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A Note on Rational Approximation with Respect to Metrizable Compactifications of the Plane

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Abstract: In the present note we examine possible extensions of Runge, Mergelyan and Arakelian Theorems, when the uniform approximation is meant with respect to the metric ϱ of a metrizable compactification (S, ϱ) of the complex plane $\mathbb{C}$.

Key words: compactification, Arakelian's theorem, Mergelyan's theorem, Runge's theorem, uniform approximation in the complex domain.

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1. Introduction

It is well known that the class of uniform limits of polynomials in $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ coincides with the disc algebra $A(D)$. A function $f : \overline{D} \to \mathbb{C}$ belongs to $A(D)$ if and only if it is continuous on $\overline{D}$ and holomorphic in the open unit disc D. It is less known (see [3, 7]) what is the corresponding class when the uniform convergence is not meant with respect to the usual Euclidean metric on $\mathbb{C}$, but it is meant with respect to the chordal metric χ on $\mathbb{C} \cup \{\infty\}$. The class of χ–uniform limits of polynomials on $\overline{D}$ is denoted by $\widetilde{A}(D)$ and contains $A(D)$. A function $f : \overline{D} \to \mathbb{C} \cup \{\infty\}$ belongs to $\widetilde{A}(D)$ if and only if $f \equiv \infty$, or it is continuous on $\overline{D}$, $f(D) \subset \mathbb{C}$ and $f_{|_D}$ is holomorphic. The function $f(z) = \frac{1}{1-z}$, $z \in D$, belongs to $\widetilde{A}(D)$, but not to $A(D)$; thus, it cannot be uniformly approximated on D, by polynomials with respect to the usual Euclidean metric on $\mathbb{C}$, but it can be uniformly approximated by polynomials with respect to the chordal metric χ.

More generally, if $K \subset \mathbb{C}$ is a compact set with connected complement, then according to Mergelyan's theorem [10] polynomials are dense in $A(K)$ with respect to the usual Euclidean metric on $\mathbb{C}$. We recall that a function $f : K \to \mathbb{C}$ belongs to $A(K)$ if and only if it is continuous on K and holomorphic in the interior K° of K.

An open problem is to characterize the class $\widetilde{A}(K)$ of χ–uniform limits of polynomials on K.

Conjecture. ([1, 6]) Let $K \subset \mathbb{C}$ be a compact set with connected complement K^c. A function $f : K \to \mathbb{C} \cup \{\infty\}$ belongs to $\widetilde{A}(K)$ if and only if it is continuous on K and for each component V of K°, either $f(V) \subset \mathbb{C}$ and $f_{|_V}$ is holomorphic, or $f_{|_V} \equiv \infty$.

Extensions of this result have been obtained in [5] when K^c has a finite number of components and K is bounded by a finite set of disjoint Jordan curves. In this case, the χ–uniform approximation is achieved using rational functions with poles out of K instead of polynomials. Furthermore, extensions of Runge's theorem are also proved in [5]. Finally a first result has been obtained in [5] concerning an extension of the approximation theorem of Arakelian ([2]).

Instead of considering the one point compactification $\mathbb{C} \cup \{\infty\}$ of the complex plane $\mathbb{C}$, we can consider an arbitrary metrizable compactification (S, ϱ) of $\mathbb{C}$ and investigate the analogues of all previous results. This is the content of the present paper.

2. Preliminaries

We say that (S, ϱ) is a *metrizable compactification of the plane* $\mathbb{C}$, if ϱ is a metric on S, S is compact, $S \supset \mathbb{C}$ and $\mathbb{C}$ is an open dense subset of S. Obviously, $S \backslash \mathbb{C}$ is a closed subset of S. We say that the points in $S \backslash \mathbb{C}$ are the points at infinity.

Let (S, ϱ) be a metrizable compactification of $\mathbb{C}$ with metric ϱ. Many such compactifications can be found in [1]. The one point compactification $\mathbb{C} \cup \{\infty\}$ with the chordal metric χ is a distinct one of them. We note that in this case, the continuous function $\pi : S \to \mathbb{C} \cup \{\infty\}$, such that $\pi(c) = c$, for every $c \in \mathbb{C}$ and $\pi(x) = \infty$, for every $x \in S \backslash \mathbb{C}$, is useful.

