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On Small Combination of Slices in Banach Spaces

Sudeshna Basu, T. S. S. R. K. Rao

Department of Mathematics, George Washington University,
Washington DC 20052, USA

sbasu@gwu.edu, sudeshna66@gmail.com

Stat-Math Division, Indian Statistical Institute, R. V. College,
P. O. Bangalore 560059, India

tss@isibang.ac.in, srin@fulbrightmail.org

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Abstract: The notion of Small Combination of Slices (SCS) in the unit ball of a Banach space was first introduced in [4] and subsequently analyzed in detail in [12] and [13]. In this work, we introduce the notion of BSCSP, which can be seen as a generalization of dentability in terms of SCS. We study certain stability results for the w^*-BSCSP leading to a discussion on BSCSP in the context of ideals of Banach spaces. We prove that the w^*-BSCSP can be lifted from a M-ideal to the whole Banach Space. We also prove similar results for strict ideals and U-subspaces of a Banach space. We note that the space $C(K, X)^*$ has w^*-BSCSP when K is dispersed and X^* has the w^*-BSCSP.

Key words: Small combination of slices, M-Ideals, Strict ideals, U-Subspaces.

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1. Introduction

Let X be a real Banach space and X^* its dual. We will denote by B_X, S_X and $B_X(x, r)$ the closed unit ball, the unit sphere and the closed ball of radius $r > 0$ and center x. We refer to the monograph [2] for notions of convexity theory that we will be using here.

Definition 1. (i) We say $A \subseteq B_{X^*}$ is a norming set for X if $\|x\| = \sup\{|x^*(x)| : x^* \in A\}$, for all $x \in X$. A closed subspace $F \subseteq X^*$ is a norming subspace if B_F is a norming set for X.
(ii) Let $f \in X^*$, $\alpha > 0$ and $C \subseteq X$. Then the set

$$S(C, f, \alpha) = \{x \in C : f(x) > \sup f(C) - \alpha\}$$

is called the open slice determined by f and α. We assume without loss of generality that $\|f\| = 1$. One can analogously define w^* slices in X^*

(iii) A point $x \neq 0$ in a convex set $K \subseteq X$ is called a SCS (small combination of slices) point of K, if for every $\varepsilon > 0$, there exist slices S_i of K, and a convex combination $S = \sum_{i=1}^{n} \lambda_i S_i$ such that $x \in S$ and $\text{diam}(S) < \varepsilon$. One can analogously define w^*-SCS point in X^*.

We introduce the following definition analogous to that of a unit ball being dentable, see [2].

Definition 2. A Banach Space is said to have Ball-Small Combination of Slices Property (BSCSP) if the unit ball has small combination of slices of arbitrarily small diameter. Analogously we can define w^*-BSCSP in a dual space.

Remark 3. (i) It is clear that if B_X has a SCS point, then it has BSCSP.
(ii) Strongly Regular spaces studied in [4] and [13] were referred to as Small Combination of Slices Property (SCSP) in [12].

SCS points were first introduced in [4] as a "slice generalization" of the notion PC (i.e. points for which the identity mapping on the unit ball, from weak topology to norm topology is continuous). It was proved in [4] that X is strongly regular (respectively, X^* is w^*-strongly regular) if and only if every non empty bounded convex set K in X (respectively K in X^*) is contained in the norm closure (respectively, w^*-closure) of $SCS(K)$ (respectively w^*-SCS(K)), i.e. the SCS points (w^*-SCS points) of K. Later, it was proved in [13] that Banach space has Radon Nikodym Property (RNP) if and only if it is strongly regular and has the Krein-Milman Property (KMP). Subsequently, the concept of SCS points was used in [12] to investigate the structure of non dentable closed bounded convex sets in Banach spaces. In this work, we study certain stability results for w^*-BSCSP leading to a discussion on BSCSP in the context of ideals of Banach spaces, see [5] and [12]. We use various techniques from the geometric theory of Banach spaces to achieve this. The spaces that we will be considering have been well studied in the literature. A large class of function spaces like the Bloch spaces, Lorentz and Orlicz spaces, spaces of vector-valued functions and spaces of compact operators are examples of the spaces we will be considering: for details, see [6]. We provide some descriptions of w^*-SCS points in Banach spaces in different contexts. We need the following definition.

