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Modified Nonparametric Tests for the Umbrella Alternative with Known Peak in a Mixed Design

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Abstract

The Mack-Wolfe and Kim-Kim statistics are two of the most commonly used non-parametric tests for the umbrella alternative problem when the underlying designs follow a CRD or an RCBD, respectively. In this paper, modifications of the Mack-Wolfe and Kim-Kim test are proposed to developtest statistics for the umbrella alternative with known peak when the data are mixture of a randomized complete block and a completely randomized design. The two proposed test statistics are compared to each other and some other existing tests. Results are given.

Keywords: Completely randomized; Randomized complete block; Mixed design; Mack-Wolfe test; Kim-Kim test; Umbrella alternative; Peak known.

1 Introduction

There are many cases in which the researchers may want to use an umbrella alternative. For example, in testing the reaction or the effectiveness of increasing the dosage level of the drug, patients might have a positive reaction or increasing effects, but after a certain dosage level, the patients' reactions might not be as positive and the effects would start decreasing. Another example of using the umbrella alternative is that with increasing age, a person's performance tends to improve up to a point, but after a certain age, performance will start to diminish(Kim & Kim, 1992). The need for an umbrella alternative may also appear in testing the effect of fertilization on the rate of crop yield or growth. When we increase the amount of fertilizer, there might an increase in the crop growth rate, but this rate of crop growth might start to decrease after reaching some given amount of fertilizer. In these instances, the null hypothesis of interest is:

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k$$

against the alternative

$$H_1: \mu_1 \leq \cdots \leq \mu_{p-1} \leq \mu_p \geq \mu_{p+1} \geq \cdots \geq \mu_k$$

with at least one strict inequality. Due to the pictorial configuration of the μ's (the means μ's have up and down ordering), the label of umbrella was given to this alternative by Mack and Wolfe (1981). Here, p is called the turning point or the peak point by Mack and Wolfe (1981). Of note, on one side of the peak p, the means are non-decreasing, and on the other side of the peak, they are non-increasing. Based on a completely randomized design, the procedure for testing the umbrella alternative for the nonparametric case, in which the underlying distributions are unknown, was developed by Mack and Wolfe (1981). The Mack-Wolfe test statistic uses the pairwise Mann and Whitney (1947) statistics, U_{uv}, (Daniel, 1990). The form of the Mack-Wolfe test statistic, A_p, is a sum of two Jonckheere-Terpstra test statistics where the Jonckheere-Terpstra test is the first nonparametric test designed to analyze ordered data (Mack &Wolfe, 1981). The Mack-Wolfe test is given in (1):

$$A_p = \sum_{u=1}^{v-1}\sum_{v=2}^{p} U_{uv} + \sum_{u=p}^{v-1}\sum_{v=p+1}^{k} U_{vu}. \qquad (1)$$

Under the null hypothesis that all population means are equal, the expected value and variance of A_p are given in (2):

$$E_0(A_p) = \frac{N_1^2 + N_2^2 - \sum_{i=1}^{k} n_i^2 - n_p^2}{4}$$

and (2)

$$var_0(A_p) = \frac{1}{72}\{2(N_1^3 + N_2^3) + 3(N_1^2 + N_2^2) - \sum_{i=1}^{k} n_i^2(2n_i + 3) - n_p^2(2n_p + 3) + 12n_pN_1N_2 - 12n_p^2N\},$$

where $N_1 = \sum_{i=1}^{p} n_i$, $N_2 = \sum_{i=p}^{k} n_i$ and $N = \sum_{i=1}^{k} n_i$; n_p is the sample size associated with the peak population with n_i denoting the sample size for population i.

The standardized version of the Mack-Wolfe test, A_p^*, given in (3) has an asymptotic standard normal distribution when H_0 is true:

$$A_p^* = \frac{A_p - E_0(A_p)}{\sqrt{var_0(A_p)}} \qquad (3)$$

The null hypothesis is rejected when $A_p^* \geq Z_\alpha$.

In some cases when a blocking factor is introduced, the researchers maybe interested in testing for the umbrella alternative, and thus a randomized complete block design is used (RCBD). For instance, when we examine the effect of a drug, a blocking factor could be the patients. Similarly, in testing the impact of advancing age on someone's performance, the person, their weight or their athletic status could be a blocking factor. Furthermore, in examining the effectiveness of increasing the amount of fertilizer on the rate of crop growth, a location or plot could be a blocking factor. Kim and Kim (1992) extended the Mack-Wolfe test to an RCBD. The Kim-Kim test statistic, A, is the sum of the Mack-Wolfe statistics over all blocks. It is given in (4):

$$A = \sum_{i=1}^{b} A_{ip} \qquad (4)$$

where A_{ip} is the Mack-Wolfe statistic of the i^{th} block. No interaction is assumed between blocks and treatments.

Under the null hypothesis that all population means are equal, the expected value and variance of A are given in (5):

$$E_0(A) = \frac{\sum_{i=1}^{b}\{N_1^2 + N_2^2 - \sum_{i=1}^{k} n_i^2 - n_p^2\}}{4}$$

and (5)

$$var_0(A) = \sum_{i=1}^{b} \{ 2(N_{i1}^3 + N_{i2}^3) + 3(N_{i1}^2 + N_{i2}^2)$$
$$- \sum_{j=1}^{k} n_{ij}^2 (2n_{ij} + 3) - n_{ip}^2 (2n_{ip} + 3) + 12n_{ip} N_{i1} N_{i2} - 12n_{ip}^2 N_i \}/72,$$

where $N_{i1} = \sum_{j=1}^{p} n_{ij}$, $N_{i2} = \sum_{j=p}^{k} n_{ij}$ and $N_i = \sum_{j=1}^{k} n_{ij}$.

