Constructing Banach Ideals Using Upper ℓ_p-estimates

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Abstract: Using upper ℓ_p-estimates for normalized weakly null sequence images, we describe a new family of operator ideals $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ with parameters $1 \leq p \leq \infty$ and $1 \leq \xi \leq \omega_1$. These classes contain the completely continuous operators, and are distinct for all choices $1 \leq p \leq \infty$ and, when $p \neq 1$, for infinitely many $1 \leq \xi \leq \omega_1$. For the case $\xi = 1$, there exists an ideal norm $\|\cdot\|_{(p,1)}$ on the class $\mathcal{WD}_{\ell_p}^{(\infty,1)}$ under which it forms a Banach ideal. We also prove that each space $\mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(X,Y)$ is the intersection of the spaces $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$ over all $1 \leq \xi < \omega_1$.

Key words: Banach spaces, operator ideals.

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1. INTRODUCTION

The roots of the theory of operator ideals extend at least as far back as 1941 when J.W. Calkin observed that if $\mathcal{H}$ is a Hilbert space, then the subspaces of finite-rank operators, compact operators, and Hilbert-Schmidt operators all form multiplicative ideals in the space $\mathcal{L}(\mathcal{H})$ of continuous linear operators on $\mathcal{H}$ ([7]). However, the concept of an ideal as a class of operators between arbitrary Banach spaces developed more recently, with the first thorough treatment of the subject, a monograph by Albrecht Pietsch, appearing in 1978 ([15]).

In this paper we define and study a new family of operator ideals $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ with parameters $1 \leq p \leq \infty$ and $1 \leq \xi \leq \omega_1$, where ω_1 denotes the first uncountable ordinal. For any fixed value of ξ, these ideals are distinct for all choices of p, which is to say that for any $1 \leq p < q \leq \infty$ there exist

Banach spaces X and Y for which the components satisfy $\mathcal{WD}_{\ell_p}^{(\xi,\infty)}(X,Y) \neq \mathcal{WD}_{\ell_q}^{(\xi,\infty)}(X,Y)$. It remains an open question whether, for fixed $1 < p \leq \infty$, the ideals are distinct for all choices of ξ. However, we do obtain a partial positive answer by finding, for any fixed $p \neq 1$, a sequence $(\xi_n)_{n=1}^\infty$ of countable ordinals $1 \leq \xi_n < \omega_1$, $n \in \mathbb{N}$, such that, as classes,

$$\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\omega_1)} \subsetneq \overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_n)} \subsetneq \overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_m)}$$

for all $m < n$. We shall also see that $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ always strictly includes the ideal of completely continuous operators $\mathcal{V}$, which shows that they are distinct from some other notable families of operator ideals with a parameter related to the ℓ_p spaces. For instance, let $\mathcal{N}_p$, $\mathcal{I}_p$, and Π_p denote the ideals of p-nuclear, p-integral, and absolutely p-summing operators, respectively. Then

$$\mathcal{N}_p \subsetneq \mathcal{I}_p \subsetneq \Pi_p \subsetneq \mathcal{V} \subsetneq \mathcal{WD}_{\ell_p}^{(\infty,\xi)}$$

(cf., e.g., Proposition 22 in [16] together with Theorem 2.17 in [9]).

Of special interest are the those operator ideals whose components are always norm-closed. For instance, given arbitrary Banach spaces X and Y, the compact, weakly compact, and completely continuous operators from X into Y are always norm-closed in $\mathcal{L}(X,Y)$, whereas the finite-rank operators are not. We shall see that, when $p \neq 1$, there always exist separable spaces X for which $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X)$ fails to be norm-closed in $\mathcal{L}(X)$, and when $p \neq \infty$, we can choose X to be reflexive. Nevertheless, in the case $\xi = 1$, we can construct an ideal norm $\|\cdot\|_{(p,1)}$ for the class $\mathcal{WD}_{\ell_p}^{(\infty,1)}$ so that it forms a Banach ideal, that is, a "nice" norm assignment for each component space $\mathcal{WD}_{\ell_p}^{(\infty,1)}(X,Y)$ under which it becomes a Banach space.

The ideas for the construction of this family originate with [6] and [4]. In [6], the authors defined the subset $\mathcal{WS}(X,Y)$ of (w_n)-*singular* operators in $\mathcal{L}(X,Y)$ as those operators T for which, given any normalized basic sequence (x_n) in X, the image sequence (Tx_n) fails to dominate (w_n). Here, (w_n) is taken to be some normalized 1-spreading basis for some fixed Banach space W. They showed that when (w_n) is the summing basis for c_0, the unit vector basis for ℓ_1, or the unit vector basis for c_0, the resulting classes $\mathcal{WS}$ are the norm-closed ideals of weakly compact, Rosenthal, or compact operators, respectively. Meanwhile, in [4] the authors constructed and studied classes of operators based on Schreier family support. In particular, they defined $\mathcal{SS}_\xi$, the $\mathcal{S}_\xi$-*strictly singular* operators, as the class of all continuous linear Banach

space operators T for which if (x_n) is any normalized basic sequence in the domain space, for any $\epsilon > 0$ there exists some $z \in [x_n]$ with support lying in the ξth Schreier family $\mathcal{S}_\xi$, and satisfying $\|Tz\| < \epsilon\|z\|$.

In this paper, we use similar ideas to produce operator ideals with certain nice properties. However, whereas classes $\mathcal{WS}$ and $\mathcal{SS}_\xi$ were constructed using normalized basic sequences and singular estimates on their images, for the classes $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ we instead use normalized weakly null sequences and uniform upper estimates. Since continuous linear operators preserve weak convergence, the choice of weakly null sequences in place of basic sequences allows us to show that the classes $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ are indeed multiplicative ideals between arbitrary Banach spaces. The choice of uniform upper estimates instead of singular estimates then gives us a natural way to show that each class $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ is closed under addition.

We shall also show that the ideals $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ are just quantized versions of the ideal $\mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}$, in the sense that

$$\mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(X,Y) = \bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$$

for all choices of Banach spaces X and Y. Somewhat analogous results can be found, for instance, in [3], where it was shown that whenever X is separable,

$$\mathcal{SS}(X^*,Y) = \bigcup_{1 \leq \xi < \omega_1} \mathcal{SS}_\xi(X^*,Y),$$

where $\mathcal{SS}$ denotes the ideal of strictly singular operators. Similar quantizations appeared in [6] for the weakly compact and Rosenthal operators. However the quantized classes do not always form operator ideals themselves. In particular, in [13] the authors showed that $\mathcal{SS}_1$ fails to be closed under addition, and hence is not an operator ideal. The results here can therefore be viewed as somewhat nicer quantizations than have typically been obtained for operator ideals.

Now we shall take a moment to recall some essential definitions and basic facts relevant to our project. Let $\mathcal{J}$ be a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces, such that for each pair of Banach spaces X and Y, $\mathcal{J}(X,Y) := \mathcal{L}(X,Y) \cap \mathcal{J}$ is a linear subspace containing all the finite-rank operators from X into Y. We call $\mathcal{J}$ an *operator ideal* if whenever W,X,Y,Z are Banach spaces and $T \in \mathcal{J}(X,Y)$, then for all operators $A \in \mathcal{L}(W,X)$ and $B \in \mathcal{L}(Y,Z)$ we have $BTA \in \mathcal{J}(W,Z)$. An *ideal norm* with

respect to an operator ideal $\mathcal{J}$ is a rule ρ that assigns to every $T \in \mathcal{J}(X, Y)$, a nonnegative real value $\rho(T)$, and satisfying the following conditions for all Banach spaces W, X, Y, and Z. First, $\rho(x^* \otimes y) = \|x^*\|\|y\|$ for all $x^* \in X^*$ and $y \in Y$, where $x^* \otimes y$ is viewed as the 1-dimensional operator $x \mapsto x^*(x)y$ lying in $\mathcal{J}(X, Y)$; second, $\rho(S + T) \leq \rho(S) + \rho(T)$ for all $S, T \in \mathcal{J}(X, Y)$; and third, $\rho(BTA) \leq \|B\|\rho(T)\|A\|$ for all $T \in \mathcal{J}(X, Y)$, $A \in \mathcal{L}(W, X)$, and $B \in \mathcal{L}(Y, Z)$. A *Banach ideal* is then an operator ideal $\mathcal{J}$ equipped with an an ideal norm ρ such that all components $\mathcal{J}(X, Y)$ are complete with respect to the norm on that space induced by ρ.

We will also need to use the *Schreier families*. These are denoted $\mathcal{S}_\xi$ for each countable ordinal $0 \leq \xi < \omega_1$, and we must define them as follows. Put $\mathcal{S}_0 := \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\mathcal{S}_1 := \{F \subset \mathbb{N} : \#F \leq \min F\} \cup \{\emptyset\}$. Now fix a countable ordinal $1 \leq \xi < \omega_1$. In case $\xi = \zeta + 1$ for some countable ordinal $1 \leq \zeta < \omega_1$ we define $\mathcal{S}_\xi$ as the set containing $\emptyset$ together with all $F \subset \mathbb{N}$ such that there exist $n \in \mathbb{N}$ and $F_1 < \cdots < F_n \in \mathcal{S}_\zeta$ satisfying $\{\min F_k\}_{k=1}^n \in \mathcal{S}_1$ and $F = \bigcup_{k=1}^n F_k$. In case ξ is a limit ordinal we fix a strictly increasing sequence (ζ_n) of non-limit-ordinals satisfying $\sup_n \zeta_n = \xi$, and define $\mathcal{S}_\xi := \bigcup_{n=1}^\infty \{F \in \mathcal{S}_{\zeta_n} : n \leq F\}$.

