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Weyl Type Theorems for Restrictions of Bounded Linear Operators

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Abstract: In this paper we give sufficient conditions for which Weyl's theorems for a bounded linear operator T, acting on a Banach space X, can be reduced to the study of Weyl's theorems for some restriction of T.

Key words: Weyl's theorem, a-Weyl's theorem, semi-Fredholm operator, pole of the resolvent.

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1. Introduction

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in L(X)$, we denote by $N(T)$ the null space of T and by $R(T) = T(X)$ the range of T. We denote by $\alpha(T) := \dim N(T)$ the nullity of T and by $\beta(T) := \operatorname{codim} R(T) = \dim X/R(T)$ the defect of T. Other two classical quantities in operator theory are the *ascent* $p = p(T)$ of an operator T, defined as the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$ (if such an integer does not exist, we put $p(T) = \infty$), and the *descent* $q = q(T)$, defined as the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$ (if such an integer does not exist, we put $q(T) = \infty$). It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Furthermore, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent, see [12, Proposition 50.2]. An operator $T \in L(X)$ is said to be *Fredholm* (respectively, *upper semi -Fredholm*, *lower semi-Fredholm*), if $\alpha(T)$, $\beta(T)$ are both finite (respectively, $R(T)$ closed and $\alpha(T) < \infty$, $\beta(T) < \infty$). $T \in L(X)$ is said to be *semi-Fredholm* if T is either an upper semi-Fredholm or a lower semi-Fredholm operator. If T is semi-Fredholm the *index* of T defined by $\operatorname{ind} T := \alpha(T) - \beta(T)$. Other two important classes of operators in Fredholm theory are the classes of semi-Browder operators. These classes are defined as follows, $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if T is a Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm) and both $p(T)$, $q(T)$ are finite (respectively, $p(T) < \infty$, $q(T) < \infty$). A bounded operator $T \in L(X)$ is said to be *upper semi-Weyl* (respectively, *lower semi-Weyl*) if T is upper Fredholm operator (respectively, lower semi-Fredholm) and index $\operatorname{ind} T \leq 0$ (respectively, $\operatorname{ind} T \geq 0$). $T \in L(X)$ is said to be *Weyl* if T is both upper and lower semi-Weyl, i.e. T is a Fredholm operator having index 0. The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\},$$

and

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

Since every Browder operator is Weyl then $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Analogously, The *upper semi-Browder spectrum* and the *upper semi-Weyl spectrum* are defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$

and

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\}.$$

In the sequel we need the following basic result:

LEMMA 1.1. *If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:*

(i) *There exists $n \geq p + 1$ such that $T^n(X)$ is closed;*

(ii) *$T^n(X)$ is closed for all $n \geq p$.*

Proof. Define $c'_i(T) := \dim(N(T^i)/N(T^{i+1}))$. Clearly, $p = p(T) < \infty$ entails that $c'_i(T) = 0$ for all $i \geq p$, so $k_i(T) := c'_i(T) - c'_{i+1}(T) = 0$ for all $i \geq p$. The equivalence easily follows from [13, Lemma 12]. ∎

Now, we introduce an important property in local spectral theory. The localized version of this property has been introduced by Finch [11], and in the framework of Fredholm theory this property has been characterized in several ways, see [1, Chapter 3]. A bounded operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at

λ_0), if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0 the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \qquad \text{for all } \lambda \in \mathbb{D}_{\lambda_0}\,,$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \qquad \Rightarrow \qquad T \text{ has SVEP at } \lambda\,, \tag{1.1}$$

and dually

$$q(\lambda I - T) < \infty \qquad \Rightarrow \qquad T^* \text{ has SVEP at } \lambda\,. \tag{1.2}$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum* defined by

$$\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}\,.$$

Note that if $\sigma_{\mathrm{s}}(T)$ denotes the *surjectivity spectrum*

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}\,,$$

then $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*)$ and $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ap}}(T^*)$.

