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Virtually $(r; r_1, \dots, r_n; s)$ -nuclear multilinear operators

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Abstract: In this paper, the space of virtually $(r; r_1, \dots, r_n; s)$ -nuclear multilinear operators between Banach spaces is introduced, some of its properties are described and its topological dual is characterized as a Banach space of multiple absolutely $(r'; r'_1, \dots, r'_n; s')$ -summing multilinear operators.

Key words: Multilinear operators, nuclear operators, summing operators.

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1. INTRODUCTION

The nuclear operators between Banach spaces appeared in [5] when the author studied an infinite dimensional extension of the Malgrange theorem on existence and approximation of solutions for convolution equations (see also [7]). The concept of nuclear multilinear operators was extended and studied in [8]. For other related results we mention [9] and [10]. Matos [9] studied virtually $(r; r_1, \dots, r_n)$ -nuclear n -linear operators from $X_1 \times \dots \times X_n$ into Y , and proved that, if the spaces X_k^* 's ($k = 1, \dots, n$) have the λ_k -bounded approximation property; then for $r, r_1, \dots, r_n \in [1, +\infty[$ the topological dual of the space of these operators, endowed with a natural linear topology, is isomorphic isometrically to the space of all absolutely (r', r'_1, \dots, r'_n) -summing operators from $X_1^* \times \dots \times X_n^*$ into Y^* with $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{r_k} + \frac{1}{r'_k} = 1$; for r, r_k and $s \in [1, +\infty]$, $k = 1, \dots, n$.

In [3] Cerna established the definition of $(r; r_1, \dots, r_n; s)$ -nuclear multilinear operators, which are the natural generalization of the concept of (r, p, s) -nuclear linear operator introduced by Lapresté [6] (see also [11]).

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Motivated by these ideas and developments, in this paper we introduce and study the virtually $(r; r_1, \dots, r_n; s)$ -nuclear n -linear operators and we will prove a relation between the topological dual of virtually $(r; r_1, \dots, r_n; s)$ -nuclear n -linear operators and the multiple $(r'; r'_1, \dots, r'_n; s')$ -summing operators [2]. As a consequence we get the same result between the topological dual of the space of $(r; r_1, \dots, r_n; s)$ -nuclear n -linear operators from $X_1 \times \dots \times X_n$ into Y [4] and to the space of all absolutely $(r', r'_1, \dots, r'_n, s')$ -summing operators from $X_1^* \times \dots \times X_n^*$ into Y^* [1], for r, r_k and $s \in [1, +\infty]$, $k = 1, \dots, n$.

The definitions and notations used in this paper are, in general, standard. Let $n \in \mathbb{N}$. As usual, an element j from \mathbb{N}^n will be represented by (j_1, \dots, j_n) with $j_k \in \mathbb{N}$ and $k = 1, \dots, n$. We also consider the finite families $(y_j)_{j \in \mathbb{N}_m^n}$ of elements of a Banach space with $\mathbb{N}_m = \{1, \dots, m\}$. If $n = 1$, we omit \mathbb{N}^n in the preceding notations. Let $X_1, \dots, X_n; Y$ be Banach spaces over \mathbb{K} (either \mathbb{C} or \mathbb{R}). The space of all continuous n -linear operators $T : X_1 \times \dots \times X_n \rightarrow Y$ will be denoted by $\mathcal{L}(X_1, \dots, X_n; Y)$. It becomes a Banach space with the natural norm

$$\|T\| = \sup_{\|x^k\| \leq 1, k=1, \dots, n} \|T(x^1, \dots, x^n)\|.$$

We recall that a n -linear mappings $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is said to be of finite type if it has a finite representation of the form

$$T = \sum_{i=1}^m \lambda_i \varphi_i^1 \times \dots \times \varphi_i^n b_i,$$

where $\lambda_i \in \mathbb{K}$, $\varphi_i^k \in X_k^*$, $k = 1, \dots, n$, $b_i \in Y$, $i = 1, \dots, m$. We denote by $\mathcal{L}_f(X_1, \dots, X_n; Y)$ the vector subspace of $\mathcal{L}(X_1, \dots, X_n; Y)$ of all n -linear mappings of finite type.