Usually in the literature, the family of finite subsets of natural numbers is denoted $[\mathbb{N}]^{<\infty}$, or $\mathcal{P}_{<\infty}(\mathbb{N})$. However, for convenience, let us abuse our notation and write this family as if it were the "ω_1th Schreier family." In other words, we set $\mathcal{S}_{\omega_1} := \{F \subset \mathbb{N} : \#F < \infty\}$. This will greatly simplify the writing.

The sets $\mathcal{S}_\xi$ ($1 \leq \xi \leq \omega_1$) have some very nice properties, most especially that each is *spreading*. This means that if $\{m_1 < \cdots < m_k\} \in \mathcal{S}_\xi$ and $\{n_1 < \cdots < n_k\}$ satisfies $m_i \leq n_i$ for all $1 \leq i \leq k$, then $\{n_1 < \cdots < n_k\} \in \mathcal{S}_\xi$ also holds. They are also *hereditary*, which means that if $E \in \mathcal{S}_\xi$ and $F \subseteq E$ then $F \in \mathcal{S}_\xi$. Contrary to what we might expect, though, the Schreier families are *not* increasing under the inclusion relation. However, it is easily seen that, for all $1 \leq \xi \leq \omega_1$, we have $\mathcal{S}_1 \subseteq \mathcal{S}_\xi$, and in particular we have $\{k\} \in \mathcal{S}_\xi$ for all $k \in \mathbb{N}$. Moreover, the Schreier families do behave somewhat nicely under the inclusion relation in the sense that, if $1 \leq \zeta < \xi \leq \omega_1$ are ordinals, then there is $d = d(\zeta, \xi) \in \mathbb{N}$ such that for all $F \in \mathcal{S}_\zeta$ satisfying $d \leq \min F$ we have $F \in \mathcal{S}_\xi$.

We will appeal several times to the Bessaga-Pełczyński Selection Principle. However, the version that we need is slightly stronger than typically stated in the literature. More specifically, we need a small uniform bound on the equivalence constant. The proof is practically identical to the stan-

dard small perturbations and gliding hump arguments found, for instance, in Theorem 1.3.9 and Proposition 1.3.10 from [1].

THEOREM 1.1. (Uniform Bessaga-Pełczyński Selection Principle) *Suppose X is a Banach space with a basis (e_i), and corresponding coefficient functionals $(e_i^*) \subset X^*$. Let $(x_n) \subset X$ be a sequence satisfying $\lim_{n\to\infty}\|x_n\| = 1$ and $\lim_{n\to\infty} e_i^*(x_n) = 0$ for all $i \in \mathbb{N}$. Then for any $\epsilon > 0$, there exists a basic subsequence (x_{n_k}) which is $(1 + \epsilon)$-congruent to a normalized block basis of (e_i).*

We divide the remainder of this paper into Sections 2 and 3. In Section 2 we define the classes $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$, and establish that, for the nontrivial case $p \neq 1$, they fail to be norm-closed, but as long as $\xi = 1$ they form Banach ideals. Then, in Section 3 we discuss the significance of the parameters p and ξ.

2. THE OPERATOR IDEALS $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$

Let us state formally the definition of classes $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$.

DEFINITION 2.1. Let X and Y be Banach spaces, and fix some constants $0 \leq C < \infty$ and $1 \leq p \leq \infty$, and some ordinal $1 \leq \xi \leq \omega_1$. Put $\mathcal{A}_\xi := \{(\alpha_k) \in c_{00} : \operatorname{supp}(\alpha_k) \in \mathcal{S}_\xi\}$, the set of all scalar sequences with support in the ξth Schreier family. Then we denote by $\mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ the set of all operators $T \in \mathcal{L}(X,Y)$ for which, given $\epsilon > 0$, each normalized weakly null sequence $(x_n) \subset X$ admits a subsequence (x_{n_k}) such that for all $(\alpha_k) \in \mathcal{A}_\xi$, the estimate $\|\sum \alpha_k T x_{n_k}\| \leq (C + \epsilon)\|(\alpha_k)\|_{\ell_p}$ holds. Then we set $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y) := \bigcup_{C \geq 0} \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$.

Immediate from the definitions and the inequality $\|(\alpha_k)\|_{\ell_p} \leq \|(\alpha_k)\|_{\ell_q}$ for all $(\alpha_k) \in c_{00}$ and $1 \leq q \leq p \leq \infty$, we get the following relations.

PROPOSITION 2.2. *Let X and Y be Banach spaces, and fix some constants $0 \leq C \leq D \leq \infty$ and $1 \leq q \leq p \leq \infty$, and some ordinal $1 \leq \xi \leq \omega_1$. Then $\mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y) \subseteq \mathcal{WD}_{\ell_q}^{(C,\xi)}(X,Y)$ and $\mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y) \subseteq \mathcal{WD}_{\ell_p}^{(D,\xi)}(X,Y)$.*

When checking that an operator satisfies the definition of $\mathcal{WD}_{\ell_p}^{(C,\xi)}$, the following propositions will come in handy.

PROPOSITION 2.3. *Let X and Y be Banach spaces, and fix constants $0 \leq C < \infty$ and $1 \leq p \leq \infty$, and some ordinal $1 \leq \xi \leq \omega_1$. Then $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ if and only if for all $\epsilon > 0$ and every seminormalized weakly null sequence $(x_n) \subset X$ which admits a subsequence (x_{n_k}) satisfying $\|x_{n_k}\| \to 1$, there exists a further subsequence (x'_{n_k}) such that for all $(\alpha_k) \in \mathcal{A}_\xi$, the estimate $\|\sum \alpha_k T x'_{n_k}\| \leq (C+\epsilon)\|(\alpha_k)\|_{\ell_p}$ holds.*

Proof. We need only prove the "only if" part since the "if" part is obvious. Suppose $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$. Let (x_n) be a seminormalized weakly null sequence with a subsequence tending to 1 in norm, and pick $\epsilon > 0$. Let $1 < \delta < 1 + \frac{\epsilon}{2C}$, which gives us $C\delta + \frac{\epsilon}{2} < C + \epsilon$, and pass to a further subsequence so that $\|x_{n_k}\| \leq \delta$ for all k. By definition of $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ we can pass to yet a further subsequence so that $(\frac{x_{n_k}}{\|x_{n_k}\|})$ satisfies $\|\sum \alpha_k T \frac{x_{n_k}}{\|x_{n_k}\|}\| \leq (C + \frac{\epsilon}{2\delta})\|(\alpha_k)\|_{\ell_p}$ for all $(\alpha_k) \in \mathcal{A}_\xi$. Since also $(\|x_{n_k}\|\alpha_k) \in \mathcal{A}_\xi$ for each $(\alpha_k) \in \mathcal{A}_\xi$, we get

$$\left\| \sum \alpha_k T x_{n_k} \right\| = \left\| \sum \|x_{n_k}\| \alpha_k T \frac{x_{n_k}}{\|x_{n_k}\|} \right\| \leq \left(C + \frac{\epsilon}{2\delta} \right) \|(\|x_{n_k}\|\alpha_k)\|_{\ell_p}$$
$$\leq \left(C\delta + \frac{\epsilon}{2} \right) \|(\alpha_k)\|_{\ell_p} \leq (C+\epsilon)\|(\alpha_k)\|_{\ell_p}. \qquad \blacksquare$$

PROPOSITION 2.4. *Let X and Y be Banach spaces, and fix constants $0 \leq C < \infty$ and $1 \leq p \leq \infty$. If $(x_n) \subset X$ is a sequence for which (Tx_n) has a norm-null subsequence, then given $\epsilon > 0$, there exists a further subsequence (x_{n_k}) for which the estimate $\|\sum \alpha_k T x_{n_k}\| \leq (C+\epsilon)\|(\alpha_k)\|_{\ell_p}$ holds for all $(\alpha_k) \in c_{00}$.*

Proof. Pick a subsequence so that $\|Tx_{n_k}\| \leq \epsilon 2^{-k}$ and hence, by Hölder, if q is conjugate to p so that $\frac{1}{p} + \frac{1}{q} = 1$, $\|\sum \alpha_k T x_{n_k}\| \leq \epsilon\|(2^{-k}\alpha_k)\|_{\ell_1} \leq \epsilon\|(2^{-k})\|_{\ell_q}\|(\alpha_k)\|_{\ell_p} \leq (C+\epsilon)\|(\alpha_k)\|_{\ell_p}$ for any sequence $(\alpha_k) \in c_{00}$. $\blacksquare$

Recall that a linear operator between Banach spaces X and Y is called *completely continuous* just in case it always sends weakly null sequences into norm-null ones. We write $\mathcal{V}(X,Y)$ for the space of these completely continuous operators. (As mentioned previously, $\mathcal{V}$ is a norm-closed operator ideal.) Thus, Proposition 2.4 yields the following.

PROPOSITION 2.5. *Let X and Y be Banach spaces, and let $1 \leq p \leq \infty$, $0 \leq C \leq \infty$, and $1 \leq \xi \leq \omega_1$. Then $\mathcal{V}(X,Y) \subseteq \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$.*

Let us observe, via several steps, that the class $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ is indeed an operator ideal.

