An Introduction to
Physics of Fluids

Colección manuales uex - 102

Vicente
Garzó Puertos

102

ÍNDICE

AN INTRODUCTION TO
PHYSICS OF FLUIDS

ÍNDICE

MANUALES UEX

102

ÍNDICE

VICENTE GARZÓ

AN INTRODUCTION TO PHYSICS OF FLUIDS

UNIVERSIDAD DE EXTREMADURA

2016

ÍNDICE

UNIVERSIDAD DE EXTREMADURA

Edita:

Universidad de Extremadura. Servicio de Publicaciones
C/ Caldereros, 2 - Planta 2ª. 10071 Cáceres (España)
Tel. 927 257 041 ; Fax 927 257 046
E-mail: publicac@unex.es
http://www.unex.es/publicaciones

ISSN 1135-870-X
ISBN de méritos 978-84-608-7478-2

Maquetación: Control P - Cáceres - 927 233 223 - www.control-p.eu

ÍNDICE

CONTENTS

ÍN
DI
CE

INICIO

ÍN
DI
CE

INICIO

CHAPTER 1
INTRODUCTION

1.1 Solids, liquids and gases

The defining property of fluids, embracing both liquids and gases, lies in the characteristic which they may be deformed. A piece of solid material has a definite shape, and that shape changes only when there is a change in the external conditions. A portion of fluid does not have a preferred shape, and different elements of a homogeneous fluid may be rearranged freely without affecting the macroscopic properties of the portion of fluid.

The distinction between solids and fluids is not a sharp one, since there are many materials which in some situations behave like a solid and in other situations behave like a fluid. A *simple* solid may be seen as a material of which the shape, and the relative positions of the constituents elements, change by a small amount only when there is a small change in the forces acting on it. A *simple* fluid might be defined as a material such that the relative positions of the elements of the material change by an amount which is not small when suitably chosen forces are applied to the material.

The distinction between liquids and gases is much less fundamental. For reasons related to the nature of intermolecular forces, most substances can exist in either of two stable phases which exhibit the property of fluidity. The density of a substance in the liquid phase is normally much larger than that in the gaseous phase, but this is not in fact a relevant difference between both systems. The most important difference between the mechanical properties between liquids and gases lies in their bulk elasticity, namely, in their compressibility. Gases can be compressed much more readily than liquids, and as a consequence any motion involving appreciable variations in pressure will be accompanied by much larger changes in specific volume in the case of a gas than in the case of a liquid.

In any case, the relevant properties of solids, liquids and gases are directly related to their molecular structure and to the nature of the forces between their constituents.

1.2 The continuum hypothesis

The molecules of a gas are separated by vacuous regions with linear dimensions much larger than those of the molecules themselves. Even in a liquid, in which the molecules are nearly closely packed as the strong short-range repulsive forces will allow, the mass of the material is concentrated in the nuclei of the atoms composing a molecule and is very far from being smeared uniformly over the volume occupied by the liquid. However, fluid mechanics is normally concerned with the behavior of matter in the large, on a macroscopic scale large compared with the distance between molecules and usually the molecular structure of the fluid will be not accounted for explicitly. Thus, we will assume that the behavior of fluids is the same as if they were perfectly *continuous* in structure; and physical quantities such as the mass and momentum associated with the matter contained within a given small volume will be regarded as being spread uniformly over that volume instead of being concentrated in a small region of it. The validity of this *continuum approach* under the conditions of everyday experience is evident. Indeed the structure and properties of air and water are so obviously continuous and smoothly-varying that no different assumption would seem natural.

To put this hypothesis in a proper context, let us consider for instance the density of mass ρ in a given point. To define the density at the point $\mathbf{r}$ we take a very small volume around this point and evaluate the density as the ratio between the sum of the masses of the particles contained in this volume and the volume δV:

$$\rho(\delta V) = \frac{\sum_i m_i}{\delta V}.$$

In the case that δV is very small, the value of $\rho(\delta V)$ will change quickly when we move from one point to another point, although both points are quite close each other since its value will depend on the number of particles considered in δV. On the other hand, as the value of δV becomes more and more big, it is possible then that the density ρ reaches a practically constant value where the addition of more particles into the volume does not affect the value of the density. Let us denote V_0 the value of the volume such that $\rho \approx$ const. If V_0 is much smaller than the macroscopic size of the system, then the value of the density in V_0 can be considered as *local*, i.e.,

$$\rho(\mathbf{r}) \rightarrow \rho(V_0).$$

On the other hand, when V_0 is larger (or the same order) than the macroscopic size of the system, the continuum hypothesis fails and we should consider a microscopic description such as, the kinetic theory of gases.

1.3 Classical Thermodynamics

In our subsequent discussion of the dynamics of fluids we shall need to make use of some of the concepts of classical thermodynamics. As you know, classical thermodynamics analyzes states in which all local mechanical, physical an thermal quantities are independent of both position and time. Thermodynamic results may be applied directly to fluids at rest when their properties are uniform. However, little is known of the thermodynamics of non-equilibrium states. Observation shows that results for equilibrium states are approximately valid for non-equilibrium non-uniform states common in practical in fluid dynamics when the departures from equilibrium in a moving fluid are apparently small in their effect on thermodynamic relationships.

The purpose of this section is to summarize briefly the laws and results of equilibrium thermodynamics. The state of a given mass of fluid in equilibrium under the simplest possible conditions is specified uniquely through two parameters, which for convenience may be chosen as the specific volume $v = 1/\rho$ and the (hydrostatic) pressure p. [1] All other quantities describing the state of the fluid are thus functions of these two *parameters of state*. One of the most important quantities is the temperature T. The relation between

---

[1] The pressure p is defined in the following way. Let us consider an element of fluid of volume dV. The surrounding fluid exerts a force $d\mathbf{F}$ on the surface element $d\mathbf{S}$ given by

$$d\mathbf{F} = -p d\mathbf{S},$$

where p is the pressure at equilibrium.

the parameters of state and the temperature defines the *equation of state*:

$$f(p, v, T) = 0.$$

Another important quantity describing the state of the fluid is the *internal energy* per unit mass u. Work and heat can be regarded as equivalent forms of energy and the change in the internal energy of a mass of fluid at rest gives rise to a change of state. Thus, if the state of a given uniform of mass of fluid is changed by a gain of heat of amount Q per unit mass and by the performance of work on the fluid of amount W per unit mass, the increase in the internal energy per unit mass is

$$\Delta u = Q + W.$$

This is first law of Thermodynamics which is a consequence of the conservation of total energy in the system. The internal energy Δu depends only on the initial and final states (function of state) but Q and W depend on the particular way in which the process between the initial and final states is made. In the case of a reversible process, the differential form of first law is

$$\delta u = \delta Q - \delta W = \delta Q - p\delta v.$$

A practical quantity of some importance is the *specific heat* of the fluid, that is, the amount of heat given to unit mass of the fluid to rise a given amount the temperature. It is defined as

$$c = \frac{\delta Q}{\delta T}.$$

For a process at constant pressure, $c \to c_p$ where

$$c_p = \left(\frac{\delta Q}{\delta T}\right)_{\delta p=0} = \left(\frac{\partial u}{\partial T}\right)_p + p\left(\frac{\partial v}{\partial T}\right)_p.$$

For a process at constant volume, $c \to c_v$ where

$$c_v = \left(\frac{\delta Q}{\delta T}\right)_{\delta v=0} = \left(\frac{\partial u}{\partial T}\right)_v.$$

The second law of Thermodynamics implies the existence of another extensive property of the fluid in equilibrium: the *entropy* S. The entropy per unit mass of a fluid s is defined by the relation

$$T\delta s = \delta Q.$$

For a reversible process,

$$T\delta s = \delta u + p\delta v.$$

The entropy is also a function of state. Another convenient function of state is the *entalphy*. The entalphy per unit mass h is defined as

$$h = u + pv,$$

and has the dimensions of energy per unit mass. Note that

$$\delta h = \delta u + p\delta v + v\delta p = T\delta s + v\delta p.$$

1.4 Euler and Lagrange description

The study of dynamics of fluids can be carried out by considering two different points of view. First, we can pay attention to the different fluid particles[2] and analyze their time evolution. This is called the Lagrangian frame of fluid. Another perspective is to analyze what happens in any point of the system at different times (Eulerian frame). Let us see the relationship between both descriptions. Let us consider the fluid particle ξ which is located in $\mathbf{r}$ at instant t. Let $\mathcal{P}(\mathbf{r}, t)$ be a property of the fluid particle in $(\mathbf{r}, t)$. Thus,

$$\mathcal{P}(\xi, t) = \mathcal{P}(\mathbf{r}(\xi), t). \tag{1.1}$$

This means that the value of $\mathcal{P}$ of fluid particle ξ at t coincides with the value of the property $\mathcal{P}$ at the point $\mathbf{r}$ where is located the particle ξ at instant t.

The time derivative of Eq. (1.1) is

$$\begin{aligned}\frac{D\mathcal{P}}{Dt} \equiv \frac{d\mathcal{P}}{dt} &= \left(\frac{\partial \mathcal{P}(\mathbf{r}(\xi, t), t)}{\partial t}\right)_{\mathbf{r}} + \frac{\partial \mathcal{P}(\mathbf{r}(\xi), t)}{\partial \mathbf{r}} \left(\frac{\partial \mathbf{r}(\xi, t)}{\partial t}\right)_{\xi} \\ &= \left(\frac{\partial \mathcal{P}}{\partial t}\right)_{\mathbf{r}} + \mathbf{v}(\mathbf{r}, t) \cdot \frac{\partial \mathcal{P}}{\partial \mathbf{r}}, \end{aligned} \tag{1.2}$$

where $\mathbf{v}(\mathbf{r}, t)$ is the velocity of fluid particle ξ at t. In other words, it is the velocity at the point $\mathbf{r}$ which is occupied by particle ξ at t. Here, $\partial \mathcal{P}/\partial t|_{\mathbf{r}}$ denotes the local variation of $\mathcal{P}$.

---

[2] Note that here *fluid particles* means a very small volume around a given point (continuum assumption)

CHAPTER 2

IDEAL FLUIDS

2.1 Introduction

Fluid dynamics is devoted to the study of the motion of fluids (liquids and gases). As said in the previous chapter, since the system is seen as *macroscopic*, then the fluid is regarded as a *continuous* medium. This means that any small volume element in the fluid is always supposed so large that it still contains a very great number of molecules. Thus, when we talk on the displacement of some fluid particle, we mean not the motion of an individual particle, but that of a volume element containing many molecules, though still regarded as a point.

From a mathematical point of view, all the hydrodynamic quantities are in general functions of the point $\mathbf{r}$ and of the time t. The state of the fluid is described by the fluid velocity $\mathbf{v}(x, y, z, t)$ and two additional thermodynamic quantities $[\rho(x, y, z, t)$ and $T(x, y, z, t)$ or $p(x, y, z, t)]$. All the thermodynamic quantities are determined by the values of two of them, together the equation of state. We recall again that $\mathbf{v}(x, y, z, t)$ or $\rho(x, y, z, t)$ refers to the values of these quantities at a given point $\mathbf{r}$ of the space and a given time t and not to specific particles of the fluid. We shall now derive the fundamental equations of fluid dynamics. For the sake of simplicity, let us consider first in this Chapter the case of ideal fluids, namely, fluids with zero viscosity and without thermal conduction.

ÍNDICE

2.2 The equation of continuity

We want to express the equation of the conservation of matter. Let V_0 a volume in the space. The mass of fluid in this volume V_0 is

$$\int_{V_0} \rho dV$$

where ρ is the fluid density and the integration is taken over the volume V_0. Let S_0 be the surface bounding V_0. The mass of fluid flowing in unit time through an element $d\mathbf{S}$ bounding the volume is

$$\rho\mathbf{v} \cdot d\mathbf{S}$$

where the direction of $d\mathbf{S}$ is along the perpendicular direction to the surface element. By convention, we take $d\mathbf{S}$ along the outward normal, so that $\rho\mathbf{v} \cdot d\mathbf{S} > 0$ ($\rho\mathbf{v} \cdot d\mathbf{S} < 0$) if the fluid is flowing out (flowing into) the volume. The total mass flowing out the volume per unit time is

$$\oint \rho\mathbf{v} \cdot d\mathbf{S} \tag{2.1}$$

where the integration is taken over the whole of the closed surface surrounding the volume.

Thus, the decrease per unit time in the mass of fluid in V_0 is given by

$$-\frac{\partial}{\partial t}\int \rho dV. \tag{2.2}$$

Equating (2.1) and (2.2), one gets

$$\frac{\partial}{\partial t}\int \rho dV = -\oint \rho\mathbf{v} \cdot d\mathbf{S} = -\int \nabla \cdot (\rho\mathbf{v}) dV, \tag{2.3}$$

where in the last equality we have applied Gauss's theorem. Thus,

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v})\right] dV = 0.$$

Since this equation holds for any volume, the integrand must vanish, i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0. \tag{2.4}$$

This the *equation of continuity*. This can be also written as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0,$$

or equivalently,

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{2.5}$$

The vector $\mathbf{j} = \rho \mathbf{v}$ is called the *mass flux density*. Its direction is that of the motion of the fluid. It is important to remark that the continuity equation (2.5) holds also for non ideal (viscous) fluids.

2.3 Euler's equation

We consider a given element in the fluid. The total force acting on this volume is

$$-\oint p d\mathbf{S},$$

where the pressure p is taken over the surface bounding the volume. Note that here we are neglecting the existence of shearing effects in the fluid since we are assuming an ideal fluid. Transforming the above surface integral to a volume integral, we have

$$-\oint p\, d\mathbf{S} = -\int (\nabla p)\ dV.$$

We see that the fluid surrounding any volume element dV exerts on that volume a force $-dV(\nabla p)$.

We can now write the equation of motion of a volume element in the fluid:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p. \tag{2.6}$$

As mentioned in Chapter 1, $d\mathbf{v}/dt$ means the rate of change of the fluid velocity of a given fluid particle which is moving in the space. Since

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,$$

then Eq. (2.6) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\rho^{-1} \nabla p,$$

or, equivalently,

$$\frac{\partial v_i}{\partial t} + v_j \nabla_j v_i = -\rho^{-1} \nabla_i p, \tag{2.7}$$

where $\partial_t \equiv \partial/\partial t$ and $\nabla_i \equiv \partial/\partial r_i$. The equation of motion (2.7) was first obtained by L. Euler in 1755. It is called *Euler's equation* and is one of the fundamental equations in fluid dynamics. In deriving this equation we have not accounted for processes of energy dissipation (internal friction due to viscosity and heat exchange between different parts of the system).

If the fluid is in a gravitational field, an additional force $\rho \mathbf{g}$ acts on any unit of volume. Here, $\mathbf{g}$ is the acceleration due to gravity. This force must be added to the right hand side of the above equation, and so one finally gets

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\rho^{-1} \nabla p + \mathbf{g}. \tag{2.8}$$

Since the motion does not involve any heat exchange, then the motion is adiabatic and consequently, the entropy remains constant. Denoting s the entropy per unit mass, then $ds/dt = 0$, or,

$$\partial_t s + \mathbf{v} \cdot \nabla s = 0. \tag{2.9}$$

Using Eq. (2.4), then

$$\partial_t(\rho s) = s\partial_t \rho + \rho \partial_t s = \rho \partial_t s - s \nabla \cdot (\rho \mathbf{v}) = -\nabla \cdot (\rho s \mathbf{v}).$$

Therefore, Eq. (2.9) can be rewritten as

$$\partial_t(\rho s) + \nabla \cdot (\rho s \mathbf{v}) = 0. \tag{2.10}$$

This is an "equation of continuity" for the entropy. Here, $\rho s \mathbf{v}$ is the entropy flux density.

Since the motion is *isentropic* ($s \equiv$ constant), then

$$dh = dq + vdp = dq + \frac{dp}{\rho} = Tds + \frac{dp}{\rho} = \frac{dp}{\rho},$$

where we have taken into account that $ds = 0$. Thus, in this case, $\nabla h = \rho^{-1} \nabla p$ and the Euler equation (2.8) (in the absence of gravity) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla h. \tag{2.11}$$

The Euler equation can be also written in a different form that only involves the velocity field. In order to do it, let us consider the relation

$$\frac{1}{2}\nabla v^2 = \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{v}.$$

Thus, Eq. (2.11) can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 - 2\mathbf{v} \times \boldsymbol{\omega} = -\nabla h, \tag{2.12}$$

where $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$ is the vorticity. Taking the vector product $(\nabla\times)$ in both sides of Eq. (2.12), one finally gets

$$\partial_t(\mathrm{rot}\mathbf{v}) = \mathrm{rot}(\mathbf{v} \times \mathrm{rot}\mathbf{v}), \tag{2.13}$$

where

$$\mathrm{rot}\mathbf{A} \equiv \nabla \times \mathbf{A}.$$

2.4 Hydrostatics

For a fluid at rest in a uniform gravitational field, Euler's equation takes the form

$$\nabla p = \rho \mathbf{g}. \tag{2.14}$$

If there are no external forces, then $p \equiv$ constant (mechanical equilibrium of fluid). In the case that $\rho \approx$ constant and $\mathbf{g} = -g\hat{\mathbf{k}}$, then Eq. (2.14) becomes

$$\partial_x p = \partial_y p = 0, \quad \partial_z p = -\rho g, \tag{2.15}$$

and hence,

$$p(z) = -\rho g z + \text{constant}. \tag{2.16}$$

If $p = p_0$ at $z = h_0$ (free surface which an external pressure p_0), then

$$p(z) = p_0 + \rho g(h_0 - z). \tag{2.17}$$

This equation does not hold for a gas since ρ depends on p.

2.5 The condition that convection is absent

A fluid can be in mechanical equilibrium (i.e., exhibit no macroscopic motion) without being in thermal equilibrium. Equation (2.14) can be satisfied even if the temperature is not constant throughout the fluid. However, a good question is to analyze the stability of this *equilibrium* state. As we will show later, it is found that the equilibrium is *stable* only when a certain conditions are satisfied. Otherwise, the equilibrium is *unstable* and this yields the presence of currents which tend to mix the fluid to equalize the temperature. This motion is called *convection*. Thus, the condition for a mechanical equilibrium to be stable is the condition that convection be absent.

Let us consider an element of fluid at height z with a specific volume $v(p,s)$, where p and s are the pressure and entropy at z. We assume that this element of volume undergoes an adiabatic upward displacement along a small interval ξ. In this case, its specific volume becomes $v(p',s)$, where p' is the pressure at height $z+\xi$. The equilibrium is stable if (though not in general sufficient) the resulting force acting on this element tends to return it to its original position. This means that the element must be heavier than the fluid which it "displaces" to its new position. The specific volume of the latter is $v(p',s')$ where s' is the entropy at $z+\xi$. The stability condition is

$$v(p',s') > v(p',s)$$

Since $s' = s + \xi ds/dz$, then

$$v(p',s') \simeq v(p',s) + \left(\frac{\partial v}{\partial s}\right)_p (s'-s) = v(p',s) + \left(\frac{\partial v}{\partial s}\right)_p \xi \frac{ds}{dz}.$$

Therefore, the above stability condition yields the result

$$\left(\frac{\partial v}{\partial s}\right)_p \frac{ds}{dz} > 0. \tag{2.18}$$

Now, we use the thermodynamic relation

$$\left(\frac{\partial v}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial v}{\partial T}\right)_p,$$

where c_p is the specific heat at constant pressure. Since $c_p > 0$ and $T > 0$, then Eq. (2.18) can be written as

$$\left(\frac{\partial v}{\partial T}\right)_p \frac{ds}{dz} > 0. \tag{2.19}$$

In most of the substances, $(\partial v/\partial T)_p > 0$ so that Eq. (2.19) simply becomes

$$\frac{ds}{dz} > 0. \tag{2.20}$$

The entropy must increase with the height to be absent the convection. We want to determine the condition for the temperature gradient. To do it, let us express the derivative ds/dT in terms of dT/dt as

$$\frac{ds}{dz} = \left(\frac{\partial s}{\partial T}\right)_p \frac{dT}{dz} + \left(\frac{\partial s}{\partial p}\right)_T \frac{dp}{dz}.$$

Taking into the thermodynamic relations

$$\left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p}{T}, \quad \left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p,$$

the condition (2.20) becomes

$$\frac{dT}{dz} > -\frac{gT}{c_p v}\left(\frac{\partial v}{\partial T}\right)_p \equiv -\frac{g\beta T}{c_p}. \tag{2.21}$$

Upon deriving (2.21) use has been made of the relation $dp/dz = -g/v$. In summary, convection occurs if these conditions are not satisfied, i.e., if the temperature decreases upwards with a gradient which magnitude exceeds the value $g\beta T/c_p$.

