Defining a Zeroth Homotopy Invariant for Graphs

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Abstract - We define a zeroth homotopy $\pi_0(G)$ for a graph G. Our definition is a variation on the usual set of connected components and has the structure of a graph, and not just a set. We prove that our π_0 is functorial and respects products: $\pi_0(G \times H) \cong \pi_0(G) \times \pi_0(H)$, a property that the set of components fails to have.

Keywords : graph; graph product; functor; connected component

Mathematics Subject Classification (2020) **:** 05C38; 05C40; 18A99; 55Q05

1 Introduction

In this paper, we introduce a new definition of a zeroth homotopy set $\pi_0(G)$ for graphs G. Traditionally, π_0 refers to the set of connected components of a space, and it is possible to make a similar definition for graphs. However, unlike the case of topological spaces, the set of connected components for graphs does not respect products. Therefore we introduce a new $\pi_0(G)$ which is itself a new graph, albeit a very simple one. This construction builds on the set of connected components, but includes more structure. We prove that our π_0 construction is functorial, meaning that in addition to producing a graph $\pi_0(G)$ associated to any graph, we also produce a map from $\pi_0(G)$ to $\pi_0(H)$ associated to a graph map from G to H, and this association respects composition of graph maps. We also show that $\pi_0(G)$ respects graph products.

This work grew out of a senior capstone course for undergraduate mathematics majors at Fort Lewis College. During the semester course and continuing into the next year, we investigated which graphs can be described as products of other graphs, a problem we referred to as graph factorization. As our results highlight, this is a problem which cannot be reduced to examining graphs component by component. In the course of understanding this peculiarity, we developed conditions for when the product of connected graphs fails to be connected, and eventually expanded these ideas to the current $\pi_0(G)$ definition. The condition on connectivity of products is not new, see [6] for the original proof; but we developed it independently, and we hope that a more categorical approach to this result may yield further insights.

Our paper is organized as follows: Section 2 contains background definitions and notation, Section 3 gives the definition of our $\pi_0(G)$ and shows that it defines a functor from the graph category to itself, Section 4 shows that $\pi_0(G)$ respects products of connected graphs, and Section 5 generalizes this result to disconnected graphs. We conclude with an example of a family of non-unique graph factorizations in Section 6.

2 Background

In this section, we give definitions and set notation that will be used later in the paper. We will use standard graph theory terminology following [1], and category theory following [3]. We work in the category Gph of finite undirected graphs, where we allow loops but have at most one edge connecting two vertices. To avoid special cases, we also assume that our graphs have no isolated vertices.

Definition 2.1 *The category of graphs* Gph *is defined as follows:*

- *An object is a graph G, consisting of a finite set of vertices $V(G) = \{v_i\}$ and a finite set $E(G)$ of edges connecting them, where each edge is given by an unordered set of two vertices. We assume that every vertex is connected to at least one other vertex (possibly itself). If two vertices are connected by an edge, we will use notation $v_1\_v_2 \in E(G)$, or just $v_1\_v_2$ if the parent graph is clear.*
- *An arrow in the category* Gph *is a graph morphism $f : G \to H$. Specifically, this is defined by a set map $f : V(G) \to V(H)$ such that if $v_1\_v_2 \in E(G)$ then $f(v_1)\_f(v_2) \in E(H)$.*

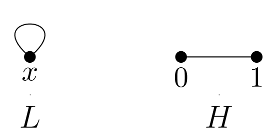
We will work in this category throughout this paper, and assume that "graph" always refers to an object in Gph. In particular, all of our graphs will have finite vertex (and hence edge) sets.

Definition 2.2 *Two graphs G and H are* **isomorphic** *when there are graph morphisms $\varphi : G \to H$ and $\psi : H \to G$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps on their respective domains. In this case, we will use notation $G \cong H$ and consider the two isomorphic graphs to be fairly interchangable.*

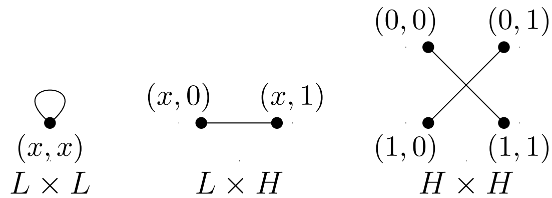
Definition 2.3 *[2, 5] Let G, H be graphs. The (categorical)* **product graph** $G \times H$ *(also called the tensor or Kronecker product) is defined by:*