PROPOSITION 2.6. *Let W, X, Y, and Z be Banach spaces, and fix constants $1 \leq p \leq \infty$ and $0 \leq C < \infty$, and some ordinal $1 \leq \xi \leq \omega_1$. Suppose $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ with $A \in \mathcal{L}(W,X)$ and $B \in \mathcal{L}(Y,Z)$. Then $TA \in \mathcal{WD}_{\ell_p}^{(C\|A\|,\xi)}(W,Y)$ and $BT \in \mathcal{WD}_{\ell_p}^{(C\|B\|,\xi)}(X,Z)$.*

Proof. Let's first show that $TA \in \mathcal{WD}_{\ell_p}^{(C\|A\|,\xi)}(W,Y)$. Recall that an operator is weak-to-weak continuous if and only if it is norm-to-norm continuous. Thus if (w_n) is a normalized weakly null sequence in W, we get that (Aw_n) is weakly null in X. If it contains a norm-null subsequence then so does TAw_n, and so by Proposition 2.4 we are done. Otherwise, we can pass to a subsequence if necessary so that $\|Aw_n\| \to \delta$ for some $0 < \delta \leq \|A\|$. Hence $\|\delta^{-1}Aw_n\| \to 1$, and by Proposition 2.3 we get, for any $\epsilon > 0$, a subsequence (n_k) satisfying $\|\sum \alpha_k T\delta^{-1}Aw_{n_k}\| \leq (C + \frac{\epsilon}{\delta})\|(\alpha_k)\|_{\ell_p}$ and hence $\|\sum \alpha_k TAw_{n_k}\| \leq (C\delta + \epsilon)\|(\alpha_k)\|_{\ell_p} \leq (C\|A\| + \epsilon)\|(\alpha_k)\|_{\ell_p}$ for all $(\alpha_k) \in \mathcal{A}_\xi$.

Next, we show that $BT \in \mathcal{WD}_{\ell_p}^{(C\|B\|,\xi)}(X,Z)$. Fix a normalized weakly null sequence $(x_n) \subset X$, and let $\epsilon > 0$. To make things nontrivial, we may assume $B \neq 0$. Then we can find a subsequence (x_{n_k}) such that for all $(\alpha_k) \in \mathcal{A}_\xi$ we get $\|\sum \alpha_k Tx_{n_k}\| \leq \left(C + \frac{\epsilon}{\|B\|}\right)\|(\alpha_k)\|_{\ell_p}$ and hence $\|\sum \alpha_k BTx_{n_k}\| \leq \|B\|\|\sum \alpha_k Tx_{n_k}\| \leq (C\|B\| + \epsilon)\|(\alpha_k)\|_{\ell_p}$. ∎

By "pushing out" a scalar sequence $(\alpha_k) \in \mathcal{A}_\xi$, and using the spreading property of $\mathcal{S}_\xi$, we obtain the following obvious lemma.

LEMMA 2.7. *Let Y be a Banach space, and fix an ordinal $1 \leq \xi \leq \omega_1$. Suppose (y_n) and (y'_k) are sequences in Y such that $(y'_k)_{k \geq m}$ is a subsequence of $(y_n)_{n \geq m}$ for some $m \in \mathbb{N}$. If $(\alpha_k) \in \mathcal{A}_\xi$ satisfies $\min \operatorname{supp}(\alpha_k) \geq m$ then there exists $(\beta_n) \in \mathcal{A}_\xi$ which satisfies $\sum \alpha_k y'_k = \sum \beta_n y_n$ and $\|(\alpha_k)\|_{\ell_p} = \|(\beta_n)\|_{\ell_p}$ for all $1 \leq p \leq \infty$.*

PROPOSITION 2.8. *Let X and Y be Banach spaces, and fix constants $0 \leq C, D < \infty$ and $1 \leq p \leq \infty$. Then for any $S \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ and $T \in \mathcal{WD}_{\ell_p}^{(D,\xi)}(X,Y)$ we have $S + T \in \mathcal{WD}_{\ell_p}^{(C+D,\xi)}(X,Y)$.*

Proof. Let $\epsilon > 0$ and pick a normalized weakly null sequence $(x_n) \subset X$. By definition of $S \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ applied to $\frac{\epsilon}{2} > 0$ and (x_n) we get a

subsequence (x_{n_k}) such that for all $(\alpha_k) \in \mathcal{A}_\xi$, the estimate $\|\sum \alpha_k S x_{n_k}\| \leq (C + \frac{\epsilon}{2})\|(\alpha_k)\|_{\ell_p}$ holds. Next, apply the definition of $T \in \mathcal{WD}_{\ell_p}^{(D,\xi)}(X,Y)$ to $\frac{\epsilon}{2} > 0$ and (x_{n_k}) to find a further subsequence (k_i) such that for all $(\alpha_i) \in \mathcal{A}_\xi$ we get $\|\sum \alpha_i T x_{n_{k_i}}\| \leq (D + \frac{\epsilon}{2})\|(\alpha_i)\|_{\ell_p}$. Notice that since $(x_{n_{k_i}})$ is a subsequence of (x_{n_k}), then for each scalar sequence $(\alpha_i) \in \mathcal{A}_\xi$, by Lemma 2.7, $\|\sum \alpha_i (S+T)x_{n_{k_i}}\| \leq \|\sum \alpha_i S x_{n_{k_i}}\| + \|\sum \alpha_i T x_{n_{k_i}}\| \leq (C+D+\epsilon)\|(\alpha_i)\|_{\ell_p}$. $\blacksquare$

From Propositions 2.5, 2.6, and 2.8, it now follows immediately that $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ is an operator ideal. In fact, the same combination of propositions shows that $\mathcal{WD}_{\ell_p}^{(0,\xi)}$ is an operator ideal, but it turns out that, using Proposition 2.5 along with the fact that every family $\mathcal{S}_\xi$ contains all the singletons, regardless of our choice of p or ξ we always get $\mathcal{WD}_{\ell_p}^{(0,\xi)} = \mathcal{V}$, the completely continuous operators.

Let us now construct two important examples.

EXAMPLE 2.9. Let X be a Banach space which fails to contain a copy of ℓ_1. (This is true in particular if X is reflexive.) Fix constants $1 \leq q < p \leq \infty$ and $0 \leq C \leq \infty$, and some ordinal $1 \leq \xi \leq \omega_1$. Then $\mathcal{WD}_{\ell_p}^{(C,\xi)}(X,\ell_q) = \mathcal{K}(X,\ell_q)$.

Proof. Assume $0 \leq C < \infty$. By Proposition 2.5 we already have $\mathcal{K}(X,\ell_q) \subseteq \mathcal{V}(X,\ell_q) \subseteq \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,\ell_q)$, and so it suffices to prove $\mathcal{WD}_{\ell_p}^{(C,\xi)}(X,\ell_q)$ contains only compact operators. For suppose towards a contradiction that there exists $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,\ell_q)$ which is not compact. Then we can find a seminormalized sequence $(x_n) \subset X$ for which (Tx_n) fails to have a convergent subsequence. Since X fails to contain a copy of ℓ_1, we can apply Rosenthal's ℓ_1 Theorem to pass to a subsequence so that (x_n) is weak Cauchy. Hence we can pass to a further subsequence if necessary so that $(x_{2n} - x_{2n+1})$ and $(Tx_{2n} - Tx_{2n+1})$ are both weakly null and seminormalized. This means the sequence (x'_n) defined by $x'_n := (x_{2n} - x_{2n+1})/\|x_{2n} - x_{2n+1}\|$ is normalized and weakly null, whereas the sequence (Tx'_n) is seminormalized and weakly null. By passing to yet another subsequence if necessary, by Proposition 2.1.3 in [1] we can assume (Tx'_n) is K-equivalent, $K \geq 1$, to the unit vector basis of ℓ_q. Thus, by this equivalence together with the definition of $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$, for any $\epsilon > 0$ we can find a subsequence (n_k) such that $\|(\alpha_k)\|_{\ell_q} \leq K\|\sum \alpha_k T x'_{n_k}\| \leq K(C+\epsilon)\|(\alpha_k)\|_{\ell_p}$ for all $(\alpha_k) \in \mathcal{A}_\xi$. Due to $\mathcal{S}_1 \subseteq \mathcal{S}_\xi$, the above inequality holds also for all $(\alpha_k) \in \mathcal{A}_1$. Notice that every $(\beta_k) \in c_{00}$ induces a corresponding "spread out" sequence $(\alpha_k) \in \mathcal{A}_1$ satisfying $\|(\alpha_k)\|_{\ell_r} = \|(\beta_k)\|_{\ell_r}$ for all $1 \leq$

$r \leq \infty$. Thus we obtain the impossible estimate $\|(\beta_k)\|_{\ell_q} = \|(\alpha_k)\|_{\ell_q} \leq K(C + \epsilon)\|(\alpha_k)\|_{\ell_p} = K(C + \epsilon)\|(\beta_k)\|_{\ell_p}$ for all $(\beta_k) \in c_{00}$. $\blacksquare$

EXAMPLE 2.10. Let X and Y be Banach spaces, and fix numbers $1 \leq p \leq q < \infty$ and an ordinal $1 \leq \xi \leq \omega_1$. Suppose $T \in \mathcal{L}(X, Y)$ is an operator such that TX has a K-embedding, $K \geq 1$, into ℓ_q. (In other words, suppose there is an operator $Q \in \mathcal{L}(TX, \ell_q)$ which satisfies $K^{-1}\|y\| \leq \|Qy\| \leq K\|y\|$ for all $y \in TX$.) Then $T \in \mathcal{WD}_{\ell_p}^{(K^2\|T\|, \xi)}(X, Y)$, and the same result holds if $1 \leq p \leq \infty$ and TX has a K-embedding into c_0. Thus, for $1 \leq p \leq q < \infty$ we have $\mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, \ell_q) = \mathcal{L}(X, \ell_q)$, and for $1 \leq p \leq \infty$ we have $\mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, c_0) = \mathcal{L}(X, c_0)$.