2.6 Bernoulli's equation

Let us consider a *steady* flow ($\partial_t \to 0$). In this case, Euler's equation (2.12) becomes

$$\frac{1}{2}\nabla v^2 - 2\mathbf{v}\times\boldsymbol{\omega} = -\nabla h. \tag{2.22}$$

The *streamlines* are defined as

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}.$$

In steady states, the streamlines do not change with time. Let $\boldsymbol{\ell}$ the unit vector tangent to the streamline in each point. Let us take the scalar product of Eq. (2.22) with $\boldsymbol{\ell}$. Since, $\mathbf{v} \times \mathrm{rot}\mathbf{v} \perp \mathbf{v}$, then the projection of (2.22) on $\boldsymbol{\ell}$ is

$$\frac{\partial}{\partial \ell}\left(\frac{1}{2}v^2 + h\right) = 0. \tag{2.23}$$

Thus, the combination $\frac{1}{2}v^2 + h$ is constant along a streamline, i.e,

$$\frac{1}{2}v^2 + h \equiv \text{const.} \tag{2.24}$$

Equation (2.24) is called *Bernoulli's equation.* If there is some gravitational field, then $\mathbf{g} = -\nabla(gz)$ and Bernoulli's equation becomes

$$\frac{1}{2}v^2 + h + gz \equiv \text{const.} \tag{2.25}$$

2.7 The energy flux

Let us consider some volume element fixed in space and find the variation with time of the energy of the fluid contained in this element. The energy of unit volume of fluid is $\frac{1}{2}\rho v^2 + \rho u$ where the first term is the kinetic energy and the second is the internal energy; u is the internal energy per unit mass. We want to calculate the time evolution of this quantity. The first term is

$$\partial_t(\frac{1}{2}\rho v^2) = \frac{1}{2}v^2 \partial_t \rho + \rho \mathbf{v} \cdot \partial_t \mathbf{v}$$

Using the continuity equation (2.4) and Euler's equation (2.6), one gets

$$\partial_t(\frac{1}{2}\rho v^2) = -\frac{1}{2}v^2 \nabla_i(\rho v_i) - v_i \nabla_i p - \rho v_i v_j \nabla_j v_i.$$

This relation can be written in a more compact form when one takes into account the results

$$\nabla_i v^2 = 2v_j \nabla_i v_j, \quad \nabla_i p = \rho \nabla_i h - \rho T \nabla_i s.$$

Thus,

$$\begin{aligned} \partial_t(\frac{1}{2}\rho v^2) &= -\frac{1}{2}v^2 \nabla_i(\rho v_i) - \rho v_i \nabla_i h + \rho T v_i \nabla_i s - \frac{1}{2}\rho v_i \nabla_i v^2 \\ &= -\frac{1}{2}v^2 \nabla_i(\rho v_i) - \rho v_i \nabla_i \left(\frac{1}{2}v^2 + h\right) + \rho T v_i \nabla_i s. \end{aligned} \tag{2.26}$$

The other derivative is $\partial_t(\rho u)$. The first Thermodynamics Law gives

$$du = Tds - pd(\rho^{-1}) = Tds + \frac{p}{\rho^2}d\rho,$$

and so,

$$d(\rho u) = ud\rho + \rho Tds + \frac{p}{\rho}d\rho = hd\rho + \rho Tds.$$

Taking into account these results, one gets

$$\partial_t(\rho u) = h\partial_t\rho + \rho T\partial_t s = -h\nabla\cdot(\rho\mathbf{v}) - \rho T\mathbf{v}\cdot\nabla s. \tag{2.27}$$

Combining Eqs. (2.26) and (2.27), after some algebra one achieves

$$\begin{aligned}\partial_t\left(\frac{1}{2}\rho v^2 + \rho u\right) &= -\frac{1}{2}v^2\nabla_i(\rho v_i) - \rho v_i\nabla_i\left(\frac{1}{2}v^2 + h\right) + \rho T v_i\nabla_i s \\ &\quad -h\nabla_i(\rho v_i) - \rho T v_i\nabla_i s \\ &= -\left(\frac{1}{2}v^2 + h\right)\nabla\cdot(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\nabla\left(\frac{1}{2}v^2 + h\right) \\ &= -\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)\right]. \qquad (2.28)\end{aligned}$$

In order to see the physical meaning of Eq. (2.28), let us integrate it over some volume:

$$\begin{aligned}\frac{\partial}{\partial t}\int\left(\frac{1}{2}\rho v^2 + \rho u\right)dV &= -\int\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)\right]dV \\ &= -\oint\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)\cdot d\mathbf{S}, \qquad (2.29)\end{aligned}$$

where use has been made of Gauss's theorem in the last step. The left hand side of Eq. (2.29) is the rate of change of energy of the fluid in some given volume while the right hand side is the amount of energy flowing out in unit time through S. Therefore, the quantity $\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)$ may be called the *energy flux density* vector. Its magnitude is the amount of energy crossing in unit time through unit area perpendicular to the direction of the velocity. Thus, any unit mass of fluid carries with it during its motion an amount of energy $\frac{1}{2}v^2 + h$.

The fact that the enthalpy h appears in Eq. (2.29) instead of u has a simply physical significance. Putting $h = u + p/\rho$, we can write the flux of energy through a closed surface in the form

$$-\oint \rho\mathbf{v}\cdot\left(\frac{1}{2}v^2+h\right)d\mathbf{S} = -\oint \rho\mathbf{v}\cdot\left(\frac{1}{2}v^2+u\right)d\mathbf{S} - \oint p\mathbf{v}\cdot d\mathbf{S}. \quad (2.30)$$

The first term is the energy (kinetic and internal) transported through the surface in unit time by the mass of fluid. The second term is the work done by pressure forces on the fluid within the surface.

2.8 The momentum flux

We shall now give a similar series of arguments for the momentum of the fluid. The momentum of unit volume is $\rho\mathbf{v}$. Let us determine its rate of change $\partial_t(\rho\mathbf{v})$. We shall use tensor notation. We have

$$\partial_t(\rho v_i) = \rho\partial_t v_i + v_i\partial_t\rho.$$

As before, using the equation of continuity

$$\partial_t\rho = -\nabla_i(\rho v_i),$$

and Euler's equation

$$\partial_t v_i = -v_j\nabla_j v_i - \rho^{-1}\nabla_i p,$$

we obtain

$$\partial_t(\rho v_i) = -\rho v_j\nabla_j v_i - \nabla_i p - v_i\nabla_j(\rho v_j) = -\nabla_i p - \nabla_j(\rho v_i v_j).$$

We write the first term on the right hand side in the form

$$\nabla_i p = \delta_{ij}\nabla_j p,$$

and finally obtain

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial\Pi_{ij}}{\partial r_j}, \quad (2.31)$$

where the tensor Π_{ij} is defined as

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j. \quad (2.32)$$

This tensor is symmetric, i.e., $\Pi_{ij} = \Pi_{ji}$.

In order to see the physical meaning of the tensor Π_{ij}, we integrate Eq. (2.31) over some volume:

$$\partial_t \int \rho v_i dV = - \int \left(\nabla_j \Pi_{ij}\right) \, dV.$$

The integral on the right hand side is transformed into a surface integral:

$$\partial_t \int \rho v_i dV = - \oint \Pi_{ij} dS_j. \tag{2.33}$$

The left hand side of Eq. (2.33) is the rate of change of the ith component of the momentum contained in the volume considered. The surface integral on the right is therefore the amount of momentum flowing out through the bounding surface in unit time. Consequently, $\Pi_{ij} dS_j$ is the ith component of the momentum flowing through the surface element dS. If we write $dS_j = n_j dS$ (where $\hat{\mathbf{n}}$ is a unit vector along the outward normal), then $\Pi_{ij} n_j$ is the flux of the ith component of momentum through unit surface area. Note that $\Pi_{ij} n_j = p n_i + \rho v_i v_j n_j$. Thus, Π_{ij} is the ith component of the amount of momentum flowing in unit time through unit area perpendicular to the j axis. The tensor Π_{ij} is called the *momentum flux density tensor.* The momentum flux is determined by a tensor of rank two since the momentum itself is a vector.

In vector form, the quantity $\Pi_{ij} n_j$ can be written as

$$p\hat{\mathbf{n}} + \rho \mathbf{v}(\mathbf{v} \cdot \hat{\mathbf{n}}).$$

This expression gives the momentum flux in the direction of $\hat{\mathbf{n}}$. If we takes $\hat{\mathbf{n}} || \mathbf{v}$, we find that only the longitudinal component of momentum is transported in this direction and its flux density is $p + \rho v^2$. In a direction perpendicular to $\mathbf{v}$, only the transverse component to $\mathbf{v}$ of momentum is transported, its flux density is simply p.

2.9 Incompressible fluids

In many of the usual situations in the flow of liquids (and also for gases), their density may be supposed constant throughout the volume of the fluid and throughout its motion (no significant compression or expansion of the fluids in such cases). We speak of *incompressible flow.*

In this case ($\rho \approx$ const.), the equation of continuity takes the simple form

$$\nabla \cdot \mathbf{v} = 0, \tag{2.34}$$

and Euler's equation reduces to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \left(\frac{p}{\rho} \right) + \mathbf{g}. \tag{2.35}$$

Thus, the fundamental system of equations in fluid dynamics for an incompressible fluid only involves the velocity field since ρ is no longer unknown.

Bernoulli's equation can be also written in a more simple form since $\nabla(p/\rho) = \nabla h$, and so it reduces to

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz \equiv \text{const.} \tag{2.36}$$

CHAPTER 3

VISCOUS FLUIDS

3.1 The equations of motion of a viscous fluid

We want to analyze in this Chapter the effect of energy dissipation (by internal viscous friction) occurring during the motion of a fluid particle. This process is the result of the thermodynamic irreversibility of the motion. This irreversibility occurs essentially due to internal friction (viscosity) and thermal conduction. Here, we will pay attention to the former process.

In order to include these effects on the motion of a viscous fluid, we have to include some additional terms in the equation of motion of an ideal fluid. The equation of continuity holds for any fluid, whether viscous or not. On the other hand, Euler's equation (2.7) requires modification. To extend Euler's equation to viscous fluids, let us write first it in the form

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial r_k}, \tag{3.1}$$

where Π_{ik} is the momentum flux density tensor. It is given by

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k. \tag{3.2}$$

The expression (3.2) for Π_{ij} represents a completely reversible transfer of momentum, due simply to the mechanical transport of the different particles of fluid from one place to other place and to the pressure forces acting in the fluid. The viscosity (internal friction) causes another irreversible transfer of momentum from points where the velocity is large to those where it is small.

The equation of motion of a viscous fluid may therefore be obtained by adding to the "ideal" momentum flux (3.2) a term $-\sigma'_{ik}$ which accounts for the *irreversible* viscous transfer of momentum in the fluid. Thus, we write the momentum flux density tensor in a viscous fluid in the form

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \sigma'_{ik} \equiv -\sigma_{ik} + \rho v_i v_k. \tag{3.3}$$

The tensor

$$\sigma_{ik} = -p\delta_{ik} + \sigma'_{ik} \tag{3.4}$$

is called the *stress tensor*. This tensor gives the part of the momentum flux that is not due to the direct transfer of momentum with the mass of moving fluid.

We should establish the general form of σ'_{ik}. It is quite evident that processes of internal friction occur in a fluid only when different fluid particles move with different velocities, so that there is a relative motion between various parts of the fluid. Hence, σ'_{ik} must depend on the space derivatives of the velocity. If the velocity gradient is small, we can assume that σ'_{ik} is proportional to the first spatial derivatives of the velocity. There can be no terms in σ'_{ik} independent of $\partial v_i/\partial r_k$ since σ'_{ik} must vanish when $\mathbf{v} \equiv$ constant. Next, we notice that σ'_{ik} must also vanish when the whole fluid is in uniform rotation since it is clear that in such motion no internal friction occurs in the fluid. In uniform rotation with angular velocity $\boldsymbol{\Omega}$, the velocity $\mathbf{v}$ is equal to the vector product $\boldsymbol{\Omega} \times \mathbf{r}$. The sums

$$\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i}$$

are linear combinations of the spatial derivatives $\partial v_i/\partial r_k$ and vanish when $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$. Hence, σ'_{ik} must contain just these symmetrical combinations of the derivatives $\partial v_i/\partial r_k$.

The most general tensor of rank two satisfying the above two conditions is

$$\sigma'_{ik} = a\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i}\right) + b\delta_{ik}\frac{\partial v_\ell}{\partial r_\ell},$$

where a and b are independents of $\mathbf{v}$. It is more convenient to write this equation in the form

$$\sigma'_{ik} = \eta\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\ell}{\partial r_\ell}\right) + \zeta\delta_{ik}\frac{\partial v_\ell}{\partial r_\ell}, \tag{3.5}$$

where η and ζ are independents of $\mathbf{v}$. Since the fluid is isotropic, its properties must be described by scalar quantities only (in this case η and ζ). Note that the tensor

$$\Delta_{ik} \equiv \frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\ell}{\partial r_\ell}$$

is traceless ($\Delta_{kk} \equiv \mathrm{Tr}\boldsymbol{\Delta} = \Delta_{xx} + \Delta_{yy} + \Delta_{zz} = 0$). The constants η and ζ are called *coefficients of viscosity*; η is usually called *shear viscosity* and ζ is the *bulk viscosity.* As we shall show later, both coefficients are *positive.*

The equations of motion of a viscous fluid can now be obtained by simply adding the term $\partial\sigma'_{ik}/\partial r_k$ to the right hand side of Euler's equation (3.1). Therefore, after taking into account the equation of continuity, one gets

$$\begin{aligned}\rho\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial r_k}\right) &= -\frac{\partial p}{\partial r_i} + \frac{\partial \sigma'_{ik}}{\partial r_k} \\ &= -\frac{\partial p}{\partial r_i} + \frac{\partial}{\partial r_k}\left[\eta\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\ell}{\partial r_\ell}\right)\right] \\ &\quad + \frac{\partial}{\partial r_i}\left(\zeta\frac{\partial v_\ell}{\partial r_\ell}\right). \qquad (3.6)\end{aligned}$$

This is the most general form of the equations of motion of a viscous fluid. The quantities η and ζ are functions of the pressure and temperature. In general, p and T (and so, η and ζ) are not constants throughout the fluid, so that η and ζ cannot be taken outside the gradient operator.

In most cases, however, the viscosity coefficients do not change significantly in the fluid and they may be considered as constants. In this case, and in vector form, Eq. (3.6) becomes

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p + \eta\nabla^2\mathbf{v} + (\zeta + \frac{1}{3}\eta)\nabla(\nabla\cdot\mathbf{v}), \qquad (3.7)$$

where use has been made of the partial result

$$\nabla_k\sigma'_{ik} = \eta\nabla^2 v_i + (\zeta + \frac{1}{3}\eta)\nabla_i\nabla_k v_k.$$

Equation (3.7) is called the *Navier-Stokes* equation. It has been deduced in the absence of a gravity field. The Navier-Stokes equation becomes simpler if the fluid is incompressible ($\nabla\cdot\mathbf{v} = 0$). In this case, the last term on the right hand side of Eq. of (3.7) vanishes and one gets

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{v}, \qquad (3.8)$$

where

$$\nu = \frac{\eta}{\rho} \tag{3.9}$$

is the *kinematic viscosity.* The stress tensor in an incompressible fluid takes the simpler form

$$\sigma_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i}\right). \tag{3.10}$$

The pressure can be eliminated in Eq. (3.8) by taking the rotational in both sides of this equation. The result is

$$\frac{\partial}{\partial t}(\mathrm{rot}\mathbf{v}) = \mathrm{rot}(\mathbf{v} \times \mathrm{rot}\mathbf{v}) + \nu\nabla^2(\mathrm{rot}\mathbf{v}). \tag{3.11}$$

In the presence of the gravity field, the Navier-Stokes equation (3.7) becomes

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p + \rho\mathbf{g} + \eta\nabla^2\mathbf{v} + (\zeta + \frac{1}{3}\eta)\nabla(\nabla\cdot\mathbf{v}). \tag{3.12}$$

3.2 Energy dissipation in an incompressible fluid

The presence of viscosity gives rise to the dissipation of energy, which is finally transformed in heat. The evaluation of the energy dissipation is relatively simple for an incompressible fluid.

For an incompressible fluid, the total kinetic energy is

$$E_{\mathrm{kin}} = \frac{1}{2}\rho\int v^2\, dV$$

We take the time derivative of the kinetic energy $\frac{1}{2}\rho v^2$ by using the Navier-Stokes equation:

$$\begin{aligned}
\partial_t(\frac{1}{2}\rho v^2) &= \rho v_i \partial_t v_i \\
&= -\rho v_i v_k \nabla_k v_i - v_i \nabla_i p + v_i \nabla_k \sigma'_{ik} \\
&= -\rho v_k \nabla_k\left(\frac{1}{2}v^2 + \frac{p}{\rho}\right) + \nabla_k(v_i\sigma'_{ik}) - \sigma'_{ik}\nabla_k v_i \\
&= -\nabla_k\left[\rho v_k\left(\frac{1}{2}v^2 + \frac{p}{\rho}\right) - v_i\sigma'_{ik}\right] - \sigma'_{ik}\nabla_k v_i,
\end{aligned} \tag{3.13}$$

where in the last step use has been made of the result $\nabla \cdot \mathbf{v} = 0$. In vector form,

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2) = -\nabla \cdot \left[\rho \mathbf{v}\left(\frac{1}{2}v^2 + \frac{p}{\rho}\right) - \mathbf{v}\cdot\boldsymbol{\sigma}'\right] - \boldsymbol{\sigma}' : \nabla \mathbf{v}. \tag{3.14}$$

Here, we have introduced the notation $\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ji}$. In Eq. (3.14), the expression in brackets is just the energy flux density in the fluid: the term $\rho\mathbf{v}(\frac{1}{2}v^2 + p/\rho)$ is the energy flux due to the actual transfer of fluid mass and is the same as the energy flux in an ideal fluid. The second term $\mathbf{v}\cdot\boldsymbol{\sigma}'$ is the energy flux due to processes of internal friction. The presence of viscosity gives rise to a momentum flux σ'_{ik}. A transfer of momentum always involves a transfer of energy and the energy flux is clearly equal to the scalar product of the momentum flux and the velocity.

Let us integrate Eq. (3.14) over some volume V. The result is

$$\frac{\partial}{\partial t}\int \frac{1}{2}\rho v^2 dV = -\oint \left[\rho \mathbf{v}\left(\frac{1}{2}v^2 + \frac{p}{\rho}\right) - \mathbf{v}\cdot\boldsymbol{\sigma}'\right]\cdot d\mathbf{S} - \int \sigma'_{ik}\nabla_k v_i dV. \tag{3.15}$$

The first term on the right hand side gives the rate of change of the kinetic energy of the fluid in V due to the energy flux through the surface bounding V. The integral in the second term gives the decrease of the kinetic energy per unit time due to viscous dissipation. We extend the integration in Eq. (3.15) to the whole volume of the fluid. In this case, since the velocity vanishes at infinity (or for a fluid enclosed in a finite volume, the surface integral vanishes because the velocity at surface vanishes), then the surface integral vanishes and so, the energy dissipated per unit time in the whole fluid is

$$\frac{dE_{\text{kin}}}{dt} = -\int \sigma'_{ik}\nabla_k v_i dV = -\frac{1}{2}\int \sigma'_{ik}\left(\nabla_k v_i + \nabla_i v_k\right) dV,$$

since $\sigma'_{ik} = \sigma'_{ki}$. For an incompressible fluid,

$$\sigma'_{ij} = \eta\left(\nabla_i v_k + \nabla_k v_i\right),$$

and so,

$$\sigma'_{ik}\left(\nabla_k v_i + \nabla_i v_k\right) = \eta(\nabla_i v_k + \nabla_k v_i)^2.$$

Therefore, one gets the important final result

$$\frac{dE_{\text{kin}}}{dt} = -\frac{1}{2}\eta\int dV\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i}\right)^2. \tag{3.16}$$

Since the dissipation yields a decrease in the mechanical energy, then

$$\frac{dE_{\rm kin}}{dt} < 0.$$

Consequently, the shear viscosity coefficient η is always *positive.*

3.3 Flow in a pipe

We shall now consider some simple problems of motion of an incompressible viscous fluid.