- *A vertex is a pair (v, w) where $v \in V(G)$ and $w \in V(H)$.*
- *There is an edge $(v_1, w_1)\_(v_2, w_2) \in E(G \times H)$ whenever $v_1\_v_2 \in E(G)$ and $w_1\_w_2 \in E(H)$.*

Example 2.4 Let L be given by a single looped vertex x, and $H = K_2$ be the graph with two vertices $\{0, 1\}$ connected by an edge:

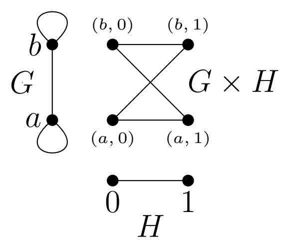


Then the products are illustrated below:



Even in this simple example, we see that we have a product of two connected graphs which is disconnected.

Example 2.5 Let G be the graph on two adjacent looped vertices: $V(G) = \{a, b\}$ and $E(G) = \{a\_a, b\_b, a\_b\}$. Let $H = K_2$ with $V(H) = \{0, 1\}$ and $E(H) = \{0\_1\}$. Then $G \times H$ is isomorphic to the cyclic graph C_4:



Definition 2.6 *If we have a graph G such that $G \cong H \times K$, we say that H and K are* **factors** *of the graph G.*

Thus the above examples show that the single loop graph L is a factor of every graph, and that two disjoint copies of K_2 can be factored as $K_2 \times K_2$.

Other concepts from graph theory we will be using are included here for completeness:

Definition 2.7 *A walk in a graph G is a list of vertices $v_0v_1v_2 \dots v_n$ such that $v_i\_v_{i+1} \in E(G)$. A closed walk is a walk where $v_0 = v_n$. The concatenation of two walks $v_0v_1 \dots v_n$ and $w_0w_1 \dots w_m$ is defined by $v_0v_1 \dots v_nw_1 \dots w_m$ when $v_n = w_0$.*

Definition 2.8 *A graph G is bipartite if $V(G)$ can be written as a disjoint union $V(G) = U \cup W$, where no vertices in U are connected to each other, and no vertices in W are connected to each other.*

3 Defining a Zeroth Homotopy for Graphs

In this section, we give a definition of $\pi_0(G)$ for any graph $G \in \mathsf{Gph}$, and show that this defines a functor $\mathsf{Gph} \to \mathsf{Gph}$.

We begin with the following definition.

Definition 3.1 *Let G be a graph, and $v, w \in V(G)$. Define $v \sim w$ if there exists a walk of even length from v to w in G.*

Lemma 3.2 *The relation $\sim$ of Definition 3.1 is an equivalence relation on $V(G)$.*

Proof. We check the following:
Reflexive: Given $v \in V(G)$, there is a length 0 walk from v to v, which is of even length.
Symmetric: If $v \sim w$, then there exists a walk of length 2ℓ from v to w. Then writing the vertices of this walk in reverse order gives a walk of length 2ℓ from w to v.
Transitive: If $u \sim v$ and $v \sim w$ then there is a walk of length 2ℓ from u to v and a walk of length $2n$ from v to w, and these walks may be concatenated to give a walk of $2\ell + 2n$ from x to z. $\square$

In order to define the graph $\pi_0(G)$, we need the following result.

Lemma 3.3 *If $v \sim w$ and there is an odd length walk from w to x, then there is also an odd length walk from v to x.*

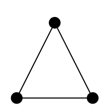
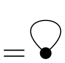
Proof. If $v \sim w$, then there is a walk of length 2ℓ from v to w. Additionally, there is a walk of length $2n + 1$ from w to x. Then by the concatenation that gives transitivity, there is a walk of length $2\ell + 2n + 1 = 2(\ell + n) + 1$. Therefore, there is an odd length walk from v to x. $\square$

Definition 3.4 *We define the graph $\pi_0(G)$ as follows:*
The vertex set $V(\pi_0(G)))$ is the set of equivalence classes

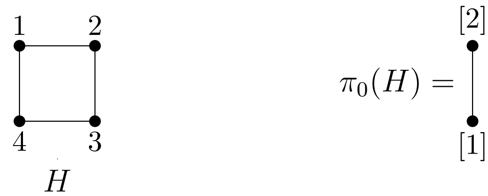
$$[v] = \{w : w \sim v\} \quad \text{for} \quad v, w \in V(G).$$

There is an edge between vertices $[v], [w]$ if there exists an odd length walk between $v, w \in V(G)$. Lemma 3.3 shows that this is well-defined.