It is easily seen from definition of localized SVEP that

$$\lambda \notin \operatorname{acc} \sigma_{\mathrm{ap}}(T) \qquad \Rightarrow \qquad T \text{ has SVEP at } \lambda\,, \tag{1.3}$$

where $\operatorname{acc} K$ means the set of all accumulation points of $K \subseteq \mathbb{C}$, and if T^* denotes the dual of T, then

$$\lambda \notin \operatorname{acc} \sigma_{\mathrm{s}}(T) \qquad \Rightarrow \qquad T \text{ has SVEP at } \lambda\,. \tag{1.4}$$

Remark 1.2. The implications (1.1), (1.2), (1.3) and (1.4) are actually equivalences whenever $T \in L(X)$ is semi-Fredholm (see [1, Chapter 3]).

Denote by $\operatorname{iso} K$ the set of all isolated points of $K \subseteq \mathbb{C}$. Let $T \in L(X)$, define

$$\pi_{00}(T) = \{\lambda \in \operatorname{iso} \sigma(T) \,:\, 0 < \alpha(\lambda I - T) < \infty\},$$
$$\pi_{00}^{a}(T) = \{\lambda \in \operatorname{iso} \sigma_{\mathrm{ap}}(T) \,:\, 0 < \alpha(\lambda I - T) < \infty\}.$$

Clearly, for every $T \in L(X)$ we have $\pi_{00}(T) \subseteq \pi_{00}^{a}(T)$.

Let $T \in L(X)$ be a bounded operator. Following Coburn [8], T is said to satisfy *Weyl's theorem*, in symbol (W), if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. According to Rakočević [15], T is said to satisfy *a-Weyl's theorem*, in symbol (aW), if $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^{a}(T)$.

Note that

$$a\text{-Weyl's theorem} \quad \Rightarrow \quad \text{Weyl's theorem},$$

see for instance [1, Chapter 3]. The converse of these implication in general does not hold.

Weyl type theorems have been recently studied by several authors ([2], [3], [5], [6], [8], [9], [10], [15] and [16]). In these papers several results are obtained, by considering an operator $T \in L(X)$ in the whole space X. In this paper we give sufficient conditions for which Weyl type theorems holds for T, if and only if there exists $n \in \mathbb{N}$ such that the range $R(T^n)$ of T^n is closed and Weyl type theorems holds for T_n, where T_n denote the restriction of T on the subspace $R(T^n) \subseteq X$.

2. Preliminaries

In this section we establish several lemmas that will be used throughout the paper. We begin examinig some algebraic relations between T and T_n, T_n viewed as a operator from the space $R(T^n)$ in to itself.

LEMMA 2.1. *Let $T \in L(X)$ and T_n, $n \in \mathbb{N}$, be the restriction of the operator T on the subspace $R(T^n) = T^n(X)$. Then, for all $\lambda \neq 0$, we have:*

(i) $N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$, *for any m;*

(ii) $R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n)$, *for any m;*

(iii) $\alpha(\lambda I - T_n) = \alpha(\lambda I - T)$;

(iv) $p(\lambda I - T_n) = p(\lambda I - T)$;

(v) $\beta(\lambda I - T_n) = \beta(\lambda I - T)$.

Proof. (i) For $m = 0$,

$$N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$$

holds trivially. Let $x \in N((\lambda I - T)^m)$, $m \geq 1$, then

$$\begin{aligned} 0 \;=\; (\lambda I - T)^m x \;&=\; \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}(-1)^k \lambda^{m-k} T^k x \\ &= \lambda^m x + \sum_{k=1}^{m} \frac{m!}{k!(m-k)!}(-1)^k \lambda^{m-k} T^k x\,. \end{aligned}$$

Thus $0 = \lambda^m x + h(T)x$, where

$$h(T) = \sum_{k=1}^{m} \frac{m!}{k!(m-k)!}(-1)^k \lambda^{m-k} T^k.$$

Hence $-\lambda^m x = h(T)x$, and since $\lambda \neq 0$, then $x = -\lambda^{-m}h(T)x$. From this equality, it follows that

$$\begin{aligned} (-\lambda^{-m}h(T))^2 x \;&=\; -\lambda^{-m}h(T)(-\lambda^{-m}h(T)x) \\ &= \; -\lambda^{-m}h(T)x \;=\; x\,. \end{aligned}$$