Let the fluid be enclosed between two parallel plates moving with a constant relative velocity $\mathbf{U}$. We also neglect the influence of gravity. We assume that the y-axis is orthogonal to the plates while the flow is in the x-direction. Under these conditions, $v_x(y) \neq 0$ and $v_y = v_z = 0$. The boundary conditions are $v_x(h) = U$ and $v_x(0) = 0$. It is clear that all the quantities depend on space only through the y coordinate. For steady flow, the equation of continuity is satisfied identically while the Navier-Stokes equation (3.8) reads

$$\left(v_y \frac{\partial}{\partial y}\right) v_j = -\rho^{-1} \nabla_j p + \nu \nabla^2 v_j.$$

For $j = x$, $\partial_y^2 v_x = 0$ while for $j = y$, $\partial_y p = 0$. Then, $p \equiv$ const. and $v_x(y) = ay + b$. Boundary conditions imply $b = 0$ and $a = U/h$. Thus, $v_x(y) = (U/h)y$.

Let us consider now a steady flow between two parallel plates at rest in the presence of a pressure gradient. We choose the coordinates as before; the x-axis is in the direction of motion of fluid. Since the velocity clearly depends only on the coordinate y, the Navier-Stokes equations give

$$\partial_y^2 v_x = \eta^{-1} \partial_x p, \quad \partial_y p = 0, \tag{3.17}$$

where we have also assumed the absence of gravity. The last identity in Eq. (3.17) implies that p does not depend on y. Consequently, the combination $\eta^{-1}\partial_x p$ is only a function of x. Since $v_x(y)$ depends only on y, then both terms are constant:

$$\partial_x p \equiv \text{const.}, \quad \partial_y^2 v_x \equiv \text{const.}$$

The x- component of the velocity field is

$$v_x(y) = \frac{1}{2\eta}\frac{dp}{dx}y^2 + ay + b.$$

Boundary conditions: $v_x(0) = v_x(h) = 0$. Thus, $b = 0$ and

$$a = -\frac{h}{2\eta}\frac{dp}{dx}.$$

The final result is

$$v_x(y) = \frac{1}{2\eta}\frac{dp}{dx}y^2 - \frac{1}{2\eta}\frac{dp}{dx}hy = -\frac{1}{2\eta}\frac{dp}{dx}\left[\frac{1}{4}h^2 - \left(y - \frac{1}{2}h\right)^2\right]. \tag{3.18}$$

Equation (3.18) clearly shows a parabolic spatial variation of the velocity field. Its maximum value $v_{x,\max}$ is located at the middle of the pipe, i.e., at $y = h/2$:

$$v_{x,\max} = -\frac{1}{8\eta}\frac{dp}{dx}h^2.$$

3.4 The law of similarity

In the study of motion of viscous fluids, one can obtain a number of important results from simple dimensional arguments of various physical quantities. Let us consider for instance, the motion of a body of some definite shape through a fluid. If the body is not spherical, we have also to specify its direction of motion, e.g., the motion of an ellipsoid in the direction of its greatest or least axis. Thus, we can say that bodies of the same shape are *geometrically similar*.

We are going to consider steady flows. We discuss for instance flow past a solid body where the velocity of the main stream must therefore be constant. We shall suppose the fluid incompressible. Regarding parameters of the fluid, only the kinematic viscosity $\nu = \eta/\rho$ appears in the Navier-Stokes equations. The unknown functions which have to determine by solving the Navier-Stokes equations are the velocity $\mathbf{v}$ and the ratio between the pressure and the mass density p/ρ (since ρ is assumed to be constant). Moreover, the flow depends (through the boundary conditions) on the shape and the dimensions of the body moving through the fluid and on its velocity. Since the shape of the body is supposed given, its geometrical properties are determined by one linear dimension, which we denote by ℓ. Let the velocity of the main stream be u. Then, any flow is specified by three parameters: $\{\nu, \ell, u\}$. These quantities have the following dimensions:

$$[\nu] = L^2T^{-1}, \quad [\ell] = L, \quad [u] = LT^{-1}.$$

The only *dimensionless* quantity built from the above three quantities is $u\ell/\nu$. This combination is called the *Reynolds number* Re:

$$\mathrm{Re} \equiv \frac{\rho u \ell}{\eta} = \frac{u\ell}{\nu}. \tag{3.19}$$

Any other dimensionless parameter can be written as a function of Re.

Therefore, we will measure lengths in terms of ℓ and velocities in terms of u. In other words, we introduce the dimensionless quantities $\mathbf{r}/\ell$ and $\mathbf{v}/u$. Since the only dimensionless parameter is the Reynolds number, it is evident that the velocity field obtained after solving the equations of incompressible flow is given by a function having the form

$$\mathbf{v} = u\mathbf{f}(\mathbf{r}/\ell, \mathrm{Re}), \tag{3.20}$$

where $\mathbf{f}$ is an unknown function. It is seen from Eq. (3.20) that, in two different flows of the same type (for example, flow past spheres with different radii by fluids with different viscosities), the velocities $\mathbf{v}/u$ are the same functions of the ratio $\mathbf{r}/\ell$ if the Reynolds number is the same in each flow. Flows which can be obtained from one another by simply changing the unit of measurement of coordinates and velocities are said to be *similar*. Thus, flows of the same type with the same Reynolds number are similar. This is called the *law of similarity* (O. Reynolds 1883).

Let us see the dynamical similarity law in a more explicit way for steady incompressible flows. In this case, the motion of the fluid when $\rho \equiv$ const. is governed by the equations

$$\nabla \cdot \mathbf{v} = 0, \tag{3.21}$$

$$v_j \frac{\partial v_i}{\partial r_j} = -\rho^{-1}\frac{\partial p}{\partial r_i} + \nu \frac{\partial^2 v_i}{\partial r_j \partial r_j}. \tag{3.22}$$

We propose now to assess the effect on the flow of changes in the (uniform) values of ρ and ν. To this end it is useful to write Eqs. (3.21) and (3.22) in dimensionless form. Since no parameters with the dimensions of length and velocity appear in the above equations, we have to look for in the boundary conditions to find quantities to define dimensionless variables.

Let us suppose that the specification of the boundary conditions for a particular flow involves some representative length L and some representative velocity U. In this case, we define the following dimensionless space and velocity variables:

$$\mathbf{r}' \equiv \frac{\mathbf{r}}{\ell}, \quad \mathbf{v}' \equiv \frac{\mathbf{v}}{u}. \tag{3.23}$$

In addition, the pressure p can be scaled with a given reference pressure $p_0 = \rho u^2$, such that

$$p' = \frac{p}{p_0}$$

is also a dimensionless quantity. It is straightforward to see that

$$\frac{\partial^n}{\partial r_i^n} = \left(\frac{1}{\ell}\right)^n \frac{\partial^n}{\partial r_i'^n}$$

so that, in terms of the dimensionless variables $\mathbf{r}'$ and $\mathbf{v}'$, Eqs. (3.21) and (3.22) become

$$\frac{\partial v_i'}{\partial r_i'} = 0, \tag{3.24}$$

$$v_j' \frac{\partial v_i'}{\partial r_j'} = -\frac{\partial p'}{\partial r_i'} + \frac{1}{\mathrm{Re}} \frac{\partial^2 v_i'}{\partial r_j' \partial r_j'}. \tag{3.25}$$

These equations contain only explicitly the (dimensionless) Reynolds number Re. The solution for the dependent variables $\mathbf{v}'$ and p' that satisfies the boundary conditions can depend *only* on $\mathbf{r}'$ and the parameter Re. Thus, once a solution for a particular flow field is known and is expressed in dimensionless form, a family of solutions can be obtained from it by choosing the values of ρ, ℓ, u and η in such a way that the value of the Reynolds number Re remains *unchanged*. All those flows satisfying the same boundary and initial conditions and for which the values of ρ, ℓ, u, and η differ but share the same value of the (dimensionless) combination $\rho\ell u/\nu$ are described by the same non-dimensional solution. As we mentioned before, all these flows are said to be *dynamically similar*.

This principle of similarity is used widely as a means to get information about an unknown flow field from "model tests", namely, from experiments carried out under physical conditions more convenient than those of the unknown flow field. This is one of most practical applications of the above principle.

3.5 Reynolds equations for turbulent flows

For any problem of viscous flow under certain conditions there must in principle exist an exact solution to the equations of fluid dynamics. These solutions exist for all Reynolds numbers. However, even if one finds an exact solution to the equations of motion, it is possible that this solution does not occur in Nature. This means that not only such a solution must obey the equations of motion of fluid dynamics, but also this solution must be *stable*. Any small perturbation which arise must decrease in time. Otherwise, the small perturbations which inevitably happen in the flow tend to increase in time so that, the flow is unstable and cannot exist.

From an experimental point of view, the phenomenon of *turbulence* was discovered many years ago and it has been widely observed in many realistic situations. When one analyzes with detail the motion of a fluid particle, one see that, under certain conditions, the hydrodynamic variables have small and quick fluctuations around a mean value. Fluctuations means small, fast and disordered variations of the hydrodynamic fields. These fluctuations are produced by the presence of external inputs that eventually change the evolution of flow. In fact, they can slightly modify the fluid conditions of the flow.

Under some conditions, the fluctuations vanish quickly and the system goes back to the initial situation. In this case, we can say that fluctuations do not play a significant role in the fluid motion and they are a consequence of our limitations to monitor the external conditions of the flow. On the other hand, under certain conditions, it is also possible that those fluctuations do not disappear and the flow has random properties. In this latter case, the source of fluctuations is not external and they must be accounted for in the dynamics of the system.

According to this picture, we can say that there are essentially two different regimes of flows. In the former case (no significant fluctuations), the flow is in *laminar* regime while in the latter case (significant fluctuations) the flow is *turbulent*. There is no a very well delimited separation between both regimes, although it is admitted that if the fluctuations are below 2%

ÍNDICE

of its mean value, then the regime is laminar; otherwise the regime becomes turbulent. To be a bit more precise, let us consider a hydrodynamic variable or field $A(t)$ at a given point. The behavior of the mean value $\overline{A}(t)$ is different in both regimes since for instance the velocity and energy fluctuations modify significantly the transport of momentum and energy in the fluid. In other words, the spatial and time variations of $\overline{A}(t)$ are quite different in both regimes. Thus, for instance, the flow in a cylindrical pipe is parabolic in laminar conditions while it is logarithmic in turbulent conditions.

Using empirical arguments, Reynolds was able in 1883 to classify the laminar and turbulent regimes in terms of the Reynolds number Re. Specifically, when $0 < \mathrm{Re} < 10^3$, the flow is laminar; when $10^3 < \mathrm{Re} < 10^4$, there is not well defined, while when $\mathrm{Re} > 10^4$ the flow is turbulent.

3.5.1 Reynolds equations

As said before, the motion of fluid in turbulent conditions is characterized by small fluctuations around a mean value. These fluctuations tend to increase in time. Since the solution to the Navier-Stokes equations has a *stochastic* component, the problem is in general quite intricate due to the mathematical difficulties involved in it. Thus, our goal is to get the average values of the hydrodynamic quantities. To be more precise, let us consider the quantity $a(\mathbf{r}, t)$. Its value at a given time t is

$$a(\mathbf{r}, t) = \overline{a}(\mathbf{r}, t) + a'(\mathbf{r}, t),$$

where

$$\overline{a}(\mathbf{r}, t) \equiv \langle a(\mathbf{r}, t)\rangle = \frac{1}{\tau}\int_t^{t+\tau} a(\mathbf{r}, t')dt',$$

where τ is longer than the fluctuation time but is much more shorter than the characteristic evolution time of $\overline{a}(\mathbf{r}, t)$. The magnitude of the fluctuation is defined as

$$\overline{a'^2} = \frac{1}{\tau}\int_t^{t+\tau} a'^2(\mathbf{r}, t')dt'.$$

The correlation function $\overline{a'b'}$ between two fluctuations a' and b' is defined as

$$\overline{a'b'} = \frac{1}{\tau}\int_t^{t+\tau} a'(\mathbf{r}, t')b'(\mathbf{r}, t')dt'.$$

Note that $\overline{a'} = 0$ by definition. Moreover, in the case that a' and b' are statistically independent, then $\overline{a'b'} = \overline{a'}\,\overline{b'} = 0$.

For the sake of simplicity, let us assume that ρ and η are constants and there are no thermal effects in the fluid. In this case, the equations of continuity and momentum are sufficient to determine $\mathbf{v}$ and p. They are given by

$$\nabla_i v_i = 0,$$

and

$$\rho \frac{dv_i}{dt} = -\nabla_i p + \rho g_i + \eta \nabla^2 v_i.$$

Reynolds assumes that the hydrodynamic fields can be written as

$$v_i = \overline{v_i} + v_i', \quad p = \overline{p} + p'.$$

Now we substitute these forms into the above equations and take time averages at a given point. The continuity equation becomes

$$\langle \nabla_i v_i \rangle = \langle \nabla_i (\overline{v_i} + v_i') \rangle = \nabla_i \overline{v_i} + \nabla_i \langle v_i' \rangle = \nabla_i \overline{v_i} = 0. \tag{3.26}$$

Equation (3.26) is identical to the one derived in the laminar case.

Now we consider the Navier-Stokes equation. Let's denote the three components of the velocity $\mathbf{v}$ as (u, v, w). For the x-component, one has

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \rho g_x + \eta \nabla^2 u.$$

The total derivative is

$$\begin{aligned} \frac{du}{dt} = \partial_t u + \mathbf{v} \cdot \nabla u &= \partial_t \overline{u} + \partial_t u' + \overline{u} \partial_x \overline{u} + \overline{u} \partial_x u' + u' \partial_x \overline{u} + u' \partial_x u' \\ &+ \overline{v} \partial_y \overline{u} + \overline{v} \partial_y u' + v' \partial_y \overline{u} + v' \partial_y u' \\ &+ \overline{w} \partial_y \overline{u} + \overline{w} \partial_y u' + w' \partial_y \overline{u} + w' \partial_y u'. \end{aligned} \tag{3.27}$$

In Eq. (3.27), note that

$$\begin{aligned} u' \partial_x u' + v' \partial_y u' + w' \partial_z u' &= \partial_x u^{'2} + \partial_y (u'v') + \partial_z (w'u') \\ -u'(\partial_x u' + \partial_y v' + \partial_z w') & \\ &= \partial_x u^{'2} + \partial_y (u'v') + \partial_z (w'u'), \end{aligned} \tag{3.28}$$

where use has been made of the continuity equation in the last step. Now, we take the time average of Eq. (3.27):

$$\begin{aligned} \langle \frac{du}{dt} \rangle &= \partial_t \overline{u} + \partial_x \langle u^{'2} \rangle + \partial_y \langle u'v' \rangle + \partial_z \langle u'w' \rangle \\ &+ \overline{u} \partial_x \overline{u} + \overline{v} \partial_y \overline{u} + \overline{w} \partial_z \overline{u} \\ &= \frac{d\overline{u}}{dt} + \partial_x \langle u^{'2} \rangle + \partial_y \langle u'v' \rangle + \partial_z \langle u'w' \rangle. \end{aligned} \tag{3.29}$$

Here, we have accounted for that $\langle \overline{u}\partial_x u' \rangle = \langle u'\partial_x \overline{u} \rangle = \ldots = 0$. Moreover,

$$\frac{d}{dt} \equiv \partial_t + +\overline{u}\partial_x + \overline{v}\partial_y + \overline{w}\partial_z.$$

Taking into account all the above results, the evolution equation for $\overline{u}$ can be written as

$$\begin{aligned} \rho\frac{d\overline{u}}{dt} &= -\frac{\partial}{\partial x}\overline{p} + \rho g_x + \frac{\partial}{\partial x}\left(\eta\frac{\partial\overline{u}}{\partial x} - \rho\overline{u'^2}\right) \\ &\quad + \frac{\partial}{\partial y}\left(\eta\frac{\partial\overline{u}}{\partial y} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial\overline{u}}{\partial z} - \rho\overline{u'w'}\right). \end{aligned} \tag{3.30}$$

Similar equations to Eq. (3.27) can be written for the components $\overline{v}$ and $\overline{w}$. In a compact form, they are given by

$$\rho\frac{d\overline{v_i}}{dt} = \rho g_i - \frac{\partial\overline{p}}{\partial r_i} + \frac{\partial}{\partial r_j}\left(\eta\frac{\partial\overline{v_j}}{\partial r_j} - \rho\overline{v_i'v_j'}\right). \tag{3.31}$$

Equation (3.31) is the Reynolds equation for the momentum. It is formally identical to the Navier-Stokes equation for the average values, except for the presence of the contributions $\rho\overline{v_i'v_j'}$ to the viscous force. The term $\rho\overline{v_i'v_j'}$ is called turbulent stress. The impact of these stresses on the fluid motion depends on the problem. For instance, we can say that in turbulent regime the turbulent stresses are much larger than the laminar stresses (i.e., those related to $\eta\partial\overline{v_i}/\partial r_j$).

CHAPTER 4

THERMAL CONDUCTION IN FLUIDS

4.1 The general equation of heat transfer

It is evident that a complete system of equations of fluid dynamics requires to know the evolution equation for the internal energy or temperature. In other words, the complete system of equations must contain five equations. For fluids with internal friction and thermal conduction, one of these equations is the equation of continuity and the other is the Navier-Stokes equation. The fifth equation for an ideal gas is the equation of conservation of entropy

$$\partial_t s + \mathbf{v} \cdot \nabla s = 0.$$

In a viscous fluid this equation does not hold, of course, due to irreversible processes of energy dissipation.

In the case of an ideal fluid, the law of conservation of energy is given by

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho u\right) = -\nabla \cdot \left[\rho \mathbf{v}\left(\frac{1}{2}v^2 + h\right)\right]. \tag{4.1}$$

The left hand side gives the rate of change of energy in unit volume of the fluid, while on the right is the divergence of the energy flux density. In a viscous fluid the law of conservation of energy still holds, of course: the change per unit time in the total energy of the fluid in any volume must still be equal to the total flux of energy through the surface bounding that volume. The energy flux density, however, now has a different form. Apart from the flux $\rho\mathbf{v}(\frac{1}{2}v^2 + h)$ due to the simple transfer of mass by the motion of fluid, there is also a flux due to processes of internal friction. This latter flux is given by the vector $\mathbf{v}\cdot\boldsymbol{\sigma}'$ with components $v_j\sigma_{ji}$ [see Eq. (3.14)]. Moreover, there is also another term that must be included in the energy flux. If the temperature of the fluid is not constant throughout its volume, there will be also a transfer of heat or energy called *thermal conduction*. Thus, there is a transfer of energy from points where the temperature is high to those where it is low. It does not involve macroscopic motion, and occurs even in a fluid at rest.

Let us denote by $\mathbf{q}$ the heat flux density due to thermal conduction. The heat flux is related to the variation of the temperature throughout the system. If the magnitude of the thermal gradient is *small*, we assume that $\mathbf{q}$ is proportional to ∇T (of course, $\mathbf{q}=\mathbf{0}$ if $\nabla T=0$) and we obtain the so-called *Fourier law* for the heat flux

$$\mathbf{q}=-\kappa\nabla T, \tag{4.2}$$

where κ is called the *thermal conductivity* coefficient. It is always positive, since the energy flux moves from points at high temperature to those at low temperature, i.e., $\mathbf{q}$ and ∇T must be in opposite directions. The coefficient κ is in general a function of temperature and pressure.

