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A Generalizing of the Fractional Sub-Equation Method to Solve Systems of the Space-Time Fractional Differential Equation

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Abstract

In the present paper, we construct the analytical solutions of some Space-Time nonlinear fractional order systems involving Jumarie's modified Riemann-Liouville derivative in mathematical physics : such that Space-Time fractional Whitham-Broer-Kaup equations, Space-Time fractional Breaking Soliton equations, Space-Time fractional Coupled Boussinesq- Burgers equations and Space-Time fractional Coupled Burgers Equations by using The fractional sub-equation method . this method is very powerful mathematical technique for finding exact solutions of nonlinear ordinary differential equations.

Key words: fractional sub-equation method, modified Riemann-Liouville derivative, Mittage-Leffler function.

1 Introduction :

Fractional differential equations are generalizations of classical differential equations of integer order. In recent years, nonlinear fractional differential equations (FDEs) have been attracted great interest. It is caused by both the development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, engineering, and biology [1–7]. For better understanding the mechanisms of the complicated nonlinear physical phenomena as well as applying them in practical life, the solution of fractional differential equation is much involved [8–15].

Recently, Zhang and Zhang [16] introduced a new method called fractional sub-equation method to look for traveling wave solutions of nonlinear FDEs. The method is based on the homogeneous balance principle [17] and Jumarie’s modified Riemann-Liouville derivative [18-19]. By using fractional sub-equation method, Zhang et al. successfully obtained traveling wave solutions of nonlinear time fractional biological population model and (4 + 1) dimensional space-time fractional Fokas equation. More recently, Guo et al. [20] and Lu [21] improved Zhang et al.’s work [16] and obtained exact solutions of some nonlinear FDEs.

In the present paper, the fractional sub-equation method will be employed on some systems involving Jumaries modified Riemann-Liouville derivative to find the exact solution for some Space-Time nonlinear fractional systems, such that Space-Time fractional Whitham-Broer-Kaup equations, Space-Time fractional Breaking Soliton equations, Space-Time fractional Coupled Boussinesq- Burgers equations and Space-Time fractional Coupled Burgers Equations .The fractional sub-equation method is very powerful mathematical technique for finding exact solutions of nonlinear ordinary differential equations.

The rest of this paper is organized as follows. In Section 2, we will describe the Modified Riemann-Liouville derivative and give the main steps of method here. In Section 3, we give four applications of the proposed method . In Section 4, some conclusion are given.

2 Descriptions the Modified Riemann-Liouville Derivative and the Proposed Method.

The Jumarie's Modified Riemann-Liouville Derivative of order α is defined by the expression [18]:

$$D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}\big(f(\xi)-f(0)\big)d\xi \quad ,\alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}\big(f(\xi)-f(0)\big)d\xi \, ,0 < \alpha < 1 \\ \{f^{(n)}(x)\}^{(\alpha-n)} \qquad ,n \le \alpha < n+1 <, n \ge 1 \end{cases} \tag{1}$$

Some properties of the proposed Modified Riemann-Liouville Derivative are listed in [18] as follows:

a) $D_x^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \gamma > 0$ (2)

b) $D_x^{\alpha}(f(x)g(x)) = f(x) D_x^{\alpha}(g(x)) + g(x) D_x^{\alpha}(f(x))$ (3)

c) $D_x^{\alpha} f[g(x)] = f'_g[g(x)] D_x^{\alpha} g(x) = D_x^{\alpha} f[g(x)] (g'(x))^{\alpha}$ (4)

d) $D_x^{\alpha} M = 0$, where M is any constant.

These equations play an important role in calculus.

We present the main steps of the generalizing fractional sub-equation method as:

Step 1: Give nonlinear FDEs with independent variables X= ($x_1, x_2, x_3, \ldots, x_m$, t) and dependent variable u :

$$P\big(u, u_t, u_{x_1}, u_{x_2}, u_{x_3}, \ldots, u_{x_m}, D_t^{\alpha} u, D_{x_1}^{\alpha} u, u, D_{x_3}^{\alpha} u, \ldots, D_{x_m}^{\alpha} u\big) = 0, \quad 0 < \alpha < 1 \tag{5}$$

Where $D_t^{\alpha} u, D_{x_1}^{\alpha} u, D_{x_2}^{\alpha} u, D_{x_3}^{\alpha} u$ and are modified fractional Riemann-liouville derivative of u with respect to t $, x_1, x_2, x_3, \ldots, x_m$ respectively and $u=(x_1, x_2, x_3, \ldots, x_m, t)$ is unknown function , P is polynomial in u and its various partial derivative , in which the highest order derivatives and nonlinear terms are involved.

Step 2: By using the traveling wave transformation:

$$u(x_1, x_2, x_3 \ldots, x_m, t)=U(\xi) \quad , \xi=k_1 x_1+k_2 x_2+k_3 x_3+\ldots\ldots+k_m x_m+ct \tag{6}$$

Where $c, k_1, k_2, k_3, \ldots, k_m$, are constants to be determined later, the FDE (5) is reduced to the nonlinear fractional ordinary differential equation (FODE)for U= U(ξ)

$$P\left(u, cu', k_1 u', k_2 u', \ldots, k_m u', c^{\alpha} D_{\xi}^{\alpha} u, k_1{}^{\alpha} D_{\xi}^{\alpha} u, k_2{}^{\alpha} D_{\xi}^{\alpha} u, \ldots, k_m{}^{\alpha} D_{\xi}^{\alpha} u\right) = 0, 0 < \alpha < 1 \tag{7}$$

Step 3: Suppose the reduced equation obtained in Step 2 has a solution in the form

$$U(\xi)=\sum_{i=0}^{n} a_i \varphi^i \tag{8}$$

Where $a_i \ (i = 0,1,2,\ldots,n)$ are constants to be determined later, n is a positive integer Determined by balancing the highest order derivative and nonlinear terms in Eq. (5) or Eq.(7)and $\varphi = \varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_{\xi}^{\alpha}\varphi = \sigma + \varphi^2, 0 < \alpha \leq 1, \text{ where } \sigma \text{ is a constant} \tag{9}$$

The following solutions of fractional Riccati equation (9):

$$\varphi(\xi)=\begin{cases} -\sqrt{-\sigma}\tanh_{\alpha}\left(\sqrt{-\sigma}\,\xi\right), \sigma < 0 \\ -\sqrt{-\sigma}\coth_{\alpha}\left(\sqrt{-\sigma}\,\xi\right) \quad ,\sigma < 0 \\ \sqrt{\sigma}\tan_{\alpha}\left(\sqrt{\sigma}\,\xi\right) \quad ,\sigma > 0 \\ -\sqrt{\sigma}\cot_{\alpha}\left(\sqrt{\sigma}\,\xi\right), \sigma > 0 \\ -\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}. w = \text{constant}. \sigma = 0 \end{cases} \tag{10}$$

Where the generalized hyperbolic and trigonometric functions are defined by the Mittage-Leffler function :

$$\sinh_{\alpha}(x) = \frac{E_{\alpha}(x^{\alpha}) - E_{\alpha}(-x^{\alpha})}{2}, \qquad \cosh_{\alpha}(x) = \frac{E_{\alpha}(x^{\alpha}) + E_{\alpha}(-x^{\alpha})}{2}, \quad \tanh_{\alpha}(x) = \frac{\sinh_{\alpha}(x)}{\cosh_{\alpha}(x)},$$

$$\sin_\alpha(x) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i}, \qquad \cos_\alpha(x) = \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2i}$$

$$\tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)}, \qquad \cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)}, \qquad \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)}$$

Remark:

We define the degree of u(ξ) as D[u(ξ)] = n, which gives rise to the degrees of other expressions, as follows:

$D[\frac{d^q u}{d^q \xi}]=n+q, D\left[u^p \left(\frac{d^q u}{d^q \xi}\right)^r\right]=np+r(q+n)$.

So, the value of n can be obtained for the equation (5 or7).

Step 4: Substituting Eq.(8) along with Eq.(9) into Eq.(7), we can get a polynomial

in $\varphi(\xi)$,setting all the coefficients of $\varphi^k (k = 0,1,\dots n)$ to zero, yields a set of overdetermined nonlinear algebraic equations for c,k_i and $a_i (i = 0,1,2,\dots,n)$.

Step 5: Assuming that the constants c, k,$a_i (i = 0,1,2,\dots,n)$ can be obtained by solving the algebraic equations in step 4, substituting these constants and the solutions of Eq.(9) into Eq. (8) we can obtain the explicit solutions of Eq.(5) immediately.

3 Application of the Method:

In this section , we want to solve some systems of non-linear fractional derivative differential equations by applying the technique of generalizing fractional sub-equation method.