Consequently $x = (-\lambda^{-m}h(T))^2 x$. By repeating successively the same argument, we obtain that $x = (-\lambda^{-m}h(T))^j x$, for all $j \in \mathbb{N}$. But since $-\lambda^{-m}h(T)x \in R(T)$, then $(-\lambda^{-m}h(T))^j x \in R(T^j)$, for all $j \in \mathbb{N}$. Therefore $x = (-\lambda^{-m}h(T))^n x \in R(T^n)$, and since $R(T^n)$ is T-invariant subspace, we conclude that

$$\begin{aligned} 0 \;=\; (\lambda I - T)^m x \;&=\; \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}(-1)^k \lambda^{m-k} T^k x \\ &= \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}(-1)^k \lambda^{m-k} (T_n)^k x \;=\; (\lambda I - T_n)^m x\,. \end{aligned}$$

So $x \in N((\lambda I - T_n)^m)$, and we get the inclusion

$$N((\lambda I - T)^m) \subseteq N((\lambda I - T_n)^m)\,.$$

On the other hand, since T_n is the restriction of T on $R(T^n)$, and $R(T^n)$ is invariant under T, it then follows the inclusion

$$N((\lambda I - T_n)^m) \subseteq N((\lambda I - T)^m)\,.$$

From which, we obtain that $N((\lambda I - T_n)^m) = N((\lambda I - T)^m)$.

(ii) Since T_n is the restriction of T on $R(T^n)$, and $R(T^n)$ is invariant under T, then

$$R((\lambda I - T_n)^m) \subseteq R((\lambda I - T)^m) \cap R(T^n)\,.$$

Now, we show the inclusion $R((\lambda I - T)^m) \cap R(T^n) \subseteq R((\lambda I - T_n)^m)$. For this, it will suffice to show that for $m \in \mathbb{N}$, the implication

$$(\lambda I - T)^m x \in R(T^n) \qquad \Rightarrow \qquad x \in R(T^n)\,,$$

holds. For $m = 1$. Let $y \in R(\lambda I - T) \cap R(T^n)$, then there exists $x \in X$ such that $\lambda x - Tx = (\lambda I - T)x = y \in R(T^n)$, so $\lambda^2 x - \lambda Tx = \lambda y \in R(T^n)$. But since $\lambda Tx - T^2 x = Ty \in R(T^n)$, because $\lambda x - Tx = y$ and $R(T^n)$ is invariant under T, we have that $\lambda^2 x - \lambda Tx$, $\lambda Tx - T^2 x \in R(T^n)$. Then

$$\lambda^2 x - T^2 x = \lambda^2 x - \lambda Tx + \lambda Tx - T^2 x \in R(T^n)\,.$$

Thus $\lambda^2 x - T^2 x \in R(T^n)$. Hence $\lambda^3 x - \lambda T^2 x = \lambda(\lambda^2 x - T^2 x) \in R(T^n)$, and since $\lambda T^2 x - T^3 x = T^2 y \in R(T^n)$, we have that $\lambda^3 x - \lambda T^2 x$, $\lambda T^2 x - T^3 x \in R(T^n)$. From which,

$$\lambda^3 x - T^3 x = \lambda^3 x - \lambda T^2 x + \lambda T^2 x - T^3 x \in R(T^n)\,.$$

That is, $\lambda^3 x - T^3 x \in R(T^n)$. Now, suppose that $\lambda^j x - T^j x \in R(T^n)$, for some $j \in \mathbb{N}$. From this, $\lambda^{j+1} x - \lambda T^j x = \lambda(\lambda^j x - T^j x) \in R(T^n)$, and $\lambda T^j x - T^{j+1} x = T^j y \in R(T^n)$, thus $\lambda^{j+1} x - \lambda T^j x$, $\lambda T^j x - T^{j+1} x \in R(T^n)$. From which,

$$\lambda^{j+1} x - T^{j+1} x = \lambda^{j+1} x - \lambda T^j x + \lambda T^j x - T^{j+1} x \in R(T^n)\,.$$

