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On Finding Coefficient Of Generating Function

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Abstract

This paper reviews the method of determining the coefficients of a generating function. Generating functions are a convenient tool for handling special constraints in selection and arrangement problems. It can be used in recurrence relations, inclusion exclusion events study, and polya's enumeration formula. It may also help to solve some other combinatorial problems. Generating functions are a kind of abstract problem-solving technique once we understand it may easy to model a broad spectrum of combinatorial problems. In this paper, we will use some vivid examples to demonstrate both the theoretical and applicable results of generating function.

Keywords and Phases: A sum of 25 when 10 dice are rolled, Generating functions, Inclusion exclusion events study, Polya's enumeration formula, Recurrence relation, Selection and arrangement problems, Two equal unordered piles.

1. Introduction

We consider three objects labeled $x_1, x_2, and\ x_3$. From the algebraic Product $(1+x_1t)(1+x_2t)(1+x_3t)$ multiplied out and arranged in powers of t, that is $1+(x_1+x_2+x_3)t+(x_1x_2+x_1x_3+x_2x_3)t^2+(x_1x_2x_3)t^3$ or, in the notation of symmetric function

$1 + a_1 t + a_2 t^2 + a_3 t^3$ where $a_1, a_2, and\ a_3$ are the elementary symmetric functions of the three variables $x_1, x_2, and\ x_3$. These symmetric functions are identified by the equation ahead, and it will be noticed that a_r, $r=1,2,3,$ contains one term for each combination of the three things taken r at a time. Hence, the number of such combinations is obtained by setting each x_i to unity, that is, $(1+t)^3 = \sum_{r=0}^{3} a_r(1,1,1)t^r$. In the case of n distinct things labeled x_1 to x_n, it is clear that $(1+x_1 t)(1+x_2 t)(1+x_3 t)........(1+x_n t)$

$= 1 + a_1(x_1, x_2, ... x_n)t + ... + a_2(x_1, x_2, ... x_n)t^2. + ... + a_n(x_1, x_2, ... x_n)t^n$ and

$(1+t)^n = \sum_{r=0}^{n} a_r(1,1,.....1)t^r = \sum_{r=0}^{n} C(n,r)t^r$. where $C(n,r)$ is binomial coefficient. The number of ways to select an r-subsets from an n set. The expression $(1+t)^n$ is called the enumerating generating function or simply the enumerator, of combinations of n distinct things. The problem of determining the coefficient of t^r when we multiply a group of polynomial factors together can be stated in terms of exponents. For example, what is the coefficient of t^4 in the expansion of $(1+t)^3(1+t+t^2)^2$. It is the number of different formal products whose sum of exponents is 4. Determining this coefficient can be modeled as an integer solution to an equation problem. The coefficient of t^4 in that form is the number of integer solution to $e_1 + e_2 + e_3 + e_4 + e_5 = 4$ where $0 \le e_1, e_2, e_3 \le 1$, and $0 \le e_4, e_5 \le 2$. The variable e_i stands for the value of exponents of the ith term in the formal product of the two polynomials. The integer solution to an equation problem is equivalent to the problem of selecting four objects from a collection of five types of objects, where there is one object of the first type, one of the second type, one of the third type, and two of each of fourth and fifth types. It is also equivalent to the problem of distributing four identical objects into five distinct boxes with at most one object in each of the first three boxes and at most two objects in the last two boxes. More generally, the coefficient of t^r in $(1+t)^3(1+t+t^2)^2$ will be the number of integer solutions to $e_1 + e_2 + e_3 + e_4 + e_5 = r$ where $0 \le e_1, e_2, e_3 \le 1$, and $0 \le e_4, e_5 \le 2$. We could repeat all the above statements but replace 4 by r. Thus $(1+t)^3(1+t+t^2)^2$ is the generating function for a_r, the number of ways to select r objects from the given collection of five types or to perform the equivalent distribution. In this paper, our objection is break down distinct generating function and finding the a_r number. In each different case, we demonstrate by some examples.

2. Generating Functions

We are developing algebraic techniques for calculating the coefficients of generating functions. All these methods seek to reduce a given generating function to a either simple binomial type or negative binomial type generating function. It might also be the product of these two types generating functions. The example (2.1), (2.2) and (2.3) demonstrate the standard of these three applications.