Another metrizable compactification is the one defined in [8] and constructed as follows: consider the map

$$\begin{array}{rcl} \phi : \mathbb{C} & \longrightarrow & D = \{\lambda \in \mathbb{C} \,:\, |\lambda < 1\} \\ z & \longmapsto & \dfrac{z}{1 + |z|} \end{array} ,$$

which is a homeomorphism. A compactification of the image D of ϕ is $\overline{D}$, the closure of D, with the usual metric. This leads to the following compactification of $\mathbb{C}$

$$S_1 := \mathbb{C} \cup \{\infty e^{i\vartheta} : 0 \le \vartheta \le 2\pi\}, \tag{2.1}$$

with metric d given by

$$\begin{aligned} d(z,w) &= \left| \frac{z}{1+|z|} - \frac{w}{1+|w|} \right| && \text{if } z,w \in \mathbb{C}\,, \\ d(z,\infty e^{i\vartheta}) &= \left| \frac{z}{1+|z|} - e^{i\vartheta} \right| && \text{if } z \in \mathbb{C},\ \vartheta \in \mathbb{R}\,, \\ d(\infty e^{i\vartheta}, \infty e^{i\varphi}) &= \left| e^{i\vartheta} - e^{i\varphi} \right| && \text{if } \vartheta, \varphi \in \mathbb{R}\,. \end{aligned} \tag{2.2}$$

In what follows, with *a compactification* (S,ϱ) *of* $\mathbb{C}$, we shall always mean *a metrizable compactification.*

An important question for a given compactification of $\mathbb{C}$ is, whether for $c \in \mathbb{C}$ and $x \in S\backslash\mathbb{C}$, the addition $c+x$ is well defined. In other words, having two convergent sequences $\{z_n\}, \{w_n\}$ in $\mathbb{C}$, such that $z_n \to c$ and $w_n \to x$ does the sequence $\{z_n + w_n\}$ have a limit in S?

If the answer is positive for any such sequences $\{z_n\}, \{w_n\}$ in $\mathbb{C}$, then the limit $y \in S$ of the sequence $\{z_n + w_n\}$ is uniquely determined and we write $c + x = y = x + c$. We are interested in compactifications (S,ϱ), where $c+x$ is well defined for any $c \in \mathbb{C}$ and $x \in S$ (it suffices to take $x \in S\backslash\mathbb{C}$). In this case, the map $\mathbb{C} \times S \to S$, $(c,x) \mapsto c + x$, is automatically continuous.

Indeed, let $x \in S\backslash\mathbb{C}$, $y \in \mathbb{C}$ and $w = x + y \in S\backslash\mathbb{C}$. Let $\{z_n\}$ in S and $\{y_n\}$ in $\mathbb{C}$, such that $z_n \to x$ and $y_n \to y$. If all but finitely many z_n belong to $\mathbb{C}$, then by our assumption $z_n + y_n \to x + y$. Suppose that infinitely many z_n belong to $S\backslash\mathbb{C}$. Without loss of generality we may assume that all z_n belong to $S\backslash\mathbb{C}$ and by compactness we can assume that $z_n + y_n \to l \ne w = x + y$.

Let $d = \varrho(l,w) > 0$. Then there exists $n_0 \in \mathbb{N}$, such that

$$\varrho(z_n + y_n, l) < \frac{d}{2} \qquad \text{for all } n \ge n_0\,.$$

Fix $n \ge n_0$. Since, $z_n + y_n$ is well defined, there exists $z'_n \in \mathbb{C}$, such that

$$\varrho(z_n, z'_n) < \frac{1}{n} \qquad \text{and} \qquad \varrho(z_n + y_n, z'_n + y_n) < \frac{1}{n}\,.$$

It follows that

$$\varrho(z'_n, x) \le \varrho(z'_n, z_n) + \varrho(z_n, x) < \frac{1}{n} + \varrho(z_n, x) \ \to\ 0\,.$$