Consequently, by mathematical induction, we obtain that $\lambda^j x - T^j x \in R(T^n)$ for all $j \in \mathbb{N}$. In particular, $\lambda^n x - T^n x \in R(T^n)$, and since $\lambda \neq 0$, then

$$x = \lambda^{-n}((\lambda^n x - T^n x) + T^n x) \in R(T^n)\,.$$

By the above reasoning, we conclude that, for $m = 1$, the implication

$$(\lambda I - T)x \in R(T^n) \qquad \Rightarrow \qquad x \in R(T^n)$$

holds.

Now, suppose that for $m \geq 1$,

$$(\lambda I - T)^m x \in R(T^n) \qquad \Rightarrow \qquad x \in R(T^n)\,.$$

If $(\lambda I-T)^{m+1}x \in R(T^n)$, then $(\lambda I-T)((\lambda I-T)^m x) \in R(T^n)$. From the proof of case $m = 1$, we conclude that $(\lambda I - T)^m x \in R(T^n)$. Therefore by inductive hypothesis, $x \in R(T^n)$. Then, by mathematical induction, we conclude that for all $m \in \mathbb{N}$

$$(\lambda I - T)^m x \in R(T^n) \qquad \Rightarrow \qquad x \in R(T^n)$$

holds.

Finally, if $y \in R((\lambda I - T)^m) \cap R(T^n)$ there exists $x \in X$ such that $(\lambda I - T)^m x = y \in R(T^n)$, then $(\lambda I - T)^m x \in R(T^n)$. As the above proof, we conclude that $x \in R(T^n)$. Thus

$$\begin{aligned} y \;&=\; (\lambda I - T)^m x \;=\; \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \lambda^{m-k} T^k x \\ &=\; \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \lambda^{m-k} (T_n)^k x \;=\; (\lambda I - T_n)^m x\,, \end{aligned}$$

then $y \in R((\lambda I - T_n)^m)$. This shows that,

$$R((\lambda I - T)^m) \cap R(T^n) \subseteq R((\lambda I - T_n)^m)\,.$$

Consequently, $R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n)$.

(iii) and (iv), it follows immediately from the equality

$$N((\lambda I - T_n)^m) = N((\lambda I - T)^m) \qquad \text{for all } m \in \mathbb{N}\,.$$

(v) Observe that $R(\lambda I - T_n)$ is a subspace of $R(T^n)$. Let M be a subspace of $R(T^n)$ such that $R(T^n) = R(\lambda I - T_n) \oplus M$. Since $R(\lambda I - T_n) = R(\lambda I - T) \cap R(T^n)$, we have

$$\begin{aligned} R(\lambda I - T) \cap M \;&=\; R(\lambda I - T) \cap R(T^n) \cap M \\ &=\; R(\lambda I - T_n) \cap M \;=\; 0\,. \end{aligned}$$

Thus $R(\lambda I - T) \cap M = \{0\}$. Now, we show that $X = R(\lambda I - T) + M$.

Let $\mu \in \mathbb{C}$ such that $\mu I - T$ is invertible in $L(X)$, then $(\mu I - T)^j$ is invertible in $L(X)$, for all $j \in \mathbb{N}$. In particular $(\mu I - T)^m$ is invertible in $L(X)$, for all $m \geq n$. Thus, if $y \in X$ there exists $x \in X$ such that $y = (\mu I - T)^m x$. Thus,

$$\begin{aligned} y \;&=\; (\mu I - T)^m x \;=\; \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} (-1)^j \mu^{m-j} T^j x \\ &=\; \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!} (-1)^j \mu^{m-j} T^j x \;+\; \sum_{j=n}^{m} \frac{m!}{j!(m-j)!} (-1)^j \mu^{m-j} T^j x\,. \end{aligned}$$