The standardized version of the Kim-Kim test, A^*, given in (6) has an asymptotic standard normal distribution when H_0 is true:

$$A^* = \frac{A - E_0(A)}{\sqrt{var_0(A)}} \tag{6}$$

The null hypothesis is rejected when $A^* \geq Z_\alpha$.

Dubnicka, Blair, and Hettmansperger (2002) consider a mixed design experiment consisting of observations that were paired, and observations from two population that were independent. Dubnicka et al. (2002) proposed a new test statistic for this design that combines the Wilcoxon-signed rank test statistic for paired data and the Mann-Whitney test statistic for two independent samples. Magel, Terpatra, Canonizado, and Park (2010)extended the idea of Dubnicka, et al. (2002) to propose test statistics for umbrella alternative, in the situation where the peak p is known, considering 3 or more mixed samples consisting of observations from a completely randomized design and observations from a randomized complete block design. Magel et al. (2010)'s first proposed test,A_p^{**}, is given in (7):

$$A_p^{**} = A_p^* + A^* \tag{7}$$

where A_p^* is the standardized version of the usual Mack-Wolfe test for the completely randomized design and A^* is the standardized version of Kim-Kim test for the randomized complete block design. Under H_0, since the distribution of each of the test statistics of A_p^* and A^* is an asymptotically standard normal distribution, the asymptotic distribution of $A_p^{**}$is normal with mean zero and variance 2. The standardized version of their first proposed test, A^{**}, given in (8) has an asymptotic standard normal distribution:

$$A^{**} = \frac{A_p^{**} - 0}{\sqrt{2}} \tag{8}$$

The null hypothesis is rejected when $A^{**} \geq Z_\alpha$. Their second proposed test, A_p^{***}, is given in (9):

$$A_p^{***} = A_p + A \tag{9}$$

where A_p and A are the usual Mack-Wolfe test (1981) and Kim-Kim test (1992), respectively.Under the H_0, the expected value and the variance of $A_p^{***}$are given below:

$$E_0(A_p^{***}) = E_0(A_p) + E_0(A)$$

and (10)

$$var_0(A_p^{***}) = var_0(A_p) + var_0(A)$$

where $E_0(A_p)$, $E_0(A)$, $var_0(A_p)$ and $var_0(A)$ are the expected values and variance of the usual Mack-Wolfe test and the Kim-Kim test, respectively. The standardized version of their second proposed test, A^{***}, given in (11) has an asymptotic standard normal distribution under H_0:

$$A^{***} = \frac{A_p^{***} - E_0(A_p^{***})}{\sqrt{var_0(A_p^{***})}} \tag{11}$$

The null hypothesis is rejected when $A^{**} \geq Z_\alpha$.

Hettmansperger and Norton (1987) also proposed a class of rank test versus the patterned alternative. Shi (1988) went on to propose a rank test statistic comparable to the test statistic proposed by Hettmansperger and Norton (1987) using various weighting schemes. Both of these test statistics could be used for the umbrella alternative.

Neuhauser, Liu, and Hothorn (1998) proposed a test statistic for the non-decreasing ordered alternative, considering different weights of the Mann-Whitney statistics extended to Jonckheere (1954) and Terpstra (1952)'s statistic. They found that their proposed test statistic had better in power than the Jonckheere (1954) and Terpstra (1952) test statistic in some cases. Following the results of Neuhauser et al. (1998), Esra and Fikri (2016) developed tests for the umbrella alternative for the completely randomized design. They applied a similar modification to the Mack and Wolfe (1981)'s test as Neuhauser, et al. (1998) did for the Jonchkeere- Terpstra test. Esra and Fikri's proposed test is just the sum of two modified Jonckheere-Terpstra test statistics as introduced by Neuhauser et al. (1998), namely,

$$mA_p = \sum_{u=1}^{v-1}\sum_{v=2}^{p}(v-u)U_{uv} + \sum_{u=p}^{v-1}\sum_{v=p+1}^{k}(u-v)U_{vu}. \tag{12}$$

Under the null hypothesis that all population means are equal, the expected value and variance of mA_p are given in (13):

$$E_0(mA_p) = \frac{n^2}{2}\left[\binom{p+1}{3} + \binom{k-p-2}{3}\right]$$

and (13)

$$var_0(mA_p) = \frac{n^2p^2(p^2-1)(np+1) + n^2(k-p+1)^2[(k-p+1)^2-1][n(k-p+1)+1]}{144} + \frac{n^3p(p-1)(k-p)(k-p+1)}{24}.$$

The standardized version of the Esra and Fikri test, mA_p^*, given in (14) has an asymptotic standard normal distribution when H_0 is true:

$$mA_p^* = \frac{mA_p - E_0(mA_p)}{\sqrt{var_0(mA_p)}} \tag{14}$$

The null hypothesis is rejected when $mA_p^* \geq Z_\alpha$.

2 Two Proposed Tests for the Mixed Design

In this paper, motivated by the idea of Dubnicka, et al. (2002) and Magel et al. (2010) of combining test statistics and by the weighting modification suggested by Esra and Fikri (2016), we propose two different versions of the test statistics for the mixed design to be used for the umbrella alternative when the peak is known . The tests will be introduced and then will be compared on the basis of estimated powers for a variety of situations.