Example 2.1: Find the coefficient of x^{12} in $\frac{x+3}{1-3x+x^2}$.

$$\frac{x+3}{1-3x+x^2}=\frac{A}{x-\frac{3+\sqrt{5}}{2}}+\frac{B}{x-\frac{3-\sqrt{5}}{2}} \qquad x+3=A\left(x-\frac{3-\sqrt{5}}{2}\right)+B\left(x-\frac{3+\sqrt{5}}{2}\right)$$

To determine constant A and B, we let $x=\frac{3+\sqrt{5}}{2}$ or $x=\frac{3-\sqrt{5}}{2}$

Solve A and B, we get $A=\frac{1}{2}+\frac{9}{2\sqrt{5}}$ and $B=\frac{1}{2}-\frac{9}{2\sqrt{5}}$ Therefore,

$$\frac{x+3}{1-3x+x^2}=\frac{\frac{1}{2}+\frac{9}{2\sqrt{5}}}{-\frac{3+\sqrt{5}}{2}}\frac{1}{1-\frac{x}{\frac{3+\sqrt{5}}{2}}}+\frac{\frac{1}{2}-\frac{9}{2\sqrt{5}}}{-\frac{3-\sqrt{5}}{2}}\frac{1}{1-\frac{x}{\frac{3-\sqrt{5}}{2}}},\quad a_1=\frac{\frac{1}{2}+\frac{9}{2\sqrt{5}}}{-\frac{3+\sqrt{5}}{2}}\quad c_1=\frac{1}{\frac{3+\sqrt{5}}{2}}\quad a_2=\frac{\frac{1}{2}-\frac{9}{2\sqrt{5}}}{-\frac{3-\sqrt{5}}{2}}\quad c_2=\frac{1}{\frac{3-\sqrt{5}}{2}}$$

$$\frac{x+3}{1-3x+x^2}=a_1(1-c_1x)^{-1}+a_2(1-c_2x)^{-1}$$

$$=a_1\{1+\binom{1+1-1}{1}(c_1x)+\ldots\ldots+ +(-1)^r\binom{r+1-1}{r}(c_1x)^r+\ldots.\}$$
$$a_2\{1+\binom{1+1-1}{1}(c_2x)+\ldots\ldots+ +(-1)^r\binom{r+1-1}{r}(c_2x)^r+\ldots.\}$$

, we are seeking coefficient of x^{12} is given by,

$$\frac{\frac{1}{2}+\frac{9}{2\sqrt{5}}}{-\frac{3+\sqrt{5}}{2}}(-1)^{12}\binom{12}{12}\left(\frac{2}{3+\sqrt{5}}\right)^{12}+\frac{\frac{1}{2}-\frac{9}{2\sqrt{5}}}{-\frac{3-\sqrt{5}}{2}}(-1)^{12}\binom{12}{12}\left(\frac{2}{3-\sqrt{5}}\right)^{12}$$

$$=(-0.959675)*0.381966)^{12}+(3.95967)*(2.6180339)^{12}\approx 0+3.959675*103682=410547$$

Alternative way, we can use the recurrence relation,

$$\frac{x+3}{1-3x+x^2} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots$$

$$x + 3 = (1 - 3x + x^2)(a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots)$$

$$= a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1 + a_0)x^2 + (a_3 - 3a_2 + a_1)x^3 + \ldots\ldots + \ldots$$

$$a_0 = 3,\ a_1 - 3a_0 = 1,\ a_1 = 10,\ a_2 - 3a_1 + a_0 = 0,\ a_2 = 3*10 - 3 = 27$$

$$in\ general,\ a_n - 3a_{n-1} + a_{n-2} = 0$$

$$a_3 = 3a_2 - a_1 = 3*27 - 10 = 71,\ a_4 = 3a_3 - a_2 = 3*71 - 27 = 186,$$

$$a_5 = 3a_4 - a_3 = 3*186 - 71 = 487, a_6 = 3a_5 - a_4 = 3*487 - 186 = 1275,$$

$$a_7 = 3a_6 - a_5 = 3*1275 - 487 = 3338,\ a_8 = 3a_7 - a_6 = 3*3338 - 1275 = 8739,$$

$$a_9 = 3a_8 - a_7 = 3*8739 - 3338 = 22879,\ a_{10} = 3a_9 - a_8 = 3*22879 - 8739 = 59898,$$

$$a_{11} = 3a_{10} - a_9 = 3*59898 - 22879 = 156815,\ a_{12} = 3a_{11} - a_{10} = 3*156815 - 59898 = 410547,$$