Thus, the total energy flux in a fluid when there is viscosity and thermal conduction is

$$\rho\mathbf{v}\left(\frac{1}{2}v^2+h\right)-\mathbf{v}\cdot\boldsymbol{\sigma}'-\kappa\nabla T.$$

Accordingly, the general law of conservation of energy is given by the equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho u\right)=-\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2+h\right)-\mathbf{v}\cdot\boldsymbol{\sigma}'-\kappa\nabla T\right]. \tag{4.3}$$

This equation completes the system of fluid-mechanical equations of a viscous fluid. It is more convenient, however, to write it in another form. To do so, let us evaluate the time derivative of the energy in unit volume of fluid, starting from the equations of motion. We have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho u\right)=\frac{1}{2}v^2\frac{\partial\rho}{\partial t}+\rho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}+\rho\frac{\partial u}{\partial t}+u\frac{\partial\rho}{\partial t}.$$

Now, we take into account the equation of continuity and the Navier-Stokes equation (in the absence of gravity field, for the sake of simplicity):

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho u\right) &= -\frac{1}{2}v^2\nabla\cdot(\rho\mathbf{v})-\rho\mathbf{v}\cdot\nabla(\frac{1}{2}v^2)-\mathbf{v}\cdot\nabla p\\ &\quad +v_i\frac{\partial\sigma'_{ij}}{\partial r_j}+\rho\frac{\partial u}{\partial t}-u\nabla\cdot(\rho\mathbf{v}).\end{aligned}\tag{4.4}$$

Using the thermodynamic relation $du=Tds+(p/\rho^2)d\rho$, we find

$$\frac{\partial u}{\partial t}=T\frac{\partial s}{\partial t}+\frac{p}{\rho^2}\frac{\partial\rho}{\partial t}=T\frac{\partial s}{\partial t}-\frac{p}{\rho^2}\nabla\cdot(\rho\mathbf{v}).$$

Substituting this into Eq. (4.4) and introducing the enthalpy $h=u+p/\rho$, one gets

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho u\right) &= -\left(\frac{1}{2}v^2+h\right)\nabla\cdot(\rho\mathbf{v})-\rho\mathbf{v}\cdot\nabla(\frac{1}{2}v^2)-\mathbf{v}\cdot\nabla p\\ &+\ \rho T\frac{\partial s}{\partial t}+v_i\frac{\partial\sigma'_{ij}}{\partial r_j}.\end{aligned}\tag{4.5}$$

Next, the thermodynamic relation $dh=Tds+dp/\rho$ leads to

$$\nabla p=\rho\nabla h-\rho T\nabla s.$$

In addition, the last term on the right hand side of Eq. (4.5) can be written as

$$v_i\frac{\partial\sigma'_{ij}}{\partial r_j}=\frac{\partial}{\partial r_j}(v_i\sigma'_{ij})-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}.$$

Substitution of the above equations into Eq. (4.5) and adding and subtracting the term $\nabla(\kappa\nabla T)$, we finally get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho u\right) &= -\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2+h\right)-\mathbf{v}\cdot\boldsymbol{\sigma}'-\kappa\nabla T\right]\\ &+\ \rho T\left(\frac{\partial s}{\partial t}+\mathbf{v}\cdot\nabla s\right)-\sigma'_{ij}\frac{\partial v_i}{\partial r_j}-\nabla\cdot(\kappa\nabla T).\end{aligned}\tag{4.6}$$

Comparing Eqs. (4.3) and (4.6), we determine the rate of change of the specific entropy of a viscous fluid with thermal conduction:

$$\rho T\left(\frac{\partial s}{\partial t}+\mathbf{v}\cdot\nabla s\right)=\sigma'_{ij}\frac{\partial v_i}{\partial r_j}+\nabla\cdot(\kappa\nabla T).\tag{4.7}$$

This equation is called the *general equation of heat transfer.* If there is no viscosity or thermal conduction, the right hand side is zero and one recovers the equation of conservation of entropy (2.9) for an ideal fluid.

For practical purposes, it is more convenient sometimes to use the total energy density per unit mass

$$e = u + \frac{1}{2}v^2$$

instead of using the specific entropy s. The derivation of this energy equation follows similar steps as those made before. Let us start from Eq. (4.3):

$$\partial_t(\rho e) = -\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)\right] + \nabla\cdot(\mathbf{v}\cdot\boldsymbol{\sigma}') + \nabla\cdot(\kappa\nabla T).$$

Using the continuity equation and the thermodynamic relation $h = u + p/\rho$, one achieves

$$\rho\frac{de}{dt} = -\nabla\cdot(p\mathbf{v}) + \nabla\cdot(\mathbf{v}\cdot\boldsymbol{\sigma}') + \nabla\cdot(\kappa\nabla T). \tag{4.8}$$

It is easy to show that Eq. (4.8) still holds in the presence of a gravity field ($\mathbf{g} = -g\hat{\mathbf{k}}$). In this case,

$$\rho e = \rho u + \frac{1}{2}\rho v^2 + \rho g z.$$

On the other hand, using the Navier-Stokes equation to eliminate the term $\nabla\cdot\boldsymbol{\sigma}'$ in Eq. (4.8), one can derive a conservation equation for the internal energy per unit mass u. It is given by

$$\rho\frac{du}{dt} = -p\nabla\cdot\mathbf{v} + \nabla\cdot(\kappa\nabla T) + \Phi, \tag{4.9}$$

where

$$\Phi \equiv \sigma'_{ij}\frac{\partial v_i}{\partial r_j} \tag{4.10}$$

is called the *viscosity dissipation function.* If the fluid is perfect, then $u = c_v T$ and so, Eq. (4.9) reduces to

$$\rho c_v\frac{dT}{dt} = -p\nabla\cdot\mathbf{v} + \nabla\cdot(\kappa\nabla T) + \Phi. \tag{4.11}$$

4.2 Rate of increase of entropy

We want here to give an interpretation of Eq. (4.7). The quantity ds/dt gives the rate of change of the entropy of a unit mass of fluid and $\rho T ds/dt$ is the amount of heat gained by unit of volume. We see from Eq. (4.7) that the amount of heat gained by unit volume of the fluid is therefore

$$\sigma'_{ij}\nabla_j v_i + \nabla\cdot(\kappa\nabla T).$$

The first term here is the energy dissipated into heat by viscosity while the second is the heat conducted into the volume concerned. We expand the term $\sigma'_{ij}\nabla_j v_i$ in (4.7) by substituting the expression (3.2) for σ'_{ij}. Thus, one obtains

$$\sigma'_{ij}\frac{\partial v_i}{\partial r_j} = \eta\frac{\partial v_i}{\partial r_j}\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}\right)+\zeta\frac{\partial v_i}{\partial r_j}\delta_{ij}(\nabla\cdot\mathbf{v}).$$

The first term can be written as

$$\frac{1}{2}\eta\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}\right)^2,$$

where use has been made of the result

$$\frac{\partial v_i}{\partial r_j}\frac{\partial v_i}{\partial r_j}+\frac{\partial v_i}{\partial r_j}\frac{\partial v_j}{\partial r_i}-\frac{2}{3}(\nabla\cdot\mathbf{v})^2=\frac{1}{2}\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}\right)^2.$$

The second term is simply

$$\zeta\frac{\partial v_i}{\partial r_j}\delta_{ij}\nabla\cdot\mathbf{v}=\zeta(\nabla\cdot\mathbf{v})^2.$$

Thus, Eq. (4.7) becomes

$$\rho T\left(\frac{\partial s}{\partial t}+\mathbf{v}\cdot\nabla s\right)=\nabla\cdot(\kappa\nabla T)+\frac{1}{2}\eta\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}\right)^2+\zeta(\nabla\cdot\mathbf{v})^2. \tag{4.12}$$

The entropy of the fluid increases as a result of the irreversible processes of thermal conduction and internal friction. Here, we are not talking about the entropy of each volume element, but the total entropy of the whole fluid. It is given by

$$\int \rho s\, dV.$$

The change of entropy per unit time is

$$\frac{d}{dt}\int \rho s\, dV = \int \frac{\partial}{\partial t}(\rho s) dV.$$

Using the equation of continuity and Eq. (4.12), one gets

$$\begin{aligned}\frac{\partial}{\partial t}(\rho s) &= \rho\frac{\partial s}{\partial t}+s\frac{\partial \rho}{\partial t} = -s\nabla\cdot(\rho\mathbf{v})-\rho\mathbf{v}\cdot\nabla s+\frac{1}{T}\nabla\cdot(\kappa\nabla T)\\ &+ \frac{\eta}{2T}\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}\right)^2+\frac{\zeta}{T}(\nabla\cdot\mathbf{v})^2. \qquad (4.13)\end{aligned}$$

The first two terms on the right give $-\nabla\cdot(\rho s\mathbf{v})$. Now, we integrate both sides of Eq. (4.13) into the whole volume of the system. In this case, the volume integral of the entropy flux $\rho s\mathbf{v}$ is transformed into a surface integral. If we consider an unbounded volume of fluid at rest at infinity, the bounding surface can be removed to infinity and the integrand in the surface integral is zero. The integral of the third term on the right hand side of Eq. (4.13) is transformed as follows:

$$\int \frac{1}{T}\nabla\cdot(\kappa\nabla T)dV = \int \nabla\cdot\left(\frac{\kappa\nabla T}{T}\right)dV+\int\frac{\kappa(\nabla T)^2}{T^2}dV.$$

Assuming that the fluid temperature quickly tends to a constant value at infinity, then we can transform the first integral into one over an infinitely remote surface integral, on which $\nabla T=0$ and the integral therefore vanishes. The result for the change of entropy per unit time is

$$\begin{aligned}\frac{d}{dt}\int \rho s\, dV &= \int \frac{\kappa(\nabla T)^2}{T^2}dV+\int\frac{\eta}{2T}\left(\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}\right)^2\\ &+ \int\frac{\zeta}{T}(\nabla\cdot\mathbf{v})^2. \qquad (4.14)\end{aligned}$$

The left hand side of Eq. (4.14) gives the time variation of entropy ds/dt. According to Thermodynamic's second law

$$\frac{ds}{dt}\geq 0.$$

The first term on the right hand side of Eq. (4.14) is the rate of increase of entropy due to thermal conduction while the other two terms give the rate of increase due to internal friction. Since the entropy can only increase then each term separately must always be *positive*. Consequently,

$$\kappa > 0, \quad \eta > 0, \quad \zeta > 0.$$

It is important to note that in the form (4.2) for the heat flux we have implicitly assumed that $\mathbf{q}$ does not depend on the pressure gradient ∇p. This assumption, which is not evident a priori, can be justified now as follows. If $\mathbf{q}$ contained a term proportional to ∇p, the expression (4.14) for the rate of change of entropy would include another term having the product $\nabla p \cdot \nabla T$ in the integrand. Since the latter could be negative or positive (since its signature is not well defined), then the time derivative ds/dt would not necessarily be positive, which is inconsistent with the second law of Thermodynamics (minimum entropy production).

4.3 Thermal conduction in an incompressible fluid

If the fluid velocity is small compared with the velocity of sound, the pressure and density variations across the system may be neglected. However, a non-uniformly heated fluid is still not completely incompressible in the sense used previously. The reason is that the density varies with the temperature; this variation cannot in general be neglected, and therefore, even at small velocities, the density of a non-uniformly heated fluid cannot be supposed constant. In this case, $p \approx$ const. but $\nabla \rho \neq 0$. Thus, we have

$$\frac{\partial s}{\partial t} = \left(\frac{\partial s}{\partial T}\right)_p \frac{\partial T}{\partial t}, \quad \nabla s = \left(\frac{\partial s}{\partial T}\right)_p \nabla T,$$

and, since $T(\partial s/\partial T)_p$ is the specific heat at constant pressure, c_p, we obtain

$$T\frac{\partial s}{\partial t} = c_p \frac{\partial T}{\partial t}, \quad T\nabla s = c_p \nabla T.$$

Therefore, Eq. (4.7) reads

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = \nabla \cdot (\kappa \nabla T) + \sigma'_{ij} \frac{\partial v_i}{\partial r_j}. \tag{4.15}$$

Equation (4.15) is equivalent to Eq. (4.11) when $\nabla \cdot \mathbf{v} = 0$ and $c_p \approx c_v$. The last identity holds for liquids. If the density is to be supposed constant in the equation of motion for a non-uniformly heated fluid, it is necessary that the fluid velocity is small compared with that of sound, and also that the temperature differences in the fluid are small. In this case, the fluid may be assumed to be incompressible in the usual sense; in particular the equation of continuity is simply $\nabla \cdot \mathbf{v} = 0$. Supposing the temperature differences small, we neglect also the temperature variation of the transport coefficients and so, they are considered as constants. Writing the term

$$\sigma'_{ij}\frac{\partial v_i}{\partial r_j} = \frac{\eta}{2}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right)^2,$$

one finally obtains the heat transfer equation for an incompressible fluid:

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \chi \nabla^2 T + \frac{\nu}{2c_p}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right)^2, \qquad (4.16)$$

where

$$\chi \equiv \frac{\kappa}{\rho c_p}$$

is the *thermometric conductivity.*

If the incompressible fluid is at rest, the transfer of energy takes place entirely by thermal conduction:

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T. \qquad (4.17)$$

This equation is called *Fourier's equation.* It can be derived in a more simple way without using the general equation of heat transfer in a moving fluid. According to the law of conservation of energy, the amount of heat absorbed in some volume in unit time must equal the total heat flux into this volume through the surface surrounding. This is an equation of continuity for the amount of heat. This equation is obtained by equating the amount of heat absorbed in unit volume in unit time to minus the divergence of the heat flux density. The former is $\rho c_p \partial_t T$ and the latter is $-\nabla \cdot \mathbf{q}$ and hence, one gets Eq. (4.17).

4.4 Reynolds equation for the temperature

As said in Chapter 3, for sufficiently high Reynolds numbers, the flow becomes turbulent and one has to modify the equations of fluid motion. Here, we will display the equation for the temperature in turbulent flows in the simple case of incompressible fluids ($\nabla \cdot \mathbf{v} = 0$). It is given by

$$\rho c_p \frac{d\overline{T}}{dt} = \frac{\partial}{\partial r_j}\left(\kappa \frac{\partial \overline{T}}{\partial r_j} - \rho c_p \overline{v_j' T'}\right) + \Phi(\overline{\mathbf{v}}) + \overline{\Phi(\mathbf{v}')}, \tag{4.18}$$

where $T = \overline{T} + T'$ and Φ is defined by Eq. (4.10). We see that there are two new terms with respect to the laminar case, Eq. (4.11): (i) a term due to the turbulent viscous dissipation $\overline{\Phi(\mathbf{v}')}$; and (ii) a term due to the correlation between the velocity and temperature fluctuations (heat flux turbulent).

Reynolds equations for the momentum and heat fluxes are in general generalizations of the Navier-Stokes equations. In the laminar regime, one can neglect the fluctuations and the average values of the hydrodynamic fields coincide with their instantaneous ones. Thus, the turbulent viscous dissipation vanishes and the stress and turbulent fluxes can also be neglected. In this situation the only contributions to the viscous force and the heat flux come from the ones obtained in the laminar regime.

Let us write the complete set of Reynolds equations for steady incompressible flows. If the transport coefficients can be considered as constants, then the above equations can be written as

$$\nabla \cdot \mathbf{v} = 0, \tag{4.19}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\sigma}', \tag{4.20}$$

$$\rho c_p \frac{dT}{dt} = -\nabla \cdot \mathbf{q} + \Phi^*, \tag{4.21}$$

where

$$\sigma'_{ij} = \eta \left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right) - \rho \overline{v_i' v_j'}, \tag{4.22}$$

$$q_i = -\kappa \frac{\partial T}{\partial r_i} + \rho c_p \overline{v_i' T'}, \quad \Phi^* = \Phi(\mathbf{v}) + \overline{\Phi(\mathbf{v}')}. \tag{4.23}$$

Equations (4.19)–(4.23) apply for both laminar and turbulent flows. Note that in the above equations, the terms without prime denote the average

ÍNDICE

values of the hydrodynamic fields. It must remarked that while the Navier-Stokes equations are *closed* equations for ρ, $\mathbf{v}$, and T, the transport equations for turbulent flows are not closed unless one finds constitutive equations for the turbulent transport terms. It is still an open problem and some theories and statistical models have been developed in the past years to close the corresponding hydrodynamic equations.

4.5 The similarity law for heat transfer

The processes of heat transfer in a fluid are more complex than those in solids, because the fluid may be in motion. A heated body immersed in a moving fluid cools considerably more rapidly than one in a fluid at rest, where the heat transfer is accomplished only by conduction. As we know, the motion of a non-uniformly heated fluid is called *convection*.

Let us assume that the temperature differences in the fluid are so small that its physical properties may be supposed independent of temperature, but are at the same time so large that the temperature changes coming from the energy dissipation by internal friction are much smaller than those due to the thermal conduction. Thus, $\Phi \ll \kappa \nabla T$ and the equation of heat transfer (4.16) for an incompressible fluid reduces to

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \chi \nabla^2 T, \tag{4.24}$$

where $\chi \equiv \kappa / \rho c_p$. Equation (4.24) along with the equation of continuity and the Navier-Stokes equation completely determines the convection in the conditions considered.

In what follows, we shall be interested only in steady incompressible flows (in this case is necessary that the solid bodies adjoining the fluid should contain some sources of heat to maintain them at constant temperature). Then, all the time derivatives are zero, and we have the following fundamental equations:

$$\mathbf{v} \cdot \nabla T = \chi \nabla^2 T, \tag{4.25}$$

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla(p/\rho) + \nu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0. \tag{4.26}$$

The unknowns of these equations are $\mathbf{v}$, T and p/ρ while the set contains only two constant parameters, ν and χ. Moreover, the solution of the set of equations depends also, through the boundary conditions, on some characteristic length ℓ, velocity u, and temperature difference $T_1 - T_0$. The first two of these are given as usual by the dimension of the solid bodies which appear in the problem and the velocity of the main stream. The third quantity is given by the temperature difference between the fluid and these bodies.

The question arises of the dimensions given for the temperature, which is determined by Eq. (4.25). This is a homogeneous linear differential equation in T. Hence, the temperature can be multiplied by any constant and still satisfies the equation. In other words, the unit of temperature can be arbitrarily chosen. Thus, the flow is characterized by five parameters, whose dimensions are $[\nu] = [\chi] = L^2T^{-1}$, $[u] = LT^{-1}$, $[\ell] = L$, and $[T_1 - T_0] = \text{deg}$. From these quantities one can form two independent dimensionless combinations. The first is the well-known Reynolds number, Re$= u\ell/\nu$. The second is a new number called the *Prandtl number*, defined as

$$\text{Pr} = \frac{\nu}{\chi} = c_p \frac{\eta}{\kappa}. \tag{4.27}$$

Any other dimensionless combination can be expressed in terms of Re and Pr.

Let us see the above similarity law for the energy equation (4.25) by considering more quantitative arguments. As in Sec. 3.4, we introduce first the (dimensionless) quantities $\mathbf{r}' \equiv \mathbf{r}/\ell$ and $\mathbf{v}' \equiv \mathbf{v}/u$, ℓ and u being a reference length and velocity, respectively. In terms of the above scaled variables, Eq. (4.25) can be written as

$$v_i' \frac{\partial T}{\partial r_i'} = \frac{\chi}{u\ell} \frac{\partial^2 T}{\partial r_i' \partial r_i'} = \frac{1}{\text{PrRe}} \frac{\partial^2 T}{\partial r_i' \partial r_i'}. \tag{4.28}$$

Thus, the solution to Eq. (4.28) can be written as $\mathbf{v}' = \mathbf{f}(\mathbf{r}', \text{Pr}, \text{Re})$ and hence, flows sharing the same values of the Reynolds and Prandtl numbers are dynamically similar.

In contrast to the Reynolds number, the Prandtl number is just a constant of the material and does not depend on the properties of the flow. For gases it is always of the order of unity. The value of Pr for liquids varies more widely. For very viscous liquids, it may be very large.

The dimensionless function which gives the temperature distribution depends on both Re and Pr as parameters. However, the velocity distribution depends only on Re, since it is determined by Eqs. (4.26), which do not involve the thermal conductivity coefficient κ. Two convective flows are similar if their Reynolds and Prandtl numbers are the same.

