constant $C_1 > 0$ such that

$$|\partial^l u(x+z) - \partial^l u(y+z) - \sum_{|j|=0}^m \frac{A_j}{|j|!} (\partial^{l+j} u(x) - \partial^{l+j} u(y)) z^j| \leq C_1 [u]_{r,\alpha;\Omega} |z|^{m+\alpha}, \quad z \in B_{\eta}(0).$$

Using the fact $m + |k| = 2p$, it follows that

$$\begin{aligned} |J_1| &\leq \left| \int_{B_{\eta}(0)} \frac{C_1 [u]_{r,\alpha;\Omega} |z|^{2p+\alpha}}{|z|^{n+s+2p}} dz \right| \\ &\leq C_2 [u]_{r,\alpha;\Omega} \eta^{\alpha-s} = C_2 [u]_{r,\alpha;\Omega} |x - y|^{\alpha-s} \end{aligned}$$

for some constant $C_2 > 0$. On the other hand, we also have

$$\begin{aligned} & |\partial^l u(x+z) - \partial^l u(y+z) - \sum_{|j|=0}^{|j|=m} \frac{A_j}{|j|!} (\partial^{l+j} u(x) - \partial^{l+j} u(y)) z^j| \\ & \leq C_3[u]_{r,\alpha;\Omega} \left[\sum_{i=0}^m |x-y|^{m+\alpha-i} |z|^i + \sum_{i=1}^m |z|^{m+\alpha-i} |x-y|^i \right], \quad z \in B_\eta^c(0), \end{aligned}$$

where $C_3 > 0$ is a constant. It follows that

$$\begin{aligned} |J_2| & \leq \left| \int_{B_\eta^c(0)} \frac{C_3[u]_{r,\alpha;\Omega} (\sum_{i=0}^m |x-y|^{m+\alpha-i} |z|^i + \sum_{i=1}^m |z|^{m+\alpha-i} |x-y|^i)}{|z|^{n+s+m}} dz \right| \\ & \leq C[u]_{r,\alpha;\Omega} \left(\sum_{i=0}^m \eta^{i-m-s} |x-y|^{m+\alpha-i} + \sum_{i=1}^m \eta^{\alpha-s-i} |x-y|^i \right) \leq C_4[u]_{r,\alpha;\Omega} |x-y|^{\alpha-s} \end{aligned}$$

for some constant $C_4 > 0$. Hence

$$|J| \leq |J_1| + |J_2| \leq (C_2 + C_4)[u]_{r,\alpha;\Omega} |x-y|^{\alpha-s}. \quad (25)$$

Let I_0 be as in Lemma 2.1.8. Then Lemma 2.1.8 together with (25) imply that

$$|I_0(x) - I_0(y)| \leq C_5[u]_{r,\alpha;\Omega} |x-y|^{\alpha-s}$$

for some constant $C_5 > 0$. With this estimate, the proof is now complete by Lemma 2.1.8 and Lemma 2.1.9 a).

(ii) In this case, we let $|l| + m = r - 1$ and $m + |k| = 2p$ in J_1, J_2 . Since $m + 1 + |k| = 2p + 1$ is an odd number, we have, for any $j \in \mathbb{N}^n$ with $|j| = m + 1$, any $w \in \Omega$, and any $\rho < d_w$ that

$$\int_{B_\rho(w)} \frac{z^{j+k}}{|z|^{n+s+2p}} dy = 0.$$

It follows that

$$\begin{aligned} J_1 &= \int_{B_\eta(0)} \left[\frac{\partial^l u(x+z) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} z^j \partial^{l+j} u(x)}{|z|^{n+s+2p}} \right. \\ &\quad \left. - \frac{\partial^l u(y+z) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} z^j \partial^{l+j} u(y)}{|z|^{n+s+2p}} \right] z^k dz, \\ J_2 &= \int_{B_{\delta\eta}(0)} \left[\frac{\partial^l u(x+z) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} z^j \partial^{l+j} u(x)}{|z|^{n+s+2p}} \right. \\ &\quad \left. - \frac{\partial^l u(y+z) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} z^j \partial^{l+j} u(y)}{|z|^{n+s+2p}} \right] z^k dz. \end{aligned}$$

The rest of the proof is similar to that of (i). We only note that using facts $|l| + m = r - 1$ and $u \in C^{r,\alpha}(\Omega)$, the estimate of J_1 follows from the inequality

$$\begin{aligned} & |\partial^l u(x+z) - \partial^l u(y+z) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} (\partial^{l+j} u(x) - \partial^{l+j} u(y)) z^j| \\ & \leq C_5 [u]_{r,\alpha;\Omega} |z|^{m+1+\alpha}, \end{aligned}$$

$z \in B_\eta(0)$, where $C_5 > 0$ is a constant, while, using facts $m + |k| = 2p$ and $s < 1 + \alpha$, the estimate of J_2 follows from the inequality

$$\begin{aligned} & |\partial^l u(x+z) - \partial^l u(y+z) - \sum_{|j|=0}^{|j|=m+1} \frac{A_j}{|j|!} (\partial^{l+j} u(x) - \partial^{l+j} u(y)) z^j| \\ & \leq C_6 [u]_{r,\alpha;\Omega} \left(\sum_{i=0}^{m+1} |x-y|^{m+1+\alpha-i} |z|^i + \sum_{i=1}^{m+1} |z|^{m+1+\alpha-i} |x-y|^i \right), \quad z \in B_\eta^c(0), \end{aligned}$$

where $C_6 > 0$ is a constant. □

2.2 Schauder estimates

In this section, we will show the Schauder estimates for the regional fractional Laplacian using those for the fractional Laplacian.

2.2.1 Schauder estimates for the fractional Laplacian

Recall that the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ is well-defined in \mathcal{S} , the Schwartz space of rapidly decreasing C^∞ functions in \mathbb{R}^n , and we can then extend its definition to the space $L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+s}})$ by

$$\langle (-\Delta)^{\frac{s}{2}} u, \varphi \rangle_{\mathbb{R}^n} = \int_{\mathbb{R}^n} u(y) (-\Delta)^{\frac{s}{2}} \varphi(y) dy, \quad \forall \varphi \in \mathcal{S}, \quad (26)$$

for any $u \in L^1(\mathbb{R}^n, \frac{dx}{(1+|x|)^{n+s}})$. In the following Lemma 2.2.1 and 2.2.2, the definition of $(-\Delta)^{\frac{s}{2}}$ is understood in the sense of (26). We refer the reader to [72] for a more general definition of the fractional Laplacian.

Lemma 2.2.1. *Let $0 < \alpha \leq 1$ and $0 < s < 2$. If, for some $w \in C^\alpha(\bar{\Omega})$, $u \in L^\infty(\mathbb{R}^n)$ solves the equation $(-\Delta)^{\frac{s}{2}} u = w$ in Ω , then for any $\delta > 0$ sufficiently small there exists a constant $C > 0$ depending only on n, s, δ and α such that*

$$\|u\|_{C^{\alpha+s}(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{C^\alpha(\bar{\Omega})}).$$

Proof. The proof follows from that of [72, Proposition 2.8]. □

Lemma 2.2.2. *Let $0 < s < 2$. Suppose that, for some $w \in L^\infty(\Omega)$, $u \in L^\infty(\mathbb{R}^n)$ solves the equation $(-\Delta)^{\frac{s}{2}}u = w$ in Ω . Then, for any sufficiently small $\delta > 0$, there exists a constant $C > 0$ depending only on n , s and δ such that the following holds:*

(i) *If $s \neq 1$, then*

$$\|u\|_{C^s(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\Omega)}).$$

(ii) *If $s = 1$, then*

$$\|u\|_{\Lambda_*(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\Omega)}).$$

Proof. We first use the argument in the proof of [72, Proposition 2.8]. By covering and rescaling arguments, we only need to consider the case $\Omega_\delta = B_{\frac{1}{2}}(0)$ and $\Omega = B_1(0)$. Let $\eta \in C_c^\infty(\mathbb{R})$ be such that $\text{range}(\eta) \subset [0, 1]$, $\text{supp}(\eta) \subset B_1(0)$, and $\eta(x) = 1$ for any $x \in B_{\frac{3}{4}}(0)$. Denote

$$u_0(x) := \mathcal{A}(n, s) \int_{\mathbb{R}^n} \frac{\eta(y)w(y)}{|x - y|^{n-s}} dy = (-\Delta)^{-\frac{s}{2}}\eta w(x).$$

Then $(-\Delta)^{\frac{s}{2}}u_0 = w = (-\Delta)^{\frac{s}{2}}u$ in $B_{\frac{3}{4}}(0)$. It follows that $u - u_0 \in C^2(\bar{B}_{\frac{1}{2}}(0))$ and

$$\|u - u_0\|_{C^2(\bar{B}_{\frac{1}{2}}(0))} \leq C\|u - u_0\|_{L^\infty(\mathbb{R}^n)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(B_1(0))}),$$

where $C > 0$ is a constant depending only on n . We note that $C^2(\bar{B}_{\frac{1}{2}}(0)) = C^{1,1}(\bar{B}_{\frac{1}{2}}(0)) \neq C^{2,0}(\bar{B}_{\frac{1}{2}}(0))$. The lemma now follows from [15, Proposition 5.2]. \square

2.2.2 Schauder estimates for the regional fractional Laplacian

It is easy to see that the regional fractional Laplacian $\Delta_\Omega^{\frac{s}{2}}$ is well-defined for functions $u \in C^2(\Omega) \cap L^\infty(\Omega)$. We can then extend the definition of the regional fractional Laplacian to the space $L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}})$. For any $u \in L^1(\Omega, \frac{dx}{(1+|x|)^{n+s}})$, we define

$$\langle \Delta_\Omega^{\frac{s}{2}}u, \varphi \rangle_\Omega := \int_\Omega u(y) \Delta_\Omega^{\frac{s}{2}}\varphi(y) dy, \quad \text{for any } \varphi \in C_c^\infty(\Omega). \quad (27)$$

In the following Theorem 2.2.3 and 2.2.4, the definition of $\Delta_\Omega^{\frac{s}{2}}$ is understood in the sense of (27).

Theorem 2.2.3. *Let $0 < s < 2$. Suppose that, for some $w \in L^\infty(\Omega)$, $u \in L^\infty(\Omega)$ solves the equation $\Delta_\Omega^{\frac{s}{2}}u = w$ in Ω . Then, for any sufficiently small $\delta > 0$, there exists a constant $C > 0$ depending only on n , s , and δ such that the following holds:*

(i) *If $s \neq 1$, then*

$$\|u\|_{C^s(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}).$$

(ii) If $s = 1$, then

$$\|u\|_{\Lambda_*(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}).$$

Proof. Let $0 < s < 2$, $w \in L^\infty(\Omega)$, and $u \in L^\infty(\Omega)$ solves the equation $\Delta_{\Omega}^{\frac{s}{2}} u = w$ in Ω . Also let $\bar{u} \in L^\infty(\mathbb{R}^n)$ be such that $\bar{u} \equiv u$ in Ω and $\bar{u} \equiv 0$ outside of Ω . Then for any $\varphi \in \mathcal{S}$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(-\Delta_{\Omega, \epsilon}^{\frac{s}{2}} \bar{u}(x) + \mathcal{A}(n, -s) \bar{u}(x) \int_{\Omega^c \setminus B_\epsilon(x)} \frac{1}{|x-y|^{n+s}} dy \right) \varphi(x) dx \\ &= \mathcal{A}(n, -s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x) - \bar{u}(y)}{|x-y|^{n+s}} dy \varphi(x) dx \\ &= \mathcal{A}(n, -s) \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x) \varphi(x)}{|x-y|^{n+s}} dy dx - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(y)} \frac{\bar{u}(y) \varphi(x)}{|x-y|^{n+s}} dx dy \right) \\ &= \mathcal{A}(n, -s) \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x) \varphi(x)}{|x-y|^{n+s}} dy dx - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\bar{u}(x) \varphi(y)}{|x-y|^{n+s}} dy dx \right) \\ &= \mathcal{A}(n, -s) \int_{\mathbb{R}^n} \bar{u}(x) \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} dy dx. \end{aligned}$$

In particular, for any $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} & - \int_{\Omega} u(x) \Delta_{\Omega, \epsilon}^{\frac{s}{2}} \varphi(x) dx + \mathcal{A}(n, -s) \int_{\text{supp}(\varphi)} u(x) \varphi(x) \int_{\Omega^c \setminus B_\epsilon(x)} \frac{1}{|x-y|^{n+s}} dy dx \\ &= \int_{\mathbb{R}^n} \left(-\Delta_{\Omega, \epsilon}^{\frac{s}{2}} \bar{u}(x) + \mathcal{A}(n, -s) \bar{u}(x) \int_{\Omega^c \setminus B_\epsilon(x)} \frac{1}{|x-y|^{n+s}} dy \right) \varphi(x) dx \\ &= \mathcal{A}(n, -s) \int_{\mathbb{R}^n} \bar{u}(x) \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} dy dx. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ in the above, we easily obtain that, for any $x \in \Omega$,

$$(-\Delta)^{\frac{s}{2}} \bar{u}(x) = -w(x) + \mathcal{A}(n, -s) u(x) \int_{\Omega^c} \frac{1}{|x-y|^{n+s}} dy. \quad (28)$$

Let $\delta > 0$ be sufficiently small. We have by Lemma 2.2.2 and (28) that there exists a constant C depending on n , s , and δ such that

$$\begin{aligned} \|u\|_{C^s(\bar{\Omega}_\delta)} &\leq C(\|\bar{u}\|_{L^\infty(\mathbb{R}^n)} + \|-w + \mathcal{A}(n, -s)u \int_{\Omega^c} \frac{1}{|\cdot-y|^{n+s}} dy\|_{L^\infty(\Omega_\delta^{\frac{\delta}{2}})}) \\ &\leq C(\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}) \end{aligned}$$

when $s \neq 1$. Similarly,

$$\|u\|_{\Lambda_*(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}).$$

when $s = 1$. □

Remark 1. *The notions of $C^{\alpha+s}(\bar{\Omega}_\delta)$ and $C^\alpha(\bar{\Omega})$ we defined earlier have unified different cases for $\alpha, \alpha+s$ being or not being natural numbers in the statement of Theorem C. We note that $C^1(\bar{\Omega}_\delta) = C^{0,1}(\bar{\Omega}_\delta) \neq C^{1,0}(\bar{\Omega}_\delta)$, $C^1(\bar{\Omega}) = C^{0,1}(\bar{\Omega}) \neq C^{1,0}(\bar{\Omega})$, and $C^2(\bar{\Omega}_\delta) = C^{1,1}(\bar{\Omega}_\delta) \neq C^{2,0}(\bar{\Omega}_\delta)$.*

Theorem 2.2.4. *Let $0 < \alpha \leq 1$ and $0 < s < 2$. If, for some $w \in C^\alpha(\bar{\Omega})$, $u \in L^\infty(\Omega)$ solves the equation $\Delta_{\bar{\Omega}}^{\frac{s}{2}} u = w$ in Ω , then*

$$\|u\|_{C^{\alpha+s}(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\Omega)} + \|w\|_{C^\alpha(\bar{\Omega})}),$$

where $\delta > 0$ is sufficiently small and C is a constant depending only on n, s, δ and α .

Proof. Using bootstrap arguments, Lemma 2.2.1 and (28) for both cases of $\alpha, \alpha+s$ being or not being natural numbers, we have, similarly to the above, that

$$\|u\|_{C^{\alpha+s}(\bar{\Omega}_\delta)} \leq C(\|u\|_{L^\infty(\Omega)} + \|w\|_{C^\alpha(\bar{\Omega})}),$$

where $C > 0$ is a constant depending on n, s, δ and α . This proves Theorem 2.2.4. \square

CHAPTER III

UNIQUENESS OF VISCOSITY SOLUTIONS FOR A CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

In this chapter, we will study uniqueness and comparison principle of viscosity solutions for a class of integro-differential equations. This is a joint work with Prof. Andrzej Swiech, see [63].

3.1 Definitions and assumptions

Suppose that G is continuous and (2), (3), (8) hold. We recall two equivalent definitions of a viscosity solution of (7). In order to do it, we introduce two associated operators $I^{1,\delta}$ and $I^{2,\delta}$,

$$I^{1,\delta}[x, p, u] = \int_{|z| < \delta} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)p \cdot z] \mu_x(dz),$$

$$I^{2,\delta}[x, p, u] = \int_{|z| \geq \delta} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)p \cdot z] \mu_x(dz).$$

Definition 1. *A function $u \in BUC(\mathbb{R}^n)$ is a viscosity subsolution of (7) if whenever $u - \varphi$ has a maximum over \mathbb{R}^n at $x \in \Omega$ for some test function $\varphi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$, then*

$$G(x, u(x), I[x, \varphi]) \leq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity supersolution of (7) if whenever $u - \varphi$ has a minimum over \mathbb{R}^n at $x \in \Omega$ for a test function $\varphi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$, then

$$G(x, u(x), I[x, \varphi]) \geq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (7) if it is both a viscosity subsolution and viscosity supersolution of (7).

It is easy to see that Definition 1 is equivalent to the definition in which the requirement that $\varphi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ is replaced by the requirement that $\varphi \in C^2(B_\delta(x)) \cap BUC(\mathbb{R}^n)$ for some $\delta > 0$. The equivalence of Definition 1 and Definition 2 is also standard.

Definition 2. A function $u \in BUC(\mathbb{R}^n)$ is a viscosity subsolution of (7) if whenever $u - \varphi$ has a maximum over $B_\delta(x)$ at $x \in \Omega$ for a test function $\varphi \in C^2(B_\delta(x))$, $\delta > 0$, then

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity supersolution of (7) if whenever $u - \varphi$ has a minimum over $B_\delta(x)$ at $x \in \Omega$ for a test function $\varphi \in C^2(B_\delta(x))$, $\delta > 0$, then

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \geq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (7) if it is both a viscosity subsolution and viscosity supersolution of (7).

We make the following assumptions on the nonlinearity G and the family of Lévy measures $\{\mu_x\}$.

(H1) For each $\Omega' \subset \subset \Omega$, there is a nondecreasing continuous function $w_{\Omega'}$ satisfying $w_{\Omega'}(0) = 0$ and a non-negative constant $\Lambda_{\Omega'}$ such that

$$G(y, r, l_2) - G(x, r, l_1) \leq \Lambda_{\Omega'}(l_1 - l_2) + w_{\Omega'}(|x - y|)$$

for any $x, y \in \Omega'$ and $r, l_1, l_2 \in \mathbb{R}$.

(H2) For every $x \in \Omega$ the measure μ_x is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^n , i.e. $\mu_x(dz) = a(x, z)dz$, where $a(x, \cdot) \geq 0$ is measurable, and there exist two constants $0 < \theta \leq 1$, $0 < \sigma < 2$ and a positive constant C such that, for any $x, y \in \Omega$, we have

$$|a(x, z) - a(y, z)| \leq C \frac{|x - y|^\theta}{|z|^{n+\sigma}} \quad \text{in } B_1(0),$$

$$a(x, z) \leq \frac{C}{|z|^{n+\sigma}} \quad \text{in } B_1(0),$$

$$\int_{\mathbb{R}^n \setminus B_1(0)} |a(x, z) - a(y, z)| dz \leq C|x - y|^\theta,$$

$$\int_{\mathbb{R}^n \setminus B_1(0)} \mu_x(dz) \leq C.$$

3.2 Uniqueness of viscosity solutions of (7) for $\gamma > 0$

In this section we prove the main comparison theorem which will be a basis for other comparison results.

Theorem 3.2.1. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (7) is continuous and satisfies (2) with $\gamma > 0$ and (H1). Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2). Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1 - r\}$, u is a viscosity subsolution of (7), v is a viscosity supersolution of (7), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, we have $u \leq v$ in \mathbb{R}^n .*

Proof. Without loss of generality we assume that $u \in C^r(\Omega)$. The proof is divided into two cases.

Case 1: $0 < \sigma \leq 1$.

Without loss of generality we can assume in this case that $0 < r < 1$. Suppose that $\max_{\Omega}(u - v) = \nu > 0$. Let $K \subset \Omega$ be a compact neighborhood of the set of maximum points of $u - v$ in Ω . Then (see Proposition 3.7 of [22]), for ϵ sufficiently small, there are $\hat{x}, \hat{y} \in K$ such that

$$u(\hat{x}) - v(\hat{y}) - \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 = \sup_{x,y} \left\{ u(x) - v(y) - \frac{1}{2\epsilon}|x - y|^2 \right\} \geq \nu.$$

Moreover, we can assume that there is $0 < c < 1$ such that $B_{2c}(\hat{x}) \cup B_{2c}(\hat{y}) \subset \Omega$. Since

$$u(x) - v(y) - \frac{1}{2\epsilon}|x - y|^2 \leq u(\hat{x}) - v(\hat{y}) - \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2,$$

for any $x, y \in \mathbb{R}^n$. Putting $x = y = \hat{y}$, we thus have

$$\frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 \leq u(\hat{x}) - u(\hat{y}) \leq C|\hat{x} - \hat{y}|^r$$

for some $C > 0$ independent of ϵ , which gives us

$$\frac{|\hat{x} - \hat{y}|^{2-r}}{2\epsilon} \leq C. \quad (29)$$

By the definition of viscosity subsolutions and supersolutions, we have for $0 < \delta < c$,

$$G\left(\hat{x}, u(\hat{x}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, \frac{|\cdot - \hat{y}|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, u(\cdot)\right]\right) \leq 0,$$

$$G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, -\frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, v(\cdot)\right]\right) \geq 0.$$

Therefore, by (2) and assumption (H1), we have

$$\begin{aligned} \gamma(u(\hat{x}) - v(\hat{y})) &\leq G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, -\frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, v(\cdot)\right]\right) \\ &\quad - G\left(\hat{x}, v(\hat{y}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, \frac{|\cdot - \hat{y}|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, u(\cdot)\right]\right) \end{aligned}$$

$$\begin{aligned}
&\leq \Lambda_K \left\{ I^{1,\delta}[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, \frac{|\cdot - \hat{y}|^2}{2\epsilon}] + I^{2,\delta}[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon}, u(\cdot)] \right. \\
&\quad \left. - \left(I^{1,\delta}[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, -\frac{|\hat{x} - \cdot|^2}{2\epsilon}] + I^{2,\delta}[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon}, v(\cdot)] \right) \right\} + w_K(|\hat{x} - \hat{y}|) \\
&\leq \Lambda_K \left\{ \int_{|z| < \delta} [\frac{1}{2\epsilon}|\hat{x} - \hat{y} + z|^2 - \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 - \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] \mu_{\hat{x}}(dz) \right. \\
&\quad + \int_{|z| < \delta} [\frac{1}{2\epsilon}|\hat{x} - \hat{y} - z|^2 - \frac{1}{2\epsilon}|\hat{x} - \hat{y}|^2 + \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] \mu_{\hat{y}}(dz) \\
&\quad + \int_{|z| \geq \delta} [u(\hat{x} + z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] \mu_{\hat{x}}(dz) \\
&\quad \left. - \int_{|z| \geq \delta} [v(\hat{y} + z) - v(\hat{y}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] \mu_{\hat{y}}(dz) \right\} + w_K(|\hat{x} - \hat{y}|) \\
&= \Lambda_K \left\{ \int_{|z| < \delta} \frac{|z|^2}{2\epsilon} \mu_{\hat{x}}(dz) + \int_{|z| < \delta} \frac{|z|^2}{2\epsilon} \mu_{\hat{y}}(dz) \right. \\
&\quad + \int_{|z| \geq \delta} [u(\hat{x} + z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad \left. + \int_{|z| \geq \delta} [u(\hat{x} + z) - u(\hat{x}) - v(\hat{y} + z) + v(\hat{y})] \mu_{\hat{y}}(dz) \right\} + w_K(|\hat{x} - \hat{y}|).
\end{aligned}$$

Since $u(x) - v(y) - \frac{1}{2\epsilon}|x - y|^2$ attains a global maximum at (\hat{x}, \hat{y}) , we have

$$u(\hat{x} + z) - u(\hat{x}) \leq v(\hat{y} + z) - v(\hat{y}), \quad \text{for any } z \in \mathbb{R}^n.$$

Moreover, by assumption (H2) and the boundedness of u , we have

$$\begin{aligned}
&\int_{|z| \geq \delta} [u(\hat{x} + z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\leq \int_{|z| \geq 1} [u(\hat{x} + z) - u(\hat{x})] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad + \int_{1 > |z| \geq c} [u(\hat{x} + z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad + \int_{c > |z| \geq \delta} [u(\hat{x} + z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\leq \int_{c > |z| \geq \delta} [u(\hat{x} + z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad + C|\hat{x} - \hat{y}|^\theta + C \frac{|\hat{x} - \hat{y}|^{1+\theta}}{\epsilon}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \gamma(u(\hat{x}) - v(\hat{y})) \\
\leq & \Lambda_K \left\{ \int_{|z|<\delta} \frac{|z|^2}{2\epsilon} \mu_{\hat{x}}(dz) + \int_{|z|<\delta} \frac{|z|^2}{2\epsilon} \mu_{\hat{y}}(dz) \right. \\
& + \int_{c>|z|\geq\delta} [u(\hat{x}+z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
& \left. + C|\hat{x} - \hat{y}|^\theta + C \frac{|\hat{x} - \hat{y}|^{1+\theta}}{\epsilon} \right\} + w_K(|\hat{x} - \hat{y}|). \tag{30}
\end{aligned}$$

Now by assumption (H2), we have for some $C > 0$

$$\int_{|z|<\delta} \frac{|z|^2}{2\epsilon} \mu_{\hat{x}}(dz) + \int_{|z|<\delta} \frac{|z|^2}{2\epsilon} \mu_{\hat{y}}(dz) \leq C \frac{\delta^{2-\sigma}}{\epsilon}, \tag{31}$$

$$\begin{aligned}
& \int_{c>|z|\geq\delta} [u(\hat{x}+z) - u(\hat{x}) - \frac{1}{\epsilon}(\hat{x} - \hat{y}) \cdot z] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
\leq & \int_{c>|z|\geq\delta} \frac{C(|z|^r |\hat{x} - \hat{y}|^\theta + \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} |z|)}{|z|^{n+\sigma}} dz \\
\leq & \begin{cases} C|\hat{x} - \hat{y}|^\theta \delta^{r-1} - C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} \ln \delta & \text{if } r < \sigma = 1, \\ C|\hat{x} - \hat{y}|^\theta \delta^{r-\sigma} + C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} & \text{if } r < \sigma < 1, \\ -C|\hat{x} - \hat{y}|^\theta \ln \delta + C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} & \text{if } r = \sigma < 1, \\ C|\hat{x} - \hat{y}|^\theta + C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} & \text{if } \sigma < r < 1. \end{cases}
\end{aligned}$$

In the rest of the proof we will only consider the case $r < \sigma$. The case $\sigma \leq r < 1$ is easier and can be handled similarly. Let $\delta = n^{-\alpha}$ and $\epsilon = n^{-\beta}$. By (29), we have

$$|\hat{x} - \hat{y}| \leq C n^{-\frac{\beta}{2-r}}.$$

If $r < \sigma < 1$, we have

$$\begin{aligned}
C \frac{\delta^{2-\sigma}}{\epsilon} &= C n^{\alpha(\sigma-2)+\beta}, \\
C|\hat{x} - \hat{y}|^\theta \delta^{r-\sigma} &\leq C n^{-\frac{\theta\beta}{2-r} + \alpha(\sigma-r)}, \\
C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} &\leq C n^{\frac{\beta}{2-r}(1-r-\theta)}.
\end{aligned}$$

Thus, if

$$\beta < (2 - \sigma)\alpha, \tag{32}$$

$$\alpha(\sigma - r) < \frac{\theta\beta}{2 - r}, \tag{33}$$

$$\theta > 1 - r, \tag{34}$$

it follows

$$C \frac{\delta^{2-\sigma}}{\epsilon} \rightarrow 0, \quad (35)$$

$$C |\hat{x} - \hat{y}|^\theta \delta^{r-\sigma} \rightarrow 0, \quad (36)$$

$$C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} \rightarrow 0. \quad (37)$$

It remains to find proper $\alpha > 0$, $\beta > 0$, and $0 < r_0 < \sigma$ so that (32) and (33) hold. We set $\beta = 1$ and $\alpha > 1/(2 - \sigma)$ so that (32) is satisfied. Then obviously there exists a positive constant $r_0 < \sigma$ such that (33) is satisfied if $r_0 < r < \sigma$.

If $r < \sigma = 1$, we have

$$\begin{aligned} C \frac{\delta^{2-\sigma}}{\epsilon} &= C n^{\beta-\alpha}, \\ C |\hat{x} - \hat{y}|^\theta \delta^{r-\sigma} &\leq C n^{-\frac{\theta\beta}{2-r} + \alpha(1-r)}, \\ C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} (-\ln \delta) &\leq C n^{\frac{\beta}{2-r}(1-r-\theta)} \ln(n). \end{aligned}$$

Thus, if

$$\beta < \alpha, \quad (38)$$

$$\alpha(1-r) < \frac{\theta\beta}{2-r}, \quad (39)$$

$$\theta > 1-r, \quad (40)$$

we have

$$C \frac{\delta^{2-\sigma}}{\epsilon} \rightarrow 0, \quad (41)$$

$$C |\hat{x} - \hat{y}|^\theta \delta^{r-\sigma} \rightarrow 0, \quad (42)$$

$$C \frac{1}{\epsilon} |\hat{x} - \hat{y}|^{1+\theta} (-\ln \delta) \rightarrow 0. \quad (43)$$

Using the same strategy as before, for any $\theta > 0$, we set $\beta = 1, \alpha > 1$, and then choose $0 < r_0 < \sigma$ such that (39) is satisfied if $r_0 < r < 1$.

Therefore, using (35)-(37), (41)-(43) in (30), we conclude

$$\gamma\nu \leq \limsup_{n \rightarrow +\infty} \gamma(u(\hat{x}) - v(\hat{y})) \leq 0$$

if $r_0 < r < \sigma$. This contradiction thus implies that we must have $u \leq v$ in \mathbb{R}^n .

Case 2: $1 < \sigma < 2$.

We assume that $r > 1$. Suppose that $\max_\Omega(u - v) = \nu > 0$. Let $K \subset \Omega$ be a compact neighborhood of the set of maximum points of $u - v$ in Ω . There is a sequence of $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ functions $\{\psi_n\}_n$ such that

$$u - \psi_n \rightarrow 0 \quad \text{as } n \rightarrow +\infty \quad \text{uniformly on } \mathbb{R}^n, \quad (44)$$

and

$$\begin{cases} |Du - D\psi_n| \leq Cn^{1-r} & \text{on } K, \\ |D^2\psi_n| \leq Cn^{2-r} & \text{on } K, \\ |D^2\psi_n(x) - D^2\psi_n(y)| \leq Cn^{3-r}|x - y| & \text{on } K, \end{cases} \quad (45)$$

where C is a positive constant (see [22]). Let ρ be a modulus of continuity of u and v .

Let $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ be a maximum point of

$$(u(x) - \psi_n(x)) - (v(y) - \psi_n(y)) - \frac{1}{2\epsilon}|x - y|^2$$

over $\mathbb{R}^n \times \mathbb{R}^n$. Again it is standard to notice (see Proposition 3.7 of [22]) that

$$\lim_{\epsilon \rightarrow 0} (u(\hat{x}) - v(\hat{y})) = \max_{\Omega} (u - v), \quad (46)$$

and there must exist $0 < c < 1$ such that $B_c(\hat{x}) \cup B_c(\hat{y}) \subset K$ if ϵ is sufficiently small. Moreover, since $u(\cdot) - \psi_n(\cdot) - (v(\hat{y}) - \psi_n(\hat{y})) - \frac{1}{2\epsilon}|\cdot - \hat{y}|^2$ has a global maximum at \hat{x} , we have

$$Du(\hat{x}) - D\psi_n(\hat{x}) = \frac{\hat{x} - \hat{y}}{\epsilon}.$$

Thus, we get

$$\frac{|\hat{x} - \hat{y}|}{\epsilon} \leq Cn^{1-r}. \quad (47)$$

By the definition of viscosity subsolutions and supersolutions, we have, for any $0 < \delta < c$,

$$\begin{aligned} G\left(\hat{x}, u(\hat{x}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{x}), \frac{|\cdot - \hat{y}|^2}{2\epsilon} + \psi_n(\cdot)\right] + \psi_n(\cdot)\right) + I^{2,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{x}), u(\cdot)\right] &\leq 0, \\ G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{y}), \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right) + I^{2,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{y}), v(\cdot)\right] &\geq 0. \end{aligned}$$

Therefore, by (2) and assumption (H1), we have

$$\begin{aligned} &\gamma(u(\hat{x}) - v(\hat{y})) \\ &\leq G\left(\hat{y}, v(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{y}), \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right) + I^{2,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{y}), v(\cdot)\right] \\ &\quad - G\left(\hat{x}, v(\hat{y}), I^{1,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{x}), \frac{|\cdot - \hat{y}|^2}{2\epsilon} + \psi_n(\cdot)\right] + \psi_n(\cdot)\right) + I^{2,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{x}), u(\cdot)\right] \\ &\leq \Lambda_K \left\{ I^{1,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{x}), \frac{|\cdot - \hat{y}|^2}{2\epsilon} + \psi_n(\cdot)\right] + I^{2,\delta}\left[\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{x}), u(\cdot)\right] \right. \\ &\quad \left. - \left(I^{1,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{y}), \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon} + D\psi_n(\hat{y}), v(\cdot)\right] \right) \right\} \\ &\quad + w_K(|\hat{x} - \hat{y}|) \end{aligned}$$

$$\begin{aligned}
&\leq \Lambda_K \left\{ \int_{|z|<\delta} \left[\psi_n(\hat{x}+z) + \frac{1}{2\epsilon} |\hat{x}-\hat{y}+z|^2 - (\psi_n(\hat{x}) + \frac{1}{2\epsilon} |\hat{x}-\hat{y}|^2) \right. \right. \\
&\quad \left. \left. - \left(\frac{1}{\epsilon} (\hat{x}-\hat{y}) + D\psi_n(\hat{x}) \right) \cdot z \right] \mu_{\hat{x}}(dz) \right. \\
&\quad - \int_{|z|<\delta} \left[\psi_n(\hat{y}+z) - \frac{1}{2\epsilon} |\hat{x}-\hat{y}-z|^2 - (\psi_n(\hat{y}) - \frac{1}{2\epsilon} |\hat{x}-\hat{y}|^2) \right. \\
&\quad \left. \left. - \left(\frac{1}{\epsilon} (\hat{x}-\hat{y}) + D\psi_n(\hat{y}) \right) \cdot z \right] \mu_{\hat{y}}(dz) \right. \\
&\quad + \int_{|z|\geq\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \left(\frac{1}{\epsilon} (\hat{x}-\hat{y}) + D\psi_n(\hat{x}) \right) \cdot z \right] \mu_{\hat{x}}(dz) \\
&\quad \left. - \int_{|z|\geq\delta} \left[v(\hat{y}+z) - v(\hat{y}) - \mathbb{1}_{B_1(0)}(z) \left(\frac{1}{\epsilon} (\hat{x}-\hat{y}) + D\psi_n(\hat{y}) \right) \cdot z \right] \mu_{\hat{y}}(dz) \right\} + w_K(|\hat{x}-\hat{y}|) \\
&\leq \Lambda_K \left\{ \int_{|z|<\delta} \left[\frac{1}{2\epsilon} |z|^2 + \psi_n(\hat{x}+z) - \psi_n(\hat{x}) - D\psi_n(\hat{x}) \cdot z \right] \mu_{\hat{x}}(dz) \right. \\
&\quad - \int_{|z|<\delta} \left[-\frac{1}{2\epsilon} |z|^2 + \psi_n(\hat{y}+z) - \psi_n(\hat{y}) - D\psi_n(\hat{y}) \cdot z \right] \mu_{\hat{y}}(dz) \\
&\quad + \int_{|z|\geq\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \left(\frac{1}{\epsilon} (\hat{x}-\hat{y}) + D\psi_n(\hat{x}) \right) \cdot z \right] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\quad + \int_{|z|\geq\delta} \left[u(\hat{x}+z) - u(\hat{x}) - v(\hat{y}+z) + v(\hat{y}) - \mathbb{1}_{B_1(0)}(z) (D\psi_n(\hat{x}) - D\psi_n(\hat{y})) \cdot z \right] \mu_{\hat{y}}(dz) \Big\} \\
&\quad + w_K(|\hat{x}-\hat{y}|).
\end{aligned}$$

Since (\hat{x}, \hat{y}) is a global maximum point of $(u(x) - \psi_n(x)) - (v(y) - \psi_n(y)) - \frac{1}{2\epsilon} |x - y|^2$, we have

$$u(\hat{x}+z) - u(\hat{x}) - v(\hat{y}+z) + v(\hat{y}) \leq \psi_n(\hat{x}+z) - \psi_n(\hat{x}) - \psi_n(\hat{y}+z) + \psi_n(\hat{y}), \quad \text{for all } z \in \mathbb{R}^n.$$

Thus, by (45) and the uniform continuity of u, v , we have

$$\begin{aligned}
&\int_{|z|\geq\delta} \left[u(\hat{x}+z) - u(\hat{x}) - v(\hat{y}+z) + v(\hat{y}) - \mathbb{1}_{B_1(0)}(z) (D\psi_n(\hat{x}) - D\psi_n(\hat{y})) \cdot z \right] \mu_{\hat{y}}(dz) \\
&\leq \int_{c \geq |z| \geq \delta} \left[(\psi_n(\hat{x}+z) - \psi_n(\hat{x}) - D\psi_n(\hat{x}) \cdot z) \right. \\
&\quad \left. - (\psi_n(\hat{y}+z) - \psi_n(\hat{y}) - D\psi_n(\hat{y}) \cdot z) \right] \mu_{\hat{y}}(dz) + C|\hat{x}-\hat{y}|^{r-1} + C\rho(|\hat{x}-\hat{y}|).
\end{aligned}$$

Moreover, by assumption (H2), the boundedness of u and $Du(\hat{x}) = \frac{1}{\epsilon}(\hat{x}-\hat{y}) + D\psi_n(\hat{x})$ (in n and ϵ), we have

$$\begin{aligned}
&\int_{|z|\geq\delta} \left[u(\hat{x}+z) - u(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \left(\frac{1}{\epsilon} (\hat{x}-\hat{y}) + D\psi_n(\hat{x}) \right) \cdot z \right] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
&\leq \int_{c \geq |z| \geq \delta} + \int_{|z| \geq c} \\
&\leq \int_{c \geq |z| \geq \delta} \left[u(\hat{x}+z) - u(\hat{x}) - \left(\frac{1}{\epsilon} (\hat{x}-\hat{y}) + D\psi_n(\hat{x}) \right) \cdot z \right] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) + C|\hat{x}-\hat{y}|^\theta.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \gamma(u(\hat{x}) - v(\hat{y})) \\
& \leq \Lambda_K \left\{ \int_{|z| < \delta} \left[\frac{1}{2\epsilon} |z|^2 + \psi_n(\hat{x} + z) - \psi_n(\hat{x}) - D\psi_n(\hat{x}) \cdot z \right] \mu_{\hat{x}}(dz) \right. \\
& \quad - \int_{|z| < \delta} \left[-\frac{1}{2\epsilon} |z|^2 + \psi_n(\hat{y} + z) - \psi_n(\hat{y}) - D\psi_n(\hat{y}) \cdot z \right] \mu_{\hat{y}}(dz) \\
& \quad + \int_{c \geq |z| \geq \delta} \left[u(\hat{x} + z) - u(\hat{x}) - \left(\frac{1}{\epsilon} (\hat{x} - \hat{y}) + D\psi_n(\hat{x}) \right) \cdot z \right] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
& \quad + \int_{c \geq |z| \geq \delta} \left[(\psi_n(\hat{x} + z) - \psi_n(\hat{x}) - D\psi_n(\hat{x}) \cdot z) - (\psi_n(\hat{y} + z) - \psi_n(\hat{y}) - D\psi_n(\hat{y}) \cdot z) \right] \\
& \quad \left. \mu_{\hat{y}}(dz) \right\} + C\rho(|\hat{x} - \hat{y}|) + C|\hat{x} - \hat{y}|^{r-1} + C|\hat{x} - \hat{y}|^\theta + w_K(|\hat{x} - \hat{y}|). \tag{48}
\end{aligned}$$

Estimate (31) holds. Moreover, by (H2) and (45), we have

$$\begin{aligned}
& \left| \int_{|z| < \delta} [\psi_n(\hat{x} + z) - \psi_n(\hat{x}) - D\psi_n(\hat{x}) \cdot z] \mu_{\hat{x}}(dz) \right| \leq Cn^{2-r} \delta^{2-\sigma}, \\
& \left| \int_{|z| < \delta} [\psi_n(\hat{y} + z) - \psi_n(\hat{y}) - D\psi_n(\hat{y}) \cdot z] \mu_{\hat{y}}(dz) \right| \leq Cn^{2-r} \delta^{2-\sigma}, \\
& \int_{c \geq |z| \geq \delta} \left[u(\hat{x} + z) - u(\hat{x}) - \left(\frac{1}{\epsilon} (\hat{x} - \hat{y}) + D\psi_n(\hat{x}) \right) \cdot z \right] (\mu_{\hat{x}}(dz) - \mu_{\hat{y}}(dz)) \\
& \leq C \int_{c \geq |z| \geq \delta} \frac{|z|^r |\hat{x} - \hat{y}|^\theta}{|z|^{n+\sigma}} dz \leq \begin{cases} C\delta^{r-\sigma} |\hat{x} - \hat{y}|^\theta & \text{if } r < \sigma, \\ -C|\hat{x} - \hat{y}|^\theta \ln \delta & \text{if } r = \sigma, \\ C|\hat{x} - \hat{y}|^\theta & \text{if } \sigma < r < 2. \end{cases}
\end{aligned}$$

We recall a simple identity. If $f \in C^2(\mathbb{R}^n)$ then for every $x, z \in \mathbb{R}^n$

$$f(x + z) = f(x) + Df(x) \cdot z + \int_0^1 \int_0^1 D^2 f(x + stz) z \cdot z t ds dt.$$

Using it, (H2), and recalling that $B_c(\hat{x}) \cup B_c(\hat{y}) \subset K$, we obtain

$$\begin{aligned}
& \int_{c \geq |z| \geq \delta} \left[(\psi_n(\hat{x} + z) - \psi_n(\hat{x}) - D\psi_n(\hat{x}) \cdot z) - (\psi_n(\hat{y} + z) - \psi_n(\hat{y}) - D\psi_n(\hat{y}) \cdot z) \right] \mu_{\hat{y}}(dz) \\
& = \int_{c \geq |z| \geq \delta} \int_0^1 \int_0^1 [D^2 \psi_n(\hat{x} + stz) - D^2 \psi_n(\hat{y} + stz)] z \cdot z t ds dt \mu_{\hat{y}}(dz) \\
& \leq C \int_{c \geq |z| \geq \delta} n^{3-r} |\hat{x} - \hat{y}| \frac{|z|^2}{|z|^{n+\sigma}} dz \leq Cn^{3-r} |\hat{x} - \hat{y}|.
\end{aligned}$$

In the remainder of the proof we will only consider the case $r < \sigma$. The case $\sigma \leq r < 2$ is easier and can be done similarly (see also Remark 3). Assume then that $1 < r < \sigma$. Let again $\delta = n^{-\alpha}$ and $\epsilon = n^{-\beta}$. By (47), we have

$$C \frac{\delta^{2-\sigma}}{\epsilon} = Cn^{\alpha(\sigma-2)+\beta},$$

$$\begin{aligned}
Cn^{2-r}\delta^{2-\sigma} &= Cn^{2-r+\alpha(\sigma-2)}, \\
C|\hat{x} - \hat{y}|^\theta \delta^{r-\sigma} &\leq Cn^{-\theta[(r-1)+\beta]-\alpha(r-\sigma)}, \\
Cn^{3-r}|\hat{x} - \hat{y}| &\leq Cn^{-(r-1)-\beta+(3-r)}.
\end{aligned}$$

Thus, if

$$\beta < (2 - \sigma)\alpha, \quad (49)$$

$$2 - r < \alpha(2 - \sigma), \quad (50)$$

$$\alpha(\sigma - r) < \theta(r - 1 + \beta), \quad (51)$$

$$(4 - 2r) < \beta, \quad (52)$$

we have

$$C \frac{\delta^{2-\sigma}}{\epsilon} \rightarrow 0, \quad (53)$$

$$Cn^{2-r}\delta^{2-\sigma} \rightarrow 0, \quad (54)$$

$$C|\hat{x} - \hat{y}|^\theta \delta^{r-\sigma} \rightarrow 0, \quad (55)$$

$$Cn^{3-r}|\hat{x} - \hat{y}| \rightarrow 0. \quad (56)$$

We need to find $\alpha > 0$, $\beta > 0$, and $1 \leq r_0 < \sigma$ so that (49)-(52) are satisfied if $r_0 < r < \sigma$. First fix β such that (52) is satisfied. Then, fix α such that (49) and (50) are satisfied. It is then clear that there exists a positive constant $1 \leq r_0 < \sigma$ such that (51) is satisfied if $r_0 < r < \sigma$.