Hence, $z'_n \to x$, $y_n \to y$ and $z'_n, y_n \in \mathbb{C}$. By our assumption, it follows that $z'_n + y_n \to x + y = w$. But

$$\begin{aligned}\varrho(z'_n + y_n, l) &\leq \varrho(z'_n + y_n, z_n + y_n) + \varrho(z_n + y_n, l)\\ &\leq \frac{1}{n} + \varrho(z_n + y_n, l) < \frac{1}{n} + \frac{d}{2} \ \to \ \frac{d}{2}\,.\end{aligned}$$

Thus, for all n large enough we have

$$\varrho(z'_n + y_n, l) \leq \frac{3d}{4} < d = \varrho(l, w)\,.$$

It follows that $\varrho(z'_n + y_n, w) \geq \frac{d}{4}$, for all n large enough. Therefore, we cannot have $z'_n + y_n \to w$.

Consequently, one concludes that the addition map is continuous at every (x, y) with $x \in S\backslash\mathbb{C}$ and $y \in \mathbb{C}$. Obviously, it is also continuous at every (x, y) with x and y in $\mathbb{C}$. Thus, addition is continuous on $S \times \mathbb{C}$. Furthermore, the following holds:

Let $K \subset \mathbb{C}$ be compact. Obviously, the map $K \times S \to S$, $(c, x) \mapsto c + x$, is uniformly continuous.

Remark 1. The preceding certainly holds for the compactification (S_1, d) (see (2.1)), since

$$c + \infty e^{i\vartheta} = \infty e^{i\vartheta} \qquad \text{for all } \ c \in \mathbb{C} \ \text{ and } \ \vartheta \in \mathbb{R}\,,$$

and we have continuity.

Remark 2. If we identify $\mathbb{R}$ with the interval $(-1, 1)$, up to a homeomorphism, then $\mathbb{C} \cong \mathbb{R}^2$ is identified with the square $(-1, 1) \times (-1, 1)$. An obvious compactification of $\mathbb{C}$ is then the closed square with the usual metric. The points at infinity are those on the boundary of the square, for instance, those points on the side $\{1\} \times [-1, 1]$. If $x \in \{1\} \times (-1, 1)$ and $c \in \mathbb{C}$, then $c + x$ is a point in the same side; if $\operatorname{Im} c \neq 0$, then $c + x \neq x$. If $x = (1, 1)$ and $c \in \mathbb{C}$, then $x + c = x$. If $\operatorname{Im} c > 0$, then $c + x$ lies higher than x in the side $\{1\} \times (-1, 1)$.

In this example, the addition is well defined and continuous, but the points at infinity are not stabilized as in Remark 1.

Question. Is there a metrizable compactification of $\mathbb{C}$ such that the addition $c + x$ is not well defined for some $c \in \mathbb{C}$ and $x \in S\backslash\mathbb{C}$?

The answer is "yes". An example comes from the previous square in Remark 2, if we identify all the points of $\{1\} \times [-\frac{1}{2}, \frac{1}{2}]$ and make them just one point.

3. Runge and Mergelyan type theorems

In this section using a compactification of $\mathbb{C}$ satisfying all properties discussed in the Preliminaries, we obtain the following theorem, that extends [5, Theorem 3.3].

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$ be a bounded domain, whose boundary consists of a finite set of pairwise disjoint Jordan curves. Let $K = \overline{\Omega}$ and A a set containing one point from each component of $(\mathbb{C} \cup \{\infty\}) \backslash K$. Let (S, ϱ) be a compactification of $\mathbb{C}$, such that the addition $+ : \mathbb{C} \times S \to S$ is well defined. Let $f : K \to S$ be a continuous function, such that $f(\Omega) \subset \mathbb{C}$ and $f \restriction_\Omega$ is holomorphic. Let $\varepsilon > 0$. Then, there exists a rational function R with poles only in A and such that $\varrho(f(z), R(z)) < \varepsilon$, for all $z \in K$.*

