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## Operator Valued Versions of the Radon-Nikodym Theorem and of the F. and M. Riesz Theorem \*

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*Abstract:* For an absolutely continuous operator valued measure in weak sense, we give a necessary and sufficient condition to have a density in strong sense. This result is used to obtain an operator valued version of the F. and M. Riesz theorem. We also give some related counterexamples.

*Key words:* Operator valued measure, weakly measurable, Fourier transform, absolutely continuous measure.

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### 1. PRELIMINARIES

A basic result of the classical harmonic analysis is the theorem of F. and M. Riesz, which states that if the Fourier-Stieltjes coefficients of a Borel measure  $\mu$  on  $[0, 2\pi)$  satisfy  $\hat{\mu}(n) = 0$  for all  $n < 0$ , then  $\mu$  is absolutely continuous with respect to the Lebesgue measure.

The Radon-Nikodym theorem is a fundamental result in measure theory. It establishes that, given a measurable space  $(\Omega, \Sigma)$ , if a  $\sigma$ -finite measure  $\nu$  on  $(\Omega, \Sigma)$  is absolutely continuous with respect to a  $\sigma$ -finite measure  $\mu$  on  $(\Omega, \Sigma)$ , then there exists a measurable function  $f$  on  $\Omega$  and taking values in  $[0, \infty)$ ,

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such that

$$\nu(A) = \int_A f d\mu$$

for any set  $A \in \Sigma$ .

Several extensions of this two results have been given, see for example [7, 8, 11, 10, 9] for extensions of the F. and M. Riesz theorem and [1, 4, 5] for extensions of the Radon-Nikodym theorem.

In [2] Arocena and Cotlar considered some vectorial moment and weighted problems related with dilation of generalized Toeplitz kernels and they commented about the possibility of extending the F. and M. Riesz theorem for operator valued measures. This paper was motivated by that comment, for more details see Section 4.

In Section 2 of this paper we give an operator valued extension of the Radon-Nikodym theorem and in Section 4 we use this result to establish an operator valued version of the F. and M. Riesz theorem.

## 2. HILBERT SPACE OPERATOR VALUED MEASURES AND A VERSION OF THE RADON-NIKODYM THEOREM.

Let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra of  $[0, 2\pi)$  and let  $m$  be the Lebesgue measure on  $[0, 2\pi)$ . By  $dx$  we denote the differential of the Lebesgue measure, with  $L^p$  the usual Lebesgue spaces and with  $H^p$  the usual Hardy spaces, for  $1 \leq p < \infty$ .

Throughout this paper  $(\mathcal{G}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$  is a separable Hilbert space,  $\|\cdot\|_{\mathcal{G}}$  is the associated norm and  $L(\mathcal{G})$  stands for the space of the continuous linear operators on  $\mathcal{G}$ .

**DEFINITION 2.1.** Let  $\mu : \mathcal{B} \rightarrow L(\mathcal{G})$  be a function.

We say that  $\mu$  is an  $L(\mathcal{G})$ -valued measure on  $[0, 2\pi)$ , in weak sense, if for each  $\zeta, \xi \in \mathcal{G}$  the function  $\mu_{\zeta\xi} : \mathcal{B} \rightarrow \mathbb{C}$  given by

$$\mu_{\zeta\xi}(\Delta) = \langle \mu(\Delta)\zeta, \xi \rangle_{\mathcal{G}}$$

is a scalar finite Radon measure on  $[0, 2\pi)$ .

We say that  $\mu$  is an  $L(\mathcal{G})$ -valued measure on  $[0, 2\pi)$ , in strong sense, if

$$\mu \left( \bigcup_{n=1}^{+\infty} \Delta_n \right) = \sum_{n=1}^{+\infty} \mu(\Delta_n),$$

convergence in norm, for any disjoint sequence  $\{\Delta_n\} \subset \mathcal{B}$ . This property is also known as *strong additivity*.