Following Esra and Fikri's (2016) modification to the Mack-Wolfe (1981) statistic given in (12), we propose a similar modification of the Kim-Kim, mA, as follows:

$$mA = \sum_{i=1}^{b} mA_{ip}$$

$$mA_{ip} = \sum_{i=1}^{b}\left\{\sum_{u=1}^{v-1}\sum_{v=2}^{p}(v-u)U_{iuv} + \sum_{u=p}^{v-1}\sum_{v=p+1}^{k}(u-v)U_{ivu}\right\}. \tag{15}$$

where mA_{ip} denotes the modified Mack-Wolfe test statistic of the i^{th}block, $(v-u)U_{iuv}$ is the weighted Mann-Whitney test statistic applied to the observations in cell $(i,u)$and (i,v), k is the number of treatments, p is the known peak and the number of blocks is b. At α level of significance, we reject $H_0$for the large value of mA.

When the sample sizes for each treatment per block are equal to one($n_{11} = \cdots = n_{bk} = n = 1$) and under the null hypothesis that all population means are equal, the expected value and variance of mA are given in (16):

$$E_0(mA) = \sum_{i=1}^{b} \left\{ \frac{1}{2}\left[\binom{p+1}{3} + \binom{k-p-2}{3} \right] \right\}$$

and (16)

$$var_0(mA) = \sum_{i=1}^{b} \left\{ \frac{p^2(p^2-1)(p+1) + (k-p+1)^2[(k-p+1)^2-1][(k-p+1)+1]}{144} + \frac{p(p-1)(k-p)(k-p+1)}{24} \right\}.$$

The standardized version of the modified Kim-Kim test, mA^*, given in (17) has an asymptotic standard normal distribution when H_0 is true:

$$mA^* = \frac{mA - E_0(mA)}{\sqrt{var_0(mA)}} \tag{17}$$

The null hypothesis is rejected when $mA_p^* \geq Z_\alpha$.

The first proposed test, mA_p^{**}, is given in (18):

$$mA_p^{**} = mA_p^* + mA^* \tag{18}$$

where mA_p^* is the standardized version of the modified Mack-Wolfe test for the completely randomized design and mA^* is the standardized version of modified Kim-Kim test for the randomized complete block design. Under the H_0 and since the distribution of each test statistics of mA_p^* and mA^* is an asymptotically standard normal distribution, the asymptotic distribution of mA_p^{**} should be normal with mean zero and variance 2. The standardized version of the first proposed test, mA^{**}, given in (19) has an asymptotic standard normal distribution:

$$mA^{**} = \frac{mA_p^{**} - 0}{\sqrt{2}} \tag{19}$$

The null hypothesis is rejected when $mA^{**} \geq Z_\alpha$.

The second proposed test, mA_p^{***}, is given in (20):

$$mA_p^{***} = mA_p + mA \tag{20}$$

where mA_p and mA are the unstandardized version of modified Mack-Wolfe test (1981) and Kim-Kim test (1992), respectively. Under the H_0, the expected value and the variance of $mA_p^{***}$are given below:

$$E_0(mA_p^{***}) = E_0(mA_p) + E_0(mA)$$

and (21)

$$var_0(mA_p^{***}) = var_0(mA_p) + var_0(mA)$$

where $E_0(mA_p)$, $E_0(mA)$, $var_0(mA_p)$ and $var_0(mA)$ are the expected values and variance of the modified Mack-Wolfe test and the Kim-Kim test, respectively. The standardized version of the second proposed test, mA^{***}, given in (22) has an asymptotic standard normal distribution under H_0:

$$mA^{***} = \frac{mA_p^{***} - E_0(mA_p^{***})}{\sqrt{var_0(mA_p^{***})}} \tag{22}$$

The null hypothesis is rejected when $mA^{**} \geq Z_\alpha$.

In the first proposed test, the modified Mack-Wolfe and modified Kim-Kim test statistics are standardized first and then add together. In the second proposed test, the unstandardized versions of the modified Mack-Wolfe and modified Kim-Kim test statistics are added together first and then standardized.

3 The Exact Mean and Variance

Esra and Fikri (2016) derived a general formula of the exact mean and variance for the modified Mack-Wolfe test. Accordingly, we derived a general formula of the exact mean and variance for the modified Kim-Kim test. These formulas are preferred in building the simulation code since they are in a general form. However, in some cases when the data are at hand and we need to calculate the mean and variance for a test statistic, it is complicated to use those formulas manually. Hence, we derived formulas of extracting the exact mean and variance for the modified Mack-Wolfe test and modified Kim-Kim test for every possible peak in three, four and five populations. In this paper, we assume that all the sample sizes are equal $n = n_1 = n_2 = \cdots = n_k$ where k is the number of treatments in the completely randomized design portion. Also, in the randomized complete block design portion, we assume that $n = n_{ij} = 1$ when $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, k$ where i is the number of blocks, and j is the number of treatments. To note, the means and variances when having three treatments where the peak is 1 are equal to those when the peak is 3. The mean and variance for four treatments where the peak is 1 are equal to those when the peak is 4; also, the same pattern happens when the peak is 3 or 4 in the four treatments. This fact of equality has been noted for five treatments where the peak is 1 or 5, and also, the same symmetry occurs when the peak is 2 or 4. Again, for the RCBD, we consider the case when $(n = n_{ij} = 1)$ where $i \in \{1, \ldots, b\}$ and $j \in \{1, 2, \ldots, k\}$