Thus, letting $n \rightarrow +\infty$ in (48) and using (46) and (53)-(56), we obtain $\gamma\nu \leq 0$ which is a contradiction. Therefore, $u \leq v$ in \mathbb{R}^n . \square

Remark 2. *It follows from the proof of Theorem 3.2.1 that if the kernel functions $a(x, \cdot)$ are symmetric, the requirement $\theta > \max\{0, 1 - r\}$ can be replaced by a weaker requirement $\theta > 0$. The same remark applies to Theorems 3.2.2, 3.3.1, 3.3.2, 3.4.4, Lemmas 3.4.1, 3.4.2, and Corollaries 1, 2, 3, 4.*

Corollary 1. *Let the assumptions of Theorem 3.2.1 be satisfied, $0 < \sigma < 2, \theta > \max\{0, 1 - r\}, 0 < r < 2$. If u is a viscosity subsolution, v is a viscosity supersolution of (7), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, then:*

- (i) *For $0 < \sigma \leq 1$, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$, we have $u \leq v$ in \mathbb{R}^n .*
- (ii) *For $1 < \sigma < 2$ and $r > 1$, if $\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, we have $u \leq v$ in \mathbb{R}^n .*

Proof. (i) Let $\beta = 1$ and $\alpha = 1/(2 - \sigma) + \eta$, where $\eta > 0$. Then (32) and (34) hold and (33) will be satisfied if

$$\left(\frac{1}{2 - \sigma} + \eta \right) (\sigma - r) < \frac{\theta}{2 - r}.$$

An easy calculation shows that the above will be true for some $\eta > 0$ if

$$\sigma < \frac{\theta(2-r)}{2-r+\theta} + r.$$

(ii) Set

$$\beta = 4 - 2r + \eta_1, \quad \alpha = \frac{4 - 2r + \eta_1}{2 - \sigma} + \eta_2,$$

where $\eta_1, \eta_2 > 0$. Then (49), (50) and (52) are satisfied, and (51) will be satisfied if

$$\left(\frac{4 - 2r + \eta_1}{2 - \sigma} + \eta_2 \right) (\sigma - r) < \theta(r - 1 + 4 - 2r + \eta_1)$$

for some $\eta_1, \eta_2 > 0$. Again a simple calculation yields that this inequality will be satisfied for some $\eta_1, \eta_2 > 0$ if

$$\sigma < 2 - 2 \frac{(2-r)^2}{\theta(3-r) + (4-2r)}.$$

□

Let us consider another important fully nonlinear integro-PDE appearing in the study of stochastic optimal control and stochastic differential games for processes with jumps, namely the Bellman-Isaacs equation (9)

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-I_{\alpha\beta}[x, u] + f_{\alpha\beta}(x)\} = 0, \quad \text{in } \Omega,$$

where $I_{\alpha\beta}[x, u] = \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(z) \cdot z] \mu_x^{\alpha\beta}(dz)$ and $\{\mu_x^{\alpha\beta}\}$ is a family of Lévy measures with indices α and β ranging in some sets \mathcal{A} and \mathcal{B} . Equation (9) is not of the same form as (7), which means that the following theorem and corollary are not corollaries of Theorem 3.2.1 and Corollary 1, however the proofs follow the same arguments. Similar results would be true if we included other typical purely local first and second order terms in (9).

Theorem 3.2.2. *Let Ω be a bounded domain. Suppose that $\gamma > 0$, the family of Lévy measures $\{\mu_x^{\alpha\beta}\}$ satisfies assumption (H2) uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, and $f_{\alpha\beta}$ are uniformly bounded in Ω and uniformly continuous in every compact subset $K \subset \Omega$, uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$. Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1 - r\}$, u is a viscosity subsolution of (9), v is a viscosity supersolution of (9), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, we have $u \leq v$ in \mathbb{R}^n .*

Corollary 2. *Let the assumptions of Theorem 3.2.2 be satisfied, $0 < \sigma < 2, \theta > \max\{0, 1 - r\}, 0 < r < 2$. If u is a viscosity subsolution of (9), v is a viscosity supersolution of (9), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, then:*

(i) *For $0 < \sigma \leq 1$, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$, we have $u \leq v$ in \mathbb{R}^n .*

(ii) *For $1 < \sigma < 2$ and $r > 1$, if $\sigma < 2 - 2 \frac{(2-r)^2}{\theta(3-r) + (4-2r)}$, we have $u \leq v$ in \mathbb{R}^n .*

Remark 3. Suppose that the kernel function $a(x, z)$ satisfies the second condition of (H2). If $r > \max(\sigma, 1)$, or if $r > \sigma$ and the kernels $a(x, \cdot)$ are symmetric, then a viscosity subsolution/supersolution of (7) which is in $C^r(\Omega)$ can be considered to be a classical subsolution/supersolution of (7). In such a case comparison theorem is standard and we do not need the full assumptions of Theorem 3.2.1. The same remark applies to Theorem 3.2.2, and Theorems 3.3.1 and 3.3.2 if condition (H3) is satisfied.

3.3 Uniqueness of viscosity solutions of (7) for $\gamma = 0$

In this section we investigate uniqueness of viscosity solutions of (7) when $\gamma = 0$ in (2). As always we assume that G is continuous and (3), (2), (8) hold. To compensate for the fact that $\gamma = 0$, we will assume that the nonlinearity G is uniformly elliptic with respect to a class of linear nonlocal operators \mathcal{L} . A class \mathcal{L} is a set of linear nonlocal operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x) \cdot z] \mu^L(dz),$$

where the Lévy measures μ^L are symmetric and satisfy $\sup_{L \in \mathcal{L}} \int_{\mathbb{R}^n} \min\{1, |z|^2\} \mu^L(dz) < +\infty$. We say that the nonlinearity G in (7) is uniformly elliptic with respect to \mathcal{L} if for every $\varphi, \psi \in C^2(B_\delta(x)) \cap BUC(\mathbb{R}^n)$, $x \in \Omega$, $r \in \mathbb{R}$, $\delta > 0$,

$$M_{\mathcal{L}}^-(\psi - \varphi)(x) \leq G(x, r, I[x, \varphi]) - G(x, r, I[x, \psi]) \leq M_{\mathcal{L}}^+(\psi - \varphi)(x),$$

where

$$\begin{aligned} M_{\mathcal{L}}^+ \varphi(x) &= \sup_{L \in \mathcal{L}} L\varphi(x), \\ M_{\mathcal{L}}^- \varphi(x) &= \inf_{L \in \mathcal{L}} L\varphi(x). \end{aligned}$$

In order to have a comparison principle for the case $\gamma = 0$, we need to impose an additional minimal ellipticity condition on the class \mathcal{L} . We will assume that the following condition holds.

(H3) There exist a non-negative function $\varphi \in C^2(\Omega) \cap BUC(\mathbb{R}^n)$ and $\delta_0 > 0$, such that $L\varphi > \delta_0$ in Ω for every $L \in \mathcal{L}$.

Theorem 3.3.1. Let Ω be a bounded domain and let a class \mathcal{L} satisfy (H3). Suppose that the nonlinearity G in (7) is continuous and uniformly elliptic with respect to \mathcal{L} , and satisfies (2) with $\gamma = 0$ and (H1). Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2). Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1 - r\}$, u is a

viscosity subsolution of (7), v is a viscosity supersolution of (7), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, we have $u \leq v$ in \mathbb{R}^n .

Proof. By (H3), there is a positive constant $M > 0$ such that $\varphi \leq M$ in \mathbb{R}^n . For any $\epsilon > 0$, let $\varphi_\epsilon = \epsilon(1 - \frac{1}{M}\varphi)$ in \mathbb{R}^n . Obviously, we have $0 \leq \varphi_\epsilon \leq \epsilon$ in \mathbb{R}^n and $M_{\mathcal{L}}^-(-\varphi_\epsilon) = M_{\mathcal{L}}^-(\frac{\epsilon}{M}\varphi) \geq \frac{\epsilon\delta_0}{M}$ in Ω .

We claim that $v + \varphi_\epsilon$ is a viscosity supersolution of $G = \frac{\epsilon\delta_0}{M}$ in Ω . Suppose that $x \in \Omega$, $\delta > 0$ and $\psi \in C^2(B_\delta(x)) \cap BUC(\mathbb{R}^n)$ are such that $v + \varphi_\epsilon - \psi$ has a minimum over $B_\delta(x)$ at x . Thus, there exists a positive constant $\delta' > 0$ such that $B_{\delta'}(x) \subset \Omega \cap B_\delta(x)$. Since v is a viscosity supersolution of (7), we have $G(x, v(x), I[x, \psi - \varphi_\epsilon]) \geq 0$. By (2) and the uniform ellipticity, we get

$$\begin{aligned} G(x, v(x) + \varphi_\epsilon(x), I[x, \psi]) &\geq G(x, v(x) + \varphi_\epsilon(x), I[x, \psi]) - G(x, v(x), I[x, \psi - \varphi_\epsilon]) \\ &\geq G(x, v(x), I[x, \psi]) - G(x, v(x), I[x, \psi - \varphi_\epsilon]) \\ &\geq M_{\mathcal{L}}^-(-\varphi_\epsilon) \geq \frac{\epsilon\delta_0}{M}. \end{aligned}$$

Therefore, the proof of the claim is complete.

We notice that $u \leq v + \varphi_\epsilon$ in Ω^c . We can now repeat the proof of Theorem 3.2.1 to obtain $u \leq v + \varphi_\epsilon \leq v + \epsilon$ in \mathbb{R}^n . (Instead of the contradiction $\gamma\nu \leq 0$ we will now get a contradiction $\frac{\epsilon\delta_0}{M} \leq 0$.) Letting $\epsilon \rightarrow 0^+$, we thus conclude that $u \leq v$ in \mathbb{R}^n . \square

Combining the proofs of Corollary 1 and Theorem 3.3.1, we have the following corollary.

Corollary 3. *Let the assumptions of Theorem 3.3.1 be satisfied, $0 < \sigma < 2, \theta > \max\{0, 1 - r\}, 0 < r < 2$. If u is a viscosity subsolution, v is a viscosity supersolution of (7), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, then:*

- (i) *For $0 < \sigma \leq 1$, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$, we have $u \leq v$ in \mathbb{R}^n .*
- (ii) *For $1 < \sigma < 2$ and $r > 1$, if $\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, we have $u \leq v$ in \mathbb{R}^n .*

The same techniques also produce the following two results for equation (9).

Theorem 3.3.2. *Let Ω be a bounded domain. Suppose that $\gamma = 0$, the family of Lévy measures $\{\mu_x^{\alpha\beta}\}$ satisfies assumption (H2) uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, and $f_{\alpha\beta}$ are uniformly bounded in Ω and uniformly continuous in every compact subset $K \subset \Omega$, uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, and the class $\{I_{\alpha\beta}\}$ satisfies (H3). Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1 - r\}$, u is a viscosity subsolution of (9), v is a viscosity supersolution of (9), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, we have $u \leq v$ in \mathbb{R}^n .*

Corollary 4. *Let the assumptions of Theorem 3.3.2 be true, $0 < \sigma < 2$, $\theta > \max\{0, 1 - r\}$ and $0 < r < 2$. If u is a viscosity subsolution of (9), v is a viscosity supersolution of (9), $u \leq v$ in Ω^c , and either u or v is in $C^r(\Omega)$, then:*

(i) *For $0 < \sigma \leq 1$, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$, we have $u \leq v$ in \mathbb{R}^n .*

(ii) *For $1 < \sigma < 2$ and $r > 1$, if $\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, we have $u \leq v$ in \mathbb{R}^n .*

3.4 Regularization by sup/inf-convolutions

In this section we show how techniques of Section 3.2 can be adapted to regularize viscosity sub/supersolutions by sup/inf-convolutions. It is a generally expected principle in the theory of viscosity solutions of PDE that whenever one is able to prove a comparison principle then one should be able to prove that a sup-convolution of a viscosity subsolution (respectively, inf-convolution of a viscosity supersolution) is a viscosity subsolution (respectively, supersolution) of a slightly perturbed equation. The same principle also seems to work for viscosity sub/supersolutions of integro-PDE under standard assumptions, see e.g. [45] for a proof for a standard Bellman-Isaacs equation. Here the situation is a bit more complicated. Since in our case the proof of comparison principle uses auxiliary functions ψ_n , we have to introduce a notion of sup/inf-convolution that depends on a parameter $\epsilon > 0$ and on a function ψ . Such sup/inf convolutions have been used in [41]. We will also show that if G is uniformly elliptic with respect to a class \mathcal{L} of linear nonlocal operators, u is a viscosity subsolution of (7) and v is a viscosity supersolution of (7), then $u - v$ satisfies $M_{\mathcal{L}}^-(v - u) \leq 0$ in the viscosity sense. Similar results can also be proved for equation (9).

We will always assume that G is continuous and satisfies (3), (2), (8). We first give yet another equivalent definition of viscosity solutions of (7).

Definition 3. *A function φ is said to be $C^{1,1}$ at the point x , and we write $u \in C^{1,1}(x)$, if there are a vector $p \in \mathbb{R}^n$, a constant $M > 0$ and a neighborhood N_x of x such that*

$$|\varphi(y) - \varphi(x) - p \cdot (y - x)| \leq M|y - x|^2 \quad \text{for } y \in N_x.$$

The definition implies that $D\varphi(x) = p$.

Definition 4. *A function $u \in BUC(\mathbb{R}^n)$ is a viscosity subsolution of (7) if for any test function $\varphi(x) \in C^{1,1}(x) \cap BUC(B_\delta(x))$ such that $u - \varphi$ has a maximum over $B_\delta(x)$ at $x \in \Omega$,*

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity supersolution of (7) if for any test function $\varphi \in C^{1,1}(x) \cap BUC(B_\delta(x))$ such that $u - \varphi$ has a minimum over $B_\delta(x)$ at $x \in \Omega$,

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \geq 0.$$

A function $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (7) if it is both a viscosity subsolution and viscosity supersolution of (7).

Proposition 1. *Let G be continuous and (3), (2), (8) hold. Then Definition 2 is equivalent to Definition 4.*

Proof. Obviously if u is a viscosity sub/supersolution in the sense of Definition 4, it is a viscosity sub/supersolution in the sense of Definition 2. Assume now that u is a viscosity subsolution in the sense of Definition 2. Let $\varphi \in C^{1,1}(x) \cap BUC(B_\delta(x))$ and $u - \varphi$ have a maximum over $B_\delta(x)$ at x . Then $I^{1,\delta}[x, D\varphi(x), \varphi], I^{2,\delta}[x, D\varphi(x), u]$ are well defined. Also because φ is $C^{1,1}(x)$, there exist a sequence of $C^2(B_\delta(x))$ functions $\{\varphi_n\}_n$ and a positive constant C such that $\varphi - \varphi_n$ has a maximum point at x over $B_\delta(x)$, $\varphi_n \geq \varphi$, $\varphi_n \rightarrow \varphi$ uniformly in $B_\delta(x)$ and $|\varphi_n(x+z) - \varphi_n(x) - D\varphi_n(x) \cdot z| \leq C|z|^2$. Thus $u - \varphi_n$ has a maximum at x over $B_\delta(x)$ and $D\varphi(x) = D\varphi_n(x)$. Therefore, by Definition 2,

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi_n] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

Letting $n \rightarrow +\infty$ and using the Lebesgue Dominated Convergence Theorem we thus conclude

$$G(x, u(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

□

Definition 5 (see [41]). *Given $u, \psi \in BUC(\mathbb{R}^n)$, $\epsilon > 0$, the ψ -sup-convolution $u^{\psi,\epsilon}$ of u is defined by*

$$u^{\psi,\epsilon}(x) := (u - \psi)^\epsilon(x) + \psi(x) = \sup_{y \in \mathbb{R}^n} \left\{ u(y) - \psi(y) - \frac{|x - y|^2}{2\epsilon} \right\} + \psi(x),$$

and the ψ -inf-convolution $u_{\psi,\epsilon}$ of u is defined by

$$u_{\psi,\epsilon}(x) := (u - \psi)_\epsilon(x) + \psi(x) = \inf_{y \in \mathbb{R}^n} \left\{ u(y) - \psi(y) + \frac{|x - y|^2}{2\epsilon} \right\} + \psi(x).$$

Remark 4. *The functions $u^{0,\epsilon}$ and $u_{0,\epsilon}$ are the usual sup- and inf-convolutions of u respectively, and we will denote them by u^ϵ and u_ϵ (see [22]).*

Remark 5. *$u^{\psi_{\alpha,\epsilon}}(x), u_{\psi_{\alpha,\epsilon}}(x) \rightarrow u(x)$ uniformly for $x \in \mathbb{R}^n$ and $\alpha \in \mathcal{A}$ as $\epsilon \rightarrow 0$ if the functions $\{\psi_\alpha\}_{\alpha \in \mathcal{A}} \subset BUC(\mathbb{R}^n)$ have a uniform modulus of continuity.*

Lemma 3.4.1. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (7) is continuous and $G(x, \cdot, l)$ is uniformly continuous, uniformly for $x \in \Omega, l \in \mathbb{R}$. Assume moreover that G satisfies (2) with $\gamma = 0$ and (H1), and the family of Lévy measures*

$\{\mu_x\}$ satisfies assumption (H2). Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1 - r\}$, $\Omega' \subset\subset \Omega$ is an open set, $u \in C^r(\Omega)$ is a viscosity subsolution of (7), then there are a sequence of $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ functions $\{\psi_n\}_n$ with a uniform modulus of continuity, a sequence of positive numbers $\{\epsilon_n\}_n$ with $\epsilon_n \rightarrow 0$, and a modulus ρ such that u^{ψ_n, ϵ_n} is a viscosity subsolution of

$$G(x, u^{\psi_n, \epsilon_n}, I[x, u^{\psi_n, \epsilon_n}]) = \rho\left(\frac{1}{n}\right) \quad \text{in } \Omega'. \quad (57)$$

Proof. Case 1: $0 < \sigma \leq 1$.

As in the proof of Theorem 3.2.1, without loss of generality, we can assume that $0 < r < 1$. For any $\hat{x} \in \Omega'$ and $B_{\hat{\delta}}(\hat{x}) \subset \Omega'$, suppose that there is a test function $\varphi \in C^2(B_{\hat{\delta}}(\hat{x}))$ such that $u^\epsilon - \varphi$ has a maximum (equal 0) at \hat{x} over $B_{\hat{\delta}}(\hat{x})$. Since $u \in BUC(\mathbb{R}^n)$, there exists a point $\hat{y} \in \Omega'$ such that $u^\epsilon(\hat{x}) = u(\hat{y}) - \frac{|\hat{x} - \hat{y}|^2}{2\epsilon}$ if ϵ is sufficiently small. Thus $u(y) - \frac{1}{2\epsilon}|x - y|^2 - \varphi(x)$ has a maximum at (\hat{y}, \hat{x}) over $\mathbb{R}^n \times B_{\hat{\delta}}(\hat{x})$ and $u(\hat{y}) \geq u^\epsilon(\hat{x})$. Therefore, we have

$$\frac{|\hat{x} - \hat{y}|^{2-r}}{2\epsilon} \leq C$$

for some $C > 0$ independent of ϵ . Notice that u^ϵ is semi-convex, which implies that there is a paraboloid touching its graph from below at \hat{x} . Since $\varphi \in C^2(B_{\hat{\delta}}(\hat{x}))$ touches the graph of u^ϵ from above at \hat{x} , we get $u^\epsilon \in C^{1,1}(\hat{x}) \cap BUC(\mathbb{R}^n)$. For any $0 < \delta < \min\{\hat{\delta}, 1\}$ and small $\epsilon > 0$, we have by (2) and (H1),

$$\begin{aligned} & G(\hat{x}, u^\epsilon(\hat{x}), I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]) \\ & - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, u\right]\right) \\ & \leq G(\hat{x}, u(\hat{y}), I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]) \\ & - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}\left[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, u\right]\right) \\ & \leq \Lambda_{\Omega'} \left\{ I^{1,\delta}\left[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, \frac{|\hat{x} - \cdot|^2}{2\epsilon}\right] + I^{2,\delta}\left[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, u\right] \right. \\ & \quad \left. - (I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]) \right\} + w_{\Omega'}(|\hat{x} - \hat{y}|) \\ & \leq \Lambda_{\Omega'} \left\{ \int_{|z| < \delta} \left[\frac{1}{2\epsilon} |\hat{x} - \hat{y} - z|^2 - \frac{1}{2\epsilon} |\hat{x} - \hat{y}|^2 - \frac{1}{\epsilon} (\hat{y} - \hat{x}) \cdot z \right] \mu_{\hat{y}}(dz) \right. \\ & \quad - \int_{|z| < \delta} \left[u^\epsilon(\hat{x} + z) - u^\epsilon(\hat{x}) - \frac{1}{\epsilon} (\hat{y} - \hat{x}) \cdot z \right] \mu_{\hat{x}}(dz) \\ & \quad \left. + \int_{|z| \geq \delta} \left[u(\hat{y} + z) - u(\hat{y}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon} (\hat{y} - \hat{x}) \cdot z \right] \mu_{\hat{y}}(dz) \right\} \end{aligned}$$

$$- \int_{|z| \geq \delta} \left[u^\epsilon(\hat{x} + z) - u^\epsilon(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon} (\hat{y} - \hat{x}) \cdot z \right] \mu_{\hat{x}}(dz) \Bigg\} + w_{\Omega'}(|\hat{x} - \hat{y}|). \quad (58)$$

Since $\frac{\hat{y} - \hat{x}}{\epsilon} = Du^\epsilon(\hat{x})$ and $u^\epsilon(z) + \frac{|z|^2}{2\epsilon}$ is convex, we have

$$-\frac{|z|^2}{2\epsilon} \leq u^\epsilon(\hat{x} + z) - u^\epsilon(\hat{x}) - \frac{1}{\epsilon} (\hat{y} - \hat{x}) \cdot z. \quad (59)$$

Thus, by (58) and (59),

$$\begin{aligned} & G(\hat{x}, u^\epsilon(\hat{x}), I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]) \\ & - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, \frac{|\hat{x} - \cdot|^2}{2\epsilon}] + I^{2,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, u]\right) \\ \leq & \Lambda_{\Omega'} \left\{ \int_{|z| < \delta} \frac{1}{2\epsilon} |z|^2 (\mu_{\hat{y}}(dz) + \mu_{\hat{x}}(dz)) \right. \\ & + \int_{|z| \geq \delta} \left[u(\hat{y} + z) - u(\hat{y}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon} (\hat{y} - \hat{x}) \cdot z \right] (\mu_{\hat{y}}(dz) - \mu_{\hat{x}}(dz)) \\ & \left. + \int_{|z| \geq \delta} \left[u(\hat{y} + z) - u(\hat{y}) - (u^\epsilon(\hat{x} + z) - u^\epsilon(\hat{x})) \right] \mu_{\hat{x}}(dz) \right\} + w_{\Omega'}(|\hat{x} - \hat{y}|). \end{aligned} \quad (60)$$

By the definition of u^ϵ , we have

$$u^\epsilon(\hat{x} + z) \geq u(\hat{y} + z) - \frac{|\hat{x} - \hat{y}|^2}{2\epsilon},$$

which implies

$$u^\epsilon(\hat{x} + z) - u^\epsilon(\hat{x}) \geq u(\hat{y} + z) - u(\hat{y}). \quad (61)$$

Thus, by (60) and (61), it follows

$$\begin{aligned} & G(\hat{x}, u^\epsilon(\hat{x}), I^{1,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon] + I^{2,\delta}[\hat{x}, Du^\epsilon(\hat{x}), u^\epsilon]) \\ & - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, \frac{|\hat{x} - \cdot|^2}{2\epsilon}] + I^{2,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon}, u]\right) \\ \leq & \Lambda_{\Omega'} \left\{ \int_{|z| < \delta} \frac{1}{2\epsilon} |z|^2 (\mu_{\hat{y}}(dz) + \mu_{\hat{x}}(dz)) \right. \\ & + \int_{|z| \geq \delta} \left[u(\hat{y} + z) - u(\hat{y}) - \mathbb{1}_{B_1(0)}(z) \frac{1}{\epsilon} (\hat{y} - \hat{x}) \cdot z \right] (\mu_{\hat{y}}(dz) - \mu_{\hat{x}}(dz)) \Bigg\} \\ & + w_{\Omega'}(|\hat{x} - \hat{y}|). \end{aligned} \quad (62)$$

We now let $\delta_n = n^{-\alpha}$ and $\epsilon_n = n^{-\beta}$, and use the same estimates as in Case 1 of the proof of Theorem 3.2.1 to show that, we can find $\alpha > 0$, $\beta > 0$, and $0 < r_0 < \sigma$ such that, if $r_0 < r < 1$ and $\theta > 1 - r$, then

$$G(\hat{x}, u^{\epsilon_n}(\hat{x}), I^{1, \delta_n}[\hat{x}, Du^{\epsilon_n}(\hat{x}), u^{\epsilon_n}] + I^{2, \delta_n}[\hat{x}, Du^{\epsilon_n}(\hat{x}), u^{\epsilon_n}]) \\ - G\left(\hat{y}, u(\hat{y}), I^{1, \delta_n}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon_n}, \frac{|\hat{x} - \cdot|^2}{2\epsilon_n}] + I^{2, \delta_n}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon_n}, u]\right) \leq \rho\left(\frac{1}{n}\right)$$

for some modulus ρ . Since u is a viscosity subsolution of (7), this implies

$$G(\hat{x}, u^{\epsilon_n}(\hat{x}), I[\hat{x}, u^{\epsilon_n}]) = G(\hat{x}, u^{\epsilon_n}(\hat{x}), I^{1, \delta_n}[\hat{x}, Du^{\epsilon_n}(\hat{x}), u^{\epsilon_n}] + I^{2, \delta_n}[\hat{x}, Du^{\epsilon_n}(\hat{x}), u^{\epsilon_n}]) \\ \leq \rho\left(\frac{1}{n}\right).$$

Case 2: $1 < \sigma < 2$.

We take $r > 1$. Let $\{\psi_n\}_n$ be a sequence of $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ functions which are uniformly bounded and have a uniform (in n) modulus of continuity h , which satisfy (44) and (45) with K replaced by Ω' .

Let $\hat{x} \in \Omega'$, $B_{\hat{\delta}}(\hat{x}) \subset \Omega'$, and suppose that there is a test function $\varphi \in C^2(B_{\hat{\delta}}(\hat{x}))$ such that $u^{\psi_n, \epsilon} - \varphi$ has a maximum (equal 0) at \hat{x} over $B_{\hat{\delta}}(\hat{x})$. Since $u \in BUC(\mathbb{R}^n)$ and $\psi_n \in BUC(\mathbb{R}^n)$, there exists a point $\hat{y} \in \Omega'$ such that $u^{\psi_n, \epsilon}(\hat{x}) = u(\hat{y}) - \psi_n(\hat{y}) + \psi_n(\hat{x}) - \frac{|\hat{x} - \hat{y}|^2}{2\epsilon}$ if ϵ is sufficiently small. Thus $u(y) - \psi_n(y) + \psi_n(x) - \frac{|x - y|^2}{2\epsilon} - \varphi(x)$ has a maximum at (\hat{y}, \hat{x}) over $\mathbb{R}^n \times B_{\hat{\delta}}(\hat{x})$ and $u(\hat{y}) - \psi_n(\hat{y}) + \psi_n(\hat{x}) \geq u^{\psi_n, \epsilon}(\hat{x})$. Since $u(\cdot) - \psi_n(\cdot) - \frac{|\hat{x} - \cdot|^2}{2\epsilon}$ has a maximum at \hat{y} over \mathbb{R}^n , we have

$$Du(\hat{y}) - D\psi_n(\hat{y}) = \frac{\hat{y} - \hat{x}}{\epsilon}.$$

Thus, by (45),

$$\frac{|\hat{y} - \hat{x}|}{\epsilon} \leq Cn^{1-r}.$$

Since $u^{\psi_n, \epsilon}$ is semi-convex, there is a paraboloid touching its graph from below at \hat{x} . Since $\varphi \in C^2(B_{\hat{\delta}}(\hat{x}))$ touches the graph of $u^{\psi_n, \epsilon}$ from above at \hat{x} , we obtain that $u^{\psi_n, \epsilon} \in C^{1,1}(\hat{x}) \cap BUC(\mathbb{R}^n)$. Thus, for any $0 < \delta < \min\{\hat{\delta}, 1\}$ and small $\epsilon > 0$, we have, by (2), (H1), (45), uniform continuity of the ψ_n and the continuity properties of G ,

$$G\left(\hat{x}, u^{\psi_n, \epsilon}(\hat{x}), I^{1, \delta}[\hat{x}, Du^{\psi_n, \epsilon}(\hat{x}), u^{\psi_n, \epsilon}] + I^{2, \delta}[\hat{x}, Du^{\psi_n, \epsilon}(\hat{x}), u^{\psi_n, \epsilon}]\right) \\ - G\left(\hat{y}, u(\hat{y}), I^{1, \delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x} - \cdot|^2}{2\epsilon} + \psi_n] + I^{2, \delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), u]\right) \\ \leq G\left(\hat{x}, u(\hat{y}) - \psi_n(\hat{y}) + \psi_n(\hat{x}), I^{1, \delta}[\hat{x}, Du^{\psi_n, \epsilon}(\hat{x}), u^{\psi_n, \epsilon}] + I^{2, \delta}[\hat{x}, Du^{\psi_n, \epsilon}(\hat{x}), u^{\psi_n, \epsilon}]\right) \\ - G\left(\hat{y}, u(\hat{y}), I^{1, \delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x} - \cdot|^2}{2\epsilon} + \psi_n] + I^{2, \delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), u]\right)$$

$$\begin{aligned}
&\leq G\left(\hat{x}, u(\hat{y}), I^{1,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] + I^{2,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}]\right) + \rho_1\left(\frac{1}{n}\right) \\
&\quad - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x} - \cdot|^2}{2\epsilon} + \psi_n] + I^{2,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), u]\right) \\
&\leq \Lambda_{\Omega'} \left\{ I^{1,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x} - \cdot|^2}{2\epsilon} + \psi_n] + I^{2,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), u] \right. \\
&\quad \left. - \left(I^{1,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] + I^{2,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] \right) \right\} + \rho_1\left(\frac{1}{n}\right) \\
&\leq \Lambda_{\Omega'} \left\{ \int_{|z|<\delta} \left[\left(\frac{1}{2\epsilon} |\hat{x} - \hat{y} - z|^2 + \psi_n(\hat{y} + z) \right) - \left(\frac{1}{2\epsilon} |\hat{x} - \hat{y}|^2 + \psi_n(\hat{y}) \right) \right. \right. \\
&\quad \left. \left. - \left(\frac{1}{\epsilon} (\hat{y} - \hat{x}) + D\psi_n(\hat{y}) \right) \cdot z \right] \mu_{\hat{y}}(dz) \right. \\
&\quad - \int_{|z|<\delta} \left[u^{\psi_n,\epsilon}(\hat{x} + z) - u^{\psi_n,\epsilon}(\hat{x}) - \left(\frac{1}{\epsilon} (\hat{y} - \hat{x}) + D\psi_n(\hat{x}) \right) \cdot z \right] \mu_{\hat{x}}(dz) \\
&\quad + \int_{|z|\geq\delta} \left[u(\hat{y} + z) - u(\hat{y}) - \mathbb{1}_{B_1(0)}(z) \left(\frac{1}{\epsilon} (\hat{y} - \hat{x}) + D\psi_n(\hat{y}) \right) \cdot z \right] \mu_{\hat{y}}(dz) \\
&\quad \left. - \int_{|z|\geq\delta} \left[u^{\psi_n,\epsilon}(\hat{x} + z) - u^{\psi_n,\epsilon}(\hat{x}) - \mathbb{1}_{B_1(0)}(z) \left(\frac{1}{\epsilon} (\hat{y} - \hat{x}) + D\psi_n(\hat{x}) \right) \cdot z \right] \mu_{\hat{x}}(dz) \right\} \\
&\quad + \rho_1\left(\frac{1}{n}\right) \tag{63}
\end{aligned}$$

for some modulus ρ_1 independent of δ, ϵ .

Since $\frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{x}) = Du^{\psi_n,\epsilon}(\hat{x})$ and $u^{\psi_n,\epsilon}(z) + \frac{|z|^2}{2\epsilon} + (\sup_{\Omega'} |D^2\psi_n|)|z|^2$ is convex on $B_{\hat{\delta}}(\hat{x})$, we have for $|z| < \hat{\delta}$

$$-\frac{|z|^2}{2\epsilon} - (\sup_{\Omega'} |D^2\psi_n|)|z|^2 \leq u^{\psi_n,\epsilon}(\hat{x} + z) - u^{\psi_n,\epsilon}(\hat{x}) - \left(\frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{x}) \right) \cdot z. \tag{64}$$

Moreover, by the definition of $u^{\psi_n,\epsilon}$,

$$u^{\psi_n,\epsilon}(\hat{x} + z) \geq u(\hat{y} + z) - \psi_n(\hat{y} + z) + \psi_n(\hat{x} + z) - \frac{|\hat{x} - \hat{y}|^2}{2\epsilon},$$

which gives

$$u(\hat{y} + z) - u(\hat{y}) - (u^{\psi_n,\epsilon}(\hat{x} + z) - u^{\psi_n,\epsilon}(\hat{x})) \leq \psi_n(\hat{y} + z) - \psi_n(\hat{y}) - (\psi_n(\hat{x} + z) - \psi_n(\hat{x})). \tag{65}$$

Thus, by (63), (64) and (65), we have

$$\begin{aligned}
&G\left(\hat{x}, u^{\psi_n,\epsilon}(\hat{x}), I^{1,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}] + I^{2,\delta}[\hat{x}, Du^{\psi_n,\epsilon}(\hat{x}), u^{\psi_n,\epsilon}]\right) \\
&\quad - G\left(\hat{y}, u(\hat{y}), I^{1,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), \frac{|\hat{x} - \cdot|^2}{2\epsilon} + \psi_n] + I^{2,\delta}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon} + D\psi_n(\hat{y}), u]\right) \\
&\leq \Lambda_{\Omega'} \left\{ \int_{|z|<\delta} \left[\frac{1}{2\epsilon} |z|^2 + (\psi_n(\hat{y} + z) - \psi_n(\hat{y}) - D\psi_n(\hat{y}) \cdot z) \right] \mu_{\hat{y}}(dz) \right. \\
&\quad \left. + \int_{|z|<\delta} \left[\frac{1}{2\epsilon} |z|^2 + (\sup_{\Omega'} |D^2\psi_n|)|z|^2 \right] \mu_{\hat{x}}(dz) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \int_{|z| \geq \delta} \left[u(\hat{y} + z) - u(\hat{y}) - \mathbb{1}_{B_1(0)}(z) \left(\frac{1}{\epsilon} (\hat{y} - \hat{x}) + D\psi_n(\hat{y}) \right) \cdot z \right] (\mu_{\hat{y}}(dz) - \mu_{\hat{x}}(dz)) \\
& + \int_{|z| \geq \delta} \left[\psi_n(\hat{y} + z) - \psi_n(\hat{y}) - \mathbb{1}_{B_1(0)}(z) D\psi_n(\hat{y}) \cdot z \right. \\
& \quad \left. - (\psi_n(\hat{x} + z) - \psi_n(\hat{x}) - \mathbb{1}_{B_1(0)}(z) D\psi_n(\hat{x}) \cdot z) \right] \mu_{\hat{x}}(dz) \Big\} + \rho_1 \left(\frac{1}{n} \right).
\end{aligned}$$

We now again set $\delta_n = n^{-\alpha}$ and $\epsilon_n = n^{-\beta}$ and use the same estimates as these in Case 2 of the proof of Theorem 3.2.1, to obtain that for any $\theta > 0$, we can find $\alpha > 0$, $\beta > 0$, and $1 \leq r_0 < \sigma$ such that, if $r_0 < r < 2$, then

$$\begin{aligned}
& G(\hat{x}, u^{\psi_n, \epsilon_n}(\hat{x}), I^{1, \delta_n}[\hat{x}, Du^{\psi_n, \epsilon_n}(\hat{x}), u^{\psi_n, \epsilon_n}] + I^{2, \delta_n}[\hat{x}, Du^{\psi_n, \epsilon_n}(\hat{x}), u^{\psi_n, \epsilon_n}]) \\
& - G(\hat{y}, u(\hat{y}), I^{1, \delta_n}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon_n} + D\psi_n(\hat{y}), \frac{|\hat{x} - \cdot|^2}{2\epsilon_n} + \psi_n] + I^{2, \delta_n}[\hat{y}, \frac{\hat{y} - \hat{x}}{\epsilon_n} + D\psi_n(\hat{y}), u]) \\
& \leq \rho \left(\frac{1}{n} \right)
\end{aligned}$$

for some modulus ρ . Since u is a viscosity subsolution of (7), this implies

$$\begin{aligned}
& G(\hat{x}, u^{\psi_n, \epsilon_n}(\hat{x}), I[\hat{x}, u^{\psi_n, \epsilon_n}]) \\
& = G(\hat{x}, u^{\psi_n, \epsilon_n}(\hat{x}), I^{1, \delta_n}[\hat{x}, Du^{\psi_n, \epsilon_n}(\hat{x}), u^{\psi_n, \epsilon_n}] + I^{2, \delta_n}[\hat{x}, Du^{\psi_n, \epsilon_n}(\hat{x}), u^{\psi_n, \epsilon_n}]) \leq \rho \left(\frac{1}{n} \right).
\end{aligned}$$

□

The same proof gives the following result for viscosity supersolutions.

Lemma 3.4.2. *Suppose that the assumptions of Lemma 3.4.1 are true. Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1 - r\}$, $\Omega' \subset\subset \Omega$ is an open set, $u \in C^r(\Omega)$ is a viscosity supersolution of (7), then there are a sequence of $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ functions $\{\tilde{\psi}_n\}_n$ with a uniform modulus of continuity, a sequence of positive numbers $\{\tilde{\epsilon}_n\}_n$ with $\tilde{\epsilon}_n \rightarrow 0$, and a modulus $\tilde{\rho}$ such that $u^{\tilde{\psi}_n, \tilde{\epsilon}_n}$ is a viscosity supersolution of*

$$G\left(x, u^{\tilde{\psi}_n, \tilde{\epsilon}_n}, I[x, u^{\tilde{\psi}_n, \tilde{\epsilon}_n}]\right) = -\tilde{\rho}\left(\frac{1}{n}\right) \quad \text{in } \Omega'. \quad (66)$$

We remark that it is clear from the proofs of Lemmas 3.4.1 and 3.4.2 that we can always have $\epsilon_n = \tilde{\epsilon}_n$.

The next lemma is standard and can be deduced from Lemmas 4.2 and 4.5 of [12].