We often need to go through some computations of the generating function to find the required coefficients. The most common situation is multiplication. If

$$h(x)=f(x)*g(x),\ \text{where } f(x)=a_0+a_1x+a_2x^2+\ldots\ldots;\ g(x)=b_0+b_1x+b_2x^2+\ldots\ldots$$

$$\text{then } h(x)=a_0b_0+(a_1b_0+a_0b_1)x+(a_2b_0+a_1b_1+a_0b_2)x^2+\ldots\ldots+(a_rb_0+a_{r-1}b_1+a_{r-2}b_2+\ldots+a_0b_r)x^r+\ldots$$

We give some examples to demonstrate the usefulness of this product formula.

Example 2.2. How many ways are there to get a sum of 25 when 10 dice are rolled?

The generating function is giving by,

$$g(x)=(x+x^2+x^3+x^4+x^5+x^6)^{10}=\left(\frac{x}{1-x}\right)^{10}(1-x^6)^{10}=x^{10}(1-x)^{-10}(1-x^6)^{10}$$

We need coefficient of x^{15} in $(1-x)^{-10}(1-x^6)^{10}$

$$let\ f(x)=(1-x)^{-10} = 1+\binom{1+10-1}{1}x+\binom{2+10-1}{2}x^2+\ldots\ldots+\binom{r+10-1}{r}x^r+\ldots.$$

$$let\ f(x)=(1-x^6)^{10} = 1-\binom{10}{1}x^6+\binom{10}{2}(x^6)^2+\ldots\ldots+(-1)^r\binom{10}{r}(x^6)^r+\ldots.(-1)^n\binom{n}{n}(x^6)^{10}$$

coefficients of x^{15} are $\binom{15+10-1}{15}-\binom{10}{1}\binom{9+10-1}{9}+\binom{10}{2}\binom{3+10-1}{3}=831204$

Example 2.3. How many ways are there to divide five pears, five apples, five doughnuts, five lollipops, five chocolate cats, and five candy rocks into two equal unordered piles?

The generating function of six piles is given by

$$g(x) = (1+x+x^2+x^3+x^4+x^5)^6 = (\frac{1-x^6}{1-x})^6 = (1-x)^{-6}(1-x^6)^6$$

$$= \{1+\binom{1+6-1}{1}x+\binom{2+6-1}{2}x^2+.....\binom{r+6-1}{r}x^r+.....\}\{1-\binom{6}{1}x^6+\binom{6}{2}x^{12}+.....(-1)^r\binom{6}{r}x^{6r}+..+(-1)^6\binom{6}{6}x^{36}\}$$

coefficient of x^{15} are $a_{15}b_0+a_9b_1+a_3b_2=\binom{15+6-1}{15}-\binom{6}{1}\binom{9+6-1}{9}+\binom{6}{2}\binom{3+6-1}{3}=4332$

so left or right pile is $\frac{1}{2}$ of coefficient x^{15}.

We may sometimes meet the situation there are more than two product factors. The next example will give a technique how to handle the Problem.