Table1. The exact mean and variance for every peak.

k	p	Design	Test	Expected Mean	Variance
3	1 or 3	CRD	mA_p	$2n^2$	$\frac{1}{2}[n^2(2n+1)+n^3]$
		RCBD	mA	$2b$	$\frac{1}{2}[4]b$
	2	CRD	mA_p	n^2	$\frac{1}{6}[n^2(2n+1)+n^3]$
		RCBD	mA	b	$\frac{1}{6}[4]b$
4	1 or 4	CRD	mA_p	$5n^2$	$\frac{1}{3}[5\,n^2(2n+1)+10\,n^3]$
		RCBD	mA	$5b$	$\frac{1}{3}[25]b$
	2 or 3	CRD	mA_p	$\frac{5}{2}n^2$	$\frac{1}{12}[7\,n^2(2n+1)+12\,n^3]$
		RCBD	mA	$\frac{5}{2}b$	$\frac{1}{12}[33]b$
5	1 or 5	CRD	mA_p	$10n^2$	$\frac{1}{6}[25\,n^2(2n+1)+75\,n^3]$
		RCBD	mA	$10b$	$\frac{1}{6}[150]b$
	2 or 4	CRD	mA_p	$\frac{11}{2}n^2$	$\frac{1}{12}[21\,n^2(2n+1)+52\,n^3]$
		RCBD	mA	$\frac{11}{2}b$	$\frac{1}{12}[115]b$
	3	CRD	mA_p	$4n^2$	$\frac{1}{2}[2\,n^2(2n+1)+5\,n^3]$
		RCBD	mA	$4b$	$\frac{1}{2}[11]b$

4 Simulation Study

The simulation study is designed via Monte Carlo Simulation and implemented in SAS version 9.4. It is conducted to investigate the type I error and the performance of the proposed test statistics. The observations are assumed to follow three different underlying distributions, which are included in this study: standard exponential, standard normal and t distribution with three degrees of freedom. These distributions were used to give two symmetric distributions, with one having a larger variance, and one skewed distribution. In this research, the data are generated from a mixed design consisting of a CRD portion and an RCBD portion.

All the simulations used to estimate the alpha values and powers are based on $5{,}000$ iterations. The initial stage of the simulation is to estimate the level of significance of the proposed test statistics, namely when all the location parameters were the same. For all the proposed tests, the significance level α is stated to be 0.05. When the null hypothesis is true, all of the test statistics have an asymptotic standard normal distribution. The estimated level of significance α is obtained by counting the number of times that the null hypothesis is rejected and then divided by the number of samples generated $(5{,}000)$ for each test of the proposed test statistics. The second stage of the simulation is to estimate and compare the powers of the proposed test statistics under various situations (different location parameter configurations), namely after adding location parameters. For each situation, we also use $5{,}000$samples to estimate the power by counting the number of times that the proposed test statistic is rejected divided by the number of samples generated $(5{,}000)$.

In this paper, three, four and five populations are considered with the assumption that the peak p is known. For three treatments, the peak is at the second population. For the four populations, the peaks are considered to be at the second and third populations. When there are five populations, the peaks are considered at the second, third and fourth populations. For every considered distribution, equal sample sizes for the CRD portion are selected so that the sample size, n, is $6, 10, 16$ and 20. The number of blocks (complete blocks) for the RCBD is considered to be half, equal and twice the sample size for each treatment in the CRD.

For the estimated powers, a variety of location parameter configurations (treatment effects shifts) are considered.

For 3 populations with peak at 2, powers were estimated in the following cases:

1. The peak is distinct and there is equal spacing between parameters; for example, (0.0, 1.0, 0.0).
2. The peak is distinct and there is unequal spacing between parameters; for example, (0.0, 1.0, 0.5) and (0.5, 1.0, 0.0).
3. One additional parameter equals the peak; for example, (0.5, 0.5, 0.0) and (0.0, 0.5, 0.5).

For 4 populations with peak at 2, powers were estimated in the following cases:

1. The peak is distinct, and the other parameters are the same; for example, (0.0, 0.5, 0.0, 0.0) and (0.5, 1.0, 0.5, 0.5).
2. Two population parameters are the same, but different from the peak and less than the first treatment; for example, (0.5, 1.0, 0.2, 0.2).
3. Two population parameters are the same, but different from the peak and greater than the last treatment; for example, (0.25, 0.5, 0.25, 0.0).
4. One additional parameter equals the peak and the other parameters are different from the peak, but equal to each other; for example, (0.5, 0.5, 0.0, 0.0) and (0.0, 0.5, 0.5, 0.0).
5. One additional parameter equals the peak and the other parameters are different from the peak and different from each other; for example, (0.5, 0.5, 0.2, 0.0).
6. Two additional parameters are equal to the peak; for example, (0.0, 0.5, 0.5, 0.5) and (0.5, 0.5, 0.5, 0.0).
7. There is unequal spacing between parameters; for example, (0.0, 1.0, 0.75, 0.2) and (0.0, 0.75, 0.5, 0.25).

For 4 populations with peak at 3, powers were estimated in the following cases:

1. The peak is distinct, and the other parameters are the same; for example, (0.0, 0.0, 0.5, 0.0) and (0.5, 0.5, 1.0, 0.5).
2. Two population parameters are the same, but different from the peak and less than the last treatment; for example, (0.2, 0.2, 1.0, 0.5).
3. Two population parameters are the same, but different from the peak and greater than the first treatment; for example, (0.0, 0.25, 0.5, 0.25).
4. One additional parameter equals the peak and the other parameters are different from the peak, but equal to each other; for example, (0.0, 0.0, 0.5, 0.5) and (0.0, 0.5, 0.5, 0.0).