1. Space-Time (1+1) fractional derivative Whitham-Broer-Kaup equations (WBK): [22,23]

We consider fractional derivative Whitham -Broer- Kaup equations in the form:

$$\begin{cases} D_t^\alpha u + u D_x^\alpha u + D_x^\alpha v + \beta D_x^{2\alpha} u = 0 \\ D_t^\alpha v + D_x^\alpha (uv) - \beta D_x^{2\alpha} v + \gamma D_x^{3\alpha} u = 0 \end{cases} \quad 0 < \alpha \leq 1 \qquad (11)$$

By considering the traveling wave transformation,

u(x,t)=U(ξ), v(x,t)=V(ξ), where ξ=kx+ct

Eq. (11) can be reduced to the following nonlinear fractional ODE:

$$\begin{cases} C^{\alpha}D_{\xi}^{\alpha}U + K^{\alpha}UD_{\xi}^{\alpha}U + K^{\alpha}D_{\xi}^{\alpha}V + \beta K^{2\alpha}D_{\xi}^{2\alpha}U = 0 \\ C^{\alpha}D_{\xi}^{\alpha}V + K^{\alpha}D_{\xi}^{\alpha}(UV) - \beta K^{2\alpha}D_{\xi}^{2\alpha}V + \gamma K^{3\alpha}D_{\xi}^{3\alpha}U = 0 \end{cases} \quad 0 < \alpha \leq 1 \qquad (12)$$

We suppose that Eq.(4.2) has the following general solution:

$$\begin{cases} U(\xi) = \sum_{i=0}^{n} a_i \varphi^i \\ V(\xi) = \sum_{j=0}^{m} b_j \varphi^j \end{cases}$$

By balancing the highest order derivative term and nonlinear term in Eq.(12), $D_{\xi}^{\alpha}V.\ UD_{\xi}^{\alpha}UandD_{\xi}^{3\alpha}U.\ D_{\xi}^{\alpha}(UV)$,we have m+1=2n+1 then n=1,3+n=n+m+1 then m=2.

Then we suppose that Eq.(12) has the following formal solution:

$$\begin{cases} U(\xi) = a_0 + a_1\varphi \\ V(\xi) = b_0+b_1\varphi+b_2\varphi^2 \end{cases} \qquad (13)$$

Where $\varphi(\xi)$satisfies fractional Riccati equation

$$D_{\xi}^{\alpha}\varphi = \sigma + \varphi^2$$

(14)

Substituting (13) along with (14) into (12) and setting the coefficient of $\varphi(\xi)^i$(i=0,1,2,3)

to zero, we can obtain aset of algebraic equation for c,k,b_0, b_1, b_2, a_0 and a_1 as follows

From the first equation of previous system (13):

$\varphi(\xi)^0: C^{\alpha}a_1\sigma + K^{\alpha}a_0a_1\sigma + K^{\alpha}\sigma b_1 = 0$

$\varphi(\xi)^1: 2k^{\alpha}b_2\sigma + 2\beta k^{2\alpha}a_1\sigma + k^{\alpha}a_1\text{^}2\sigma = 0$

$\varphi(\xi)^2: C^{\alpha}a_1 + b_1K^{\alpha} + a_1k^{\alpha}a_0 = 0$

$\varphi(\xi)^3: k^{\alpha}a_1\text{^}2 + 2k^{\alpha}b_2 + 2k^{2\alpha}\beta a_1 = 0$

From the second equation of previous system (13):

$\varphi(\xi)^0 : \sigma c^\alpha b_1 + k^\alpha \sigma(b_1 a_0 + b_0 a_1) + 2\gamma k^{3\alpha} a_1 \sigma^\wedge 2 - 2\beta k^{2\alpha} b_2 \sigma^\wedge 2 = 0$

$\varphi(\xi)^1 : 2\sigma c^\alpha b_2 + k^\alpha (2b_2 a_0 \sigma + 2b_1 a_1 \sigma) - 2\beta k^{2\alpha} b_1 \sigma = 0$

$\varphi(\xi)^2 : c^\alpha b_1 + k^\alpha (a_0 b_1 + 2a_1 b_2 \sigma + b_0 a_1) + 8\gamma k^{3\alpha} a_1 \sigma - 8\beta k^{2\alpha} b_2 \sigma = 0$

$\varphi(\xi)^3 : 2c^\alpha b_2 + 2k^\alpha b_2 a_0 + 2k^\alpha a_1 b_1 - 2\beta k^{2\alpha} b_1 = 0$

By using Mathematica:

Case 1: $\{a_1 \to 0, b_1 \to 0, b_2 \to 0\}$, where a_0 and b_0 are arbitrary.

$$\begin{cases} U_1 = a_0 \\ V_1 = b_0 \end{cases}$$

■

Case 2: $\{a_0 \to -c^\alpha k^{-\alpha}, a_1 \to 2k^\alpha \sqrt{\beta^2 + \gamma}, b_0 \to 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\sigma\right), b_1 \to 0, b_2 \to 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\right)\}$

Where $\sigma < 0$, ξ=xk+ct

$$\begin{cases} U_2 = -c^\alpha k^{-\alpha} - 2k^\alpha \sqrt{\beta^2 + \gamma}(\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\,\xi)) \\ V_2 = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\right)\sigma\tanh_\alpha{}^2(\sqrt{-\sigma}\,\xi) \end{cases}$$

$$\begin{cases} U_3 = -c^\alpha k^{-\alpha} - 2k^\alpha \sqrt{\beta^2 + \gamma}(\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\,\xi)) \\ V_3 = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\right)\sigma\coth_\alpha{}^2(\sqrt{-\sigma}\,\xi) \end{cases}$$

Where $\sigma > 0$, ξ= xk+ct

$$\begin{cases} U_4 = -c^\alpha k^{-\alpha} + 2k^\alpha \sqrt{\beta^2 + \gamma}(\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\,\xi)) \\ V_4 = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\right)\sigma\tan_\alpha{}^2(\sqrt{\sigma}\,\xi) \end{cases}$$

$$\begin{cases} U_5 = -c^\alpha k^{-\alpha} - 2k^\alpha \sqrt{\beta^2 + \gamma}(\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\,\xi)) \\ V_5 = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma - k^{2\alpha}\beta\sqrt{\beta^2 + \gamma}\right)\sigma\cot_\alpha{}^2(\sqrt{\sigma}\,\xi) \end{cases}$$

Where $\sigma = 0$, ξ=xk+ct, w is constant.

$$\begin{cases} U_6 = -c^{\alpha}k^{-\alpha} - 2k^{\alpha}\sqrt{\beta^2+\gamma}\left(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}\right) \\ V_6 = 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma - k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\right)\left(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}\right)^2 \end{cases} \blacksquare$$

Case3:

$\{a_0 \to -c^{\alpha}k^{-\alpha}, a_1 \to -2k^{\alpha}\sqrt{\beta^2+\gamma}, b_0 \to 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\sigma\right), b_1 \to 0, b_2 \to 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\right)\}$

Where $\sigma < 0$, ξ=xk+ct

$$\begin{cases} U_7 = -c^{\alpha}k^{-\alpha} + 2k^{\alpha}\sqrt{\beta^2+\gamma}\left(\sqrt{-\sigma}\tanh_{\alpha}\left(\sqrt{-\sigma}\,\xi\right)\right) \\ V_7 = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\right)\left(\sigma \tanh_{\alpha}{}^2\left(\sqrt{-\sigma}\,\xi\right)\right) \end{cases}$$

$$\begin{cases} U_8 = -c^{\alpha}k^{-\alpha} + 2k^{\alpha}\sqrt{\beta^2+\gamma}\left(\sqrt{-\sigma}\coth_{\alpha}\left(\sqrt{-\sigma}\,\xi\right)\right) \\ V_8 = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\right)\left(\sigma \coth^2{}_{\alpha}\left(\sqrt{-\sigma}\,\xi\right)\right) \end{cases}$$

Where $\sigma > 0$, ξ=xk+ct

$$\begin{cases} U_9 = -c^{\alpha}k^{-\alpha} - 2k^{\alpha}\sqrt{\beta^2+\gamma}\left(\sqrt{\sigma}\tan_{\alpha}\left(\sqrt{\sigma}\,\xi\right)\right) \\ V_9 = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\right)\left(\sigma \tan_{\alpha}{}^2\left(\sqrt{\sigma}\,\xi\right)\right) \end{cases}$$

$$\begin{cases} U_{10} = -c^{\alpha}k^{-\alpha} + 2k^{\alpha}\sqrt{\beta^2+\gamma}\left(\sqrt{\sigma}\cot_{\alpha}\left(\sqrt{\sigma}\,\xi\right)\right) \\ V_{10} = 2\left(-k^{2\alpha}\beta^2\sigma - k^{2\alpha}\gamma\sigma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\sigma\right) + 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma + k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\right)\left(\sigma \cot^2{}_{\alpha}\left(\sqrt{\sigma}\,\xi\right)\right) \end{cases}$$

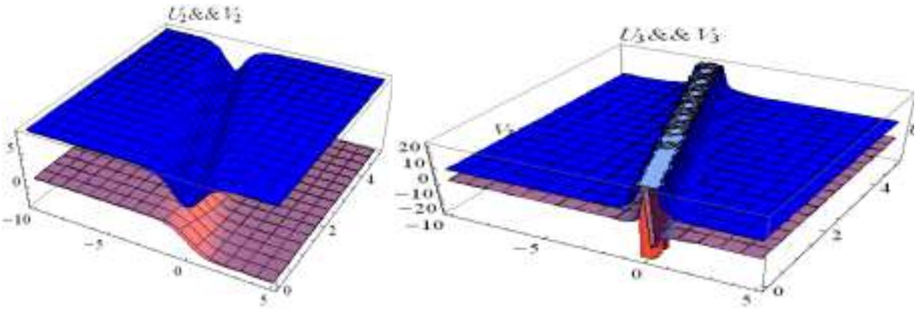
Where $\sigma = 0$, ξ=xk+ct, w is constant.