Since $R(T^j) \subseteq R(T^n)$, for $n \leq j \leq m$, then we can write $y = u + v$, where:

$$u = \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!} (-1)^j \mu^{m-j} T^j x \in X\,,$$

$$v = \sum_{j=n}^{m} \frac{m!}{j!(m-j)!} (-1)^j \mu^{m-j} T^j x \in R(T^n)\,.$$

Now, from the above decomposition and for any $\lambda \neq 0$, we obtain a sequence $(y_k)_{k=0}^{\infty}$, where $y_k = \lambda^{-k-1}(\lambda I - T)T^k u$, for $k = 0, 1, \ldots,$ such that

$$u = y_0 + y_1 + \cdots + y_{n-1} + \lambda^{-n} T^n u \in R(\lambda I - T) + R(T^n)\,,$$

because $y_k = \lambda^{-k-1}(\lambda I - T)T^k u \in R(\lambda I - T)$ and $\lambda^{-n} T^n u \in R(T^n)$.

On the other hand,

$$v + \lambda^{-n} T^n u \in R(T^n) + R(T^n) = R(T^n) = R(\lambda I - T_n) + M\,.$$

Thus $v + \lambda^{-n} T^n u = z + m$, where $z \in R(\lambda I - T_n)$ and $m \in M$. From this, and since $R(\lambda I - T_n) \subseteq R(\lambda I - T)$, we obtain that

$$\begin{aligned} y \;=\; u + v \;&=\; y_0 + y_1 + \cdots + y_{n-1} + \lambda^{-n} T^n u + v \\ &=\; y_0 + y_1 + \cdots + y_{n-1} + z + m \\ &=\; (y_0 + y_1 + \cdots + y_{n-1} + z) + m \;\in\; R(\lambda I - T) + M\,. \end{aligned}$$

Therefore, we have that $X \subseteq R(\lambda I - T) + M$, consequently $X = R(\lambda I - T) + M$. But since $R(\lambda I - T) \cap M = \{0\}$, and hence it follows that $X = R(\lambda I - T) \oplus M$, which implies that

$$\beta(\lambda I - T) = \dim M = \beta(\lambda I - T_n)\,.$$

This shows that $\beta(\lambda I - T) = \beta(\lambda I - T_n)$. ∎

The following result concerning the ranges of the powers of $\lambda I - T$, where $\lambda \in \mathbb{C}$ and $T \in L(X)$, plays an important role in this paper. In the proof of this corollary we use the notion of paraclosed (or paracomplete) subspace and the Neubauer Lemma (see [14]).

Lemma 2.2. *If $R(T^n)$ is closed in X and $R((\lambda I - T_n)^m)$ is closed in $R(T^n)$, then there exists $k \in \mathbb{N}$ such that $R((\lambda I - T)^k)$ is closed in X.*

Proof. Observe that for $\lambda = 0$,

$$R((0I - T_n)^m) = R((T_n)^m) = R(T^{m+n}).$$

Then $R(T^{m+n})$ is a closed subspace of $R(T^n)$. Since $R(T^n)$ is closed, we have that $R((0I - T)^{m+n}) = R(T^{m+n})$ is closed. On the other hand, if $\lambda \neq 0$ and $R((\lambda I - T_n)^m)$ is a closed subspace of $R(T^n)$, since $R(T^n)$ is closed in X, we have that $R((\lambda I - T_n)^m)$ is closed in X. But, from the incise (ii) in Lemma 2.1,

$$R((\lambda I - T_n)^m) = R((\lambda I - T)^m) \cap R(T^n).$$

Thus $R((\lambda I - T)^m) \cap R(T^n)$ is closed in X. Also, if $\lambda \neq 0$ the polynomials $(\lambda - z)^m$ and z^n have no common divisors, so there exist two polynomials u and v such that $1 = (\lambda - z)^m u(z) + z^n v(z)$, for all $z \in \mathbb{C}$. Hence $I = (\lambda I - T)^m u(T) + T^n v(T)$ and so $R((\lambda I - T)^m) + R(T^n) = X$. Since both $R((\lambda I - T)^m)$ and $R(T^n)$ are paraclosed subspaces, and $R((\lambda I - T)^m) \cap R(T^n)$ and $R((\lambda I - T)^m) + R(T^n)$ are closed, using Neubauer Lemma [14, Proposition 2.1.2], we have that $R((\lambda I - T)^m)$ is closed. ∎

Recall that for an operator $T \in L(X)$, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T (see [12, Proposition 50.2]).