5. One additional parameter equals the peak and the other parameters are different from the peak and different from each other; for example, (0.0, 0.2, 0.5, 0.5).
6. Two additional parameters are equal to the peak; for example, (0.0, 0.5, 0.5, 0.5) and (0.5, 0.5, 0.5, 0.0).
7. There is unequal spacing between parameters; for example, (0.2, 0.75, 1.0, 0.0) and (0.25, 0.5, 0.75, 0.0).

For 5 populations with peak at 2, powers were estimated in the following cases:

1. The peak is distinct, and the other parameters are the same; for example, (0.0, 0.5, 0.0, 0.0, 0.0).
2. The peak is distinct and each two of the other parameters are equal to each other; for example, (0.0, 0.5, 0.2, 0.2, 0.0) and (0.2, 0.5, 0.2, 0.0, 0.0).
3. The peak is distinct, two of the other parameters are equal to each other and the other two parameters are different from each other; for example, (0.0, 0.6, 0.4, 0.4, 0.2), and (0.2, 0.6, 0.4, 0.4, 0.0)
4. The peak is distinct, three of the other parameters are equal to each other where two of them on the edges; for example, (0.0, 0.5, 0.2, 0.0, 0.0).
5. The peak is distinct, and the other parameters are different from each other; for example, (0.2, 0.8, 0.6, 0.4, 0.0), (0.0, 0.8, 0.6, 0.4, 0.2).
6. One additional parameter equals the peak and the other parameters are different from the peak, but equal to each other; for example, (0.5, 0.5, 0.0, 0.0, 0.0) and (0.2, 0.5, 0.5, 0.2, 0.2).
7. One additional parameter equals the peak, the two other parameters are equal to each other but not equal to the peak and one parameter is different; for example, (0.5, 0.5, 0.2, 0.2, 0.0), and (0.5, 0.5, 0.2, 0.0, 0.0).
8. One additional parameter equals the peak, and the other parameters are different from the peak and different from each other; for example, (0.8, 0.8, 0.5, 0.2, 0.0) and (0.2, 0.7, 0.7, 0.3, 0.0).
9. Two additional parameters are equal to the peak, and the other parameters are equal to each other; for example, (0.5, 0.5, 0.5, 0.0, 0.0) and (0.2, 0.5, 0.5, 0.5, 0.2).
10. Two additional parameters are equal to the peak, and the other parameters are different from each other; for example, (0.5, 0.5, 0.5, 0.2, 0.0), and (0.0, 0.5, 0.5, 0.5, 0.2).
11. Three additional parameters are equal to the peak; for example, (0.5, 0.5, 0.5, 0.5, 0.0) and (0.0, 0.5, 0.5, 0.5, 0.5).
12. Three parameters are different from the peak but are equal to each other and less than the first treatment; for example, (0.2, 0.5, 0.0, 0.0, 0.0).
13. Three parameters are different from the peak but are equal to each other and greater than the first treatment; for example, (0.0, 0.6, 0.2, 0.2, 0.2).

For 5 populations with peak at 3, powers were estimated in the following cases:

1. The peak is distinct, and the other parameters are the same; for example, (0.0, 0.0, 0.5, 0.0, 0.0).
2. The peak is distinct and each two of the other parameters are equal to each other; for example, (0.0, 0.0, 0.5, 0.2, 0.2), and (0.2, 0.2, 0.5, 0.0, 0.0).
3. The peak is distinct, two of the other parameters are equal to each other and the other two parameters are different from each other; for example, (0.0, 0.4, 0.7, 0.2, 0.0), and (0.0, 0.4, 0.7, 0.4, 0.2).
4. The peak is distinct, and the other parameters are different from each other; for example, (0.0, 0.2, 0.8, 0.5, 0.3), (0.0, 0.8, 0.6, 0.4, 0.2).
5. One additional parameter equals the peak and the other parameters are different from the peak, but equal to each other; for example, (0.0, 0.5, 0.5, 0.0, 0.0) and (0.2, 0.5, 0.5, 0.2, 0.2).
6. One additional parameter equals the peak, two other parameters are equal to each other but not equal to the peak and one parameter is different; for example, (0.0, 0.5, 0.5, 0.2, 0.0) and (0.2, 0.5, 0.5, 0.2, 0.0).
7. One additional parameter equals the peak, and the other parameters are different from the peak and different from each other; for example, (0.0, 0.8, 0.8, 0.5, 0.2) and (0.2, 0.5, 0.8, 0.8, 0.0).
8. Two additional parameters are equal to the peak, and the other parameters are equal to each other; for example, (0.0, 0.5, 0.5, 0.5, 0.0), and (0.5, 0.5, 0.5, 0.0, 0.0).
9. Two additional parameters are equal to the peak, and the other parameters are different from each other; for example, (0.0, 0.5, 0.5, 0.5, 0.2), and (0.5, 0.5, 0.5, 0.2, 0.0).
10. Three additional parameters are equal to the peak; for example, (0.0, 0.5, 0.5, 0.5, 0.5) and (0.5, 0.5, 0.5, 0.5, 0.0).