In both cases  $\mu$  is *bounded* if

$$\sup_{\Delta \in \mathcal{B}} \|\mu(\Delta)\|_{L(\mathcal{G})} < +\infty$$

and  $\mu$  has *finite variation* if

$$\sum_{n=1}^{+\infty} \|\mu(\Delta_n)\|_{L(\mathcal{G})} < +\infty,$$

for any disjoint sequence  $\{\Delta_n\} \subset \mathcal{B}$ .

The set of the bounded  $L(\mathcal{G})$ -valued measures on  $[0, 2\pi)$  in weak sense and the set of the bounded  $L(\mathcal{G})$ -valued measures on  $[0, 2\pi)$  in strong sense, will be denoted by  $\mathfrak{M}_w(L(\mathcal{G}))$  and  $\mathfrak{M}_s(L(\mathcal{G}))$ , respectively.

It is clear that  $\mathfrak{M}_s(L(\mathcal{G})) \subset \mathfrak{M}_w(L(\mathcal{G}))$ , it also holds that  $\mathfrak{M}_s(L(\mathcal{G})) \subsetneq \mathfrak{M}_w(L(\mathcal{G}))$  as the following example shows.

Let  $\mathcal{G} = L^2$  and let  $\mu : \mathcal{B} \rightarrow L(\mathcal{G})$  be defined by  $\mu(\Delta)\xi = \xi 1_\Delta$ , where  $1_\Delta$  is the characteristic function of the set  $\Delta$ .

For  $\zeta, \xi \in \mathcal{G}$  and  $\Delta \in \mathcal{B}$  we have

$$\mu_{\zeta\xi}(\Delta) = \int_{\Delta} \zeta(x) \overline{\xi(x)} dx.$$

So  $\mu$  is an  $L(\mathcal{G})$ -valued measure on  $[0, 2\pi)$ , in weak sense.

On the other hand, if  $\Delta, \Delta' \in \mathcal{B}$  are such that  $\Delta' \subset \Delta$  and the Lebesgue measure of  $\Delta \setminus \Delta'$  is positive then

$$\|\mu(\Delta) - \mu(\Delta')\|_{L(\mathcal{G})}^2 = \sup_{\|\xi\|_{\mathcal{G}}=1} \|\mu(\Delta)\xi - \mu(\Delta')\xi\|_{\mathcal{G}}^2 = \sup_{\|\xi\|_{\mathcal{G}}=1} \int_{\Delta \setminus \Delta'} |\xi(x)|^2 dx = 1.$$

So  $\mu$  is not strongly additive.

*Remark 2.2.* For  $\mu \in \mathfrak{M}_s(L(\mathcal{G}))$  it holds that  $\mu$  has finite variation if and only if

$$\sup \sum_{n=1}^{+\infty} \|\mu(\Delta_n)\|_{L(\mathcal{G})} < +\infty,$$

where the supremum is taken over all the partitions  $\{\Delta_n\}$  of  $[0, 2\pi)$  such that  $\{\Delta_n\} \subset \mathcal{B}$ .

This property is usually expressed by saying that  $\mu$  is of *bounded variation*.

DEFINITION 2.3. A function  $F : [0, 2\pi) \rightarrow L(\mathcal{G})$  is said to be *weakly measurable* if for every  $\zeta, \xi \in \mathcal{G}$  the complex valued function

$$x \mapsto \langle F(x)\zeta, \xi \rangle_{\mathcal{G}}$$

is measurable.

For a function  $F : [0, 2\pi) \rightarrow L(\mathcal{G})$  it holds that

$$\|F(x)\|_{L(\mathcal{G})} = \sup_{\|\zeta\|_{\mathcal{G}}=\|\xi\|_{\mathcal{G}}=1} |\langle F(x)\zeta, \xi \rangle_{\mathcal{G}}|,$$

for  $x \in [0, 2\pi)$ . Since we are considering a separable Hilbert space  $\mathcal{G}$ , if  $F$  is weakly measurable, we have that the function

$$x \mapsto \|F(x)\|_{L(\mathcal{G})}$$

is measurable in the usual sense.