LEMMA 2.3. *If 0 is not a pole of the resolvent of $T \in L(X)$ and $R(T^n)$ is closed, then $\pi_{00}(T) \subseteq \pi_{00}(T_n)$.*

Proof. By Lemma 2.1, $\sigma(T_n) \setminus \{0\} = \sigma(T) \setminus \{0\}$. Also, $0 \notin \sigma(T)$ implies T bijective, thus $T = T_n$. Hence $\sigma(T_n) \subseteq \sigma(T)$. Moreover, $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(T_n)$. Since, if $\lambda \in \operatorname{iso}\sigma(T)$, then $\sigma(T) \cap \mathbb{D}_\lambda = \{\lambda\}$ for some open disc $\mathbb{D}_\lambda \subseteq \mathbb{C}$ centered at λ. Thus,

$$\sigma(T_n) \cap \mathbb{D}_\lambda \subseteq \sigma(T) \cap \mathbb{D}_\lambda = \{\lambda\}.$$

Consequently $\sigma(T_n) \cap \mathbb{D}_\lambda = \{\lambda\}$ or $\sigma(T_n) \cap \mathbb{D}_\lambda = \emptyset$. If $\sigma(T_n) \cap \mathbb{D}_\lambda = \emptyset$, then $\lambda \notin \sigma(T_n)$, so that $p(\lambda I - T_n) = \beta(\lambda I - T_n) = 0$. For the case $\lambda \neq 0$, from Lemma 2.1, $p(\lambda I - T) = 0$ and $\beta(\lambda I - T) = 0$, then $\lambda \notin \sigma(T)$ a contradiction. In the case where $\lambda = 0$, $p(T_n) = q(T_n) = 0$ implies, by [7, Lemma 2 and Lemma 3] and [12, Proposition 38.6], that $0 < p(T) = q(T) < \infty$, which is impossible, because 0 is not a pole of the resolvent of T. Consequently, $\sigma(T_n) \cap \mathbb{D}_\lambda = \{\lambda\}$, so we have that $\lambda \in \operatorname{iso}\sigma(T_n)$.

Now, the following argument shows that $\pi_{00}(T) \subseteq \pi_{00}(T_n)$. If $\lambda \in \pi_{00}(T)$, we have that $\lambda \in \operatorname{iso}\sigma(T_n)$, because $\lambda \in \operatorname{iso}\sigma(T)$. On the other hand, for

$\lambda \neq 0$, Lemma 2.1 implies that $\alpha(\lambda I-T)=\alpha(\lambda I-T_n)$, so $0<\alpha(\lambda I-T_n)<\infty$. For $\lambda=0$, we claim that $\alpha(T_n)>0$. If $\alpha(T_n)=0$, we have that $p(T_n)=0$. By [7, Lemma 2], $p(T)<\infty$. Moreover [7, Remark 1],

$$p(T)=\inf\{k\in\mathbb{N} : T_k \text{ is injective}\}\leq n\,.$$

Thus, by Lemma 1.1, T_n is bounded below, because T_n is injective and $R(T_n)=R(T^{n+1})$ is closed, so T_n is semi-Fredholm. Also $(T_n)^*$ has SVEP at 0, because $0\in \operatorname{iso}\sigma(T_n)$, then $q(T_n)<\infty$ ([1, Chapter 3]), which implies that $q(T)<\infty$ ([7, Lemma 3]). Hence $0<p(T)=q(T)<\infty$, a contradiction, since 0 is not a pole of the resolvent of T. Thus $0<\alpha(T_n)=\alpha(0I-T_n)$. Finally, since $N(T_n)\subseteq N(T)$ and $\alpha(T)<\infty$ it then follows the equality $\alpha(T_n)=\alpha(0I-T_n)<\infty$. Thus, $0\in\operatorname{iso}\sigma(T_n)$ and $0<\alpha(0I-T_n)<\infty$. Consequently $\lambda\in\pi_{00}(T_n)$, for each $\lambda\in\pi_{00}(T)$, so we have the inclusion $\pi_{00}(T)\subseteq\pi_{00}(T_n)$. ∎

The result of Lemma 2.3 may be extended as follows.