For 5 populations with peak at 4, powers were estimated in the following cases:

1. The peak is distinct, and the other parameters are the same; for example, (0.0, 0.0, 0.0, 0.5, 0.0).

2. The peak is distinct and each two of the other parameters are equal to each other; for example, (0.0, 0.2, 0.2, 0.5, 0.0) and (0.0, 0.0, 0.2, 0.5, 0.2).
3. The peak is distinct, two of the other parameters are equal to each other and the other two parameters are different from each other; for example, (0.2, 0.4, 0.4, 0.6, 0.0), and (0.0, 0.4, 0.4, 0.6, 0.2).
4. The peak is distinct, three of the other parameters are equal to each other where two of them on the edges; for example, (0.0, 0.0, 0.2, 0.5, 0.0).
5. The peak is distinct, and the other parameters are different from each other; for example, (0.0, 0.4, 0.6, 0.8, 0.2), (0.2, 0.4, 0.6, 0.8, 0.0).
6. One additional parameter equals the peak and the other parameters are different from the peak, but equal to each other; for example, (0.0, 0.0, 0.0, 0.5, 0.5) and (0.2, 0.2, 0.5, 0.5, 0.2).
7. One additional parameter equals the peak, the two other parameters are equal to each other but not equal to the peak and one parameter is different; for example, (0.0, 0.2, 0.2, 0.5, 0.5), and (0.0, 0.0, 0.2, 0.5, 0.5).
8. One additional parameter equals the peak, and the other parameters are different from the peak and different from each other; for example, (0.0, 0.2, 0.5, 0.8, 0.8) and (0.0, 0.3, 0.7, 0.7, 0.2).
9. Two additional parameters are equal to the peak, and the other parameters are equal to each other; for example, (0.0, 0.0, 0.5, 0.5, 0.5) and (0.2, 0.5, 0.5, 0.5, 0.2).
10. Two additional parameters are equal to the peak, and the other parameters are different from each other; for example, (0.0, 0.2, 0.5, 0.5, 0.5), and (0.2, 0.5, 0.5, 0.5, 0.0).
11. Three additional parameters are equal to the peak; for example, (0.0, 0.5, 0.5, 0.5, 0.5) and (0.5, 0.5, 0.5, 0.5, 0.0).
12. Three parameters are different from the peak but are equal to each other and less than the first treatment; for example, (0.0, 0.0, 0.0, 0.5, 0.2).
13. Three parameters are different from the peak but are equal to each other and greater than the first treatment; for example, (0.2, 0.2, 0.2, 0.6, 0.0).

5 Results from the Simulation Study

In this section, we present results for the two proposed test statistics (modified tests) in this research and the two test statistics that were proposed by Magel et al. (2010) (non-modified tests). These test statistics are for analyzing data in a mixed design comprised of a completely randomized design (CRD) portion and a randomized complete block design (RCBD) portion in the situation of having a known umbrella peak. In an equivalent manner, by conducting the simulation study, the results in this research, as far as how the two proposed tests performed relative to one another for all the underlying distributions, are similar to Magel et al. (2010)'s results. The asymptotic distribution of these test statistics are all standard normal under the null hypothesis, and the stated alpha value for each test conducted is 0.05. All test statistics considered maintained their stated alpha value.

5.1. Three Treatments with peak 2:

In this case, we find that neither of the proposed tests (modified tests) in this research are better than the test statistics proposed by Magel et al. (2010) (non-modified tests) since the associated weight (distance modification) for both the Mack-Wolfe (1981) and Kim-Kim (1992) test statistics is just one in the two proposed tests of this research. They are exactly the same. Overall, the Standardized First version is more powerful than Standardized Second version.

5.2. Four Treatments with peak 2:

Selected results are given in Tables 2-4for four treatments at peak 2 for the exponential and normal distributions. Results show that the Standardized First is more powerful than the Standardized Second for both proposed tests and Magel et al. (2010)'s tests statistics, and the results are consistent when the proportions between the number of blocks in the RCBD and sample size for each treatment in the CRD change.

Table 2. Estimated rejection percentages of tests for mixed design under the exponential distribution for 4 treatments at peak 2: Blocks= 5, n= 10.

Location Parameter	Standardized	Non Modification	Distance Modification
(0.0, 0.0, 0.0,0.0)	First	0.0486	0.0464
	Second	0.0436	0.0434
(0.5, 0.5, 0.0,0.0)	First	0.3416	0.4128
	Second	0.2928	0.3582
(0.5, 0.5, 0.2,0.0)	First	0.3934	0.4520
	Second	0.3436	0.3976
(0.5, 0.5, 0.5,0.0)	First	0.3702	0.4338
	Second	0.3226	0.3800
(0.25,0.5, 0.25,0.0)	First	0.5518	0.5778
	Second	0.4798	0.5072
(0.0, 0.75, 0.5,0.25)	First	0.7742	0.7270
	Second	0.6950	0.6534

Table 3. Estimated rejection percentages of tests for mixed design under the exponential distribution for 4 treatments at peak 2: Blocks= 10, n= 10.