DEFINITION 4. Let X be a Banach space.

(i) A linear projection P on X is called an M-*projection* if

$$\|x\| = \max\{\|Px\|, \|x - Px\|\},$$

for all $x \in X$; A linear projection P on X is called an L-*projection* if

$$\|x\| = \|Px\| + \|x - Px\|$$

for all $x \in X$.

(ii) A subspace $M \subseteq X$ is called an M-summand if it is the range of an M-projection. A closed subspace $M \subseteq X$ is called an L-summand if it is the range of an L-projection.

(iii) A subspace $M \subseteq X$ is called an M-ideal if $M^\perp$ is the kernel of an L-projection in X^*

We recall from [6, Chapter I] that when $M \subset X$ is an M-ideal, elements of M^* have unique norm-preserving extension to X^* and one has the identification, $X^* = M^* \oplus_1 M^\perp$. Several examples from among function spaces and spaces of operators that satisfy these geometric properties can be found in the monograph [6], see also [8]. First, we prove that for an L-summand $M \subset X$, if a SCS point of B_X has a non-zero component $m \in M$, then m is a SCS point of B_M. For an M- ideal $M \subset X$, this yields: any w^*-SCS point of B_{X^*}, if its restriction to M, say m^*, has the same norm, then m^* it is a w^*-SCS point of B_{M^*} . We prove a similar result for a U-subspace of a Banach space of X. We prove a converse statement for a strict ideal $Y \subset X$ (see Section 2 for the definition) i.e., we prove that a w^*-SCS point of a strict ideal continues to be so in the bigger space. We also prove corresponding results for the BSCSP.

2. Stability results

We will use the standard notation of $\oplus_1$, $\oplus_\infty$ to denote the ℓ^1 and ℓ^∞-direct sum of two or more Banach spaces.

PROPOSITION 5. *Suppose X, Y, Z are Banach spaces such that $Z = X \oplus_1 Y$; suppose $z_0 = (x_0, y_0) \in B_Z$ is a SCS point of B_Z with both the components non-zero, then x_0 and y_0 are SCS points of B_X and B_Y respectively.*

Proof. Since z_0 is a SCS point of B_Z, we have for any $\varepsilon > 0$, $z_0 = \sum_{i=1}^n \lambda_i z_i$, where $z_i \in S_i$ and for $z_i^* = (x_i^*, y_i^*)$ with $1 = \|z_i^*\| = \max\{\|x_i^*\|, \|y_i^*\|\}$, $S_i = \{z \in B_Z / z_i^*(z) > 1 - \varepsilon_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i) < \varepsilon$,

$$S_i = \{z \in B_Z / z_i^*(x, y) > 1 - \varepsilon_i\} = \{z \in B_Z / x_i^*(x) + y_i^*(y) > 1 - \varepsilon_i\}.$$

Since $z_i = (x_i, y_i) \in S_i$, then $x_i^*(x_i) + y_i^*(y_i) > 1 - \varepsilon_i$.

Case 1 : $\|z_i^*\| = \|x_i^*\| = 1$. Then,

$$\begin{aligned} x_i^*(x_i) + y_i^*(y_i) > 1 - \varepsilon_i = \|x_i^*\| - \varepsilon_i, \\ &\implies x_i^*(x_i) > \|x_i^*\| - \varepsilon_i - y_i^*(y_i), \\ &\implies 1 \geq x_i^*(x_i) > \|x_i^*\| - \beta_i, \text{ where } \beta_i = \varepsilon_i + y_i^*(y_i), \\ &\implies \varepsilon_i + y_i^*(y_i) > 0. \end{aligned}$$

So we have, $x_i \in S_{iX} = \{x \in B_X / x_i^*(x) > 1 - \beta_i\}$. Then $(x_i, y_i) \in S_{iX} \times \{y_i\} \subseteq S_i$.