$$\begin{cases} U_{11} = -c^{\alpha}k^{-\alpha} + 2k^{\alpha}\sqrt{\beta^2+\gamma}\left(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}\right) \\ V_{11} = 2\left(-k^{2\alpha}\beta^2 - k^{2\alpha}\gamma - k^{2\alpha}\beta\sqrt{\beta^2+\gamma}\right)\left(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}\right)^2 \end{cases} \blacksquare$$

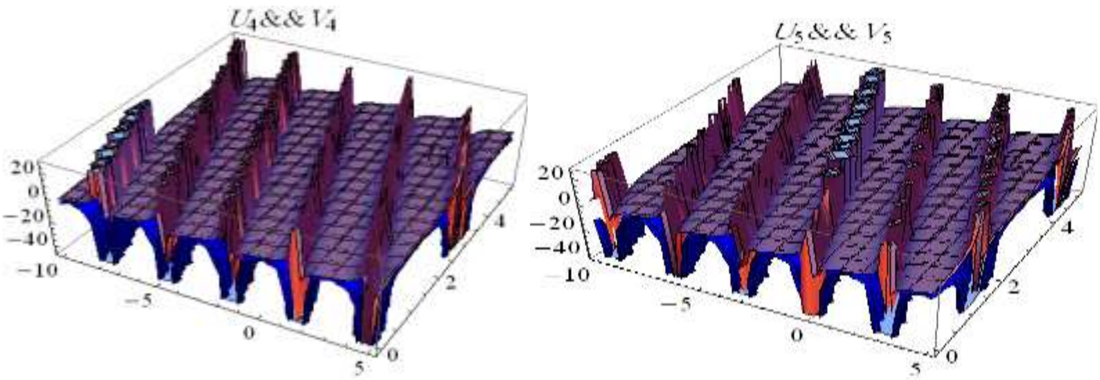
1.1 Figures of Space-Time (1+1) fractional derivative Whitham -Broer- Kaup equations:

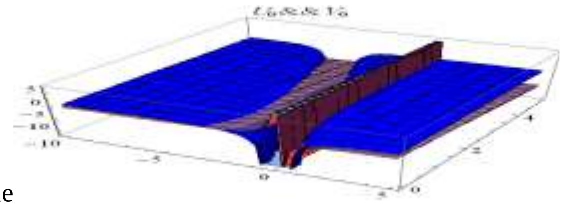
Case 2:

Where $\sigma < 0$, Let $\alpha = c = k = \beta = 1, \sigma = -1, \gamma = 0$,then:



Where $\sigma > 0$, Let $\alpha = c = k = \beta = \sigma = 1, \gamma = 0$, then:

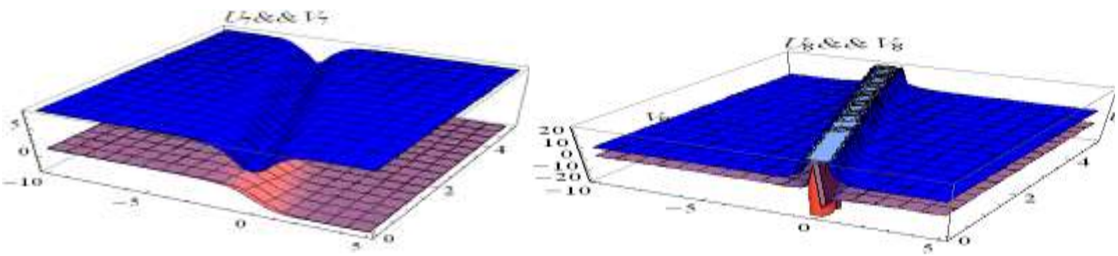




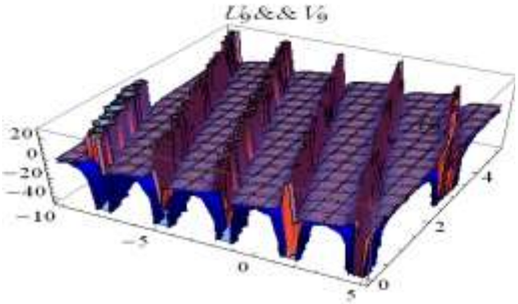
Where $\sigma = 0$, Let $\alpha = c = k = \beta = 1, w = \gamma = 0$, the

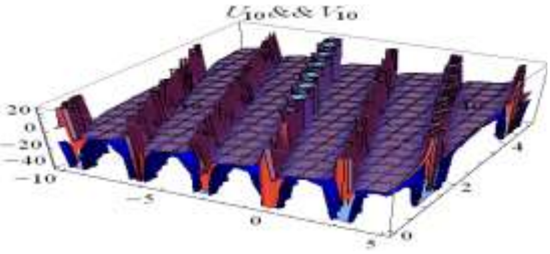
Case 3:

Where $\sigma < 0$, Let $\alpha = c = k = \beta = 1, \sigma = -1, \gamma = 0$, then:

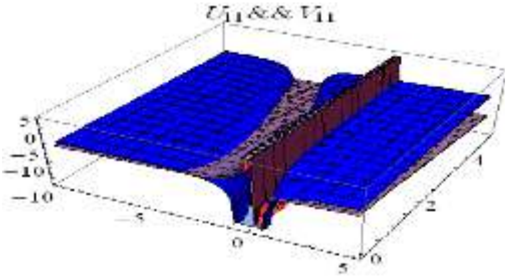


Where $\sigma > 0$, Let $\alpha = c = k = \beta = \sigma = 1, \gamma = 0$,then:





Where $\sigma = 0$,Let $\alpha = c = k = \beta = 1, \gamma = w = 0$ then:



2:Space-Time (2+1) fractional derivative Breaking Soliton equations:[18]

We consider the Space-time (2+1) fractional Breaking Soliton equations:

$$\begin{cases} D_t^{\alpha}u + aD_x^{2\alpha}D_y^{\alpha}u + 4auD_x^{\alpha}v + 4avD_x^{\alpha}u = 0 \\ D_y^{\alpha}u - D_x^{\alpha}v = 0 \end{cases} \qquad 0 < \alpha \leq 1 \qquad (15)$$

By considering the traveling wave transformation

$u(x,t)= U(\xi)$, $v(x,t) = V(\xi)$, Where $\xi=k_1x+ k_2y+ct$

Eq.(15), can be reduced to the following nonlinear fractional ODE:

$$\begin{cases} C^{\alpha}D_{\xi}^{\alpha}U + ak_1{}^{2\alpha}k_2{}^{\alpha}D_{\xi}^{3\alpha}U + 4ak_1{}^{\alpha}UD_{\xi}^{\alpha}V + 4ak_1{}^{\alpha}VD_{\xi}^{\alpha}U = 0 \\ k_2{}^{\alpha}D_{\xi}^{\alpha}U - k_1{}^{\alpha}D_{\xi}^{\alpha}V = 0 \end{cases} \qquad 0 < \alpha \leq 1 \qquad (16)$$

We suppose that Eq.(16) has the following general solution:

$$\begin{cases} U(\xi) = \sum_{i=0}^{n} a_i \varphi^i \\ V(\xi) = \sum_{j=0}^{m} b_j \varphi^j \end{cases}$$

By balancing the highest order derivative term and nonlinear term in Eq.(16), $D_\xi^{3\alpha}U$,$UD_\xi^{\alpha}V$and$D_\xi^{\alpha}U$ and$D_\xi^{\alpha}V$, we have n+3=n+m+1 then m=2 ,1+n=m+1 then m=n=2.

Then we suppose that Eq.(4.6) has the following formal solution:

$$\begin{cases} U(\xi) = a_0 + a_1\varphi + a_2\varphi^2 \\ V(\xi) = b_0 + b_1\varphi + b_2\varphi^2 \end{cases} \quad (17)$$

Substituting (17) along with (14) into (16) and setting the coefficient of $\varphi(\xi)^i$(i=0,1,2,3,4,5) to zero, we can obtain

a set of algebraicequation for c,k,b_1, b_0, b_2, a_0, a_1 and a_2 as follows:

From the first equation of previous system (15):

$\varphi(\xi)^0$: $c^\alpha a_1\sigma + 2ak_1^{2\alpha}k_2^{\alpha}a_1\sigma\hat{}2 + 4k_1^{\alpha}a(b_1\sigma a_0 + a_1\sigma b_0) = 0$

$\varphi(\xi)^1$: $2c^\alpha a_2\sigma + 16ak_1^{2\alpha}k_2^{\alpha}a_2\sigma^2 + 4ak_1^{\alpha}(2b_2\sigma a_0 + 2a_1\sigma b_1 + 2b_0\sigma a_2) = 0$

$\varphi(\xi)^2$: $c^\alpha a_1 + 8ak_1^{2\alpha}k_2^{\alpha}a_1\sigma + 4ak_1^{\alpha}(a_0b_1 + 3a_1b_2\sigma + 3a_2b_1\sigma + b_0a_1) = 0$

$\varphi(\xi)^3$: $2a_2c^\alpha + 40ak_1^{2\alpha}k_2^{\alpha}a_2\sigma + 4k_1^{\alpha}a(2b_2a_0 + 2b_1a_1 + 2a_2b_2\sigma + 2b_0a_2 + 2a_2b_2\sigma) = 0$

$\varphi(\xi)^4$: $6ak_1^{2\alpha}k_2^{\alpha}a_1 + 4k_1^{\alpha}a(3b_2a_1 + 3b_1a_2) = 0$