Lemma 3.4.3. *Let $\{u_n\}_n$ be a sequence of bounded and uniformly continuous functions on \mathbb{R}^n such that:*

- (i) u_n is a viscosity subsolution of $M_{\mathcal{L}}^+(u_n) = -f_n$ in Ω .
- (ii) The sequence $\{u_n\}$ converges to u uniformly in \mathbb{R}^n for some $u \in BUC(\mathbb{R}^n)$.
- (iii) The sequence $\{f_n\}$ converges to f uniformly in Ω for some $f \in C(\Omega)$.

Then u is a viscosity subsolution of $M_{\mathcal{L}}^+(u) = -f$ in Ω .

Theorem 3.4.4. *Let the assumptions of Lemma 3.4.1 be satisfied and let G be uniformly elliptic with respect to \mathcal{L} . Then, for any $0 < \sigma < 2$, there exists a constant $0 \leq r_0 < \sigma$ ($r_0 \geq 1$ if $\sigma > 1$) such that if $r_0 < r < 2$, $\theta > \max\{0, 1 - r\}$, $u \in C^r(\Omega)$ is a viscosity subsolution of (7) and $v \in C^r(\Omega)$ is a viscosity supersolution of (7), then $u - v$ is a viscosity subsolution of*

$$M_{\mathcal{L}}^-(v - u) = 0 \quad (67)$$

in $\Omega \cap \{u - v > 0\}$. If $G(x, r, l)$ is independent of the second variable r , then (67) holds in Ω .

Proof. For any $\Omega' \subset\subset \Omega$, let $x \in \Omega'$, $u^{\psi_n, \epsilon_n}(x) > v_{\tilde{\psi}_n, \epsilon_n}(x)$, and let φ be a $C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ be a test function touching the graph of $u^{\psi_n, \epsilon_n} - v_{\tilde{\psi}_n, \epsilon_n}$ from above at x . Since u^{ψ_n, ϵ_n} and $-v_{\tilde{\psi}_n, \epsilon_n}$ are semi-convex in a neighborhood of x , each of them has a paraboloid touching its graph from below at x . Therefore, u^{ψ_n, ϵ_n} and $-v_{\tilde{\psi}_n, \epsilon_n}$ must be in $C^{1,1}(x) \cap BUC(\mathbb{R}^n)$. Thus, by Proposition 1 and Lemmas 3.4.1 and 3.4.2, we have

$$G(x, u^{\psi_n, \epsilon_n}(x), I[x, u^{\psi_n, \epsilon_n}]) \leq \rho\left(\frac{1}{n}\right)$$

and

$$G(x, v_{\tilde{\psi}_n, \epsilon_n}(x), I[x, v_{\tilde{\psi}_n, \epsilon_n}]) \geq -\rho\left(\frac{1}{n}\right)$$

for some modulus ρ . Thus, by (2) and the uniform ellipticity, we obtain

$$M_{\mathcal{L}}^-(v_{\tilde{\psi}_n, \epsilon_n} - u^{\psi_n, \epsilon_n})(x) \leq 2\rho\left(\frac{1}{n}\right).$$

Thus, we have

$$M_{\mathcal{L}}^-\varphi(x) \leq 2\rho\left(\frac{1}{n}\right).$$

Therefore, we have proved that $u^{\psi_n, \epsilon_n} - v_{\tilde{\psi}_n, \epsilon_n}$ is a viscosity subsolution of

$$M_{\mathcal{L}}^-(v_{\tilde{\psi}_n, \epsilon_n} - u^{\psi_n, \epsilon_n}) = 2\rho\left(\frac{1}{n}\right)$$

in $\Omega' \cap \{u^{\psi_n, \epsilon_n} - v_{\tilde{\psi}_n, \epsilon_n} > 0\}$.

By Remark 5, we have that $u^{\psi_n, \epsilon_n} - v_{\tilde{\psi}_n, \epsilon_n}$ converges uniformly to $u - v$ in \mathbb{R}^n . Thus, for any $\epsilon > 0$, there exists a sufficiently large n_ϵ such that $\Omega' \cap \{u - v > \epsilon\} \subset \Omega' \cap \{u^{\psi_n, \epsilon_n} - v_{\tilde{\psi}_n, \epsilon_n} > 0\}$ if $n > n_\epsilon$. Therefore, $u^{\psi_n, \epsilon_n} - v_{\tilde{\psi}_n, \epsilon_n}$ is a viscosity subsolution of $M_{\mathcal{L}}^-(v_{\tilde{\psi}_n, \epsilon_n} - u^{\psi_n, \epsilon_n}) = 2\rho(\frac{1}{n})$ in $\Omega' \cap \{u - v > \epsilon\}$ if $n > n_\epsilon$, and hence, by Lemma 3.4.3, $u - v$ is a viscosity subsolution of $M_{\mathcal{L}}^-(v - u) = 0$ in $\Omega' \cap \{u - v > \epsilon\}$. Since $\Omega' \subset\subset \Omega$ and $\epsilon > 0$ are arbitrary, $u - v$ is a viscosity subsolution of $M_{\mathcal{L}}^-(v - u) = 0$ in $\Omega \cap \{u - v > 0\}$. \square

Remark 6. *Theorem 3.4.4, combined with an Alexandrov-Bakelman-Pucci estimate of [34], can be used as an alternative way to prove comparison theorem when $\gamma = 0$, at least for some class of equations which are independent of the u variable.*

3.5 Regularity

In this section we recall some regularity results for nonlocal equations. We first recall regularity results proved in [6] and [8]. Here, we only state their simplified versions applicable for our equations, which can be deduced from the results and techniques of [6, 8]. The full theorems of [6] and [8] are much more general. An equivalent of Theorem 3.5.2 has not been stated in [6, 8] but it can be deduced easily from the proofs there. We impose here an additional requirement $\theta > \max\{0, 1 - \sigma\}$. It is possible that Theorems 3.5.1 and 3.5.2 are true without this assumption but it would require some more substantial changes in the proofs on [6, 8].

Theorem 3.5.1. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (7) is continuous and satisfies (2) with $\gamma = 0$ and (H1) with $\Lambda_{\Omega'} > 0$ for each $\Omega' \subset\subset \Omega$. Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2) with $\theta > \max\{0, 1 - \sigma\}$ and, there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathbb{S}^{n-1}$, $\eta \in (0, 1)$, $\delta \in (0, 1)$,*

$$\int_{\{z: |z| \leq \delta, |d \cdot z| \geq (1-\eta)|z|\}} |z|^2 \mu_x(dz) \geq C \eta^{\frac{n-1}{2}} \delta^{2-\sigma}. \quad (68)$$

Then, we have:

- (1) *If $0 < \sigma \leq 1$, any viscosity solution u of (7) is $C^r(\Omega)$ for any $r < \sigma$.*
- (2) *If $1 < \sigma$, any viscosity solution u of (7) is $C^{0,1}(\Omega)$.*

Theorem 3.5.2. *Let Ω be a bounded domain. Suppose that $\gamma \geq 0$ in (9), the family of Lévy measures $\{\mu_x^{\alpha\beta}\}$ satisfies assumption (H2) with $\theta > \max\{0, 1 - \sigma\}$, uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, and $f_{\alpha\beta}$ are uniformly continuous in Ω , uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$. Suppose that there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathbb{S}^{n-1}$, $\eta \in (0, 1)$, $\delta \in (0, 1)$, $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$,*

$$\int_{\{z: |z| \leq \delta, |d \cdot z| \geq (1-\eta)|z|\}} |z|^2 \mu_x^{\alpha\beta}(dz) \geq C \eta^{\frac{n-1}{2}} \delta^{2-\sigma}.$$

Then, we have:

- (1) *If $0 < \sigma \leq 1$, any viscosity solution u of (9) is $C^r(\Omega)$ for any $r < \sigma$.*
- (2) *If $1 < \sigma$, any viscosity solution u of (9) is $C^{0,1}(\Omega)$.*

Let us now introduce some definitions and regularity theorems from [13, 48, 70]. Consider the following nonlocal equations

$$\gamma u - I[x, u] = f(x) \quad \text{in } \Omega, \quad (69)$$

where $\gamma \geq 0$, Ω is a bounded domain, f is bounded and continuous in Ω , and $I[x, u]$ is a nonlocal operator of the form

$$I[x, u] = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} I_{\alpha\beta}[x, u] := \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] K_{\alpha\beta}(x, z) dz.$$

We will denote

$$I_{\alpha\beta,x_0}[x, u] := \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x) \cdot z] K_{\alpha\beta}(x_0, z) dz.$$

Remark 7. *It is easy to see that if $K_{\alpha\beta}(x, z) = \frac{a_{\alpha\beta}(x, z)}{|z|^{n+\sigma}}$, $\lambda \leq a_{\alpha\beta}(x, z) \leq \Lambda$ and $|a_{\alpha\beta}(x_1, z) - a_{\alpha\beta}(x_2, z)| \leq h(|x_1 - x_2|)$ for some modulus h for any $x, x_1, x_2 \in \Omega, z \in \mathbb{R}^n, \alpha \in \mathcal{A}, \beta \in \mathcal{B}$, then the nonlocal operator $I[x, u]$ satisfies the following properties:*

- (1) $I[x, u]$ is well defined as long as $u \in C^{1,1}(x)$ and $u \in L^1(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma}})$.
- (2) If $u \in C^2(\Omega) \cap L^1(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma}})$, then $I(x, u)$ is continuous in Ω as a function of x .

Thus $I[x, u]$ falls into the class of nonlocal operators considered in [13, 48, 70] which was a little more general. Moreover the definition of viscosity sub/supersolutions in [13, 48, 70] was slightly different from Definition 2 as they allowed viscosity sub/supersolutions to be unbounded (as long as they are in the domain of definition of the nonlocal operator I) and they did not required them to be uniformly continuous.

We say that the nonlocal operator I above is uniformly elliptic with respect to a class \mathcal{L} of linear nonlocal operators if

$$M_{\mathcal{L}}^-(u - v)(x) \leq I[x, u] - I[x, v] \leq M_{\mathcal{L}}^+(u - v)(x).$$

The norm $\|I\|$ of a nonlocal operator I is defined in the following way.

Definition 6.

$$\|I\| : = \sup \left\{ \frac{|I[x, u]|}{1 + M} : x \in \Omega, u \in C^{1,1}(x), \|u\|_{L^1(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma}})} \leq M, \right. \\ \left. |u(x+z) - u(x) - Du(x) \cdot z| \leq M|z|^2 \text{ for any } z \in B_1(0) \right\}.$$

The following classes of linear nonlocal operators $\mathcal{L}_0(\lambda, \Lambda, \sigma)$ and $\mathcal{L}_\kappa(\lambda, \Lambda, \sigma), 0 < \kappa \leq 2$ were introduced in [13, 70]. Let $0 < \lambda \leq \Lambda$ be fixed constants. A linear nonlocal operator $L \in \mathcal{L}_0(\lambda, \Lambda, \sigma)$ if

$$Lu = \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z)Du(x) \cdot z] K(z) dz, \quad (70)$$

where the kernel K is symmetric and satisfies for all $z \in \mathbb{R}^n \setminus \{0\}$

$$(2 - \sigma) \frac{\lambda}{|z|^{n+\sigma}} \leq K(z) \leq (2 - \sigma) \frac{\Lambda}{|z|^{n+\sigma}}. \quad (71)$$

Since K is symmetric, we have

$$Lu = \int_{\mathbb{R}^n} [u(x+z) + u(x-z) - 2u(x)] K(z) dz.$$

Lemma 3.5.3. *The class $\mathcal{L}_0(\lambda, \Lambda, \sigma)$ satisfies (H3) for any $0 < \sigma < 2$.*

Proof. We will be using the form of L in (70). Let R be such that $R^{\frac{3}{2}} > \max\{3R, 1 + R\}$ and $\Omega \subset B_R(0)$. We define $\varphi(x) = \min(R^3, |x|^2)$ (see Assumption 5.1 in [12]). By the definition of R , the fact that K is symmetric, we now have for every $x \in \Omega \subset B_R(0)$

$$\begin{aligned} L\varphi(x) &\geq \int_{B_1(0)} |z|^2 K(z) dz + \int_{1 \leq |z| < R^{\frac{3}{2}} - R} (|z|^2 + 2x \cdot z) K(z) dz \\ &\quad + \int_{\{\varphi(x+z) < R^3\} \cap \{|z| \geq R^{\frac{3}{2}} - R\}} (|z|^2 - 2|x||z|) K(z) dz \\ &\quad + \int_{\{\varphi(x+z) = R^3\} \cap \{|z| \geq R^{\frac{3}{2}} - R\}} (R^3 - R^2) K(z) dz \\ &\geq (2 - \sigma) \lambda \int_{B_1(0)} |z|^{-n-\sigma+2} dz. \end{aligned}$$

□

The class $\mathcal{L}_\kappa(\lambda, \Lambda, \sigma)$ is a subclass of $\mathcal{L}_0(\lambda, \Lambda, \sigma)$ of kernels K such that

$$[K]_{C^\kappa(B_\rho)} \leq \Lambda(2 - \sigma) \rho^{-n-\sigma-\kappa} \quad \text{if } B_{2\rho} \subset \mathbb{R}^n \setminus \{0\}$$

for any balls $B_\rho, B_{2\rho}$ of radii $\rho, 2\rho > 0$. We notice that the classes $\mathcal{L}_0(\lambda, \Lambda, \sigma)$ and $\mathcal{L}_\kappa(\lambda, \Lambda, \sigma)$ have scale σ . A class $\mathcal{L} \subset \mathcal{L}_0(\lambda, \Lambda, \sigma)$ has scale σ if whenever a nonlocal operator with kernel $K(z)$ is in \mathcal{L} , then the one with kernel $\nu^{n+\sigma} K(\nu z)$ is also in \mathcal{L} for any $\nu < 1$. The following definition of a distance between two nonlocal operators takes scaling of order σ into account.

Definition 7. *For any $0 < \sigma < 2$ and any nonlocal operator I , we define the rescaled operator*

$$I_{\mu, \nu}[x, u] = \nu^\sigma \mu I[\nu x, \mu^{-1} u(\nu^{-1} \cdot)].$$

The norm of scale σ is defined as

$$\|I^{(1)} - I^{(2)}\|_\sigma = \sup_{\nu < 1} \|I_{1, \nu}^{(1)} - I_{1, \nu}^{(2)}\|.$$

The following regularity theorems for nonlocal equations were proved in [13, 48, 70]. We only state their simplified versions which are suitable for our purposes.

Theorem 3.5.4 (Theorem 2.6 of [13]). *Assume that $0 < \sigma_0 < \sigma < 2$. Let u solve*

$$M_{\mathcal{L}_0}^+ u \geq -C_0 \quad \text{in } B_1(0),$$

$$M_{\mathcal{L}_0}^- u \leq C_0 \quad \text{in } B_1(0)$$

in the viscosity sense for some $C_0 \geq 0$. Then there exists a constant $0 < r < 1$, depending only on λ , Λ , n and σ_0 , such that $u \in C^r(B_{\frac{1}{2}}(0))$ and

$$\|u\|_{C^r(B_{\frac{1}{2}}(0))} \leq C \left(\|u\|_{L^\infty(B_1(0))} + \|u\|_{L^1(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma_0}})} + C_0 \right)$$

for some constant $C > 0$ which depends on σ_0 , λ , Λ and n .

Theorem 3.5.5 (Theorem 4.1 of [48]). Assume $1 < \sigma_0 < \sigma < 2$. Let

$$I = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} I_{\alpha\beta}$$

be a nonlocal operator such that $\{I_{\alpha\beta, x_0} : \alpha \in \mathcal{A}, \beta \in \mathcal{B}, x_0 \in B_1(0)\} \subset \mathcal{L}_0(\lambda, \Lambda, \sigma)$. Denote $I_{x_0} = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} I_{\alpha\beta, x_0}$. There exist constants $r > 1, \eta > 0$ such that if for any $x_0 \in B_{\frac{1}{2}}(0)$,

$$\|I - I_{x_0}\|_\sigma < \eta,$$

and u is a viscosity solution of

$$I[x, u] = f(x) \quad \text{in } B_1(0)$$

for some bounded continuous function f , then $u \in C^r(B_{\frac{1}{2}}(0))$ and

$$\|u\|_{C^r(B_{\frac{1}{2}}(0))} \leq C \left(\|u\|_{L^\infty(B_1(0))} + \|u\|_{L^1(\mathbb{R}^n, \frac{1}{1+|z|^{n+\sigma_0}})} + \|f\|_{L^\infty(B_1(0))} \right)$$

for some absolute constant $C > 0$.

Theorem 3.5.6 (Theorem 1.2 and Remark 1.3 of [70]). Let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be a class of linear nonlocal operators

$$I_\alpha[x, u] = \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] K_\alpha(x, z) dz$$

such that $\{I_{\alpha, x_0} : \alpha \in \mathcal{A}, x_0 \in B_1(0)\} \subset \mathcal{L}_\kappa(\lambda, \Lambda, \sigma)$ for some $\kappa > 0$ and $0 < \sigma < 2$. Suppose that for all $x_1, x_2 \in B_1(0), z \in \mathbb{R}^n \setminus \{0\}, \alpha \in \mathcal{A}$,

$$|K_\alpha(x_1, z) - K_\alpha(x_2, z)| \leq |x_1 - x_2|^\theta \frac{\Lambda(2-\sigma)}{|z|^{n+\sigma}}.$$

Then there exists $\bar{r} > 0$ such that if $\kappa \in (0, \bar{r}]$, $\theta \in (0, \kappa)$ and u is a viscosity solution of

$$I[x, u] = \inf_{\alpha \in \mathcal{A}} I_\alpha[x, u] = 0 \quad \text{in } B_1(0),$$

then $u \in C^{\sigma+\theta}(B_{\frac{1}{2}}(0))$ and

$$\|u\|_{C^{\sigma+\theta}(B_{\frac{1}{2}}(0))} \leq C \|u\|_{L^\infty(\mathbb{R}^n)}$$

for some absolute constant $C > 0$.

Theorem 3.5.7 (Theorem 5.2 of [13]). Assume $1 < \sigma_0 < \sigma < 2$. Let $I^0 = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} I_{\alpha\beta}^0$ be a nonlocal operator such that $\{I_{\alpha\beta}^0\}_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \subset \mathcal{L}$, where $\mathcal{L} \subset \mathcal{L}_0(\lambda, \Lambda, \sigma)$ has scale σ and interior $C^{\bar{r}}$ estimates for some $\bar{r} > 1$. Let

$$I = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} I_{\alpha\beta}$$

be a nonlocal operator uniformly elliptic with respect to $\mathcal{L}_0(\lambda, \Lambda, \sigma)$. Then for every $r < \min\{\bar{r}, \sigma_0\}$ there is $\eta > 0$ such that if

$$\|I^0 - I\|_{\sigma} < \eta$$

and u is a viscosity solution of

$$I[x, u] = f(x) \quad \text{in } B_1(0)$$

for some bounded and continuous function f , then $u \in C^r(B_{\frac{1}{2}}(0))$ and

$$\|u\|_{C^r(B_{\frac{1}{2}}(0))} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1(0))})$$

for some absolute constant $C > 0$.

Corollary 5. Let $0 < \sigma < 2$ and let u be a viscosity solution of (69) in $B_1(0)$, where $\gamma \geq 0$, $f \in C(\overline{B_1(0)})$ and $I[x, u] = \inf_{\alpha \in \mathcal{A}} (2 - \sigma) \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] \frac{a_\alpha(x, z)}{|z|^{n+\sigma}} dz$. Assume that $a_\alpha(x, \cdot)$ is symmetric, $\lambda \leq a_\alpha(x, z) \leq \Lambda$, $\frac{a_\alpha(x, \cdot)}{|\cdot|^{n+\sigma}} \in \mathcal{L}_\kappa(\lambda, \Lambda, \sigma)$ and $|a_\alpha(x_1, z) - a_\alpha(x_2, z)| \leq C|x_1 - x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $x, x_1, x_2 \in B_1(0)$, $z \in \mathbb{R}^n \setminus \{0\}$, and some constants $\kappa > 0, \theta > \max\{0, 1 - \sigma\}$. Then, for any $r < \sigma$, $u \in C^r(B_{\frac{1}{2}}(0))$.

Proof. For $0 < \sigma \leq 1$, since $\lambda \leq a_\alpha(x, z) \leq \Lambda$ for any $x \in B_1(0)$ and $z \in \mathbb{R}^n$, it follows that the family of Lévy measures $\{\frac{a_\alpha(x, z)}{|z|^{n+\sigma}} dz\}_{x, \alpha}$ satisfies (68) (see Example 1 in [6]). Thus, by Theorem 3.5.2, the proof is complete for the case $0 < \sigma \leq 1$.

For $\sigma > 1$, if we fix $x_0 \in B_{\frac{1}{2}}(0)$, then the operator $I_{\alpha, x_0} u = (2 - \sigma) \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] \frac{a_\alpha(x_0, z)}{|z|^{n+\sigma}} dz$ is in $\mathcal{L}_\kappa(\lambda, \Lambda, \sigma)$. Thus, by Theorem 3.5.6, it has interior $C^{\bar{r}}$ estimates for some $\bar{r} > \sigma$. By the Hölder continuity of $a_\alpha(\cdot, z)$ for fixed $z \in \mathbb{R}^n \setminus \{0\}$, we can find a small ball $B_{r_0}(x_0)$ such that $|a_\alpha(x, z) - a_\alpha(x_0, z)| < \eta$. Thus, by a simple calculation (see the proof of Theorem 6.1 in [13]), we can derive that $\|I - I_{x_0}\|_{\sigma} < C\eta$ in $B_{r_0}(x_0)$ where C is a positive constant and $I_{x_0} = \inf_{\alpha \in \mathcal{A}} I_{\alpha, x_0}$. Finally, we apply Theorem 3.5.7 with $I^0 = I_{x_0}$ and $f := f - \gamma u$, scaled in $B_{r_0}(x_0)$. \square

Corollary 6. Let $0 < \sigma < 2$. Let u be a viscosity solution of

$$\gamma u - \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{I_{\alpha\beta}[x, u]\} = f(x) \quad \text{in } B_1(0),$$

where $\gamma \geq 0$, $f \in C(\overline{B_1(0)})$ and $I_{\alpha\beta}[x, u] = (2-\sigma) \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] \frac{a_{\alpha\beta}(x, z)}{|z|^{n+\sigma}} dz$. Assume that $a_{\alpha\beta}(x, \cdot)$ is symmetric, $\lambda \leq a_{\alpha\beta}(x, z) \leq \Lambda$, and $|a_{\alpha\beta}(x_1, z) - a_{\alpha\beta}(x_2, z)| \leq |x_1 - x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $x, x_1, x_2 \in B_1(0)$, $z \in \mathbb{R}^n \setminus \{0\}$ and some constant $\theta > \max\{0, 1 - \sigma\}$. Then, if $\sigma > 1$, $u \in C^r(B_{\frac{1}{2}}(0))$, where r is from Theorem 3.5.5, and if $\sigma \leq 1$, $u \in C^r(B_{\frac{1}{2}}(0))$ for every $r < \sigma$.

Proof. For $0 < \sigma \leq 1$, the proof is the same as for Corollary 5. For $\sigma > 1$, by the Hölder continuity of $a_{\alpha\beta}(\cdot, z)$ for fixed $z \in \mathbb{R}^n \setminus \{0\}$, we can find a small ball $B_{r_0}(x_0)$ such that $|a_{\alpha\beta}(x, z) - a_{\alpha\beta}(x_0, z)| < \eta$. Thus, like in the proof of Corollary 5, we can obtain $\|I - I_{x_0}\|_\sigma < C\eta$ in $B_{r_0}(x_0)$ for some constant $C > 0$. We then apply Theorem 3.5.5 with $f := f - \gamma u$, scaled in $B_{r_0}(x_0)$. \square

3.6 Applications

In this section, we provide several concrete applications when we have uniqueness of viscosity solutions.

3.6.1 Nonlinear convex equations with variable coefficients

Theorem 3.6.1. *Let Ω be a bounded domain. Consider the following nonlinear non-local equations*

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \{-I_\alpha[x, u]\} = f(x) \quad \text{in } \Omega, \quad (72)$$

where $\gamma \geq 0$, $0 < \sigma < 2$, $f \in C(\overline{\Omega})$ and $I_\alpha[x, u] = (2 - \sigma) \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] \frac{a_\alpha(x, z)}{|z|^{n+\sigma}} dz$. Assume that $a_\alpha(x, \cdot)$ is symmetric, $\lambda \leq a_\alpha(x, z) \leq \Lambda$, $\frac{a_\alpha(x, \cdot)}{|\cdot|^{n+\sigma}} \in \mathcal{L}_\kappa(\lambda, \Lambda, \sigma)$ and $|a_\alpha(x_1, z) - a_\alpha(x_2, z)| \leq C|x_1 - x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $x, x_1, x_2 \in \Omega$, $z \in \mathbb{R}^n \setminus \{0\}$ and some $\kappa > 0, \theta > 0$. Suppose that $\theta > \max\{0, 1 - \sigma\}$. Then, if u is a viscosity solution of (72), v is a viscosity supersolution (respectively, subsolution) of (72) and $u \leq v$ (respectively, $u \geq v$) in Ω^c , we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .

Proof. The theorem follows from Theorem 3.3.2, Corollary 5 and Lemma 3.5.3 since we can take r arbitrarily close to σ . \square

3.6.2 Nonlinear non-convex equations with variable coefficients

Theorem 3.6.2. *Let Ω be a bounded domain. Consider the following nonlinear non-local equations*

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-I_{\alpha\beta}[x, u]\} = f(x) \quad \text{in } \Omega, \quad (73)$$

where $\gamma \geq 0$, $0 < \sigma < 2$, $f \in C(\overline{\Omega})$ and $I_{\alpha\beta}[x, u] = (2 - \sigma) \int_{\mathbb{R}^n} [u(x+z) - u(x) - \mathbb{1}_{B_1(0)}(z) Du(x) \cdot z] \frac{a_{\alpha\beta}(x, z)}{|z|^{n+\sigma}} dz$. Assume that $a_{\alpha\beta}(x, \cdot)$ is symmetric, $\lambda \leq a_{\alpha\beta}(x, z) \leq \Lambda$

and $|a_{\alpha\beta}(x_1, z) - a_{\alpha\beta}(x_2, z)| \leq C|x_1 - x_2|^\theta$ for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $x, x_1, x_2 \in \Omega$ and $z \in \mathbb{R}^n \setminus \{0\}$. Then, if u is a viscosity solution of (73), v is a viscosity supersolution (respectively, subsolution) of (73) and $u \leq v$ (respectively, $u \geq v$) in Ω^c , we have:

- (i) For $0 < \sigma \leq 1$, if $\theta > 1 - \sigma$, we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .
- (ii) For $1 < \sigma < 2$, if $\sigma < 2 - 2\frac{(2-r)^2}{\theta(3-r)+(4-2r)}$, where $r < 2$ is given by Corollary 6, we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .

Proof. The theorem follows from Theorem 3.3.2, Corollary 4, Lemma 3.5.3, and Corollary 6. \square

3.6.3 General nonlocal uniformly elliptic equations with respect to \mathcal{L}_0

Theorem 3.6.3. *Let Ω be a bounded domain and $1 \geq \sigma > 0$. Suppose that the nonlinearity G in (7) is continuous and uniformly elliptic with respect to \mathcal{L}_0 , and satisfies (2) with $\gamma \geq 0$ and (H1). Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2). Suppose that u is a viscosity solution of (7), v is a viscosity supersolution (respectively, subsolution) of (7) and $u \leq v$ (respectively, $u \geq v$) in Ω^c . Then, if $\sigma < \frac{\theta(2-r)}{2-r+\theta} + r$ and $\theta > 1 - r$, where $r < 1$ is given by Theorem 3.5.4, we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .*

Proof. The theorem follows from Corollary 3(i), Theorem 3.5.4 and Lemma 3.5.3. \square

3.6.4 General nonlocal equations with a family of Lévy measures satisfying (68)

Theorem 3.6.4. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (7) is continuous and satisfies (2) with $\gamma > 0$ and (H1) with $\Lambda_{\Omega'} > 0$ for each $\Omega' \subset\subset \Omega$. Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2), and there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathcal{S}^{n-1}$, $\eta, \delta \in (0, 1)$, we have (68). If u is a viscosity solution of (7), v is a viscosity supersolution (respectively, subsolution) of (7) and $u \leq v$ (respectively, $u \geq v$) in Ω^c , then:*

- (i) For $0 < \sigma \leq 1$, if $\theta > 1 - \sigma$, we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .
- (ii) For $1 < \sigma < 2$, if $0 < \theta \leq 1$ and $\sigma < 2 - \frac{1}{1+\theta}$, we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .

Proof. The theorem follows from Theorem 3.2.1, Corollary 1 and Theorem 3.5.1. \square

Theorem 3.6.5. *Let Ω be a bounded domain. Suppose that the nonlinearity G in (7) is continuous and uniformly elliptic with respect to \mathcal{L}_0 , and satisfies (2) with $\gamma = 0$ and (H1) with $\Lambda_{\Omega'} > 0$ for each $\Omega' \subset\subset \Omega$. Suppose that the family of Lévy measures $\{\mu_x\}$ satisfies assumption (H2) and, there exists a constant $C > 0$ such that, for any $x \in \Omega$, $d \in \mathcal{S}^{n-1}$, $\eta, \delta \in (0, 1)$, we have (68). If u is a viscosity solution of (7), v*

is a viscosity supersolution (respectively, subsolution) of (7) and $u \leq v$ (respectively, $u \geq v$) in Ω^c , then:

- (i) For $0 < \sigma \leq 1$, if $\theta > 1 - \sigma$, we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .
- (ii) For $1 < \sigma < 2$, if $0 < \theta \leq 1$ and $\sigma < 2 - \frac{1}{1+\theta}$, we have $u \leq v$ (respectively, $u \geq v$) in \mathbb{R}^n .

Proof. This theorem follows from Theorem 3.3.1, Corollary 3, Theorem 3.5.1 and Lemma 3.5.3. □

CHAPTER IV

PERRON'S METHOD FOR INTEGRO-DIFFERENTIAL EQUATIONS

In this chapter, we will study existence of viscosity solutions for the following two classes of integro-differential equations.

$$\begin{cases} G(x, u, I[x, u]) = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega^c, \end{cases} \quad (74)$$

and

$$\begin{cases} \gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-I_{\alpha\beta}[x, u] + f_{\alpha\beta}(x)\} = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega^c. \end{cases} \quad (75)$$

where Ω is a bounded domain, $I[x, u]$ and $I_{\alpha\beta}[x, u]$ are of Lévy type, g is a bounded continuous function in \mathbb{R}^n .

4.1 Notation and definitions

We will use the following notations: if u is a function on Ω , then, for any $x \in \Omega$,

$$u^*(x) = \limsup_{r \rightarrow 0} \{u(y); y \in \Omega \text{ and } |y - x| \leq r\},$$

$$u_*(x) = \liminf_{r \rightarrow 0} \{u(y); y \in \Omega \text{ and } |y - x| \leq r\}.$$

The function u^* is called the upper semicontinuous envelope of u and u_* is called the lower semicontinuous envelope of u . The following notion of a discontinuous viscosity solution of (74) will be used in this chapter.

Definition 8. *A bounded function u is a discontinuous viscosity subsolution of (74) if u^* is a viscosity subsolution of $G = 0$ and $u^* \leq g$ in Ω^c . A bounded function u is a discontinuous viscosity supersolution of (74) if u_* is a viscosity supersolution of $G = 0$ and $u_* \geq g$ in Ω^c . A function u is a discontinuous viscosity solution of (74) if it is both a discontinuous viscosity subsolution and discontinuous viscosity supersolution of (74).*

Remark 8. *If u is a discontinuous viscosity solution of (74) and u is continuous in \mathbb{R}^n , then u is a viscosity solution of (74).*

4.2 Perron's method

In this section, we discuss Perron's method for discontinuous viscosity solutions of (74).

Lemma 4.2.1. *Suppose that the nonlinearity G in (74) is continuous and satisfies (2), (3), (8). Let \mathcal{F} be a family of viscosity subsolutions of $G = 0$ in Ω . Let $w(x) = \sup\{u(x) : u \in \mathcal{F}\}$ in \mathbb{R}^n and assume that $w^*(x) < \infty$ for $x \in \mathbb{R}^n$. Then w is a discontinuous viscosity subsolution of $G = 0$ in Ω .*

Proof. Suppose that φ is a $C_b^2(\mathbb{R}^n)$ function such that $w^* - \varphi$ has a strict maximum (equal 0) at $x_0 \in \Omega$ over \mathbb{R}^n . We can construct a uniformly bounded sequence of $C^2(\mathbb{R}^n)$ functions $\{\varphi_m\}_m$ such that $\varphi_m = \varphi$ in $B_1(x_0)$, $\varphi \leq \varphi_m$ in \mathbb{R}^n , $\sup_{x \in B_2^c(x_0)} \{w^*(x) - \varphi_m(x)\} \leq -\frac{1}{m}$ and $\varphi_m \rightarrow \varphi$ pointwise. Thus, for any positive integer m , $w^* - \varphi_m$ has a strict maximum (equal 0) at x_0 over \mathbb{R}^n . Therefore, $\sup_{x \in B_1^c(x_0)} \{w^*(x) - \varphi_m(x)\} = \epsilon_m < 0$. By the definition of w^* , we have, for any $u \in \mathcal{F}$, $\sup_{x \in B_1^c(x_0)} \{u(x) - \varphi_m(x)\} \leq \epsilon_m < 0$. Again, by the definition of w^* , we have, for any $\epsilon_m < \epsilon < 0$, there exist $u_\epsilon \in \mathcal{F}$ and $\bar{x}_\epsilon \in B_1(x_0)$ such that $u_\epsilon(\bar{x}_\epsilon) - \varphi(\bar{x}_\epsilon) > \epsilon$. Since $u_\epsilon \in USC(\mathbb{R}^n)$ and $\varphi_m \in C_b^2(\mathbb{R}^n)$, there exists $x_\epsilon \in B_1(x_0)$ such that $u_\epsilon(x_\epsilon) - \varphi_m(x_\epsilon) = \sup_{x \in \mathbb{R}^n} \{u_\epsilon(x) - \varphi(x)\} \geq u_\epsilon(\bar{x}_\epsilon) - \varphi_m(\bar{x}_\epsilon) > \epsilon$. Since $w^* - \varphi_m$ attains a strict maximum (equal 0) at x_0 over \mathbb{R}^n and $u \leq w^*$ for any $u \in \mathcal{F}$, then $u_\epsilon(x_\epsilon) \rightarrow w^*(x_0)$ and $x_\epsilon \rightarrow x_0$ as $\epsilon \rightarrow 0^-$. Since u_ϵ is a viscosity subsolution of $G = 0$, we have

$$G(x_\epsilon, u_\epsilon(x_\epsilon), I[x_\epsilon, \varphi_m]) \leq 0. \quad (76)$$

Since $\{x_\epsilon\}_\epsilon \subset B_1(x_0)$ and $x_\epsilon \rightarrow x_0$ as $\epsilon \rightarrow 0^-$, there exists a sufficiently small $\delta_0 > 0$ such that $B_{\delta_0}(x_\epsilon) \subset B_1(0)$ for any $\epsilon \in (-\delta_0, 0)$. By the choice of φ_m , we can rewrite (76) as

$$G(x_\epsilon, u_\epsilon(x_\epsilon), I^{1, \delta_0}[x_\epsilon, D\varphi(x_\epsilon), \varphi] + I^{2, \delta}[x_\epsilon, D\varphi(x_\epsilon), \varphi_m]) \leq 0. \quad (77)$$

Since $x_\epsilon \rightarrow x_0$, $u_\epsilon(x_\epsilon) \rightarrow w^*(x_0)$, $\varphi_m \rightarrow \varphi$ pointwise as $\epsilon \rightarrow 0^-$, $\varphi \in C_b^2(\mathbb{R}^n)$ and G is continuous, we have, letting $\epsilon \rightarrow 0^-$ in (77),

$$G(x_0, w^*(x_0), I[x_0, \varphi]) \leq 0.$$

Therefore, w^* is a discontinuous viscosity subsolution of $G = 0$. □

Theorem 4.2.2. *Suppose that the nonlinearity G in (74) is continuous and satisfies (2), (3), (8). Let \underline{u}, \bar{u} be bounded continuous functions and be respectively a viscosity subsolution and a viscosity supersolution of $G = 0$ in Ω . Assume moreover that $\bar{u} = \underline{u} = g$ in Ω^c and $\underline{u} \leq \bar{u}$ in \mathbb{R}^n . Then*

$$w(x) = \sup_{u \in \mathcal{F}} u(x),$$

where

$\mathcal{F} = \{u \in C^0(\mathbb{R}^n); \underline{u} \leq u \leq \bar{u} \text{ in } \mathbb{R}^n \text{ and } u \text{ is a viscosity subsolution of } G = 0 \text{ in } \Omega\}$,
is a discontinuous viscosity solution of (74).

Proof. Since $\underline{u} \in \mathcal{F}$, then $\mathcal{F} \neq \emptyset$. Thus, w is well defined, $\underline{u} \leq w \leq \bar{u}$ in \mathbb{R}^n and $w = \bar{u} = \underline{u}$ in Ω^c . By Lemma 4.2.1, w is a discontinuous viscosity subsolution of $G = 0$ in Ω . We claim that w is a discontinuous viscosity supersolution of $G = 0$ in Ω . If not, without loss of generality, we assume that $0 \in \Omega$ and there exists a function $\varphi \in C_b^2(\mathbb{R}^n)$ such that $w_* - \varphi$ has a strict minimum (equal 0) at point 0 over \mathbb{R}^n and

$$G(0, w_*(0), I[0, \varphi]) < -\epsilon_0,$$

where ϵ_0 is a positive constant. Thus, we can find sufficiently small constants $\epsilon_1 > 0$ and $\delta_0 > 0$ such that $B_{\delta_0}(0) \subset \Omega$ and there exists a $C_b^2(\mathbb{R}^n)$ function φ_{ϵ_1} satisfying that $\varphi_{\epsilon_1} = \varphi$ in $B_{\delta_0}(0)$, $\varphi_{\epsilon_1} \leq \varphi$ in \mathbb{R}^n , $\inf_{x \in B_{2\delta_0}^c(0)} \{w_*(x) - \varphi_{\epsilon_1}(x)\} \geq \epsilon_1 > 0$ and

$$G(0, \varphi_{\epsilon_1}(0), I[0, \varphi_{\epsilon_1}]) < -\frac{\epsilon_0}{2}. \quad (78)$$

Thus, by the Dominated Convergence Theorem, there exists $\delta_1 < \delta_0$ such that, for any $x \in B_{\delta_1}(0)$,

$$G(x, \varphi_{\epsilon_1}(x), I[x, \varphi_{\epsilon_1}]) < -\frac{\epsilon_0}{4}. \quad (79)$$

By the definition of w , we have $\varphi_{\epsilon_1} \leq w_* \leq \bar{u}$ in \mathbb{R}^n . If $\varphi_{\epsilon_1}(0) = w_*(0) = \bar{u}(0)$, then $\bar{u} - \varphi_{\epsilon_1}$ has a strict minimum at point 0 over \mathbb{R}^n . Since \bar{u} is a viscosity supersolution of $G = 0$ in Ω , we have

$$G(0, \varphi_{\epsilon_1}(0), I[0, \varphi_{\epsilon_1}]) \geq 0,$$

which contradicts with (78). Thus, we have $\varphi_{\epsilon_1}(0) < \bar{u}(0)$. Since \bar{u} and φ_{ϵ_1} are continuous function in \mathbb{R}^n , we have $\varphi_{\epsilon_1}(x) < \bar{u}(x) - \epsilon_2$ in $B_{\delta_2}(0)$ for some $0 < \delta_2 < \delta_1$ and $\epsilon_2 > 0$. We define

$$\Delta_r = \sup_{x \in B_r^c(0)} \{\varphi_{\epsilon_1}(x) - w_*(x)\}.$$

Since $\inf_{x \in B_{2\delta_0}^c(0)} \{w_*(x) - \varphi_{\epsilon_1}(x)\} \geq \epsilon_1 > 0$, $w_* - \varphi_{\epsilon_1}$ has a strict minimum (equal 0) at point 0 and $-w_* \in USC(\mathbb{R}^n)$, we have $\Delta_r < 0$ for each $r > 0$. For any $y \in \bar{\Omega} \setminus B_r(0)$, there exists a function $v_y \in \mathcal{F}$ such that $v_y(y) - \varphi_{\epsilon_1}(y) \geq -\frac{3\Delta_r}{4}$. Since v_y and φ_{ϵ_1} are continuous in \mathbb{R}^n , there exists a positive constant δ_y such that $\inf_{x \in B_{\delta_y}(y)} \{v_y(x) - \varphi_{\epsilon_1}(x)\} \geq -\frac{\Delta_r}{2}$. Since $\bar{\Omega} \setminus B_r(0)$ is a compact set in \mathbb{R}^n , there exists a finite set $\{y_i\}_{i=1}^{n_r} \subset \bar{\Omega} \setminus B_r(0)$ such that $\bar{\Omega} \setminus B_r(0) \subset \cup_{i=1}^{n_r} B_{\delta_{y_i}}(y_i)$. Thus, we define

$$v_r(x) = \sup_{1 \leq i \leq n_r} \{v_{y_i}(x)\}, \quad x \in \mathbb{R}^n.$$

By Lemma 4.2.1 and the definition of v_r , we have $v_r \in \mathcal{F}$ and $\inf_{x \in \bar{\Omega} \setminus B_r(0)} \{v_r(x) - \varphi_{\epsilon_1}(x)\} \geq -\frac{\Delta_r}{2}$. Let α_r be a constant such that $0 < \alpha_r < \frac{1}{2}$ and $-\alpha_r \Delta_r < \epsilon_2$. Thus, we define

$$U(x) = \begin{cases} \max\{\varphi_{\epsilon_1}(x) - \alpha \Delta_r, v_r(x)\}, & x \in B_r(0), \\ v_r(x), & x \in B_r^c(0), \end{cases}$$

where $0 < r < \delta_2$ and $0 < \alpha < \alpha_r$. By the definition of U , we obtain $U \in C^0(\mathbb{R}^n)$, $\underline{u} \leq U \leq \bar{u}$ in \mathbb{R}^n , and there exists a sequence $\{x_n\}_n \subset B_r(0)$ such that $x_n \rightarrow 0$ as $n \rightarrow +\infty$ and $U(x_n) > w(x_n)$.