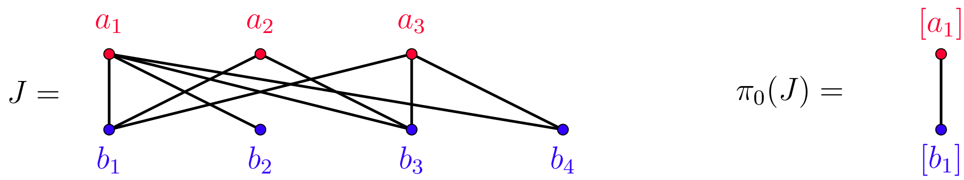
Example 3.5 Suppose that $G = C_3$ with $V(G) = \{1, 2, 3\}$ and $E(G) = \{1\_2, 2\_3, 3\_1\}$. Then every vertex has an even length walk to every other, so $\pi_0(G)$ has only one vertex $[1] = [2] = [3]$; and every vertex has an odd length walk to every other, so $[1]\_[1] \in E(\pi_0(G))$.

$G =$  $\pi_0(G) =$ 

Example 3.6 Suppose that $H = C_4$ with $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{1\_2, 2\_3, 3\_4, 4\_1\}$. Then $1 \sim 3$ and $2 \sim 4$; so $V(\pi_0(H)) = \{[1], [2]\}$. There is an odd length walk from 1 to 2 and so $[1]\_[2] \in E(\pi_0(H))$. There are no odd length walks between any vertex and itself or any other equivalent vertex, so there are no loops in $E(\pi_0(H))$. So $\pi_0(H) \cong K_2$.



Example 3.7 Suppose that J is the graph drawn below. We see that there are even length walks between a_i and a_j, and between b_i and b_j, but not between a_i and b_j. So $V(\pi_0(J)) = \{[a_1], [b_1]\}$. And there is an odd length walk between a_1 and b_1, but not between any a_i and a_j, or between any b_i and b_j. Thus $[a_1]\_[b_1] \in E(\pi_0(J))$ is the only edge in $\pi_0(J)$, and $\pi_0(J) \cong K_2$.



Next, we show that π_0 defines a functor from the category Gph to itself.

Definition 3.8 *[3] A functor $\mathcal{F} : \mathsf{Gph} \to \mathsf{Gph}$ is defined by:*

- *for any graph $G \in \mathsf{Gph}$, $\mathcal{F}(G)$ defines a graph in Gph;*
- *for any graph homomorphism $f : G \to H$, there is a graph homomorphsim $\mathcal{F}(f) : \mathcal{F}(G) \to \mathcal{F}(H)$ such that*
  - *$\mathcal{F}(id_G) = id_{\mathcal{F}(G)}$ where id_G denotes the identity map;*
  - *if $f : G \to H$ and $g : H \to K$, then $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.*

We have already shown that $\pi_0(G)$ defines a graph in Gph. So we need to consider graph morphisms.

Proposition 3.9 *If $f : G \to H$ is a graph homomorphism, then there is a graph homomorphism $f_0 : \pi_0(G) \to \pi_0(H)$ defined by $f_0([x]) = [f(x)]$*

Proof. We first show that f_0 is well-defined. Let $x, y \in V(G)$ and suppose $[x] = [y]$. Then there exists an even walk from x to y defined by $xv_1v_2 \ldots v_{n-1}y$. Because f is a graph homomorphism, $f(x)f(v_1)f(v_2) \ldots f(v_{n-1})f(y)$ defines an even walk from $f(x)$ to $f(y)$. Therefore $[f(x)] = [f(y)]$.

Now we show that f_0 is a graph homomorphism. Suppose that $[x]\_[y] \in E(\pi_0(G))$. This means that there is an odd walk from x to y in G; applying f to each vertex of the walk gives an odd walk from $f(x)$ to $f(y)$, so $[f(x)]\_[f(y)] \in E(\pi_0(H))$.

$\square$

Proposition 3.10 *π_0 defines a functor from Gph to Gph.*

Proof. If id is the identity map $G \to G$, then for $x \in V(G)$, we have $id(x) = x$ and so $id_0 : \pi_0(G) \to \pi_0(G)$ is defined by $id_0([x]) = [id(x)] = [x]$, which is the identity map on $\pi_0(G)$. Next, we check that π_0 respects composition. We want to show that $(g \circ f)_0 = g_0 \circ f_0$: if $x \in V(G)$, $(g \circ f)_0(x) = [(g \circ f)(x))] = [g(f(x))]$, while $g_0 \circ f_0 = g_0([f(x)]) = [g(f(x))]$. Thus π_0 defines a functor.