With  $L_{L(\mathcal{G})}^{1,w}$  we will denote the set of all weakly measurable functions  $F : [0, 2\pi) \rightarrow L(\mathcal{G})$  such that

$$\int_0^{2\pi} \|F(x)\|_{L(\mathcal{G})} dx < \infty.$$

Note that for  $F \in L_{L(\mathcal{G})}^{1,w}$ , the sesquilinear functional  $B_F : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}$  defined by

$$B_F(\zeta, \xi) = \int_0^{2\pi} \langle F(x)\zeta, \xi \rangle_{\mathcal{G}} dx$$

is bounded. Therefore, from the Riesz representation theorem, it follows that there exists a unique bounded linear operator  $I_F : \mathcal{G} \rightarrow \mathcal{G}$  such that

$$\langle I_F\zeta, \xi \rangle_{\mathcal{G}} = \int_0^{2\pi} \langle F(x)\zeta, \xi \rangle_{\mathcal{G}} dx,$$

for all  $\zeta, \xi \in \mathcal{G}$ . We say that  $I_F$  is the *weak integral of  $F$* .

*Remark 2.4.* Note that if  $\mathcal{G}$  is infinite dimensional, then  $L(\mathcal{G})$  is nonseparable, so this integral is not necessarily the Bochner integral.

If  $F : [0, 2\pi) \rightarrow L(\mathcal{G})$  is a weakly measurable function, for each  $\zeta, \xi \in \mathcal{G}$  we define the function  $F_{\zeta\xi} : [0, 2\pi) \rightarrow \mathbb{C}$  by

$$F_{\zeta\xi}(x) = \langle F(x)\zeta, \xi \rangle_{\mathcal{G}}.$$

Let  $\mu \in \mathfrak{M}_w(L(\mathcal{G}))$  be absolutely continuous with respect to the Lebesgue measure, that is if  $m(A) = 0$ , then  $\mu(A) = 0$  for  $A \in \mathcal{B}$ . In this case for each  $\zeta, \xi \in \mathcal{G}$  there exists a, unique a.e., Lebesgue integrable function  $h^{\zeta\xi} : [0, 2\pi) \rightarrow \mathbb{C}$  such that

$$d\mu_{\zeta\xi}(x) = h^{\zeta\xi}(x) dx.$$

A natural question is to determine under what conditions there exists a weakly measurable function  $F : [0, 2\pi) \rightarrow L(\mathcal{G})$  such that for all  $\zeta, \xi \in \mathcal{G}$

$$h^{\zeta\xi}(x) = F_{\zeta\xi}(x).$$

Theorem 2.5 gives a necessary and sufficient condition for an affirmative answer.

**THEOREM 2.5.** *Let  $\mu \in \mathfrak{M}_w(L(\mathcal{G}))$  be absolutely continuous with respect to the Lebesgue measure. For  $\zeta, \xi \in \mathcal{G}$ , let  $h^{\zeta\xi}$  be the integrable function such that  $d\mu_{\zeta\xi}(x) = h^{\zeta\xi}(x) dx$ .*

*Then the following conditions are equivalent*

- (a)  $\mu \in \mathfrak{M}_s(L(\mathcal{G}))$  and  $\mu$  has finite variation.
- (b) There exists a function  $F \in L_{L(\mathcal{G})}^{1,w}$ , unique a.e., such that

$$h^{\zeta\xi}(x) = F_{\zeta\xi}(x) \quad \text{a.e.}(x).$$

- (c) There exists an integrable function  $y : [0, 2\pi) \rightarrow [0, +\infty)$  such that for each  $\zeta, \xi \in \mathcal{G}$

$$|h^{\zeta\xi}(x)| \leq y(x) \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}} \quad \text{a.e.}(x).$$

*Proof.* (a)  $\Rightarrow$  (b) This part follows from a result of Alvarez de Araya (Theorem 2.4 of [1]), see also the book of Diestel and Uhl [5].