Proof. If Ω is a disk, the proof has been given in [1]. If Ω is the interior of a Jordan curve, the proof is given again in [1], but also in [6]. In the general case, we imitate the proof of [5, Theorem 3.3]. Namely, we consider the Laurent decomposition of f, given by $f = f_0 + f_1 + \cdots + f_N$ (see [4]). The function f_0 is defined on a simply connected domain, bounded by a Jordan curve, and it can be uniformly approximated by a polynomial or a rational function R_0 with pole in the unbounded component. Similarly, f_1 is approximated by a rational function R_1 with pole in A and so on. Thus, the function $R_0 + R_1 + \cdots + R_N$ approximates, with respect to ϱ, the function $f = f_0 + f_1 + \cdots + f_N$. This is due to the fact that at every point z all the f_i's, $i = 1, 2, \cdots, N$, except maybe one, take values in $\mathbb{C}$ and the one, maybe has as a value, an infinity point in $S \backslash \mathbb{C}$. In this way, the addition map $\mathbb{C} \times S \to S$, $(c, x) \mapsto c + x$, is well defined and uniformly continuous on compact sets and so we are done. ∎

Another Runge–type theorem is the following, where we do not need any assumption for the compactification S, or the addition map $+ : \mathbb{C} \times S \to S$.

Theorem 3.2. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ be holomorphic and (S, ϱ) a compactification of $\mathbb{C}$. Let A be a set containing one point from each component of $(\mathbb{C} \cup \infty) \backslash \Omega$. Let $\varepsilon > 0$ and $L \subset \Omega$ compact. Then, there* *exists a rational function R with poles in A, such that $\varrho(f(z), R(z)) < \varepsilon$ for all $z \in L$.*

Proof. Clearly the subset $f(L)$ of $\mathbb{C}$ is compact. Then, from the classical theorem of Runge, there exist rational functions $\{R_n\}$, with poles only in A, converging uniformly to f on L, with respect to the Euclidean metric $|\cdot|$. Hence, there is a positive integer n_0 and a compact K, such that

$$f(L) \subset K \subset \mathbb{C} \quad \text{and} \quad R_n(L) \subset K \qquad \text{for all } \ n \geq n_0\,.$$

But on K the metrics $|\cdot|$ and ϱ are uniformly equivalent. Therefore, $R_n \to f$ uniformly on L, with respect to ϱ. To conclude the proof, it suffices to put $R = R_n$, for n large enough. ∎

Theorem 3.2 easily yields the following

Corollary 3.3. *Under the assumptions of Theorem 3.2 there exists a sequence $\{R_n\}$ of rational functions with poles in A, such that $R_n \to f$, ϱ–uniformly, on each compact subset of Ω.*

Remark. According to Corollary 3.3, some of the ϱ–uniform limits, on compacta, of rational functions with poles in A, are the holomorphic functions $f : \Omega \to \mathbb{C}$. Those are limits of the finite type. The other limits of sequences $\{R_n\}$ as above may be functions $f : \Omega \to S\backslash\mathbb{C}$ of infinite type, continuous (but maybe not all of them, as the Example (S_1, d) shows; cf. [8]).

Question. Is a characterization possible for such limits $f : \Omega \to S_1\backslash\mathbb{C}$?

An imitation of the arguments in [8, p. 1007] gives that f must be of the form $f(z) = \infty e^{i\vartheta(z)}$, $z \in \Omega$, where ϑ is a multivalued harmonic function.

The following extends [5, Section 5].

Theorem 3.4. *Let $\Omega \subset \mathbb{C}$ be open and f a meromorphic function on Ω. Let B denote the set of poles of f. Let (S, ϱ) be a compactification of $\mathbb{C}$, such that the addition $+ : \mathbb{C} \times S \to S$ is well defined. Let $\varepsilon > 0$ and $K \subset \Omega$ be a compact set. Then, there is a rational function g, such that $\varrho(f(z), g(z)) < \varepsilon$, for every $z \in K\backslash B$.*

Proof. Since $B \cap K$ is a finite set, the function f decomposes to $f = h + w$, where h is a rational function with poles in $B \cap K$ and w is holomorphic on an open set containing K. By Runge's theorem there exists a rational function R

with poles off K, such that $|w(z)-R(z)|<\varepsilon'$ on K. Since $w(K)$ is a compact subset of $\mathbb{C}$ and the addition $+:\mathbb{C}\times S\to S$ is well defined, a suitable choice of ε' gives

$$\varrho\big(\,[h(z)+w(z)]\,,\,[h(z)+R(z)]\,\big)<\varepsilon \qquad \text{on}\ \ K\backslash B\,.$$