□

4 Products of Connected Graphs

We will prove that the π_0 construction respects products. In this section we consider connected graphs; in the following section we will extend this result to more general graphs.

Theorem 4.1 *For any connected graphs $G, H \in \mathsf{Gph}$, we have*

$$\pi_0(G \times H) \cong \pi_0(G) \times \pi_0(H).$$

It turns out that for connected graphs, π_0 detects whether the graph is bipartite or not. So our strategy for proving Theorem 4.1 will be to work through various cases of bipartite and non-bipartite graphs. We begin with the following result about bipartite graphs. This result is standard in many graph theory sources, but we present a proof here for completeness and to introduce ideas we will be using to prove Theorem 4.1.

Theorem 4.2 *[1] Let $G \in \mathsf{Gph}$ be a connected graph. Then the following are equivalent:*

1. *G is bipartite.*
2. *G does not contain an odd closed walk.*
3. *For any two vertices v, w of G, there is either an odd or even length walk connecting them, but not both.*

Proof.

(1) $\Rightarrow$ (2): Suppose that G is bipartite. Then $V(G)$ is the disjoint union of two subsets U and W such that no vertices of U are connected to each other, and no vertices of W are connected to each other. Therefore any walk in G must alternate between vertices of U and vertices of W, and hence any closed walk which begins and ends at the same vertex must be of even length.

(2) $\Rightarrow$ (3) : We show the contrapositive. If there are two vertices $v, w \in V(G)$ with no walk connecting them, then G is not connected. If there are two vertices v, w of G such that there is both an odd length and even length walk from v to w, then concatenating these walks gives an odd length walk from v to v, and hence an odd closed walk.

(3) $\Rightarrow$ (1): Select a vertex $x \in V(G)$. We can partition $V(G)$ into two sets U, W where $U(G) = \{v \in G \mid$ there is an even walk from $x\}$ and
$W(G) = \{v \in G \mid$ there is an odd walk from $x\}$. The hypothesis of (3) shows that these

sets are disjoint, and the fact that G is connected ensures that every vertex has a walk to x and hence that the union of these sets is all of $V(G)$. If $v_1, v_2 \in U(G)$, there cannot be an edge from v_1 to v_2, since this would result in both an even and odd walk from x to v_1: the even walk is assumed, and the odd walk would be the concatenation of the even walk to v_2 and the edge to v_1. Similarly, no vertices of $U(G)$ can be connected.

□

Observation 4.3 We have actually proved a stronger relationship between conditions (1) and (3) above: we have shown that if we choose any vertex $x \in V(G)$, then the bipartite partition is exactly the two sets of vertices with odd and even walks from x.

We will also use the following form of this result:

Corollary 4.4 *Let G be a connected graph. Then the following are equivalent:*

1. *G is not bipartite.*
2. *G contains an odd closed walk.*
3. *For any two vertices v, w of G, there is both an odd and even length walk connecting them.*

We use these results to show that if G is connected, there are only two possible options for the $\pi_0(G)$ graph.

Proposition 4.5 *If a graph $G \in \mathsf{Gph}$ is connected and bipartite, then*

$$\pi_0(G) \cong K_2 = \begin{tikzcd} \overset{A}{\bullet} \arrow[r, dash] & \overset{B}{\bullet} \end{tikzcd}$$

where the equivalence classes A and B coincide with the partition of the bipartite graph into disjoint subsets of vertices.

Proof. Since G is bipartite, $V(G)$ is the disjoint union of two subsets $A(G)$ and $B(G)$ such that no vertices from A are connected to each other, and no vertices from B are connected to each other. Choose a vertex $a \in A(G)$. By Condition (3) of Theorem 4.2, for each vertex $v \in V(G)$, either there is an even walk from a or an odd walk from a, but not both. Let b be a vertex connected to a (recall that we are not allowing isolated vertices, so such a b exists). Then $b \in B(G)$.

If v has an even walk from a, then $[v] = [a]$. If a vertex v has an odd walk from a, then the concatenation of the walk from v to a with the edge from a to b gives an even walk from v to b, and so $[v] = [b]$. Thus $V(\pi_0(G)) = \{[a], [b]\}$, and these are exactly the subsets $A(G)$ and $B(G)$. The edge from a to b provides an edge from $[a]$ to $[b]$ in $\pi_0(G)$. If v, w are both in $A(G)$, then conditon (2) disallows an odd length walk connecting them, and so there is no loop from $[a]$ to $[a]$ in $\pi_0(G)$. Similarly, there is no loop from $[b]$ to $[b]$. Thus $\pi_0(G)$ is the graph above.