We claim that U is a viscosity subsolution of $G = 0$ in Ω . For any $y \in \Omega$, suppose that there is a test function $\psi \in C_b^2(\mathbb{R}^n)$ such that $U - \psi$ has a maximum (equal 0) at y over \mathbb{R}^n . We then divide the proof into two cases.

Case 1: $U(y) = v_r(y)$.

Since $v_r \leq U \leq \psi$ in \mathbb{R}^n , then $v_r - \psi$ has a maximum (equal 0) at y over \mathbb{R}^n . We recall that v_r is a viscosity subsolution of $G = 0$ in Ω . Therefore, we have

$$G(y, U(y), I[y, \psi]) \leq 0.$$

Case 2: $U(y) = \varphi_{\epsilon_1}(y) - \alpha \Delta_r$.

We first notice that $y \in B_r(0)$. Since $\varphi_{\epsilon_1} - \alpha \Delta_r \leq U \leq \psi$ in $B_r(0)$, then $\varphi_{\epsilon_1} - \alpha \Delta_r - \psi \leq 0$ in $B_r(0)$. By the definition of U , we have $\psi \geq U = v_r$ in $B_r^c(0)$. Thus, $\varphi_{\epsilon_1} - \alpha \Delta_r - \psi \leq \varphi_{\epsilon_1} - \alpha \Delta_r - v_r \leq \frac{\Delta_r}{2} - \alpha \Delta_r \leq 0$ in $B_r^c(0)$. Therefore, we have $\varphi_{\epsilon_1} - \alpha \Delta_r - \psi$ has a maximum (equal 0) at $y \in B_r(0) \subset B_{\delta_1}(0)$ over \mathbb{R}^n . Since (79) holds and G is a continuous function, we can choose sufficiently small α independent of ψ such that

$$G(y, \psi(y), I[y, \psi]) \leq G(y, \varphi_{\epsilon_1}(y) - \alpha \Delta_r, I[y, \varphi_{\epsilon_1}]) \leq 0.$$

Based on the two cases, we have U is a viscosity subsolution of $G = 0$ in Ω . Therefore, $U \in \mathcal{F}$, which contradicts with the definition of w . Thus, w is a discontinuous viscosity supersolution of $G = 0$ in Ω . Therefore, w is a discontinuous viscosity solution of $G = 0$ in Ω . \square

Theorem 4.2.3. *Let \underline{u}, \bar{u} be bounded continuous functions and be respectively a viscosity subsolution and a viscosity supersolution of*

$$\gamma u + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \{-I_{\alpha\beta}[x, u] + f_{\alpha\beta}(x)\} = 0, \quad \text{in } \Omega, \quad (80)$$

where $\gamma \geq 0$, $f_{\alpha\beta}$ is a continuous function and $I_{\alpha\beta}[x, u]$ is of Lévy type. Assume moreover that $\bar{u} = \underline{u} = g$ in Ω^c and $\underline{u} \leq \bar{u}$ in \mathbb{R}^n . Then

$$w(x) = \sup_{u \in \mathcal{F}} u(x),$$

where

$\mathcal{F} = \{u \in C^0(\mathbb{R}^n); \underline{u} \leq u \leq \bar{u} \text{ in } \mathbb{R}^n \text{ and } u \text{ is a viscosity subsolution of (80) in } \Omega\}$,
is a discontinuous viscosity solution of (75).

4.3 Regularity

In this section we give Hölder estimates of the discontinuous viscosity solution constructed by Perron's method. As always we assume that G is continuous and (2), (3), (8) hold. To have Hölder estimates, we will assume that the nonlinearity G is uniformly elliptic with respect to the class of linear nonlocal operators $\mathcal{L}_0(\sigma, \lambda, \Lambda)$, where $0 < \sigma < 2$ and $0 < \lambda \leq \Lambda$, and $G(x, 0, 0)$ is bounded in \mathbb{R}^n .

The following lemma we borrow from [12] is crucial in our proof of the Hölder estimates.

Lemma 4.3.1. *Let $u \geq 0$ in \mathbb{R}^n and u is a viscosity supersolution of $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- u = C_0$ in $B_{2r}(0)$ for positive constants C_0 and r . Assume $\sigma \geq \sigma_0$ for some $\sigma_0 > 0$. Then*

$$|\{u > t\} \cap B_r(0)| \leq Cr^n(u(0) + C_0 r^\sigma)^\epsilon t^{-\epsilon} \quad \text{for any } t > 0,$$

where the positive constants ϵ and C depends on λ, Λ, n and σ_0 .

Theorem 4.3.2. *Assume that $\sigma > \sigma_0$ for some $\sigma_0 > 0$. Let \mathcal{F} be a class of bounded continuous functions in \mathbb{R}^n such that, for any $u \in \mathcal{F}$, we have $-\frac{1}{2} \leq u \leq \frac{1}{2}$ in \mathbb{R}^n , u is a viscosity subsolution of $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+ u = -\frac{\epsilon_0}{2}$ in $B_1(0)$ and $w = \sup_{u \in \mathcal{F}} u$ is a discontinuous viscosity supersolution of $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- w = \frac{\epsilon_0}{2}$ in $B_1(0)$ where ϵ_0 is a sufficiently small positive constant. Then there exist positive constants $\alpha > 0$ and $C > 0$ depending on λ, Λ, n and σ_0 such that*

$$-C|x|^\alpha \leq w_*(x) - w^*(0) \leq w^*(x) - w_*(0) \leq C|x|^\alpha.$$

Proof. We claim that there exist an increasing sequence $\{m_k\}_k$ and a decreasing sequence $\{M_k\}_k$ such that $M_k - m_k = 8^{-\alpha k}$ and $m_k \leq \inf_{B_{8^{-k}}(0)} w_* \leq \sup_{B_{8^{-k}}(0)} w^* \leq M_k$. We will prove this claim by induction.

For $k = 0$, we can choose $m_0 = -\frac{1}{2}$ and $M_0 = \frac{1}{2}$ since $-\frac{1}{2} \leq u \leq \frac{1}{2}$ for any $u \in \mathcal{F}$. Assume that we have the sequences up to m_k and M_k . In $B_{8^{-k-1}}(0)$, we have either

$$|\{w_* \geq \frac{M_k + m_k}{2}\} \cap B_{8^{-k-1}}(0)| \geq \frac{|B_{8^{-k-1}}(0)|}{2}, \quad (81)$$

or

$$|\{w_* \leq \frac{M_k + m_k}{2}\} \cap B_{8^{-k-1}}(0)| \geq \frac{|B_{8^{-k-1}}(0)|}{2}. \quad (82)$$

We first assume that (81) holds. We define

$$v(x) := \frac{w_*(8^{-k}x) - m_k}{\frac{M_k - m_k}{2}}.$$

Thus, $v \geq 0$ in $B_1(0)$ and

$$|\{v \geq 1\} \cap B_{\frac{1}{8}}(0)| \geq \frac{|B_{\frac{1}{8}}(0)|}{2}.$$

Since w is a discontinuous viscosity supersolution of $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- w = \frac{\epsilon_0}{2}$ in $B_1(0)$, then v is a viscosity supersolution of

$$M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v = \epsilon_0 \quad \text{in } B_{8^k}(0), \quad \text{if } \alpha < \sigma_0.$$

By the inductive assumption, we have, for any $k \geq j \geq 0$,

$$v \geq \frac{m_{k-j} - m_k}{\frac{M_k - m_k}{2}} \geq \frac{m_{k-j} - M_{k-j} + M_k - m_k}{\frac{M_k - m_k}{2}} = 2(1 - 8^{\alpha j}) \quad \text{in } B_{8^j}(0). \quad (83)$$

Moreover, we have

$$v \geq 2 \cdot 8^{\alpha k} \left[-\frac{1}{2} - \left(\frac{1}{2} - 8^{-\alpha k} \right) \right] = 2(1 - 8^{\alpha k}) \quad \text{in } B_{8^k}^c(0). \quad (84)$$

By (83) and (84), we have

$$v(x) \geq -2(|8x|^\alpha - 1), \quad \text{for any } x \in B_1^c(0).$$

For any $x \in B_{\frac{3}{4}}(0)$, we can choose sufficiently small $\alpha < \sigma_0$ such that

$$\begin{aligned} M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v^+(x) &\leq M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v(x) + M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+ v^-(x) \\ &\leq M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v(x) - \Lambda(2 - \sigma) \int_{\mathbb{R}^n \setminus \{v(x+y) < 0\}} \frac{v(x+y)}{|y|^{n+\sigma}} dy \\ &\leq M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v(x) - \Lambda(2 - \sigma) \int_{B_{\frac{1}{4}}^c(0)} \frac{\min\{-2(|8(x+y)|^\alpha - 1), 0\}}{|y|^{n+\sigma}} dy \\ &\leq M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v(x) + \epsilon_0 \leq 2\epsilon_0. \end{aligned}$$

where $v^+(x) := \max\{v(x), 0\}$ and $v^-(x) := -\min\{v(x), 0\}$. Given any point $x \in B_{\frac{1}{8}}(0)$, we can apply Lemma 4.3.1 in $B_{\frac{1}{4}}(x)$ to obtain

$$C(v^+(x) + 2\epsilon_0)^\epsilon \geq |\{v^+ > 1\} \cap B_{\frac{1}{4}}(x)| \geq |\{v^+ > 1\} \cap B_{\frac{1}{8}}(0)| \geq \frac{|B_{\frac{1}{8}}(0)|}{2}.$$

Thus, we can choose sufficiently small ϵ_0 such that $v^+ \geq \theta$ in $B_{\frac{1}{8}}(0)$ for some $\theta > 0$. Therefore,

$$v(x) = \frac{w_*(8^{-k}x) - m_k}{\frac{M_k - m_k}{2}} \geq \theta \quad \text{in } B_{\frac{1}{8}}(0).$$

If we set $m_{k+1} = m_k + \theta \frac{M_k - m_k}{2}$ and $M_{k+1} = M_k$, we must have $m_{k+1} \leq \inf_{B_{8^{-k-1}}(0)} w_* \leq \sup_{B_{8^{-k-1}}(0)} w^* \leq M_{k+1}$. Moreover, $M_{k+1} - m_{k+1} = (1 - \frac{\theta}{2})8^{-\alpha k}$. Therefore, we can choose α and θ sufficiently small such that $(1 - \frac{\theta}{2}) = 8^{-\alpha}$. Then we have $M_{k+1} - m_{k+1} = 8^{-\alpha(k+1)}$.

We then assume that (82) holds. For any $u \in \mathcal{F}$, we obtain that $u \in C^0(\mathbb{R}^n)$ is a viscosity subsolution of $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+ u = -\frac{\epsilon_0}{2}$ in $B_1(0)$ and $u \leq w_*$ in \mathbb{R}^n . Thus, we have

$$|\{u \leq \frac{M_k + m_k}{2}\} \cap B_{8^{-k-1}}(0)| \geq \frac{|B_{8^{-k-1}}(0)|}{2}.$$

We define

$$v_u(x) := \frac{M_k - u(8^{-k}x)}{\frac{M_k - m_k}{2}}.$$

Thus, $v_u \geq 0$ in $B_1(0)$ and

$$|\{v_u \geq 1\} \cap B_{\frac{1}{8}}(0)| \geq \frac{|B_{\frac{1}{8}}(0)|}{2}.$$

Since u is a viscosity subsolution of $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+ u = -\frac{\epsilon_0}{2}$ in $B_1(0)$, then v_u is a viscosity supersolution of

$$M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v_u \leq \epsilon_0 \quad \text{in } B_{8^k}(0), \quad \text{if } \alpha < \sigma_0.$$

By the inductive assumption, we have, for any $k \geq j \geq 0$,

$$v_u \geq \frac{M_k - M_{k-j}}{\frac{M_k - m_k}{2}} \geq \frac{M_k - m_k + m_{k-j} - M_{k-j}}{\frac{M_k - m_k}{2}} = 2(1 - 8^{\alpha j}) \quad \text{in } B_{8^j}(0). \quad (85)$$

Moreover, we have

$$v_u \geq 2 \cdot 8^{\alpha k} \left(-\frac{1}{2} + 8^{-\alpha k} - \frac{1}{2}\right) = 2(1 - 8^{\alpha k}) \quad \text{in } B_{8^k}^c(0). \quad (86)$$

By (85) and (86), we have

$$v_u(x) \geq -2(|8x|^\alpha - 1), \quad \text{for any } x \in B_1^c(0).$$

For any $x \in B_{\frac{3}{4}}(0)$, we can choose sufficiently small $\alpha < \sigma_0$ such that

$$M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^- v_u^+(x) \leq 2\epsilon_0, \quad (87)$$

where $v_u^+(x) := \max\{v_u(x), 0\}$. Given any point $x \in B_{\frac{1}{8}}(0)$, we can apply Lemma 4.3.1 in $B_{\frac{1}{4}}(x)$ to obtain

$$C(v_u^+(x) + 2\epsilon_0)^\epsilon \geq |\{v_u^+ > 1\} \cap B_{\frac{1}{4}}(x)| \geq |\{v_u^+ > 1\} \cap B_{\frac{1}{8}}(0)| \geq \frac{|B_{\frac{1}{8}}(0)|}{2}.$$

Thus, we can choose sufficiently small ϵ_0 such that $v_u^+ \geq \theta$ in $B_{\frac{1}{8}}(0)$ for some $\theta > 0$. Therefore,

$$v_u(x) = \frac{M_k - u(8^{-k}x)}{\frac{M_k - m_k}{2}} \geq \theta \quad \text{in } B_{\frac{1}{8}}(0),$$

which implies

$$u(8^{-k}x) \leq M_k - \theta \frac{M_k - m_k}{2} \quad \text{in } B_{\frac{1}{8}}(0).$$

By the definition of w , we have

$$w^*(8^{-k}x) \leq M_k - \theta \frac{M_k - m_k}{2} \quad \text{in } B_{\frac{1}{8}}(0).$$

If we set $m_{k+1} = m_k$ and $M_{k+1} = M_k - \theta \frac{M_k - m_k}{2}$, we must have $m_{k+1} \leq \inf_{B_{8^{-k-1}}(0)} w_* \leq \sup_{B_{8^{-k-1}}(0)} w^* \leq M_{k+1}$. Moreover, $M_{k+1} - m_{k+1} = (1 - \frac{\theta}{2})8^{-\alpha k}$. Therefore, we can choose α and θ sufficiently small such that $(1 - \frac{\theta}{2}) = 8^{-\alpha}$. Then we have $M_{k+1} - m_{k+1} = 8^{-\alpha(k+1)}$. \square

Corollary 7. Assume that $\sigma > \sigma_0$ for some $\sigma_0 > 0$ and $G(x, 0, 0)$ is bounded in \mathbb{R}^n . Assume that G is uniformly elliptic with respect to $\mathcal{L}_0(\sigma, \lambda, \Lambda)$. Let u be the bounded discontinuous viscosity solution of $G = 0$ in Ω constructed in Theorem 4.2.2. Then there exists a positive constant $\alpha > 0$ depending on λ, Λ, n and σ_0 such that $u \in C^\alpha(\Omega)$.

Corollary 8. Assume that $\{f_{\alpha\beta}\}_{\alpha,\beta}$ is a set of uniformly continuous and bounded functions in Ω , $\gamma \geq 0$ and $I_{\alpha\beta}$ is of Lévy type and uniformly elliptic with respect to $L_0(\lambda, \Lambda, \sigma)$ for some $2 > \sigma > \sigma_0 > 0$. Let u be the bounded discontinuous viscosity solution of (80) constructed in Theorem 4.2.3. Then there exists a positive constant $\alpha > 0$ depending on λ, Λ, n and σ_0 such that $u \in C^\alpha(\Omega)$.

4.4 A sub/supersolution and existence of a solution

In this section we construct a subsolution and a supersolution that are needed in proving the existence of a viscosity solution by Perron's method. For the construction, we first follow the ideas in [65] to construct a class of barrier functions. We define $v_\alpha(x) = ((x_1 - 1)^+)^alpha$ where $x_1 = x \cdot e_1$.

Lemma 4.4.1. Given any $\sigma \in (0, 2)$, there exists a sufficiently small $\alpha > 0$ such that $M_{L_0(\lambda, \Lambda, \sigma)}^+ v_\alpha((1+r)e_1) = -\epsilon_0 r^{\alpha-\sigma}$ for any $r > 0$ where ϵ_0 is some positive constant.

Proof. For any $\alpha > 0$ and $r > 0$,

$$\begin{aligned} & M_{L_0(\lambda, \Lambda, \sigma)}^+ v_\alpha((1+r)e_1) \\ &= (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda(\delta v_\alpha((1+r)e_1), y)^+ - \lambda(\delta v_\alpha((1+r)e_1), y)^-}{|y|^{n+\sigma}} dy \end{aligned}$$

$$\begin{aligned}
&= (2 - \sigma) \left[\int_{\mathbb{R}^n} \frac{\Lambda(((r + y_1)^+)^{\alpha} + ((r - y_1)^+)^{\alpha} - 2r^{\alpha})^+}{|y|^{n+\sigma}} dy \right. \\
&\quad \left. - \int_{\mathbb{R}^n} \frac{\lambda(((r + y_1)^+)^{\alpha} + ((r - y_1)^+)^{\alpha} - 2r^{\alpha})^-}{|y|^{n+\sigma}} dy \right] \\
&= (2 - \sigma)r^{\alpha-\sigma} \left[\int_{\mathbb{R}^n} \frac{\Lambda(((1 + y_1)^+)^{\alpha} + ((1 - y_1)^+)^{\alpha} - 2)^+}{|y|^{n+\sigma}} dy \right. \\
&\quad \left. - \int_{\mathbb{R}^n} \frac{\lambda(((1 + y_1)^+)^{\alpha} + ((1 - y_1)^+)^{\alpha} - 2)^-}{|y|^{n+\sigma}} dy \right].
\end{aligned}$$

By the Dominated Convergence Theorem, we have

$$\lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R}^n} \frac{(((1 + y_1)^+)^{\alpha} + ((1 - y_1)^+)^{\alpha} - 2)^+}{|y|^{n+\sigma}} dy = 0$$

and

$$\lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R}^n} \frac{(((1 + y_1)^+)^{\alpha} + ((1 - y_1)^+)^{\alpha} - 2)^-}{|y|^{n+\sigma}} dy > \int_{y_1 < -1} \frac{1}{|y|^{n+\sigma}} dy > 0.$$

Therefore, for some sufficiently small fixed α , there exists a positive constant $\epsilon_0 > 0$ such that

$$M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+ v_{\alpha}((1 + r)e_1) \leq -\epsilon_0 r^{\alpha-\sigma}, \quad \text{for any } r > 0.$$

□

Lemma 4.4.2. *Assume that $\sigma \in (0, 2)$. Then there are $\alpha > 0$ and $r_0 > 0$ sufficiently small so that the function $u_{\alpha}(x) = (|x| - 1)^{\alpha}$ satisfies $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+ u_{\alpha} \leq -1$ in $\bar{B}_{1+r_0}(0) \setminus \bar{B}_1(0)$.*

Proof. We notice that u_{α} and $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+$ are rotation-invariant. Then we only need to prove that $M_{\mathcal{L}_0(\lambda, \Lambda, \sigma)}^+ u_{\alpha}((1 + s)e_1) \leq -1$ for any $s \in (0, r_0]$ where $\alpha > 0$ and $r_0 > 0$ are sufficiently small. Note that, $\forall s > 0$, $u_{\alpha}((1 + s)e_1) = v_{\alpha}((1 + s)e_1)$ and that, $\forall y \in B_1(0)$,

$$|((1 + s)e_1 + y| - 1)^+ - (s + y_1)^+| \leq C|y'|^2,$$

where $y = (y_1, y')$. Therefore, we have

$$0 \leq (u_{\alpha} - v_{\alpha})((1 + s)e_1 + y) \leq \begin{cases} Cs^{\alpha-1}|y'|^2, & y \in B_{\frac{s}{2}}(0), \\ C|y'|^{2\alpha}, & y \in B_1(0) \setminus B_{\frac{s}{2}}(0), \\ C|y|^{\alpha}, & y \in \mathbb{R}^n \setminus B_1(0). \end{cases}$$

Therefore, we have, $\forall L \in \mathcal{L}_0(\lambda, \Lambda, \sigma)$,

$$\begin{aligned}
0 &\leq L(u_{\alpha} - v_{\alpha})((1 + s)e_1) \\
&= (2 - \sigma) \int_{\mathbb{R}^n} [(u_{\alpha} - v_{\alpha})((1 + s)e_1 + y) + (u_{\alpha} - v_{\alpha})((1 + s)e_1 - y)] K(y) dy \\
&\leq C(2 - \sigma) \Lambda \left(\int_{B_{\frac{s}{2}}} \frac{s^{\alpha-1}|y'|^2}{|y|^{n+\sigma}} dy + \int_{B_1(0) \setminus B_{\frac{s}{2}}(0)} \frac{|y'|^{2\alpha}}{|y|^{n+\sigma}} dy + \int_{\mathbb{R}^n \setminus B_1(0)} \frac{|y|^{\alpha}}{|y|^{n+\sigma}} dy \right) \\
&\leq C(s^{\alpha-\sigma+1} + s^{2\alpha-\sigma} + 1).
\end{aligned}$$

Thus, we have $M_{L_0(\lambda, \Lambda, \sigma)}^+(u_\alpha - v_\alpha)((1+s)e_1) \leq C(s^{\alpha-\sigma+1} + s^{2\alpha-\sigma} + 1)$. Therefore, by Lemma 4.4.1, there exists a sufficiently small $\alpha > 0$ such that

$$\begin{aligned} M_{L_0(\lambda, \Lambda, \sigma)}^+ u_\alpha((1+s)e_1) &\leq M_{L_0(\lambda, \Lambda, \sigma)}^+(u_\alpha - v_\alpha)((1+s)e_1) + M_{L_0(\lambda, \Lambda, \sigma)}^+ v_\alpha((1+s)e_1) \\ &\leq C(s^{\alpha-\sigma+1} + s^{2\alpha-\sigma} + 1) - \epsilon_0 s^{\alpha-\sigma}. \end{aligned}$$

Thus, there exists a sufficiently small $r_0 > 0$ such that we have $M_{L_0(\lambda, \Lambda, \sigma)}^+ u_\alpha(e_1 + se_1) \leq -1$ for any $s \in (0, r_0]$. \square

In the rest of this section, we assume that Ω is a bounded domain satisfying uniform exterior ball condition with uniform radius $r_\Omega (< 1)$. Without loss of generality, we can assume that $\Omega \subset \subset \{x | x_1 < 0\}$. For any $x \in \partial\Omega$ and any $0 < r < r_\Omega$, there exists $y_x^r \in \Omega^c$ such that $\bar{B}_r(y_x^r) \cap \bar{\Omega} = \{x\}$.

Lemma 4.4.3. *Assume that $\sigma \in (0, 2)$. There exists an $\epsilon_0 > 0$ such that, for any $x \in \partial\Omega$ and $0 < r < r_\Omega$, there is a continuous function $\varphi_{x,r}$ satisfying*

$$\begin{cases} \varphi_{x,r} \equiv 0, & \text{in } \bar{B}_r(y_x^r), \\ \varphi_{x,r} > 0, & \text{in } \bar{B}_r^c(y_x^r), \\ \varphi_{x,r} \equiv 2, & \text{in } B_{2r}^c(y_x^r), \\ M_{L_0(\lambda, \Lambda, \sigma)}^+ \varphi_{x,r} \leq -\epsilon_0, & \text{in } \Omega. \end{cases}$$

Proof. We define a uniformly continuous function φ in \mathbb{R}^n such that $1 \leq \varphi \leq 2$ and

$$\begin{cases} \varphi(y) = 1, & \text{in } y_1 > 1, \\ \varphi(y) = 2, & \text{in } y_1 \leq 0. \end{cases}$$

We pick some sufficiently large $C > \frac{2}{r_0^\alpha}$ and we define $\varphi_{x,r}(y) = \min\{\varphi(y), Cu_\alpha(\frac{y-y_x^r}{r})\}$ where α and r_0 are defined in Lemma 4.4.2. It is easy to verify that $\varphi_{x,r} \equiv 0$ in $\bar{B}_r(y_x^r)$, $\varphi_{x,r} > 0$ in $\bar{B}_r^c(y_x^r)$, and $\varphi_{x,r} \equiv 2$ in $B_{2r}^c(y_x^r)$. By Lemma 4.4.2, we have $M_{L_0(\lambda, \Lambda, \sigma)}^+ u_\alpha \leq -1$ in $\bar{B}_{1+r_0}(0) \setminus \bar{B}_1(0)$. It is obvious that, for any $y \in \bar{B}_{(1+r_0)r}(y_x^r) \setminus \bar{B}_r(y_x^r)$, we have $(M_{L_0(\lambda, \Lambda, \sigma)}^+ u_\alpha(\frac{\cdot - y_x^r}{r}))(y) \leq -\frac{1}{r^\sigma}$.

For any $y \in \bar{B}_{(1+(\frac{2}{C})^\frac{1}{\alpha})r}(y_x^r) \setminus \bar{B}_r(y_x^r)$, then we have $\varphi_{x,r}(y) = Cu_\alpha(\frac{y-y_x^r}{r})$. Suppose that there exists a test function $\Psi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ touches $\varphi_{x,r}$ from below at y . Thus, $\frac{\Psi}{C}$ touches $u_\alpha(\frac{\cdot - y_x^r}{r})$ from below at y . Thus, $M_{L_0(\lambda, \Lambda, \sigma)}^+ \Psi(y) \leq -\frac{C}{r^\sigma}$. For any $y \in \Omega \cap \bar{B}_{(1+(\frac{2}{C})^\frac{1}{\alpha})r}(y_x^r)$, we have $\varphi_{x,r}(y) = \varphi(y) = \max_{\mathbb{R}^n} \varphi_{x,r} = 2$. Suppose that there exists a test function $\Psi \in C^2(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$ touches $\varphi_{x,r}$ from below at y .

Therefore,

$$\begin{aligned}
(M_{L_0(\lambda, \Lambda, \sigma)}^+ \Psi)(y) &= (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda \delta \Psi(y, z)^+ - \lambda \delta \Psi(y, z)^-}{|z|^{n+\sigma}} dz \\
&= -\lambda(2 - \sigma) \int_{\mathbb{R}^n} \frac{\delta \Psi(y, z)^-}{|z|^{n+\sigma}} dz \\
&\leq -\lambda(2 - \sigma) \int_{\mathbb{R}^n} \frac{(\Psi(y + z) - 2)^-}{|z|^{n+\sigma}} dz \\
&\leq -\lambda(2 - \sigma) \int_{\{z|z_1 > -y_1 + 1\}} \frac{1}{|z|^{n+\sigma}} dz \\
&\leq -\lambda(2 - \sigma) \int_{\{z|z_1 > -\min\{y_1|y \in \Omega\} + 1\}} \frac{1}{|z|^{n+\sigma}} dz.
\end{aligned}$$

Based on the above argument, if we set $\epsilon_0 = \min\{\frac{C}{r_\Omega^\sigma}, \lambda(2 - \sigma) \int_{z_1 > -\min\{y_1|y \in \Omega\} + 1} \frac{1}{|z|^{n+\sigma}} dz\}$, we have

$$M_{L_0(\lambda, \Lambda, \sigma)}^+ \varphi_{x,r} \leq -\epsilon_0, \quad \text{in } \Omega.$$

□

Theorem 4.4.4. Assume that $0 < \sigma < 2$ and $G(x, 0, 0)$ is bounded in \mathbb{R}^n . Suppose that G is uniformly elliptic with respect to $L_0(\lambda, \Lambda, \sigma)$ and g is a bounded continuous function in \mathbb{R}^n . Then (74) admits a viscosity supersolution \bar{u} and a viscosity subsolution \underline{u} and $\bar{u} = \underline{u} = g$ in Ω^c .

Proof. For any $x \in \bar{\Omega}^c$, we let \tilde{u}_x be a bounded continuous function touches g from above at x and $\tilde{u}_x \geq 2C(\|g\|_{L^\infty(\mathbb{R}^n)} + 1)$ in Ω for some sufficiently large $C(> 1)$ we determine later. Thus, we define $u_x = \min\{C(\|g\|_{L^\infty(\mathbb{R}^n)} + 1)\varphi, \tilde{u}_x\}$ where φ is defined in Lemma 4.4.3. It is obvious that $u_x \geq g$ in \mathbb{R}^n , $u_x(x) = g(x)$ and $u_x = C\{\|g\|_{L^\infty(\mathbb{R}^n)} + 1\}\varphi = 2C(\|g\|_{L^\infty(\mathbb{R}^n)} + 1) = \max_{\mathbb{R}^n} u_x$ in Ω . For any $y \in \Omega$, we have

$$\begin{aligned}
(M_{L_0(\lambda, \Lambda, \sigma)}^+ u_x)(y) &= (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda \delta u_x(y, z)^+ - \lambda \delta u_x(y, z)^-}{|z|^{n+\sigma}} dz \\
&= -\lambda(2 - \sigma) \int_{\mathbb{R}^n} \frac{\delta u_x(y, z)^-}{|z|^{n+\sigma}} dz \\
&\leq -\lambda(2 - \sigma) \int_{\mathbb{R}^n} \frac{(u_x(y + z) - 2C(\|g\|_{L^\infty(\mathbb{R}^n)} + 1))^-}{|z|^{n+\sigma}} dz \\
&\leq -\lambda(2 - \sigma)C(\|g\|_{L^\infty(\mathbb{R}^n)} + 1) \int_{\{z|z_1 > -y_1 + 1\}} \frac{1}{|z|^{n+\sigma}} dz \\
&\leq -\lambda(2 - \sigma)C(\|g\|_{L^\infty(\mathbb{R}^n)} + 1) \int_{\{z|z_1 > -\min\{y_1|y \in \Omega\} + 1\}} \frac{1}{|z|^{n+\sigma}} dz \\
&\leq -\|G(x, 0, 0)\|_{L^\infty(\mathbb{R}^n)},
\end{aligned}$$

where C is chosen sufficiently large such that the last inequality holds. Although u_x does not depend on r , we define $u_{x,r} = u_x$ for any $0 < r < r_\Omega$.

Since g is a continuous function, let ρ_R be a modulus of continuity of g in $B_R(0)$. Let R_0 be a sufficiently large constant such that $\Omega \subset B_{R_0-1}(0)$. For any $x \in \partial\Omega$, we let $u_{x,r} = \rho_{R_0}(3r) + g(x) + \max\{\|g\|_{L^\infty(\mathbb{R}^n)}, \frac{\|G(x,0,0)\|_{L^\infty(\mathbb{R}^n)}}{\epsilon_0}\}\varphi_{x,r}$ where $\varphi_{x,r}$ is defined in Lemma 4.4.3. It is obvious that $u_{x,r}(x) = \rho_{R_0}(3r) + g(x)$, $u_{x,r} \geq g$ in \mathbb{R}^n and $M_{L_0(\lambda,\Lambda,\sigma)}^+ u_{x,r} \leq -\|G(x,0,0)\|_{L^\infty(\mathbb{R}^n)}$ in Ω .

Now we define $\bar{u} = \inf_{x \in \Omega^c, 0 < r < r_\Omega} \{u_{x,r}\}$. Therefore, $\bar{u} = g$ in Ω^c and $\bar{u} \geq g$ in \mathbb{R}^n . For any $x \in \partial\Omega$ and $y \in \mathbb{R}^n$, we have $g(y) - g(x) \leq \bar{u}(y) - \bar{u}(x) = \bar{u}(y) - g(x) \leq \rho_{R_0}(3r) + \max\{\|g\|_{L^\infty(\mathbb{R}^n)}, \frac{\|G(x,0,0)\|_{L^\infty(\mathbb{R}^n)}}{\epsilon_0}\}\varphi_{x,r}(y)$ for any $0 < r < r_\Omega$. Therefore, \bar{u} is continuous on $\partial\Omega$. For any $y \in \Omega$, we define $d_y = \text{dist}(y, \partial\Omega) > 0$. If $r < \frac{d_y}{2}$, then we have, for any $z \in B_{\frac{d_y}{2}}(y)$,

$$u_{x,r}(z) = \begin{cases} \rho_{R_0}(3r) + g(x) + 2 \max\{\|g\|_{L^\infty(\mathbb{R}^n)}, \frac{\|G(x,0,0)\|_{L^\infty(\mathbb{R}^n)}}{\epsilon_0}\}, & x \in \partial\Omega, \\ 2C(\|g\|_{L^\infty(\mathbb{R}^n)} + 1), & x \in \bar{\Omega}^c. \end{cases}$$

Thus, we have, for any $z \in B_{\frac{d_y}{2}}(y)$,

$$\inf_{x \in \partial\Omega, \frac{d_y}{2} < r < r_\Omega} \{u_{x,r}(z) - u_{x,r}(y), 0\} \leq \bar{u}(z) - \bar{u}(y) \leq \sup_{x \in \partial\Omega, \frac{d_y}{2} < r < r_\Omega} \{u_{x,r}(z) - u_{x,r}(y), 0\}.$$

Since $\{u_{x,r}\}_{x \in \partial\Omega, \frac{d_y}{2} < r < r_\Omega}$ has a uniform modulus of continuity, then \bar{u} is continuous in Ω . Therefore, \bar{u} is a bounded continuous function in \mathbb{R}^n and $\bar{u} = g$ in Ω^c .

By Lemma 4.2.1, we have $M_{L_0(\lambda,\Lambda,\sigma)}^+ \bar{u} \leq -\|G(x,0,0)\|_{L^\infty(\mathbb{R}^n)}$ in Ω . Therefore, for any $x \in \Omega$, $G(x, \bar{u}, I[x, \bar{u}]) - G(x, 0, 0) \geq M_{L_0(\lambda,\Lambda,\sigma)}^-(-\bar{u})(x) = -M_{L_0(\lambda,\Lambda,\sigma)}^+(\bar{u})(x) \geq \|G(\cdot, 0, 0)\|_{L^\infty(\mathbb{R}^n)}$. Thus, $G(x, \bar{u}, I[x, \bar{u}]) \geq 0$ in Ω . \square

Similarly, we can construct a subsolution and a supersolution of (75).

Theorem 4.4.5. *Assume that $\{f_{\alpha\beta}\}_{\alpha,\beta}$ is a set of uniformly continuous and bounded functions in Ω , g is a bounded continuous function in \mathbb{R}^n , $\gamma \geq 0$ and $I_{\alpha\beta}$ is of Lévy type and uniformly elliptic with respect to $L_0(\lambda, \Lambda, \sigma)$ for some $2 > \sigma > 0$. Then (75) admits a viscosity supersolution \bar{u} and a viscosity subsolution \underline{u} and $\bar{u} = \underline{u} = g$ in Ω^c .*

Now we have enough ingredients to conclude

Theorem 4.4.6. *Assume that $0 < \sigma < 2$, $G(x, 0, 0)$ is bounded in \mathbb{R}^n and g is a bounded continuous function. Suppose that G is uniformly elliptic with respect to $L_0(\lambda, \Lambda, \sigma)$. Then (74) admits a viscosity solution u .*

Proof. The result follows from Theorem 4.4.4 and Corollary 7. \square

Theorem 4.4.7. *Assume that $\{f_{\alpha\beta}\}_{\alpha,\beta}$ is a set of uniformly continuous and bounded functions in Ω , $\gamma \geq 0$, g is a bounded continuous function and $I_{\alpha\beta}$ is of Lévy type and uniformly elliptic with respect to $L_0(\lambda, \Lambda, \sigma)$ for some $2 > \sigma > 0$. Then (75) admits a viscosity solution u .*

Proof. The result follows from Theorem 4.4.5 and Corollary 8. □

CHAPTER V

SEMICONCAVITY OF VISCOSITY SOLUTIONS FOR A CLASS OF DEGENERATE ELLIPTIC INTEGRO-DIFFERENTIAL EQUATIONS IN \mathbb{R}^N

In this chapter, we will study semiconcavity of viscosity solutions for a class of degenerate elliptic integro-differential equations in \mathbb{R}^n , see [60].

5.1 Notation and Definitions

We recall the definition of a viscosity solution of (1). In order to do it, we introduce two associated operators $I^{1,\delta}$ and $I^{2,\delta}$,

$$I^{1,\delta}[x, p, u] = \int_{|\xi| < \delta} [u(x + j(x, \xi)) - u(x) - \mathbb{1}_{B_1(0)}(\xi)p \cdot j(x, \xi)] \mu(d\xi),$$

$$I^{2,\delta}[x, p, u] = \int_{|\xi| \geq \delta} [u(x + j(x, \xi)) - u(x) - \mathbb{1}_{B_1(0)}(\xi)p \cdot j(x, \xi)] \mu(d\xi).$$

Definition 9. *A bounded function $u \in USC(\mathbb{R}^n)$ is a viscosity subsolution of (1) if whenever $u - \varphi$ has a maximum over $B_\delta(x)$ at $x \in \mathbb{R}^n$ for a test function $\varphi \in C^2(B_\delta(x))$, $\delta > 0$, then*

$$G(x, u(x), D\varphi(x), D^2\varphi(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \leq 0.$$

A bounded function $u \in LSC(\mathbb{R}^n)$ is a viscosity supersolution of (1) if whenever $u - \varphi$ has a minimum over $B_\delta(x)$ at $x \in \mathbb{R}^n$ for a test function $\varphi \in C^2(B_\delta(x))$, $\delta > 0$, then

$$G(x, u(x), D\varphi(x), D^2\varphi(x), I^{1,\delta}[x, D\varphi(x), \varphi] + I^{2,\delta}[x, D\varphi(x), u]) \geq 0.$$

A function u is a viscosity solution of (1) if it is both a viscosity subsolution and viscosity supersolution of (1).

5.2 Hölder and Lipschitz continuity

In this section we prove the Hölder and Lipschitz continuity of viscosity solutions of (1) and (6). We start with equation (1). We make the following assumptions on the nonlinearity G and the function $j(x, \xi)$.

(H1) There are a constant $0 < \theta \leq 1$, a non-negative constant Λ and two positive constants C_1, C_2 such that, for any $x, y \in \mathbb{R}^n$, $r, l_x, l_y \in \mathbb{R}$, $X, Y \in \mathbb{S}^n$ and $L, \eta > 0$, we have

$$\begin{aligned} & G(y, r, L\theta|x-y|^{\theta-2}(x-y), Y, l_y) - G(x, r, L\theta|x-y|^{\theta-2}(x-y) + 2\eta x, X, l_x) \\ & \leq \Lambda(l_x - l_y) + C_1(1+L)|x-y|^\theta + C_2\eta(1+|x|^2), \end{aligned}$$

if

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq L|x-y|^{\theta-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\eta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

(H2) For any $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} |j(x, \xi) - j(y, \xi)| & \leq |x-y|\rho(\xi) \quad \text{for } \xi \in \mathbb{R}^n, \\ |j(0, \xi)| & \leq \rho(\xi) \quad \text{for } \xi \in \mathbb{R}^n. \end{aligned}$$

The following lemma is a nonlocal version of the Jensen-Ishii lemma we borrow from [40], Theorem 4.9. The reader can consult [9] for a more general Jensen-Ishii lemma for integro-differential equations, which allows for arbitrary growth of solutions at infinity. Before giving the lemma, we notice that our Definition 9 corresponds to the alternative definition of a viscosity solution in [40], see Lemma 4.8.

Lemma 5.2.1. *Suppose that the nonlinearity G in (1) is continuous and satisfies (12)-(3). Let u, v be bounded functions and be respectively a viscosity subsolution and a viscosity supersolution of*

$$G(x, u, Du, D^2u, I[x, u]) = 0 \quad \text{and} \quad G(x, v, Dv, D^2v, I[x, v]) = 0 \quad \text{in } \mathbb{R}^n.$$

Let $\psi \in C^2(\mathbb{R}^{2n})$ and $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that

$$(x, y) \mapsto u(x) - v(y) - \psi(x, y)$$

has a global maximum at (\hat{x}, \hat{y}) . Furthermore, assume that in a neighborhood of (\hat{x}, \hat{y}) there are continuous functions $g_0 : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $g_1 : \mathbb{R}^n \rightarrow \mathbb{S}^n$ with $g_0(\hat{x}, \hat{y}) > 0$, satisfying

$$D^2\psi(x, y) \leq g_0(x, y) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} g_1(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, for any $0 < \delta < 1$ and $\epsilon_0 > 0$, there are $X, Y \in \mathbb{S}^n$ satisfying

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} - \begin{pmatrix} g_1(\hat{x}) & 0 \\ 0 & 0 \end{pmatrix} \leq (1 + \epsilon_0)g_0(\hat{x}, \hat{y}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

such that

$$\begin{aligned} & G(\hat{x}, u(\hat{x}), D_x\psi(\hat{x}, \hat{y}), X, I^{1,\delta}[\hat{x}, D_x\psi(\hat{x}, \hat{y}), \psi(\cdot, \hat{y})] + I^{2,\delta}[\hat{x}, D_x\psi(\hat{x}, \hat{y}), u(\cdot)]) \leq 0, \\ & G(\hat{y}, v(\hat{y}), -D_y\psi(\hat{x}, \hat{y}), Y, I^{1,\delta}[\hat{y}, -D_y\psi(\hat{x}, \hat{y}), -\psi(\hat{x}, \cdot)] + I^{2,\delta}[\hat{y}, -D_y\psi(\hat{x}, \hat{y}), v(\cdot)]) \geq 0. \end{aligned}$$

Remark 9. The statement of Lemma 5.2.1 is weaker than Theorem 4.9 in [40]. By Theorem 4.9 in [40], the same result as Lemma 5.2.1 is also true for Bellman-Isaacs equations (6).