Case 2: $\|z_i^*\| = \|y_i^*\| = 1$. We may assume that $0 < \|x_i^*\| < 1$, and let $\delta_i = \|y_i^*\| - \|x_i^*\|$. Then,

$$\begin{aligned} x_i^*(x_i) + y_i^*(y_i) > 1 - \varepsilon_i = \|y_i^*\| - \varepsilon_i = \|x_i^*\| + \delta_i - \varepsilon_i \\ &\implies x_i^*(x_i) > \|x_i^*\| + \delta_i - \varepsilon_i - y_i^*(y_i), \\ &\implies \|x_i^*\| \geq x_i^*(x_i) > \|x_i^*\| - r_i, \text{ where } r_i = \delta_i - \varepsilon_i - y_i^*(y_i) > 0, \\ &\implies x_i \in S_{iX} = \{x \in B_X / x_i^*(x) > 1 - r_i\}. \end{aligned}$$

Then $(x_i, y_i) \in S_{iX} \times \{y_i\} \subseteq S_i$.

Let $x_0 = \sum_{i=1}^n \lambda_i x_i$ and $y_0 = \sum_{i=1}^n \lambda_i y_i$. Now $x_0 \in \sum_{i=1}^n \lambda_i S_{iX}$. Also,

$$\begin{aligned} \sum_{i=1}^n \lambda_i [S_{iX} \times y_i] \subseteq \sum_{i=1}^n \lambda_i S_i, \\ &\implies \sum_{i=1}^n \lambda_i [S_{iX}] \times \{y_0\} \subseteq \sum_{i=1}^n \lambda_i [S_{iX} \times y_i] \subseteq \sum_{i=1}^n \lambda_i S_i, \\ &\implies \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_{iX}\right) < \varepsilon, \\ &\implies x_0 \text{ is a SCS point of } B_X. \end{aligned}$$

Similarly it follows that y_0 is a SCS point of B_Y. ∎

Arguments similar to the ones given above in the context of a ℓ^∞-sum yield the following corollary.

Corollary 6. *Suppose X, Y, Z are Banach spaces such that $Z = X \oplus_\infty Y$, suppose $z^* = (x^*, y^*) \in B_{Z^*}$ is a w^*-SCS point of B_{Z^*} with both the components non-zero, then x^* and y^* are w^*-SCS points of B_{X^*} and B_{Y^*} respectively.*

Remark 7. Since in the sequence space ℓ^∞ any weakly open set has norm diameter 2, by taking $X = c_0$ and $Y = \ell^1$, $Z = X \oplus_\infty Y$, any w^*-SCS point of B_{Z^*} has its second component 0. We thank the referee for this observation.

Definition 8. We recall that a closed subspace Y of a Banach space X is called a U-subspace if for $y^* \in Y^*$ there exists a unique norm preserving extension of y^* in X^*. We continue to denote the unique extension also by y^*.

See the discussion on [6, page 44] and the references in that monograph for several examples of U-subspaces from among classical function spaces and spaces of operators.

Before the next result we also need a definition from [5]. See also [11] for more information and several examples from spaces of operators and tensor product spaces.

Definition 9. A closed subspace Y of a Banach Space X is said to be an ideal of X if there is a linear projection $P : X^* \to X^*$ of norm one such that $\ker(P) = Y^\perp$.

For $x^* \in X^*$ since $P(x^*) - x^* = 0$ on Y, as $\|P\| = 1$, we see that $P(x^*)$ is a norm-preserving extension of $x^*|Y$.