□

Proposition 4.6 *If H is a connected graph which is not bipartite, then*

$$\pi_0(H) \cong$$ 

Proof. If G is connected, then there exists a walk connecting any two vertices. By Corollary 4.4, if G is not bipartite then there is both an even and an odd walk between every pair of vertices v, w. Thus, $[v] = [w]$ and $\pi_0(H)$ consists of a single vertex, and there is an edge from $[v]$ to $[w]$ in $\pi_0(H)$. $\square$

Observation 4.7 To prove Theorem 4.1, we have three cases to consider: the product of two bipartite graphs, the product of two non-bipartite graphs, and the product of a bipartite graph with a non-bipartite graph.

We will use the following fact about walks in product graphs for all of our cases.

Proposition 4.8 *There is an even [resp. odd] length walk from (v, w) to (v', w') in $G \times H$ if and only if there is an even [resp. odd] length walk from v to v' in G and an even [resp. odd] length walk w to w' in H.*

Proof. A walk in $G \times H$ is given by a sequence of vertices

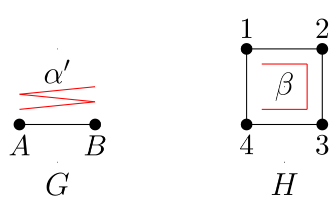
$$(v_0, w_0)(v_1, w_1)(v_2, w_2) \ldots (v_n, w_n)$$

where each successive pair is connected in $G \times H$. By the definition of the product graph, (v_i, w_i) is connected to (v_{i+1}, w_{i+1}) if and only if v_i and v_{i+1} are connected in G, and (w_i) and (w_{i+1}) are connected in H. Thus a walk in $G \times H$ corresponds exactly to two simultaneous walks in G and H of equal length, and we see that an even [resp. odd] length walk in $G \times H$ gives even [odd] length walks in each of G and H.

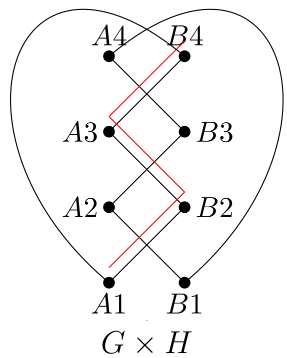
Conversely, suppose that there is a walk of length $2n$ from v to v' in G, and a walk of length $2m$ from w to w' in H. If $n = m$ we can combine the two walks to get a walk from (v, w) to (v', w'). Now suppose $n \neq m$, and suppose without loss of generality that $n < m$. Then we can extend the walk $v = v_0 v_1 v_, \ldots, v_{2n-1} v_{2n} = v'$ to be a walk of length $2m$ by adding $v_{2n-1} v_{2n}$ to the end an appropriate number of times. A similar argument works for a pair of odd length walks. $\square$

Example 4.9 Suppose that $G = K_2$ with $V(G) = \{A, B\}$ and $E(G) = \{A\_B\}$, and $H = C_4$ with $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{1\_2, 2\_3, 3\_4, 4\_1\}$.

Let α be the length 1 walk AB in G, and β be the length 3 walk 1234 in H. In order to define a walk in $G \times H$, we extend α to be the walk $\alpha' = ABAB$:



Then we can combine α' and β to give a walk in $G \times H$ defined by $A1\, B2\, A3\, B4$:



We now consider the following cases.

Proposition 4.10 *If G, H are connected non-bipartite graphs, then $G \times H$ is a connected non-bipartite graph.*

Proof. Suppose G, H are connected non-bipartite graphs. Then for any vertices v_1, v_2 in $V(G)$ and u_1, u_2 in $V(H)$, there is both an odd and even length walk from v_1 to v_2 and an even and an odd walk from u_1 to u_2. Then there are both even and odd length walks (v_1, u_1) to (v_2, u_2) in $G \times H$ by Proposition 4.8. Thus $G \times H$ is connected and by Corollary 4.4, $G \times H$ is non-bipartite.