CHAPTER 5

MULTICOMPONENT FLUID SYSTEMS

5.1 The equations of fluid dynamics for a mixture of fluids

So far, we have derived the equations of fluid dynamics for a simple or monocomponent fluid. Now we want to extend these equations to the case of a mixture of fluids whose composition is different at different points. As we will see the corresponding equations are considerably modified.

We shall discuss here only mixtures constituted by two components. The generalization to an arbitrary number of components can be easily made by following similar steps as those carried out in the present chapter. The composition of the mixture is described by the concentration $c = \rho_1/\rho$. It is defined as the ratio of the mass of one component (say for instance, component 1) to the total mass of the fluid in a given volume element. The distribution of the concentration through the fluid will in general change. This change occurs in two ways. Firstly, when there is a macroscopic motion of the fluid, any given small portion of it moves as a whole and its composition remains unchanged. This is due to the pure mechanical mixing of the fluid. In other words, although the composition of each moving portion of it is unchanged, the concentration of the fluid at any point varies with time. If we ignore any process of thermal conduction and internal friction (which may be also taking place) this change in concentration is a thermodynamically reversible process, and does not give rise to any dissipation of energy. Secondly, a change in composition can occur by the molecular transfer of the components from one part of the fluid to another. The tendency of the system to equilibrate the concentration of each component is called *diffusion*. Diffusion is an irreversible process and is, like thermal conduction and viscosity, one of the sources of energy dissipation in a fluid mixture.

We denote by ρ the total mass density of the fluid. As before, its equation of continuity is

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0. \tag{5.1}$$

This means that the total mass of fluid in any volume can change only by the movement of fluid into or out that volume. Strictly speaking, we must emphasize that the concept of velocity must be redefined for a mixture of fluids. Here, $\rho\mathbf{v}$ is the total momentum per unit volume of fluid. The Navier-Stokes equation (3.7) is also unchanged. We shall derive the remaining equations of fluid dynamics for a fluid mixture.

In the absence of diffusion, the composition of any given element of fluid would remain unchanged during its motion. This means that the total derivative dc/dt would be zero, namely,

$$\frac{dc}{dt}=\frac{\partial c}{\partial t}+\mathbf{v}\cdot\nabla c=0.$$

Using Eq. (5.1), the above equation can be written as

$$\frac{\partial}{\partial t}(\rho c)+\nabla\cdot(\rho c\mathbf{v})=0. \tag{5.2}$$

This is an equation of continuity for each component of the mixture (ρc being the mass of that component in unit volume). In the integral form, it is given by

$$\frac{\partial}{\partial t}\int \rho c dV=-\oint \rho c\mathbf{v}\cdot d\mathbf{S}.$$

It shows that the rate of change of the amount of this component in any volume is equal to the amount of the component transported through the surface of that volume by the motion of fluid.

When diffusion occurs, apart from the flux $\rho c\mathbf{v}$ of the component in question, there is another flux which results in the transfer of the components even when the fluid as a whole is at rest. Let $\mathbf{j}_i$ be the density of this diffusion flux, i.e., the amount of the component transported by diffusion through unit area in unit time. Since the sum of the flux densities of the two components must be $\rho\mathbf{v}$, the flux density for one component is $\rho c\mathbf{v}+\mathbf{j}_i$ while for the other is $\rho(1-c)\mathbf{v}-\mathbf{j}_i$. Thus, the rate of change of the amount of the component in any volume is

$$\frac{\partial}{\partial t}\int \rho c dV = -\oint \rho c\mathbf{v}\cdot d\mathbf{S} - \oint \mathbf{j}_i\cdot d\mathbf{S},$$

or in differential form

$$\frac{\partial}{\partial t}(\rho c) = -\nabla\cdot(\rho c\mathbf{v}) - \nabla\cdot\mathbf{j}_i.$$

Using Eq. (5.1), we can rewrite this equation of continuity for one component in the form

$$\rho\left(\frac{\partial c}{\partial t}+\mathbf{v}\cdot\nabla c\right) = -\nabla\cdot\mathbf{j}_i. \qquad (5.3)$$

Let us derive the equation for the energy. We note that for a binary mixture the thermodynamic quantities of the fluid are also functions of the concentration. We want to evaluate the derivative $\partial_t(\frac{1}{2}\rho v^2+\rho u)$. To do it, we use the thermodynamic relations

$$du = Tds + \frac{p}{\rho^2}d\rho + \mu dc, \quad dh = Tds + \frac{1}{\rho}dp + \mu dc,$$

where μ is the chemical potential of the mixture. It is given by

$$\mu = \frac{\mu_1}{m_1} - \frac{\mu_2}{m_2},$$

where μ_i is the chemical potential of the species or component i and m_i is the mass of a fluid particle of species i. Note that for species 1, $c = n_1 m_1 = \rho_1/\rho$ where n_i is the number of particles i contained in 1 gr of the mixture fluid. In this case, an additional term $\rho\mu\partial c/\partial t$ appears in the derivative $\rho\partial u/\partial t$. Writing the second thermodynamic relation in the form

$$dp = \rho dh - \rho T ds - \rho\mu dc,$$

we see that the term $-\mathbf{v}\cdot\nabla p$ in Eq. (4.5) will contain the additional term $\rho\mu\mathbf{v}\cdot\nabla c$.

Thus, if we go to the derivation of Eq. (4.6), then to the above expression we must add the term

$$\rho\mu\left(\frac{\partial c}{\partial t}+\mathbf{v}\cdot\nabla c\right) = -\mu\nabla\cdot\mathbf{j}_i.$$

Consequently, the final result is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho u\right) &= -\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2+h\right)-\mathbf{v}\cdot\boldsymbol{\sigma}'+\mathbf{q}\right]\\ &+ \rho T\left(\frac{\partial s}{\partial t}+\mathbf{v}\cdot\nabla s\right)-\sigma'_{k\ell}\frac{\partial v_k}{\partial r_\ell}+\nabla\cdot\mathbf{q}-\mu\nabla\cdot\mathbf{j}_i.\end{aligned} \tag{5.4}$$

We have replaced the term $-\kappa\nabla T$ by a heat flux $\mathbf{q}$ since the latter can also depend on the concentration gradient (apart from its dependence on ∇T). The last two terms in Eq. (5.4) on the right hand side can be written as

$$\nabla\cdot\mathbf{q}-\mu\nabla\cdot\mathbf{j}_i=\nabla\cdot(\mathbf{q}-\mu\mathbf{j}_i)+\mathbf{j}_i\cdot\nabla\mu.$$

The expression

$$\rho\mathbf{v}\left(\frac{1}{2}v^2+h\right)-\mathbf{v}\cdot\boldsymbol{\sigma}'+\mathbf{q}$$

appearing in the divergence operator in Eq. (5.4) can be identified as the total energy flux in the fluid. The first term is the *reversible* energy flux (due simply to the movement of the fluid as a whole) while the sum $-\mathbf{v}\cdot\boldsymbol{\sigma}'+\mathbf{q}$ is the *irreversible* part of the energy flux. When there is no macroscopic motion, the viscosity flux $\mathbf{v}\cdot\boldsymbol{\sigma}'$ is zero and the energy flux is simply $\mathbf{q}$.

The equation of conservation of energy is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho u\right)=-\nabla\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2+h\right)-\mathbf{v}\cdot\boldsymbol{\sigma}'+\mathbf{q}\right]. \tag{5.5}$$

Comparison between Eqs. (5.4) and (5.5) yields the required equation for the specific entropy:

$$\rho T\left(\frac{\partial s}{\partial t}+\mathbf{v}\cdot\nabla s\right)=\sigma'_{k\ell}\frac{\partial v_k}{\partial r_\ell}-\nabla\cdot(\mathbf{q}-\mu\mathbf{j}_i)-\mathbf{j}_i\cdot\nabla\mu. \tag{5.6}$$

This equation generalizes (4.7) to fluid mixtures.

We have thus obtained a complete system of equations of fluid dynamics for a two-component fluid mixture. The number of equations in this system is one more than for a single fluid, since there is one more unknown function, namely the concentration. The equations are the equation of continuity (5.1), the Navier-Stokes equations (which are the same as those obtained for a single gas), the equation of continuity (5.3) for one component, and Eq. (5.6) for the change of entropy. They are not closed equations for the unknowns unless one gives the corresponding constitutive equations for $\mathbf{j}_i$ and $\mathbf{q}$ in terms of the gradients of concentration and temperature. These equations will be displayed in the next section.

5.2 Constitutive equations for the diffusion and heat fluxes

The diffusion flux $\mathbf{j}_i$ and the heat flux $\mathbf{q}$ are due to the presence of concentration and temperature gradients in the fluid. In the case of a two-component fluid system, both fluxes depend on both gradients. If the concentration and temperature gradients are small, one can assume that $\mathbf{j}_i$ and $\mathbf{q}$ are *linear* functions of $\nabla\mu$ and ∇T. The fluxes are independent of the pressure gradient for the same reason as that given with the case of the heat flux for a single fluid. Accordingly, we write these fluxes as

$$\mathbf{j}_i = -\alpha\nabla\mu - \beta\nabla T, \quad \mathbf{q} - \mu\mathbf{j}_i = -\delta\nabla\mu - \gamma\nabla T.$$

However, there is a simple relation between the transport coefficients β and δ, which is a consequence of a *symmetry principle* of the kinetic coefficients (Onsager's reciprocity relations). If we rewrite the above equations as

$$\mathbf{j}_i = -\alpha T\left(\frac{\nabla\mu}{T}\right) - \beta T^2\left(\frac{\nabla T}{T^2}\right),$$

$$\mathbf{q} - \mu\mathbf{j}_i = -\delta T\left(\frac{\nabla\mu}{T}\right) - \gamma T^2\left(\frac{\nabla T}{T^2}\right),$$

then, Onsager's relations yields $\beta T^2 = \delta T$ or $\delta = \beta T$. We can therefore rewrite the heat flux as

$$\begin{aligned}
\mathbf{q} &= \mu\mathbf{j}_i - \beta T\nabla\mu - \gamma\nabla T = \left(\mu + \frac{\beta T}{\alpha}\right)\mathbf{j}_i - \frac{\beta T}{\alpha}\mathbf{j}_i - \beta T\nabla\mu - \gamma\nabla T \\
&= \left(\mu + \frac{\beta T}{\alpha}\right)\mathbf{j}_i + \beta T\nabla\mu + \frac{\beta^2 T}{\alpha}\nabla T - \beta T\nabla\mu - \gamma\nabla T \\
&= \left(\mu + \frac{\beta T}{\alpha}\right)\mathbf{j}_i - \kappa\nabla T, \qquad (5.7)
\end{aligned}$$

where

$$\kappa \equiv \gamma - \frac{\beta^2 T}{\alpha}. \tag{5.8}$$

Thus, the mass and heat fluxes can be finally written as

$$\mathbf{j}_i = -\alpha \nabla \mu - \beta \nabla T, \quad \mathbf{q} = \left(\mu + \frac{\beta T}{\alpha} \right) \mathbf{j}_i - \kappa \nabla T. \tag{5.9}$$

In the case that the diffusion flux is zero, we have pure thermal conduction. In this case, T and μ must satisfy the condition $\alpha \nabla \mu + \beta \nabla T = 0$ or $\alpha d\mu + \beta dT = 0$. The integration of this equation gives a relation of the form $f(c, T) = 0$. This relation determines the dependence of c on T which must hold if there is no diffusion flux. Moreover, when $\mathbf{j}_i = \mathbf{0}$ the heat flux becomes

$$\mathbf{q} = -\kappa \nabla T,$$

and so κ is just the thermal conductivity coefficient.

Let us now change to the usual variables p, T, and c. This is essentially motivated because the spatial gradients of the above quantities are more accessible in the experimental conditions. Thus, we have

$$\nabla \mu = \left(\frac{\partial \mu}{\partial c} \right)_{p,T} \nabla c + \left(\frac{\partial \mu}{\partial T} \right)_{c,p} \nabla T + \left(\frac{\partial \mu}{\partial p} \right)_{c,T} \nabla p.$$

In the last term we use the thermodynamic relation

$$\left(\frac{\partial \mu}{\partial p} \right)_{c,T} = \left(\frac{\partial v}{\partial c} \right)_{p,T},$$

where $v = \rho^{-1}$ is the specific volume. Moreover, we introduce the coefficients

$$D \equiv \frac{\alpha}{\rho} \left(\frac{\partial \mu}{\partial c} \right)_{T,p},$$

$$\frac{\rho D}{T} \kappa_T \equiv \alpha \left(\frac{\partial \mu}{\partial T} \right)_{c,p} + \beta,$$

$$\kappa_p \equiv p \frac{\left(\frac{\partial v}{\partial c} \right)_{T,p}}{\left(\frac{\partial \mu}{\partial c} \right)_{T,p}}.$$

Substitution of these relations into the first relation of Eq. (5.9) yields the expression

$$\begin{aligned}
\mathbf{j}_i &= -\alpha\left(\frac{\partial\mu}{\partial c}\right)_{T,p}\nabla c-\alpha\left(\frac{\partial\mu}{\partial T}\right)_{c,p}\nabla T-\alpha\left(\frac{\partial\mu}{\partial p}\right)_{c,T}\nabla p-\beta\nabla T\\
&= -\rho D\nabla c-\left(\frac{\rho D}{T}\kappa_T-\beta\right)\nabla T-\alpha\left(\frac{\partial v}{\partial c}\right)_{p,T}\nabla p-\beta\nabla T\\
&= -\rho D\nabla c-\frac{\rho D}{p}\kappa_p\nabla p-\frac{\rho D}{T}\kappa_T\nabla T. \qquad (5.10)
\end{aligned}$$

The heat flux can be also rewritten in a more convenient form when one takes into account the identity

$$\frac{\beta T}{\alpha}=\kappa_T\left(\frac{\partial\mu}{\partial c}\right)_{T,p}-T\left(\frac{\partial\mu}{\partial T}\right)_{c,p}.$$

Therefore, the constitutive equations of the mass and heat fluxes can be expressed in terms of the spatial gradients of concentration, pressure and temperature as

$$\mathbf{j}_i=-\rho D\left[\nabla c+\frac{\kappa_T}{T}\nabla T+\frac{\kappa_p}{p}\nabla p\right], \qquad (5.11)$$

$$\mathbf{q}=\left[\kappa_T\left(\frac{\partial\mu}{\partial c}\right)_{T,p}-T\left(\frac{\partial\mu}{\partial T}\right)_{c,p}+\mu\right]\mathbf{j}_i-\kappa\nabla T. \qquad (5.12)$$

Here, D is called the *diffusion coefficient* or mass transfer coefficient (it gives the diffusion flux when only a concentration gradient is present), $\kappa_T D$ is the *thermal diffusion coefficient* (it gives the diffusion flux due to the temperature gradient) and the dimensionless coefficient κ_T is called the *thermal diffusion ratio.* The last term of the right hand side of Eq. (5.11) takes into account the effect of a pressure gradient on the diffusion flux; the coefficient $\kappa_p D$ is called the *barodiffusion coefficient.* In a single fluid, where there is no diffusion flux, then $\kappa_T=\kappa_p=0$.

The condition that the entropy must increase in time gives some restrictions for the transport coefficients appearing in the forms of $\mathbf{j}_i$ and $\mathbf{q}$. Omitting for brevity the viscous terms, the time derivative of the entropy is

$$\begin{aligned}
\frac{\partial}{\partial t}\int \rho s dV &= -\oint \rho s \mathbf{v}\cdot d\mathbf{S} - \int \frac{\nabla\cdot(\mathbf{q}-\mu\mathbf{j}_i)}{T}dV - \int \frac{\mathbf{j}_i\cdot\nabla\mu}{T}dV + \cdots \\
&= -\oint \rho s \mathbf{v}\cdot d\mathbf{S} - \int \nabla\cdot\left(\frac{\mathbf{q}-\mu\mathbf{j}_i}{T}\right)dV \\
&\quad -\int \frac{(\mathbf{q}-\mu\mathbf{j}_i)\cdot\nabla T}{T^2}dV - \int \frac{\mathbf{j}_i\cdot\nabla\mu}{T}dV + \cdots \\
&= -\oint \rho s \mathbf{v}\cdot d\mathbf{S} - \oint \frac{1}{T}(\mathbf{q}-\mu\mathbf{j}_i)\cdot d\mathbf{S} - \int \frac{\kappa(\nabla T)^2}{T^2}dV \\
&\quad + \int \frac{\alpha}{T}(\nabla\mu)^2 dV + \cdots \\
&= -\int \frac{\kappa(\nabla T)^2}{T^2}dV + \int \frac{\alpha}{T}(\nabla\mu)^2 dV + \cdots . \qquad (5.13)
\end{aligned}$$

Note that all the terms involving surface integrals vanish when one takes convenient boundary conditions for an infinite system. It is apparent that, besides the condition $\kappa > 0$ which we already know, we must have also $\alpha > 0$. Bearing in mind that the derivative $(\partial\mu/\partial c)_{p,T}$ is always positive according to one of the thermodynamic quantities, we therefore find that the diffusion coefficient must be positive: $D > 0$. The quantities κ_T and κ_p, however, may be either positive or negative.

We consider now the more simple case where $\nabla p = 0$ and the magnitude of the concentration and temperature gradients is quite small so that the transport coefficients can be considered as constants. In addition, we also assume that there is no macroscopic motion in the fluid ($\mathbf{v} = 0$), except that is produced by ∇c and ∇T. We also neglect terms which are of second order in spatial gradients. Under these conditions, the balance equations reduce to

$$\rho\frac{\partial c}{\partial t} + \nabla\cdot\mathbf{j}_i = 0, \qquad (5.14)$$

$$\rho T\frac{\partial s}{\partial t} + \nabla\cdot(\mathbf{q}-\mu\mathbf{j}_i) = 0, \qquad (5.15)$$

where $\mathbf{j}_i$ and $\mathbf{q}$ are given by Eqs. (5.11) and (5.12), respectively, with $\nabla p = 0$. The derivative $\partial s/\partial t$ can be transformed as follows:

$$\frac{\partial s}{\partial t} = \left(\frac{\partial s}{\partial T}\right)_{c,p}\frac{\partial T}{\partial t} + \left(\frac{\partial s}{\partial c}\right)_{T,p}\frac{\partial c}{\partial t} = \frac{c_p}{T}\frac{\partial T}{\partial t} - \left(\frac{\partial\mu}{\partial T}\right)_{c,p}\frac{\partial c}{\partial t}.$$

Taking into account this result and the identity

$$\mathbf{q} - \mu \mathbf{j}_i = \frac{\beta T}{\alpha} \mathbf{j}_i - \kappa \nabla T,$$

Eq. (5.15) can be expressed as

$$\frac{\partial T}{\partial t} - \left[\frac{T}{c_p} \left(\frac{\partial \mu}{\partial T} \right)_{c,p} + \frac{\beta T}{\alpha c_p} \right] \frac{\partial c}{\partial t} = \chi \nabla^2 T. \tag{5.16}$$

Equation (5.16) can be simplified when one takes into account the relation

$$\left(\frac{\partial \mu}{\partial T} \right)_{c,p} = \frac{\rho D}{T \alpha} \kappa_T - \frac{\beta}{\alpha} = \left(\frac{\partial \mu}{\partial c} \right)_{T,p} \frac{\kappa_T}{T} - \frac{\beta}{\alpha}.$$

Thus, Eqs. (5.14) and (5.15) can be finally written as

$$\frac{\partial c}{\partial t} = D \left(\nabla^2 c + \frac{\kappa_T}{T} \nabla^2 T \right), \tag{5.17}$$

$$\frac{\partial T}{\partial t} - \frac{\kappa_T}{c_p} \left(\frac{\partial \mu}{\partial c} \right)_{T,p} \frac{\partial c}{\partial t} = \chi \nabla^2 T. \tag{5.18}$$

There is a particularly important situation when the concentration c is quite small. In this limit ($c \to 0$), the diffusion coefficient tends to a finite constant, but the thermal diffusion coefficient $(\kappa_T / T) D$ tends to zero. Thus, Eq. (5.17) becomes

$$\frac{\partial c}{\partial t} = D \nabla^2 c. \tag{5.19}$$

This diffusion equation is similar to that of thermal conduction by replacing $c \leftrightarrow T$. The solution to Eq. (5.19) can be written as

$$c(r, t) = \frac{M}{8 \rho (\pi D t)^{3/2}} e^{-r^2/4Dt}, \tag{5.20}$$

where M is the total amount of the solute. Equation (5.20) gives the distribution of the solute at any time, if at time $t = 0$ it is all concentrated at the origin.