If $r \in]0, +\infty[$, we denote by $l_r(Y; \mathbb{N}^n)$ or $(l_r(\mathbb{N}^n); Y)$, if $Y = \mathbb{K}$, the vector space of all families $(y_j)_{j \in \mathbb{N}^n}$ of elements of Y such that

$$\|(y_j)_{j \in \mathbb{N}^n}\|_r = \left(\sum_{j \in \mathbb{N}^n} \|y_j\|_Y^r \right)^{\frac{1}{r}} < \infty.$$

We observe that $\|\cdot\|_r$ is a norm (r -norm, if $r < 1$) on $l_r(Y; \mathbb{N}^n)$ and defines a complete metrizable linear topology on it. We denote by $l_\infty(Y; \mathbb{N}^n)$ (or $l_\infty(\mathbb{N}^n)$, if $Y = \mathbb{K}$) the Banach space of all bounded families $(y_j)_{j \in \mathbb{N}^n}$ of

elements of Y , with the norm

$$\left\| (y_j)_{j \in \mathbb{N}^n} \right\|_\infty = \sup_{j \in \mathbb{N}^n} \|y_j\|.$$

The Banach subspace of all families $(y_j)_{j \in \mathbb{N}^n}$ such that

$$\lim_{j_k \rightarrow +\infty, k=1, \dots, n} \|y_j\| = 0$$

is denoted by $c_0(Y; \mathbb{N}^n)$ (or $c_0(\mathbb{N}^n)$, if $Y = \mathbb{K}$).

If $0 < s \leq \infty$, we will write $l_r^w(Y; \mathbb{N}^n)$ (or $l_r^w(\mathbb{N}^n)$, if $Y = \mathbb{K}$) for the vector space of all families $(y_j)_{j \in \mathbb{N}^n}$ of elements of Y such that

$$\left\| (y_j)_{j \in \mathbb{N}^n} \right\|_{w,s} := \sup_{\|\psi\|_{Y^*} \leq 1} \left(\sum_{j \in \mathbb{N}^n} |\psi(y_j)|^s \right)^{\frac{1}{s}} = \sup_{\|\psi\|_{Y^*} \leq 1} \left\| (\psi(y_j))_{j \in \mathbb{N}^n} \right\|_s < \infty,$$

where Y^* denotes the topological dual of Y .

It is well-known that for $1 \leq s < \infty$ and $(\varphi_j)_{j \in \mathbb{N}^n} \in l_s^w(Y^*; \mathbb{N}_m^n)$, we have

$$\left\| (\varphi_j)_{j \in \mathbb{N}^n} \right\|_{w,s} = \sup_{\phi \in B_{Y^{**}}} \left(\sum_{j \in \mathbb{N}^n} |\phi(\varphi_j)|^s \right)^{\frac{1}{s}} = \sup_{y \in B_Y} \left\| (\varphi_j(y))_{j \in \mathbb{N}^n} \right\|_s.$$

Let $0 < r, 1 < p, s \leq \infty$ such that

$$\frac{1}{t} = \frac{1}{r} + \frac{1}{p} + \frac{1}{s}, \text{ with } t \in]0, 1].$$

An operator $T \in \mathcal{L}(X; Y)$ is $(r; p; s)$ -nuclear (see, e.g., [6, 11]) if it has a representation of the form

$$T = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i, \tag{1}$$

with $(\lambda_i)_i \in l_r$, if $r < \infty$ (or $(\lambda_i)_i \in c_0$, if $r = +\infty$), $(x_i)_i \in l_p^w(X^*)$ and $(y_i)_i \in l_s^w(Y)$. The vector space of all such operators is denoted by $\mathcal{N}_{(r;p;s)}(X; Y)$ and it is a complete metrizable topological vector space under the t -norm

$$\mu_{(r;p;s)}(T) = \inf \left\{ \left\| (\lambda_i)_i \right\|_r \left\| (x_i)_i \right\|_{w,p} \left\| (y_i)_i \right\|_{w,s} \right\}$$

where the infimum is taken over all representations of T as in (1).