Lemma 2.4. *If 0 is not a pole of the resolvent of $T\in L(X)$ and $R(T^n)$ is closed, then $\pi^a_{00}(T)\subseteq\pi^a_{00}(T_n)$.*

Proof. If $\lambda\notin\sigma_{\rm ap}(T)$, then $\lambda I-T$ is injective and $R(\lambda I-T)$ is closed. Now, here we consider the two different cases $\lambda\neq 0$ and $\lambda=0$. If $\lambda\neq0$, by Lemma 2.1, $N(\lambda I-T_n)=N(\lambda I-T)$ and $R(\lambda I-T_n)=R(\lambda I-T)\cap R(T^n)$ is closed. Hence $\lambda I-T_n$ is bounded below, and so $\lambda\notin\sigma_{\rm ap}(T_n)$. In the other case, $-T$ bounded below implies that $0=p(T)=p(T_n)$ and $R(T)$ is closed. Thus T_n is inyective and, by Lemma 1.1, $R(T_n)=R(T^{n+1})$ is closed. From this we obtain that T_n is bounded below. Consequently, $\sigma_{\rm ap}(T_n)\subseteq\sigma_{\rm ap}(T)$. Similarly, as in the proof of Lemma 2.3 and taking into account Lemma 2.2, we can prove that $\operatorname{iso}\sigma_{\rm ap}(T)\subseteq\operatorname{iso}\sigma_{\rm ap}(T_n)$.

Finally, to show $\pi^a_{00}(T)\subseteq\pi^a_{00}(T_n)$. Observe that, if $\lambda\in\pi^a_{00}(T)$ then $\lambda\in\operatorname{iso}\sigma_{\rm ap}(T)$ and $0<\alpha(\lambda I-T)<\infty$. Thus $\lambda\in\operatorname{iso}\sigma(T_n)$. For $\lambda\neq0$, by Lemma 2.1, $\alpha(\lambda I-T)=\alpha(\lambda I-T_n)$, and so $0<\alpha(\lambda I-T_n)<\infty$. In the case $\lambda=0$, $p(T_n)=0$ and $R(T^n)$ is closed. Similarly to the case $p(T_n)=0$ and $R(T^n)$ closed in the proof of Lemma 2.3, one shows that $0<\alpha(0I-T_n)<\infty$. Consequently $\pi^a_{00}(T)\subseteq\pi^a_{00}(T_n)$. ∎

3. Weyl's theorems and restrictions

In this section we give conditions for which Weyl's theorem (resp. a-Weyl's theorem) for an operator $T\in L(X)$ is equivalent to Weyl's theorem (resp. a-

Weyl's theorem) for certain restriction T_n of T.

It is well known that if λ is a pole of the resolvent of T, then λ is an isolated point of the spectrum $\sigma(T)$. Thus, the following result is an immediate consequence of Lemma 2.1 and Lemma 2.3.

THEOREM 3.1. *Suppose that* 0 *is not an isolated point of* $\sigma(T)$. *Then* T *satisfies (W) if and only if there exists* $n \in \mathbb{N}$ *such that* $R(T^n)$ *is closed and* T_n *satisfies (W).*

Proof. (Necessity) Assume that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (W). Let $\lambda \in \pi_{00}(T)$, i.e. $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \alpha(\lambda I - T) < \infty$. By hypothesis and Lemma 2.3, $0 \neq \lambda \in \pi_{00}(T_n) = \sigma(T_n) \setminus \sigma_{\mathrm{w}}(T_n)$. Then $\alpha(\lambda I - T_n) = \beta(\lambda I - T_n) < \infty$ since $\lambda I - T_n$ is a Weyl operator, and so by Lemma 2.1

$$\alpha(\lambda I - T) = \alpha(\lambda I - T_n) = \beta(\lambda I - T_n) = \beta(\lambda I - T) < \infty\,.$$