5.3 Diffusion of particles suspended in a fluid

Under the influence of the molecular motion in a fluid, particles suspended in the fluid move randomly (called the *Brownian motion*). Let us assume that such particle is located at the origin at the initial instant. Its subsequent motion may be regarded as a diffusion, in which the concentration can be interpreted as the probability of finding the particle in any particular volume element. To determine this probability we can use the solution (5.20) to the diffusion equation. We can employ this solution since $c \ll 1$ so that the particles of the solute do not affect practically each other and the motion of each solute particle can be considered independently.

Let $w(r,t)dr$ be the probability of finding the particle at a distance between r and $r+dr$ from the origin at time t. Putting in Eq. (5.20) $M/\rho \approx 1$ (since the probability of finding the particle at the origin at $t=0$ is equal to one), the corresponding probability reads

$$w(r,t)dr = \int_0^\pi d\theta\ \sin\theta \int_0^{2\pi} d\varphi\ r^2 c(r,t)\ dr. \tag{5.21}$$

Since $c(r,t)$ does not depend on the angles θ and φ, then one simply gets

$$w(r,t)dr = \frac{e^{-r^2/4Dt}}{2\sqrt{\pi D^3 t^3}} r^2 dr. \tag{5.22}$$

Let us determine the mean square displacement from the origin at time t. We have

$$\overline{r^2} = \int_0^\infty r^2 w(r,t)dr.$$

Using Eq. (5.22), the result is

$$\overline{r^2} = 6Dt. \tag{5.23}$$

Thus, the mean distance traveled by the particle during any time is proportional to the square root of the time, i.e.,

$$\sqrt{\overline{r^2}} \approx \overline{r} \sim \sqrt{t}.$$

The diffusion coefficient for particles suspended in a fluid can be calculated from what is called the *mobility*. Let us suppose that some constant external force **f** (the force of gravity, for example) acts on the particles. In a steady state, the force acting on each particle must be balanced by the drag force exerted by the fluid on a moving particle. When the velocity **v** is small, the drag force is proportional to $\mathbf{v}/b$ where b is a constant. Equating this to the external force **f**, we have

$$\mathbf{v} = b\mathbf{f},$$

i.e., the velocity acquired by the particle under the action of an external force is proportional to that force. The constant b is called the *mobility*. This coefficient can be calculated from the equations of fluid dynamics. For example, for spherical particles with radius R, the drag force is $6\pi\eta R v$ and therefore, the mobility is

$$b = (6\pi\eta R)^{-1}.$$

The mobility b is related to the diffusion coefficient D. To derive this relation, we write the diffusion flux, which contains the term $-\rho D\nabla c$ due to the concentration gradient (we assume that the temperature is constant) plus a term containing the velocity acquired by the particle due to the external force. This latter term is $\rho c\mathbf{v} = \rho c b\mathbf{f}$. Thus,

$$\mathbf{j}_i = -\rho D\nabla c + \rho c b\mathbf{f}.$$

This equation can be rewritten as

$$\mathbf{j}_i = -\frac{\rho D c}{k_B T}\nabla\mu + \rho c b\mathbf{f},$$

where we have taken into account the relation

$$(\nabla\mu)_T = \frac{k_B T}{c}\nabla c,$$

k_B being the Boltzmann constant [it will be defined in Chapter 7, see Eq. (7.30)].

In thermodynamic equilibrium, there is no diffusion, and $\mathbf{j}_i$ must be zero. On the other hand, when an external field is present, the condition of equilibrium requires $\mu + U = \text{const.}$, where U is the potential energy of a suspended particle in that field. Then, $\nabla\mu = -\nabla U = \mathbf{f}$, and the equation $\mathbf{j}_i = 0$ gives

$$D = k_B T b. \qquad (5.24)$$

This is *Einstein's relation* between the diffusion coefficient and the mobility. In the case of spherical particles, one has

$$D = \frac{k_B T}{6\pi\eta R}.$$

CHAPTER 6

STABILITY OF THE HYDRODYNAMIC EQUATIONS

6.1 Hydrodynamic equations

The goal of this Chapter is to solve the hydrodynamic equations for the densities of mass, momentum and energy in a very limiting case: when the system is close to equilibrium. Let us first write the hydrodynamic equations. They can be written as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0, \tag{6.1}$$

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial r_j} \right) = -\frac{\partial p}{\partial r_i} + \frac{\partial}{\partial r_j} \left[\eta \left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_\ell}{\partial r_\ell} \right) \right] + \frac{\partial}{\partial r_i} \left(\zeta \frac{\partial v_\ell}{\partial r_\ell} \right), \tag{6.2}$$

$$\rho c_v \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = -T \left(\frac{\partial p}{\partial T} \right)_\rho \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T) + \sigma'_{ij} \frac{\partial v_i}{\partial r_j}. \tag{6.3}$$

Note that Eq. (6.3) differs from Eq. (4.11) by the term $\partial_T p$ appearing on the right hand side of (6.3). For an ideal fluid, $\partial_T p = p/T$ and Eq. (6.3) reduces to Eq. (4.11). Thus, Eq. (6.3) applies beyond an ideal fluid.

The system of Eqs. (6.1)–(6.3) is very complicated to solve due essentially to its *nonlinear* character. This set of equations has an extremely rich variety of solutions since for instance it describes different problems such as the turbulence of liquids and the onset of instabilities in stars. On the other hand, this system admits a very simple solution: the absolute equilibrium state described by

$$\mathbf{v} = \mathbf{0}, \quad \rho(\mathbf{r}, t) \equiv \text{const.}, \quad T(\mathbf{r}, t) \equiv \text{const.}$$

An interesting problem is to analyze the stability of the equilibrium solution. In other words, we want to see if small deviations from the equilibrium solution decay or not in time. To analyze this problem, we then write the hydrodynamic fields as

$$\rho(\mathbf{r}, t) = \rho + \delta\rho(\mathbf{r}, t), \tag{6.4}$$

$$\mathbf{v}(\mathbf{r}, t) = \delta\mathbf{v}(\mathbf{r}, t), \tag{6.5}$$

$$T(\mathbf{r}, t) = T + \delta T(\mathbf{r}, t), \tag{6.6}$$

where henceforth we will use the convention that when the arguments $(\mathbf{r}, t)$ of a hydrodynamic variable are not explicitly displayed, it means that this quantity is defined in the (homogeneous) equilibrium state.

The next step is to substitute Eqs. (6.4)–(6.6) into the hydrodynamic equations (6.1)–(6.3), neglect nonlinear terms in the perturbations and assume that the transport coefficients are constant since they are defined in terms of the equilibrium state. Let us work out each hydrodynamic equation separately.

The equation of continuity (6.1) reads

$$\partial_t \delta\rho + \nabla \cdot [(\rho + \delta\rho)\delta\mathbf{v}] = 0.$$

To first order in $\delta\rho$, one gets

$$\partial_t \delta\rho(\mathbf{r}, t) + \rho \nabla \cdot \delta\mathbf{v}(\mathbf{r}, t) = 0. \tag{6.7}$$

We consider now the equation for the velocity field (6.2). First, the pressure p can be written as

$$p(\mathbf{r}, t) = p + \delta p(\mathbf{r}, t) = p + \left(\frac{\partial p}{\partial \rho}\right)_T \delta\rho(\mathbf{r}, t) + \left(\frac{\partial p}{\partial T}\right)_\rho \delta T(\mathbf{r}, t),$$

while the term on the right hand side of Eq. (6.2) becomes

$$\frac{\partial}{\partial r_j}\left[\eta\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_\ell}{\partial r_\ell}\right)\right] + \frac{\partial}{\partial r_i}\left(\zeta\frac{\partial v_\ell}{\partial r_\ell}\right) \to \eta\nabla^2\delta v_i + \left(\frac{\eta}{3} + \zeta\right)\nabla_i\nabla\cdot\delta\mathbf{v}.$$

With these results, the corresponding linearized version of the hydrodynamic equation (6.2) is

$$\partial_t \delta v_i + \rho^{-1} \left(\frac{\partial p}{\partial T}\right)_\rho \nabla_i \delta T + \rho^{-1} \left(\frac{\partial p}{\partial \rho}\right)_T \nabla_i \delta\rho = \frac{\eta}{\rho} \nabla^2 \delta v_i + \rho^{-1} \left(\frac{\eta}{3} + \zeta\right) \nabla_i \nabla \cdot \delta \mathbf{v}, \tag{6.8}$$

where the explicit dependence on $(\mathbf{r}, t)$ has been omitted for the sake of brevity. The linearized hydrodynamic equation for the temperature can be derived by using similar mathematical steps as before. It is given by

$$\partial \delta T(\mathbf{r}, t) + \frac{T}{\rho c_v} \left(\frac{\partial p}{\partial T}\right)_\rho \nabla \cdot \delta \mathbf{v}(\mathbf{r}, t) = \frac{\kappa}{\rho c_v} \nabla^2 \delta T(\mathbf{r}, t). \tag{6.9}$$

6.2 Linear stability analysis

The next step is to look for a solution to the set of linearized hydrodynamic equations (6.7), (6.8) and (6.9) for a system of infinite volume, assuming that the deviations of the macroscopic variables from equilibrium vanish at large distances. We introduce now the Fourier transform of the above set of equations. For instance, in the case of the mass density we have

$$\rho_{\mathbf{q}} = \int d\mathbf{r}\; e^{-i\mathbf{q}\cdot\mathbf{r}}\; \delta\rho(\mathbf{r}, t), \tag{6.10}$$

and similar formulas for the velocity field $\mathbf{v}_{\mathbf{q}}$ and temperature $T_{\mathbf{q}}$. Moreover, without loss of generality, let us take a reference frame where $\mathbf{q}$ is oriented along the x-axis, namely,

$$\mathbf{q} = q\hat{\mathbf{i}}. \tag{6.11}$$

We take now the Fourier transform of Eqs. (6.7), (6.8) and (6.9). Using the well-known correspondence $\partial/\partial\mathbf{r} \Longrightarrow i\mathbf{q}$ for Fourier transforms, one has for instance

$$\begin{aligned} \int d\mathbf{r} e^{-iqx} \nabla \cdot \delta \mathbf{v} &= \int d\mathbf{r} e^{-iqx} \frac{\partial \delta v_x}{\partial x} \\ &= \frac{\partial}{\partial r_j} \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \delta v_j + iq\delta_{jx} \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \delta v_j \\ &= iq v_{\mathbf{q},\|} = i\mathbf{q} \cdot \mathbf{v}_{\mathbf{q}}, \end{aligned} \tag{6.12}$$

where $v_{\mathbf{q},\parallel}$ denotes the component of the velocity field parallel to the perturbation. Moreover,

$$\int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \nabla^2 \delta \mathbf{v} = -q^2 \mathbf{v}_{\mathbf{q}}, \tag{6.13}$$

$$\int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \nabla\nabla \cdot \delta \mathbf{v} = -\mathbf{q}(\mathbf{q}\cdot\mathbf{v}_{\mathbf{q}}). \tag{6.14}$$

With these results, the Fourier transform of Eqs. (6.7), (6.8) and (6.9) yields

$$\partial_t \rho_{\mathbf{q}}(t) = -i\rho \mathbf{q}\cdot\mathbf{v}_{\mathbf{q}}, \tag{6.15}$$

$$\partial_t \mathbf{v}_{\mathbf{q}}(t) = -i\alpha_1 \mathbf{q}\rho_{\mathbf{q}} - i\beta_1 \mathbf{q} T_{\mathbf{q}} - \nu q^2 \mathbf{v}_{\mathbf{q}} - \delta_1 \mathbf{q}(\mathbf{q}\cdot\mathbf{v}_{\mathbf{q}}), \tag{6.16}$$

$$\partial_t T_{\mathbf{q}}(t) = -i\mu_1 \mathbf{q}\cdot\mathbf{v}_{\mathbf{q}} - \xi_1 q^2 T_{\mathbf{q}}, \tag{6.17}$$

where we have introduced the abbreviations

$$\alpha_1 \equiv \rho^{-1}\left(\frac{\partial p}{\partial \rho}\right)_T, \quad \beta_1 \equiv \rho^{-1}\left(\frac{\partial p}{\partial T}\right)_\rho, \nu \equiv \frac{\eta}{\rho}, \tag{6.18}$$

$$\delta_1 \equiv \rho^{-1}\left(\frac{\eta}{3}+\zeta\right), \quad \mu_1 \equiv \frac{T}{\rho c_v}\left(\frac{\partial p}{\partial T}\right)_\rho, \quad \xi_1 \equiv \frac{\kappa}{\rho c_v}. \tag{6.19}$$

The set of equations (6.15)–(6.17) clearly show that the transversal component of the velocity field

$$\mathbf{v}_{\mathbf{q},\perp} = \mathbf{v}_{\mathbf{q}} - \frac{\mathbf{q}(\mathbf{q}\cdot\mathbf{v}_{\mathbf{q}})}{q^2} \tag{6.20}$$

is decoupled from the remaining fields. Its evolution equation can be obtained from Eq. (6.16):

$$\partial_t \mathbf{v}_{\mathbf{q},\perp}(t) = -\nu q^2 \mathbf{v}_{\mathbf{q},\perp}(t). \tag{6.21}$$

The solution to Eq. (6.21) is

$$\mathbf{v}_{\mathbf{q},\perp}(t) = \mathbf{v}_{\mathbf{q},\perp}(0) e^{-\nu q^2 t}, \tag{6.22}$$

where $\mathbf{v}_{\mathbf{q},\perp}(0)$ denotes the initial value of $\mathbf{v}_{\mathbf{q},\perp}$. Since the shear viscosity is positive, then $\nu > 0$ and $\mathbf{v}_{\mathbf{q},\perp}(t)$ goes to zero for long times. Consequently, the transversal shear mode is linearly *stable*.

The remaining longitudinal fields $(\rho_{\mathbf{q}}, v_{\mathbf{q},\parallel}, T_{\mathbf{q}})$ obey a *coupled* system of ordinary differential equations. For the sake of completeness, let us write in matrix form the complete system of equations for the five hydrodynamic fields (including the transversal components). The set of differential equations can be written as

$$\partial_t \mathbf{\Psi}_q(t) = \mathbf{M}_q \cdot \mathbf{\Psi}_q(t), \tag{6.23}$$

where

$$\mathbf{\Psi}_q(t) = \begin{pmatrix} \rho_{\mathbf{q}}(t) \\ v_{\mathbf{q},x}(t) \\ v_{\mathbf{q},y}(t) \\ v_{\mathbf{q},z}(t) \\ T_{\mathbf{q}}(t) \end{pmatrix}, \tag{6.24}$$

$$\mathbf{M}_q = \begin{pmatrix} 0 & -i\rho q & 0 & 0 & 0 \\ -i\alpha_1 q & -(\nu+\delta_1)q^2 & 0 & 0 & -i\beta_1 q \\ 0 & 0 & -\nu q^2 & 0 & 0 \\ 0 & 0 & 0 & -\nu q^2 & 0 \\ 0 & -i\mu_1 q & 0 & 0 & -\xi_1 q^2 \end{pmatrix}. \tag{6.25}$$

Although $\mathbf{M}_q$ is non-Hermitian, explicit calculations show that it can be diagonalized, or equivalently, that the eigenvalue problem

$$\mathbf{M}_q \cdot \Phi_\alpha^q = \lambda_\alpha^q \Phi_\alpha^q \tag{6.26}$$

has five linearly independent solutions. In this case, the eigenvalues λ_α^q provide the time evolution of $\mathbf{\Psi}_q(t)$:

$$\mathbf{\Psi}_q(t) = \sum_{\alpha=1}^{5} c_\alpha^q(t)\Phi_\alpha^q = \sum_{\alpha=1}^{5} c_\alpha^q(0) e^{\lambda_\alpha^q t}\Phi_\alpha^q. \tag{6.27}$$

Hence the solution of the linearized hydrodynamic equations is the superposition of five *hydrodynamic modes*, each describing a coherent motion of the five hydrodynamic variables, with a simple time behavior.

6.3 Hydrodynamic modes

The eigenvalues λ_q^α of the eigenvalue problem (6.26) are the solutions of the secular determinant

$$\|\mathbf{M}_q - \lambda_\alpha^q \mathbf{1}\| = 0, \tag{6.28}$$

where **1** denotes the matrix identity. Equation (6.28) is a fifth-degree equation that decouples into a third-degree equation and a second degree-equation. It is given by

$$\left(\nu q^2+\lambda_\alpha^q\right)^2\left\{\lambda_\alpha^{q3}+(\nu+\delta_1+\xi_1)q^2\lambda_\alpha^{q2}+\left[(\beta_1\mu_1+\alpha_1\rho)q^2\right.\right.$$
$$\left.\left.+\xi_1(\nu+\delta_1)q^4\right]\lambda_\alpha^q+\alpha_1\xi_1\rho q^4\right\}=0. \tag{6.29}$$

As expected, the modes corresponding to the transversal components of the velocity field are decoupled. They are given by

$$\lambda_3^q=\lambda_4^q=-\nu q^2 \tag{6.30}$$

The other three modes are given in terms of the solution of a cubic equation. Although they have an explicit form, we note that hydrodynamics only applies for phenomena varying slowly in space. In the Fourier language, this means that the only relevant coefficients in Eq. (6.29) correspond to small q. Thus, we look for solutions in the limit $q\to 0$. In this limit, we write

$$\lambda_\alpha^q=a_\alpha q+b_\alpha q^2+\mathcal{O}(q^3). \tag{6.31}$$

Thus, one has

$$\lambda_\alpha^{q3}=(a_\alpha q+b_\alpha q^2)^3\to a_\alpha^3q^3+3a_\alpha^2b_\alpha q^4,$$
$$\lambda_\alpha^{q2}=(a_\alpha q+b_\alpha q^2)^2\to a_\alpha^2q^2+b_\alpha^2q^4+2a_\alpha b_\alpha q^3.$$

Substituting these expansions into the cubic equation

$$\lambda_\alpha^{q3}+(\nu+\delta_1+\xi_1)q^2\lambda_\alpha^{q2}+\left[(\beta_1\mu_1+\alpha_1\rho)q^2+\xi_1(\nu+\delta_1)q^4\right]\lambda_\alpha^q+\alpha_1\xi_1\rho q^4=0, \tag{6.32}$$

and setting the different terms of the same power in q equal to zero, one gets several relations for the coefficients. In the order q^3, one gets

$$a_\alpha\left(a_\alpha^2+\beta_1\mu_1+\rho\alpha_1\right)=0, \tag{6.33}$$

while the relation in the order q^4 is

$$3a_\alpha^2b_\alpha+a_\alpha^2(\nu+\delta_1+\xi_1)+b_\alpha(\beta_1\mu_1+\rho\alpha_1)+\rho\alpha_1\xi_1=0. \tag{6.34}$$

Equation (6.34) allows us to express b_α in terms of a_α as

$$b_\alpha=-\frac{a_\alpha^2(\nu+\delta_1+\xi_1)+\rho\alpha_1\xi_1}{3a_\alpha^2+\beta_1\mu_1+\rho\alpha_1}. \tag{6.35}$$

The solutions to Eq. (6.34) are $a_5 = 0$ and

$$a_1 = -a_2 = i\sqrt{\beta_1\mu_1 + \rho\alpha_1}. \tag{6.36}$$

The corresponding forms for b_α are

$$b_5 = -\frac{\rho\alpha_1\xi_1}{\beta_1\mu_1 + \rho\alpha_1}, \tag{6.37}$$

$$\begin{aligned} b_1 = b_2 &= \frac{\rho\alpha_1\xi_1 - (\nu + \delta_1 + \xi_1)(\beta_1\mu_1 + \rho\alpha_1)}{2(\beta_1\mu_1 + \rho\alpha_1)} \\ &= -\frac{1}{2}\left(\nu + \delta_1 + \xi_1 - \frac{\rho\alpha_1\xi_1}{\beta_1\mu_1 + \rho\alpha_1}\right). \end{aligned} \tag{6.38}$$

Using the thermodynamic relation

$$c_p - c_v = \frac{T\left(\frac{\partial p}{\partial T}\right)_\rho^2}{\rho^2\left(\frac{\partial p}{\partial \rho}\right)_T},$$

and the explicit forms (6.18) and (6.19), one gets the expressions

$$\rho\alpha_1\xi_1 = \frac{\kappa}{\rho c_v}\left(\frac{\partial p}{\partial \rho}\right)_T, \tag{6.39}$$

$$\beta_1\mu_1 + \rho\alpha_1 = \frac{T}{\rho^2 c_v}\left(\frac{\partial p}{\partial T}\right)_\rho^2 + \left(\frac{\partial p}{\partial \rho}\right)_T = \left(\frac{c_p - c_v}{c_v} + 1\right)\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{c_p}{c_v}\left(\frac{\partial p}{\partial \rho}\right)_T. \tag{6.40}$$

Thus, the coefficients b_5 and b_1 can be explicitly obtained when one uses Eqs. (6.39) and (6.40) in the definitions (6.37) and (6.38):

$$b_5 = -\frac{\kappa}{\rho c_p}, \tag{6.41}$$

$$b_1 = -\frac{1}{2\rho}\left[\frac{4}{3}\eta + \zeta + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right]. \tag{6.42}$$

The eigenvalues associated to the longitudinal modes can be easily determined when one takes into account Eqs. (6.41) and (6.42). The results are

$$\lambda_{1,2}^q = \mp i c_\mathrm{s} q - \Gamma_\mathrm{s} q^2, \tag{6.43}$$

$$\lambda_5^q = -\frac{\kappa}{\rho c_p} q^2, \tag{6.44}$$

where

$$c_\mathrm{s} = \sqrt{\frac{c_p}{c_v}\left(\frac{\partial p}{\partial \rho}\right)_T} \tag{6.45}$$

is the sound velocity and

$$\Gamma_\mathrm{s} = \frac{1}{2\rho}\left[\frac{4}{3}\eta + \zeta + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right] \tag{6.46}$$

is the sound-absorption coefficient.