The definition of the virtually $(r; r_1, \dots, r_n)$ -nuclear operators below was first given in [9].

We consider $r \in]0, +\infty]$, $r_k \in [1, +\infty]$, such that $r \leq r_k$, $k = 1, \dots, n$ and

$$1 \leq \frac{1}{t_n} = \frac{1}{r} + \frac{1}{r'_1} + \dots + \frac{1}{r'_n}.$$

DEFINITION 1.1. An operator $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is said to be virtually $(r; r_1, \dots, r_n)$ -nuclear if there is a representation of the form

$$T = \sum_{j \in \mathbb{N}^n} \lambda_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n b_j \tag{2}$$

with $(\lambda_j)_{j \in \mathbb{N}^n} \in l_r(\mathbb{N}^n)$, if $r < \infty$ (or $(\lambda_j)_{j \in \mathbb{N}^n} \in c_0(\mathbb{N}^n)$, if $r = +\infty$), $(\phi_i^k)_{i=1}^\infty \in l_{r'_k}^w(X_k^*)$, for $k = 1, \dots, n$ and $(b_j)_{j \in \mathbb{N}^n} \in l_\infty(Y; \mathbb{N}^n)$.

The vector space of these operators is denoted by $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n)}(X_1, \dots, X_n; Y)$ and we consider on it the t_n -norm

$$\|T\|_{VN, (r; r_1, \dots, r_n)} = \inf \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_r \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_\infty \prod_{k=1}^n \left\| (\phi_i^k)_{i=1}^\infty \right\|_{w, r'_k},$$

where the infimum is taken over all representations of T as in (2).

The notion of absolutely $(r; r_1, \dots, r_n; s)$ -summing multilinear operators was introduced by the first author in [1].

DEFINITION 1.2. For $0 < r, r_1, \dots, r_n < \infty$ and $0 < s \leq \infty$ with $\frac{1}{r} \leq \frac{1}{r_1} + \dots + \frac{1}{r_n} + \frac{1}{s}$, an n -linear operator $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is $(r; r_1, \dots, r_n; s)$ -summing if there is a constant $C > 0$ such that for any $x_1^k, \dots, x_m^k \in X_k$, ($1 \leq k \leq n$), and any $\varphi_1, \dots, \varphi_m \in Y^*$, we have

$$\left(\sum_{i=1}^m \left| \varphi_i(T(x_i^1, \dots, x_i^n)) \right|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^n \left\| (x_i^k)_{i=1}^m \right\|_{w, r_k} \left\| (\varphi_i)_{i=1}^m \right\|_{w, s}.$$

We denote the vector space of these operators by $\mathcal{L}_{as, (r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$ and the smallest C satisfying the above inequality by $\pi_{(r; r_1, \dots, r_n; s)}^n(T)$ which defines a norm (r -norm if $r < 1$) on $\mathcal{L}_{as, (r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$.

The following multilinear generalization of $(r; r_1, \dots, r_n; s)$ -summing operators was recently introduced by Bernardino et al. in [2].

DEFINITION 1.3. Let $n \in \mathbb{N}$, $r, s, r_1, \dots, r_n \geq 1$ and X_1, \dots, X_n, Y be Banach spaces. A continuous multilinear operator $T : X_1 \times \dots \times X_n \rightarrow Y$ is multiple $(r; r_1, \dots, r_n; s)$ -summing if there is a $C > 0$ such that

$$\left(\sum_{j \in \mathbb{N}_m^n} \left| \varphi_j (T(x_{j_1}^1, \dots, x_{j_n}^n)) \right|^r \right)^{\frac{1}{r}} \leq C \left\| (\varphi_j)_{j \in \mathbb{N}_m^n} \right\|_{w,s} \prod_{k=1}^n \left\| (x_i^k)_{i=1}^m \right\|_{w,r_k}$$

where $\frac{1}{r} \leq \frac{1}{r_1} + \dots + \frac{1}{r_n} + \frac{1}{s}$, $x_1^k, \dots, x_m^k \in X_k$, $k = 1, \dots, n$ and $(\varphi_j)_{j \in \mathbb{N}_m^n} \in l_s^w(Y^*; \mathbb{N}_m^n)$.