Furthermore, $\lambda \in \sigma(T)$ because $\lambda \in \sigma(T_n) \subseteq \sigma(T)$. Thus $\lambda I - T$ is Weyl, and hence $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. But since $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) \subseteq \pi_{00}(T)$, it then follows that $\pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$, which implies that T satisfies (W).

(Sufficiency) Suppose that T satisfies (W). Then for $n = 0$, $R(T^0) = X$ is closed and $T_0 = T$ satisfies (W). ∎

In the same way as in Theorem 3.1, we have the following characterization of a-Weyl theorem for an operator throughout a-Weyl theorem for some restriction of the operator.

THEOREM 3.2. *Suppose that* 0 *is not an isolated point of* $\sigma(T)$. *Then* T *satisfies (aW) if and only if there exists* $n \in \mathbb{N}$ *such that* $R(T^n)$ *is closed and* T_n *satisfies (aW).*

Proof. (Necessity) Suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (aW). Let $\lambda \in \pi^a_{00}(T)$, by hypothesis and Lemma 2.4, $\lambda \in \pi^a_{00}(T_n) = \sigma_{\mathrm{ap}}(T_n) \setminus \sigma_{\mathrm{uw}}(T_n)$. Thus $\lambda I - T_n$ is a upper semi-Fredholm operator, because $\lambda I - T_n$ is a upper semi-Weyl operator. Since $\lambda I - T_n$ is upper semi-Fredholm, it follows that $R((\lambda I - T_n)^m)$ is closed in $R(T^n)$ for all $m \in \mathbb{N}$, and so by Lemma 2.2, there exists $k \in \mathbb{N}$ such that $R((\lambda I - T)^k)$ is closed. But since $\alpha(\lambda I - T) < \infty$, then $\alpha((\lambda I - T)^k) < \infty$. That is, $(\lambda I - T)^k$ is a upper semi-Fredholm operator, which implies that $\lambda I - T$ is upper semi-Fredholm. Furthermore, T has SVEP at λ because $\lambda \in \operatorname{iso}\sigma_{\mathrm{ap}}(T)$. Consequently, if

$\lambda \in \pi^a_{00}(T)$ then $\lambda I - T$ is upper semi-Fredholm and $p(\lambda I - T) < \infty$. Hence $\lambda I - T$ is upper semi-Weyl and $\lambda \in \sigma_{\rm ap}(T)$, thus $\lambda \in \sigma_{\rm ap}(T) \setminus \sigma_{\rm uw}(T)$, and we obtain the inclusion $\pi^a_{00}(T) \subseteq \sigma_{\rm ap}(T) \setminus \sigma_{\rm uw}(T)$. But since $\sigma_{\rm ap}(T) \setminus \sigma_{\rm uw}(T) \subseteq \pi^a_{00}(T)$, it then follows that $\pi^a_{00}(T) = \sigma_{\rm ap}(T) \setminus \sigma_{\rm uw}(T)$, which implies that T satisfies (aW).

(Sufficiency) If T satisfies (aW). Then for $n = 0$, trivially $R(T^0) = X$ is closed and $T_0 = T$ satisfies (aW). ∎

Clearly, T has SVEP at every isolated point of $\sigma(T)$. Thus, by Theorem 3.1 and Theorem 3.2, we have the following corollary.

Corollary 3.3. *If T does not have SVEP at 0, then:*

(i) *there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (W) if and only if T satisfies (W).*

(ii) *there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (aW) if and only if T satisfies (aW).*

Remark 3.4. There are more alternative ways to express Corollary 3.3. We may replace the assumption T does not have SVEP at 0 by: $0 \notin \partial\sigma(T)$, $p(T) = \infty$ or $q(T) = \infty$.

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