Location Parameter	Standardized	Non Modification	Distance Modification
(0.0,0.0,0.0,0.0)	First	0.0514	0.0502
	Second	0.0526	0.0506
(0.5,0.5,0.0,0.0)	First	0.4528	0.5274
	Second	0.3246	0.3928
(0.5,0.5,0.2,0.0)	First	0.4912	0.5470
	Second	0.3616	0.4160
(0.5,0.5,0.5,0.0)	First	0.4572	0.5204
	Second	0.3408	0.3918
(0.25,0.5,0.25, 0.0)	First	0.6764	0.6848
	Second	0.5204	0.5274
(0.0, 0.75, 0.5,0.25)	First	0.8772	0.8402
	Second	0.7218	0.6752

It is important to note that for each of the considered proportions between the sample size in the CRD and the number of blocks in the RCBD, we found that the two proposed test statistics (modified tests) in this research provide the highest values of the estimated powers compared to the Magel et al. (2010)'s test statistics (non-modified tests) in the following cases:

1. The two population parameters on either side of the peak were equal to the peak or equal to each other such as the following: (0.5,0.5,0.5,0.0) and (0.25,0.5,0.25,0.0)
2. The difference between the parameter before the peak and the peak was less than the distance between the parameter after the peak and the peak such as the following:
   (0.5,0.5,0.2,0.0.

Table 4. Estimated rejection percentages of tests for mixed design under the normal distribution for 4 treatments at peak 2: Blocks= 20, n= 10.

Location Parameter	Standardized	Non Modification	Distance Modification
(0.0,0.0,0.0,0.0)	First	0.0484	0.0500
	Second	0.0516	0.0454
(0.5,0.5,0.0,0.0)	First	0.3488	0.4114
	Second	0.2362	0.2728
(0.5,0.5,0.2,0.0)	First	0.3524	0.4170
	Second	0.2430	0.2866
(0.5,0.5,0.5,0.0)	First	0.3606	0.4054
	Second	0.2530	0.2802
(0.25,0.5,0.25, 0.0)	First	0.4862	0.5048
	Second	0.3294	0.3390
(0.0, 0.75, 0.5,0.25)	First	0.9590	0.9368
	Second	0.7908	0.7412

5.3. Four Treatments with peak 3:

In the case of having four treatments and a known peak at the third population, we found that the modified test statistics had greater power than the non-modified test statistics under the following situations.

1. The two population parameters on either side of the peak were equal to the peak or equal to each other such as the following: (0.0,0.5,0.5,0.5) and (0.0,0.25,0.50,0.25);
2. The difference between the parameter before the peak and the peak was greater than the distance between the parameter after the peak and the peak such as the following:
(0.0,0.2,0.5,0.5).

5.4. Five Treatments with peak 2:

Selected results for five treatments at peak two are given in Tables 5-7 for the exponential, normal and t distribution with three degrees of freedom. For all the distributions, results show that the Standardized First is more powerful than the Standardized Second for both proposed tests and Magel et al. (2010)'s tests statistics and the results are consistent when the proportions are changed between the number of blocks in the RCBD and sample size for each treatment in the CRD.

It is important to note that the two proposed test statistics (modified tests) in this research provide the highest values of estimated powers compared to the Magel et al. (2010)'s test statistics (non-modified tests) in the following cases:

1. The two population parameters on either side of the peak parameter were equal to the peak or equal to each other such as the following: (0.4, 0.8, 0.4, 0.2, 0.0)
2. The difference between the parameter before the peak and the peak was less than the distance between the parameter after the peak and the peak such as the following:
(0.5, 0.5, 0.0, 0.0, 0.0) or (0.5, 0.5, 0.2, 0.0, 0.0)

Table 5. Estimated rejection percentages of tests for mixed design under the exponential distribution for 5 treatments at peak 2: Blocks= 5, n= 10

LocationParameter	Standardized	Non Modification	Distance Modification
(0.0,0.0,0.0,0.0,0.0)	First	0.0504	0.0560
	Second	0.0522	0.0546
(0.2,0.8,0.6,0.2,0.0)	First	0.9466	0.9424
	Second	0.9036	0.8896
(0.4,0.8,0.4,0.2,0.0)	First	0.9020	0.9032
	Second	0.8474	0.8392
(0.5,0.5,0.0,0.0,0.0)	First	0.3972	0.4638
	Second	0.3362	0.4002
(0.5,0.5,0.2,0.2,0.0)	First	0.4464	0.4932
	Second	0.3854	0.4084
(0.2,0.5,0.5,0.2,0.2)	First	0.4678	0.4510
	Second	0.4076	0.3856

Table 6. Estimated rejection percentages of tests for mixed design under the normal distribution for 5 treatments at peak 2: Blocks= 20, n= 10

Location Parameter	Standardized	Non Modification	Distance Modification
(0.0,0.0,0.0,0.0,0.0)	First	0.0464	0.0530
	Second	0.0494	0.0506
(0.2,0.8,0.6,0.2,0.0)	First	0.9350	0.9362
	Second	0.7500	0.7388
(0.4,0.8,0.4,0.2,0.0)	First	0.8688	0.8902
	Second	0.6512	0.6784
(0.5,0.5,0.0,0.0,0.0)	First	0.3844	0.4592
	Second	0.2532	0.3014
(0.5,0.5,0.2,0.2,0.0)	First	0.3916	0.4534
	Second	0.2546	0.2940
(0.2,0.5,0.5,0.5,0.2)	First	0.3004	0.2708
	Second	0.2018	0.1924

Table7.Estimatedrejectionpercentagesoftestsformixeddesignunderthetdistributionfor 5 treatments

at peak 2: Blocks= 20, n=10.