Proof. Fix a normalized weakly null sequence $(x_n) \subset X$, and denote by $Q \in \mathcal{L}(TX, \ell_q)$ (resp., $Q \in \mathcal{L}(TX, c_0)$) the K-embedding. If (Tx_n) contains a norm-null subsequence then we are done by Proposition 2.4. Otherwise let $\epsilon > 0$, and find a subsequence so that $\|QTx_{n_k}\| \to r$ with $0 < r \leq K\|T\|$, and quickly enough so that by the uniform version of Bessaga-Pełczyński combined with Lemma 2.1.1 in [1], we can pass to a further subsequence if necessary so that $(\frac{1}{r}QTx_{n_k})$ is $(1 + \frac{\epsilon}{Kr})$-equivalent to the unit vector basis of ℓ_q (resp. c_0). This gives us, in the ℓ_q case,

$$\left\| \sum \alpha_k Tx_{n_k} \right\| \leq K \left\| \sum \alpha_k QTx_{n_k} \right\| = Kr \left\| \sum \alpha_k \frac{1}{r} QTx_{n_k} \right\|$$
$$\leq Kr \left(1 + \frac{\epsilon}{Kr} \right) \|(\alpha_k)\|_{\ell_q} \leq \left(K^2\|T\| + \epsilon \right) \|(\alpha_k)\|_{\ell_q}$$
$$\leq \left(K^2\|T\| + \epsilon \right) \|(\alpha_k)\|_{\ell_p}$$

for all $(\alpha_k) \in c_{00}$, and a similar inequality holds in the c_0 case. $\blacksquare$

We must lay some groundwork aimed at showing that, in case $\xi = 1$, the class $\mathcal{WD}_{\ell_p}^{(\infty, 1)}$ forms a Banach ideal. We begin by defining a seminorm on the linear space $\mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y)$.

DEFINITION 2.11. Let X and Y be Banach spaces, and fix a constant $1 \leq p \leq \infty$ and an ordinal $1 \leq \xi \leq \omega_1$. For each $T \in \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y)$, we define $C_{(p,\xi)}(T) := \inf \left\{ 0 \leq C < \infty : T \in \mathcal{WD}_{\ell_p}^{(C, \xi)}(X, Y) \right\}$.

PROPOSITION 2.12. *Let X and Y be Banach spaces, and fix a constant $1 \leq p \leq \infty$ and an ordinal $1 \leq \xi \leq \omega_1$. If $T \in \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y)$ then $T \in$*

$\mathcal{WD}_{\ell_p}^{(C_{(p,\xi)}(T),\xi)}(X,Y)$. Furthermore, $T \mapsto C_{(p,\xi)}(T)$ defines a seminorm on the linear space $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$.

Proof. The first part of the proposition is clear from applying the definition of $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ for $C_{(p,\xi)}(T) < C < \infty$, and absolute homogeneity is similarly obvious. The only thing nontrivial to show is that the triangle inequality holds. Indeed, let $S, T \in \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$, and suppose (x_n) is a normalized weakly null sequence. Let $\epsilon > 0$. Then we can apply the definition of $S \in \mathcal{WD}_{\ell_p}^{(C_{(p,\xi)}(S),\xi)}(X,Y)$ to (x_n) and $\frac{\epsilon}{2} > 0$ to find a subsequence (n_k) satisfying $\left\| \sum \alpha_k S x_{n_k} \right\| \leq \left(C_{(p,\xi)}(S) + \frac{\epsilon}{2}\right)\|(\alpha_k)\|_{\ell_p}$ for all $(\alpha_k) \in \mathcal{A}_\xi$. Then, we successively apply the definition of $T \in \mathcal{WD}_{\ell_p}^{(C_{(p,\xi)}(T),\xi)}(X,Y)$ to (x_{n_k}) and $\frac{\epsilon}{2} > 0$ to find to a further subsequence (n_{k_j}) so that $\left\| \sum \alpha_j T x_{n_{k_j}} \right\| \leq \left(C_{(p,\xi)}(T) + \frac{\epsilon}{2}\right)\|(\alpha_j)\|_{\ell_p}$ for all $(\alpha_j) \in \mathcal{A}_\xi$. Thus, by these facts together with Lemma 2.7,

$$\left\| \sum \alpha_j (S+T) x_{n_{k_j}} \right\| \leq \left\| \sum \alpha_j S x_{n_{k_j}} \right\| + \left\| \sum \alpha_j T x_{n_{k_j}} \right\|$$

$$= \left(C_{(p,\xi)}(S) + \frac{\epsilon}{2}\right) \|(\alpha_j)\|_{\ell_p} + \left(C_{(p,\xi)}(T) + \frac{\epsilon}{2}\right) \|(\alpha_j)\|_{\ell_p}$$

$$= \left(C_{(p,\xi)}(S) + C_{(p,\xi)}(T) + \epsilon\right) \|(\alpha_j)\|_{\ell_p}.$$

Thus, $C_{(p,\xi)}(S+T) \leq C_{(p,\xi)}(S) + C_{(p,\xi)}(T)$, and we are done. ∎

Next we show that $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ fails to be norm-closed (as a class) whenever $p \neq 1$. The main idea toward this end proceeds from the following lemma.

LEMMA 2.13. *Fix constants $1 < p \leq \infty$ and $1 \leq q < \infty$, and an ordinal $1 \leq \xi \leq \omega_1$. Let (X_m) and (Y_m) be sequences of Banach spaces, and for each $m \in \mathbb{N}$, let $T_m \in \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X_m,Y_m)$ be an operator satisfying $\|T_m\| = 1$. If $C_{(p,\xi)}(T_m) \to \infty$ then there exists a subsequence (m_j) and a sequence of operators $S_j \in \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$ for which $S_j \to S \in \mathcal{L}(X,Y)$ but $S \notin \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$, where we define $X := (\bigoplus_{j=1}^{\infty} X_{m_j})_{\ell_q}$ and $Y := (\bigoplus_{j=1}^{\infty} Y_{m_j})_{\ell_q}$.*

Proof. Define the subsequence by letting (m_j) be an increasing sequence satisfying $C_j := C_{(p,\xi)}(T_{m_j}) > j2^j$ for every j. Next, set $S := \bigoplus_{j=1}^{\infty} 2^{-j} T_{m_j} \in \mathcal{L}(X,Y)$. For each i, let $P_i \in \mathcal{L}(Y)$ denote the continuous linear projection onto the first i coordinates of Y, and set $S_i := P_i S$. It's easy to see that $S_i \to S$.

Next, we claim that each $S_i \in \mathcal{WD}_{\ell_p}^{(M_i, \xi)}(X,Y) \subseteq \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X,Y)$, where we set $M_i := \|(2^{-j}C_j)_{j=1}^i\|_{\ell_q}$. Indeed fix any $i \in \mathbb{N}$, and let (x_n) be a normalized weakly null sequence in X. Pick any $\epsilon > 0$. For each j, let $\widetilde{X}_j$ be the obvious isometrically isomorphic copy of X_j contained in X, and let $U_j : \widetilde{X}_j \to X_j$ be the corresponding isometric isomorphism. For each n, write $x_n = (x_{n,j})_j \in X$. Then $(x_{n,j})_n$ is a sequence in X_j which is bounded by 1. If $(x_{n,j})_n$ has a norm-null subsequence, then by Proposition 2.4 we can find a subsequence (n_k) such that, for all $(\alpha_k) \in c_{00}$,

$$\left\| \sum_k \alpha_k T_{m_j} x_{n_k,j} \right\| \leq \left(C_j + \frac{\epsilon 2^j}{i^{1/q}} \right) \|(x_{n_k,j})_k\|_{\ell_p} \tag{2.1}$$

Otherwise we can find a subsequence (n_k) so that $\|x_{n_k,j}\|_{X_j} \to r$ as $k \to \infty$ for some $0 < r \leq 1$. Clearly, $(x_{n,j})_n$ is weakly null in X_j, and so by Propositions 2.3 and 2.12, we can pass to a further subsequence if necessary so that, for all $(\alpha_k) \in \mathcal{A}_\xi$,

$$\left\| \sum_k \alpha_k T_{m_j} x_{n_k,j} \right\| = r \left\| \sum_k \alpha_k T_{m_j} \frac{x_{n_k,j}}{r} \right\|$$

$$\leq r \left(C_j + \frac{\epsilon 2^j}{i^{1/q}r} \right) \|(x_{n_k,j})_k\|_{\ell_p} \leq \left(C_j + \frac{\epsilon 2^j}{i^{1/q}} \right) \|(x_{n_k,j})_k\|_{\ell_p}.$$

In either case, for each j and any subsequence of $(x_{n,j})_n$, we can pass to a further subsequence so that the inequality (2.1) holds for all $(\alpha_k) \in \mathcal{A}_\xi$.