The meaning of the hydrodynamic modes is clear. The modes $\lambda_{1,2}^q$ describe damped sound wave propagations; the modes $\lambda_{3,4}^q$ correspond diffusion of the transverse velocity and λ_5^q describes heat diffusion. Given that the real part of *all* the modes is negative, then the state of absolute equilibrium is (linearly) *stable*.

CHAPTER 7

MICROSCOPIC DESCRIPTION: BALANCE EQUATIONS

7.1 Introduction

In the previous Chapters we have derived the balance equations of mass, momentum and energy by considering a continuum description of the fluid. In this Chapter, we will obtain the above balance equations by considering the macroscopic fluid system constituted by particles. These particles interact each other so that the collisions among them gives rise to the transfer of momentum and energy across the system.

The hydrodynamic description of nonequilibrium fluids is based on three major stages. In the first stage, one introduces local densities of mass, momentum, and energy, which are conserved quantities. The rate of change of these quantities can be expressed in terms of the divergence of the corresponding mass, momentum, and energy fluxes. These *balance* equations are not exact since they do not constitute a closed set of equations. In the second stage, the irreversible parts of the momentum and energy fluxes are assumed to be *linear* functions of the flow velocity and temperature gradients, respectively. These linear relationships between fluxes and gradients define the *Navier-Stokes constitutive equations*: Newton's friction law for the stress tensor and Fourier's law for heat conduction. The associated coefficients are the *transport coefficients* of the fluid, namely, the shear η and bulk ζ viscosities and the thermal conductivity κ, respectively. When these phenomenological laws for the fluxes are inserted into the balance equations, the resulting set of equations is still not closed because one needs to determine the hydrostatic pressure p and the internal energy density u in terms of the local mass density ρ and the local temperature T. This is the done in the third stage, where the equilibrium equation of state relating p and u to ρ and T are extended to the nonequilibrium local quantities (*local equilibrium hypothesis*). The three-stage process yields the closed set of Navier-Stokes hydrodynamic equations. This scheme is essentially the same for fluid mixtures, except that the number of transport coefficients is larger than in the case of a simple fluid.

The prototype system that embodies the most relevant physical properties of ordinary fluids consists of a low-density gas (*dilute* gas) of particles interacting via a pair-wise, central short-ranged potential. Under these conditions, the system admits a kinetic theory description rather than a complete non-equilibrium statistical-mechanical treatment. If the gas is rarefied (namely, it is sufficiently dilute), a simple picture grasped by the fathers of kinetic theory arises: every particle moves most of the time freely (or under the action of an external force, if it exists), except when eventually its trajectory approaches to that of another particle and, as a result of their mutual interactions, both particles abruptly change their respective velocities. Thus, in a dilute gas the interactions among the particles reduce to a sequence of uncorrelated binary collisions. The collisions can be considered as events well localized in space and time because the typical distance traveled between two successive collisions is much larger than the range of the interaction, and analogously, the typical time elapsed between two collisions is much larger than the duration of a collision.

The simplifications inherent to the low-density limit allow for a huge reduction of the microscopic description of the system: from a general statistical-mechanical description to a *contracted* kinetic theory description, from the N-body phase space probability density to the one-body velocity distribution function, from the *reversible* Liouville equation to the *irreversible* Boltzmann equation. The Boltzmann equation is the master equation for a low-density gas and describes the rate of change of the velocity distribution function as produced by two uncoupled mechanisms: free motion and localized, instantaneous binary collisions.

7.2 Velocity distribution function

We consider a *dilute* monatomic gas of N identical particles of mass m interacting through a pair-wise central interaction potential of finite range. We also assume that quantum-mechanical effects are not relevant or significant and hence, we use classical mechanics. At a kinetic level *all* the relevant information on the state of the system is contained in the *one-body velocity distribution function* (vdf) $f(\mathbf{r},\mathbf{v},t)$. It is defined so that $f(\mathbf{r},\mathbf{v},t)d\mathbf{r}d\mathbf{v}$ is the *most probable* (or average) number of particles which at time t lie in the volume element $d\mathbf{r}$ centered at the point $\mathbf{r}$ and moving with velocities in the range $d\mathbf{v}$ about $\mathbf{v}$. Accordingly, the distribution function $f(\mathbf{r},\mathbf{v},t)$ must satisfy the condition [1]

$$N=\int d\mathbf{r}\int d\mathbf{v}f(\mathbf{r},\mathbf{v},t), \tag{7.1}$$

or equivalently,

$$n(\mathbf{r},t)=\int d\mathbf{v}f(\mathbf{r},\mathbf{v},t). \tag{7.2}$$

Here, $n(\mathbf{r},t)$ is the average number of particles per unit volume, namely, it is the local number density. The mass density per unit volume $\rho(\mathbf{r},t)=mn(\mathbf{r},t)$.

The average velocity of those particles located around the point $\mathbf{r}$ at time t defines the local flow velocity $\mathbf{U}(\mathbf{r},t)$ as

$$\mathbf{U}(\mathbf{r},t)=\frac{1}{n(\mathbf{r},t)}\int d\mathbf{v}\ \mathbf{v}\ f(\mathbf{r},\mathbf{v},t). \tag{7.3}$$

In general, if $\chi(\mathbf{v})$ is a given function of the velocity $\mathbf{v}$ of a particle, we can define its average value as

$$\overline{\chi(\mathbf{r},t)}\equiv\langle\chi(\mathbf{r},t)\rangle=\frac{1}{n(\mathbf{r},t)}\int d\mathbf{v}\ \chi(\mathbf{v})\ f(\mathbf{r},\mathbf{v},t). \tag{7.4}$$

In particular, the mean value of $\chi(\mathbf{v})=\frac{1}{2}mv^2$ defines the average kinetic energy of each particle. Note that the definition (7.4) for the average values only involves properties associated with only one particle.

---

[1] Note that here $\mathbf{v}$ denotes the random velocity of a particle while in the previous Chapters $\mathbf{v}$ means the velocity of a fluid particle (which is equivalent to the flow mean velocity $\mathbf{U}$).

7.3 Microscopic fluxes

Once we have introduced the concept of the velocity distribution function, as an application it is interesting to determine the flux of the dynamical quantity χ. This flux will be denoted as $\mathcal{F}(\mathbf{r},t)$. We want to evaluate the net flow of $\chi(\mathbf{r},\mathbf{v})$ per unit area and time due to the net flow of molecules across the surface element $d\mathbf{S}$. If $d\mathbf{S} \parallel \widehat{\mathbf{k}}$, then $\mathcal{F}_z(\mathbf{r},t)$ gives the net flow of χ which is transferred at the time t and the point $\mathbf{r}$ per unit time and unit area perpendicular to the z-axis. As usual in fluid dynamics, we are only interested in the *irreversible* transfer processes accounting for a given quantity from one side to another side of the system. If the gas is not at rest ($\mathbf{U} \neq \mathbf{0}$), there is a reversible convective flow due to the motion of the fluid. Our goal here is to determine the internal flow, even when the fluid is at rest. To do it, we assume that the surface element is moving with the flow velocity $\mathbf{U}$ at this point. This is equivalent to refer the velocities of the particles to a frame moving at the velocity $\mathbf{U}(\mathbf{r},t)$ and introduce the peculiar velocity

$$\mathbf{V}(\mathbf{r},t) = \mathbf{v} - \mathbf{U}(\mathbf{r},t). \tag{7.5}$$

Let us consider a given surface element dA perpendicular to the z-axis. In this case, the number of particles crossing this surface from the negative to the positive side with velocities between $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$ in the time interval between t and $t+dt$ is

$$dA|V_z dt| f(\mathbf{r},\mathbf{v},t) d\mathbf{v}. \tag{7.6}$$

In this relation, $V_z > 0$. Thus, the net flow of $\chi(\mathbf{v})$ across dA in ascendent direction is

$$\mathcal{F}_z^{(+)}(\mathbf{r},t) = \int_{V_z>0} d\mathbf{v}\ |V_z|\ \chi(\mathbf{v})\ f(\mathbf{r},\mathbf{v},t). \tag{7.7}$$

Analogously,

$$\begin{aligned}\mathcal{F}_z^{(-)}(\mathbf{r},t) &= \int_{V_z<0} d\mathbf{v}\ |V_z|\ \chi(\mathbf{v})\ f(\mathbf{r},\mathbf{v},t) \\ &= -\int_{V_z<0} d\mathbf{v}\ V_z\ \chi(\mathbf{v})\ f(\mathbf{r},\mathbf{v},t).\end{aligned} \tag{7.8}$$

The net flow is then

$$\begin{aligned}\mathcal{F}_z &= \mathcal{F}_z^{(+)} - \mathcal{F}_z^{(-)} \\ &= \int_{V_z>0} d\mathbf{v}\, V_z\, \chi(\mathbf{v})\, f(\mathbf{r},\mathbf{v},t) + \int_{V_z<0} d\mathbf{v}\, V_z\, \chi(\mathbf{v})\, f(\mathbf{r},\mathbf{v},t) \\ &= \int d\mathbf{v}\, V_z\, \chi(\mathbf{v})\, f(\mathbf{r},\mathbf{v},t). \end{aligned} \tag{7.9}$$

In general, we define the flux vector $\boldsymbol{\mathcal{F}}(\mathbf{r},t)$ as

$$\boldsymbol{\mathcal{F}}(\mathbf{r},t) = \int d\mathbf{v}\, \mathbf{V}\, \chi(\mathbf{v})\, f(\mathbf{r},\mathbf{v},t) = n(\mathbf{r},t)\langle \mathbf{V}\chi(\mathbf{v})\rangle. \tag{7.10}$$

7.4 Momentum and heat fluxes

We consider here some examples of irreversible fluxes.

7.4.1 Flux of kinetic energy

This flux is usually measured in the Lagrangian frame moving with the flow velocity $\mathbf{U}$ of the gas. In this case,

$$\chi(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m(\mathbf{v}-\mathbf{U}(\mathbf{r},t))^2 = \frac{1}{2}mV(\mathbf{r},t)^2.$$

The corresponding flux is the *heat flux* $\mathbf{q}$:

$$\mathbf{q}(\mathbf{r},t) = \int d\mathbf{v}\, \frac{1}{2}mV^2\, \mathbf{V}\, f(\mathbf{v}). \tag{7.11}$$

7.4.2 Flux of momentum

In this case χ is a vector and so, its corresponding flux will be a tensor of second rank. It is given by

$$\begin{aligned} P_{ij}(\mathbf{r},t) = \int d\mathbf{v}\, V_i m v_j\, f(\mathbf{v}) &= \int d\mathbf{v}\, V_i m(V_j+U_j)\, f(\mathbf{v}) \\ &= \int d\mathbf{v}\, mV_iV_j\, f(\mathbf{v}) = P_{ji}(\mathbf{r},t), \end{aligned} \tag{7.12}$$

where use has been made of the property

$$\langle V_i \rangle = \langle v_i \rangle - U_i = U_i - U_i = 0.$$

The tensor P_{ij} is called the *pressure tensor*. It is symmetric. At equilibrium, the one-particle distribution function $f(v)$ is isotropic in velocity and so, it depends on $\mathbf{v}$ only through its magnitude. Thus,

$$\mathbf{U} = \langle \mathbf{v} \rangle = \mathbf{0}, \quad \mathbf{q} = \frac{1}{2} mn \langle v^2 \mathbf{v} \rangle = \mathbf{0}, \tag{7.13}$$

$$P_{ij} = mn\langle v_i v_j \rangle = mn\langle v_x^2 \rangle \delta_{ij} = \frac{1}{3} mn \langle v^2 \rangle \delta_{ij}. \tag{7.14}$$

Note that the diagonal elements of the pressure tensor represent normal forces per unit area. Out of equilibrium, in general $P_{xx} \neq P_{yy} \neq P_{zz}$. However, one defines still the hydrostatic pressure as

$$p = \frac{1}{3} \text{Tr}\, \mathbf{P} = \frac{1}{3} \left(P_{xx} + P_{yy} + \cdots \right) = \frac{1}{3} mn \langle V^2 \rangle. \tag{7.15}$$

7.5 Evolution equation for the velocity distribution function

Our aim now is to get the equation for the rate of change of f with time. We will restrict ourselves to the case of a low-density monatomic gas, although we will assume that the gas is not isolated and is subjected to the action of an external conservative force. In general, there are two sources of temporal change of f: the streaming and interactions, namely,

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t} \right)_{\text{str}} + \left(\frac{\partial f}{\partial t} \right)_{\text{int}}. \tag{7.16}$$

The first term on the right hand side represents the change of f due to the free motion of the particles under the action, in general, of an *external* force $\mathbf{F}$. The second term refers to the change due to the mutual interactions among the particles. In the case of a dilute gas, we can assume that both mechanisms are *decoupled*.

To get the free streaming contribution, let us assume for the moment that the interactions are absent. Such a system is usually called a Knudsen gas. In that case, the change during a time interval dt of the average number of particles which are located at the point $\mathbf{r}$ with velocity $\mathbf{v}$ is due to two reasons:

- Since the particles are moving, then the particles located in the point $\mathbf{r}$ at instant t will move to the point $\mathbf{r}+\mathbf{v}\delta t$ at $t+\delta t$, where δt is an infinitesimal time interval. Analogously, the particles which are located in positions $\mathbf{r}'$ close to the point $\mathbf{r}$ will move at the position $\mathbf{r}$ after a time interval δt.
- Due to the action of the external force $\mathbf{F}(\mathbf{r},t)$ on each particle, the particles with a velocity $\mathbf{v}$ at t will have a different velocity $\mathbf{v}+m^{-1}\mathbf{F}\delta t$ at $t+\delta t$. Analogously, the particles with velocities $\mathbf{v}'$ near $\mathbf{v}$ will be accelerated to have the velocity $\mathbf{v}$ after a time interval δt.

Therefore, the particles that at time t are located at the point $\mathbf{r}$ and moving with velocity $\mathbf{v}$ will be located at

$$\mathbf{r}'=\mathbf{r}+\mathbf{v}\delta t$$

having a velocity

$$\mathbf{v}'=\mathbf{v}+\frac{\mathbf{F}}{m}\delta t$$

at a later time $t'=t+\delta t$. Consequently,

$$d\mathbf{r}'=d\mathbf{r}+d\mathbf{v}\delta t, \quad d\mathbf{v}'=d\mathbf{v}+\frac{\partial\mathbf{F}}{\partial\mathbf{r}}\cdot\frac{d\mathbf{r}}{m}\delta t. \tag{7.17}$$

Moreover, in the absence of collisions,

$$f(\mathbf{r},\mathbf{v},t)d\mathbf{r}d\mathbf{v}=f(\mathbf{r}',\mathbf{v}',t')d\mathbf{r}'d\mathbf{v}'. \tag{7.18}$$

The Jacobian of the transformation $(\mathbf{r},\mathbf{v})\to(\mathbf{r}',\mathbf{v}')$ is defined as

$$d\mathbf{r}'d\mathbf{v}'=|J|d\mathbf{r}d\mathbf{v},$$

where

$$\begin{aligned} |J| &= \left|\frac{\partial(\mathbf{r}',\mathbf{v}')}{\partial(\mathbf{r},\mathbf{v})}\right| \\ &= \left|\frac{\partial(x',y',z',v_x',v_y',v_z')}{\partial(x,y,z,v_x,v_y,v_z)}\right| = 1+O\left((\delta t)^2\right)\simeq 1. \end{aligned} \tag{7.19}$$

Combination of Eqs. (7.18) and (7.19) yields

$$f(\mathbf{r}',\mathbf{v}',t')=f(\mathbf{r},\mathbf{v},t). \tag{7.20}$$

In addition, retaining only linear terms in δt, the Taylor expansion of $f(\mathbf{r}', \mathbf{v}', t')$ is

$$\begin{aligned} f(\mathbf{r}', \mathbf{v}', t') &= f(\mathbf{r}, \mathbf{v}, t) + \frac{\partial f}{\partial \mathbf{r}} \cdot (\mathbf{r}' - \mathbf{r}) + \frac{\partial f}{\partial \mathbf{v}} \cdot (\mathbf{v}' - \mathbf{v}) + \frac{\partial f}{\partial t}(t' - t) \\ &= f(\mathbf{r}, \mathbf{v}, t) + \frac{\partial f}{\partial \mathbf{r}} \cdot \mathbf{v}\delta t + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\mathbf{F}}{m}\delta t + \frac{\partial f}{\partial t}\delta t. \end{aligned} \tag{7.21}$$

Taking into account Eq. (7.21), Eq. (7.20) leads to

$$\left(\frac{\partial f}{\partial t}\right)_{\text{str}} = -\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}}. \tag{7.22}$$

Inserting Eq. (7.22) into Eq. (7.16), one finally gets

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{int}}. \tag{7.23}$$

The interaction term is much more difficult to deal with that the free streaming term. An explicit form for this term can be derived under some *statistical* hypothesis that are justified in the low-density limit. The resulting closed equation is the well-known Boltzmann kinetic equation. Although its explicit derivation goes beyond the scope of the present course, let us formally write this nonlinear equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = J[\mathbf{v}|f, f], \tag{7.24}$$

where $J[f, f]$ is the Boltzmann collision operator. This operator depends in a bilinear way on the distribution function f. Consequently, the Boltzmann equation (7.24) is an integro-nonlinear differential equation for f. Due to the mathematical difficulties embodied in the Boltzmann collision operator, for the sake of convenience, it is usual to consider simplified versions of $J[f, f]$. These approaches are mathematically more simple than the true Boltzmann equation but retain its most physical relevant properties. One of the most well-known kinetic models is the BGK equation where $J[f, f]$ is replaced by the relaxation term

$$J[f, f] \to -\nu(\mathbf{r}, t)\left(f(\mathbf{r}, \mathbf{v}, t) - f_{\text{LE}}(\mathbf{r}, \mathbf{v}, t)\right), \tag{7.25}$$

where

$$f_{\text{LE}} = n\left(\frac{m}{2\pi k_{\text{B}} T}\right)^{3/2} e^{-\frac{m}{2k_{\text{B}}T}V^2} \tag{7.26}$$

is the local-equilibrium distribution. Here, ν is an effective collision frequency of the gas which is velocity independent. Note that the BGK model is actually a highly nonlinear equation since f_{LE} is a functional of f through its dependence on n, $\mathbf{U}$, and T. In fact, f and f_{LE} share the same first few velocity moments, namely,

$$\int d\mathbf{v}\ f = \int d\mathbf{v}\ f_{\text{LE}} = n, \tag{7.27}$$

$$\int d\mathbf{v}\ \mathbf{v}\ f = \int d\mathbf{v}\ \mathbf{v}\ f_{\text{LE}} = n\mathbf{U}, \tag{7.28}$$

$$\int d\mathbf{v}\ (\mathbf{v}-\mathbf{U})^2\ f = \int d\mathbf{v}\ (\mathbf{v}-\mathbf{U})^2\ f_{\text{LE}} = 3\frac{nk_{\text{B}}T}{m}, \tag{7.29}$$

where

$$k_{\text{B}} = 1,381 \times 10^{-23}\text{J/K} \tag{7.30}$$

is the Boltzmann constant. It is related with the universal constant of gases R as $R = N_{\text{A}}k_{\text{B}} = 8,314$ J/mol, where $N_{\text{A}} = 6,022 \times 10^{23}$ is the Avogrado number. A consequence of Eqs. (7.27)–(7.29) is that the quantities 1, $\mathbf{v}$ and v^2 do not change during the collisions (*collisional invariants*):

$$\int d\mathbf{v}\ \left\{1, v_i, v^2\right\}\left(\frac{\partial f}{\partial t}\right)_{\text{int}} = \{0,0,0\}\,. \tag{7.31}$$

7.6 Solution to the BGK model for states close to equilibrium

Before considering the derivation of the balance equations, it is quite instructive to determine the explicit microscopic forms of the shear viscosity η and the thermal conductivity κ. Let us evaluate each coefficient separately.