Theorem 10. *Suppose Y is an ideal which is also a U-subspace of X. If $y^* \in S_{Y^*}$ is a w^*-SCS point of B_{X^*}, then y^* is a w^*-SCS point of B_{Y^*}.*

Proof. Let $y_0^* \in S_{Y^*}$ be a w^*-SCS point of B_{X^*} , hence for any $\varepsilon > 0$ there exist w^* slices S_i, $0 \le \lambda_i \le 1$, $i = 1, 2, \ldots, n$, $S_i = \{x^* \in B_{X^*}/x^*(x_i) > 1 - \alpha_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i) < \varepsilon$ and $y_0^* = \sum \lambda_i x_{0i}^*$. Since $y_0^* \in S_{Y^*}$ and Y is a U-subspace, y_0^* has unique norm preserving extension in X^*. Let $P : X^* \longrightarrow X^*$ be the canonical projection. Then $\|P(y_0^*)\| = \|y_0^*\| = 1$, Also,

$$1 = \|y_0^*\| = \left\| \sum_{i=1}^n \lambda_i x_{0i}^* \right\| \le \sum_{i=1}^n \lambda_i \|P(x_{0i}^*)\| \le 1.$$

This implies $\|P(x_{0i}^*)\| = \|x_{0i}{}^*\| = 1$ for all $i = 1, \ldots, n$. Thus by hypothesis, $P(x_{0i}^*)$ and the restriction of x_{0i}^* to Y are denoted by y_{0i}^*. Now $y_{0i}^* \in S_i$, then $y_{0i}^*(x_i) > 1 - \alpha_i$. Also, since Y is an ideal, there exists an operator $T : \text{span}\{x_i\} \longrightarrow Y$ such that $\|T(x_i)\| \leq (1+\varepsilon)\|x_i\| = 1 + \varepsilon$.

Let $y_i = T(x_i)$. Hence,

$$\begin{aligned} y_{0i}^*(x_i) > 1 - \alpha_i &\implies y_{0i}^*(y_i - y_i + x_i) > 1 - \alpha_i, \\ &\implies y_{0i}^*(y_i) + y_{0i}^*(x_i - y_i) > 1 - \alpha_i, \\ &\implies y_{0i}^*(y_i) > 1 - \alpha_i - y_{0i}^*(x_i - y_i). \end{aligned}$$

Case 1: $\|y_i\|$ =1. So we have

$$\begin{aligned} 1 > y_{0i}^*(y_i) > 1 - \alpha_i - y_{0i}^*(x_i - y_i) = 1 - \beta_i, \\ \implies y_{0i}^* \in S_{iY} = \{y^* \in B_{Y^*} / y^*(y_i) > 1 - \beta_i\}. \end{aligned}$$

Case 2: $\|y_i\| < 1$. Let $\|y_i\| = 1 - \delta_i$. Then

$$\begin{aligned} \|y_i\| > y_{0i}^*(y_i) > \|y_i\| + \delta_i - \beta_i = \|y_i\| - (\beta_i - \delta_i) = \|y_i\| - \gamma_i, \gamma_i > 0, \\ \implies y_{0i}^* \in S_{iY} = \{y^* \in B_{Y^*} / y^*(y_i) > \|y_i\| - \gamma_i\}. \end{aligned}$$

Case 3: $\|y_i\| = 1 + \delta_i$. Then

$$\begin{aligned} 1 + \delta_i > y_{0i}^*(y_i) > 1 - \beta_i = 1 + \delta_i - (\beta_i + \delta_i), \\ \implies y_{0i}^* \in S_{iY} = \{y^* \in B_{Y^*} / y^*(y_i) > \|y_i\| - (\beta_i + \delta_i\}. \end{aligned}$$

Hence

$$y_0^* = \sum_{i=1}^n \lambda_i y_{0i}^* \in \sum_{i=1}^n \lambda_i S_{iY} \subseteq \sum_{i=1}^n \lambda_i S_i.$$

Hence

$$\operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_{iY}\right) < \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i\right) < \varepsilon.$$

Thus y_0^* is w^*-SCS point of B_{Y^*}. ∎

Let $M \subseteq X$ be an M-ideal. It follows from the results in [6, Chapter I] that any $x^* \in X^*$, if $\|m^*\| = \|x^*|_M\| = \|x^*\|$, then x^* is the unique norm preserving extension of m^*. For notational convenience we denote both the functionals by m^*. Clearly any M-ideal is also an ideal. Thus the following corollary answers a natural question in this context for w^*-SCS points of the unit sphere. We omit its easy proof.