Thus, by successively passing to further subsequences for $j = 1, \ldots, i$, due to Lemma 2.7, we get a subsequence (n_k) such that (2.1) holds for all $j = 1, \ldots, i$ and all $(\alpha_k) \in \mathcal{A}_\xi$. In particular, this means

$$\left\| \sum_k \alpha_k S_i x_{n_k} \right\| = \left(\sum_{j=1}^i 2^{-jq} \left\| \sum_k \alpha_k T_{m_j} x_{n_k,j} \right\|^q \right)^{1/q}$$

$$\leq \left(\sum_{j=1}^i 2^{-jq} \left(C_j + \frac{\epsilon 2^j}{i^{1/q}} \right)^q \|(\alpha_k)\|_{\ell_p}^q \right)^{1/q} = \left\| \left(2^{-j}C_j + \frac{\epsilon}{i^{1/q}} \right)_{j=1}^i \right\|_{\ell_q} \|(\alpha_k)\|_{\ell_p}$$

$$\leq \left(\left\| (2^{-j}C_j)_{j=1}^i \right\|_{\ell_q} + \left\| \left(\frac{\epsilon}{i^{1/q}} \right)_{j=1}^i \right\|_{\ell_q} \right) \|(\alpha_k)\|_{\ell_p} = (M_i + \epsilon)\|(\alpha_k)\|_{\ell_p},$$

which proves the claim that $S_i \in \mathcal{WD}_{\ell_p}^{(M_i, \xi)}(X, Y) \subseteq \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y)$.

However, it cannot be that $S \in \mathcal{WD}_{\ell_p}^{(C, \xi)}(X, Y)$ for any $0 \le C < \infty$. To show why not, fix $i \in \mathbb{N}$, and let (x_n) be a normalized weakly null sequence in X_i. Then let $\epsilon > 0$ be such that, for any subsequence (n_k), there exists $(\alpha_k) \in \mathcal{A}_\xi$ with $\|\sum_k \alpha_k T_{m_i} x_{n_k}\| > (i2^i + \epsilon)\|(\alpha_k)\|_{\ell_p}$. Let $Q_i : X_i \to X$ be the canonical embedding of X_i into X, and observe that $(Q_i x_n)_n$ is a normalized weakly null sequence in X. However, for every subsequence (n_k) there exists $(\alpha_k) \in \mathcal{A}_\xi$ with

$$\left\| \sum_k \alpha_k S Q_i x_{n_k} \right\| = 2^{-i} \left\| \sum_k \alpha_k T_{m_i} x_{n_k} \right\|$$
$$> 2^{-i}(i2^i + \epsilon)\|(\alpha_k)\|_{\ell_p} \ge (i + 2^{-i}\epsilon)\|(\alpha_k)\|_{\ell_p}.$$

It follows that $S \notin \mathcal{WD}_{\ell_p}^{(i, \xi)}(X, Y)$ for any i, and hence $S \notin \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y)$. ∎

EXAMPLE 2.14. Fix a constant $1 < p \le \infty$ and an ordinal $1 \le \xi \le \omega_1$. There exists a Banach space X for which $\mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X)$ is not norm-closed. If $p \ne \infty$, then we can choose X to be reflexive.

Proof. For convenience in writing, let us consider the case where $p \ne \infty$. The case where $p = \infty$ uses c_0 in place of ℓ_p, and the resulting proof is nearly identical, except that the resulting space X is not reflexive.

Let (e_n) denote the unit vector basis of ℓ_p. For each finite $E \subset \mathbb{N}$, define the functional $f_E \in \ell_p^*$ by the rule $f_E(e_n) = 1$ if $n \in E$ and $f_E(e_n) = 0$ otherwise. Now, fix $m \in \mathbb{N}$, and define the norming set $\mathcal{B}_m := B_{\ell_p^*} \cup \{f_E : E \subset \mathbb{N}, \#E = m\}$, where $B_{\ell_p^*}$ denotes the closed unit ball of $\ell_p^* = \ell_q$. Notice that for every $E \subset \mathbb{N}$ of size m, we have $|f_E(\sum \alpha_k e_k)| \le \|(\alpha_k)_{k \in E}\|_{\ell_1} \le m^{1 - \frac{1}{p}}\|(\alpha_k)\|_{\ell_p}$ so that $\|f_E\|_{\ell_p^*} \le m^{1 - \frac{1}{p}}$. So $\mathcal{B}_m$ is a bounded subset of ℓ_p^* containing $B_{\ell_p^*}$. Due to the identity $\|x\|_{\ell_p} = \sup_{f \in B_{\ell_p^*}} |f(x)|$, we can now define an equivalent norm $\|\cdot\|_m$ on ℓ_p by the rule $\|x\|_m := \sup_{f \in \mathcal{B}_m} |f(x)|$. Put $X_m := (\ell_p, \|\cdot\|_m)$, and notice that for all n and E, we have $|f_E(e_n)| \le 1$. Hence (e_n) is still normalized in X_m, and weakly null since X_m is isomorphic to ℓ_p. Furthermore, this isomorphism also means the identity map $I_m \in \mathcal{L}(X_m)$ is a norm-1 operator which lies in $\mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X_m)$ by Example 2.10. However, we will show that $C_{(p, \xi)}(I_m) \ge m^{1 - \frac{1}{p}}$.

Suppose $C < m^{1-\frac{1}{p}}$, and let $0 < \epsilon < m^{1-\frac{1}{p}} - C$. Then, let (e_{n_k}) be any subsequence of (e_n), which we have previously observed is normalized and weakly null in X_m. Pick $E = (m+1 < m+2 < \cdots < 2m) \in \mathcal{S}_1 \subseteq \mathcal{S}_\xi$ of size m, and define $F := (n_{m+1} < n_{m+2} < \cdots < n_{2m})$. Since $\mathcal{S}_\xi$ is spreading, we have $F \in \mathcal{S}_\xi$, and also of size m. Next, define $(\alpha_k) \in \mathcal{A}_\xi$ by letting $\alpha_k = 1$ for all $k \in E$ and $\alpha_k = 0$ otherwise. Then

$$\left\| \sum \alpha_k e_{n_k} \right\|_m \geq \left| f_F \left(\sum \alpha_k e_{n_k} \right) \right| = \left| f_F \left(\sum_{n \in F} e_n \right) \right| = m$$

$$= m^{1-\frac{1}{p}} \|(\alpha_k)\|_{\ell_p} > (C + \epsilon) \|(\alpha_k)\|_{\ell_p}.$$

Thus, the identity map I_m does not lie in $\mathcal{WD}_{\ell_p}^{(C,\xi)}(X_m)$, and $C_{(p,\xi)}(I_m) \geq m^{1-\frac{1}{p}}$ as claimed.

We have therefore constructed a sequence (X_m) of Banach spaces and a corresponding sequence $I_m \in \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X_m)$ of norm-1 operators with $C_{(p,\infty)}(I_m) \to \infty$. By Lemma 2.13, there exists a space X for which $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X)$ fails to be norm-closed, and in case $p \neq \infty$, we can choose it to be reflexive. ∎

Even though $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ is not a norm-closed operator ideal, when $\xi = 1$ its components are F_σ-subsets of $\mathcal{L}(X,Y)$, as the following Proposition shows.

PROPOSITION 2.15. *Let X and Y be Banach spaces, and fix constants $0 \leq C < \infty$ and $1 \leq p \leq \infty$. We consider the case $\xi = 1$. Then $\mathcal{WD}_{\ell_p}^{(C,1)}(X,Y)$ is a norm-closed subset of $\mathcal{L}(X,Y)$.*

Proof. Let (T_j) be a sequence in $\mathcal{WD}_{\ell_p}^{(C,1)}(X,Y)$ converging to some $T \in \mathcal{L}(X,Y)$. Fix any $\epsilon > 0$, and let $(x_n) \subset X$ be a normalized weakly null sequence in X. Without loss of generality we may assume $\|T - T_j\| < \epsilon/(2j^{1-\frac{1}{p}})$ for all j. Let (x_{n_k}) be a subsequence formed by a diagonal argument using the T_j's with $\frac{\epsilon}{2} > 0$. In other words, begin with a subsequence $(x_{n_{1,k}})$ given by the definition of $T_1 \in \mathcal{WD}_{\ell_p}^{(C,1)}(X,Y)$, corresponding to $\frac{\epsilon}{2} > 0$. Then find a *further* subsequence $(x_{n_{2,k}}) \subset (x_{n_{1,k}})$ given by the definition of $T_2 \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$, also corresponding to $\frac{\epsilon}{2} > 0$, and so on. Finally, for each k, put $n_k := n_{k,k}$.

Let $(\alpha_k) \in \mathcal{A}_1$, and set $m := \min \operatorname{supp}(\alpha_k) \leq \#\operatorname{supp}(\alpha_k)$. Notice that $(x_{n_k})_{k \geq m}$ is a subsequence of $(x_{n_{m,i}})_{i \geq m}$ so that by Lemma 2.7,

$$\left\| \sum \alpha_k T x_{n_k} \right\| \le \left\| \sum_k \alpha_k T_m x_{n_k} \right\| + \| T - T_m \| \left\| \sum \alpha_k x_{n_k} \right\|$$

$$< \left(C + \frac{\epsilon}{2} \right) \|(\alpha_k)\|_{\ell_p} + m^{1-\frac{1}{p}} \left(\frac{\epsilon}{2m^{1-\frac{1}{p}}} \right) \|(\alpha_k)\|_{\ell_p}$$

$$= (C + \epsilon)\|(\alpha_k)\|_{\ell_p}$$

$\blacksquare$

DEFINITION 2.16. Fix Banach spaces X and Y, along with a constant $1 \le p \le \infty$ and an ordinal $1 \le \xi \le \omega_1$. We define the norm $\|\cdot\|_{(p,\xi)}$ on the space $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$ by the rule $\|T\|_{(p,\xi)} := \|T\|_{\mathcal{L}(X,Y)} + C_{(p,\xi)}(T)$.