Location Parameter	Standardized	Non Modification	Distance Modification
(0.0,0.0,0.0,0.0,0.0)	First	0.0514	0.0500
	Second	0.0530	0.0500
(0.2,0.8,0.6,0.2,0.0)	First	0.8284	0.8154
	Second	0.5922	0.5966
(0.4,0.8,0.4,0.2,0.0)	First	0.7198	0.7428
	Second	0.4974	0.5192
(0.5,0.5,0.0,0.0,0.0)	First	0.2988	0.3370
	Second	0.2062	0.2356
(0.5,0.5,0.2,0.2,0.0)	First	0.2950	0.3404
	Second	0.2040	0.2240
(0.2,0.5,0.5,0.5,0.2)	First	0.2360	0.2228
	Second	0.1624	0.1648

5.5. Five Treatments with peak 4:

In the case of having five treatments and a known peak at the third population, we found that the modified test statistics had greater power than the non-modified test statistics under the following situations.

1. The two population parameters on either side of the peak were equal to the peak or equal to each other such as the following: (0.0,0.0,0.5,0.5,0.5) and (0.0,0.0,0.25,0.50, 0.25);
2. The difference between the parameter before the peak and the peak was greater than the distance between the parameter after the peak and the peak such as the following:
   (0.0, 0.0,0.2,0.5,0.5).

5.6. Five Treatments with peak 3:

Selected results for five treatments at peak three may be found in Tables 8-9 for exponential and normal distributions. Again, the Standardized First is more powerful than the Standardized Second for both the proposed tests and Magel et al. (2010)'s tests statistics.

Importantly, the results from the two proposed tests (modified tests) and the test statistics proposed by Magel et al. (2010) (non-modified tests) vary from configuration to one another and from distribution to one another. Accordingly, it is difficult to emphasize whether the distance modification provides highest power for the test statistics or not. No pattern was found as to when the modified versions of the test statistics did better the non-modified versions of the test statistics did better when the peak was at 3. At times, the modified versions would have higher power, and at other times, the non-modified version would have higher power.

Table 8. Estimated rejection percentages of tests for mixed design under the exponential distribution for 5 treatments at peak 3: Blocks= 5, n= 10.

Location Parameter	Standardized	Non Modification	Distance Modification
(0.0,0.0,0.0,0.0,0.0)	First	0.0544	0.0462
	Second	0.0536	0.0490
(0.2,0.2,0.5,0.0,0.0)	First	0.5190	0.5292
	Second	0.4458	0.4524
(0.0,0.2,0.5,0.2,0.0)	First	0.7146	0.7130
	Second	0.6284	0.6274
(0.0,0.4,0.7,0.2,0.0)	First	0.9132	0.9078
	Second	0.8598	0.8454
(0.0,0.8,0.8,0.5,0.2)	First	0.8706	0.8918
	Second	0.7998	0.8164
(0.0,0.5,0.5,0.5,0.0)	First	0.6660	0.6654
	Second	0.5804	0.5878
(0.0,0.5,0.5,0.5,0.2)	First	0.5222	0.5232
	Second	0.4558	0.4594
(0.0,0.0,0.5,0.5,0.5)	First	0.2488	0.2542
	Second	0.2144	0.2142
(0.0,0.2,0.5,0.5,0.5)	First	0.2686	0.2670
	Second	0.2390	0.2240

Table 9. Estimated rejection percentages of tests for mixed design under the normal distribution for 5 treatments at peak 3: Blocks= 10, n= 10.

Location Parameter	Standardized	Non Modification	Distance Modification
(0.0,0.0,0.0,0.0,0.0)	First	0.0474	0.0476
	Second	0.0496	0.0464
(0.2,0.2,0.5,0.0,0.0)	First	0.3742	0.3906
	Second	0.2916	0.2852
(0.0,0.2,0.5,0.2,0.0)	First	0.5052	0.5230
	Second	0.3844	0.3932
(0.0,0.4,0.7,0.2,0.0)	First	0.7594	0.7604
	Second	0.6020	0.6122
(0.0,0.8,0.8,0.5,0.2)	First	0.7442	0.7556
	Second	0.5946	0.5954
(0.0,0.5,0.5,0.5,0.0)	First	0.5178	0.4962
	Second	0.3928	0.3768
(0.0,0.5,0.5,0.5,0.2)	First	0.3612	0.3806
	Second	0.2928	0.2882
(0.0,0.0,0.5,0.5,0.5)	First	0.2086	0.2082

	Second	0.1606	0.1610
(0.0,0.2,0.5,0.5,0.5)	First	0.1938	0.2030
	Second	0.1518	0.1572

Conclusion and Discussion

Results showed that regardless of the underlying distribution, the proportions between the CRD and RCBD portions in the mixed design, and the peak p, the alpha values for all test statistics are approximately 0.05. We conclude that the Standardized First versions of the test statistics were generally better than the Standardized Last versions of the test statistics.

When the study is comprised of four treatments with a known peak at the second population, the modified versions of the test statistics have more power than the non-modified versions in the following situations: there is about the same difference between the peak parameter and the parameters on either side of the peak parameter; or there is less of difference between the parameter before the peak and the peak parameter than between the parameter after the peak and the peak parameter. When the study is comprised of four treatments with known peak at 3, the modified test statistics generally do better under the following situations: there is about the same difference between the parameter before the peak and the peak parameter and the parameter after the peak and the peak parameter; or there is more of a difference between the parameter before the peak and the peak parameter than there is between the parameter after the peak and the peak parameter. When the study is comprised of five treatments with a known peak at the second or fourth populations, the modified test statistics have greater power under the same circumstances as described when there are four populations with known peaks of 2 and 3, respectively. When the study is comprised of five treatments with a known peak at the third population, it is difficult to determine in what situations it is better to use the modification. In this case, one could use either the modified, or non-modified test statistics.

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