Lemma 5.2.2. Suppose that a Lévy measure μ satisfies (12) and $j(x, \xi)$ satisfies assumption (H2). Then we have

$$M_1 : = \sup_{x \neq y} \left\{ |x - y|^{-\theta} \int_{\mathbb{R}^n} \left[|x - y + j(x, \xi) - j(y, \xi)|^\theta - |x - y|^\theta - \mathbb{1}_{B_1(0)}(\xi) \theta |x - y|^{\theta-2} (x - y) \cdot (j(x, \xi) - j(y, \xi)) \right] \mu(d\xi) \right\} < +\infty. \quad (88)$$

Proof. We first define

$$\phi(x, y) = |x - y|^\theta. \quad (89)$$

By calculation, we have

$$D\phi(x, y) = \theta |x - y|^{\theta-2} \begin{pmatrix} x - y \\ y - x \end{pmatrix}, \quad (90)$$

$$\begin{aligned} D^2\phi(x, y) &= \theta |x - y|^{\theta-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \theta(\theta - 2) |x - y|^{\theta-4} \begin{pmatrix} x - y \\ y - x \end{pmatrix} \otimes \begin{pmatrix} x - y \\ y - x \end{pmatrix} \\ &\leq \theta |x - y|^{\theta-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \end{aligned} \quad (91)$$

Since $\lim_{\xi \rightarrow 0} \rho(\xi) = 0$, there exists a positive constant $\delta_1 < 1$ such that $\sup_{\xi \in B_{\delta_1}(0)} \rho(\xi) \leq \frac{1}{2}$. By (12), (90), (91) and (H2), we have, for any $x, y \in \mathbb{R}^n$ and $x \neq y$

$$\begin{aligned} & |x - y|^{-\theta} \int_{\mathbb{R}^n} \left[|x - y + j(x, \xi) - j(y, \xi)|^\theta - |x - y|^\theta - \mathbb{1}_{B_1(0)}(\xi) \theta |x - y|^{\theta-2} (x - y) \cdot (j(x, \xi) - j(y, \xi)) \right] \mu(d\xi) \\ & \leq |x - y|^{-\theta} \theta \int_{B_{\delta_1}(0)} \left(\sup_{0 \leq t \leq 1} |x - y + t(j(x, \xi) - j(y, \xi))|^{\theta-2} |j(x, \xi) - j(y, \xi)|^2 \right) \mu(d\xi) \\ & \quad + |x - y|^{-\theta} \int_{\mathbb{R}^n \setminus B_{\delta_1}(0)} \left[|x - y + j(x, \xi) - j(y, \xi)|^\theta - |x - y|^\theta - \mathbb{1}_{B_1(0)}(\xi) \theta |x - y|^{\theta-2} (x - y) \cdot (j(x, \xi) - j(y, \xi)) \right] \mu(d\xi) \\ & \leq 2^{2-\theta} \theta \int_{B_{\delta_1}(0)} \rho(\xi)^2 \mu(d\xi) + \int_{\mathbb{R}^n \setminus B_{\delta_1}(0)} \rho(\xi)^\theta \mu(d\xi) + \theta \int_{B_1(0) \setminus B_{\delta_1}(0)} \rho(\xi) \mu(d\xi) < +\infty. \end{aligned} \quad (92)$$

□

Theorem 5.2.3. *Suppose that the nonlinearity G in (1) is continuous, and satisfies (12)-(3) and (H1). Suppose that $j(x, \xi)$ satisfies assumption (H2). Then, if $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (1) and $\gamma > \Lambda M_1 + C_1$ where M_1 is defined in (88), we have $u \in C^{0,\theta}(\bar{\mathbb{R}}^n)$.*

Proof. Let $\Phi(x, y) = u(x) - u(y) - \psi(x, y)$ where $\psi(x, y) = L\phi(x, y) + \eta|x|^2$ and $\phi(x, y)$ is defined in (89). We want to prove, for any $\eta > 0$, we have $\Phi(x, y) \leq 0$ for all $x, y \in \mathbb{R}^n$ and some fixed sufficiently large L . Otherwise, there exists a positive constant η_0 such that $\sup_{x, y \in \mathbb{R}^n} \Phi(x, y) > 0$ if $0 < \eta < \eta_0$. By boundedness of u , there is a point (\hat{x}, \hat{y}) such that $\Phi(\hat{x}, \hat{y}) = \sup_{x, y \in \mathbb{R}^n} \Phi(x, y) > 0$. Therefore, we have

$$\max\{\eta|\hat{x}|^2, L|\hat{x} - \hat{y}|^\theta\} < u(\hat{x}) - u(\hat{y}). \quad (93)$$

By (90) and (91), we obtain

$$D^2\psi(\hat{x}, \hat{y}) \leq \theta L|\hat{x} - \hat{y}|^{\theta-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\eta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

By Lemma 5.2.1, since $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (1), for any $0 < \delta < 1$ and $\epsilon_0 > 0$, there are $X, Y \in \mathbb{S}^n$ satisfying

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} - 2\eta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \leq (1 + \epsilon_0)\theta L|\hat{x} - \hat{y}|^{\theta-2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (94)$$

such that

$$\begin{aligned} G(\hat{x}, u(\hat{x}), LD_x\phi(\hat{x}, \hat{y}) + 2\eta\hat{x}, X, l_{\hat{x}}) &\leq 0, \\ G(\hat{y}, u(\hat{y}), -LD_y\phi(\hat{x}, \hat{y}), Y, l_{\hat{y}}) &\geq 0, \end{aligned}$$

where

$$\begin{aligned} l_{\hat{x}} &= I^{1,\delta}[\hat{x}, LD_x\phi(\hat{x}, \hat{y}) + 2\eta\hat{x}, L\phi(\cdot, \hat{y}) + \eta|\cdot|^2] + I^{2,\delta}[\hat{x}, LD_x\phi(\hat{x}, \hat{y}) + 2\eta\hat{x}, u(\cdot)], \\ l_{\hat{y}} &= I^{1,\delta}[\hat{y}, -LD_y\phi(\hat{x}, \hat{y}), -L\phi(\hat{x}, \cdot)] + I^{2,\delta}[\hat{y}, -LD_y\phi(\hat{x}, \hat{y}), u(\cdot)]. \end{aligned}$$

Thus, by (2), (93) and (H1), we have

$$\begin{aligned} \gamma L|\hat{x} - \hat{y}|^\theta &\leq \gamma(u(\hat{x}) - u(\hat{y})) \\ &\leq G(\hat{y}, u(\hat{y}), -LD_y\phi(\hat{x}, \hat{y}), Y, l_{\hat{y}}) - G(\hat{x}, u(\hat{x}), LD_x\phi(\hat{x}, \hat{y}) + 2\eta\hat{x}, X, l_{\hat{x}}) \\ &\leq \Lambda(l_{\hat{x}} - l_{\hat{y}}) + C_1(1 + L)|\hat{x} - \hat{y}|^\theta + C_2\eta(1 + |\hat{x}|^2). \end{aligned} \quad (95)$$

Now we focus on estimating the integral term $l_{\hat{x}} - l_{\hat{y}}$. Thus,

$$\begin{aligned}
l_{\hat{x}} - l_{\hat{y}} &= L \int_{B_{\delta}(0)} \left[|\hat{x} - \hat{y} + j(\hat{x}, \xi)|^{\theta} - |\hat{x} - \hat{y}|^{\theta} - \theta |\hat{x} - \hat{y}|^{\theta-2} (\hat{x} - \hat{y}) \cdot j(\hat{x}, \xi) \right] \mu(d\xi) \\
&\quad + L \int_{B_{\delta}(0)} \left[|\hat{y} - \hat{x} + j(\hat{y}, \xi)|^{\theta} - |\hat{y} - \hat{x}|^{\theta} - \theta |\hat{y} - \hat{x}|^{\theta-2} (\hat{y} - \hat{x}) \cdot j(\hat{y}, \xi) \right] \mu(d\xi) \\
&\quad + \eta \int_{B_{\delta}(0)} \left(|\hat{x} + j(\hat{x}, \xi)|^2 - |\hat{x}|^2 - 2\hat{x} \cdot j(\hat{x}, \xi) \right) \mu(d\xi) \\
&\quad + \int_{B_{\delta}^c(0)} \left[u(\hat{x} + j(\hat{x}, \xi)) - u(\hat{x}) - u(\hat{y} + j(\hat{y}, \xi)) + u(\hat{y}) \right. \\
&\quad \left. - \mathbb{1}_{B_1(0)}(\xi) (\theta L |\hat{x} - \hat{y}|^{\theta-2} (\hat{x} - \hat{y})) \cdot (j(\hat{x}, \xi) - j(\hat{y}, \xi)) \right. \\
&\quad \left. - \mathbb{1}_{B_1(0)}(\xi) 2\eta \hat{x} \cdot j(\hat{x}, \xi) \right] \mu(d\xi).
\end{aligned}$$

Since $\Phi(x, y)$ attains a global maximum at (\hat{x}, \hat{y}) , we have, for any $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
&u(\hat{x} + j(\hat{x}, \xi)) - u(\hat{x}) - u(\hat{y} + j(\hat{y}, \xi)) + u(\hat{y}) \\
&\leq L \left(|\hat{x} - \hat{y} + j(\hat{x}, \xi) - j(\hat{y}, \xi)|^{\theta} - |\hat{x} - \hat{y}|^{\theta} \right) + \eta \left(|\hat{x} + j(\hat{x}, \xi)|^2 - |\hat{x}|^2 \right). \quad (96)
\end{aligned}$$

Thus, by (91) and (96), we have

$$\begin{aligned}
l_{\hat{x}} - l_{\hat{y}} &\leq \theta L \int_{B_{\delta}(0)} \left(\sup_{0 \leq t \leq 1} |\hat{x} - \hat{y} + tj(\hat{x}, \xi)|^{\theta-2} |j(\hat{x}, \xi)|^2 \right. \\
&\quad \left. + \sup_{0 \leq t \leq 1} |\hat{y} - \hat{x} + tj(\hat{y}, \xi)|^{\theta-2} |j(\hat{y}, \xi)|^2 \right) \mu(d\xi) \\
&\quad + \eta \int_{\mathbb{R}^n} \left(|\hat{x} + j(\hat{x}, \xi)|^2 - |\hat{x}|^2 - \mathbb{1}_{B_1(0)}(\xi) 2\hat{x} \cdot j(\hat{x}, \xi) \right) \mu(d\xi) \\
&\quad + L \int_{B_{\delta}^c(0)} \left[|\hat{x} - \hat{y} + j(\hat{x}, \xi) - j(\hat{y}, \xi)|^{\theta} - |\hat{x} - \hat{y}|^{\theta} \right. \\
&\quad \left. - \mathbb{1}_{B_1(0)}(\xi) \theta |\hat{x} - \hat{y}|^{\theta-2} (\hat{x} - \hat{y}) \cdot (j(\hat{x}, \xi) - j(\hat{y}, \xi)) \right] \mu(d\xi). \quad (97)
\end{aligned}$$

We claim that $\eta |\hat{x}|^2 \rightarrow 0$ as $\eta \rightarrow 0$. Since u is bounded in \mathbb{R}^n , for any positive integer k , let (x_k, y_k) be a point such that

$$u(x_k) - u(y_k) - L\phi(x_k, y_k) \geq M - \frac{1}{k},$$

where $M := \sup_{x, y \in \mathbb{R}^n} \{u(x) - u(y) - L\phi(x, y)\} < +\infty$. Thus,

$$M - \frac{1}{k} - \eta |x_k|^2 \leq \Phi(x_k, y_k) \leq \Phi(\hat{x}, \hat{y}) \leq M. \quad (98)$$

Letting $\eta \rightarrow 0$ and then letting $k \rightarrow +\infty$ in (98), we have $\lim_{\eta \rightarrow 0} \Phi(\hat{x}, \hat{y}) = M$. If we notice that

$$\Phi(\hat{x}, \hat{y}) + \eta |\hat{x}|^2 = u(\hat{x}) - u(\hat{y}) - L\phi(\hat{x}, \hat{y}) \leq M, \quad \forall \eta > 0,$$

the claim follows. Since $u \in BUC(\mathbb{R}^n)$ and (93) holds, we have

$$\epsilon_1 \leq |\hat{x} - \hat{y}| \leq \epsilon_1^{-1},$$

where ϵ_1 is a positive constant independent of η . Letting $\delta \rightarrow 0$ and then letting $\eta \rightarrow 0$ in (95), we have, by (12), (97) and (H2),

$$\begin{aligned} \gamma L |\hat{x} - \hat{y}|^\theta &\leq \Lambda L \int_{\mathbb{R}^n} \left[|\hat{x} - \hat{y} + j(\hat{x}, \xi) - j(\hat{y}, \xi)|^\theta - |\hat{x} - \hat{y}|^\theta \right. \\ &\quad \left. - \mathbb{1}_{B_1(0)}(\xi) \theta |\hat{x} - \hat{y}|^{\theta-2} (\hat{x} - \hat{y}) \cdot (j(\hat{x}, \xi) - j(\hat{y}, \xi)) \right] \mu(d\xi) \\ &\quad + C_1(1 + L) |\hat{x} - \hat{y}|^\theta. \end{aligned}$$

Therefore, by Lemma 5.2.2,

$$\begin{aligned} \gamma &\leq \Lambda |\hat{x} - \hat{y}|^{-\theta} \int_{\mathbb{R}^n} \left[|\hat{x} - \hat{y} + j(\hat{x}, \xi) - j(\hat{y}, \xi)|^\theta - |\hat{x} - \hat{y}|^\theta \right. \\ &\quad \left. - \mathbb{1}_{B_1(0)}(\xi) \theta |\hat{x} - \hat{y}|^{\theta-2} (\hat{x} - \hat{y}) \cdot (j(\hat{x}, \xi) - j(\hat{y}, \xi)) \right] \mu(d\xi) + C_1(1 + \frac{1}{L}) \\ &\leq \Lambda M_1 + C_1(1 + \frac{1}{L}) < +\infty, \end{aligned} \tag{99}$$

where M_1 is defined in (88). It is now obvious from (99) that, if $\gamma > \Lambda M_1 + C_1$, we can find a sufficiently large L such that we have a contradiction. Therefore, we have $u \in C^{0,\theta}(\bar{\mathbb{R}}^n)$. \square

Let us consider another important fully nonlinear integro-PDE appearing in the study of stochastic optimal control and stochastic differential games for processes with jumps, namely the Bellman-Isaacs equation (6). Equation (6) is not of the same form as (1), which means that the following theorem is not a corollary of Theorem 5.2.3.

Theorem 5.2.4. *Suppose that $c_{\alpha\beta} \geq \gamma$ in \mathbb{R}^n uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$. Suppose that the Lévy measure μ satisfies (12), and the family $\{j_{\alpha\beta}(x, \xi)\}$ satisfies assumption (H2) uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$. Suppose moreover that there exist a positive constant C and $0 < \theta \leq 1$ such that*

$$\sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \max\{|\sigma_{\alpha\beta}(0)|, |b_{\alpha\beta}(0)|\} < C, \tag{100}$$

and

$$\sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \max\{[\sigma_{\alpha\beta}]_{0,1;\mathbb{R}^n}, [b_{\alpha\beta}]_{0,1;\mathbb{R}^n}, [c_{\alpha\beta}]_{0,\theta;\mathbb{R}^n}, [f_{\alpha\beta}]_{0,\theta;\mathbb{R}^n}\} < +\infty. \tag{101}$$

Then, if $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (6) and $\gamma > N_1$ where

$$\begin{aligned}
N_1 : &= \sup_{x \neq y} \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \left\{ \theta |x - y|^{-2} Tr \left[(\sigma_{\alpha\beta}(x) - \sigma_{\alpha\beta}(y)) (\sigma_{\alpha\beta}(x) - \sigma_{\alpha\beta}(y))^T \right] \right. \\
&\quad + \theta |x - y|^{-2} (b_{\alpha\beta}(y) - b_{\alpha\beta}(x)) \cdot (x - y) \\
&\quad + |x - y|^{-\theta} \int_{\mathbb{R}^n} \left[|x - y + j_{\alpha\beta}(x, \xi) - j_{\alpha\beta}(y, \xi)|^\theta - |x - y|^\theta \right. \\
&\quad \left. \left. - \mathbb{1}_{B_1(0)}(\xi) \theta |x - y|^{\theta-2} (x - y) \cdot (j_{\alpha\beta}(x, \xi) - j_{\alpha\beta}(y, \xi)) \right] \mu(d\xi) \right\} < +\infty,
\end{aligned} \tag{102}$$

we have $u \in C^{0,\theta}(\bar{\mathbb{R}}^n)$.

Proof. At the beginning of the proof, we will show that the constant N_1 has an upper bound. By (101) and the estimates in (92), we have

$$\begin{aligned}
N_1 &\leq \theta \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [\sigma_{\alpha\beta}]_{0,1;\mathbb{R}^n}^2 + \theta \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [b_{\alpha\beta}]_{0,1;\mathbb{R}^n} + 2^{2-\theta} \theta \int_{B_{\delta_1}(0)} \rho(\xi)^2 \mu(d\xi) \\
&\quad + \int_{\mathbb{R}^n \setminus B_{\delta_1}(0)} \rho(\xi)^\theta \mu(d\xi) + \theta \int_{B_1(0) \setminus B_{\delta_1}(0)} \rho(\xi) \mu(d\xi) < +\infty,
\end{aligned}$$

where δ_1 was chosen in Lemma 5.2.2.

Then we want to prove that, for any $\eta > 0$, we have $\Phi(x, y) = u(x) - u(y) - \psi(x, y) \leq 0$ for all $x, y \in \mathbb{R}^n$ and some fixed sufficiently large L where $\psi(x, y)$ is given in Theorem 5.2.3. Otherwise, there exists a positive constant η_0 such that $\sup_{x, y \in \mathbb{R}^n} \Phi(x, y) > 0$ if $0 < \eta < \eta_0$. By boundedness of u , there is a point (\hat{x}, \hat{y}) such that $\Phi(\hat{x}, \hat{y}) = \sup_{x, y \in \mathbb{R}^n} \Phi(x, y) > 0$. Therefore, we have (93). By Remark 9, since $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (6), for any $0 < \delta < 1$ and $\epsilon_0 > 0$, there are $X, Y \in \mathbb{S}^n$ satisfying (94) such that

$$\sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \left\{ -Tr(\sigma_{\alpha\beta}(\hat{x}) \sigma_{\alpha\beta}^T(\hat{x}) X) - l_{\hat{x}, \alpha\beta} + b_{\alpha\beta}(\hat{x}) \cdot D_x \psi(\hat{x}, \hat{y}) + c_{\alpha\beta}(\hat{x}) u(\hat{x}) + f_{\alpha\beta}(\hat{x}) \right\} \leq 0,$$

$$\sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \left\{ -Tr(\sigma_{\alpha\beta}(\hat{y}) \sigma_{\alpha\beta}^T(\hat{y}) Y) - l_{\hat{y}, \alpha\beta} - b_{\alpha\beta}(\hat{y}) \cdot D_y \psi(\hat{x}, \hat{y}) + c_{\alpha\beta}(\hat{y}) u(\hat{y}) + f_{\alpha\beta}(\hat{y}) \right\} \geq 0,$$

where

$$\begin{aligned}
l_{\hat{x}, \alpha\beta} &= I_{\alpha\beta}^{1,\delta}[\hat{x}, D_x \psi(\hat{x}, \hat{y}), \psi(\cdot, \hat{y})] + I_{\alpha\beta}^{2,\delta}[\hat{x}, D_x \psi(\hat{x}, \hat{y}), u(\cdot)], \\
l_{\hat{y}, \alpha\beta} &= I_{\alpha\beta}^{1,\delta}[\hat{y}, -D_y \psi(\hat{x}, \hat{y}), -\psi(\hat{x}, \cdot)] + I_{\alpha\beta}^{2,\delta}[\hat{y}, -D_y \psi(\hat{x}, \hat{y}), u(\cdot)].
\end{aligned}$$

Since (90) and (93) hold, and $c_{\alpha\beta} \geq \gamma$ in \mathbb{R}^n uniformly in $\alpha \in \mathcal{A}, \beta \in \mathcal{B}$, we have

$$\gamma L |\hat{x} - \hat{y}|^\theta \leq \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \left\{ L_{\alpha\beta} + N_{\alpha\beta} \right\}, \tag{103}$$

where

$$\begin{aligned} L_{\alpha\beta} = & \operatorname{Tr}\left(\sigma_{\alpha\beta}(\hat{x})\sigma_{\alpha\beta}^T(\hat{x})X - \sigma_{\alpha\beta}(\hat{y})\sigma_{\alpha\beta}^T(\hat{y})Y\right) + \left(b_{\alpha\beta}(\hat{y}) - b_{\alpha\beta}(\hat{x})\right) \cdot LD_x\phi(\hat{x}, \hat{y}) \\ & + \left(c_{\alpha\beta}(\hat{y}) - c_{\alpha\beta}(\hat{x})\right)u(\hat{y}) + f_{\alpha\beta}(\hat{y}) - f_{\alpha\beta}(\hat{x}) - 2\eta b_{\alpha\beta}(\hat{x}) \cdot \hat{x}, \end{aligned}$$

and

$$N_{\alpha\beta} = l_{\hat{x}, \alpha\beta} - l_{\hat{y}, \alpha\beta}.$$

By (94), (100) and (101), we see that (see also Example 3.6 in [22])

$$\begin{aligned} & \operatorname{Tr}\left(\sigma_{\alpha\beta}(\hat{x})\sigma_{\alpha\beta}^T(\hat{x})X - \sigma_{\alpha\beta}(\hat{y})\sigma_{\alpha\beta}^T(\hat{y})Y\right) \\ \leq & (1 + \epsilon_0)\theta L|\hat{x} - \hat{y}|^{\theta-2}\operatorname{Tr}\left[(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))^T\right] \\ & + 2\eta\operatorname{Tr}(\sigma_{\alpha\beta}(\hat{x})\sigma_{\alpha\beta}^T(\hat{x})) \\ \leq & (1 + \epsilon_0)\theta L|\hat{x} - \hat{y}|^{\theta-2}\operatorname{Tr}\left[(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))^T\right] \\ & + 2\eta(C + \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [\sigma_{\alpha\beta}]_{0,1;\mathbb{R}^n}|\hat{x}|)^2. \end{aligned}$$

Thus, we can estimate the local term $L_{\alpha\beta}$ easily. Using (90), (100), (101) and boundedness of u , we obtain

$$\begin{aligned} L_{\alpha\beta} \leq & (1 + \epsilon_0)\theta L|\hat{x} - \hat{y}|^{\theta-2}\operatorname{Tr}\left[(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))^T\right] \\ & + 2\eta(C + \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [\sigma_{\alpha\beta}]_{0,1;\mathbb{R}^n}|\hat{x}|)^2 + \theta L|\hat{x} - \hat{y}|^{\theta-2}\left(b_{\alpha\beta}(\hat{y}) - b_{\alpha\beta}(\hat{x})\right) \cdot (\hat{x} - \hat{y}) \\ & + \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [c_{\alpha\beta}]_{0,\theta;\mathbb{R}^n}|u|_{0;\mathbb{R}^n}|\hat{x} - \hat{y}|^\theta + \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [f_{\alpha\beta}]_{0,\theta;\mathbb{R}^n}|\hat{x} - \hat{y}|^\theta \\ & + 2\eta(C|\hat{x}| + \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [b_{\alpha\beta}]_{0,1;\mathbb{R}^n}|\hat{x}|^2). \end{aligned} \tag{104}$$

Similarly as in the proof of Theorem 5.2.3, we have $\eta|\hat{x}|^2 \rightarrow 0$ as $\eta \rightarrow 0$ and

$$\epsilon_1 \leq |\hat{x} - \hat{y}| \leq \epsilon_1^{-1},$$

where ϵ_1 is a positive constant independent of η . Letting $\delta \rightarrow 0$, $\eta \rightarrow 0$ and $\epsilon_0 \rightarrow 0$ in (103), we have, by (104) and the same estimates on the nonlocal term $N_{\alpha\beta}$ as Theorem 5.2.3,

$$\begin{aligned} \gamma L|\hat{x} - \hat{y}|^\theta \leq & \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} L\left\{\theta|\hat{x} - \hat{y}|^{\theta-2}\operatorname{Tr}\left[(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))^T\right] \right. \\ & + \theta|\hat{x} - \hat{y}|^{\theta-2}\left(b_{\alpha\beta}(\hat{y}) - b_{\alpha\beta}(\hat{x})\right) \cdot (\hat{x} - \hat{y}) \\ & + \int_{\mathbb{R}^n} \left[|\hat{x} - \hat{y} + j_{\alpha\beta}(\hat{x}, \xi) - j_{\alpha\beta}(\hat{y}, \xi)|^\theta - |\hat{x} - \hat{y}|^\theta \right. \\ & \left. - \mathbb{1}_{B_1(0)}(\xi)\theta|\hat{x} - \hat{y}|^{\theta-2}(\hat{x} - \hat{y}) \cdot (j_{\alpha\beta}(\hat{x}, \xi) - j_{\alpha\beta}(\hat{y}, \xi))\right]\mu(d\xi)\Big\} \\ & + \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [c_{\alpha\beta}]_{0,\theta;\mathbb{R}^n}|u|_{0;\mathbb{R}^n}|\hat{x} - \hat{y}|^\theta + \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [f_{\alpha\beta}]_{0,\theta;\mathbb{R}^n}|\hat{x} - \hat{y}|^\theta. \end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma &\leq \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \left\{ \theta |\hat{x} - \hat{y}|^{-2} \text{Tr} \left[(\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y})) (\sigma_{\alpha\beta}(\hat{x}) - \sigma_{\alpha\beta}(\hat{y}))^T \right] \right. \\
&\quad + \theta |\hat{x} - \hat{y}|^{-2} \left(b_{\alpha\beta}(\hat{y}) - b_{\alpha\beta}(\hat{x}) \right) \cdot (\hat{x} - \hat{y}) \\
&\quad + |\hat{x} - \hat{y}|^{-\theta} \int_{\mathbb{R}^n} \left[|\hat{x} - \hat{y} + j_{\alpha\beta}(\hat{x}, \xi) - j_{\alpha\beta}(\hat{y}, \xi)|^\theta - |\hat{x} - \hat{y}|^\theta \right. \\
&\quad \left. - \mathbb{1}_{B_1(0)}(\xi) \theta |\hat{x} - \hat{y}|^{\theta-2} (\hat{x} - \hat{y}) \cdot (j_{\alpha\beta}(\hat{x}, \xi) - j_{\alpha\beta}(\hat{y}, \xi)) \right] \mu(d\xi) \Big\} \\
&\quad + \frac{1}{L} \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [c_{\alpha\beta}]_{0, \theta; \mathbb{R}^n} |u|_{0; \mathbb{R}^n} + \frac{1}{L} \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [f_{\alpha\beta}]_{0, \theta; \mathbb{R}^n} \\
&\leq N_1 + \frac{1}{L} \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [c_{\alpha\beta}]_{0, \theta; \mathbb{R}^n} |u|_{0; \mathbb{R}^n} + \frac{1}{L} \sup_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} [f_{\alpha\beta}]_{0, \theta; \mathbb{R}^n}, \tag{106}
\end{aligned}$$

where N_1 is defined in (102). It now follows from (105) that, if $\gamma > N_1$, we can find a sufficiently large L such that we have a contradiction. Therefore, we have $u \in C^{0, \theta}(\mathbb{R}^n)$. \square

5.3 Semiconcavity

In this section we investigate the semiconcavity of viscosity solutions of (1) and (13). Again we start with equation (1). We impose the following conditions on G and $j(x, \xi)$.

($\bar{H}1$) If $\varphi \in C^{0,1}(\bar{\mathbb{R}}^n)$, there are a constant $1 < \bar{\theta} \leq 2$, a non-negative constant Λ and two positive constants C_3, C_4 such that, for any $x, y, z \in \mathbb{R}^n$, $l_x, l_y, l_z \in \mathbb{R}$, $X, Y, Z \in \mathbb{S}^n$ and $L, \eta > 0$, we have

$$\begin{aligned}
&2G(z, \varphi(z), -\frac{L}{2} D_z \phi(x, y, z), \frac{Z}{2}, l_z) \\
&- G(x, \varphi(x), L D_x \phi(x, y, z) + 2\eta x, X, l_x) - G(y, \varphi(y), L D_y \phi(x, y, z), Y, l_y) \\
&\leq -\gamma(\varphi(x) + \varphi(y) - 2\varphi(z)) + \Lambda(l_x + l_y - 2l_z) + C_3(1 + L)\phi(x, y, z) \\
&\quad + C_4\eta(1 + |x|^2), \tag{107}
\end{aligned}$$

if

$$\begin{aligned}
&\begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & -Z \end{pmatrix} \\
&\leq \frac{L}{\phi(x, y, z)} \left[\bar{\theta}(2\bar{\theta} - 1) |x - y|^{2\bar{\theta}-2} \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right]
\end{aligned}$$

$$+2\eta \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (108)$$

where γ is given by (2) and $\phi(x, y, z) = (|x - y|^{2\bar{\theta}} + |x + y - 2z|^2)^{\frac{1}{2}}$.

($\bar{H}2$) ($H2$) holds and, with the same $\bar{\theta}$ in ($\bar{H}1$) and for any $x, y \in \mathbb{R}^n$, we have

$$|j(x, \xi) + j(y, \xi) - 2j(\frac{x+y}{2}, \xi)| \leq |x - y|^{\bar{\theta}} \rho(\xi) \quad \text{for } \xi \in \mathbb{R}^n.$$

Example 5.3.1. *Since the assumption ($\bar{H}1$) is complicated, we provide a concrete example to show when it is satisfied. We consider the nonlinear convex nonlocal equation*

$$-Tr(\sigma(x)\sigma^T(x)D^2u(x)) + F(I[x, u]) + b(x) \cdot Du(x) + c(x)u(x) + f(x) = 0, \quad \text{in } \mathbb{R}^n, \quad (109)$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Suppose the following conditions are satisfied: there exists a non-negative constant Λ such that, for any $l_x, l_y \in \mathbb{R}$,

$$c \geq \gamma \text{ in } \mathbb{R}^n \text{ and } c \in C^{1, \bar{\theta}-1}(\bar{\mathbb{R}}^n),$$

$$f \text{ is } \bar{\theta}\text{-semiconvex in } \mathbb{R}^n,$$

$$\max\{[\sigma]_{0,1;\mathbb{R}^n}, [\sigma]_{1,\bar{\theta}-1;\mathbb{R}^n}, [b]_{0,1;\mathbb{R}^n}, [b]_{1,\bar{\theta}-1;\mathbb{R}^n}, [f]_{0,1;\mathbb{R}^n}\} < +\infty,$$

$$F \text{ is convex in } \mathbb{R}^n \text{ and } F(l_y) - F(l_x) \leq \Lambda(l_x - l_y). \quad (110)$$

By the estimates on the local terms in Theorem 5.3.5, if equation (109) does not contain the nonlocal term $F(I[x, u])$, then (109) satisfies ($\bar{H}1$). Thus, we only need to estimate the nonlocal terms. For any l_x, l_y, l_z , we have, by (110),

$$\begin{aligned} 2F(l_z) - F(l_x) - F(l_y) &\leq 2F(l_z) - 2F(\frac{l_x + l_y}{2}) + \left(2F(\frac{l_x + l_y}{2}) - F(l_x) - F(l_y)\right) \\ &\leq \Lambda(l_x + l_y - 2l_z). \end{aligned}$$

Therefore, equation (109) satisfies ($\bar{H}1$).

This example can be generalized to equation

$$G(x, u, Du, D^2u) + F(I[x, u]) = 0, \quad \text{in } \mathbb{R}^n, \quad (111)$$

where G satisfies (107) without the last argument if $\varphi \in C^{0,1}(\bar{\mathbb{R}}^n)$ and (108) holds, and F satisfies (110). It is obvious that ($\bar{H}1$) holds for equation (111).

Lemma 5.3.1. *Suppose that the nonlinearity G in (1) is continuous and satisfies (12)-(3). Let u, v, w be bounded functions and be respectively a viscosity subsolution, a viscosity subsolution and a viscosity supersolution of*

$$G(x, u, Du, D^2u, I[x, u]) = 0, \quad \text{in } \mathbb{R}^n,$$

$$G(x, v, Dv, D^2v, I[x, v]) = 0, \quad \text{in } \mathbb{R}^n,$$

$$G(x, w, Dw, D^2w, I[x, w]) = 0, \quad \text{in } \mathbb{R}^n.$$

Let $\psi \in C^2(\mathbb{R}^{3n})$ and $(\hat{x}, \hat{y}, \hat{z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be such that

$$(x, y, z) \mapsto u(x) + v(y) - 2w(z) - \psi(x, y, z)$$

has a global maximum at $(\hat{x}, \hat{y}, \hat{z})$. Furthermore, assume that in a neighborhood of $(\hat{x}, \hat{y}, \hat{z})$ there are continuous functions $g_0, g_1 : \mathbb{R}^{3n} \rightarrow \mathbb{R}$, $g_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $g_1(\hat{x}, \hat{y}, \hat{z}) > 0$, satisfying

$$\begin{aligned} D^2\psi(x, y, z) \leq & g_0(x, y, z) \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + g_1(x, y, z) \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \\ & + \begin{pmatrix} g_2(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Then, for any $0 < \delta < 1$ and $\epsilon_0 > 0$, there are $X, Y, Z \in \mathbb{S}^n$ satisfying

$$\begin{aligned} & \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & -Z \end{pmatrix} - \begin{pmatrix} g_2(\hat{x}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \leq & (1 + \epsilon_0) \left[g_0(\hat{x}, \hat{y}, \hat{z}) \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + g_1(\hat{x}, \hat{y}, \hat{z}) \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right], \end{aligned}$$

such that

$$G(\hat{x}, u(\hat{x}), D_x\psi(\hat{x}, \hat{y}, \hat{z}), X, I^{1,\delta}[\hat{x}, D_x\psi(\hat{x}, \hat{y}, \hat{z}), \psi(\cdot, \hat{y}, \hat{z})] + I^{2,\delta}[\hat{x}, D_x\psi(\hat{x}, \hat{y}, \hat{z}), u(\cdot)]) \leq 0,$$

$$G(\hat{y}, v(\hat{y}), D_y\psi(\hat{x}, \hat{y}, \hat{z}), Y, I^{1,\delta}[\hat{y}, D_y\psi(\hat{x}, \hat{y}, \hat{z}), \psi(\hat{x}, \cdot, \hat{z})] + I^{2,\delta}[\hat{y}, D_y\psi(\hat{x}, \hat{y}, \hat{z}), v(\cdot)]) \leq 0,$$

$$\begin{aligned} & G(\hat{z}, w(\hat{z}), -\frac{1}{2}D_z\psi(\hat{x}, \hat{y}, \hat{z}), \frac{Z}{2}, I^{1,\delta}[\hat{z}, -\frac{D_z\psi(\hat{x}, \hat{y}, \hat{z})}{2}, -\frac{\psi(\hat{x}, \hat{y}, \cdot)}{2}] \\ & \quad + I^{2,\delta}[\hat{z}, -\frac{D_z\psi(\hat{x}, \hat{y}, \hat{z})}{2}, w(\cdot)]) \geq 0. \end{aligned}$$

Proof. This lemma can be deduced from the proof of Theorem 4.9 in [40]. \square

Remark 10. Lemma 5.3.1 is also true for Bellman-Isaacs equations (6).

Lemma 5.3.2. *Suppose that a Lévy measure μ satisfies (12) and $j(x, \xi)$ satisfies assumption $(\bar{H}2)$. Then*

$$\begin{aligned}
M_2 : &= \sup_{\phi(x,y,z) \neq 0} \left\{ \phi(x,y,z)^{-1} \int_{\mathbb{R}^n} \left[\phi(x+j(x,\xi), y+j(y,\xi), z+j(z,\xi)) - \phi(x,y,z) \right. \right. \\
&\quad \mathbb{1}_{B_1(0)}(\xi) \left(D_x \phi(x,y,z), D_y \phi(x,y,z), D_z \phi(x,y,z) \right) \\
&\quad \left. \cdot (j(x,\xi), j(y,\xi), j(z,\xi)) \right] \mu(d\xi) \Big\} \\
&< +\infty,
\end{aligned} \tag{112}$$

where $\phi(x, y, z)$ is defined in $(\bar{H}1)$.