(b)  $\Rightarrow$  (c) If  $h^{\zeta\xi}(x) = F_{\zeta\xi}(x)$  where  $F \in L_{L(\mathcal{G})}^{1,w}$ , then

$$|h^{\zeta\xi}(x)| = |\langle F(x)\zeta, \xi \rangle_{\mathcal{G}}| \leq \|F(x)\|_{L(\mathcal{G})} \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}},$$

so it is enough to take  $y(x) = \|F(x)\|_{L(\mathcal{G})}$ .

(c)  $\Rightarrow$  (a) We have to show that  $\mu$  is  $\sigma$ -additive in strong sense and that  $\mu$  has finite variation.

Let  $\zeta, \xi \in \mathcal{G}$  be such that  $\|\zeta\|_{\mathcal{G}} = \|\xi\|_{\mathcal{G}} = 1$ , then for any  $\Delta \in \mathcal{B}$  we have

$$|\langle \mu(\Delta)\zeta, \xi \rangle| = \left| \int_{\Delta} h^{\zeta\xi}(x) dx \right| \leq \int_{\Delta} y(x) dx.$$

Therefore

$$\|\mu(\Delta)\|_{L(\mathcal{G})} = \sup_{\|\zeta_{-\mathcal{G}}\|=\|\xi\|_{\mathcal{G}}=1} |\langle \mu(\Delta)\zeta, \xi \rangle| \leq \int_{\Delta} y(x) dx,$$

Let  $\{A_n\} \subset \mathcal{B}$  be a disjoint sequence, let  $A = \bigcup_{n=1}^{+\infty} A_n$  and, for  $N \in \mathbb{N}$ , let  $B_N = \bigcup_{n=1}^N A_n$ . We have

$$\sum_{n=1}^{+\infty} \|\mu(A_n)\|_{L(\mathcal{G})} \leq \sum_{n=1}^{+\infty} \int_{A_n} y(x) dx = \int_A y(x) dx < +\infty.$$

We also have

$$|\langle (\mu(A) - \mu(B_N))\zeta, \xi \rangle| = \left| \int_{A \setminus B_N} h^{\zeta\xi}(x) dx \right| \leq \int_{A \setminus B_N} y(x) dx.$$

Therefore

$$\|\mu(A) - \mu(B_N)\|_{L(\mathcal{G})} = \|\mu(A \setminus B_N)\|_{L(\mathcal{G})} \leq \int_{A \setminus B_N} y(x) dx,$$

since  $y$  is integrable  $\int_{A \setminus B_N} y(x) dx \rightarrow 0$  as  $N \rightarrow \infty$ , so  $\mu \in \mathfrak{M}_s(L(\mathcal{G}))$ .

**2.1. AN EXAMPLE OF A MEASURE  $\mu \in \mathfrak{M}_w(L(\mathcal{G}))$  FOR WHICH IT DOES NOT EXIST  $F \in L_{L(\mathcal{G})}^{1,w}$  SUCH THAT  $h^{\zeta\xi}(x) = F_{\zeta\xi}(x)$  *a.e.*( $x$ )**

Consider a function  $\phi : [0, 2\pi) \rightarrow \mathbb{C}$  such that:

- (a)  $\phi \in L^1$ .
- (b)  $\phi$  is continuous on  $(0, 2\pi)$ .
- (c)  $\phi$  is not bounded.

Let  $\{x_k\}_{k=1}^{\infty} \subset (0, 2\pi)$  be a dense set and let  $\{\tau_k\}_{k=1}^{\infty}$  be an orthonormal basis of  $\mathcal{G}$ .

Let  $\zeta, \xi \in \mathcal{G}$  given by  $\zeta = \sum_{k=1}^{\infty} a_k \tau_k$ ,  $\xi = \sum_{k=1}^{\infty} b_k \tau_k$ . For a Borel set  $\Delta \subset [0, 2\pi)$ , let  $\Omega_{\Delta} : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}$  be defined by

$$\Omega_{\Delta}(\zeta, \xi) = \sum_{k=1}^{\infty} a_k \bar{b}_k \int_{\Delta} \phi(x - x_k) dx.$$