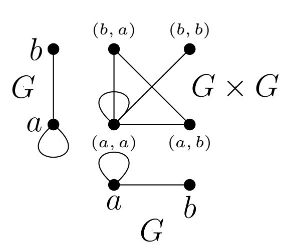
□

Corollary 4.11 *If G and H are connected non-bipartite graphs, then $\pi_0(G) \times \pi_0(H) \cong \pi_0(G \times H)$.*

Proof. By Proposition 4.10, $G \times H$ is also connected and non-bipartite. So by Proposition 4.6, all the π_0 graphs consist of a single looped vertex. As shown in Example 2.4, the product of two looped vertex graphs is another looped vertex graph.

□

Example 4.12 Let G be the graph on two adjacent vertices, one of which has a loop: $V(G) = \{a, b\}$ and $E(G) = \{a\_a, a\_b\}$. Then G is not bipartite, and there is both an even and odd length walk between every pair of vertices; so $\pi_0(G)$ is a single looped vertex. Now $G \times G$ is drawn below, and we see that there is also an even and odd length walk between every pair of vertices in $G \times G$, and so $\pi_0(G \times G) \cong \pi_0(G) \times \pi_0(G)$.



Proposition 4.13 *If G is a connected bipartite graph and H is a connected non-bipartite graph, then $G \times H$ is a connected bipartite graph.*

Proof.

Suppose that G is bipartite and H is not. To see that $G \times H$ is connected, let (v, w) and (v', w') be vertices from $G \times H$. Since G is connected, there is a walk from v to v' in G. Since H is connected and non-bipartite, there is both an even and odd length walk from w to w' in H. Thus pairing the walk in G with the walk of the same parity from H gives a walk from (v, w) to (v', w') in $G \times H$.

To see that $G \times H$ is bipartite, we observe that if $G \times H$ were non-bipartite, then there would be both odd and even length walks from (v, w) to (v', w') and hence by Proposition 4.8, both even and odd length walks in G from v to v', contradicting the bipartite condition of Proposition 4.2.

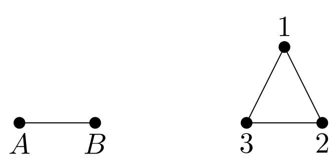
□

Corollary 4.14 *If G is a connected bipartite graph and H is a connected non-bipartite graph, then $\pi_0(G) \times \pi_0(H) \cong \pi_0(G \times H)$.*

Proof. By Proposition 4.13, $G \times H$ is connected and bipartite. So by Proposition 4.5 and 4.6, we have $\pi_0(G) = \pi_0(G \times H) \cong K_2$, and $\pi_0(H) \cong L$ where L denotes the looped vertex. As shown in Example 2.4, the product $K_2 \times L$ is isomorphic to K_2.

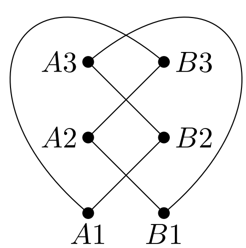
□

Example 4.15 Let $G = K_2$ with $V(G) = \{A, B\}$ and $E(G) = \{A\_B\}$. Let $H = C_3$ with $V(H) = \{1, 2, 3\}$ and $E(H) = \{1\_2, 2\_3, 3\_1\}$.



G $\qquad$ H

So $\pi_0(G) \cong K_2$ and $\pi_0(H) \cong L$ where L is a single looped vertex. Then $G \times H$ is isomorphic to the cyclic graph C_6:



$G \times H$

and $\pi_0(G \times H)$ has two vertices, $[A] = [A1] = [A2] = [A3]$ and $[B] = [B1] = [B2] = [B3]$, and an edge connecting them: $[A]\_[B] \in \pi_0(G \times H)$. So $\pi_0(G \times H) \cong K_2 \cong K_2 \times L \cong \pi_0(G) \times \pi_0(H)$.

Proposition 4.16 *If G, H are connected bipartite graphs, then $G \times H$ is a disconnected graph with two components, each of which is bipartite.*

Proof. First, observe that $G \times H$ is bipartite: if not, then $G \times H$ would contain an odd closed walk, which would correspond to odd closed walks in G and H by Proposition 4.8, a contradiction of Proposition 4.2.