Corollary 11. *Suppose $M \subseteq X$ is a M-ideal in X. If $m^* \in S_{X^*}$ is w^*-SCS point of B_{X^*}, then $m^* \in S_{M^*}$ is a w^*-SCS point of B_{M^*}.*

Remark 12. The referee has kindly pointed out an independent proof to show that for $Z = X \oplus_1 Y$, Z has the BSCSP if and only if X or Y has the BSCSP.

Arguments similar to the ones given during the proof of Proposition 5 can be used to show that for $Z = X \oplus_\infty Y$, if X^* or Y^* has the w^*-BSCSP then so does Z^*.

In the case of an M-ideal $M \subset X$, for the sake of completeness we give a detailed proof of the following result.

Proposition 13. *Let $M \subseteq X$ be a M-ideal, then if M^* has the w^*-BSCSP then X^* has the w^*-BSCSP.*

Proof. Suppose M^* has the w^*-BSCSP, then for any $\varepsilon > 0$ there exists slices S_{iM} and $0 \leq \lambda_i \leq 1, i = 1, 2, \ldots, n$, $S_{iM} = \{m^* \in B_{M^*}/m^*(m_i) > 1 - \alpha_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_{iM}) < \varepsilon$. Since M is an M- ideal, for any $x^* \in X^*$ we have the unique decomposition, $x^* = m^* + m^\perp$, where $m^* \in M^*$ and $m^\perp \in M^\perp$. Suppose we have $0 < \mu_i < \alpha_i$. Then

$$\begin{aligned} S_{iX} &= \{x^* \in B_{X^*}/x^*(m_i) > 1 - \mu_i\} \\ &= \{x^* \in B_{X^*}/m^*(m_i) + m^\perp(m_i) > 1 - \mu_i\}, \\ &\subseteq S_{iM} \times \mu_i B_{M^\perp}, \\ &\implies \sum_{i=1}^n \lambda_i S_{iX} \subseteq \sum_{i=1}^n \lambda_i S_{iM} \times \mu_i B_{M^\perp}. \end{aligned}$$

Choose $\beta_i = \min(\mu_i, \varepsilon)$. Then

$$\begin{aligned} S'_{iX} &= \{x^* \in B_{X^*}/x^*(m_i) > 1 - \beta_i\} \subseteq S_{iX} \times \beta_i B_{M^\perp}, \\ &\implies \sum_{i=1}^n \lambda_i S'_{iX} \subseteq \left(\sum_{i=1}^n \lambda_i S_{iM} \times \beta_i B_{M^\perp} \right) \\ &\implies \sum_{i=1}^n \lambda_i S'_{iX} \subseteq \left(\sum_{i=1}^n \lambda_i S_{iM} \times \beta_i B_{M^\perp} \right). \end{aligned}$$

Thus $\operatorname{diam}(\sum_{i=1}^n \lambda_i S'_{iX}) \leq \operatorname{diam}(\sum_{i=1}^n \lambda_i S_{iM}) + 2\varepsilon < \varepsilon + 2\varepsilon = 3\varepsilon$. Also, since $\|m_i\| = 1$, there exists $m_i^* \in B_{M^*}$ such that $m_i^*(m_i) > 1 - \beta_i$. Hence $m_i^* \in S'_{iX}$. Similarly, $\sum_{i=1}^n \lambda_i m_i^* \in \sum_{i=1}^n \lambda_i S'_{iX} \implies \sum_{i=1}^n \lambda_i S'_i \neq \emptyset$. ∎

Since any summand in a ℓ^∞-direct sum is in particular an M-ideal of the sum, the following corollary is easy to prove.

Corollary 14. *Suppose $X = \oplus_{\ell^\infty} X_i$. If X_i^* has the w^*-BSCSP for some i, then X^* has the w^*-BSCSP.*

The above arguments extend easily to vector-valued continuous functions. We recall that for a compact Hausdorff space K, $C(K,X)$ denotes the space of continuous X-valued functions on K, equipped with the supremum norm. We recall from [9] that dispersed compact Hausdorff spaces have isolated points.