We have that  $\Omega_\Delta$  is a sesquilinear form. Also

$$\begin{aligned} |\Omega_\Delta(\zeta, \xi)| &\leq \sum_{k=1}^{\infty} |a_k \bar{b}_k| \|\phi\|_1 \\ &\leq \|\phi\|_1 \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} |b_k|^2 \right)^{1/2} \\ &= \|\phi\|_1 \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}}. \end{aligned}$$

Therefore there exists a function  $\mu : \mathcal{B} \rightarrow L(\mathcal{G})$  such that

$$\Omega_\Delta(\zeta, \xi) = \langle \mu(\Delta)\zeta, \xi \rangle_{\mathcal{G}}.$$

From the last inequality, clearly

$$\|\mu(\Delta)\|_{L(\mathcal{G})} \leq \|\phi\|_1.$$

Now consider a sequence of disjoint Borel sets  $\Delta_1, \Delta_2, \dots \subset [0, 2\pi)$  and let

$$\Delta = \bigcup_{n=1}^{\infty} \Delta_n.$$

In order to prove that  $\mu$  is a  $L(\mathcal{G})$ -valued measure in weak sense we need to consider iterated series. We have that

$$\begin{aligned} \langle \mu(\Delta)\zeta, \xi \rangle_{\mathcal{G}} &= \sum_{k=1}^{\infty} a_k \bar{b}_k \int_{\Delta} \phi(x - x_k) dx \\ &= \sum_{k=1}^{\infty} a_k \bar{b}_k \sum_{n=1}^{\infty} \int_{\Delta_n} \phi(x - x_k) dx \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_k \bar{b}_k \int_{\Delta_n} \phi(x - x_k) dx \\ &= \sum_{n=1}^{\infty} \langle \mu(\Delta_n)\zeta, \xi \rangle_{\mathcal{G}}, \end{aligned}$$

because

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| a_k \bar{b}_k \int_{\Delta_n} \phi(x - x_k) dx \right| &\leq \sum_{k=1}^{\infty} |a_k \bar{b}_k| \sum_{n=1}^{\infty} \int_{\Delta_n} |\phi(x - x_k)| dx \\
&= \sum_{k=1}^{\infty} |a_k \bar{b}_k| \int_{\Delta} |\phi(x - x_k)| dx \\
&\leq \sum_{k=1}^{\infty} |a_k \bar{b}_k| \|\phi\|_1 \\
&\leq \|\phi\|_1 \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}}.
\end{aligned}$$

So  $\mu \in \mathfrak{M}_w(L(\mathcal{G}))$ .

From the definition of  $\mu$  it follows that

$$\mu_{\zeta\xi}(\Delta) = \sum_{k=1}^{\infty} a_k \bar{b}_k \int_{\Delta} \phi(x - x_k) dx$$

and

$$|\mu_{\zeta\xi}(\Delta)| \leq c \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}}.$$

Thus

$$d\mu_{\zeta\xi}(x) = \left( \sum_{k=1}^{\infty} a_k \bar{b}_k \phi(x - x_k) \right) dx,$$

so the corresponding function  $h^{\zeta\xi}$  for this measure is given by

$$h^{\zeta\xi}(x) = \sum_{k=1}^{\infty} a_k \bar{b}_k \phi(x - x_k).$$

Note that

$$h^{\tau_k \tau_k}(x) = \phi(x - x_k) \quad \text{a.e.}(x) \quad k = 1, 2, \dots$$

Now we will show that it does not exist  $F \in L_{L(\mathcal{G})}^{1,w}$  such that  $h^{\zeta\xi}(x) = F_{\zeta\xi}(x)$  a.e.(x).