To see that there are exactly two components, recall from Observation 4.3 that if we fix $x \in V(G)$ then the bipartite partition of $V(G)$ is exactly given by the partition of vertices into U, W where

$$U(G) = \{v \in G \,|\, \exists \text{ an even walk from } v \text{ to } x\}$$

and

$$W(G) = \{v \in G \,|\, \exists \text{ an odd walk from } v \text{ to } x\}.$$

Similarly, for $y \in V(H)$, the bipartite partition of $V(H)$ is given by $U(H)$ and $W(H)$, those vertices which are an even and odd length walk from y. Now (v, w) is connected to (x, y) in $G \times H$ if and only if there are walks x to v in G, and y to w in H, of the same parity. Thus the component of (x, y) in $G \times H$ is exactly $[U(G) \times U(H)] \cup [W(G) \times W(H)]$. The complement is $[U(G) \times W(H)] \cup [W(G) \times U(H)]$, which will be a second component, since it would have the same form for a choice of x in the other partition subset. Since the entire graph is bipartite, each component must also be bipartite.

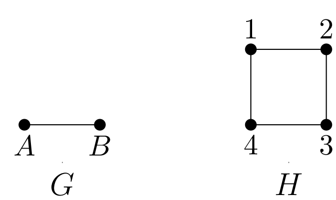
□

Corollary 4.17 *If G and H are connected bipartite graphs then $\pi_0(G) \times \pi_0(H) \cong \pi_0(G \times H)$.*

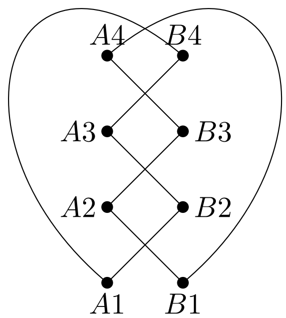
Proof. By Proposition 4.16, $G \times H$ consists of two components, each given by a connected bipartite graph. So by Proposition 4.5 , we have $\pi_0(G) \cong \pi_0(H) \cong K_2$, and $\pi_0(G \times H)$ given by two copies of K_2. As shown in Example 2.4, the product of $K_2 \times K_2$ is exactly two copies of K_2.

□

Example 4.18 Suppose that $G = K_2$ with $V(G) = \{A, B\}$ and $E(G) = \{A\_B\}$, and $H = C_4$ with $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{1\_2, 2\_3, 3\_4, 4\_1\}$.



So $\pi_0(G) \cong \pi_0(H) \cong K_2$. Now $G \times H$ is isomoprhic to the disjoint union of two copies of C_4:



$G \times H$

So $\pi_0(G \times H) \cong K_2 \times K_2 \cong \pi_0(G) \times \pi_0(H)$.

This completes the proof of Theorem 4.1. We have also recovered the following result.

Corollary 4.19 *[6] Suppose G and H are connected. Then $G \times H$ is disconnected if and only if both of the graphs G and H are bipartite.*

Observation 4.20 These results have profound implications for the question of factoring graphs as in Definition 2.6, particularly the case of two bipartite graphs of Corollary 4.17. We have shown that it is NOT sufficient to look for factors of components of the graph, working on each component seperately; instead, there may be cases where two components need to be paired and created as products of connected graphs together. In Example 4.18, we would need to look at both copies of C_4 together in order to find the given factors. This complicates the question of finding factors for graphs considerably.

5 Disconnected Graphs

In this section, we generalize to disconnected graphs. In order to do this, we will work with the categorical coproduct or disjoint union graph $G + H$ of graphs G, H, which has vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H)$ as defined in [1].

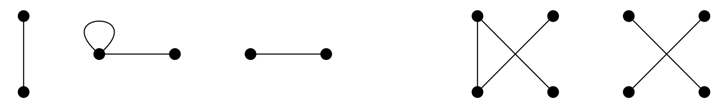
We first check that the coproduct distributes over the product.

Proposition 5.1 *For any graphs G_1, G_2 and H, we have $(G_1+G_2)\times H \cong G_1\times H+G_2\times H$*

Proof.

If $(v, w) \in V(G_1 \times H)$, then (v, w) also defines a vertex in $V((G_1 + G_2) \times H)$. Define a map $\varphi : (G_1 + G_2) \times H \to G_1 \times H + G_2 \times H$ by $\varphi(v, w) = (v, w)$. This is a graph homomorphism since if $(v, w)\_(v', w')$ in $G_1\times H$, then $v\_v' \in E(G_1)$ and hence in G_1+G_2, and $w\_w' \in E(H)$. Again, it is easily checked that this is injective and surjective, and therefore defines a graph isomoprhism. $\square$

Example 5.2



$$G \times (H_1 + H_2) \cong (G \times H_1) + (G \times H_2)$$

Next, we show that π_0 respects the coproduct.