We set $g=h+R$ and the result follows. ∎

4. Arakelian sets

A closed set $F\subset\mathbb{C}$ is said a *set of approximation* if every function $f:F\to\mathbb{C}$ continuous on F and holomorphic in F° can be approximated by entire functions, uniformly on the whole F. This is equivalent to the fact that F is an Arakelian set (see [2]), that is $(\mathbb{C}\cup\{\infty\})\backslash F$ is connected and locally connected (at ∞).

We can now ask about an extension of the Arakelian theorem in the context of metrizable compactifcations. A result in this direction is the following

Proposition 4.1. *Let $F\subset\mathbb{C}$ be a closed Arakelian set with empty interior, i.e., $F^\circ=\emptyset$. We consider the compactification (S_1,d) of $\mathbb{C}$ (see (2.1) and (2.2)) and let $f:F\to S_1$ be a continuous function. Let $\varepsilon>0$. Then, there is an entire function g such that $d(f(z),g(z))<\varepsilon$, for every $z\in F$.*

Proof. According to (1.1), the compactification S_1 is homeomorphic to $\overline{D}=\{z\in\mathbb{C}\,:\,|z|\le 1\}$. For each $0<R<1$ let us define

$$\begin{aligned}\phi_R:\overline{D}\ &\longrightarrow\ \{z\in\mathbb{C}\,:\,|z|\le R\}\subset\overline{D}\\ z\ &\longmapsto\ \begin{cases} z\,, & \text{if}\ \ |z|\le R\,,\\ \frac{Rz}{|z|}\,, & \text{if}\ \ R\le|z|\le 1\,.\end{cases}\end{aligned}$$

In other words, the whole line segment $\big[Re^{i\vartheta},e^{i\vartheta}\big]$ is mapped at the end point $Re^{i\vartheta}$. The function ϕ_R is continuous and induces a continuous function $\widetilde{\phi}_R:S_1\to S_1$. It suffices to take $\widetilde{\phi}_R:=T^{-1}\circ\phi_R\circ T$, where $T:S_1\to\{w\in\mathbb{C}\,:\,|w|\le 1\}$ is defined as follows

$$T(z):=\frac{z}{1+|z|} \qquad \text{for}\ z\in\mathbb{C}\subset S_1\,,$$

$$T\big(\infty e^{i\vartheta}\big):=e^{i\vartheta} \qquad \text{for}\ \vartheta\in\mathbb{R}\,.$$

If $\varepsilon > 0$ is given, then there exists $R_\varepsilon < 1$, such that for $R_\varepsilon \leq R < 1$ and $z \in S_1$, we have $d\big(z, \widetilde{\phi}_R(z)\big) < \frac{\varepsilon}{2}$.

Let now f be as in the statement of the Proposition 4.1. Then,

$$d\Big(f(z), \big(\widetilde{\phi}_R \circ f\big)(z)\Big) < \frac{\varepsilon}{2} \qquad \text{for all } z \in F\,.$$

Moreover, the function $\widetilde{\phi}_R \circ f : F \to \mathbb{C}$ is continuous. Since F is a closed Arakelian set, with empty interior, and $\big(\widetilde{\phi}_R \circ f\big)(F) \subset K$, is included in a compact subset K of $\mathbb{C}$, there exists g entire, such that

$$\Big|\big(\widetilde{\phi}_R \circ f\big)(z) - g(z)\Big| < \varepsilon' \qquad \text{for all } z \in F\,.$$

Since $\big(\widetilde{\phi}_R \circ f\big)(F)$ is contained in a compact subset K of $\mathbb{C}$, for a suitable choice of ε', it follows that

$$d\Big(\big(\widetilde{\phi}_R \circ f\big)(z), g(z)\Big) < \frac{\varepsilon}{2} \qquad \text{for all } z \in F\,.$$

The triangle inequality completes the proof. ∎

An analogue of Proposition 4.1 for the one point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ has been established in [5].

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