Suppose there exists a weakly measurable function  $F : [0, 2\pi) \rightarrow L(\mathcal{G})$  such that for each  $\zeta, \xi \in \mathcal{G}$

$$h^{\zeta\xi}(x) = F_{\zeta\xi}(x) \quad \text{a.e.}(x).$$

Then it would hold that

$$|h^{\zeta\xi}(x)| = |\langle F(x)\zeta, \xi \rangle_{\mathcal{G}}| \leq \|F(x)\|_{L(\mathcal{G})} \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}} \quad \text{a.e.}(x).$$



In particular,

$$|\phi(x - x_k)| = |h^{\tau_k \tau_k}(x)| \leq \|F(x)\|_{L(\mathcal{G})} \quad \text{a.e.}(x) \quad k = 1, 2, \dots$$

This is not possible because  $\|F(\cdot)\|_{L(\mathcal{G})}$  must be finite a.e.,  $\phi$  is continuous except in 0,  $\phi$  is unbounded and  $\{x_n\}_{n=1}^\infty$  is a dense set in  $(0, 2\pi)$ .

### 3. FOURIER TRANSFORM OF OPERATOR VALUED MEASURES

Let  $F \in L^1_{L(\mathcal{G})}$  and  $\zeta, \xi \in \mathcal{G}$ , then we have that  $F_{\zeta\xi} \in L^1$ . Let  $\widehat{F_{\zeta\xi}}$  be the usual Fourier transform of  $F_{\zeta\xi}$ . By linearity for every  $n \in \mathbb{Z}$  there is  $\widehat{F}(n) \in L(\mathcal{G})$  such that

$$\widehat{F_{\zeta\xi}}(n) = \langle \widehat{F}(n)\zeta, \xi \rangle_{\mathcal{G}}.$$

More generally it is possible to consider the Fourier transform of operator valued measures, as the following construction shows.

Let  $\mathcal{P}$  denote the linear space of all scalar valued trigonometric polynomials.

**PROPOSITION 3.1.** *Let  $\mu \in \mathfrak{M}_w(L(\mathcal{G}))$  and let  $c = \sup_{\Delta \in \mathcal{B}(\mathbb{T})} \|\mu(\Delta)\|_{L(\mathcal{G})}$ . There exists a unique linear operator  $T_\mu : \mathcal{P} \rightarrow L(\mathcal{G})$  such that:*

(a) *For all  $p \in \mathcal{P}$  and  $\zeta, \xi \in \mathcal{G}$  we have that*

$$\langle T_\mu(p)\zeta, \xi \rangle_{\mathcal{G}} = \int_{\mathbb{T}} p(x) d\langle \mu(x)\zeta, \xi \rangle_{\mathcal{G}}.$$

(b)  $\|T_\mu(p)\|_{\mathcal{G}} \leq 4c \|p\|_\infty$  for all  $p \in \mathcal{P}$ .

*Proof.* For  $\zeta, \xi \in \mathcal{G}$  let

$$B(\zeta, \xi) = \int_{\mathbb{T}} p(x) d\langle \mu(x)\zeta, \xi \rangle_{\mathcal{G}}.$$

Then  $B$  is a sesquilinear form on  $\mathcal{G} \times \mathcal{G}$ . Using the Hahn decomposition of the real part of  $\mu$  and of the imaginary part of  $\mu$  we obtain that

$$|B(\zeta, \xi)| \leq 4c \|p\|_\infty \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}}.$$

So the result follows from the Riesz representation theorem.

Given a measure  $\mu \in \mathfrak{M}_w(L(\mathcal{G}))$ , the *Fourier transform* of  $\mu$  is the function  $\widehat{\mu} : \mathbb{Z} \rightarrow L(\mathcal{G})$  defined by

$$\widehat{\mu}(n) = T_\mu(e_{-n}) \quad (n \in \mathbb{Z}),$$

where  $e_n(x) = e^{inx}$ .

It holds that

$$\langle \widehat{\mu}(n)\zeta, \xi \rangle_{\mathcal{G}} = \int_0^{2\pi} e^{-inx} d\langle \mu(x)\zeta, \xi \rangle_{\mathcal{G}} = \widehat{\mu_{\zeta\xi}}(n)$$

for every  $\zeta, \xi \in \mathcal{G}$  and  $n \in \mathbb{Z}$ .

*Remark 3.2.* Note that if  $d\mu(x) = F(x)dx$ , where  $F \in L_{L(\mathcal{G})}^{1,w}$ , then  $\widehat{\mu}(n) = \widehat{F}(n)$ .