Proof. By direct calculations, we have

$$D\phi(x, y, z) = \frac{1}{\phi(x, y, z)} \left[\bar{\theta} |x - y|^{2\bar{\theta}-2} \begin{pmatrix} x - y \\ y - x \\ 0 \end{pmatrix} + \begin{pmatrix} x + y - 2z \\ x + y - 2z \\ -2x - 2y + 4z \end{pmatrix} \right] \tag{113}$$

and

$$\begin{aligned}
D^2\phi(x, y, z) &= -\frac{1}{\phi(x, y, z)} D\phi(x, y, z) \otimes D\phi(x, y, z) \\
&+ \frac{1}{\phi(x, y, z)} \left[\bar{\theta} |x - y|^{2\bar{\theta}-2} \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\
&+ \bar{\theta}(2\bar{\theta} - 2) |x - y|^{2\bar{\theta}-4} \begin{pmatrix} x - y \\ y - x \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x - y \\ y - x \\ 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \Big] \\
&\leq \frac{1}{\phi(x, y, z)} \left[\bar{\theta}(2\bar{\theta} - 1) |x - y|^{2\bar{\theta}-2} \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right].
\end{aligned} \tag{115}$$

Since $\lim_{\xi \rightarrow 0} \rho(\xi) = 0$, there exists a positive constant $\delta_2 < 1$ such that

$$\sup_{\xi \in B_{\delta_2}(0)} \rho(\xi) \leq \frac{1}{4}.$$

By (113) and (115), we have, for any $x, y, z \in \mathbb{R}^n$ and $\phi(x, y, z) \neq 0$,

$$\begin{aligned}
& \phi(x, y, z)^{-1} \int_{\mathbb{R}^n} \left[\phi(x + j(x, \xi), y + j(y, \xi), z + j(z, \xi)) - \phi(x, y, z) \right. \\
& \quad \left. - \mathbb{1}_{B_1(0)}(\xi) \left(D_x \phi(x, y, z), D_y \phi(x, y, z), D_z \phi(x, y, z) \right) \cdot \left(j(x, \xi), j(y, \xi), j(z, \xi) \right) \right] \mu(d\xi) \\
\leq & \phi(x, y, z)^{-1} \left\{ \int_{B_{\delta_2}(0)} \left[\sup_{0 \leq t \leq 1} \left(j(x, \xi), j(y, \xi), j(z, \xi) \right) \right. \right. \\
& \quad \left. \left. D^2 \phi(x + tj(x, \xi), y + tj(y, \xi), z + tj(z, \xi)) \left(j(x, \xi), j(y, \xi), j(z, \xi) \right)^T \right] \mu(d\xi) \right. \\
& \quad + \int_{B_{\delta_2}^c(0)} \left[\left(|x - y + j(x, \xi) - j(y, \xi)|^{2\bar{\theta}} + |x + y - 2z + j(x, \xi) + j(y, \xi) - 2j(z, \xi)|^2 \right)^{\frac{1}{2}} \right. \\
& \quad \left. - \phi(x, y, z) - \mathbb{1}_{B_1(0)}(\xi) \frac{1}{\phi(x, y, z)} \left(\bar{\theta} |x - y|^{2\bar{\theta}-2} (x - y) \cdot (j(x, \xi) - j(y, \xi)) \right. \right. \\
& \quad \left. \left. + (x + y - 2z) \cdot (j(x, \xi) + j(y, \xi) - 2j(z, \xi)) \right) \right] \mu(d\xi) \Big\} \\
\leq & \phi(x, y, z)^{-1} \left\{ \int_{B_{\delta_2}(0)} \left[\sup_{0 \leq t \leq 1} \frac{1}{\phi(x + tj(x, \xi), y + tj(y, \xi), z + tj(z, \xi))} \right. \right. \\
& \quad \left. \left((j(x, \xi) + j(y, \xi) - 2j(z, \xi))^2 \right. \right. \\
& \quad \left. \left. + \bar{\theta}(2\bar{\theta} - 1) |x - y + t(j(x, \xi) - j(y, \xi))|^{2\bar{\theta}-2} (j(x, \xi) - j(y, \xi))^2 \right) \right] \mu(d\xi) \\
& \quad + \int_{B_{\delta_2}^c(0)} \left[\left(|x - y + j(x, \xi) - j(y, \xi)|^{2\bar{\theta}} + |x + y - 2z + j(x, \xi) + j(y, \xi) - 2j(z, \xi)|^2 \right)^{\frac{1}{2}} \right. \\
& \quad \left. - \phi(x, y, z) - \mathbb{1}_{B_1(0)}(\xi) \frac{1}{\phi(x, y, z)} \left(\bar{\theta} |x - y|^{2\bar{\theta}-2} (x - y) \cdot (j(x, \xi) - j(y, \xi)) \right. \right. \\
& \quad \left. \left. + (x + y - 2z) \cdot (j(x, \xi) + j(y, \xi) - 2j(z, \xi)) \right) \right] \mu(d\xi) \Big\}.
\end{aligned}$$

By $(\bar{H}2)$, we have

$$\begin{aligned}
& |j(x, \xi) + j(y, \xi) - 2j(z, \xi)| \\
& \leq |j(x, \xi) + j(y, \xi) - 2j(\frac{x+y}{2}, \xi)| + |2j(\frac{x+y}{2}, \xi) - 2j(z, \xi)| \\
& \leq \rho(\xi) (|x - y|^{\bar{\theta}} + |x + y - 2z|).
\end{aligned}$$

Using it, we obtain, for any $\xi \in B_{\delta_2}(0)$ and $t \in [0, 1]$,

$$\begin{aligned}
& \phi(x + tj(x, \xi), y + tj(y, \xi), z + j(z, \xi)) \\
& = \left[|x - y + t(j(x, \xi) - j(y, \xi))|^{2\bar{\theta}} + |x + y - 2z + t(j(x, \xi) + j(y, \xi) - 2j(z, \xi))|^2 \right]^{\frac{1}{2}} \\
& \geq \left[\left(\frac{3}{4} \right)^{2\bar{\theta}} |x - y|^{2\bar{\theta}} + \left(\frac{3}{4} |x + y - 2z| - \frac{1}{4} |x - y|^{\bar{\theta}} \right)^2 \right]^{\frac{1}{2}} \\
& \geq \left\{ \left[\left(\frac{3}{4} \right)^{2\bar{\theta}} - \frac{1}{16} \right] |x - y|^{2\bar{\theta}} + \frac{9}{32} |x + y - 2z|^2 \right\}^{\frac{1}{2}} \\
& \geq \frac{1}{2} \phi(x, y, z).
\end{aligned}$$

Therefore, for any $x, y, z \in \mathbb{R}^n$ and $\phi(x, y, z) \neq 0$, we have by (12),

$$\begin{aligned}
& \phi(x, y, z)^{-1} \int_{\mathbb{R}^n} \left[\phi(x + j(x, \xi), y + j(y, \xi), z + j(z, \xi)) - \phi(x, y, z) \right. \\
& \quad \left. - \mathbb{1}_{B_1(0)}(\xi) \left(D_x \phi(x, y, z), D_y \phi(x, y, z), D_z \phi(x, y, z) \right) \cdot \left(j(x, \xi), j(y, \xi), j(z, \xi) \right) \right] \mu(d\xi) \\
& \leq 2 \int_{B_{\delta_2}(0)} \left[2 + \left(\frac{5}{4} \right)^{2\bar{\theta}-2} \bar{\theta} (2\bar{\theta} - 1) \right] \rho(\xi)^2 \mu(d\xi) \\
& \quad + \int_{B_{\delta_2}^c(0)} \left\{ \sqrt{2} \left[(1 + \rho(\xi))^{\bar{\theta}} + \rho(\xi) \right] - 1 \right\} \mu(d\xi) \\
& \quad + (\bar{\theta} + \frac{3}{2}) \int_{B_1(0) \cap B_{\delta_2}^c(0)} \rho(\xi) \mu(d\xi) < +\infty.
\end{aligned} \tag{116}$$

□

Theorem 5.3.3. *Suppose that the nonlinearity G in (1) is continuous, and satisfies (12)-(3) and $(\bar{H}1)$. Suppose that $j(x, \xi)$ satisfies assumption $(\bar{H}2)$. Then, if $u \in C^{0,1}(\mathbb{R}^n)$ is a viscosity solution of (1) and $\gamma > \Lambda M_2 + C_3$ where M_2 is defined in (112), then u is $\bar{\theta}$ -semiconcave in \mathbb{R}^n .*

Proof. Let $\Phi(x, y, z) = u(x) + u(y) - 2u(z) - \psi(x, y, z)$ where $\psi(x, y, z) = L\phi(x, y, z) + \eta|x|^2$ and $\phi(x, y, z)$ is defined in $(\bar{H}1)$. We want to prove, for any $\eta > 0$, we have $\Phi(x, y, z) \leq 0$ for all $x, y, z \in \mathbb{R}^n$ and some fixed sufficiently large L . Otherwise, there exists a positive constant η_0 such that $\sup_{x, y, z \in \mathbb{R}^n} \Phi(x, y, z) > 0$ if $0 < \eta < \eta_0$. By boundedness of u , there is a point $(\hat{x}, \hat{y}, \hat{z})$ such that $\Phi(\hat{x}, \hat{y}, \hat{z}) = \sup_{x, y, z \in \mathbb{R}^n} \Phi(x, y, z) > 0$. Therefore, we have

$$\max\{\eta|\hat{x}|^2, L\phi(\hat{x}, \hat{y}, \hat{z})\} < u(\hat{x}) + u(\hat{y}) - 2u(\hat{z}). \tag{117}$$

By (113) and (115), we have

$$\begin{aligned}
& D^2\psi(\hat{x}, \hat{y}, \hat{z}) \\
& \leq \frac{L}{\phi(\hat{x}, \hat{y}, \hat{z})} \left[\bar{\theta}(2\bar{\theta} - 1) |\hat{x} - \hat{y}|^{2\bar{\theta}-2} \left(\begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right) \right] \\
& \quad + 2\eta \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

By Lemma 5.3.1, since $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (1), for any $0 < \delta < 1$

and $\epsilon_0 > 0$, there are $X, Y, Z \in \mathbb{S}^n$ satisfying

$$\begin{aligned} & \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & -Z \end{pmatrix} - 2\eta \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & \leq \frac{(1 + \epsilon_0)L}{\phi(\hat{x}, \hat{y}, \hat{z})} \left[\bar{\theta}(2\bar{\theta} - 1)|\hat{x} - \hat{y}|^{2\bar{\theta}-2} \begin{pmatrix} I & -I & 0 \\ -I & I & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix} \right], \end{aligned} \quad (118)$$

such that

$$\begin{aligned} G(\hat{x}, u(\hat{x}), LD_x \phi(\hat{x}, \hat{y}, \hat{z}) + 2\eta \hat{x}, X, l_{\hat{x}}) &\leq 0, \\ G(\hat{y}, u(\hat{y}), LD_y \phi(\hat{x}, \hat{y}, \hat{z}), Y, l_{\hat{y}}) &\leq 0, \\ G(\hat{z}, u(\hat{z}), -\frac{L}{2} D_z \phi(\hat{x}, \hat{y}, \hat{z}), \frac{Z}{2}, l_{\hat{z}}) &\geq 0, \end{aligned}$$

where

$$\begin{aligned} l_{\hat{x}} &= I^{1,\delta}[\hat{x}, LD_x \phi(\hat{x}, \hat{y}, \hat{z}) + 2\eta \hat{x}, L\phi(\cdot, \hat{y}, \hat{z}) + \eta|\cdot|^2] \\ &\quad + I^{2,\delta}[\hat{x}, LD_x \phi(\hat{x}, \hat{y}, \hat{z}) + 2\eta \hat{x}, u(\cdot)], \\ l_{\hat{y}} &= I^{1,\delta}[\hat{y}, LD_y \phi(\hat{x}, \hat{y}, \hat{z}), L\phi(\hat{x}, \cdot, \hat{z})] + I^{2,\delta}[\hat{y}, LD_y \phi(\hat{x}, \hat{y}, \hat{z}), u(\cdot)], \\ l_{\hat{z}} &= I^{1,\delta}[\hat{z}, -\frac{L}{2} D_z \phi(\hat{x}, \hat{y}, \hat{z}), -\frac{L}{2} \phi(\hat{x}, \hat{y}, \cdot)] + I^{2,\delta}[\hat{z}, -\frac{L}{2} D_z \phi(\hat{x}, \hat{y}, \hat{z}), u(\cdot)]. \end{aligned}$$

Therefore, by $(\bar{H}1)$ and (117), we have

$$\gamma L\phi(\hat{x}, \hat{y}, \hat{z}) \leq \Lambda(l_{\hat{x}} + l_{\hat{y}} - 2l_{\hat{z}}) + C_3(1 + L)\phi(\hat{x}, \hat{y}, \hat{z}) + C_4\eta(1 + |\hat{x}|^2). \quad (119)$$

We now estimate the integral term $l_{\hat{x}} + l_{\hat{y}} - 2l_{\hat{z}}$.

$$\begin{aligned} & l_{\hat{x}} + l_{\hat{y}} - 2l_{\hat{z}} \\ &= L \int_{B_{\delta}(0)} \left(\phi(\hat{x} + j(\hat{x}, \xi), \hat{y}, \hat{z}) - \phi(\hat{x}, \hat{y}, \hat{z}) - D_x \phi(\hat{x}, \hat{y}, \hat{z}) \cdot j(\hat{x}, \xi) \right) \mu(d\xi) \\ &\quad + \eta \int_{B_{\delta}(0)} \left(|\hat{x} + j(\hat{x}, \xi)|^2 - |\hat{x}|^2 - 2\hat{x} \cdot j(\hat{x}, \xi) \right) \mu(d\xi) \\ &\quad + L \int_{B_{\delta}(0)} \left(\phi(\hat{x}, \hat{y} + j(\hat{y}, \xi), \hat{z}) - \phi(\hat{x}, \hat{y}, \hat{z}) - D_y \phi(\hat{x}, \hat{y}, \hat{z}) \cdot j(\hat{y}, \xi) \right) \mu(d\xi) \\ &\quad + L \int_{B_{\delta}(0)} \left(\phi(\hat{x}, \hat{y}, \hat{z} + j(\hat{z}, \xi)) - \phi(\hat{x}, \hat{y}, \hat{z}) - D_z \phi(\hat{x}, \hat{y}, \hat{z}) \cdot j(\hat{z}, \xi) \right) \mu(d\xi) \\ &\quad + \int_{B_{\delta}^c(0)} \left[u(\hat{x} + j(\hat{x}, \xi)) - u(\hat{x}) + u(\hat{y} + j(\hat{y}, \xi)) - u(\hat{y}) - 2(u(\hat{z} + j(\hat{z}, \xi)) - u(\hat{z})) \right. \\ &\quad \left. - \mathbb{1}_{B_1(0)}(\xi) (LD_x \phi(\hat{x}, \hat{y}, \hat{z}) + 2\eta \hat{x}) \cdot j(\hat{x}, \xi) - \mathbb{1}_{B_1(0)}(\xi) LD_y \phi(\hat{x}, \hat{y}, \hat{z}) \cdot j(\hat{y}, \xi) \right. \\ &\quad \left. - \mathbb{1}_{B_1(0)}(\xi) LD_z \phi(\hat{x}, \hat{y}, \hat{z}) \cdot j(\hat{z}, \xi) \right] \mu(d\xi). \end{aligned}$$

Thus, by (113) and (115), we have

$$\begin{aligned}
& l_{\hat{x}} + l_{\hat{y}} - 2l_{\hat{z}} \\
\leq & L \int_{B_{\delta}(0)} \left[\sup_{0 \leq t \leq 1} \frac{1}{\phi(\hat{x} + tj(\hat{x}, \xi), \hat{y}, \hat{z})} \right. \\
& \quad \left. \left(\bar{\theta}(2\bar{\theta} - 1)|\hat{x} - \hat{y} + tj(\hat{x}, \xi)|^{2\bar{\theta}-2} + 1 \right) |j(\hat{x}, \xi)|^2 \right] \mu(d\xi) \\
& + L \int_{B_{\delta}(0)} \left[\sup_{0 \leq t \leq 1} \frac{1}{\phi(\hat{x}, \hat{y} + tj(\hat{y}, \xi), \hat{z})} \right. \\
& \quad \left. \left(\bar{\theta}(2\bar{\theta} - 1)|\hat{x} - \hat{y} - tj(\hat{y}, \xi)|^{2\bar{\theta}-2} + 1 \right) |j(\hat{y}, \xi)|^2 \right] \mu(d\xi) \\
& + 4L \int_{B_{\delta}(0)} \left(\sup_{0 \leq t \leq 1} \frac{1}{\phi(\hat{x}, \hat{y}, \hat{z} + tj(\hat{z}, \xi))} |j(\hat{z}, \xi)|^2 \right) \mu(d\xi) + \eta \int_{B_{\delta}(0)} |j(\hat{x}, \xi)|^2 \mu(d\xi) \\
& + \int_{B_{\delta}^c(0)} \left[u(\hat{x} + j(\hat{x}, \xi)) - u(\hat{x}) + u(\hat{y} + j(\hat{y}, \xi)) - u(\hat{y}) - 2(u(\hat{z} + j(\hat{z}, \xi)) - u(\hat{z})) \right. \\
& \quad - \mathbb{1}_{B_1(0)}(\xi) L \left(D_x \phi(\hat{x}, \hat{y}, \hat{z}), D_y \phi(\hat{x}, \hat{y}, \hat{z}), D_z \phi(\hat{x}, \hat{y}, \hat{z}) \right) \cdot \left(j(\hat{x}, \xi), j(\hat{y}, \xi), j(\hat{z}, \xi) \right) \\
& \quad \left. - \mathbb{1}_{B_1(0)}(\xi) 2\eta \hat{x} \cdot j(\hat{x}, \xi) \right] \mu(d\xi).
\end{aligned}$$

Since $\Phi(x, y, z)$ attains a global maximum at $(\hat{x}, \hat{y}, \hat{z})$, we have, for any $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
& u(\hat{x} + j(\hat{x}, \xi)) - u(\hat{x}) + u(\hat{y} + j(\hat{y}, \xi)) - u(\hat{y}) - 2(u(\hat{z} + j(\hat{z}, \xi)) - u(\hat{z})) \\
\leq & L\phi(\hat{x} + j(\hat{x}, \xi), \hat{y} + j(\hat{y}, \xi), \hat{z} + j(\hat{z}, \xi)) - L\phi(\hat{x}, \hat{y}, \hat{z}) + \eta|\hat{x} + j(\hat{x}, \xi)|^2 - \eta|\hat{x}|^2.
\end{aligned} \tag{120}$$

By (120), we have

$$\begin{aligned}
& l_{\hat{x}} + l_{\hat{y}} - 2l_{\hat{z}} \\
\leq & L \int_{B_{\delta}(0)} \left[\sup_{0 \leq t \leq 1} \frac{1}{\phi(\hat{x} + tj(\hat{x}, \xi), \hat{y}, \hat{z})} \right. \\
& \quad \left. \left(\bar{\theta}(2\bar{\theta} - 1)|\hat{x} - \hat{y} + tj(\hat{x}, \xi)|^{2\bar{\theta}-2} + 1 \right) |j(\hat{x}, \xi)|^2 \right] \mu(d\xi) \\
& + L \int_{B_{\delta}(0)} \left[\sup_{0 \leq t \leq 1} \frac{1}{\phi(\hat{x}, \hat{y} + tj(\hat{y}, \xi), \hat{z})} \right. \\
& \quad \left. \left(\bar{\theta}(2\bar{\theta} - 1)|\hat{x} - \hat{y} - tj(\hat{y}, \xi)|^{2\bar{\theta}-2} + 1 \right) |j(\hat{y}, \xi)|^2 \right] \mu(d\xi) \\
& + 4L \int_{B_{\delta}(0)} \left(\sup_{0 \leq t \leq 1} \frac{1}{\phi(\hat{x}, \hat{y}, \hat{z} + tj(\hat{z}, \xi))} |j(\hat{z}, \xi)|^2 \right) \mu(d\xi) \\
& + \eta \int_{\mathbb{R}^n} \left(|\hat{x} + j(\hat{x}, \xi)|^2 - |\hat{x}|^2 - \mathbb{1}_{B_1(0)}(\xi) 2\hat{x} \cdot j(\hat{x}, \xi) \right) \mu(d\xi) \\
& + L \int_{B_{\delta}^c(0)} \left[\phi(\hat{x} + j(\hat{x}, \xi), \hat{y} + j(\hat{y}, \xi), \hat{z} + j(\hat{z}, \xi)) - \phi(\hat{x}, \hat{y}, \hat{z}) \right. \\
& \quad \left. - \mathbb{1}_{B_1(0)}(\xi) \left(D_x \phi(\hat{x}, \hat{y}, \hat{z}), D_y \phi(\hat{x}, \hat{y}, \hat{z}), D_z \phi(\hat{x}, \hat{y}, \hat{z}) \right) \cdot \left(j(\hat{x}, \xi), j(\hat{y}, \xi), j(\hat{z}, \xi) \right) \right] \mu(d\xi).
\end{aligned} \tag{121}$$

Similarly as in the proof of Theorem 5.2.3, we have $\eta|\hat{x}|^2 \rightarrow 0$ as $\eta \rightarrow 0$ and

$$\epsilon_1 \leq \phi(\hat{x}, \hat{y}, \hat{z}) \leq \epsilon_1^{-1},$$

where ϵ_1 is a positive constant independent of η . Letting $\delta \rightarrow 0$ and then letting $\eta \rightarrow 0$ in (119), we have, by (12), (121) and ($\bar{H}2$),

$$\begin{aligned} \gamma L \phi(\hat{x}, \hat{y}, \hat{z}) &\leq \Lambda L \int_{\mathbb{R}^n} \left[\phi(\hat{x} + j(\hat{x}, \xi), \hat{y} + j(\hat{y}, \xi), \hat{z} + j(\hat{z}, \xi)) - \phi(\hat{x}, \hat{y}, \hat{z}) \right. \\ &\quad \left. - \mathbb{1}_{B_1(0)}(\xi) \left(D_x \phi(\hat{x}, \hat{y}, \hat{z}), D_y \phi(\hat{x}, \hat{y}, \hat{z}), D_z \phi(\hat{x}, \hat{y}, \hat{z}) \right) \cdot \left(j(\hat{x}, \xi), j(\hat{y}, \xi), j(\hat{z}, \xi) \right) \right] \mu(d\xi) \\ &\quad + C_3(1 + L)\phi(\hat{x}, \hat{y}, \hat{z}). \end{aligned}$$

Therefore, by Lemma 5.3.2,

$$\begin{aligned} \gamma &\leq \Lambda \phi(\hat{x}, \hat{y}, \hat{z})^{-1} \int_{\mathbb{R}^n} \left[\phi(\hat{x} + j(\hat{x}, \xi), \hat{y} + j(\hat{y}, \xi), \hat{z} + j(\hat{z}, \xi)) - \phi(\hat{x}, \hat{y}, \hat{z}) \right. \\ &\quad \left. - \mathbb{1}_{B_1(0)}(\xi) \left(D_x \phi(\hat{x}, \hat{y}, \hat{z}), D_y \phi(\hat{x}, \hat{y}, \hat{z}), D_z \phi(\hat{x}, \hat{y}, \hat{z}) \right) \cdot \left(j(\hat{x}, \xi), j(\hat{y}, \xi), j(\hat{z}, \xi) \right) \right] \mu(d\xi) \\ &\quad + C_3(1 + \frac{1}{L}) \\ &\leq \Lambda M_2 + C_3(1 + \frac{1}{L}) < +\infty, \end{aligned}$$

where M_2 is defined in (112). This yields a contradiction, if $\gamma > \Lambda M_2 + C_3$, for sufficiently large L . Therefore, u is $\bar{\theta}$ -semiconcave in \mathbb{R}^n . \square

Let us consider the semiconcavity of viscosity solutions of the Bellman equation (13). The following estimates will be frequently used in the proof of the semiconcavity.

Lemma 5.3.4. (a) *If f is $\bar{\theta}$ -semiconvex with constant C in \mathbb{R}^n and $[f]_{0,1;\mathbb{R}^n} < +\infty$, then*

$$2f(z) - f(x) - f(y) \leq C|x - y|^{\bar{\theta}} + [f]_{0,1;\mathbb{R}^n}|x + y - 2z|.$$

Moreover, if $[f]_{1,\bar{\theta}-1;\mathbb{R}^n} < +\infty$, then

$$|f(x) + f(y) - 2f(z)| \leq \frac{\sqrt{n}}{2} [f]_{1,\bar{\theta}-1;\mathbb{R}^n} |x - y|^{\bar{\theta}} + [f]_{0,1;\mathbb{R}^n} |x + y - 2z|.$$

(b) *If $f \in C^{0,1}(\bar{\mathbb{R}}^n)$, then*

$$|f(x) - f(z)| \leq 2 \max\{|f|_{0;\mathbb{R}^n}, [f]_{0,1;\mathbb{R}^n}\} \phi(x, y, z)^{\frac{1}{2}},$$

where $\phi(x, y, z)$ is defined in ($\bar{H}1$).

Proof. (a) Since f is $\bar{\theta}$ -semiconvex with constant C in \mathbb{R}^n and $[f]_{0,1;\mathbb{R}^n} < +\infty$,

$$\begin{aligned} 2f(z) - f(x) - f(y) &= 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) + \left(2f(z) - 2f\left(\frac{x+y}{2}\right)\right) \\ &\leq C|x - y|^{\bar{\theta}} + [f]_{0,1;\mathbb{R}^n}|x + y - 2z|. \end{aligned}$$

Moreover, if $[f]_{1,\bar{\theta}-1;\mathbb{R}^n} < +\infty$, then f is $\bar{\theta}$ -semiconvex and $\bar{\theta}$ -semiconcave with a constant $\frac{\sqrt{n}}{2}[f]_{1,\bar{\theta}-1;\mathbb{R}^n}$ in \mathbb{R}^n . Thus, the result follows from the above estimate.

(b) Since $g \in C^{0,1}(\bar{\mathbb{R}}^n)$, then

$$\begin{aligned} |g(x) - g(z)| &\leq |g(x) - g(\frac{x+y}{2})| + |g(\frac{x+y}{2}) - g(z)| \\ &\leq [g]_{0,1;\mathbb{R}^n} \frac{|x-y|}{2} + \left(2[g]_{0;\mathbb{R}^n}[g]_{0,1;\mathbb{R}^n} \frac{|x+y-2z|}{2}\right)^{\frac{1}{2}} \\ &\leq 2 \max\{[g]_{0;\mathbb{R}^n}, [g]_{0,1;\mathbb{R}^n}\} \phi(x, y, z)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 5.3.5. *Suppose that $c_\alpha \geq \gamma$ in \mathbb{R}^n uniformly in $\alpha \in \mathcal{A}$. There exist a positive constant C and $1 < \bar{\theta} \leq 2$ such that (100) holds and*

$$\sup_{\alpha \in \mathcal{A}} \max\{[\sigma_\alpha]_{0,1;\mathbb{R}^n}, [\sigma_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n}, [b_\alpha]_{0,1;\mathbb{R}^n}, [b_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n}, [f_\alpha]_{0,1;\mathbb{R}^n}\} < +\infty. \quad (122)$$

Suppose that the Lévy measure μ satisfies (12), the family $\{j_\alpha(x, \xi)\}$ satisfies assumption ($\bar{H}2$) uniformly in $\alpha \in \mathcal{A}$, and $c_\alpha \in C^{1,\bar{\theta}-1}(\bar{\mathbb{R}}^n)$ and $\{f_\alpha\}$ is uniformly $\bar{\theta}$ -semiconvex with constant C_5 , uniformly in $\alpha \in \mathcal{A}$. Then, if $u \in C^{0,1}(\bar{\mathbb{R}}^n)$ is a viscosity solution of (13) and $\gamma > N_2$ where

$$\begin{aligned} N_2 := & \sup_{\phi(x,y,z) \neq 0, \alpha \in \mathcal{A}} \phi(x, y, z)^{-2} \left\{ \bar{\theta}(2\bar{\theta}-1)|x-y|^{2\bar{\theta}-2} \right. \\ & \text{Tr} \left[(\sigma_\alpha(x) - \sigma_\alpha(y)) (\sigma_\alpha(x) - \sigma_\alpha(y))^T \right] \\ & + \text{Tr} \left[(\sigma_\alpha(x) + \sigma_\alpha(y) - 2\sigma_\alpha(z)) (\sigma_\alpha(x) + \sigma_\alpha(y) - 2\sigma_\alpha(z))^T \right] \\ & + \bar{\theta}|x-y|^{2\bar{\theta}-2} (x-y) \cdot (b_\alpha(y) - b_\alpha(x)) + (x+y-2z) \cdot (2b_\alpha(z) - b_\alpha(x) - b_\alpha(y)) \\ & + \phi(x, y, z) \int_{\mathbb{R}^n} \left[\phi(x + j_\alpha(x, \xi), y + j_\alpha(y, \xi), z + j_\alpha(z, \xi)) - \phi(x, y, z) \right. \\ & \left. - \mathbb{1}_{B_1(0)}(\xi) \left(D_x \phi(x, y, z), D_y \phi(x, y, z), D_z \phi(x, y, z) \right) \right. \\ & \left. \cdot (j_\alpha(x, \xi), j_\alpha(y, \xi), j_\alpha(z, \xi)) \right] \mu(d\xi) \left. \right\} \\ & < +\infty, \end{aligned} \quad (123)$$

then u is $\bar{\theta}$ -semiconcave in \mathbb{R}^n .

Proof. At the beginning of the proof, we will show that the constant N_2 has an upper bound. By (122), Lemma 5.3.4 and the estimates in (116), we have

$$\begin{aligned} N_2 &\leq \bar{\theta}(2\bar{\theta}-1) \sup_{\alpha \in \mathcal{A}} [\sigma_\alpha]_{0,1;\mathbb{R}^n}^2 + \left(\frac{\sqrt{n}}{2} \sup_{\alpha \in \mathcal{A}} [\sigma_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n} + \sup_{\alpha \in \mathcal{A}} [\sigma_\alpha]_{0,1;\mathbb{R}^n} \right)^2 \\ &\quad + \bar{\theta} \sup_{\alpha \in \mathcal{A}} [b_\alpha]_{0,1;\mathbb{R}^n} + \left(\frac{\sqrt{n}}{2} \sup_{\alpha \in \mathcal{A}} [b_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n} + \sup_{\alpha \in \mathcal{A}} [b_\alpha]_{0,1;\mathbb{R}^n} \right) \\ &\quad + 2 \int_{B_{\delta_2}(0)} \left[2 + \left(\frac{5}{4} \right)^{2\bar{\theta}-2} \bar{\theta}(2\bar{\theta}-1) \right] \rho(\xi)^2 \mu(d\xi) \end{aligned}$$

$$\begin{aligned}
& + \int_{B_{\delta_2^c}(0)} \left\{ \sqrt{2} \left[(1 + \rho(\xi))^{\bar{\theta}} + \rho(\xi) \right] - 1 \right\} \mu(d\xi) + \left(\bar{\theta} + \frac{3}{2} \right) \int_{B_1(0) \cap B_{\delta_2^c}(0)} \rho(\xi) \mu(d\xi) \\
& < +\infty,
\end{aligned}$$

where δ_2 was chosen in Lemma 5.3.2.

Then we want to prove that, for any $\eta > 0$, $\Phi(x, y, z) = u(x) + u(y) - 2u(z) - \psi(x, y, z) \leq 0$ for all $x, y, z \in \mathbb{R}^n$ and some fixed sufficiently large L , where $\psi(x, y, z)$ is given in Theorem 5.3.3. Otherwise, there exists a positive constant η_0 such that $\sup_{x, y, z \in \mathbb{R}^n} \Phi(x, y, z) > 0$ if $0 < \eta < \eta_0$. By boundedness of u , there is a point $(\hat{x}, \hat{y}, \hat{z})$ such that $\Phi(\hat{x}, \hat{y}, \hat{z}) = \sup_{x, y, z \in \mathbb{R}^n} \Phi(x, y, z) > 0$. Therefore, we have (117). By Remark 10, since $u \in BUC(\mathbb{R}^n)$ is a viscosity solution of (13), we have, for any $0 < \delta < 1$ and $\epsilon_0 > 0$, there are $X, Y, Z \in \mathbb{S}^n$ satisfying (118) such that

$$\begin{aligned}
& \sup_{\alpha \in \mathcal{A}} \left\{ -Tr(\sigma_\alpha(\hat{x})\sigma_\alpha^T(\hat{x})X) - l_{\hat{x}, \alpha} + b_\alpha(\hat{x}) \cdot D_x \psi(\hat{x}, \hat{y}, \hat{z}) + c_\alpha(\hat{x})u(\hat{x}) + f_\alpha(\hat{x}) \right\} \leq 0, \\
& \sup_{\alpha \in \mathcal{A}} \left\{ -Tr(\sigma_\alpha(\hat{y})\sigma_\alpha^T(\hat{y})Y) - l_{\hat{y}, \alpha} + b_\alpha(\hat{y}) \cdot D_y \psi(\hat{x}, \hat{y}, \hat{z}) + c_\alpha(\hat{y})u(\hat{y}) + f_\alpha(\hat{y}) \right\} \leq 0, \\
& \sup_{\alpha \in \mathcal{A}} \left\{ -Tr(\sigma_\alpha(\hat{z})\sigma_\alpha^T(\hat{z})\frac{Z}{2}) - l_{\hat{z}, \alpha} - b_\alpha(\hat{z}) \cdot \frac{D_z \psi(\hat{x}, \hat{y}, \hat{z})}{2} + c_\alpha(\hat{z})u(\hat{z}) + f_\alpha(\hat{z}) \right\} \geq 0,
\end{aligned}$$

where

$$\begin{aligned}
l_{\hat{x}, \alpha} &= I^{1, \delta}[\hat{x}, D_x \psi(\hat{x}, \hat{y}, \hat{z}), \psi(\cdot, \hat{y}, \hat{z})] + I^{2, \delta}[\hat{x}, D_x \psi(\hat{x}, \hat{y}, \hat{z}), u(\cdot)], \\
l_{\hat{y}, \alpha} &= I^{1, \delta}[\hat{y}, D_y \psi(\hat{x}, \hat{y}, \hat{z}), \psi(\hat{x}, \cdot, \hat{z})] + I^{2, \delta}[\hat{y}, D_y \psi(\hat{x}, \hat{y}, \hat{z}), u(\cdot)], \\
l_{\hat{z}, \alpha} &= I^{1, \delta}[\hat{z}, -\frac{D_z \psi(\hat{x}, \hat{y}, \hat{z})}{2}, -\frac{\psi(\hat{x}, \hat{y}, \cdot)}{2}] + I^{2, \delta}[\hat{z}, -\frac{D_z \psi(\hat{x}, \hat{y}, \hat{z})}{2}, u(\cdot)].
\end{aligned}$$

Thus, for any $\epsilon > 0$, there exists $\alpha_\epsilon \in \mathcal{A}$ such that

$$c_{\alpha_\epsilon}(\hat{x})u(\hat{x}) + c_{\alpha_\epsilon}(\hat{y})u(\hat{y}) - 2c_{\alpha_\epsilon}(\hat{z})u(\hat{z}) \leq L_{\alpha_\epsilon} + N_{\alpha_\epsilon} + \epsilon, \quad (124)$$

where

$$\begin{aligned}
L_{\alpha_\epsilon} &= Tr\left(\sigma_{\alpha_\epsilon}(\hat{x})\sigma_{\alpha_\epsilon}^T(\hat{x})X + \sigma_{\alpha_\epsilon}(\hat{y})\sigma_{\alpha_\epsilon}^T(\hat{y})Y - \sigma_{\alpha_\epsilon}(\hat{z})\sigma_{\alpha_\epsilon}^T(\hat{z})Z\right) \\
&\quad - \left(b_{\alpha_\epsilon}(\hat{x}) \cdot D_x \psi(\hat{x}, \hat{y}, \hat{z}) + b_{\alpha_\epsilon}(\hat{y}) \cdot D_y \psi(\hat{x}, \hat{y}, \hat{z}) + b_{\alpha_\epsilon}(\hat{z}) \cdot D_z \psi(\hat{x}, \hat{y}, \hat{z})\right) \\
&\quad + 2f_{\alpha_\epsilon}(\hat{z}) - f_{\alpha_\epsilon}(\hat{y}) - f_{\alpha_\epsilon}(\hat{x})
\end{aligned}$$

and

$$N_{\alpha_\epsilon} = l_{\hat{x}, \alpha_\epsilon} + l_{\hat{y}, \alpha_\epsilon} - 2l_{\hat{z}, \alpha_\epsilon}.$$

Since $c_\alpha \in C^{1,\bar{\theta}-1}(\bar{\mathbb{R}}^n)$ uniformly in $\alpha \in \mathcal{A}$ and $u \in C^{0,1}(\bar{\mathbb{R}}^n)$, using Lemma 5.3.4, we have

$$\begin{aligned}
& c_{\alpha_\epsilon}(\hat{x})u(\hat{x}) + c_{\alpha_\epsilon}(\hat{y})u(\hat{y}) - 2c_{\alpha_\epsilon}(\hat{z})u(\hat{z}) \\
= & c_{\alpha_\epsilon}(\hat{z})(u(\hat{x}) + u(\hat{y}) - 2u(\hat{z})) + (c_{\alpha_\epsilon}(\hat{x}) + c_{\alpha_\epsilon}(\hat{y}) - 2c_{\alpha_\epsilon}(\hat{z}))u(\hat{z}) \\
& + (c_{\alpha_\epsilon}(\hat{x}) - c_{\alpha_\epsilon}(\hat{z}))(u(\hat{x}) - u(\hat{z})) + (c_{\alpha_\epsilon}(\hat{y}) - c_{\alpha_\epsilon}(\hat{z}))(u(\hat{y}) - u(\hat{z})) \\
\geq & \gamma(u(\hat{x}) + u(\hat{y}) - 2u(\hat{z})) \\
& - |u|_{0;\mathbb{R}^n} \left(\frac{\sqrt{n}}{2} \sup_{\alpha \in \mathcal{A}} [c_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n} |\hat{x} - \hat{y}|^{\bar{\theta}} + \sup_{\alpha \in \mathcal{A}} [c_\alpha]_{0,1;\mathbb{R}^n} |\hat{x} + \hat{y} - 2\hat{z}| \right) \\
& - 8 \max\{|u|_{0;\mathbb{R}^n}, [u]_{0,1;\mathbb{R}^n}\} \sup_{\alpha \in \mathcal{A}} \max\{|c_\alpha|_{0;\mathbb{R}^n}, [c_\alpha]_{0,1;\mathbb{R}^n}\} \phi(\hat{x}, \hat{y}, \hat{z}). \tag{125}
\end{aligned}$$

By (100), (118) and (122), we see that

$$\begin{aligned}
& Tr \left(\sigma_{\alpha_\epsilon}(\hat{x}) \sigma_{\alpha_\epsilon}^T(\hat{x}) X + \sigma_{\alpha_\epsilon}(\hat{y}) \sigma_{\alpha_\epsilon}^T(\hat{y}) Y - \sigma_{\alpha_\epsilon}(\hat{z}) \sigma_{\alpha_\epsilon}^T(\hat{z}) Z \right) \\
\leq & \frac{(1 + \epsilon_0)L}{\phi(\hat{x}, \hat{y}, \hat{z})} \left\{ \bar{\theta}(2\bar{\theta} - 1) |\hat{x} - \hat{y}|^{2\bar{\theta}-2} Tr \left[(\sigma_{\alpha_\epsilon}(\hat{x}) - \sigma_{\alpha_\epsilon}(\hat{y})) (\sigma_{\alpha_\epsilon}(\hat{x}) - \sigma_{\alpha_\epsilon}(\hat{y}))^T \right] \right. \\
& + Tr \left[(\sigma_{\alpha_\epsilon}(\hat{x}) + \sigma_{\alpha_\epsilon}(\hat{y}) - 2\sigma_{\alpha_\epsilon}(\hat{z})) (\sigma_{\alpha_\epsilon}(\hat{x}) + \sigma_{\alpha_\epsilon}(\hat{y}) - 2\sigma_{\alpha_\epsilon}(\hat{z}))^T \right] \left. \right\} \\
& + 2\eta \left(C + \sup_{\alpha \in \mathcal{A}} [\sigma_\alpha]_{0,1;\mathbb{R}^n} |\hat{x}| \right)^2.
\end{aligned}$$

Thus, we can estimate the local term L_{α_ϵ} easily. By (100), (113), (122), uniform $\bar{\theta}$ -semiconvexity of f_α with constant C_5 and Lemma 5.3.4, we have

$$\begin{aligned}
L_{\alpha_\epsilon} \leq & \frac{(1 + \epsilon_0)L}{\phi(\hat{x}, \hat{y}, \hat{z})} \left\{ \bar{\theta}(2\bar{\theta} - 1) |\hat{x} - \hat{y}|^{2\bar{\theta}-2} Tr \left[(\sigma_{\alpha_\epsilon}(\hat{x}) - \sigma_{\alpha_\epsilon}(\hat{y})) (\sigma_{\alpha_\epsilon}(\hat{x}) - \sigma_{\alpha_\epsilon}(\hat{y}))^T \right] \right. \\
& + Tr \left[(\sigma_{\alpha_\epsilon}(\hat{x}) + \sigma_{\alpha_\epsilon}(\hat{y}) - 2\sigma_{\alpha_\epsilon}(\hat{z})) (\sigma_{\alpha_\epsilon}(\hat{x}) + \sigma_{\alpha_\epsilon}(\hat{y}) - 2\sigma_{\alpha_\epsilon}(\hat{z}))^T \right] \left. \right\} \\
& + 2\eta \left(C + \sup_{\alpha \in \mathcal{A}} [\sigma_\alpha]_{0,1;\mathbb{R}^n} |\hat{x}| \right)^2 + \frac{\bar{\theta}L |\hat{x} - \hat{y}|^{2\bar{\theta}-2}}{\phi(\hat{x}, \hat{y}, \hat{z})} (\hat{x} - \hat{y}) \cdot (b_{\alpha_\epsilon}(\hat{y}) - b_{\alpha_\epsilon}(\hat{x})) \\
& + \frac{L}{\phi(\hat{x}, \hat{y}, \hat{z})} (\hat{x} + \hat{y} - 2\hat{z}) \cdot (2b_{\alpha_\epsilon}(\hat{z}) - b_{\alpha_\epsilon}(\hat{x}) - b_{\alpha_\epsilon}(\hat{y})) \\
& + 2\eta (C|\hat{x}| + \sup_{\alpha \in \mathcal{A}} [b_\alpha]_{0,1;\mathbb{R}^n} |\hat{x}|^2) + C_5 |\hat{x} - \hat{y}|^{\bar{\theta}} + \sup_{\alpha \in \mathcal{A}} [f_\alpha]_{0,1;\mathbb{R}^n} |\hat{x} + \hat{y} - 2\hat{z}|. \tag{126}
\end{aligned}$$

Similarly as in the proof of Theorem 5.2.3, we have $\eta|\hat{x}|^2 \rightarrow 0$ as $\eta \rightarrow 0$ and

$$\epsilon_1 \leq \phi(\hat{x}, \hat{y}, \hat{z}) \leq \epsilon_1^{-1},$$

where ϵ_1 is a positive constant independent of η . Letting $\delta \rightarrow 0$, $\eta \rightarrow 0$, $\epsilon \rightarrow 0$ and $\epsilon_0 \rightarrow 0$ in (124), we have, by (117), (125), (126) and the same estimates on the

nonlocal term N_{α_ϵ} as Theorem 5.3.3

$$\begin{aligned}
& \gamma L \phi(\hat{x}, \hat{y}, \hat{z}) \\
\leq & L \sup_{\alpha \in \mathcal{A}} \phi(\hat{x}, \hat{y}, \hat{z})^{-1} \left\{ \bar{\theta} (2\bar{\theta} - 1) |\hat{x} - \hat{y}|^{2\bar{\theta}-2} \text{Tr} \left[(\sigma_\alpha(\hat{x}) - \sigma_\alpha(\hat{y})) (\sigma_\alpha(\hat{x}) - \sigma_\alpha(\hat{y}))^T \right] \right. \\
& + \text{Tr} \left[(\sigma_\alpha(\hat{x}) + \sigma_\alpha(\hat{y}) - 2\sigma_\alpha(\hat{z})) (\sigma_\alpha(\hat{x}) + \sigma_\alpha(\hat{y}) - 2\sigma_\alpha(\hat{z}))^T \right] \\
& + \bar{\theta} |\hat{x} - \hat{y}|^{2\bar{\theta}-2} (\hat{x} - \hat{y}) \cdot (b_\alpha(\hat{y}) - b_\alpha(\hat{x})) + (\hat{x} + \hat{y} - 2\hat{z}) \cdot (2b_\alpha(\hat{z}) - b_\alpha(\hat{x}) - b_\alpha(\hat{y})) \\
& + \phi(\hat{x}, \hat{y}, \hat{z}) \int_{\mathbb{R}^n} \left[\phi(\hat{x} + j_\alpha(\hat{x}, \xi), \hat{y} + j_\alpha(\hat{y}, \xi), \hat{z} + j_\alpha(\hat{z}, \xi)) - \phi(\hat{x}, \hat{y}, \hat{z}) - \mathbb{1}_{B_1(0)}(\xi) \right. \\
& \left. \left(D_x \phi(\hat{x}, \hat{y}, \hat{z}), D_y \phi(\hat{x}, \hat{y}, \hat{z}), D_z \phi(\hat{x}, \hat{y}, \hat{z}) \right) \cdot \left(j_\alpha(\hat{x}, \xi), j_\alpha(\hat{y}, \xi), j_\alpha(\hat{z}, \xi) \right) \right] \mu(d\xi) \left. \right\} \\
& + C_5 |\hat{x} - \hat{y}|^{\bar{\theta}} + \sup_{\alpha \in \mathcal{A}} [f_\alpha]_{0,1;\mathbb{R}^n} |\hat{x} + \hat{y} - 2\hat{z}| \\
& + |u|_{0;\mathbb{R}^n} \left(\frac{\sqrt{n}}{2} \sup_{\alpha \in \mathcal{A}} [c_\alpha]_{1,\bar{\theta}-1;\mathbb{R}^n} |\hat{x} - \hat{y}|^{\bar{\theta}} + \sup_{\alpha \in \mathcal{A}} [c_\alpha]_{0,1;\mathbb{R}^n} |\hat{x} + \hat{y} - 2\hat{z}| \right) \\
& + 8 \max\{|u|_{0;\mathbb{R}^n}, [u]_{0,1;\mathbb{R}^n}\} \sup_{\alpha \in \mathcal{A}} \max\{|c_\alpha|_{0;\mathbb{R}^n}, [c_\alpha]_{0,1;\mathbb{R}^n}\} \phi(\hat{x}, \hat{y}, \hat{z}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma & \leq \sup_{\alpha \in \mathcal{A}} \phi(\hat{x}, \hat{y}, \hat{z})^{-2} \left\{ \bar{\theta} (2\bar{\theta} - 1) |\hat{x} - \hat{y}|^{2\bar{\theta}-2} \text{Tr} \left[(\sigma_\alpha(\hat{x}) - \sigma_\alpha(\hat{y})) (\sigma_\alpha(\hat{x}) - \sigma_\alpha(\hat{y}))^T \right] \right. \\
& + \text{Tr} \left[(\sigma_\alpha(\hat{x}) + \sigma_\alpha(\hat{y}) - 2\sigma_\alpha(\hat{z})) (\sigma_\alpha(\hat{x}) + \sigma_\alpha(\hat{y}) - 2\sigma_\alpha(\hat{z}))^T \right] \\
& + \bar{\theta} |\hat{x} - \hat{y}|^{2\bar{\theta}-2} (\hat{x} - \hat{y}) \cdot (b_\alpha(\hat{y}) - b_\alpha(\hat{x})) + (\hat{x} + \hat{y} - 2\hat{z}) \cdot (2b_\alpha(\hat{z}) - b_\alpha(\hat{x}) - b_\alpha(\hat{y})) \\
& + \phi(\hat{x}, \hat{y}, \hat{z}) \int_{\mathbb{R}^n} \left[\phi(\hat{x} + j_\alpha(\hat{x}, \xi), \hat{y} + j_\alpha(\hat{y}, \xi), \hat{z} + j_\alpha(\hat{z}, \xi)) - \phi(\hat{x}, \hat{y}, \hat{z}) - \mathbb{1}_{B_1(0)}(\xi) \right. \\
& \left. \left(D_x \phi(\hat{x}, \hat{y}, \hat{z}), D_y \phi(\hat{x}, \hat{y}, \hat{z}), D_z \phi(\hat{x}, \hat{y}, \hat{z}) \right) \cdot \left(j_\alpha(\hat{x}, \xi), j_\alpha(\hat{y}, \xi), j_\alpha(\hat{z}, \xi) \right) \right] \mu(d\xi) \left. \right\} \\
& + \frac{C_6}{L} \leq N_2 + \frac{C_6}{L}, \tag{127}
\end{aligned}$$

where N_2 is defined in (123) and C_6 is a positive constant. Hence, if $\gamma > N_2$, we can find a sufficiently large L such that we have a contradiction in (127). Therefore, u is $\bar{\theta}$ -semiconcave in \mathbb{R}^n . \square

CHAPTER VI

INTERIOR REGULARITY FOR NONLOCAL FULLY NONLINEAR EQUATIONS WITH DINI CONTINUOUS TERMS

In this chapter, we will study C^σ estimates of viscosity solutions of nonlocal fully nonlinear equations with Dini continuous terms, see [61].