Corollary 15. *Suppose K is a compact Hausdorff space with an isolated point. If X^* has the w^*-BSCSP, then $C(K,X)^*$ has the w^*-BSCSP.*

Proof. Suppose X^* has the w^*-BSCSP. For an isolated point $k_0 \in K$, the map $F \to \chi_{k_0} F$ is an M-projection in $C(K,X)$ whose range is isometric to X. Hence we see that $C(K,X)^*$ has the w^*-BSCSP. ∎

We recall that an ideal Y is said to be a strict ideal if for a projection $P : X^* \to X^*$ with $\|P\| = 1$, $\ker(P) = Y^\perp$, one also has $B_{P(X^*)}$ is w^*-dense in B_{X^*} or in other words $B_{P(X^*)}$ is a norming set for X.

In the case of an ideal also one has that Y^* embeds (though there may not be uniqueness of norm-preserving extensions) as $P(X^*)$. Thus we continue to write $X^* = Y^* \oplus Y^\perp$. In what follows we use a result from [11], that identifies strict ideals as those for which $Y \subset X \subset Y^{**}$ under the canonical embedding of Y in Y^{**}.

Proposition 16. *Suppose Y is a strict ideal of X. If $y^* \in B_{Y^*}$ is a w^*-SCS point of B_{Y^*}, then y^* is a w^*-SCS point of B_{X^*}.*

Proof. Since $y^* \in B_{Y^*}$ is a w^*-SCS point of B_{Y^*}, for any $\varepsilon > 0$ there exists w^* slices S_i and $0 \le \lambda_i \le 1$, $i = 1, 2, \ldots, n$, $S_i = \{y^* \in B_{Y^*} / y^*(y_i) > 1 - \alpha_i\}$ and $\operatorname{diam}(\sum_{i=1}^n \lambda_i S_i) < \varepsilon$. Since Y is a strict ideal in X, we have $B_{X^*} = \overline{B_{Y^*}}^{w^*}$, hence we have the following:

$$S_i' = \{x^* \in B_{X^*} / x^*(x_i) > 1 - \alpha_i\} = \{x^* \in \overline{B_{Y^*}}^{w^*} / x^*(x_i) > 1 - \alpha_i\},$$

$$\implies \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i'\right) \subseteq \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i\right) < \varepsilon,$$

$$\implies \operatorname{diam}\left(\sum_{i=1}^n \lambda_i S_i'\right) < \varepsilon.$$

Hence y^* is a w^*-SCS point of B_{Y^*}. ∎

Arguing similarly it follows that:

Proposition 17. *Suppose Y is a strict ideal of X. If Y^* has w^*-BSCSP then X^* has w^*-BSCSP.*

Remark 18. A prime example of a strict ideal is a Banach space X under its canonical embedding in X^{**}. It is known that any w^*-denting point of $B_{X^{**}}$ is a point of X. Now let $x^{**} \in B_{X^{**}}$ be a w^*-SCS point. The referee has kindly pointed out that since B_X is weak* dense in $B_{X^{**}}$, for any $\epsilon > 0$, there is a convex combination $\sum_{i=1}^n \lambda_i x_i$ of vectors in X so that $\|x^{**} - \sum_{i=1}^n \lambda_i x_i\| \le \epsilon$. Hence $x^{**} \in X$.

We conclude the paper with a set of remarks and questions. See also the recent paper [1] for other possible geometric connections. Let us consider the following densities of w^*-SCS points of B_{X^*}.

(i) All points of S_{X^*} are w^*-SCS points of B_{X^*}.

(ii) The w^*-SCS points of B_{X^*} are dense in S_{X^*}.

(iii) B_{X^*} is contained in the closure of w^*-SCS points of B_{X^*}.

(iv) B_{X^*} is the closed convex hull of w^*-SCS points of B_{X^*}.

(v) X^* is the closed linear span of w^*-SCS points of B_{X^*}.

Questions:

(i) How can each of these properties be realized as a ball separation property considered in [3]?

(ii) What stability results will hold for these properties?

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