7.6.1 Thermal conductivity

We consider a *steady* state of a dilute gas at rest ($\mathbf{U}=\mathbf{0}$) in the absence of an external force ($\mathbf{F}=\mathbf{0}$) and with a temperature gradient along the z axis ($\partial_x T = \partial_y T = 0$ but $\partial_z T \neq 0$). Under these conditions, the BGK kinetic equation (7.24) reduces to

$$v_z\frac{\partial f}{\partial z} = -\nu\left(f - f_{\text{LE}}\right). \tag{7.32}$$

Equation (7.32) is still a difficult differential equation to solve, especially for arbitrary thermal gradients. On the other hand, when $|\partial_z T|$ is very small then one can assume that the distribution function f is *close* to the local equilibrium distribution $f_{\rm LE}$ and so,

$$f = f_{\rm LE} + \varphi, \quad \varphi \ll f_{\rm LE}. \tag{7.33}$$

In this approximation, $\partial_z f \simeq \partial_z f_{\rm LE}$ and Eq. (7.32) can be rewritten as

$$\varphi = -\frac{v_z}{\nu}\frac{\partial f_{\rm LE}}{\partial z}. \tag{7.34}$$

The goal of this section is to determine the thermal conductivity coefficient κ. To get it, we need to compute the heat flux q_z. According to Eq. (7.11), q_z is defined as

$$\begin{aligned} q_z &= \int d\mathbf{v}\, \frac{m}{2} v^2 v_z f(\mathbf{v}) \\ &= -\nu^{-1}\frac{\partial}{\partial z}\int d\mathbf{v}\, \frac{m}{2} v^2 v_z^2\, f_{\rm LE}(\mathbf{v}). \end{aligned} \tag{7.35}$$

Upon deriving Eq. (7.35) use has been made of the symmetry properties of $f_{\rm LE}$ (it is an even function of $\mathbf{v}$ and so, $\langle v^2 v_z\rangle_{\rm LE} = 0$). The integral appearing on the right hand side of (7.35) can be written in dimensionless form as

$$\begin{aligned} &\int d\mathbf{v}\, \frac{m}{2} v^2 v_z^2\, f_{\rm LE}(\mathbf{v}) = \frac{1}{3}\int d\mathbf{v}\, \frac{m}{2} v^4\, f_{\rm LE}(\mathbf{v}) \\ &= \frac{1}{3} n \left(\frac{2k_{\rm B}T}{m}\right)^2 \frac{m}{2}\pi^{-3/2}\int d\mathbf{c}\, c^4\, e^{-c^2} \\ &= \frac{1}{3} n \left(\frac{2k_{\rm B}T}{m}\right)^2 \frac{m}{2}\pi^{-3/2}\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \int_0^{\infty} dc\, c^6 e^{-c^2} \\ &= \frac{4}{3\sqrt{\pi}} n \left(\frac{2k_{\rm B}T}{m}\right)^2 \frac{m}{2}\int_0^{\infty} dc\, c^6\, e^{-c^2}, \end{aligned} \tag{7.36}$$

where $c_i = v_i/v_0$, $v_0 = \sqrt{2k_{\rm B}T/m}$ being the thermal velocity. The integral over the (dimensionless) velocity c can be easily evaluated by taking into account the general result

$$I(n) \equiv \int_0^{\infty} dx\, x^n\, e^{-ax^2} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right) a^{-\frac{n+1}{2}}, \tag{7.37}$$

where $a > 0$ and $\Gamma(z)$ is the gamma function with the properties $\Gamma(z) = (z-1)\Gamma(z-1)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. In the case that z is a positive integer then $\Gamma(z) = (z-1)!$. The expression of q_z can be easily obtained when one takes into account the identity (7.37) for $n = 6$. The result is

$$q_z = -\frac{5}{2}\frac{k_\text{B}^2}{m\nu}\frac{\partial}{\partial z}(nT^2). \tag{7.38}$$

On the other hand, to keep the system in a steady state, the presence of a thermal gradient induces a density gradient $\partial_z n \neq 0$ so that the hydrostatic pressure $p = nk_\text{B}T \equiv$ const. This constraint can be easily seen when one evaluates the mean flow velocity U_z from the distribution (7.33) since in this case $U_z \propto \partial_z(nT)$. Consistency requires that $U_z = 0$ and hence, $\partial_z(nT) = 0$. Therefore,

$$q_z = -\frac{5}{2}\frac{nk_\text{B}^2 T}{m\nu}\frac{\partial T}{\partial z} \equiv -\kappa\frac{\partial T}{\partial z}, \tag{7.39}$$

where the last identity defines the thermal conductivity coefficient κ from Fourier's law. Thus, κ can be identified as

$$\kappa = \frac{5}{2}\frac{nk_\text{B}^2 T}{m\nu}. \tag{7.40}$$

7.6.2 Shear viscosity

The evaluation of the shear viscosity coefficient η follows similar steps as those made before for the thermal conductivity κ. Now, we consider a steady state where $\nabla n = \nabla T = 0$, $U_y = U_z = 0$ and $\partial_y U_x \neq 0$. Thus, as before we assume that $f = f_\text{LE} + \varphi$ where the distribution function φ is

$$\varphi = -\frac{v_y}{\nu}\frac{\partial f_\text{LE}}{\partial y}. \tag{7.41}$$

The relevant element of the pressure tensor is P_{xy}. It is given by

$$\begin{aligned} P_{xy} &= \int d\mathbf{v}\; mV_xV_y\; f(\mathbf{v}) = \int d\mathbf{v}\; mV_xV_y\; \varphi(\mathbf{v}) \\ &= -\frac{m}{\nu}\int d\mathbf{v}\; V_xV_y^2\frac{\partial f_\text{LE}}{\partial y} = -\frac{m}{\nu}\frac{\partial U_x}{\partial y}\int d\mathbf{v}\; V_y^2 f_\text{LE}, \end{aligned} \tag{7.42}$$

where in the last step we have taken into account that $V_x = v_x - U_x(y)$ and so, $\partial V_x/\partial y = -\partial U_x/\partial y$. The integral appearing in Eq. (7.42) is

$$\int d\mathbf{v}\; V_y^2 f_\text{LE} = \frac{1}{3}\int d\mathbf{v}\; V^2 f_\text{LE} = \frac{nk_\text{B}T}{m}. \tag{7.43}$$

With this result, the element P_{xy} is given by

$$P_{xy} = -\frac{nk_{\rm B}T}{\nu}\frac{\partial U_x}{\partial y} \equiv -\eta \frac{\partial U_x}{\partial y}, \tag{7.44}$$

where the last identity defines the shear viscosity η from Newton's law. Thus, η can be identified as

$$\eta = \frac{nk_{\rm B}T}{\nu}. \tag{7.45}$$

7.7 Balance equations

The objective of this Section is to derive the general balance equations for the dilute gas from the Boltzmann equation. This derivation is based on more fundamental arguments (microscopic description) than those considered before in the context of fluid dynamics. Since we have not written the explicit form of the Boltzmann collision operator, we will consider its BGK approximation to obtain the above balance equations. Actually, although the explicit form of $J[f,f]$ is not needed in this calculation, only some specific properties of the above operator are required.

Let us assume that $\chi(\mathbf{v})$ is a collisional invariant. In this case,

$$\chi(\mathbf{v}) + \chi(\mathbf{v}_1) = \chi(\mathbf{v}') + \chi(\mathbf{v}_1'). \tag{7.46}$$

Of course, these quantities are directly related to the mass, momentum and energy. In fact, when $\chi(\mathbf{v}) \equiv \left\{m, m\mathbf{v}, \frac{m}{2}v^2\right\}$ then

$$\int d\mathbf{v}\; \chi(\mathbf{v})\; J[\mathbf{v}|f,f] = 0. \tag{7.47}$$

Equation (7.47) can be easily verified if one considers the BGK collision term (7.25). The collisions conserve the mass and the average values of momentum and energy. Thus, if one multiplies both sides of the Boltzmann equation (7.24) by $\chi(\mathbf{v})$ and integrates over velocity, one gets

$$\int d\mathbf{v}\; \chi(\mathbf{v}) \left(\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}}\right) = 0, \tag{7.48}$$

where use has been made of Eq. (7.47). Equation (7.48) can be rewritten as

$$\frac{\partial}{\partial t}\int d\mathbf{v}\,\chi f-\int d\mathbf{v}\,\frac{\partial\chi}{\partial t}f+\frac{\partial}{\partial r_i}\int d\mathbf{v}\,\chi\, v_i\, f-\int d\mathbf{v}\,\frac{\partial\chi}{\partial r_i}v_i\, f$$
$$+\frac{1}{m}\int d\mathbf{v}\;\frac{\partial}{\partial v_i}(\chi F_i f)-\frac{1}{m}\int d\mathbf{v}\;\frac{\partial\chi}{\partial v_i}F_i\, f-\frac{1}{m}\int d\mathbf{v}\;\chi\,\frac{\partial F_i}{\partial v_i}\, f=0. \tag{7.49}$$

The fifth term vanishes since $f\to 0$ when $|\mathbf{v}|\to\infty$. Moreover, the external force does not depend on the particle velocity and so, the last term in Eq. (7.49) also vanishes. Thus, taking into account the definition (7.4) for average values, Eq. (7.49) becomes

$$\frac{\partial}{\partial t}n\langle\chi\rangle-n\langle\frac{\partial\chi}{\partial t}\rangle+\frac{\partial}{\partial r_i}n\langle v_i\chi\rangle-n\langle v_i\frac{\partial\chi}{\partial r_i}\rangle-\frac{nF_i}{m}\langle\frac{\partial\chi}{\partial v_i}\rangle=0. \tag{7.50}$$

This is a general transfer equation for the average value of χ. The next step is to apply Eq. (7.50) for the particular cases $\chi=m$, $\chi=mv_i$, and $\chi=\frac{1}{2}mv^2$. This allow us to derive the balance equations for the densities of mass, momentum and energy with microscopic expressions for the irreversible fluxes of momentum (stress or pressure tensor) and heat. Let us consider each balance equation separately.

7.7.1 Mass balance equation

If $\chi\equiv m$, then Eq. (7.50) yields

$$\frac{\partial}{\partial t}(mn)+\frac{\partial}{\partial r_i}\langle mnv_i\rangle=0. \tag{7.51}$$

This equation can be rewritten as

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{7.52}$$

where $\rho(\mathbf{r},t)=mn(\mathbf{r},t)$ is the mass density. In terms of the operator d/dt, Eq. (7.52) can be written as

$$\frac{d\rho}{dt}+\mathbf{U}\cdot\nabla\rho=0. \tag{7.53}$$

Equation (7.53) is the well-known continuity equation (2.5) for the mass density.

7.7.2 Momentum balance equation

If $\chi \equiv mv_i$, then Eq. (7.50) yields

$$\frac{\partial}{\partial t}\langle mnv_i\rangle + \frac{\partial}{\partial r_j}\langle mnv_iv_j\rangle = nF_j\langle\frac{\partial v_i}{\partial v_j}\rangle. \tag{7.54}$$

The latter term is

$$F_j\langle\frac{\partial v_i}{\partial v_j}\rangle = F_j\langle\delta_{ij}\rangle = F_i.$$

Thus, Eq. (7.54) becomes

$$\frac{\partial}{\partial t}\left(\rho U_i\right) + \frac{\partial}{\partial r_j}\left(\rho\langle v_iv_j\rangle\right) = \frac{\rho}{m}F_i. \tag{7.55}$$

The second term can be explicitly obtained as

$$\langle v_iv_j\rangle = \langle (U_i+V_i)(U_j+V_j)\rangle = U_iU_j + \langle V_iV_j\rangle = U_iU_j + \frac{P_{ij}}{\rho}, \tag{7.56}$$

where use has been made of the identity $\langle V_i\rangle = 0$ and the definition (7.12) of the pressure tensor P_{ij}. Substitution of the result (7.56) into Eq. (7.55) yields

$$\frac{\partial}{\partial t}\left(\rho U_i\right) + \frac{\partial}{\partial r_j}\left(\rho U_iU_j\right) = -\frac{\partial}{\partial r_j}P_{ij} + \frac{\rho}{m}F_i. \tag{7.57}$$

We can simplify a bit more the left hand-side of this equation by using the continuity equation (7.52). The result is

$$\begin{aligned} U_i\partial_t\rho + \rho\partial_t U_i + U_i\partial_j(\rho U_j) + \rho U_j\partial_j U_i &= U_i\left[\partial_t\rho + \partial_j(\rho U_j)\right] + \rho\left[\partial_t U_i\right. \\ &\left. + U_j\partial_j U_i\right] = \rho\frac{dU_i}{dt}. \end{aligned} \tag{7.58}$$

Taking into account this last result, the balance equation for the momentum density reads

$$\rho\frac{dU_i}{dt} = -\frac{\partial}{\partial r_j}P_{ij} + \frac{\rho}{m}F_i. \tag{7.59}$$

For small spatial gradients (Navier-Stokes description), the pressure tensor P_{ij} of a dilute gas is given by

$$P_{ij} = p\delta_{ij} - \eta\left(\partial_j U_i + \partial_i U_j - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{U}\right), \tag{7.60}$$

where use has been made of the fact that the bulk viscosity coefficient $\zeta = 0$ for a dilute gas. When the constitutive equation (7.60) is substituted into Eq. (7.59) one recovers the Navier-Stokes hydrodynamic equation (3.6). This shows the consistency between both approaches (continuum and discrete descriptions).

7.7.3 Energy balance equation

For the sake of convenience, let us consider the kinetic energy of the particle measured in a frame moving at the flow velocity $\mathbf{U}$. In this case, $\chi \equiv \frac{1}{2}mV^2$, where $\mathbf{V} = \mathbf{v} - \mathbf{U}$ is the peculiar velocity. From Eq. (7.50), one gets

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \rho V^2\rangle + \frac{1}{2}\frac{\partial}{\partial r_i}\langle \rho v_i V^2\rangle - \frac{1}{2}\rho\langle v_i \frac{\partial V^2}{\partial r_i}\rangle = 0. \tag{7.61}$$

We define the *local* temperature $T(\mathbf{r}, t)$ as

$$k_{\mathrm{B}} T \equiv \theta = \frac{1}{3} m \langle V^2\rangle. \tag{7.62}$$

Let us evaluate each one of the terms appearing in Eq. (7.61). First, the second term on the left hand side of this equation can be rewritten as

$$\frac{1}{2}\langle \rho v_i V^2\rangle = \frac{1}{2}\langle \rho (v_i - U_i) V^2\rangle + \frac{1}{2}\rho U_i \langle V^2\rangle = q_i + \frac{3}{2} n\theta U_i, \tag{7.63}$$

where use has been made of the definition (7.11) of the heat flux. The third term on the left hand side of (7.61) is

$$\begin{aligned} \frac{1}{2}\rho\langle v_i \frac{\partial V^2}{\partial r_i}\rangle &= \frac{1}{2}\rho\langle v_i \frac{\partial}{\partial r_i}(v_j - U_j)(v_j - U_j)\rangle = -\rho\langle v_i (v_j - U_j)\rangle \frac{\partial U_j}{\partial r_i} \\ &= -\rho\langle (v_i - U_i)(v_j - U_j)\rangle \frac{\partial U_j}{\partial r_i} = -P_{ij}\frac{\partial U_j}{\partial r_i} = -P_{ij} D_{ji}, \end{aligned} \tag{7.64}$$

where

$$D_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial r_j} + \frac{\partial U_j}{\partial r_i}\right). \tag{7.65}$$

Upon deriving Eq. (7.64) use has been made of the symmetry property of the pressure tensor $P_{ij} = P_{ji}$. Equation (7.61) can be written in a different form when one takes into account Eqs. (7.63) and (7.64). The result is

$$\frac{3}{2}\frac{\partial}{\partial t}(n\theta) + \frac{3}{2}\frac{\partial}{\partial r_i}(n\theta U_i) + \frac{\partial q_i}{\partial r_i} + P_{ij} D_{ji} = 0. \tag{7.66}$$

The first two terms can be simplified after taking into account the equation of continuity:

$$\begin{aligned}\frac{3}{2}\frac{\partial}{\partial t}(n\theta)+\frac{3}{2}\frac{\partial}{\partial r_i}(n\theta U_i) &= \frac{3}{2}\theta\left[\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{U})\right]+\frac{3}{2}n\left(\frac{\partial\theta}{\partial t}+\mathbf{U}\cdot\nabla\theta\right)\\ &= \frac{3}{2}n\frac{d\theta}{dt}, \qquad (7.67)\end{aligned}$$

where use has been made of Eq. (7.52) in the last step. With this result, the energy balance equation can be finally written as

$$\frac{3}{2}nk_\mathrm{B}\frac{dT}{dt}=-\frac{\partial q_i}{\partial r_i}-P_{ij}D_{ji}, \qquad (7.68)$$

or in a more compact form as

$$\frac{3}{2}nk_\mathrm{B}\frac{dT}{dt}=-\nabla\cdot\mathbf{q}-\mathbf{P}:\mathbf{D}. \qquad (7.69)$$

In the limit of small spatial gradients, the constitutive equation for the pressure tensor is given by Eq. (7.60) while the heat flux obeys Fourier's law

$$\mathbf{q}=-\kappa\nabla T. \qquad (7.70)$$

When both constitutive equations are substituted into Eq. (7.68), then the (continuum) energy balance equation (4.11) is recovered. This sows again the consistency between kinetic theory and hydrodynamics.

BIBLIOGRAPHY

[1] L. D. LANDAU and E. M. LIFSHITZ, *Fluid Mechanics*, Pergamon Press (1987).

[2] F. M. WHITE, *Fluid Mechanics*, McGraw-Hill (2013).

[3] G. K. BATCHELOR, *Fluid Dynamics*, Cambridge University Press (2000).

[4] D. J. TRITTON, *Physical Fluid Dynamics*, Clarendon Press, Oxford (2010).

[5] P. RÉSIBOIS AND M. DE LEENER, *Classical Kinetic Theory of Fluids*, Wiley (1977).

[6] V. GARZÓ AND A. SANTOS, *Kinetic Theory of Gases in Shear Flows*, Kluwer Academic Publishers, (2003).

colec

UNIVERSIDAD DE EXTREMADURA

manu

102

ÍNDICE