$\varphi(\xi)^5$: $24aa_2k_1^{2\alpha}k_2^{\alpha} + 4k_1^{\alpha}a(4a_2b_2) = 0$

From the second equation of previous system (15):

$\varphi(\xi)^0$: $\sigma k_2^{\alpha}a_1 - b_1\sigma k_1^{\alpha} = 0$

$\varphi(\xi)^1: 2k_2{}^{\alpha} a_2\sigma - 2k_1{}^{\alpha} b_2\sigma = 0$

$\varphi(\xi)^2: a_1 k_2{}^{\alpha} - k_1{}^{\alpha} b_1 = 0$

$\varphi(\xi)^3: 2a_2 k_2{}^{\alpha} - 2k_1{}^{\alpha} b_2 = 0$

By using Mathematica :

Case1: $\{a_1 \to 0, a_2 \to 0, b_2 \to 0, b_1 \to 0\}$

$$\begin{cases} U_1 = a_0 \\ V_1 = b_0 \end{cases}$$

■

Case2: $\{a_1 \to 0, a_2 \to -\frac{3}{2}k_1^{2\alpha}, b_2 \to -\frac{3}{2}k_1^{\alpha}k_2^{\alpha}, b_1 \to 0, b_0 \to -\frac{k_1^{-\alpha}(c^{\alpha}+4aa_0k_2^{\alpha}+8a\sigma k_1^{2\alpha}k_2^{\alpha})}{4a}\}$

Where $\sigma < 0$, $\xi = xk_1 + yk_2 + ct$

$$\begin{cases} U_2 = a_0 - \frac{3}{2}k_1^{2\alpha}\sigma \tanh_{\alpha}{}^2(\sqrt{-\sigma}\,\xi) \\ V_2 = -\frac{k_1^{-\alpha}(c^{\alpha}+4aa_0k_2^{\alpha}+8a\sigma k_1^{2\alpha}k_2^{\alpha})}{4a} - \frac{3}{2}k_1^{\alpha}k_2^{\alpha}\sigma \tanh_{\alpha}{}^2(\sqrt{-\sigma}\,\xi) \end{cases}$$

$$\begin{cases} U_3 = a_0 - \frac{3}{2}k_1^{2\alpha}\sigma \coth_{\alpha}{}^2(\sqrt{-\sigma}\,\xi) \\ V_3 = -\frac{k_1^{-\alpha}(c^{\alpha}+4aa_0k_2^{\alpha}+8a\sigma k_1^{2\alpha}k_2^{\alpha})}{4a} - \frac{3}{2}k_1^{\alpha}k_2^{\alpha}\sigma \coth_{\alpha}{}^2(\sqrt{-\sigma}\,\xi) \end{cases}$$

Where $\sigma > 0$, $\xi = xk_1 + yk_2 + ct$

$$\begin{cases} U_4 = a_0 - \frac{3}{2}k_1^{2\alpha}\sigma \tan_{\alpha}{}^2(\sqrt{\sigma}\,\xi) \\ V_4 = -\frac{k_1^{-\alpha}(c^{\alpha}+4aa_0k_2^{\alpha}+8a\sigma k_1^{2\alpha}k_2^{\alpha})}{4a} - \frac{3}{2}k_1^{\alpha}k_2^{\alpha}\sigma \tan_{\alpha}{}^2(\sqrt{\sigma}\,\xi) \end{cases}$$

$$\begin{cases} U_5 = a_0 - \frac{3}{2}k_1^{2\alpha}\sigma \cot_{\alpha}{}^2(\sqrt{\sigma}\,\xi) \\ V_5 = -\frac{k_1^{-\alpha}(c^{\alpha}+4aa_0k_2^{\alpha}+8a\sigma k_1^{2\alpha}k_2^{\alpha})}{4a} - \frac{3}{2}k_1^{\alpha}k_2^{\alpha}\sigma \cot_{\alpha}{}^2(\sqrt{\sigma}\,\xi) \end{cases}$$

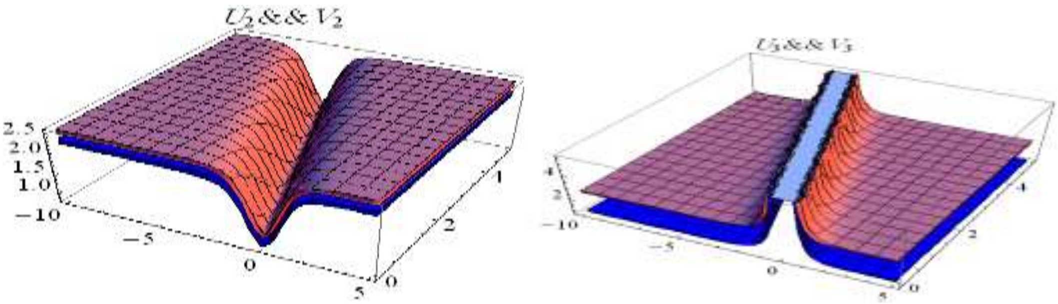
Where $\sigma = 0$,$\xi = xk_1 + yk_2 + ct$

$$\begin{cases} U_6 = a_0 - \frac{3}{2} k_1^{2\alpha}{}_2 \left(\frac{\Gamma(1+\alpha)}{\xi^\alpha + w}\right)^2 \\ V_6 = -\frac{k_1^{-\alpha}(c^\alpha + 4aa_0 k_2^\alpha)}{4a} - \frac{3}{2} k_1^\alpha k_2^\alpha \left(\frac{\Gamma(1+\alpha)}{\xi^\alpha + w}\right)^2 \end{cases}$$ ■

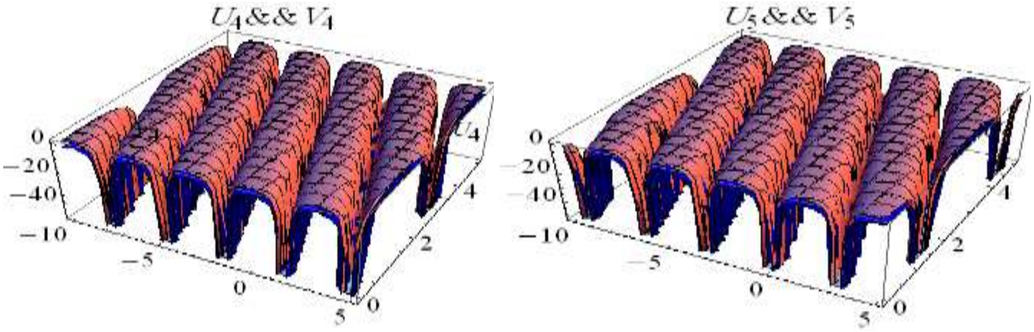
2.1: Figures of Space-Time (2+1) fractional derivative Breaking Soliton equations:

Case 2:

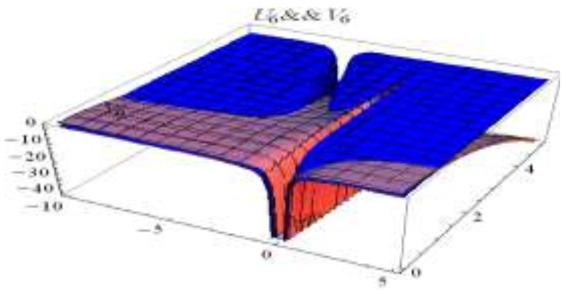
Where $\sigma < 0$,Let $a_0 = \alpha = c = k = 1, \sigma = -1, \gamma = 0$, then:



Where $\sigma > 0$, Let $\alpha = c = k = \beta = a = \sigma = 1, \gamma = 0$, then:



Where $\sigma = 0$, Let $\alpha = c = k = \beta = 1, w = \gamma = 0$, then:



3: Space-Time (1+1) fractional derivative Coupled Boussinesq-Burgers equations:

We consider the Space-Time fractional Coupled Boussinesq-Burgers equations:

$$\begin{cases} D_t^{\alpha}u - \frac{1}{2}D_x^{\alpha}v + 2uD_x^{\alpha}u = 0 \\ D_t^{\alpha}v - \frac{1}{2}D_x^{3\alpha}u + 2D_x^{\alpha}(uv) = 0 \end{cases} \quad 0 < \alpha \leq 1 \tag{18}$$

By considering the traveling wave transformation

$u(x,t)= U(\xi)$, $v(x,t)=V(\xi)$, Where $\xi= Kx+ ct$

Eq.(18) can be reduced to the following nonlinear fractional ODE:

$$\begin{cases} C^{\alpha}D_{\xi}^{\alpha}U - \frac{1}{2}K^{\alpha}D_{\xi}^{\alpha}V + 2K^{\alpha}UD_{\xi}^{\alpha}U = 0 \\ C^{\alpha}D_{\xi}^{\alpha}V - \frac{1}{2}K^{3\alpha}D_{\xi}^{3\alpha}U + 2K^{\alpha}D_{\xi}^{\alpha}(UV) = 0 \end{cases} \quad 0 < \alpha \leq 1 \tag{19}$$

We suppose that Eq.(19) has the following general solution:

$$\begin{cases} U(\xi) = \sum_{i=0}^{n} a_i\,\varphi^i \\ V(\xi) = \sum_{j=0}^{m} b_j\,\varphi^j \end{cases}$$

By balancing the highest order derivative term and nonlinear term in Eq.(19) $D_{\xi}^{\alpha}V$,$UD_{\xi}^{\alpha}U$ and

$D_{\xi}^{3\alpha}U$,$D_{\xi}^{\alpha}(VU)$, we have n+3=n+m+1 then m=2 , 1+2n= m+1 then n=1.