Example 2.4. Find the coefficient of x^r in $f(x)=(1+x+x^2)^n(1+x)^n$

This generating function can be rewritten as

$f(x)=\left(\frac{1-x^3}{1-x}\right)^n(1+x)^n=(1-x^3)^n(1-x)^{-n}(1+x)^n$, a product of three functions. It is very messy to try to apply the product rule to a multiple product repeatedly. Instead we view the two products as 2n-type factors: $\underset{(1)}{(1+x+x^2)}\underset{(2)}{(1+x+x^2)}......\underset{(n)}{(1+x+x^2)}\underset{(1)}{(1+x)}\underset{(2)}{1+x)}....\underset{(n)}{1+x)}$, we can select k of x^2-type among the first n factors in $\binom{n}{k}$ ways of exponent 2k. Since total exponents are assume r therefore there are r-2k remains for type x and there is 2n-k factor remain to be selected. There is $\binom{2n-k}{r-2k}$ way for type-x factor to be selected. The coefficient of x^r will be $a_r=\sum_{k=\max(r-2n,0)}^{n}\binom{n}{k}\binom{2n-k}{r-2k}$

A partition of a group of r identical objects divides the group into a collection of unordered subsets of various sizes. Analogously we define a partition of the integer r to be a collection of positive integers whose sum is r.

Usually we write this collection as a sum and list the integers of the partition in increasing order. We will give an example to illustrate the application.

Example 2.5. Find a generating function $g(x,y,z)$ whose coefficient of $x^ry^sz^t$ is the number of ways n people can each pick two different fruits from a bowl of apples, oranges, and bananas for a total of r apples, s oranges, and t bananas. For each person, we consider $xy+yz+zx$. Hence n people

we have the generating function $(xy+yz+zx)^n$. Let us assume that the coefficient of $x^r y^s z^t$ is a_n and partition of a positive integer n has $e_1+e_2+....+e_k=n$ where $1 \le e_1 \le e_2 + ... \le e_k$. Except a recurrence relation we will prove in the appendix, there is no general rule to help us to count the number of ways to partitions a positive integer. We give a list of the number of ways to partition the first eight positive integer in the tables below.

Table 2.1

N	1	2	3	4	5	6	7	8
R	1	2	3	5	7	11	15	22

For example, n=8, there are 22 ways to partition 8. We list
Them as follow: 8,7+1,6+1+1,5+1+1+1,4+1+1+1+1,3+1+1+1+1+1,2+1+1+1+1+1+1,
1+1+1+1+1+1+1+1,6+2,5+2+1,4+2+1+1,3+2+1+1+1,2+2+1+1+1+1,
5+3,4+3+1,3+3+1+1,2+2+2+1+1,4+4,4+2+2,3+2+2+1,3+3+2,
2+2+2+2.

3. Concluding Remarks

The concept of a generating function is a convenient tool for handling special constraints in selection and arrangement problems. They are a simpleminded and a sophisticated mathematical model for counting problems; simpleminded because polynomial multiplication is a familiar, well-understood part of high school algebra and sophisticated since with standard algebraic manipulations on generating function, we can solve complicated counting problems that usually cannot be solved by the combinatorial arguments. These algebraic manipulations automatically perform the correct combinatorial reasoning for us. In this paper, we first use the regular hand calculator to compute data. Then we recomputed data by R-package.

4. References

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5. Appendix

Let $R(r,k)$ denote the number of partitions of the integer r into k parts. Show that

$R(r,k)=R(r-1,k-1)+R(r-k,k)$

$e_1 + e_2 + \ldots. + e_k = r$ where $1 \le e_1 \le e_2 + \ldots. \le e_k$

Case 1. Assume $e_1=1$ and $e_2 + \ldots. + e_k = r-1$ where $1 \le e_2 + \ldots. \le e_k$

Hence there are $R(r-1,k-1)$ ways.

Case 2. $e_1>1$, let $e_1^{/}=e_1-1$, $e_2^{/}=e_2-1$, ……$e_k^{/}=e_k-1$,

then $e_1^{/} + .e_2^{/} \ldots + e_k^{/} = r-k$ where $1 \le e_1^{/} + e_2^{/} \ldots. \le e_k^{/}$

Hence there are $R(r-k,k)$ ways.

We further breakdown table 2.1, for each given number r

K: number of parts.

r: Given number

k: Number of parts.

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k

r\\k	1	2	3	4	5	6	7	8	Row sum
1	1								1
2	1	1							2
3	1	1	1						3
4	1	2	1	1					5
5	1	2	2	1	1				7
6	1	3	3	2	1	1			11
7	1	3	4	3	2	1	1		15
8	1	4	5	5	3	2	1	1	22