Lemma 5.3 *For any graphs G, H, $\pi_0(G + H) \cong \pi_0(G) + \pi_0(H)$.*

Proof. We can define an isomorphism $\varphi : \pi_0(G)+\pi_0(H) \to \pi_0(G+H)$. Given $v \in V(G)$, v also defines a vertex in $G + H$, so we define $\varphi[v] = [v]$. This is well-defined, since if $[v] = [v']$ in $\pi_0(G+H)$ then there is an even length walk from v to v' in $G+H$; but since $G + H$ has no edges connecting vertices of G to those of H, this walk must lie entirely in G, and so $[v] = [v'] \in V(\pi_0(G))$. Similarly we define $\varphi[w] = [w]$ for $w \in V(H)$. This is a graph homomorphism, since if $[v]\_[v'] \in E(\pi_0(G) + \pi_0(H))$, there exists an odd length walk from v to v' in G (or in H), and hence from v to v' in $G + H$. It is easy to see that this map is surjective and injective, and so defines an isomorphism of graphs. $\square$

The previous two results give the following.

Corollary 5.4 *For a disconnected graph G with components G_1, G_2 and any graph H,*

$$\pi_0(G_1 + G_2) \times \pi_0(H) \cong [\pi_0(G_1) \times \pi_0(H)] + [\pi_0(G_2) \times \pi_0(H))]$$

A straightforward induction argument extends this to multiple components. Thus we can extend Theorem 4.1 to arbitrary graphs.

Theorem 5.5 *For any graphs $G, H \in \mathsf{Gph}$, we have*

$$\pi_0(G \times H) \cong \pi_0(G) \times \pi_0(H).$$

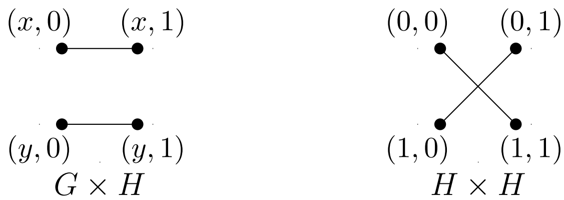
6 Factoring Graphs

Our original motivation for studying $\pi_0(G)$ was the more general question of finding factors of graphs as in Definition 2.6. We observed earlier that our results prove that the problem of finding factors can not be studied component-wise, as it may be necessary to pair up components when looking for graph factors.

Additionally, our results have produced the following example which shows the failure of unique graph factors.

Example 6.1 Let G be the disconnected graph given by two looped vertices $L + L$, and $H = K_2$ be the graph with two vertices connected by an edge.

Then $G \times H$ is isomoprhic to $H \times H$:



We show that this example represents a more general scenario.

Proposition 6.2 *Let G be the disconnected graph given by two looped vertices, and $H \cong K_2$ as in Example 6.1. Then for any connected graph $J \in \mathsf{Gph}$, $\pi_0(J) \cong K_2$ if and only if $G \times J \cong H \times J$.*

Proof. First, $G \times J \cong J_x + J_y$ for any graph J, where J_x denotes the the subgraph of vertices (v, x) and J_y those of vertices (v, y), and both J_x and J_y are isomorphic to J. Now suppose that $\pi_0(J) = K_2$, so J is a bipartite graph with vertices given by the disjoint union of vertex subsets $U = \{u_i\}$ and $V = \{v_i\}$. Then the vertices of $H \times J$ are given by $U_0 = \{(u_i, 0)\}, U_1 = \{(u_i, 1), V_0 = \{(v_i, 0)\}$ and $V_1 = \{(v_i, 1)\}$. Since there are no edges connecting $u_i\_u_j$ or $v_i\_v_j$ in J, the graph $H \times J$ can be separated into disjoint components given by $U_0 \cup V_1$ and $U_1 \cup V_0$. Since each edge in J from $u_i\_v_j$ corresponds to an edge $(u_i, 0)\_(v_j, 1)$ and an edge $(u_i, 1)\_(v_j, 0)$, each of these components is isomorphic to a copy of J. Thus $H \times J$ is also isomorphic to $J + J$.

Conversely, suppose that J is not bipartite. Then $\pi_0(J) \cong L$ where L denotes the single looped vertex, and $\pi_0(G \times J) \cong \pi_0(J + J) \cong L + L$. On the other hand, $\pi_0(H \times J) \cong \pi_0(H) \times \pi_0(J) \cong K_2 \times L \cong K_2$. So $G \times J$ is not isomorphic to $H \times J$. $\square$

Open Problem 6.3 We leave the reader with the following question to ponder: can we find other classes of graphs which can be factored in multiple ways?

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