Notice that $\|\cdot\|_{(p,\xi)}$ is indeed a norm on $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y)$, as it is the sum of a norm and a seminorm.

PROPOSITION 2.17. *Fix $1 \le p \le \infty$. In case $\xi = 1$, the rule $\|\cdot\|_{(p,1)}$ is an ideal norm which makes $\mathcal{WD}_{\ell_p}^{(\infty,1)}$ into a Banach ideal.*

Proof. As was observed earlier, that $\mathcal{WD}_{\ell_p}^{(\infty,1)}$ is an operator ideal follows from Propositions 2.5, 2.6, and 2.8. To show that $\|\cdot\|_{(p,1)}$ induces a complete norm on each component space $\mathcal{WD}_{\ell_p}^{(\infty,1)}(X,Y)$, suppose (T_n) is a $\|\cdot\|_{(p,1)}$-Cauchy sequence. Then it is $\|\cdot\|_{(p,1)}$-bounded and hence $C_{(p,1)}$-bounded, say by $M > 0$. It is also Cauchy in the operator norm so that $T_n \to T$ for some $T \in \mathcal{L}(X,Y)$. By Proposition 2.12, every T_n lies in the set $\mathcal{WD}_{\ell_p}^{(M,1)}(X,Y)$, which is closed under the operator norm by Proposition 2.15. Hence, $T \in \mathcal{WD}_{\ell_p}^{(\infty,1)}(X,Y)$ as well, and it remains only to show that $\|\cdot\|_{(p,1)}$ is indeed an ideal norm.

Since any element of the form $x^* \otimes y$ is rank-1, it is completely continuous. By Proposition 2.5, this means $C_{(p,1)}(x^* \otimes y) = 0$ and hence $\|x^* \otimes y\|_{(p,1)} = \|x^* \otimes y\|_{\mathcal{L}(X,Y)} = \sup_{x \in S_X} \|(x^* \otimes y)(x)\|_Y = \sup_{x \in S_X} |x^*(x)| \|y\|_Y = \|x^*\|_{X^*} \|y\|_Y$. The triangle inequality follows from the fact that $\|\cdot\|_{(p,1)}$ is a norm on each component space $\mathcal{WD}_{\ell_p}^{(\infty,1)}(X,Y)$. That

$$\|BTA\|_{(p,1)} \le \|B\|_{\mathcal{L}(Y,Z)} \|T\|_{(p,1)} \|A\|_{\mathcal{L}(W,X)}$$

for all $T \in \mathcal{J}(X,Y)$, $A \in \mathcal{L}(W,X)$, and $B \in \mathcal{L}(Y,Z)$, follows naturally from Propositions 2.6 and 2.12. $\blacksquare$

3. SIGNIFICANCE OF PARAMETERS

Let $1 < q < p < \infty$. By Example 2.9 we get $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(\ell_q) = \mathcal{K}(\ell_q)$, whereas by Example 2.10 we get $\mathcal{WD}_{\ell_q}^{(\infty,\xi)}(\ell_q) = \mathcal{L}(\ell_q)$. Applying Proposition 2.2 therefore gives us the following.

PROPOSITION 3.1. *Fix any ordinal* $1 \leq \xi \leq \omega_1$. *For* $1 \leq q < p \leq \infty$, *the norm-closed operator ideals* $\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi)}$ *and* $\overline{\mathcal{WD}}_{\ell_q}^{(\infty,\xi)}$ *are distinct (as classes).*

However, it is natural to also ask whether the classes $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ are distinct as ξ ranges over $1 \leq \xi \leq \omega_1$. Obviously, this is not the case for the trivial ideal $\mathcal{WD}_{\ell_1}^{(\infty,\xi)} = \mathcal{L}$. The question remains open in general for $1 < p \leq \infty$, but in this section we do give a *partial* answer by exhibiting, for each $1 < p \leq \infty$, a strictly increasing sequence (ξ_n) of countable ordinals $1 \leq \xi_n < \xi_{n+1} < \omega_1$, $n \in \mathbb{N}$, and a sequence (X_n) of Banach spaces, such that for all $m, n \in \mathbb{N}$ with $m < n$ we have
$$\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_n)}(X_m) \subsetneq \overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_m)}(X_m)$$
Our task requires a few preliminaries, which we proceed to lay out.

PROPOSITION 3.2. *Let* X *and* Y *be Banach spaces, and fix a constant* $1 \leq p \leq \infty$. *Let* $1 \leq \xi < \zeta \leq \omega_1$ *be ordinals, and* $0 \leq C \leq \infty$. *Then* $\mathcal{WD}_{\ell_p}^{(C,\zeta)}(X,Y) \subseteq \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$.

Proof. We may assume $C \neq \infty$. Suppose $T \in \mathcal{WD}_{\ell_p}^{(C,\zeta)}(X,Y)$, and let (x_n) be a normalized weakly null sequence in X, and $\epsilon > 0$. Then there exists a subsequence (n_k) such that $\|\sum \alpha_k T x_{n_k}\| \leq (C + \epsilon)\|(\alpha_k)\|_{\ell_p}$ for all $(\alpha_k) \in \mathcal{A}_\zeta$. Let $d = d(\xi,\zeta)$ be such that if $E \in \mathcal{S}_\xi$ with $\min E \geq d$ then $E \in \mathcal{S}_\zeta$. Now, let $(\alpha_k) \in \mathcal{S}_\xi$, and define the scalar sequence (β_k) as $\beta_k := \alpha_{k-d}$ for $k \geq d$ and $\beta_k := 0$ for $k < d$. By the spreading property of $\mathcal{S}_\xi$ we have $(\beta_k) \in \mathcal{A}_\xi$, and since also $\min \mathrm{supp}(\beta_k) \geq d$ we have $(\beta_k) \in \mathcal{A}_\zeta$. Thus, $\|\sum \alpha_k T x_{n_{k+d}}\| = \|\sum \beta_k T x_{n_k}\| \leq (C + \epsilon)\|(\beta_k)\|_{\ell_p} = (C + \epsilon)\|(\alpha_k)\|_{\ell_p}$. $\blacksquare$

Let $1 \leq \xi < \omega_1$ be a countable ordinal. A finite sequence $(E_i)_{i=1}^{j}$ of finite subsets of $\mathbb{N}$ is called $\mathcal{S}_\xi$-*admissible* whenever $E_1 < \cdots < E_j$ and $\{\min E_i\}_{i=1}^{j} \in \mathcal{S}_\xi$. Then the *Tsirelson-type space* $T[\frac{1}{2}, \mathcal{S}_\xi]$ is the completion of c_{00} under the norm $\|\cdot\|_T$ uniquely defined by the implicit equation $\|x\|_T = \max\{\|x\|_{\ell_\infty}, \frac{1}{2}\sup\sum_i \|E_i x\|_T\}$, where the "sup" is taken over all $j \in \mathbb{N}$ and all $\mathcal{S}_\xi$-admissible families $(E_i)_{i=1}^{j}$. Here we use the notation

$E_i x := \sum_{n \in E_i} \alpha_n e_n$ for $x := \sum \alpha_n e_n \in c_{00}$, where (e_n) are the canonical basis vectors in c_{00}. We also use the abbreviation $T = T[\frac{1}{2}, \mathcal{S}_\xi]$ when the ordinal ξ is understood from context.

It is easily seen that the canonical unit vectors in c_{00} form a normalized 1-unconditional basis for T. For $1 \le q < \infty$, its *q-convexification* $T_q[\frac{1}{2}, \mathcal{S}_\xi]$, where we again use the abbreviation $T_q = T_q[\frac{1}{2}, \mathcal{S}_\xi]$, is then usually defined in the literature by setting $T_q = T_q[\frac{1}{2}, \mathcal{S}_\xi] = \{(\alpha_n) : (|\alpha_n|^q) \in T = T[\frac{1}{2}, \mathcal{S}_\xi]\}$, which is a Banach space under the norm $\|(\alpha_n)\|_{T_q} := \||(\alpha_n|^q)\|_T^{1/q}$, and with the canonical unit vectors in c_{00} again forming a normalized 1-unconditional basis. (Notice also that if $q = 1$ then we get $T_1 = T$.) However, it will serve our purposes much better to use instead the following equivalent construction (cf. [11, p. 1062]). We inductively define a sequence $(\|\cdot\|_n)$ of norms on c_{00}. Set $\|\cdot\|_0 := \|\cdot\|_{\ell_\infty}$ and define each successive $\|\cdot\|_{n+1}$ by the rule $\|x\|_{n+1} = \max\{\|x\|_{\ell_\infty}, 2^{-1/q} \sup(\sum_i \|E_i x\|_n^q)^{1/q}\}$, where the "sup" is taken over all $j \in \mathbb{N}$ and all $\mathcal{S}_\xi$-admissible families $(E_i)_{i=1}^j$. Then $\|x\|_{T_q} := \lim_{n \to \infty} \|x\|_n$ defines a norm on c_{00}. In fact, it is easily seen (cf., e.g., the kind of argument used in the proof to Theorem 10.3.2 in [1]) that $\|\cdot\|_{T_q}$ is the unique norm on c_{00} satisfying the implicit equation $\|x\|_{T_q} = \max\{\|x\|_{\ell_\infty}, 2^{-1/q} \sup(\sum_i \|E_i x\|_{T_q}^q)^{1/q}\}$. The space T_q is just the completion of c_{00} under this norm.