6.1 Preliminaries

In this chapter, Ω is always assumed to be a bounded domain in \mathbb{R}^n . We first review some properties of

$$Lu := \int_{\mathbb{R}^n} \delta u(x, y) K(y) dy, \quad (128)$$

see [42].

Lemma 6.1.1. *Suppose that $u \in C^4(B_2(0)) \cap L^\infty(\mathbb{R}^n)$ and $L \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$. Then*

$$\|Lu\|_{C^2(B_1(0))} \leq C(\|u\|_{C^4(B_2(0))} + \|u\|_{L^\infty(\mathbb{R}^n)}),$$

where L is defined in (128) and C is a positive constant depending on n , σ_0 and Λ .

Lemma 6.1.2. *Suppose that $u \in C^{\sigma+\alpha}(\mathbb{R}^n)$, $0 \leq K(y) \leq (2 - \sigma)\Lambda|y|^{-n-\sigma}$ and $K(y) = K(-y)$. Then*

$$\|Lu\|_{C^\alpha(\mathbb{R}^n)} \leq C\|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)},$$

where L is defined in (128) and C is a positive constant depending on n , α , σ_0 and Λ .

Lemma 6.1.3. *Suppose that $u \in C^{\sigma+\alpha}(B_2(0)) \cap L^\infty(\mathbb{R}^n)$, $0 \leq K(y) \leq (2 - \sigma)\Lambda|y|^{-n-\sigma}$, $K(y) = K(-y)$ and $|DK(y)| \leq \Lambda|y|^{-n-\sigma-1}$. Then*

$$\|Lu\|_{C^\alpha(B_1(0))} \leq C(\|u\|_{C^{\sigma+\alpha}(B_2(0))} + \|u\|_{L^\infty(\mathbb{R}^n)}),$$

where L is defined in (128) and C is a positive constant depending on n , α , σ_0 and Λ .

Lemma 6.1.4. *Let $v \in C_c^{\sigma+\alpha}(B_{\frac{1}{2}}(0))$ be such that $\|v\|_{C^{\sigma+\alpha}(B_{\frac{1}{2}}(0))} \leq 1$, and $p(x)$ be the Taylor polynomial of v at $x = 0$ of degree $[\sigma + \alpha]$. For any $L \in \mathcal{L}_0(\lambda, \Lambda, \sigma)$, there exists $P \in C_c^\infty(B_{\frac{1}{2}}(0))$ such that $P(x) = p(x)$ in $B_{\frac{1}{4}}(0)$, $\|P\|_{C^4(B_{\frac{1}{2}}(0))} \leq C$ and*

$$LP(0) = Lv(0),$$

where C is a positive constant depending on $n, \lambda, \Lambda, \sigma_0$ and α .

We borrow the following two approximation lemmas from [42].

Lemma 6.1.5. [42, Lemma A.1] *For some $\sigma \geq \sigma_0 > 0$, we consider nonlocal operators I_0, I_1 and I_2 uniformly elliptic with respect to $\mathcal{L}_0(\lambda, \Lambda, \sigma)$. Assume that I_0 is translation invariant and $I_0(0) = 1$.*

Given $M > 0$, a modulus of continuity w_1 and $\epsilon > 0$, there exist $\eta_1 > 0$ and $R > 5$ such that if u, v, I_0, I_1 and I_2 satisfy

$$I_0(v, x) = 0, \quad I_1(u, x) \geq -\eta_1 \quad \text{and} \quad I_2(u, x) \leq \eta_1 \quad \text{in } B_4(0)$$

in the viscosity sense, and

$$\|I_1 - I_0\|_{B_4(0)} \leq \eta_1, \quad \|I_2 - I_0\|_{B_4(0)} \leq \eta_1,$$

$$u = v \quad \text{in } \mathbb{R}^n \setminus B_4(0),$$

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq M \quad \text{in } \mathbb{R}^n,$$

and

$$|u(x) - u(y)| \leq w_1(|x - y|) \quad \text{for any } x \in B_R(0) \setminus B_4(0) \text{ and } y \in \mathbb{R}^n \setminus B_4(0),$$

then $|u - v| \leq \epsilon$ in $B_4(0)$.

and

Lemma 6.1.6. [42, Lemma A.2] *For some $\sigma \geq \sigma_0 > 0$, we consider nonlocal operators I_0, I_1 and I_2 uniformly elliptic with respect to $\mathcal{L}_0(\lambda, \Lambda, \sigma)$. Assume that*

$$I_0 v(x) := \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta v(x, y) K_a(y) dy + h_a(x) \right\} \quad \text{in } B_4(0),$$

where each $K_a \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$ and for some constant $\beta \in (0, 1)$,

$$[h_a]_{C^\beta(B_4(0))} \leq M_0 \quad \text{and} \quad \inf_{a \in \mathcal{A}} h_a(x) = 0, \quad \text{for any } x \in B_4(0).$$

Given $M_0, M_1, M_2, M_3 > 0, R_0 > 5, 0 < \beta, \nu < 1$ and $\epsilon > 0$, there exists η_2 such that if u, v, I_0, I_1 and I_2 satisfy

$$I_0(v, x) = 0, \quad I_1(u, x) \geq -\eta_2 \quad \text{and} \quad I_2(u, x) \leq \eta_2 \quad \text{in } B_4(0),$$

in the viscosity sense and

$$\|I_1 - I_0\|_{B_4(0)} \leq \eta_2, \quad \|I_2 - I_0\|_{B_4(0)} \leq \eta_2$$

$$u = v \quad \text{in } \mathbb{R}^n \setminus B_4(0),$$

$$u = 0 \quad \text{in } \mathbb{R}^n \setminus B_{R_0}(0),$$

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq M_1,$$

$$[u]_{C^\nu(B_{R_0-\tau}(0))} \leq M_2 \tau^{-4}, \quad \text{for any } 0 < \tau < 1,$$

$$[v]_{C^{\sigma+\beta}(B_{4-\tau}(0))} \leq M_3 \tau^{-4}, \quad \text{for any } 0 < \tau < 1,$$

then $|u - v| \leq \epsilon$ in $B_4(0)$.

We now introduce a modification of Evans-Krylov theorem for concave translation invariant nonlocal fully nonlinear equations.

Theorem 6.1.7. [42, Theorem 2.1] *Assume that $K_a(y) \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$ with $2 > \sigma \geq \sigma_0 > 1$ and b_a is a constant for any $a \in \mathcal{A}$. If u is a bounded viscosity solution of*

$$\inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta u(x, y) K_a(y) dy + b_a \right\} = 0, \quad \text{in } B_1(0),$$

then $u \in C^{\sigma+\bar{\alpha}}(B_{\frac{1}{2}}(0))$ with

$$\|u\|_{C^{\sigma+\bar{\alpha}}(B_{\frac{1}{2}}(0))} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + |\inf_a b_a|),$$

where $\bar{\alpha}$ and C are positive constants depending on n, σ_0, λ and Λ .

In the rest of this chapter, $\bar{\alpha}$ will always be the constant from Theorem 6.1.7. We recall the definition of Dini modulus of continuity.

Definition 10. *We say that $w(t)$ is a Dini modulus of continuity, if it satisfies*

$$\int_0^{t_0} \frac{w(r)}{r} dr < +\infty, \quad \text{for some } t_0 > 0.$$

We will make some additional assumption on our Dini modulus of continuity $w(t)$. Let $\bar{\beta} > 0$ and $0 < \sigma < 2$.

(H1) $_{\bar{\beta}}$ There exists some $0 < \beta < \bar{\beta}$ such that

$$\lim_{\mu \rightarrow 0^+} \sup_{i \in \mathbb{N}} \frac{\mu^\beta w(\mu^i)}{w(\mu^{i+1})} = 0. \quad (129)$$

(H1) $_{\bar{\beta}, \sigma}$ There exists some $0 < \beta < \min\{2 - \sigma, \bar{\beta}\}$ such that (129) holds.

(H2) $_{\bar{\beta}, \sigma}$ Let $w(t)$ be a Dini modulus of continuity satisfying (H1) $_{\bar{\beta}, \sigma}$. There exists another Dini modulus of continuity $\tilde{w}(t)$ satisfying (H1) $_{\bar{\beta}, \sigma}$ such that, for any small $0 < s \leq 1$ and $0 \leq t \leq 1$ we have

$$w(st) \leq \eta(s) \tilde{w}(t),$$

where $\eta(s)$ is a positive function of s such that $\lim_{s \rightarrow 0^+} \eta(s) = 0$.

Remark 11. For any $\bar{\beta} > 0$ and $0 < \sigma < 2$, we define

$$\mathcal{S}_{\bar{\beta},\sigma} := \{ \text{Dini modulus of continuity satisfying } (H2)_{\bar{\beta},\sigma} \}.$$

It is obvious that $w(t) = t^\alpha \in \mathcal{S}_{\bar{\beta},\sigma}$ for any $0 < \alpha < \min\{\bar{\beta}, 2 - \sigma\}$ and $\cap_{\bar{\beta} > 0, 0 < \sigma < 2} \mathcal{S}_{\bar{\beta},\sigma}$ does not contain any modulus of $w(t) = t^\alpha$.

Lemma 6.1.8. $\cap_{\bar{\beta} > 0, 0 < \sigma < 2} \mathcal{S}_{\bar{\beta},\sigma} \neq \emptyset$.

Proof. We claim that $w(t) = (\ln \frac{1}{t})^{\kappa-1} \in \cap_{\bar{\beta} > 0, 0 < \sigma < 2} \mathcal{S}_{\bar{\beta},\sigma}$ for any $\kappa < 0$. For any fixed $\bar{\beta} > 0$ and $0 < \sigma < 2$, it is easy to verify that $w(t)$ is a Dini modulus of continuity satisfying $(H1)_{\bar{\beta},\sigma}$. Now let us prove that $w(t)$ satisfies $(H2)_{\bar{\beta},\sigma}$. For any $0 < s < 1$, we have

$$w(st) = (\ln \frac{1}{st})^{\kappa-1} = \frac{(\ln \frac{1}{st})^{\kappa-1}}{(\ln \frac{1}{t})^{\frac{\kappa}{2}-1}} (\ln \frac{1}{t})^{\frac{\kappa}{2}-1}.$$

We notice that $(\ln \frac{1}{t})^{\frac{\kappa}{2}-1}$ is also a Dini modulus of continuity satisfying $(H1)_{\bar{\beta},\sigma}$. For any $\epsilon > 0$, there exists a sufficiently small constant $\delta_0 > 0$ depending only on ϵ such that

$$\frac{(\ln \frac{1}{st})^{\kappa-1}}{(\ln \frac{1}{t})^{\frac{\kappa}{2}-1}} = \frac{(\ln \frac{1}{s} + \ln \frac{1}{t})^{\kappa-1}}{(\ln \frac{1}{t})^{\frac{\kappa}{2}-1}} < \epsilon, \quad \text{if } t < \delta_0.$$

Then there exists a sufficiently small constant $\delta_1 > 0$ depending only on ϵ such that

$$\frac{(\ln \frac{1}{st})^{\kappa-1}}{(\ln \frac{1}{t})^{\frac{\kappa}{2}-1}} < \epsilon, \quad \text{if } \delta_0 \leq t < 1 \text{ and } 0 < s < \delta_1.$$

□

6.2 A recursive Evans-Krylov theorem

The following theorem is a version of the recursive Evans-Krylov theorem we will use to prove C^σ interior regularity.

Theorem 6.2.1. Assume that $2 > \sigma \geq \sigma_0 > 0$, b_a is a constant and $K_a(y) \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$ for any $a \in \mathcal{A}$. Assume that w is a modulus of continuity which satisfies $(H1)_{\bar{\beta}}$ where $\bar{\beta}$ depends on $n, \sigma_0, \lambda, \Lambda$. For each $m \in \mathbb{N} \cup \{0\}$, let $\{v_l\}_{l=0}^m$ be a sequence of functions satisfying (18) in the viscosity sense for any $j = 0, 1, \dots, m$, where $K_a^j(x) := \rho^{j(n+\sigma)} K_a(\rho^j x)$ and $\rho \in (0, 1)$. Suppose that $\|v_l\|_{L^\infty(\mathbb{R}^n)} \leq 1$ for any $l = 0, 1, \dots, m$ and $|\inf_{a \in \mathcal{A}} b_a| \leq 1$. Then, there exist a sufficiently large constant $C > 0$ and a sufficiently small constant $\rho_0 > 0$, both of which depend on $n, \sigma_0, \lambda, \Lambda$ and w , such that $v_l \in C^{\sigma+\bar{\beta}}(B_1(0))$ and, if $\rho \leq \rho_0$, we have

$$\|v_l\|_{C^{\sigma+\bar{\beta}}(B_1(0))} \leq C, \quad \text{for any } l = 0, 1, \dots, m. \quad (130)$$

Remark 12. If $\sigma_0 > 1$, then Theorem 6.2.1 holds for $\bar{\beta} = \bar{\alpha}$.

Proof of Theorem 6.2.1. We will give the proof of Theorem 6.2.1 in the case $\sigma_0 > 1$. For the case $0 < \sigma_0 \leq 1$ the proof is similar. We adapt the approach from [42].

Let M be a sufficiently large constant to be fixed later. By normalization, we can assume that

$$\|v_l\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{M} \quad \text{and} \quad |\inf_{a \in \mathcal{A}} b_a| \leq \frac{1}{M}, \quad \text{for any } l = 0, 1, \dots, m.$$

Then we need to prove that (130) holds for $C = 1$.

We will prove Theorem 6.2.1 by induction on m . For the case of $m = 0$, (130) holds for $\bar{\beta} = \bar{\alpha}$ by Theorem 6.1.7. Now we assume that Theorem 6.2.1 is true up to $m = i$ for any positive integer i . We want to show that the theorem is also true for $m = i + 1$. Define

$$R(x) = \sum_{l=0}^i \rho^{-(i-l)\sigma} w^{-1}(\rho^i) w(\rho^l) v_l(\rho^{i-l}x),$$

and, for any function v

$$v_\rho^l(x) = \rho^{-\sigma} \frac{w(\rho^l)}{w(\rho^{l+1})} v(\rho x).$$

By (18), we have

$$\inf_{a \in \mathcal{A}} \{L_a^{i+1} R_\rho^i(x) + w^{-1}(\rho^{i+1}) b_a\} = 0, \quad \text{in } B_{\frac{5}{\rho}}(0),$$

where L_a^{i+1} is the linear operator with kernel $K_a^{i+1} \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$. Hence, there exists $\bar{a} \in \mathcal{A}$ such that

$$0 \leq L_{\bar{a}}^{i+1} R_\rho^i(0) + w^{-1}(\rho^{i+1}) b_{\bar{a}} < \rho^{\bar{\alpha}-\alpha}, \quad (131)$$

where α is given by $(H1)_{\bar{\alpha}}$. Let $\eta_0 = 1$ in $B_{\frac{1}{4}}(0)$ and $\eta_0 \in C_c^\infty(B_{\frac{1}{2}}(0))$ be a fixed cut-off function. Let

$$v_l = v_l \eta_0 + v_l(1 - \eta_0) =: v_l^1 + v_l^2,$$

and $p_l(x)$ be the Talyor polynomial of $v_l^1(x)$ at $x = 0$ of degree $[\sigma + \bar{\alpha}]$. By Lemma 6.1.4, there exists $P_l \in C_c^\infty(B_{\frac{1}{2}}(0))$ such that $P_l(x) = p_l(x)$ in $B_{\frac{1}{4}}(0)$ and $\|P_l\|_{C^4(B_{\frac{1}{2}}(0))} \leq C$ and

$$L_{\bar{a}}^l P_l(0) = L_{\bar{a}}^l v_l^1(0). \quad (132)$$

Let

$$v_l = (v_l^1 - P_l) + (v_l^2 + P_l) =: V_l^1 + V_l^2.$$

Thus, we have

$$\|V_l^1\|_{L^\infty(\mathbb{R}^n)} + \|V_l^2\|_{L^\infty(\mathbb{R}^n)} \leq C, \quad V_l^1(0) = 0,$$

$$V_l^1 \in C_c^{\sigma+\bar{\alpha}}(B_{\frac{1}{2}}(0)), \quad \|V_l^1\|_{C^{\sigma+\bar{\alpha}}(\mathbb{R}^n)} + \|V_l^2\|_{C^{\sigma+\bar{\alpha}}(B_1(0))} \leq C, \quad (133)$$

$$V_l^1 = v_l - p_l \text{ in } B_{\frac{1}{4}}(0), V_l^2 = p_l \text{ in } B_{\frac{1}{4}}(0), \|V_l^1(x)\| \leq C|x|^{\sigma+\bar{\alpha}} \text{ in } \mathbb{R}^n.$$

Decompose $R(x)$ as

$$R(x) = R^{(1)}(x) + R^{(2)}(x),$$

where

$$R^{(1)}(x) = \sum_{l=0}^i \rho^{-(i-l)\sigma} w^{-1}(\rho^i) w(\rho^l) V_l^1(\rho^{i-l}x),$$

and

$$R^{(2)}(x) = \sum_{l=0}^i \rho^{-(i-l)\sigma} w^{-1}(\rho^i) w(\rho^l) V_l^2(\rho^{i-l}x).$$

Then, we have that, for each $a \in \mathcal{A}$

$$\begin{aligned} L_a^{i+1} R_\rho^{(1)i}(x) &= \sum_{l=0}^i \int_{\mathbb{R}^n} \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) \delta V_l^1(\rho^{i+1-l}x, \rho^{i+1-l}y) K_a^{i+1}(y) dy \\ &= \sum_{l=0}^i \int_{\mathbb{R}^n} w^{-1}(\rho^{i+1}) w(\rho^l) \delta V_l^1(\rho^{i+1-l}x, y) K_a^l(y) dy \\ &= \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} (L_a^l V_l^1)(\rho^{i+1-l}x) \end{aligned} \quad (134)$$

and

$$L_a^{i+1} R_\rho^{(2)i}(x) = \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} (L_a^l V_l^2)(\rho^{i+1-l}x). \quad (135)$$

It follows from (131) and (132) that

$$L_{\bar{a}}^{i+1} R_\rho^{(1)i}(0) = 0, \quad (136)$$

$$0 \leq L_{\bar{a}}^{i+1} R_\rho^{(2)i}(0) + w^{-1}(\rho^{i+1}) b_{\bar{a}} \leq \rho^{\bar{\alpha}-\alpha}. \quad (137)$$

By $(H1)_{\bar{\alpha}}$, (133), (134), (136) and Lemma 6.1.2, we have, for any $x \in \mathbb{R}^n$

$$\begin{aligned} |L_{\bar{a}}^{i+1} R_\rho^{(1)i}(x)| &= |L_{\bar{a}}^{i+1} R_\rho^{(1)i}(x) - L_{\bar{a}}^{i+1} R_\rho^{(1)i}(0)| \\ &\leq \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} |L_a^l V_l^1(\rho^{i+1-l}x) - L_a^l V_l^1(0)| \\ &\leq C|x|^{\bar{\alpha}} \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} \rho^{(i+1-l)\bar{\alpha}} \|V_l^1\|_{C^{\sigma+\bar{\alpha}}(\mathbb{R}^n)} \\ &\leq C|x|^{\bar{\alpha}} \sum_{l=0}^i \rho^{(i+1-l)(\bar{\alpha}-\alpha)} \\ &\leq C\rho^{\bar{\alpha}-\alpha} |x|^{\bar{\alpha}}. \end{aligned} \quad (138)$$

Using $(H1)_{\bar{\alpha}}$, (133), (135) and Lemma 6.1.3, we have, for any $x \in B_5(0)$

$$\begin{aligned}
|L_{\bar{a}}^{i+1}R_{\rho}^{(2)i}(x) - L_{\bar{a}}^{i+1}R_{\rho}^{(2)i}(0)| &\leq \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} |L_{\bar{a}}^l V_l^2(\rho^{i+1-l}x) - L_{\bar{a}}^l V_l^2(0)| \\
&\leq C|x|^{\bar{\alpha}} \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} \rho^{(i+1-l)\bar{\alpha}} (\|V_l^2\|_{C^{\sigma+\bar{\alpha}}(B_1(0))} + \|V_l^2\|_{L^{\infty}(\mathbb{R}^n)}) \\
&\leq C\rho^{\bar{\alpha}-\alpha}|x|^{\bar{\alpha}}.
\end{aligned} \tag{139}$$

Thus, by (137) and (139), we have

$$|L_{\bar{a}}^{i+1}R_{\rho}^{(2)i}(x) + w^{-1}(\rho^{i+1})b_{\bar{a}}| \leq C\rho^{\bar{\alpha}-\alpha}(|x|^{\bar{\alpha}} + 1), \quad \text{for any } x \in B_5(0). \tag{140}$$

We define

$$\tilde{v}_{i+1} := v_{i+1} + R_{\rho}^{(1)i}.$$

By (133), we have

$$\begin{aligned}
|\tilde{v}_{i+1}(y)| &\leq \|v_{i+1}\|_{L^{\infty}(\mathbb{R}^n)} + |R_{\rho}^{(1)i}(y)| \\
&\leq \frac{1}{M} + \sum_{l=0}^i \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) V_l^1(\rho^{i+1-l}y) \\
&\leq \frac{1}{M} + \sum_{l=0}^i \rho^{-(i+1-l)(\sigma+\alpha)} |\rho^{i+1-l}y|^{\sigma+\bar{\alpha}} \\
&\leq \frac{1}{M} + \rho^{\bar{\alpha}-\alpha} |y|^{\sigma+\bar{\alpha}}.
\end{aligned} \tag{141}$$

By the definition of \tilde{v}_{i+1} , the following two equations are equivalent

$$\inf_{a \in \mathcal{A}} \{L_a^{i+1}(v_{i+1} + R_{\rho}^i)(x) + w^{-1}(\rho^{i+1})b_a\} = 0, \quad \text{in } B_5(0), \tag{142}$$

and

$$\inf_{a \in \mathcal{A}} \{L_a^{i+1}(\tilde{v}_{i+1} + R_{\rho}^{(2)i})(x) + w^{-1}(\rho^{i+1})b_a\} = 0, \quad \text{in } B_5(0). \tag{143}$$

By (138), (140), (142) and (143), we have

$$\begin{aligned}
L_{\bar{a}}^{i+1}v_{i+1}(x) &\geq -C\rho^{\bar{\alpha}-\alpha}, \quad \text{in } B_5(0), \\
L_{\bar{a}}^{i+1}\tilde{v}_{i+1}(x) &\geq -C\rho^{\bar{\alpha}-\alpha}, \quad \text{in } B_5(0).
\end{aligned}$$

Lemma 6.2.2. *Let K be a symmetric kernel satisfying $0 \leq K(y) \leq (2 - \sigma)\Lambda|y|^{-n-\sigma}$. Then, for any smooth function $\tilde{\eta}$ such that*

$$0 \leq \tilde{\eta}(x) \leq 1 \text{ in } \mathbb{R}^n, \quad \tilde{\eta}(x) = \tilde{\eta}(-x) \text{ in } \mathbb{R}^n, \quad \tilde{\eta}(x) = 0 \text{ in } \mathbb{R}^n \setminus B_{\frac{4}{5}}(0), \quad \tilde{\eta}(x) = 1 \text{ in } B_{\frac{3}{4}}(0),$$

we have

$$M_{\mathcal{L}_2}^+(\tilde{\eta}(x) \int_{B_1(0)} \delta \tilde{v}_{i+1}(x, y) K(y) dy) \geq -C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}), \quad \text{in } B_{\frac{3}{5}}(0).$$

Proof. Define

$$\phi_k(y) = \mathbb{1}_{B_1(0) \setminus B_{\frac{1}{k}}(0)}(y)K(y)$$

and

$$T_k v(x) = \int_{\mathbb{R}^n} \delta v(x, y) \phi_k(y) dy, \quad \text{for any function } v.$$

By (143), we have

$$L_a^{i+1} \tilde{v}_{i+1}(x) + L_a^{i+1} R_\rho^{(2)i}(x) + w^{-1}(\rho^{i+1}) b_a \geq 0, \quad \text{for any } x \in B_3(0) \text{ and } a \in \mathcal{A}.$$

It follows that, for any $x \in B_{\frac{3}{2}}(0)$

$$\begin{aligned} 0 &\leq (L_a^{i+1} \tilde{v}_{i+1} + L_a^{i+1} R_\rho^{(2)i} + w^{-1}(\rho^{i+1}) b_a) * \phi_k(x) \\ &\leq L_a^{i+1} (\tilde{v}_{i+1} * \phi_k)(x) + L_a^{i+1} R_\rho^{(2)i} * \phi_k(x) + w^{-1}(\rho^{i+1}) b_a \|\phi_k\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

It also follows from (143) that

$$\inf_{a \in \mathcal{A}} \left\{ \|\phi_k\|_{L^1(\mathbb{R}^n)} (L_a^{i+1} \tilde{v}_{i+1}(x) + L_a^{i+1} R_\rho^{(2)i}(x) + w^{-1}(\rho^{i+1}) b_a) \right\} = 0, \quad \text{for any } x \in B_3(0).$$

Thus, for any $x \in B_{\frac{3}{2}}(0)$

$$\sup_{a \in \mathcal{A}} L_a^{i+1} (\tilde{v}_{i+1} * \phi_k - \|\phi_k\|_{L^1(\mathbb{R}^n)} \tilde{v}_{i+1})(x) + \sup_{a \in \mathcal{A}} \{ L_a^{i+1} R_\rho^{(2)i} * \phi_k(x) - \|\phi_k\|_{L^1(\mathbb{R}^n)} L_a^{i+1} R_\rho^{(2)i}(x) \} \geq 0.$$

By (133), (135) and Lemma 6.1.1, we have, for any $x \in B_{\frac{3}{2}}(0)$ and $a \in \mathcal{A}$

$$\begin{aligned} &2 |L_a^{i+1} R_\rho^{(2)i} * \phi_k(x) - \|\phi_k\|_{L^1(\mathbb{R}^n)} L_a^{i+1} R_\rho^{(2)i}(x)| \\ &\leq \left| \int_{B_1(0) \setminus B_{\frac{1}{k}}(0)} \delta(L_a^{i+1} R_\rho^{(2)i})(x, y) K(y) dy \right| \\ &\leq \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} \int_{B_1(0) \setminus B_{\frac{1}{k}}(0)} \left| \delta L_a^l V_l^2(\rho^{i+1-l} x, \rho^{i+1-l} y) \right| K(y) dy \\ &\leq \sum_{l=0}^i \rho^{(i+1-l)\sigma} \frac{w(\rho^l)}{w(\rho^{i+1})} \int_{B_{\rho^{i+1-l}}(0) \setminus B_{\frac{\rho^{i+1-l}}{k}}(0)} \left| \delta L_a^l V_l^2(\rho^{i+1-l} x, y) \right| K^{-(i+1-l)}(y) dy \\ &\leq \sum_{l=0}^i \rho^{(i+1-l)\sigma} \frac{w(\rho^l)}{w(\rho^{i+1})} \int_{B_{\rho^{i+1-l}}(0)} \|L_a^l V_l^2\|_{C^2(B_{\frac{1}{8}}(0))} |y|^2 K^{-(i+1-l)}(y) dy \\ &\leq C \sum_{l=0}^i \rho^{(i+1-l)\sigma} \frac{w(\rho^l)}{w(\rho^{i+1})} (\|V_l^2\|_{C^4(B_{\frac{1}{4}}(0))} + \|V_l^2\|_{L^\infty(\mathbb{R}^n)}) \int_{B_{\rho^{i+1-l}}(0)} \frac{(2-\sigma)\Lambda|y|^2}{|y|^{n+\sigma}} dy \\ &\leq C \sum_{l=0}^i \rho^{(i+1-l)(2-\alpha)} \leq C \rho^{2-\alpha}. \end{aligned}$$

Therefore,

$$M_{\mathcal{L}_2}^+ (\tilde{v}_{i+1} * \phi_k - \|\phi_k\|_{L^1(\mathbb{R}^n)} \tilde{v}_{i+1})(x) \geq -C \rho^{2-\alpha}, \quad \text{in } B_{\frac{3}{2}}(0).$$

Thus, we have

$$M_{\mathcal{L}_2}^+(T_k \tilde{v}_{i+1})(x) \geq -C\rho^{2-\alpha}, \quad \text{in } B_{\frac{3}{2}}(0). \quad (144)$$

Let \bar{L} be any operator with kernel $\bar{K} \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$. For any $x \in B_{\frac{3}{5}}(0)$, we have

$$\begin{aligned} \bar{L}(\tilde{\eta} T_k \tilde{v}_{i+1})(x) &= \int_{\mathbb{R}^n} \delta(T_k \tilde{v}_{i+1})(x, y) \bar{K}(y) dy - \int_{\mathbb{R}^n} \delta((1 - \tilde{\eta}) T_k \tilde{v}_{i+1})(x, y) \bar{K}(y) dy \\ &= \bar{L}(T_k \tilde{v}_{i+1})(x) - 2 \int_{\mathbb{R}^n} (1 - \tilde{\eta}(x - y)) T_k \tilde{v}_{i+1}(x - y) \bar{K}(y) dy. \end{aligned} \quad (145)$$

We now estimate the second term in (145). For any $x \in B_{\frac{3}{5}}(0)$

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} T_k v_{i+1}(x - y) (1 - \tilde{\eta}(x - y)) \bar{K}(y) dy \right| \\ &= \left| \int_{\mathbb{R}^n} v_{i+1}(x - y) T_k((1 - \tilde{\eta}(x - \cdot)) \bar{K}(\cdot))(y) dy \right| \\ &\leq \|v_{i+1}\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \int_{B_1(0)} \left| (1 - \tilde{\eta}(x - y - z)) \bar{K}(y + z) \right. \\ &\quad \left. + (1 - \tilde{\eta}(x - y + z)) \bar{K}(y - z) - 2(1 - \tilde{\eta}(x - y)) \bar{K}(y) \right| K(z) dz dy \\ &\leq C \|v_{i+1}\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{M}, \end{aligned} \quad (146)$$

and, by (133) and Lemma 6.1.2,

$$\begin{aligned} |T_k R_\rho^{(1)i}(x)| &= \left| \int_{B_1(0) \setminus B_{\frac{1}{k}}(0)} \delta R_\rho^{(1)i}(x, y) K(y) dy \right| \\ &= \left| \sum_{l=0}^i \int_{B_1(0) \setminus B_{\frac{1}{k}}(0)} \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) \delta V_l^1(\rho^{i+1-l}x, \rho^{i+1-l}y) K(y) dy \right| \\ &= \left| \sum_{l=0}^i \int_{B_{\rho^{i+1-l}}(0) \setminus B_{\frac{\rho^{i+1-l}}{k}}(0)} w^{-1}(\rho^{i+1}) w(\rho^l) \delta V_l^1(\rho^{i+1-l}x, y) K^{-(i+1-l)}(y) dy \right| \\ &\leq \sum_{l=0}^i w^{-1}(\rho^{i+1}) w(\rho^l) \left| \int_{B_{\rho^{i+1-l}}(0) \setminus B_{\frac{\rho^{i+1-l}}{k}}(0)} (\delta V_l^1(\rho^{i+1-l}x, y) - \delta V_l^1(0, y)) K^{-(i+1-l)}(y) dy \right| \\ &\quad + \sum_{l=0}^i w^{-1}(\rho^{i+1}) w(\rho^l) \left| \int_{B_{\rho^{i+1-l}}(0) \setminus B_{\frac{\rho^{i+1-l}}{k}}(0)} \delta V_l^1(0, y) K^{-(i+1-l)}(y) dy \right| \\ &\leq C \sum_{l=0}^i w^{-1}(\rho^{i+1}) w(\rho^l) |\rho^{i+1-l}x|^{\bar{\alpha}} \\ &\quad + C \sum_{l=0}^i w^{-1}(\rho^{i+1}) w(\rho^l) \int_{B_{\rho^{i+1-l}}(0)} \frac{(2 - \sigma)\Lambda |y|^{\sigma + \bar{\alpha}}}{|y|^{n + \sigma}} dy \\ &\leq C \rho^{\bar{\alpha} - \alpha} (1 + |x|^{\bar{\alpha}}). \end{aligned} \quad (147)$$

Since $\sigma > \bar{\alpha}$ holds, we have, for any $x \in B_{\frac{3}{5}}(0)$

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n} (1 - \tilde{\eta}(x - y)) T_k R_\rho^{(1)i}(x - y) \bar{K}(y) dy \right| \\
&= \left| \int_{\mathbb{R}^n} (1 - \tilde{\eta}(y)) T_k R_\rho^{(1)i}(y) \bar{K}(x - y) dy \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_{\frac{3}{4}}(0)} (1 - \tilde{\eta}(y)) T_k R_\rho^{(1)i}(y) \bar{K}(x - y) dy \right| \\
&\leq C \rho^{\bar{\alpha} - \alpha} \int_{|y| > \frac{1}{64}} \frac{(2 - \sigma) \Lambda}{|y|^{n + \sigma - \bar{\alpha}}} dy \leq C \rho^{\bar{\alpha} - \alpha}.
\end{aligned} \tag{148}$$

Taking the supremum of all $\bar{K} \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$ in (145) and using (144), (146) and (148), we have, for any $x \in B_{\frac{3}{5}}(0)$

$$\begin{aligned}
M_{\mathcal{L}_2}^+(\tilde{\eta} T_k \tilde{v}_{i+1})(x) &\geq -C \rho^{2 - \alpha} - \frac{C}{M} - C \rho^{\bar{\alpha} - \alpha} \\
&\geq -C(\rho^{\bar{\alpha} - \alpha} + \frac{1}{M}).
\end{aligned}$$

By Theorem 6.1.7, we know that $\tilde{v}_{i+1} \in C^{\sigma + \bar{\alpha}}(B_4(0))$. Thus

$$\int_{B_1(0) \setminus B_{\frac{1}{k}}(0)} \delta \tilde{v}_{i+1}(x, y) K(y) dy \rightarrow \int_{B_1(0)} \delta \tilde{v}_{i+1}(x, y) K(y) dy, \quad \text{in } B_{\frac{3}{2}}(0) \text{ uniformly,}$$

as $k \rightarrow +\infty$. It is obvious that, in $L^1(\mathbb{R}^n, \frac{1}{1 + |x|^{n + \sigma}})$,

$$\tilde{\eta}(x) \int_{B_1(0) \setminus B_{\frac{1}{k}}(0)} \delta \tilde{v}_{i+1}(x, y) K(y) dy \rightarrow \tilde{\eta}(x) \int_{B_1(0)} \delta \tilde{v}_{i+1}(x, y) K(y) dy.$$

Thus, the result follows by Lemma 5 in [12]. \square

Lemma 6.2.3. *There is a constant C depending on $n, \sigma_0, \lambda, \Lambda$ such that, for any operator L with a symmetric kernel K satisfying $0 \leq K(y) \leq (2 - \sigma) \Lambda |y|^{n + \sigma}$ we have*

$$|Lv_{i+1}(x)| \leq C(\rho^{\bar{\alpha} - \alpha} + \frac{1}{M}), \quad \text{in } B_1(0).$$

Proof. The proof follows from that of Lemma 2.9 and Lemma 2.10 in [42]. \square

Lemma 6.2.4. *There is a constant C depending on $n, \sigma_0, \lambda, \Lambda$ such that*

$$\max\{|M_{\mathcal{L}_0}^+ v_{i+1}|, |M_{\mathcal{L}_0}^- v_{i+1}|\} \leq C(\rho^{\bar{\alpha} - \alpha} + \frac{1}{M}), \quad \text{in } B_1(0). \tag{149}$$

Moreover, we have

$$\|\nabla v_{i+1}\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C(\rho^{\bar{\alpha} - \alpha} + \frac{1}{M}), \tag{150}$$

and

$$\|\nabla \tilde{v}_{i+1}\|_{L^\infty(B_{\frac{1}{2}}(0))} \leq C(\rho^{\bar{\alpha} - \alpha} + \frac{1}{M}). \tag{151}$$

Proof. (149) follows directly from Lemma 6.2.3. To prove (150), we first notice that v_{i+1} satisfies

$$\inf_{a \in \mathcal{A}} \{L_a^{i+1}(v_{i+1} + R_\rho^i)(x) + w^{-1}(\rho^{i+1})b_a\} = 0, \quad \text{in } B_5(0).$$

We define

$$I^0 \cdot (x) = \inf_{a \in \mathcal{A}} \{L_a^{i+1} \cdot (x) + L_a^{i+1}R_\rho^i(0) + w^{-1}(\rho^{i+1})b_a\}.$$

By Theorem 6.1.7, we know that I^0 has $C^{\sigma+\bar{\alpha}}$ estimates. By (138) and (139), we have that v_{i+1} is a bounded function solves

$$I^0 v_{i+1}(x) \leq - \inf_{a \in \mathcal{A}} \{L_a^{i+1}R_\rho^i(x) - L_a^{i+1}R_\rho^i(0)\} \leq C\rho^{\bar{\alpha}-\alpha}, \quad \text{in } B_1(0),$$

and

$$I^0 v_{i+1}(x) \geq - \sup_{a \in \mathcal{A}} \{L_a^{i+1}R_\rho^i(x) - L_a^{i+1}R_\rho^i(0)\} \geq -C\rho^{\bar{\alpha}-\alpha}, \quad \text{in } B_1(0).$$

It follows from Theorem 5.2 in [12] that $v_{i+1} \in C^{1,\alpha_1}(B_{\frac{1}{2}}(0))$ for any $\alpha_1 < \sigma_0 - 1$ and

$$\|v_{i+1}\|_{C^{1,\alpha_1}(B_{\frac{1}{2}}(0))} \leq C\left(\frac{1}{M} + \rho^{\bar{\alpha}-\alpha}\right).$$

By (133), we have $|\nabla V_l^1(x)| \leq C|x|^{\sigma+\bar{\alpha}-1}$ in $B_{\frac{1}{2}}(0)$. Thus, for any $x \in B_{\frac{1}{2}}(0)$ we have

$$\begin{aligned} |\nabla R_\rho^{(1)i}(x)| &= \left| \nabla \sum_{l=0}^i \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) V_l^1(\rho^{i+1-l}x) \right| \\ &\leq C \sum_{l=0}^i \rho^{-(i+1-l)(\sigma+\alpha-1)} \rho^{(i+1-l)(\sigma+\bar{\alpha}-1)} \\ &\leq C \sum_{l=0}^i \rho^{(i+1-l)(\bar{\alpha}-\alpha)} \leq C\rho^{\bar{\alpha}-\alpha}. \end{aligned}$$

Thus, (151) follows. \square

Lemma 6.2.5. *There is a constant C depending on $n, \sigma_0, \lambda, \Lambda$ such that*

$$\int_{\mathbb{R}^n} |\delta v_{i+1}(x, y)| \frac{2-\sigma}{|y|^{n+\sigma}} dy \leq C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}) \quad \text{in } B_1(0).$$

Proof. By Lemma 6.2.3 and 6.2.4, it follows from the proof of Theorem 7.4 in [14]. \square

Let $\tilde{\eta}$ be the smooth function in Lemma 6.2.2. For any symmetric measurable set A , we define

$$w_A(x) := \tilde{\eta}(x) \int_{B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y)) K_A(y) dy,$$

where

$$K_A(y) = \frac{2-\sigma}{|y|^{n+\sigma}} \mathbb{1}_A(y).$$

By Lemma 6.1.2, we have for any $x \in B_1(0)$

$$\begin{aligned} & \left| \int_{B_1(0)} (\delta R_\rho^{(1)i}(x, y) - \delta R_\rho^{(1)i}(0, y)) K_A(y) dy \right| \\ &= \left| \sum_{l=0}^i \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) \right. \\ & \quad \left. \int_{B_1(0)} (\delta V_l^1(\rho^{i+1-l}x, \rho^{i+1-l}y) - \delta V_l^1(0, \rho^{i+1-l}y)) K_A(y) dy \right| \\ &= \left| \sum_{l=0}^i w^{-1}(\rho^{i+1}) w(\rho^l) \int_{B_1(0)} (\delta V_l^1(\rho^{i+1-l}x, y) - \delta V_l^1(0, y)) K_A^{l-1-i}(y) dy \right| \\ &\leq \sum_{l=0}^i \rho^{-(i+1-l)\alpha} \|V_l^1\|_{C^{\sigma+\bar{\alpha}}(\mathbb{R}^n)} \rho^{(i+1-l)\bar{\alpha}} |x|^{\bar{\alpha}} \leq C \rho^{\bar{\alpha}-\alpha} |x|^{\bar{\alpha}}. \end{aligned} \quad (152)$$

Using Lemma 6.2.5 and (152), we get

$$|w_A| \leq C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}), \quad \text{in } \mathbb{R}^n.$$

It follows from Lemma 6.2.3 and (147) that

$$\left| \int_{B_1(0)} \delta \tilde{v}_{i+1}(0, y) K_A(y) dy \right| \leq C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}).$$

By Lemma 6.2.2, we have

$$M_{\mathcal{L}_2}^+ w_A \geq -C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}), \quad \text{in } B_{\frac{3}{5}}(0) \text{ uniformly in } A.$$

We define

$$P(x) := \sup_A w_A(x) = \tilde{\eta}(x) \int_{B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y))^+ \frac{2-\sigma}{|y|^{n+\sigma}} dy,$$

and

$$N(x) := \sup_A -w_A(x) = \tilde{\eta}(x) \int_{B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y))^- \frac{2-\sigma}{|y|^{n+\sigma}} dy.$$

Lemma 6.2.6. *For any $x \in B_{\frac{1}{4}}(0)$, we have*

$$\frac{\lambda}{\Lambda} N(x) - C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x| \leq P(x) \leq \frac{\lambda}{\Lambda} N(x) + C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|. \quad (153)$$

Proof. For any $x \in B_{\frac{1}{2}}(0)$, we define $\tilde{v}_{i+1,x}(z) := \tilde{v}(x+z)$. By (143), we have

$$M_{\mathcal{L}_2}^+(\tilde{v}_{i+1,x} - \tilde{v}_{i+1})(0) \geq -\sup_{a \in \mathcal{A}} (L_a^{i+1} R_\rho^{(2)i}(x) - L_a^{i+1} R_\rho^{(2)i}(0))$$

and

$$M_{\mathcal{L}_2}^-(\tilde{v}_{i+1,x} - \tilde{v}_{i+1})(0) \leq \sup_{a \in \mathcal{A}} (L_a^{i+1} R_\rho^{(2)i}(0) - L_a^{i+1} R_\rho^{(2)i}(x)).$$

By Lemma 6.1.1 and (133),

$$\begin{aligned} |L_a^{i+1} R_\rho^{(2)i}(x) - L_a^{i+1} R_\rho^{(2)i}(0)| &= \left| \sum_{l=0}^i \frac{w(\rho^l)}{w(\rho^{i+1})} (L_a^l V_l^2(\rho^{i+1-l}x) - L_a^l V_l^2(0)) \right| \\ &\leq C \sum_{l=0}^i \rho^{-(i+1-l)\alpha} (\|V_l^2\|_{C^4(B_{\frac{1}{4}}(0))} + \|V_l^2\|_{L^\infty(\mathbb{R}^n)}) \rho^{i+1-l} |x| \\ &\leq C \rho^{1-\alpha} |x|. \end{aligned}$$

Thus, we have

$$M_{\mathcal{L}_2}^+(\tilde{v}_{i+1,x} - \tilde{v}_{i+1})(0) \geq -C \rho^{1-\alpha} |x| \text{ and } M_{\mathcal{L}_2}^-(\tilde{v}_{i+1,x} - \tilde{v}_{i+1})(0) \leq C \rho^{1-\alpha} |x|. \quad (154)$$

For any $L \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$, we have

$$\begin{aligned} L(\tilde{v}_{i+1,x} - \tilde{v}_{i+1})(0) &= \int_{\mathbb{R}^n} (\delta \tilde{v}_{i+1}(x, y) - \delta v_{i+1}(0, y)) K(y) dy \\ &= \int_{B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y)) K(y) dy \\ &\quad + \int_{\mathbb{R}^n \setminus B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y)) K(y) dy. \end{aligned}$$

By (141), (151) and $L \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$, we have, for any $x \in B_{\frac{1}{4}}(0)$

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^n \setminus B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y)) K(y) dy \\ &= \int_{\mathbb{R}^n} \tilde{v}_{i+1}(y) (K(y-x) \mathbb{1}_{B_1^c(0)}(y-x) - K(y) \mathbb{1}_{B_1^c(0)}(y)) dy \\ &\quad - (\tilde{v}_{i+1}(x) - \tilde{v}_{i+1}(0)) \int_{\mathbb{R}^n \setminus B_1(0)} K(y) dy \\ &\leq \int_{\mathbb{R}^n \setminus B_1(0)} |\tilde{v}_{i+1}(y)| |K(y-x) - K(y)| dy \\ &\quad + \|\tilde{v}_{i+1}\|_{L^\infty(B_{1+|x|}(0))} \int_{B_{1+|x|}(0) \setminus B_{1-|x|}(0)} K(y) dy + C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}) |x| \\ &\leq C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}) |x|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y)) K(y) dy \\ & \leq \int_{B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y)) K(y) dy + C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|. \end{aligned} \quad (155)$$

By (154) and (155), we obtain

$$\begin{aligned} -C\rho^{1-\alpha}|x| & \leq M_{\mathcal{L}_2}^+(\tilde{v}_{i+1,x} - \tilde{v}_{i+1})(0) \\ & \leq \sup_{\frac{\lambda(2-\sigma)}{|y|^{n+\sigma}} \leq K \leq \frac{\Lambda(2-\sigma)}{|y|^{n+\sigma}}} \int_{B_1(0)} (\delta \tilde{v}_{i+1}(x, y) - \delta \tilde{v}_{i+1}(0, y)) K(y) dy + C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|. \end{aligned}$$

Therefore, we have

$$\Lambda P(x) - \lambda N(x) \geq -C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|.$$

The second inequality of (153) follows from $M_{\mathcal{L}_2}^-(\tilde{v}_{i+1,x} - \tilde{v}_{i+1})(0) \leq C\rho^{1-\alpha}|x|$. \square

Now the proof of Theorem 6.2.1 follows from the proofs of Lemma 2.14 and Theorem 2.2 in [42].