Then we suppose that Eq .(19) has the following formal solution:

$$\begin{cases} U(\xi) = a_0 + a_1\varphi \\ V(\xi) = b_0 + b_1\varphi + b_2\varphi^2 \end{cases} \tag{20}$$

Substituting (20) along with (14) into (19) and setting the coefficient of $\varphi(\xi)^i$(i=0,1,2,3,4) to zero, we can obtain a set of algebraic equation for c,k,b_0, b_1, b_2, a_0 and a_1 as follows:

From the first equation of previous system (19):

$$\varphi(\xi)^0: c^{\alpha} * a_1 * \sigma - \frac{1}{2} * k^{\alpha} * b_1 * \sigma + 2 * k^{\alpha} * a_0 * a_1 * \sigma = 0$$

$$\varphi(\xi)^1: -k^{\alpha} * b_2 * \sigma + 2k^{\alpha} * \sigma * a_1{}^2 = 0$$

$$\varphi(\xi)^2: c^{\alpha} * a_1 - \frac{1}{2} * k^{\alpha} * b_1 + 2 * k^{\alpha} * a_0 * a_1 = 0$$

$$\varphi(\xi)^3: -k^{\alpha} * b_2 + 2 * k^{\alpha} * a_1 \hat{} 2 = 0$$

From the second equation of previous system (19):

$$\varphi(\xi)^0: c^{\alpha} * b_1 * \sigma - k^{3\alpha} * a_1 * \sigma \hat{} 2 + 2k^{\alpha} * (a_0 * b_1 * \sigma + b_0 * a_1 * \sigma) = 0$$

$$\varphi(\xi)^1: 2 * \sigma * c^{\alpha} * b_2 + 4 * a_0 * \sigma * k^{\alpha} * b_2 + 2k^{\alpha} * (a_1 * b_1 * \sigma + b_1 * a_1 * \sigma) = 0$$

$$\varphi(\xi)^2: c^{\alpha} * b_1 - 4k^{3\alpha} * a_1 * \sigma + 2k^{\alpha} * (a_0 * b_1 + 3b_2 * a_1 * \sigma + b_0 * a_1) = 0$$

$$\varphi(\xi)^3: 2 * c^{\alpha} * b_2 + 4k^{\alpha} * b_2 * a_0 + 4k^{\alpha} * a_1 * b_1 = 0$$

$$\varphi(\xi)^4: -3k^{3\alpha} * a_1 + 6k^{\alpha} * a_1 * b_2 = 0$$

By usingMathematica:

Case1: $\{b_1 \to 0, a_1 \to 0, b_2 \to 0\}$

$$\begin{cases} U_1 = a_0 \\ V_1 = b_0 \end{cases}$$ ■

Case2: $\left\{b_1 \to 0, b_0 \to \frac{1}{2} k^{2\alpha}\sigma, a_1 \to -\frac{k^{\alpha}}{2}, a_0 \to -\frac{1}{2} c^{\alpha} k^{-\alpha}, b_2 \to \frac{k^{2\alpha}}{2}\right\}$

Where $\sigma < 0$, $\xi = xk + ct$

$$\begin{cases} U_2 = -\frac{1}{2} c^{\alpha} k^{-\alpha} - \frac{k^{\alpha}}{2} \left(\sqrt{-\sigma} \tanh_{\alpha}\left(\sqrt{-\sigma}\, \xi\right)\right) \\ V_2 = \frac{1}{2} k^{2\alpha} \sigma + \frac{k^{2\alpha}}{2} \sigma \tanh_{\alpha}{}^2\left(\sqrt{-\sigma}\, \xi\right) \end{cases}$$

$$\begin{cases} U_3 = -\frac{1}{2}c^{\alpha}k^{-\alpha} - \frac{k^{\alpha}}{2}\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\,\xi) \\ V_3 = \frac{1}{2}k^{2\alpha}\sigma + \frac{k^{2\alpha}}{2}\sigma\ \coth_{\alpha}{}^{2}(\sqrt{-\sigma}\,\xi) \end{cases}$$

Where $\sigma > 0$,ξ =xk+ ct

$$\begin{cases} U_4 = -\frac{1}{2}c^{\alpha}k^{-\alpha} + \frac{k^{\alpha}}{2}\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\,\xi) \\ V_4 = \frac{1}{2}k^{2\alpha}\sigma + \frac{k^{2\alpha}}{2}\sigma\ \tan_{\alpha}{}^{2}(\sqrt{\sigma}\,\xi) \end{cases}$$

$$\begin{cases} U_5 = -\frac{1}{2}c^{\alpha}k^{-\alpha} - \frac{k^{\alpha}}{2}(\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\,\xi) \\ V_5 = \frac{1}{2}k^{2\alpha}\sigma + \frac{k^{2\alpha}}{2}\sigma\ \cot_{\alpha}{}^{2}(\sqrt{\sigma}\,\xi) \end{cases}$$

Where $\sigma = 0$, ξ=xk+ct

$$\begin{cases} U_6 = -\frac{1}{2}c^{\alpha}k^{-\alpha} - \frac{k^{\alpha}}{2}(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}) \\ V_6 = \frac{k^{2\alpha}}{2}((\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}))^2 \end{cases}$$

■

Case3: $\left\{b_1 \to 0, b_0 \to \frac{1}{2}k^{2\alpha}\sigma, a_1 \to \frac{k^{\alpha}}{2}, a_0 \to -\frac{1}{2}c^{\alpha}k^{-\alpha}, b_2 \to \frac{k^{2\alpha}}{2}\right\}$

Where $\sigma < 0$, ξ =xk+ ct

$$\begin{cases} U_7 = -\frac{1}{2}c^{\alpha}k^{-\alpha} + \frac{k^{\alpha}}{2}(\sqrt{-\sigma}\tanh_{\alpha}(\sqrt{-\sigma}\,\xi)) \\ V_7 = \frac{1}{2}k^{2\alpha}\sigma + \frac{k^{2\alpha}}{2}\sigma\ \tanh_{\alpha}{}^{2}(\sqrt{-\sigma}\,\xi) \end{cases}$$

$$\begin{cases} U_8 = -\frac{1}{2}c^{\alpha}k^{-\alpha} + \frac{k^{\alpha}}{2}\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\,\xi) \\ V_8 = \frac{1}{2}k^{2\alpha}\sigma + \frac{k^{2\alpha}}{2}\sigma\ \coth_{\alpha}{}^{2}(\sqrt{-\sigma}\,\xi) \end{cases}$$

Where $\sigma > 0$,ξ =xk+ ct

$$\begin{cases} U_9 = -\frac{1}{2}c^{\alpha}k^{-\alpha} - \frac{k^{\alpha}}{2}\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\,\xi) \\ V_9 = \frac{1}{2}k^{2\alpha}\sigma + \frac{k^{2\alpha}}{2}\sigma\ \tan_{\alpha}{}^{2}(\sqrt{\sigma}\,\xi) \end{cases}$$

$$\begin{cases} U_{10} = -\frac{1}{2}c^{\alpha}k^{-\alpha} + \frac{k^{\alpha}}{2}(\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\,\xi) \\ V_{10} = \frac{1}{2}k^{2\alpha}\sigma + \frac{k^{2\alpha}}{2}\sigma\ \cot_{\alpha}{}^{2}(\sqrt{\sigma}\,\xi) \end{cases}$$

Where $\sigma = 0$,ξ=xk+ct

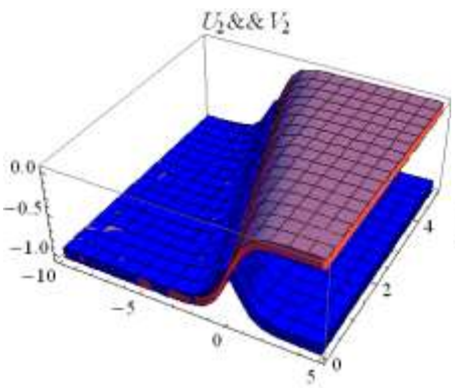
$$\begin{cases} U_{11} = -\frac{1}{2}c^{\alpha}k^{-\alpha} + \frac{k^{\alpha}}{2}(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}) \\ V_{11} = \frac{k^{2\alpha}}{2}((\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}))^{2} \end{cases}$$

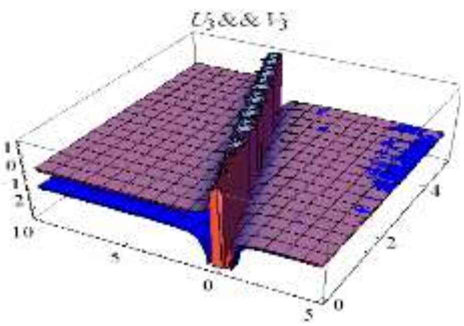
■

3.1 Figures of Space-Time (1+1) fractional derivative Coupled Boussinesq-Burgers equations:

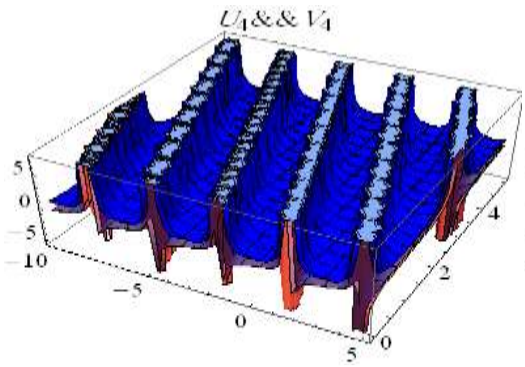
Case 2:

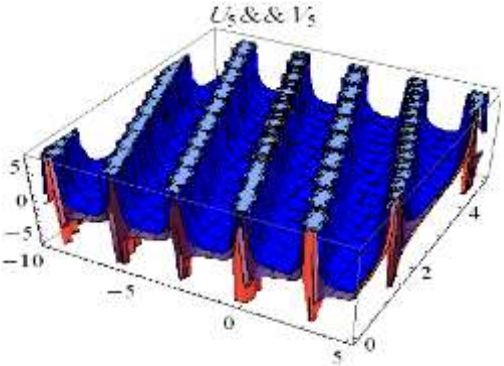
Where $\sigma < 0$,Let $\alpha = c = k = 1, \sigma = -1, w = 0$, then:



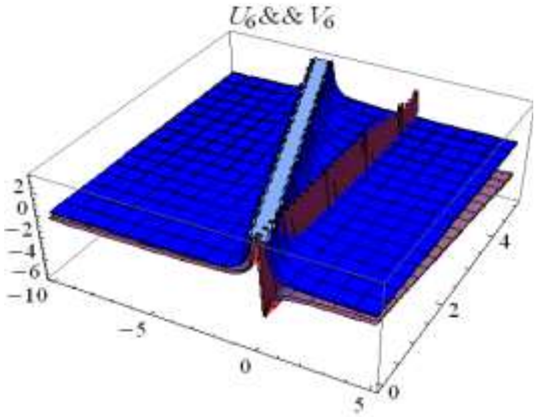


Where $\sigma > 0$, Let $\alpha = c = k = \beta = \sigma = 1, \gamma = 0$, then:



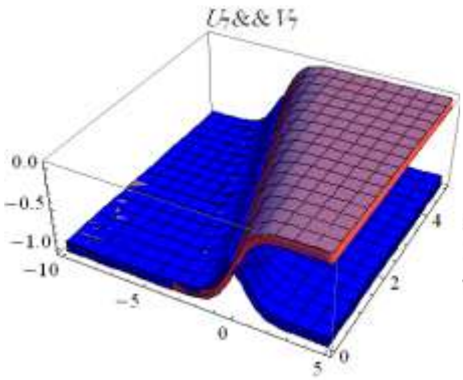


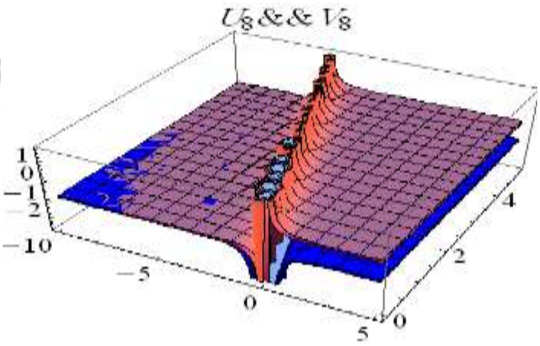
Where $\sigma = 0$, Let $\alpha = c = k = \beta = 1, w = \gamma = 0$, then:



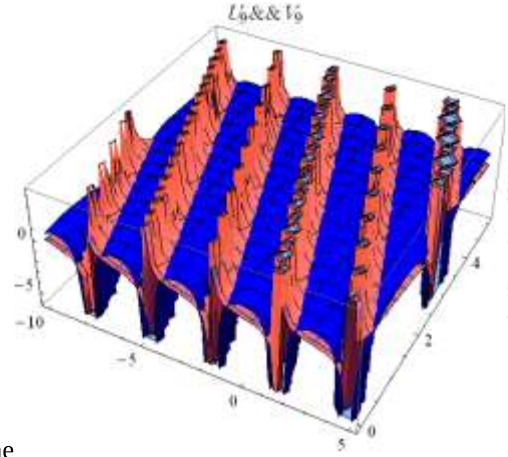
Case 3:

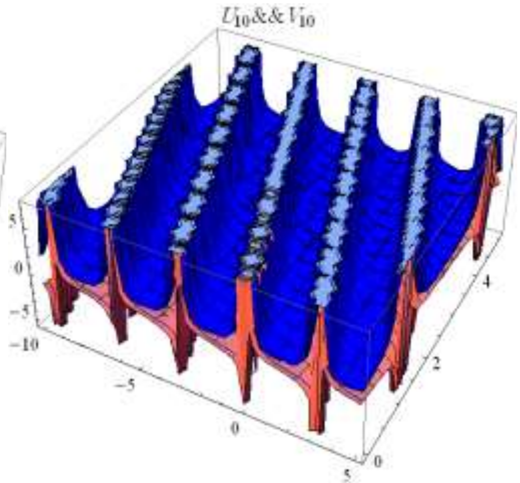
Where $\sigma < 0$, Let $\alpha = c = k = 1, \sigma = -1.$, then





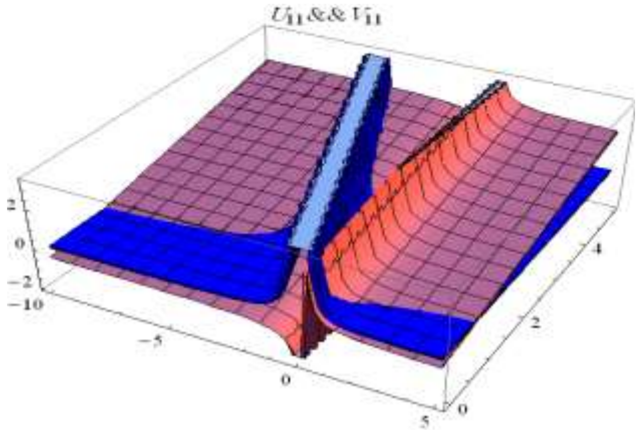
Where $\sigma > 0$, Let $\alpha = c = k = \beta = \sigma = 1, \gamma = 0$





the

Where $\sigma = 0$, Let $\alpha = c = k = \beta = 1, w = \gamma = 0$, then:



4: Space-Time (1+1) fractional derivative Coupled Burgers Equations.

We consider Space-Time fractional coupled Burgers Equations:

$$\begin{cases} D_t^{\alpha}u - D_x^{2\alpha}u + 2uD_x^{\alpha}u + pD_x^{\alpha}(uv) = 0 \\ D_t^{\alpha}v - D_x^{2\alpha}v + 2vD_x^{\alpha}v + qD_x^{\alpha}(uv) = 0 \end{cases} \quad 0 < \alpha \leq 1 \tag{21}$$

By considering the traveling wave transformation u(x,t)=U(ξ), v(x,t)=V(ξ), and ξ=xk+ct

Eq.(21) can be reduced to the following nonlinear fractional ODE:

$$\begin{cases} C^{\alpha}D_{\xi}^{\alpha}U - K^{2\alpha}D_{\xi}^{2\alpha}U + 2K^{\alpha}uD_{\xi}^{\alpha}U + pK^{\alpha}D_{\xi}^{\alpha}(UV) = 0, \; 0 < \alpha \leq 1 \\ C^{\alpha}D_{\xi}^{\alpha}V - K^{2\alpha}D_{\xi}^{2\alpha}V + 2K^{\alpha}VD_{\xi}^{\alpha}V + qK^{\alpha}D_{\xi}^{\alpha}(UV) = 0 \end{cases} \tag{22}$$

We suppose that Eq.(22) has the following general solution:

$$\begin{cases} U(\xi) = \sum_{i=0}^{n} a_i \varphi^i \\ V(\xi) = \sum_{j=0}^{m} b_j \varphi^j \end{cases}$$

By balancing the highest order derivative term and nonlinear term in Eq.(22)$D_\xi^{2\alpha}U$,$UD_\xi^{\alpha}U$and $VD_\xi^{\alpha}V$,$D_\xi^{2\alpha}V$, we have

n+2=n+n+1 then n=1,2+m= m+m+1 then m=n=1.