Due to this construction, $\|x\|_{T_q} \le \|x\|_{\ell_q}$ for each $x \in c_{00}$. Furthermore, T_q is known to be a reflexive Banach space which contains no copy of ℓ_q. When $q = 1$ this follows from Proposition 5.1 in [2]. In case $1 < q < \infty$, Remark T.1 on [11, p. 1062] tells us that T_q is an asymptotic ℓ_q space which contains no copy of ℓ_q, and thus by Remark 6.3 in [12] it is also reflexive. Therefore each dual space T_q^* is a reflexive space which fails to contain any copy of ℓ_p, $\frac{1}{p} + \frac{1}{q} = 1$. Notice that T_q^* can also be viewed as a completion of c_{00} under some norm $\|\cdot\|_{T_q^*}$, with the usual action $f(x) = \sum \alpha_n \beta_n$ for $f = (\alpha_n) \in T_q^*$ and $x = (\beta_n) \in T_q$.

In [14] was given an implicit formula for the norm of $T_1[\frac{1}{2}, \mathcal{S}_1]^*$. It is natural to conjecture that a similar formula always holds for the norm of $T_q[\frac{1}{2}, \mathcal{S}_\xi]^*$, but for our purposes we only need a crude estimate.

LEMMA 3.3. *Let $1 < p \le \infty$ and $1 \le q < \infty$ be conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Set $1 \le \xi < \omega_1$ and $T_q = T_q[\frac{1}{2}, \mathcal{S}_\xi]$. Then $\|x\|_{T_q^*} \le 2^{1/q} \big\|(\|E_i x\|_{T_q^*})\big\|_{\ell_p}$ for all $x \in c_{00}$ and $\mathcal{S}_\xi$-admissible families $(E_i)_{i=1}^j$ satisfying $x = \sum_{i=1}^j E_i x$.*

Proof. Let $y \in T_q$. Since $x = \sum_{i=1}^j E_i x$ we have $x(y) = \sum_{i=1}^j (E_i x)(E_i y)$.

Then by this fact together with Hölder and the relation

$$2^{-1/q}\left(\sum_{i=1}^{j}\|E_iy\|_{T_q}^q\right)^{1/q} \leq \|y\|_{T_q}$$

(which follows from the construction of T_q), we have

$$|x(y)| = \left|\sum_{i=1}^{j}(E_ix)(E_iy)\right| \leq \sum_{i=1}^{j}|(E_ix)(E_iy)| \leq \sum_{i=1}^{j}\|E_ix\|_{T_q^*}\|E_iy\|_{T_q}$$

$$\leq \big\|(\|E_ix\|_{T_q^*})\big\|_{\ell_p}\left(\sum_{i=1}^{j}\|E_iy\|_{T_q}^q\right)^{1/q} \leq 2^{1/q}\big\|(\|E_ix\|_{T_q^*})\big\|_{\ell_p}\|y\|_{T_q}. \qquad\blacksquare$$

LEMMA 3.4. *Let $T_q = T_q[\frac{1}{2}, \mathcal{S}_\xi]$, $1 \leq q < \infty$ and $1 \leq \xi < \omega_1$, and let (u_k) be any normalized block basic sequence in the dual space T_q^* (with respect to the canonical unit vectors in c_{00}). Then for every $(\alpha_k) \in \mathcal{A}_\xi$ we have $\|\sum \alpha_k u_k\|_{T_q^*} \leq 2^{1/q}\|(\alpha_k)\|_{\ell_p}$, where $1 < p \leq \infty$ is conjugate to q, that is, $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Write $\mathrm{supp}(\alpha_k) =: \{k_1, \ldots, k_j\} \in \mathcal{S}_\xi$, and set $E_i := \mathrm{supp}\{u_{k_i}\}$ for each $1 \leq i \leq j$. Then $x := \sum \alpha_k u_k = \sum_{i=1}^{j} E_i x$, where $\|E_i x\|_{T_q^*} = \|\alpha_{k_i} u_{k_i}\|_{T_q^*} = |\alpha_{k_i}|$ for each $1 \leq i \leq j$. Furthermore, due to $k_i \leq \min \mathrm{supp}\{u_{k_i}\} = \min E_i$ together with $\{k_1, \ldots, k_j\} \in \mathcal{S}_\xi$ and the spreading property of Schreier families, we see that $(E_i)_{i=1}^{j}$ is $\mathcal{S}_\xi$-admissible. All of this together with Lemma 3.3 means

$$\left\|\sum \alpha_k u_k\right\|_{T_q^*} \leq 2^{1/q}\big\|(\|E_ix\|_{T_q^*})_{i=1}^{j}\big\|_{\ell_p} = 2^{1/q}\big\|(\alpha_{k_i})_{i=1}^{j}\big\|_{\ell_p} = 2^{1/q}\|(\alpha_k)\|_{\ell_p}. \qquad\blacksquare$$

LEMMA 3.5. *Set $T_q = T_q[\frac{1}{2}, \mathcal{S}_\xi]$, $1 \leq q < \infty$ and $1 \leq \xi < \omega_1$. Let $1 < p \leq \infty$ denote the conjugate of q, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Then $\|x^*\|_{\ell_p} \leq \|x^*\|_{T_q^*}$ for all $x^* \in c_{00}$.*

Proof. Since $c_{00} \subseteq \ell_p = \ell_q^*$ with c_{00} dense in ℓ_q, for each $\epsilon > 0$ we can find $x \in c_{00}$ such that $|x^*(x)| \geq (\|x^*\|_{\ell_p} - \epsilon)\|x\|_{\ell_q}$. Combining this with the relation $\|x\|_{T_q} \leq \|x\|_{\ell_q}$ (which follows from the construction of T_q) we get $|x^*(x)| \geq (\|x^*\|_{\ell_p} - \epsilon)\|x\|_{\ell_q} \geq (\|x^*\|_{\ell_p} - \epsilon)\|x\|_{T_q}$ and hence $\|x^*\|_{T_q^*} \geq \|x^*\|_{\ell_p} - \epsilon$. Letting $\epsilon \to 0$ completes the proof. $\blacksquare$

EXAMPLE 3.6. Set $T_q = T_q[\frac{1}{2}, \mathcal{S}_\xi]$, $1 \leq q < \infty$ and $1 \leq \xi < \omega_1$, and let T_q^* denote its dual. Let $1 < p \leq \infty$ be conjugate to q, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Then $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(T_q^*) = \mathcal{L}(T_q^*)$, whereas $\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\omega_1)}(T_q^*) \neq \mathcal{L}(T_q^*)$.

Proof. Consider the identity operator $I : T_q^* \to T_q^*$. We claim that $I \in \mathcal{WS}_{\ell_p}^{(2^{1/q},\xi)}(T_q^*)$. Indeed, let (x_n) be a normalized weakly null sequence in T_q^*, and let $\epsilon > 0$. By the uniform version of the Bessaga-Pełczyński Selection Principle, there exists a subsquence (x_{n_k}) which is $(1 + 2^{-1/q}\epsilon)$-equivalent to a normalized block basic sequence (u_k) of the unit vector basis. Thus, by Lemma 3.4, for every $(\alpha_k) \in \mathcal{A}_\xi$ we have $\|\sum \alpha_k x_{n_k}\|_{T_q^*} \leq (1 + 2^{-1/q}\epsilon)\|\sum \alpha_k u_k\|_{T_q^*} \leq (2^{1/q} + \epsilon)\|(\alpha_k)\|_{\ell_p}$, and the claim is proved.

On the other hand, we also claim $I \notin \mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(T_q^*)$. Let (e_n) be the unit vector basis of T_q^*, which is also weakly null since T_q^* is reflexive. Recall from Lemma 3.5 that $\|(\alpha_n)\|_{\ell_p} \leq \|(\alpha_n)\|_{T_q^*}$ for all $(\alpha_n) \in c_{00}$. Hence, for any subsequence (n_k) we have $\|\sum \alpha_k e_{n_k}\|_{T_q^*} \geq \|(\alpha_k)\|_{\ell_p}$. Since T_q^* fails to contain a copy of ℓ_p, then for any $C \geq 0$ and $\epsilon > 0$ we must now be able to find some $(\alpha_k) \in c_{00}$ with $\|\sum \alpha_k e_{n_k}\|_{T_q^*} \geq (C + \epsilon)\|(\alpha_k)\|_{\ell_p}$.

Thus, $\mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(T_q^*) \neq \mathcal{L}(T_q^*)$, and it remains only to recall that the norm-closure of a proper (algebraic) ideal in a Banach algebra is again a proper ideal (cf., e.g., Corollary VII.2.4 in [8]). ∎

At this point, we have shown that for every fixed $1 < p \leq \infty$ and every countable ordinal $1 \leq \xi < \omega_1$, the classes $\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi)}$ and $\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\omega_1)}$ are distinct. We can use descriptive set theoretic methods to improve this result. For this purpose, we must recall some additional terminology.