6.3 C^σ regularity

Before introducing the main theorem, we remind that, for any $\sigma \in (0, 2)$, $[\sigma]$ denotes the largest integer which is less than or equal to σ .

Theorem 6.3.1. *Assume that $2 > \sigma \geq \sigma_0 > 0$ and $K_a(x, y) \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$ for any $a \in \mathcal{A}$. Assume that $w(t)$ is a Dini modulus of continuity satisfying $(H2)_{\bar{\beta}, \sigma}$, where $\bar{\beta}$ is given in Theorem 6.2.1. Assume that f satisfies, for some $C_f > 0$,*

$$|f(x) - f(0)| \leq C_f w(|x|) \text{ and } |f(x)| \leq C_f, \quad \text{in } B_1(0), \quad (156)$$

and $K_a(x, y)$ satisfies, for any $0 < r \leq 1$, $a \in \mathcal{A}$ and $x \in B_1(0)$,

$$\int_{\mathbb{R}^n} |K_a(x, y) - K_a(0, y)| \min\{|y|^{\min\{2, \sigma + \bar{\beta}\}}, r^{\min\{2, \sigma + \bar{\beta}\}}\} dy \leq \Lambda w(|x|) r^{\min\{2-\sigma, \bar{\beta}\}}. \quad (157)$$

If u is a bounded viscosity solution of (14), then there exists a polynomial $p(x)$ of degree $[\sigma]$ such that

$$|u(x) - p(x)| \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + C_f)|x|^\sigma \psi(|x|), \quad \text{in } B_{\frac{1}{2}}(0),$$

and

$$|D^i p(0)| \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + C_f), \quad i = 0, \dots, [\sigma],$$

where $\psi(t) := w(t) + \int_0^t \frac{w(r)}{r} dr$ and C is a constant depending on $\lambda, \Lambda, n, \sigma_0, \sigma$ and w .

Proof. By covering and rescaling arguments, we can assume (14), (156) and (157) hold in $B_5(0)$. We will give the proof of Theorem 6.3.1 in the most complicated case $\sigma_0 \geq 1$. Without loss of generality, we can assume that $w(1) > 1$.

We claim that we can find a sequence of functions $\{u_l\}_{l=0}^{l=+\infty}$ such that, for any $\rho \leq \rho_0$, $0 < \kappa \leq \sigma + \bar{\beta}$ and $i = 0, 1, 2, \dots$, we have

$$\inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \sum_{l=0}^i \delta u_l(x, y) K_a(0, y) dy \right\} = f(0), \quad \text{in } B_{4\rho^i}(0), \quad (158)$$

$$(u - \sum_{l=0}^i u_l)(\rho^i x) = 0, \quad \text{in } \mathbb{R}^n \setminus B_4(0), \quad (159)$$

$$\|u_i\|_{L^\infty(\mathbb{R}^n)} \leq \rho^{\sigma i} w(\rho^i), \quad (160)$$

$$\|u_i\|_{C^\kappa(B_{(4-\tau)\rho^i}(0))} \leq C_2 \rho^{(\sigma-\kappa)i} w(\rho^i) \tau^{-\kappa}, \quad (161)$$

$$\|u - \sum_{l=0}^i u_l\|_{L^\infty(\mathbb{R}^n)} \leq \rho^{\sigma(i+1)} w(\rho^{i+1}), \quad (162)$$

$$[u - \sum_{l=0}^i u_l]_{C^{\alpha_1}(B_{(4-3\tau)\rho^i}(0))} \leq 8C_1 \rho^{(\sigma-\alpha_1)i} w(\rho^i) \tau^{-3}, \quad (163)$$

where ρ_0 is given by Theorem 6.2.1, τ is an arbitrary constant in $(0, 1]$, α_1 and C_1 are positive constants depending on $n, \lambda, \Lambda, \sigma_0$, and C_2 is the constant in (130).

Suppose that we have (158)-(163). Then, for any $\rho^{i+1} \leq |x| < \rho^i$

$$\begin{aligned} & \left| u(x) - \sum_{l=0}^{+\infty} u_l(0) - \sum_{l=0}^{+\infty} \nabla u_l(0) \cdot x \right| \\ & \leq \left| u(x) - \sum_{l=0}^i u_l(x) \right| + \left| \sum_{l=0}^i (u_l(x) - u_l(0) - \nabla u_l(0) \cdot x) \right| \\ & \quad + \left| \sum_{l=i+1}^{+\infty} u_l(0) \right| + \left| \sum_{l=i+1}^{+\infty} \nabla u_l(0) \cdot x \right| \\ & \leq \rho^{\sigma(i+1)} w(\rho^{i+1}) + C|x|^{\min\{2, \sigma+\bar{\beta}\}} \sum_{l=0}^i \rho^{-\min\{2-\sigma, \bar{\beta}\}l} w(\rho^l) \\ & \quad + \sum_{l=i+1}^{+\infty} \rho^{\sigma l} w(\rho^l) + C|x| \sum_{l=i+1}^{+\infty} \rho^{(\sigma-1)l} w(\rho^l). \end{aligned}$$

By $(H1)_{\bar{\beta}, \sigma}$, we have, for $\rho^{i+1} \leq |x| < \rho^i$

$$\begin{aligned} & |x|^{\min\{2, \sigma+\bar{\beta}\}} \sum_{l=0}^i \rho^{-\min\{2-\sigma, \bar{\beta}\}l} w(\rho^l) \leq \rho^{i\sigma} w(\rho^i) \sum_{l=0}^i \rho^{\min\{2-\sigma, \bar{\beta}\}(i-l)} \frac{w(\rho^l)}{w(\rho^i)} \\ & \leq \rho^{i\sigma} w(\rho^i) \sum_{l=0}^i \rho^{(\min\{2-\sigma, \bar{\beta}\}-\beta)(i-l)} \end{aligned}$$

$$\begin{aligned}
&\leq \rho^{i\sigma} w(\rho^i) \sum_{l=0}^{+\infty} \rho^{(\min\{2-\sigma, \bar{\beta}\}-\beta)l} \\
&\leq C \rho^{i\sigma} w(\rho^i) \leq C \rho^{-\beta-\sigma} \rho^{(i+1)\sigma} w(\rho^{i+1}) \frac{\rho^\beta w(\rho^i)}{w(\rho^{i+1})} \\
&\leq C \rho^{(i+1)\sigma} w(\rho^{i+1}).
\end{aligned}$$

We notice that $\min\{2, \sigma + \bar{\beta}\} - \min\{2 - \sigma, \bar{\beta}\} = \sigma$. Thus, for $\rho^{i+1} \leq |x| < \rho^i$

$$\begin{aligned}
&\left| u(x) - \sum_{l=0}^{+\infty} u_l(0) - \sum_{l=0}^{+\infty} \nabla u_l(0) \cdot x \right| \\
&\leq C \rho^{\sigma(i+1)} w(\rho^{i+1}) + (\rho^{\sigma(i+1)} + C \rho^i \rho^{(\sigma-1)(i+1)}) \sum_{l=i+1}^{+\infty} w(\rho^l) \\
&\leq C \rho^{\sigma(i+1)} w(\rho^{i+1}) + C \rho^{\sigma(i+1)} \sum_{l=i+1}^{+\infty} w(\rho^l) \\
&\leq C \rho^{\sigma(i+1)} \psi(\rho^{i+1}),
\end{aligned}$$

where $\psi(t) = w(t) + \int_0^t \frac{w(r)}{r} dr$.

We first prove the claim for $i = 0$. Let u_0 be the viscosity solution of

$$\begin{cases} I_0 u_0 := \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta u_0(x, y) K_a(0, y) \right\} - f(0) = 0, & \text{in } B_4(0), \\ u_0 = u, & \text{in } B_4^c(0). \end{cases}$$

Then, by Lemma 3.1 in [42], we have

$$\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_5(0))}).$$

By normalization, we can assume that

$$\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2} \quad \text{and} \quad \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_5(0))} \leq \frac{1}{2}.$$

Using Theorem 6.2.1, we have, for any $0 < \kappa \leq \sigma + \bar{\beta}$

$$\|u_0\|_{C^\kappa(B_{4-\tau}(0))} \leq C_2 \tau^{-\kappa},$$

where C_2 is the constant in (130). Since u is a bounded viscosity solution of (14), it follows from Theorem 12.1 in [13] that there exist constants $\alpha_1 > 0$ and $C_1 > 0$, depending only on $n, \lambda, \Lambda, \sigma_0$, such that, for any $0 < \tau \leq 1$

$$\|u\|_{C^{\alpha_1}(B_{4-\tau}(0))} \leq \frac{C_1}{2} \tau^{-\alpha_1}. \quad (164)$$

Let $\epsilon := \rho^{\sigma+\bar{\beta}} \leq \rho^\sigma w(\rho)$, $M = 1$ and $w_1(r) := r^{\alpha_1}$. Then, for these w_1 , ϵ and M , there exist $\eta_1 > 0$ and $R > 5$ such that Lemma 6.1.5 holds. Without loss of generality, we can assume that, for any $0 < r \leq 1$

$$|f(x) - f(0)| \leq \gamma w(|x|), \quad \text{in } B_5(0),$$

$$\int_{B_r(0)} |K_a(x, y) - K_a(0, y)| |y|^{\min\{2, \sigma+\bar{\beta}\}} dy \leq \gamma w(|x|) r^{\min\{2-\sigma, \bar{\beta}\}}, \quad \text{in } B_5(0),$$

$$\int_{B_r^c(0)} |K_a(x, y) - K_a(0, y)| dy \leq \gamma w(|x|) r^{-\sigma}, \quad \text{in } B_5(0),$$

$$|u(x) - u(y)| \leq w_1(|x - y|), \quad \text{for any } x \in B_R(0) \setminus B_4(0) \text{ and } y \in \mathbb{R}^n \setminus B_4(0), \quad (165)$$

where γ is a sufficiently small constant we determine later. This can be achieved by scaling. For a sufficiently small $s > 0$, if we let

$$\tilde{K}_a(x, y) = s^{n+\sigma} K_a(sx, sy) \in \mathcal{L}_2(\lambda, \Lambda, \sigma),$$

$$\tilde{u}(x) = u(sx),$$

$$\tilde{f}(x) = s^\sigma f(sx),$$

then we see that

$$\tilde{I}\tilde{u}(x) := \inf_{a \in \mathcal{A}} \int_{\mathbb{R}^n} \delta \tilde{u}(x, y) \tilde{K}_a(x, y) dy = \tilde{f}(x), \quad \text{in } B_5(0).$$

It follows from $(H2)_{\bar{\beta}, \sigma}$ that, if we choose s sufficiently small, then for any $x \in B_5(0)$

$$|\tilde{f}(x) - \tilde{f}(0)| \leq C_f s^\sigma w(s|x|) \leq C_f s^\sigma \eta(s) \tilde{w}(|x|) \leq \gamma \tilde{w}(|x|),$$

$$\begin{aligned} & \int_{B_r(0)} |\tilde{K}_a(x, y) - \tilde{K}_a(0, y)| |y|^{\min\{2, \sigma+\bar{\beta}\}} dy \\ &= s^{-\min\{2-\sigma, \bar{\beta}\}} \int_{B_{sr}(0)} |K_a(sx, y) - K_a(0, y)| |y|^{\min\{2, \sigma+\bar{\beta}\}} dy \\ &\leq \Lambda w(s|x|) r^{\min\{2-\sigma, \bar{\beta}\}} \leq \Lambda \eta(s) \tilde{w}(|x|) r^{\min\{2-\sigma, \bar{\beta}\}} \leq \gamma \tilde{w}(|x|) r^{\min\{2-\sigma, \bar{\beta}\}}, \end{aligned}$$

and

$$\begin{aligned} \int_{B_r^c(0)} |\tilde{K}_a(x, y) - \tilde{K}_a(0, y)| dy &= s^\sigma \int_{B_{sr}^c(0)} |K_a(sx, y) - K_a(0, y)| dy \\ &\leq \Lambda w(s|x|) r^{-\sigma} \leq \Lambda \eta(s) \tilde{w}(|x|) r^{-\sigma} \leq \gamma \tilde{w}(|x|) r^{-\sigma}, \end{aligned}$$

where $\tilde{w}(t)$ is another Dini modulus of continuity satisfying $(H1)_{\bar{\beta}, \sigma}$ and $\eta(s)$ is a positive function of s such that $\lim_{s \rightarrow 0^+} \eta(s) = 0$. Using (164) with $\tau = 1$, we have, if we let s sufficiently small,

$$\|\tilde{u}\|_{C^{\alpha_1}(B_{2R}(0))} \leq \|\tilde{u}\|_{C^{\alpha_1}(B_{\frac{3}{s}}(0))} \leq s^{\alpha_1} \frac{C_1}{2} \leq 1.$$

Since $R > 5$ and $\|\tilde{u}\|_{C^{\alpha_1}(B_{2R}(0))} \leq 1$,

$$|\tilde{u}(x) - \tilde{u}(y)| \leq |x - y|^{\alpha_1} \quad \text{for any } x \in B_R(0) \setminus B_4(0) \text{ and } y \in B_{2R}(0) \setminus B_4(0),$$

and

$$|\tilde{u}(x) - \tilde{u}(y)| \leq 1 \leq |x - y|^{\alpha_1} \quad \text{for } x \in B_R(0) \setminus B_4(0) \text{ and } y \in B_{2R}^c(0).$$

Therefore, (165) holds for \tilde{u} .

If $x \in B_4(0)$, $h \in C^{1,1}(x)$, $\|h\|_{L^\infty(\mathbb{R}^n)} \leq M$ and $|h(y) - h(x) - (y - x) \cdot \nabla h(x)| \leq \frac{M}{2}|x - y|^2$ for any $y \in B_1(x)$, we have

$$\begin{aligned} \|I - I_0\|_{B_4(0)} &\leq \frac{1}{M+1} \left\{ \int_{\mathbb{R}^n} |\delta h(x, y)| |K_a(x, y) - K_a(0, y)| dy + f(x) - f(0) \right\} \\ &\leq \frac{M}{M+1} \left\{ \int_{B_1(0)} |y|^2 |K_a(x, y) - K_a(0, y)| dy + 4 \int_{\mathbb{R}^n \setminus B_1(0)} |K_a(x, y) - K_a(0, y)| dy \right\} \\ &\quad + f(x) - f(0) \leq 6\gamma w(|x|) \leq 6\gamma w(5). \end{aligned} \tag{166}$$

We will choose $\gamma < \min\{\frac{\eta_1}{6w(5)}, \frac{1}{(C_2+4)w(4)}\}$. By Lemma 6.1.5, we have

$$\|u - u_0\|_{L^\infty(B_4(0))} \leq \epsilon \leq \rho^\sigma w(\rho),$$

and thus

$$\|u - u_0\|_{L^\infty(\mathbb{R}^n)} \leq \|u - u_0\|_{L^\infty(B_4(0))} \leq \epsilon \leq \rho^\sigma w(\rho).$$

Let $v(x) = u(x) - u_0(x)$. Since $u_0 \in C_{\text{loc}}^{\sigma+\bar{\beta}}(B_4(0))$, v is a viscosity solution of

$$\begin{aligned} I^{(0)}v(x) : &= \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta v(x, y) K_a(x, y) + \delta u_0(x, y) K_a(x, y) dy \right\} - f(0) \\ &= f(x) - f(0) \quad \text{in } B_4(0). \end{aligned}$$

It is clear that $I^{(0)}$ is uniformly elliptic with respect to $\mathcal{L}_0(\lambda, \Lambda, \sigma)$. Since $\gamma < \frac{1}{(C_2+4)w(4)}$, we have for any $x \in B_{4-2\tau}(0)$

$$\begin{aligned} |I^{(0)}0(x)| &= \left| \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta u_0(x, y) K_a(x, y) dy \right\} - f(0) \right| \\ &\leq \sup_{a \in \mathcal{A}} \int_{\mathbb{R}^n} |\delta u_0(x, y)| |K_a(x, y) - K_a(0, y)| dy \\ &\leq \sup_{a \in \mathcal{A}} \left\{ \int_{B_\tau(0)} C_2 \tau^{-\min\{2, \sigma+\bar{\beta}\}} |y|^{\min\{2, \sigma+\bar{\beta}\}} |K_a(x, y) - K_a(0, y)| dy \right. \\ &\quad \left. + 4 \int_{\mathbb{R}^n \setminus B_\tau(0)} |K_a(x, y) - K_a(0, y)| dy \right\} \\ &\leq \gamma C_2 \tau^{-\min\{2, \sigma+\bar{\beta}\}} w(|x|) \tau^{\min\{2-\sigma, \bar{\beta}\}} + 4\gamma w(|x|) \tau^{-\sigma} \\ &= \gamma(C_2 + 4)w(|x|) \tau^{-\sigma} \leq \gamma(C_2 + 4)w(4) \tau^{-\sigma} \leq \tau^{-\sigma}. \end{aligned}$$

It follows from Theorem 12.1 in [13] that

$$\|v\|_{C^{\alpha_1}(B_{4-3\tau}(0))} \leq C_1 \tau^{-\alpha_1} (\tau^{-\sigma} + w(4)\gamma + 1) \leq 8C_1 \tau^{-3},$$

and thus

$$[u - u_0]_{C^{\alpha_1}(B_{4-3\tau}(0))} \leq 8C_1 \tau^{-3}.$$

We then assume (158)-(163) hold up to $i \geq 0$ and we will show that they hold for $i + 1$ as well. Let

$$U(x) = \rho^{-(i+1)\sigma} w^{-1}(\rho^{i+1}) \left(u - \sum_{l=0}^i u_l \right) (\rho^{i+1} x),$$

$$v_l(x) = \rho^{-l\sigma} w^{-1}(\rho^l) u_l(\rho^l x),$$

and

$$K_a^{i+1}(x, y) = \rho^{(n+\sigma)(i+1)} K_a(\rho^{i+1} x, \rho^{i+1} y).$$

Since $u_l \in C_{\text{loc}}^{\sigma+\bar{\beta}}(B_{4\rho^l}(0))$ for each $0 \leq l \leq i$, then U is a viscosity solution of

$$I^{(i+1)}U = w^{-1}(\rho^{i+1})f(\rho^{i+1}x) - w^{-1}(\rho^{i+1})f(0), \quad \text{in } B_{\frac{4}{\rho}}(0),$$

where

$$\begin{aligned} I^{(i+1)}U &:= \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} (\delta U(x, y) + \sum_{l=0}^i \rho^{-(i+1)\sigma} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1} x, \rho^{i+1} y)) K_a^{i+1}(x, y) dy \right\} \\ &\quad - w^{-1}(\rho^{i+1})f(0) \\ &= \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} (\delta U(x, y) + \sum_{l=0}^i \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) \delta v_l(\rho^{i+1-l} x, \rho^{i+1-l} y)) K_a^{i+1}(x, y) dy \right\} \\ &\quad - w^{-1}(\rho^{i+1})f(0). \end{aligned}$$

It is clear that $I^{(i+1)}$ is uniformly elliptic with respect to $\mathcal{L}_0(\lambda, \Lambda, \sigma)$. Denote

$$\begin{aligned} I_0^{(i+1)}v &:= \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} (\delta v(x, y) + \sum_{l=0}^i \rho^{-(i+1)\sigma} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1} x, \rho^{i+1} y)) K_a^{i+1}(0, y) dy \right\} \\ &\quad - w^{-1}(\rho^{i+1})f(0) \\ &= \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} (\delta v(x, y) + \sum_{l=0}^i \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) \delta v_l(\rho^{i+1-l} x, \rho^{i+1-l} y)) K_a^{i+1}(0, y) dy \right\} \\ &\quad - w^{-1}(\rho^{i+1})f(0), \end{aligned}$$

which is also uniformly elliptic with respect to $\mathcal{L}_0(\lambda, \Lambda, \sigma)$. Let v_{i+1} be the viscosity solution of

$$\begin{cases} I_0^{(i+1)}v_{i+1} = 0, & \text{in } B_4(0), \\ v_{i+1} = U, & \text{in } B_4^c(0). \end{cases}$$

With our induction assumption (158), it follows that for all $m = 0, \dots, i+1$ and $x \in B_4(0)$,

$$\inf_{a \in \mathcal{A}} \int_{\mathbb{R}^n} \left(\sum_{l=0}^m \rho^{-(m-l)\sigma} w^{-1}(\rho^m) w(\rho^l) \delta v_l(\rho^{m-l}x, \rho^{m-l}y) \right) K_a^m(0, y) dy = w^{-1}(\rho^m) f(0).$$

It follows from Theorem 6.2.1 that $v_{i+1} \in C_{\text{loc}}^{\sigma+\bar{\beta}}(B_4(0))$ and for any $0 < \kappa \leq \sigma + \bar{\beta}$

$$\|v_{i+1}\|_{C^\kappa(B_{4-\tau}(0))} \leq C_2 \tau^{-\kappa}.$$

We then want to prove that

$$\|v_{i+1}\|_{L^\infty(\mathbb{R}^n)} \leq \|U\|_{L^\infty(\mathbb{R}^n)} \leq 1.$$

Since $\|u\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2}$, (160), (164) and $u_l \in C_{\text{loc}}^{\sigma+\bar{\beta}}(B_{4\rho^l}(0))$ for any $0 \leq l \leq i$ hold, it follows from Theorem 3.2 in [12] that $v_{i+1} \in C(\overline{B_4(0)})$. Suppose that there exists $x_0 \in B_4(0)$ such that $v_{i+1}(x_0) = \max_{B_4(0)} v_{i+1} > \|U\|_{L^\infty(\mathbb{R}^n \setminus B_4(0))}$. Then

$$\sup_{a \in \mathcal{A}} \int_{\mathbb{R}^n} \delta v_{i+1}(x_0, y) K_a^{i+1}(0, y) dy < 0. \quad (167)$$

Since $I_0^{(i+1)} 0(x) = 0$ for any $x \in B_4(0)$, then we have

$$0 = I_0^{(i+1)} v_{i+1}(x) - I_0^{(i+1)} 0(x) \leq \sup_{a \in \mathcal{A}} \int_{\mathbb{R}^n} \delta v_{i+1}(x, y) K_a^{i+1}(0, y) dy, \quad \text{for any } B_4(0),$$

which contradicts (167). Similarly, we have $v_{i+1}(x) \geq -\|U\|_{L^\infty(\mathbb{R}^n \setminus B_4(0))}$ for any $x \in B_4(0)$. By induction assumptions, we have $\|U\|_{L^\infty(\mathbb{R}^n)} \leq 1$, $U = 0$ in $B_{\frac{4}{\rho}}^c(0)$ and

$$[U]_{C^{\alpha_1}(B_{\frac{4-3\tau}{\rho}}(0))} \leq 8C_1 \frac{w(\rho^i)}{w(\rho^{i+1})} \rho^{\alpha_1-\sigma} \tau^{-3} \leq 8C_1 \rho^{-3} \tau^{-3}.$$

By Lemma 6.1.3, we have, for any $x_1, x_2 \in B_4(0)$

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \sum_{l=0}^i \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) \right. \\ & \quad \left. (\delta v_l(\rho^{i+1-l}x_1, \rho^{i+1-l}y) - \delta v_l(\rho^{i+1-l}x_2, \rho^{i+1-l}y)) K_a^{i+1}(0, y) dy \right| \\ &= \left| \sum_{l=0}^i \rho^{-(i+1-l)\sigma} w^{-1}(\rho^{i+1}) w(\rho^l) \right. \\ & \quad \left. \int_{\mathbb{R}^n} (\delta v_l(\rho^{i+1-l}x_1, \rho^{i+1-l}y) - \delta v_l(\rho^{i+1-l}x_2, \rho^{i+1-l}y)) K_a^{i+1}(0, y) dy \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \sum_{l=0}^i w^{-1}(\rho^{i+1})w(\rho^l) \int_{\mathbb{R}^n} (\delta v_l(\rho^{i+1-l}x_1, y) - \delta v_l(\rho^{i+1-l}x_2, y)) K_a^l(0, y) dy \right| \\
&\leq \sum_{l=0}^i w^{-1}(\rho^{i+1})w(\rho^l) \|L_a^l v_l\|_{C^{\bar{\beta}}(B_{4\rho^{i+1-l}}(0))} \rho^{(i+1-l)\bar{\beta}} |x_1 - x_2|^{\bar{\beta}} \\
&\leq \sum_{l=0}^i w^{-1}(\rho^{i+1})w(\rho^l) C(\|v_l\|_{C^{\sigma+\bar{\beta}}(B_{5\rho^{i+1-l}}(0))} + \|v_l\|_{L^\infty(\mathbb{R}^n)}) \rho^{(i+1-l)\bar{\beta}} |x_1 - x_2|^{\bar{\beta}} \\
&\leq \sum_{l=0}^i w^{-1}(\rho^{i+1})w(\rho^l) C(C_2 + 1) \rho^{(i+1-l)\bar{\beta}} |x_1 - x_2|^{\bar{\beta}} \\
&\leq \sum_{l=0}^i C \frac{w(\rho^l)}{w(\rho^{i+1})} \rho^{(i+1-l)\bar{\beta}} |x_1 - x_2|^{\bar{\beta}} \leq C \rho^{\bar{\beta}-\beta} |x_1 - x_2|^{\bar{\beta}}.
\end{aligned}$$

Then we will show that we can choose γ sufficiently small such that

$$\|I^{(i+1)} - I_0^{(i+1)}\|_{B_4(0)} \leq \eta_2 \leq 1, \quad (168)$$

where η_2 is given in Lemma 6.1.6 depending on $\epsilon = \rho^{\sigma+\bar{\beta}}$, $R_0 = \frac{4}{\rho}$, $M_0 = C\rho^{\bar{\beta}-\beta}$, $M_1 = 1$, $M_2 = 8C_1\rho^{-3}$ and $M_3 = C_2$. For any $x \in B_4(0)$, $h \in C^{1,1}(x)$, $\|h\|_{L^\infty(\mathbb{R}^n)} \leq M$, $|h(y) - h(x) - (y - x) \cdot \nabla h(x)| \leq \frac{M}{2}|x - y|^2$ for any $y \in B_1(x)$, we have

$$\begin{aligned}
\|I^{(i+1)} - I_0^{(i+1)}\|_{B_4(0)} &\leq \frac{1}{M+1} \sup_{a \in \mathcal{A}} \left| \int_{\mathbb{R}^n} \delta h(x, y) (K_a^{i+1}(x, y) - K_a^{i+1}(0, y)) dy \right| \\
&\quad + \sum_{l=0}^i \sup_{a \in \mathcal{A}} \left| \int_{\mathbb{R}^n} \rho^{-(i+1)\sigma} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1}x, \rho^{i+1}y) (K_a^{i+1}(x, y) - K_a^{i+1}(0, y)) dy \right| \\
&= I_1 + I_2.
\end{aligned}$$

It follows from the same computation as that in (166) that

$$|I_1| \leq 5\gamma w(5).$$

By (161), we have, for any $a \in \mathcal{A}$, $l = 0, \dots, i$ and $x \in B_4(0)$

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \delta u_l(\rho^{i+1}x, \rho^{i+1}y) (K_a^{i+1}(x, y) - K_a^{i+1}(0, y)) dy \right| \\
&\leq \rho^{\sigma(i+1)} \int_{\mathbb{R}^n} |\delta u_l(\rho^{i+1}x, y)| |K_a(\rho^{i+1}x, y) - K_a(0, y)| dy \\
&\leq \rho^{\sigma(i+1)} \int_{B_{\rho^l}(0)} C_2 \rho^{-\min\{2-\sigma, \bar{\beta}\}l} w(\rho^l) |y|^{\min\{2, \sigma+\bar{\beta}\}} |K_a(\rho^{i+1}x, y) - K_a(0, y)| dy \\
&\quad + \rho^{\sigma(i+1)} \int_{\mathbb{R}^n \setminus B_{\rho^l}(0)} 4\rho^{l\sigma} w(\rho^l) |K_a(\rho^{i+1}x, y) - K_a(0, y)| dy \\
&\leq (C_2 + 4) \rho^{\sigma(i+1)} w(\rho^l) \gamma w(\rho^{i+1}|x|) \\
&\leq (C_2 + 4) \rho^{\sigma(i+1)} w(\rho^l) \gamma w(\rho^i).
\end{aligned}$$

Thus, we have

$$I_2 \leq (C_2 + 4) \frac{w(\rho^i)}{w(\rho^{i+1})} \gamma \sum_{l=0}^i w(\rho^l) \leq (C_2 + 4) \rho^{-1} \gamma \sum_{l=0}^{+\infty} w(\rho^l) < +\infty.$$

We finally choose γ such that

$$\gamma \leq \min \left\{ \frac{\eta_2}{(5w(5) + (C_2 + 4)\rho^{-1} \sum_{l=0}^{+\infty} w(\rho^l))}, \frac{\eta_1}{6w(5)}, \frac{1}{(C_2 + 4)w(4)} \right\}.$$

Therefore, (168) holds. By Lemma 6.1.6, we have

$$\|U - v_{i+1}\|_{L^\infty(\mathbb{R}^n)} = \|U - v_{i+1}\|_{L^\infty(B_4(0))} \leq \epsilon = \rho^{\sigma+\bar{\beta}} \leq \rho^\sigma \frac{w(\rho^{i+2})}{w(\rho^{i+1})}.$$

Let

$$u_{i+1}(x) = \rho^{\sigma(i+1)} w(\rho^{i+1}) v_{i+1}(\rho^{-(i+1)} x),$$

and

$$V = U - v_{i+1} = \rho^{-\sigma(i+1)} w^{-1}(\rho^{i+1}) \left(u - \sum_{l=0}^{i+1} u_l \right) (\rho^{i+1} x).$$

Then, for any $x \in B_4(0)$ we have

$$\begin{aligned} \bar{I}^{(i+1)} V : &= \inf_{a \in \mathcal{A}} \int_{\mathbb{R}^n} \delta V(x, y) + \sum_{l=0}^{i+1} \rho^{-\sigma(i+1)} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1} x, \rho^{i+1} y) K_a^{i+1}(x, y) dy \\ &\quad - w^{-1}(\rho^{i+1}) f(0) \\ &= w^{-1}(\rho^{i+1}) f(\rho^{i+1} x) - w^{-1}(\rho^{i+1}) f(0). \end{aligned}$$

Moreover, we have for any $x \in B_{4-2\tau}(0)$

$$\begin{aligned} \bar{I}^{(i+1)} 0 &= \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \sum_{l=0}^{i+1} \rho^{-\sigma(i+1)} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1} x, \rho^{i+1} y) K_a^{i+1}(x, y) dy \right\} \\ &\quad - w^{-1}(\rho^{i+1}) f(0) \\ &= \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \sum_{l=0}^{i+1} \rho^{-\sigma(i+1)} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1} x, \rho^{i+1} y) K_a^{i+1}(x, y) dy \right\} \\ &\quad - \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \sum_{l=0}^{i+1} \rho^{-(i+1)\sigma} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1} x, \rho^{i+1} y) K_a^{i+1}(0, y) dy \right\} \\ &\leq \sup_{a \in \mathcal{A}} \left\{ \sum_{l=0}^{i+1} \int_{\mathbb{R}^n} \rho^{-\sigma(i+1)} w^{-1}(\rho^{i+1}) \delta u_l(\rho^{i+1} x, \rho^{i+1} y) (K_a^{i+1}(x, y) - K_a^{i+1}(0, y)) dy \right\} \\ &\leq (C_2 + 4) \rho^{-1} \tau^{-\sigma} \gamma \sum_{l=0}^{+\infty} w(\rho^l) \leq \eta_2 \tau^{-\sigma} \leq \tau^{-\sigma}. \end{aligned}$$

It is clear that $\bar{I}^{(i+1)}$ is uniformly elliptic with respect to $\mathcal{L}_0(\lambda, \Lambda, \sigma)$. Thus, for any $x \in B_{4-2\tau}(0)$

$$\begin{aligned} M_{\mathcal{L}_0}^+ V &\geq \bar{I}^{(0)} V - \bar{I}^{(0)} 0 = w^{-1}(\rho^{i+1})(f(\rho^{i+1}x) - f(0)) - \tau^{-\sigma} \\ &\geq -\gamma w^{-1}(\rho^{i+1})w(\rho^{i+1}|x|) - \tau^{-\sigma} \\ &\geq -\gamma w^{-1}(\rho^{i+1})w(4\rho^{i+1}) - \tau^{-\sigma} \\ &\geq -\gamma\rho^{-1} - \tau^{-\sigma}, \end{aligned}$$

and similarly,

$$M_{\mathcal{L}_0}^- V \leq \gamma\rho^{-1} + \tau^{-\sigma}.$$

It follows from Theorem 12.1 of [13] that,

$$\begin{aligned} [V]_{C^{\alpha_1}(B_{4-3\tau}(0))} &\leq C_1\tau^{-\alpha_1}(\|V\|_{L^\infty(\mathbb{R}^n)} + \gamma\rho^{-1} + \tau^{-\sigma}) \\ &\leq C_1\tau^{-\alpha_1}(\epsilon + \gamma\rho^{-1} + \tau^{-\sigma}) \\ &\leq 8C_1\tau^{-3}. \end{aligned}$$

Thus, we finish the proof. \square

Corollary 9. *Assume that $2 > \sigma \geq \sigma_0 > 0$ and $K_a(x, y) \in \mathcal{L}_2(\lambda, \Lambda, \sigma)$ for any $a \in \mathcal{A}$. Assume that $w(t)$ is a Dini modulus of continuity satisfying $(H2)_{\bar{\beta}, \sigma}$, where $\bar{\beta}$ is given in Theorem 6.2.1. Assume that there exists $C_f > 0$ such that, for any $x_1, x_2 \in B_1(0)$*

$$|f(x_1) - f(x_2)| \leq C_f w(|x_1 - x_2|) \text{ and } \|f\|_{L^\infty(B_1(0))} \leq C_f$$

and $K_a(x, y)$ satisfies, for any $0 < r \leq 1$

$$\int_{\mathbb{R}^n} |K_a(x_1, y) - K_a(x_2, y)| \min\{|y|^{\min\{2, \sigma + \bar{\beta}\}}, r^{\min\{2, \sigma + \bar{\beta}\}}\} dy \leq \Lambda w(|x_1 - x_2|) r^{\min\{2 - \sigma, \bar{\beta}\}}.$$

If u is a bounded viscosity solution of (14), then there exists a constant $C > 0$ depending on $\lambda, \Lambda, n, \sigma_0, \sigma$ and w such that

$$\|u\|_{C^\sigma(B_{\frac{1}{2}}(0))} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + C_f).$$

Example 6.3.1. *Since the assumption (157) is slightly complicated, we provide several examples when it is satisfied. We first consider the kernel $K_a(x, y)$ which satisfies, for any $r > 0$*

$$\int_{B_{2r}(0) \setminus B_r(0)} |K_a(x, y) - K_a(0, y)| dy \leq \Lambda w(|x|) r^{-\sigma}, \quad \text{in } B_1(0). \quad (169)$$

Thus, for any $0 < r < 1$, $x \in B_1(0)$ and non-negative integer n , we have

$$\int_{B_{\frac{r}{2^n}}(0) \setminus B_{\frac{r}{2^{n+1}}}(0)} |K_a(x, y) - K_a(0, y)| |y|^{\min\{2, \sigma + \bar{\beta}\}} dy \leq \Lambda w(|x|) 2^{\sigma - n \min\{2 - \sigma, \bar{\beta}\}} r^{\min\{2 - \sigma, \bar{\beta}\}},$$

and

$$\int_{B_{2^{n+1}r}(0) \setminus B_{2^n r}(0)} |K_a(x, y) - K_a(0, y)| |r|^{\min\{2, \sigma + \bar{\beta}\}} dy \leq \Lambda w(|x|) 2^{-n\sigma} r^{\min\{2 - \sigma, \bar{\beta}\}}.$$

Then it is not hard to verify that (169) implies (157). Another more concrete example satisfying (157) is given by the kernel of the form

$$K_a(x, y) = \frac{k_a(x, y)}{|y|^{n+\sigma}}, \quad \text{for any } x \in B_1(0) \text{ and } y \in \mathbb{R}^n, \quad (170)$$

where $|k_a(x, y) - k_a(0, y)| \leq \Lambda w(|x|)$.

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