More details about the definition of  $\widehat{\mu}(n)$  can be seen in [3].

#### 4. AN OPERATOR VALUED VERSION OF THE F. AND M. RIESZ THEOREM

The following comment appears on a paper of Arocena and Cotlar [2]: if  $\mu \in \mathfrak{M}_w(L(\mathcal{G}))$ ,  $\widehat{\mu}(n) = 0$  for  $n < 0$  and  $\zeta, \xi \in \mathcal{G}$  then for the scalar measure  $\mu_{\zeta\xi}$  we have that  $\widehat{\mu_{\zeta\xi}}(n) = 0$  for  $n < 0$ , so there exists  $h^{\zeta\xi} \in H^1$  such that  $d\mu_{\zeta\xi}(x) = h^{\zeta\xi}(x) dx$ . But though

$$|\mu_{\zeta\xi}(\Delta)| \leq c \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}}$$

we cannot say that there exists an operator  $F(x) \in L(\mathcal{G})$  such that

$$\langle F(x)\zeta, \xi \rangle_{\mathcal{G}} = h^{\zeta\xi}(x) \quad \text{a.e.}(x).$$

This remark was a motivation for this paper.

As it is natural, we define

$$H_{L(\mathcal{G})}^{1,w} = \{F \in L_{L(\mathcal{G})}^{1,w} : \widehat{F}(n) = 0 \text{ if } n < 0\}.$$

**THEOREM 4.1.** *Let  $\mu \in \mathfrak{M}_s(L(\mathcal{G}))$  be a measure that has finite variation such that  $\widehat{\mu}(n) = 0$  if  $n < 0$ . Then there exists  $F \in H_{L(\mathcal{G})}^{1,w}$  such that*

$$d\mu(x) = F(x) dx.$$

*Proof.* It follows that, for each  $\zeta, \xi \in \mathcal{G}$ ,  $\widehat{\mu_{\zeta\xi}}(n) = 0$  for all  $n < 0$ . So from the F. and M. Riesz theorem, it follows that each of the measures  $\mu_{\zeta\xi}$  are absolutely continuous with respect to the Lebesgue measure.

Therefore  $\mu$  is absolutely continuous with respect to the Lebesgue measure. From Theorem 2.5 it follows that there exists a function  $F \in L^1_{L(\mathcal{G})}$ , unique a.e., such that

$$h^{\zeta\xi}(x) = F_{\zeta\xi}(x) \quad \text{a.e.}(x).$$

Finally it is clear that  $F \in H^{1,w}_{L(\mathcal{G})}$ .

*Remark 4.2.* The hypothesis in the last theorem can not be omitted, as is shown by the following example.

For the example given in Subsection 2.1 consider the following function

$$\phi(x) = \frac{1}{1 - e^{ix}} \left( \frac{1}{e^{ix}} \log \frac{1}{1 - e^{ix}} \right)^{-2},$$

for  $x \neq 0$  and  $x \neq 2\pi$ .

It holds that  $\phi \in H^1$ , see [6, pag. 13, exer. 3].

With the same notation of Subsection 2.1 we have that

$$h^{\zeta\xi}(x) = \sum_{k=1}^{\infty} a_k \bar{b}_k \phi(x - x_k).$$

Since

$$\sum_{k=1}^{\infty} \left\| a_k \bar{b}_k \phi(\cdot - x_k) \right\|_1 \leq \|\phi\|_1 \|\zeta\|_{\mathcal{G}} \|\xi\|_{\mathcal{G}},$$

we have that  $h^{\zeta\xi} \in H^1$  for each  $\zeta, \xi \in \mathcal{G}$ .

So the corresponding operator valued measure  $\mu$  belongs to  $\mathfrak{M}_w(L(\mathcal{G}))$  and  $\widehat{\mu}(n) = 0$  if  $n < 0$ . But, as proved in Subsection 2.1, it does not exist  $F \in L^1_{L(\mathcal{G})}$  such that  $h^{\zeta\xi}(x) = F_{\zeta\xi}(x)$  a.e.(x).

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