Denote by $\mathbb{N}^{<\mathbb{N}} := \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, where $\mathbb{N}^n$ is the set of all sequences of elements of $\mathbb{N}$ of length $n \in \mathbb{N}$, with the convention that $\mathbb{N}^0 = \{\emptyset\}$. We can define a partial order $\leq$ on $\mathbb{N}^{<\mathbb{N}}$ by writing $s \leq t$ for $s, t \in \mathbb{N}^{<\mathbb{N}}$ whenever s is an *initial segment* of t, that is, whenever the elements of s are precisely the initial elements of t, and in the same order. Write $s < t$ when the ordering is strict. Then we define a *tree* (on $\mathbb{N}$) as a subset T of $\mathbb{N}^{<\mathbb{N}}$ which is closed under taking initial segments, i.e. if $t \in T$ and $s \leq t$ then $s \in T$. Notice that this means $\emptyset \in T$ for any nonempty tree T. A sequence $(j_n)_{n=0}^\infty$ of elements in $\mathbb{N}$ is called an *infinite branch* of T just in case every $(j_n)_{n \leq k} \in T$ for all $k \in \mathbb{N}$. The tree T is called *well-founded* whenever it has no such infinite branches. We define the *derivative* T^1 of T as the tree

$$T^1 := \big\{ s \in T : s < t \text{ for some } t \in T \big\}.$$

Then, using transfinite induction, we can define for each countable ordinal $1 \leq \xi < \omega_1$ the iterated derivative T^ξ of T as follows. If $1 \leq \xi < \omega_1$ and T^ξ has been defined then we set $T^{\xi+1} = (T^\xi)^1$, and if ξ is a limit ordinal such that T^ζ has been defined for all $1 \leq \zeta < \xi$ then we set

$$T^\xi := \bigcap_{0 \leq \zeta < \xi} T^\zeta,$$

where by convention $T^0 = T$. In case $T^\xi = \emptyset$ for some countable ordinal $0 \leq \xi < \omega_1$ then we define the *order*, or *ordinal index*, of T, denoted $o(T)$, as the least such ordinal; otherwise we write $o(T) = \omega_1$. Also, by convention we define $o(\emptyset) = 0$.

It is well-known that if T is a well-founded tree on $\mathbb{N}$ then $o(T) < \omega_1$ (cf., e.g., [10, p. 4]). We also have the following fact (cf., e.g., Proposition 1.5 in [10]).

PROPOSITION 3.7. *Suppose S and T are trees on $\mathbb{N}$. Then $o(S) \leq o(T)$ if and only if there exists a map $f : S \to T$ such that for all $s_1 < s_2$ in S we have $f(s_1) < f(s_2)$.*

Let us now give two successive propositions, the first of which is due to an anonymous referee, and the second to Ryan Causey and Dan Freeman.

PROPOSITION 3.8. *Let $1 \leq p \leq \infty$, and let X and Y be Banach spaces. Then for each $0 \leq C < \infty$ we have*

$$\bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y) = \mathcal{WD}_{\ell_p}^{(C,\omega_1)}(X,Y).$$

Proof. We need only show that

$$\bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y) \subseteq \mathcal{WD}_{\ell_p}^{(C,\omega_1)}(X,Y),$$

since the reverse inequality holds by Proposition 3.2. So let us assume that $T \notin \mathcal{WD}_{\ell_p}^{(C,\omega_1)}(X,Y)$. Then we can find $\epsilon > 0$ and a normalized weakly null sequence (x_n) in X such that no subsequence of (Tx_n) is $(C + \epsilon)$-dominated

by the canonical basis of ℓ_p. Let us define a tree τ on $\mathbb{N}$ by

$$\tau = \left\{ (k_1 < \cdots < k_m) \in \mathbb{N}^{<\mathbb{N}} : \right.$$

$$\left. m \in \mathbb{N}, \ \left\| \sum_{j=1}^{m} \alpha_j T x_{k_j} \right\|_Y \leq (C + \epsilon) \left\| (\alpha_j)_{j=1}^{m} \right\|_{\ell_p} \ \forall (\alpha_j) \in c_{00} \right\}.$$

Notice that τ must be well-founded, since otherwise we could find an infinite branch $(k_j)_{j=1}^{\infty}$ such that

$$\left\| \sum_{j=1}^{m} \alpha_j T x_{k_j} \right\|_Y \leq (C + \epsilon) \left\| (\alpha_j)_{j=1}^{m} \right\|_{\ell_p}$$

for all $m \in \mathbb{N}$ and $(\alpha_j) \in c_{00}$, violating the hypothesis that the canonical basis of ℓ_p is not $(C+\epsilon)$-dominated by any subsequence of $(T x_n)$. Thus, the ordinal index of τ is countable, that is, $o(\tau) < \omega_1$. It follows that

$$\xi := o(\tau) + 1 < \omega_1$$

as well.

Assume towards a contradiction that $T \in \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$. Then we can find a subsequence (n_k) such that

$$\left\| \sum \alpha_k T x_{n_k} \right\|_Y \leq (C + \epsilon) \| (\alpha_k) \|_{\ell_p}$$

for all $(\alpha_k) \in \mathcal{A}_\xi$. Together with the spreading property of $\mathcal{S}_\xi$, this means we can define a map $f : \mathcal{S}_\xi \to \tau$ according to the rule

$$f\big((k_1, \ldots, k_m)\big) = (n_{k_1}, \ldots, n_{k_m}).$$

Recall that $o(\mathcal{S}_\xi) = \omega^\xi$ (cf., e.g., Proposition 2.1 in [5]). Thus, by Proposition 3.7 it follows that

$$\xi \leq \omega^\xi = o(\mathcal{S}_\xi) \leq o(\tau) < \xi$$

which gives us the desired contradiction. ∎

PROPOSITION 3.9. (Causey-Freeman) *Let* $1 \leq p \leq \infty$, *and let* X *and* Y *be Banach spaces. Then*

$$\bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y) = \mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(X,Y).$$

Proof. Let us suppose

$$T \in \bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y).$$

By Proposition 2.2 we can write

$$\bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y) = \bigcap_{1 \leq \xi < \omega_1} \bigcup_{n=1}^{\infty} \mathcal{WD}_{\ell_p}^{(n, \xi)}(X, Y),$$

so that for each $1 \leq \xi < \omega_1$ we have

$$T \in \mathcal{WD}_{\ell_p}^{(n_\xi, \xi)}(X, Y)$$

for some minimal $n_\xi \in \mathbb{N}$. Then, due to Proposition 3.2 together with mini-
mality, we get $n_\xi \leq n_\zeta$ whenever $1 \leq \xi \leq \zeta < \omega_1$.

We claim that $(n_\xi)_{1 \leq \xi < \omega_1}$ is bounded. Indeed, suppose otherwise, towards
a contradiction. Then we can define a sequence $(\xi_j)_{j=1}^{\infty}$ of countable ordinals
$1 \leq \xi_j < \omega_1$, $j \in \mathbb{N}$, such that $n_{\xi_j} \to \infty$. Since

$$\zeta := \sup_{j \in \mathbb{N}} \xi_j$$

is the least upper bound of a countable set of countable ordinals, we must
have $\zeta < \omega_1$. Thus n_ζ exists, and due to $(n_\xi)_{1 \leq \xi < \omega_1}$ being nondecreasing in ξ
we get

$$\infty > n_\zeta \geq \sup_{j \in \mathbb{N}} n_\xi = \infty,$$

which is impossible. This proves the claim.

Hence, there exists $N \in \mathbb{N}$ such that $n_\xi \leq N$ for all $1 \leq \xi < \omega_1$. By
Proposition 2.2 we now have

$$T \in \mathcal{WD}_{\ell_p}^{(N, \xi)}(X, Y)$$

for all $1 \leq \xi < \omega_1$. Applying Proposition 3.8 it follows that

$$T \in \bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(N, \xi)}(X, Y) = \mathcal{WD}_{\ell_p}^{(N, \omega_1)}(X, Y) \subseteq \mathcal{WD}_{\ell_p}^{(\infty, \omega_1)}(X, Y)$$

and hence

$$\bigcap_{1 \leq \xi < \omega_1} \mathcal{WD}_{\ell_p}^{(\infty, \xi)}(X, Y) \subseteq \mathcal{WD}_{\ell_p}^{(\infty, \omega_1)}(X, Y).$$

The reverse inequality is immediate from Proposition 3.2. ∎

Now we are ready to prove what we had originally set out to do in this section.

PROPOSITION 3.10. *Fix a number $1 < p \leq \infty$. There exists a strictly increasing sequence (ξ_n) of countable ordinals $1 \leq \xi_n < \xi_{n+1} < \omega_1$, $n \in \mathbb{N}$, and a sequence (X_n) of Banach spaces, such that for all $m, n \in \mathbb{N}$ with $m < n$ we have*

$$\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_n)}(X_m) \subsetneqq \overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_m)}(X_m).$$

Proof. We will define $(\xi_n)_{n=1}^\infty$ and $(X_n)_{n=1}^\infty$ inductively, such that for each $n \in \mathbb{N}$ we have

$$\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_{n+1})}(X_n) \subsetneqq \overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_n)}(X_n). \tag{3.1}$$

By Proposition 3.2, this will be sufficient to complete the proof.

We begin by setting $\xi_1 := 1$. Now suppose we have defined ξ_n for some $n \in \mathbb{N}$. By Example 3.6, there exists a Banach space $X_n = T_q[\frac{1}{2}, \mathcal{S}_{\xi_n}]^*$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(X_n) \subsetneqq \mathcal{L}(X_n) = \mathcal{WD}_{\ell_p}^{(\infty,\xi_n)}(X_n).$$

Then by Propositions 3.2 and 3.9 we can find $\xi_{n+1} > \xi_n$ such that

$$\mathcal{WD}_{\ell_p}^{(\infty,\xi_{n+1})}(X_n) \subsetneqq \mathcal{L}(X_n) = \mathcal{WD}_{\ell_p}^{(\infty,\xi_n)}(X_n).$$

Recall once more that the norm-closure of a proper (algebraic) ideal in a Banach algebra is again a proper ideal (cf., e.g., Corollary VII.2.4 in [8]). Hence, the relation (3.1) holds for this n. ∎

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