Then we suppose that Eq.(22) has the following formal solution.

$$\begin{cases} U(\xi) = a_0 + a_1\varphi \\ V(\xi) = b_0 + b_1\varphi \end{cases} \quad (23)$$

Substituting (23) along with (14) into (22) and setting the coefficient of $\varphi(\xi)^i$(i=0,1,2,3)

to zero, We can obtain a set of algebraic equation for c,k,b_0, b_1, a_0 and a_1 as follows:

From the first equation of previous system (22):

$\varphi(\xi)^0: c^\alpha a_1\sigma + 2k^\alpha a_1\sigma a_0 + pk^\alpha a_0 b_1\sigma + pk^\alpha b_0\sigma a_1 = 0$

$\varphi(\xi)^1: -2k^{2\alpha} a_1\sigma + 2k^\alpha \sigma a_1{}^2 + 2p\sigma k^\alpha b_1 a_1 = 0$

$\varphi(\xi)^2: c^\alpha a_1 + 2k^\alpha a_1 a_0 + pk^\alpha a_0 b_1 + pk^\alpha b_0 a_1 =0$

$\varphi(\xi)^3: -2k^{2\alpha} a_1 + 2k^\alpha a_1^2 + 2pk^\alpha b_1 a_1 = 0$

From the second equation of previous system (22):

$\varphi(\xi)^0: c^\alpha b_1\sigma + 2k^\alpha b_1\sigma b_0 + qk^\alpha b_0 a_1\sigma + qk^\alpha a_0\sigma b_1 = 0$

$\varphi(\xi)^1: -2\sigma k^{2\alpha} b_1 + 2k^\alpha b_1^2\sigma + 2k^\alpha q b_1 a_1\sigma =0$

$\varphi(\xi)^2: c^\alpha b_1 + 2k^\alpha b_0 b_1 + qk^\alpha b_1 a_0 + qk^\alpha b_0 a_1 = 0$

$\varphi(\xi)^3: -2k^{2\alpha} b_1 + k^\alpha 2b_1\hat{}2 + 2k^\alpha q a_1 b_1 = 0$

By using Mathematica:

Case1: $\{b_1 \to 0, a_1 \to 0\}$

$$\begin{cases} U_1 = a_0 \\ V_1 = b_0 \end{cases}$$ ■

Case2: $\left\{b_1 \to k^\alpha, b_0 \to -\frac{1}{2}c^\alpha k^{-\alpha}, a_1 \to 0, a_0 \to 0\right\}$

Where $\sigma < 0$,ξ=xk+ct

$$\begin{cases} U_2 = 0 \\ V_2 = -\frac{1}{2}c^\alpha k^{-\alpha} - k^\alpha\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\,\xi\right) \end{cases}$$

$$\begin{cases} U_3 = 0 \\ V_3 = -\frac{1}{2}c^\alpha k^{-\alpha} - k^\alpha\sqrt{-\sigma}\coth_\alpha\left(\sqrt{-\sigma}\xi\right) \end{cases}$$

Where $\sigma > 0$, $\xi = x\,k + c\,t$

$$\begin{cases} U_4 = 0 \\ V_4 = -\frac{1}{2}c^\alpha k^{-\alpha} + k^\alpha(\sqrt{\sigma}\tan_\alpha\left(\sqrt{\sigma}\,\xi\right) \end{cases}$$

$$\begin{cases} U_5 = 0 \\ V_5 = -\frac{1}{2}c^\alpha k^{-\alpha} - k^\alpha\sqrt{\sigma}\cot_\alpha\left(\sqrt{\sigma}\,\xi\right) \end{cases}$$

Where $\sigma = 0$, $\xi = x\,k + c\,t$, w is constant.

$$\begin{cases} U_6 = 0 \\ V_6 = -\frac{1}{2}c^\alpha k^{-\alpha} - \frac{k^\alpha\Gamma(1+\alpha)}{\xi^\alpha + w} \end{cases}$$ ■

Case3: $\left\{b_1 \to 0, b_0 \to 0, a_1 \to k^\alpha, a_0 \to -\frac{1}{2}c^\alpha k^{-\alpha}\right\}$

Where $\sigma < 0$, $\xi = x\,k + c\,t$

$$\begin{cases} U_7 = -\frac{1}{2}c^\alpha k^{-\alpha} - k^\alpha\sqrt{-\sigma}\tanh_\alpha\left(\sqrt{-\sigma}\,\xi\right) \\ V_7 = 0 \end{cases}$$

$$\begin{cases} U_8 = -\frac{1}{2}c^{\alpha}k^{-\alpha} - k^{\alpha}\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\,\xi) \\ V_8 = 0 \end{cases}$$

Where $\sigma > 0$, $\xi = x\,k + c\,t$

$$\begin{cases} U_9 = -\frac{1}{2}c^{\alpha}k^{-\alpha} + k^{\alpha}(\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\,\xi) \\ V_9 = 0 \end{cases}$$

$$\begin{cases} U_{10} = -\frac{1}{2}c^{\alpha}k^{-\alpha} - k^{\alpha}\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\,\xi) \\ V_{10} = 0 \end{cases}$$

Where $\sigma = 0$, $\xi = x\,k + c\,t$, w is constant.

$$\begin{cases} U_{11} = -\frac{1}{2}c^{\alpha}k^{-\alpha} - k^{\alpha}\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w} \\ V_{11} = 0 \end{cases}$$

Case 4: $\{b_1 \to \frac{k^{\alpha}(-1+q)}{-1+pq}, b_0 \to -\frac{c^{\alpha}k^{-\alpha}(-1+q)}{2(-1+pq)}, a_1 \to \frac{k^{\alpha}(-1+p)}{-1+pq}, a_0 \to -\frac{c^{\alpha}k^{-\alpha}(-1+p)}{2(-1+pq)}\}\}$

Where $\sigma < 0$, $\xi = x\,k + c\,t$

$$\begin{cases} U_{12} = -\frac{c^{\alpha}k^{-\alpha}(-1+p)}{2(-1+pq)} - \frac{k^{\alpha}(-1+p)}{-1+pq}(\sqrt{-\sigma}\tanh_{\alpha}(\sqrt{-\sigma}\,\xi) \\ V_{12} = -\frac{c^{\alpha}k^{-\alpha}(-1+q)}{2(-1+pq)} - \frac{k^{\alpha}(-1+q)}{-1+pq}(\sqrt{-\sigma}\tanh_{\alpha}(\sqrt{-\sigma}\,\xi)) \end{cases}$$

$$\begin{cases} U_{13} = -\frac{c^{\alpha}k^{-\alpha}(-1+p)}{2(-1+pq)} - \frac{k^{\alpha}(-1+p)}{-1+pq}(\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\,\xi) \\ V_{13} = -\frac{c^{\alpha}k^{-\alpha}(-1+q)}{2(-1+pq)} - \frac{k^{\alpha}(-1+q)}{(-1+pq)}(\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}\,\xi) \end{cases}$$

Where $\sigma > 0$, $\xi = x\,k + c\,t$

$$\begin{cases} U_{14} = -\frac{c^{\alpha}k^{-\alpha}(-1+p)}{2(-1+pq)} + \frac{k^{\alpha}(-1+p)}{-1+pq}(\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\,\xi)) \\ V_{14} = -\frac{c^{\alpha}k^{-\alpha}(-1+q)}{2(-1+pq)} + \frac{k^{\alpha}(-1+q)}{-1+pq}(\sqrt{\sigma}\tan_{\alpha}(\sqrt{\sigma}\,\xi)) \end{cases}$$

$$\begin{cases} U_{15} = -\frac{c^{\alpha}k^{-\alpha}(-1+p)}{2(-1+pq)} - \frac{k^{\alpha}(-1+p)}{-1+pq}(\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\,\xi)) \\ V_{15} = -\frac{c^{\alpha}k^{-\alpha}(-1+q)}{2(-1+pq)} - \frac{k^{\alpha}(-1+q)}{(-1+pq)}(\sqrt{\sigma}\cot_{\alpha}(\sqrt{\sigma}\,\xi)) \end{cases}$$

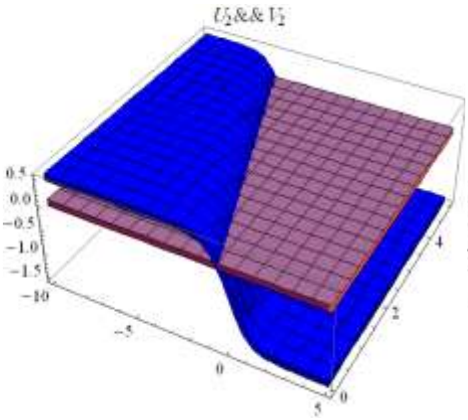
Where $w = \text{constant},\ \sigma = 0$ and $\xi = xk + ct$

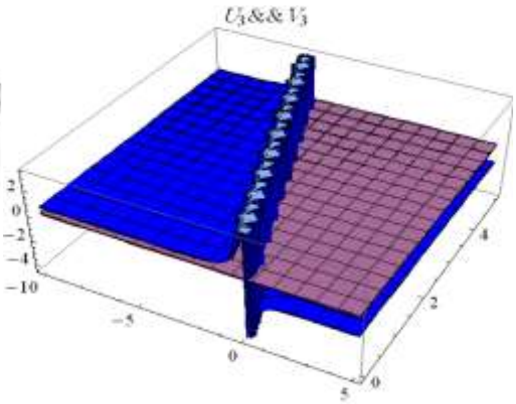
$$\begin{cases} U_{16} = -\frac{c^{\alpha}k^{-\alpha}(-1+p)}{2(-1+pq)} - \frac{k^{\alpha}(-1+p)}{-1+pq}\left(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}\right) \\ V_{16} = -\frac{c^{\alpha}k^{-\alpha}(-1+q)}{2(-1+pq)} - \frac{k^{\alpha}(-1+q)}{(-1+pq)}\left(\frac{\Gamma(1+\alpha)}{\xi^{\alpha}+w}\right) \end{cases}$$ ∎

4.1: Figures of Space-Time (1+1) fractional derivative Coupled Burgers Equations:

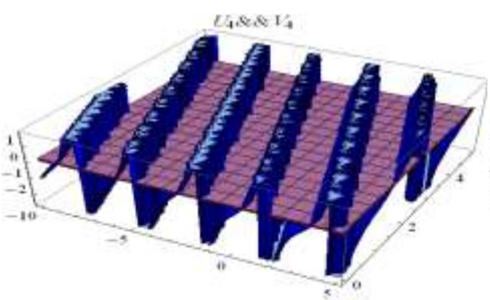
Case 2:

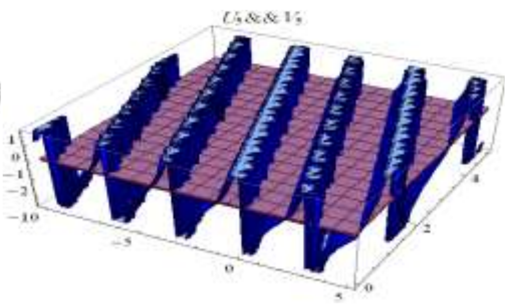
Where $\sigma < 0$, Let $\alpha = c = k = 1, \sigma = -1$, then



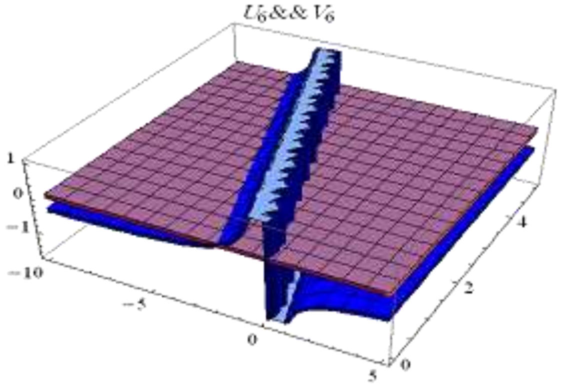


Where $\sigma > 0$, Let $\alpha = c = k = \sigma = 1$, then:



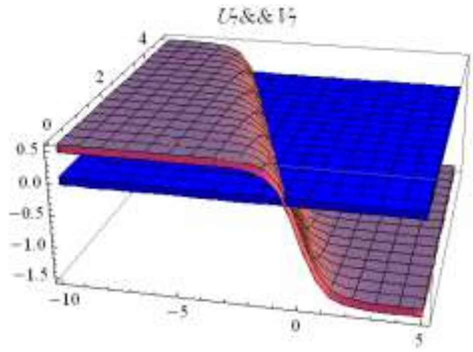


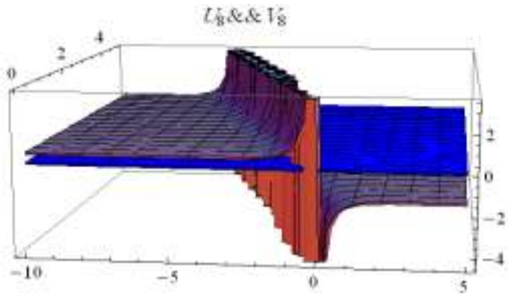
Where $\sigma = 0$, Let $\alpha = c = k = 1, w = 0$, then:



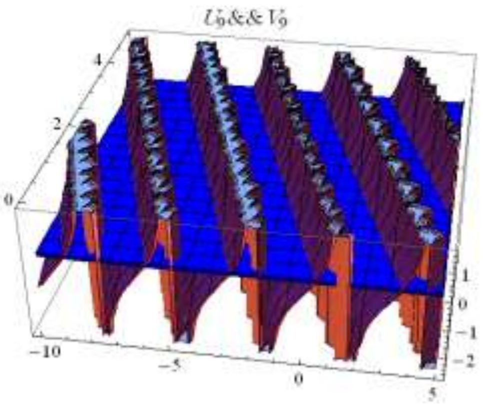
Case 3:

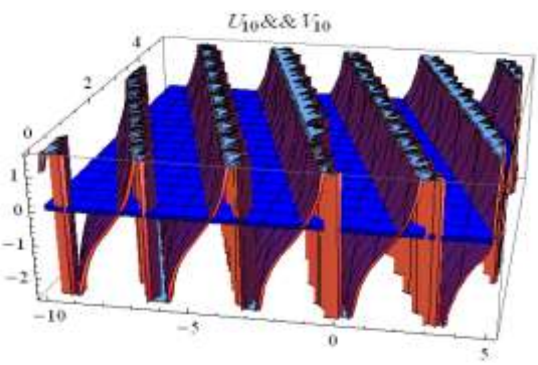
Where $\sigma < 0$, Let $\alpha = c = k = 1, \sigma = -1, w = 0$, then:



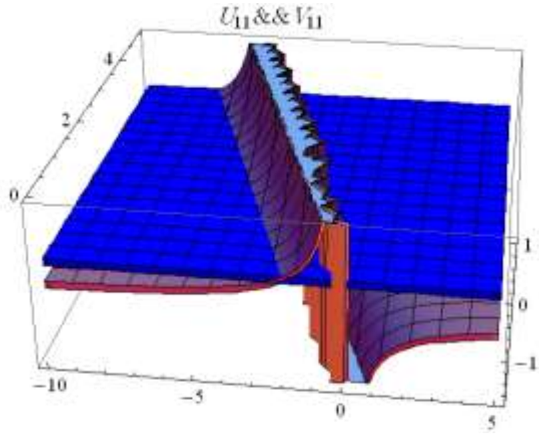


Where $\sigma > 0$, Let $\alpha = c = k = \sigma = 1, w = 0$, then:



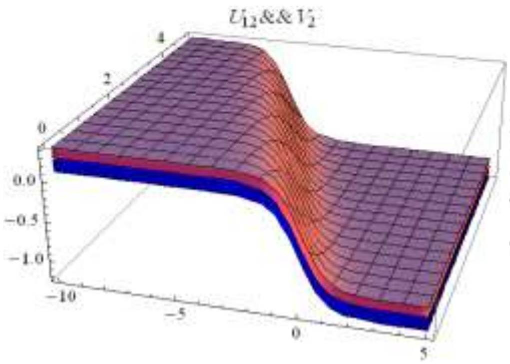


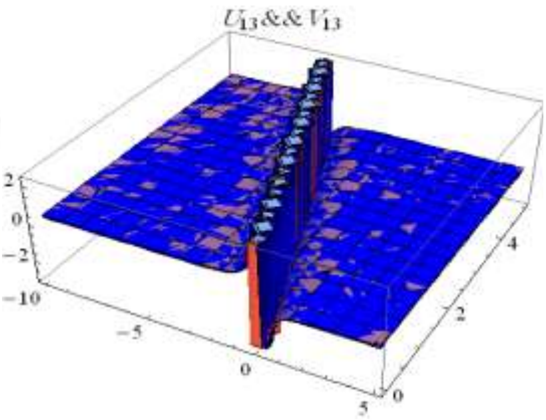
Where $\sigma = 0$, Let $\alpha = c = k = 1, w = 0$, then:



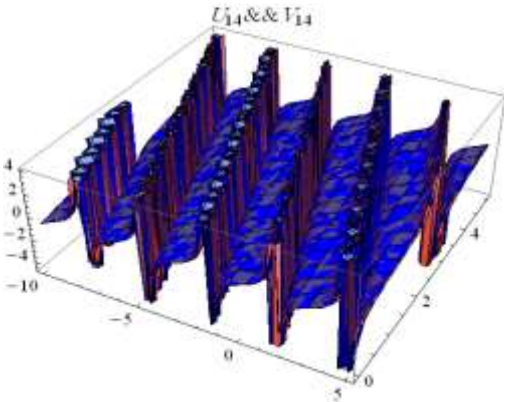
Case 4:

Where $\sigma < 0$, Let $\alpha = c = k = 1, \sigma = -1, p = q = 0.5 \&\& pq \neq 1$, then:

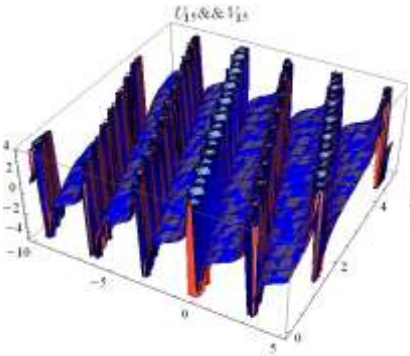


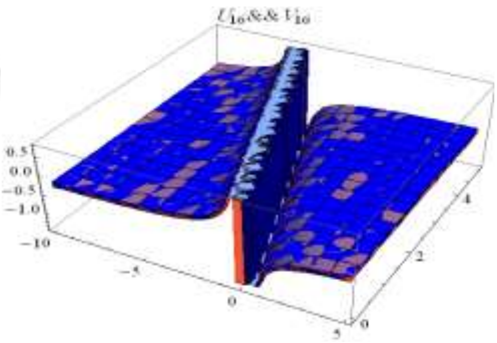


Where $\sigma > 0$, $Let\ \alpha = c = k = \sigma = 1, \&\& pq \neq 1$then:



Where $\sigma = 0$, Let $\alpha = c = k = 1, w = 0, p = q = 0.5, \&\& pq \neq 1$ then:





Conclusions

In this paper, we proposed generalizing fractional sub- equation method to construct exact solutions of space-time nonlinear fractional derivative systems: Whitham-Broer-Kaup equations, Breaking Soliton equations, coupled Boussinesq-Burgers equations and coupled Burgers equations. As this method is based on the homogenous balancing principle, so it also be applied to other space-time nonlinear fractional derivative systems where the homogeneous balancing principle is satisfied. We conclude that the fractional sub-equation method is powerful, effective and convenient for nonlinear fractional PDEs.

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