We denote by $\mathcal{L}_{mas}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$ the vector space of these operators. The smallest C satisfying the above inequality defines a norm (r -norm if $r < 1$) on $\mathcal{L}_{mas}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$; it is denoted by $\|T\|_{mas(r; r_1, \dots, r_n; s)}$.

Remark 1.4. By choosing $(s = \infty)$ in Definition 1.3, we obtain the definition of fully (or multiple) $(r; r_1, \dots, r_n)$ -summing n -linear operators presented in [9].

We also need the definition of the $(r; r_1, \dots, r_n; s)$ -nuclear n -linear operators. The ideal of $(r; r_1, \dots, r_n; s)$ -nuclear operators was introduced by Cerna [3] (see also [4]).

DEFINITION 1.5. For $0 < r \leq \infty$, $1 \leq s, r_1, \dots, r_n \leq \infty$, such that $1 \leq \frac{1}{r} + \frac{1}{r_1'} + \dots + \frac{1}{r_n'} + \frac{1}{s'}$, $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is called $(r; r_1, \dots, r_n; s)$ -nuclear if it has the form

$$T = \sum_{i=1}^{+\infty} \lambda_i \phi_i^1 \times \dots \times \phi_i^n b_i, \tag{3}$$

with $(\lambda_i)_{i \in \mathbb{N}} \in l_r(\mathbb{N})$, if $r < \infty$ (or $(\lambda_i)_{i \in \mathbb{N}} \in c_0(\mathbb{N})$, if $r = +\infty$), $(\phi_i^k)_{i \in \mathbb{N}} \in l_{r_k'}^w(X_k^*)$ for $k = 1, \dots, n$ and $(b_i)_{i \in \mathbb{N}} \in l_{s'}^w(Y)$. The set of $(r; r_1, \dots, r_n; s)$ -nuclear operators satisfying the definition is a vector space and is denoted by $\mathcal{N}_{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$. Considering that

$$N_{(r; r_1, \dots, r_n; s)}(T) = \inf \left\| (\lambda_i)_{i \in \mathbb{N}} \right\|_r \left\| (b_i)_{i \in \mathbb{N}} \right\|_{w,s'} \prod_{k=1}^n \left\| (\phi_i^k)_{i \in \mathbb{N}} \right\|_{w,r_k'}$$

where the infimum is taken over all possible representations of T described in (3), we obtain a t -norm with

$$\frac{1}{t} = \frac{1}{r} + \frac{1}{r_1'} + \dots + \frac{1}{r_n'} + \frac{1}{s'}.$$

2. VIRTUALLY $(r; r_1, \dots, r_n; s)$ -NUCLEAR n -LINEAR OPERATORS

We consider $r \in]0, +\infty]$, $s, r_k \in [1, +\infty]$, $k = 1, \dots, n$, such that $1 \leq \frac{1}{t_n} = \frac{1}{r} + \frac{1}{r'_1} + \dots + \frac{1}{r'_n} + \frac{1}{s'}$.

DEFINITION 2.1. An operator $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is said to be virtually $(r; r_1, \dots, r_n; s)$ -nuclear if there are $(\lambda_j)_{j \in \mathbb{N}^n} \in l_r(\mathbb{N}^n)$, if $r < \infty$ (or $(\lambda_j)_{j \in \mathbb{N}^n} \in c_0(\mathbb{N}^n)$, if $r = +\infty$), $(\phi_i^k)_{i=1}^\infty \in l_{r'_k}^w(X_k^*)$, for $k = 1, \dots, n$ and $(b_j)_{j \in \mathbb{N}^n} \in l_{s'}^w(Y; \mathbb{N}^n)$ such that

$$T = \sum_{j \in \mathbb{N}^n} \lambda_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n b_j. \tag{4}$$

We denote the vector space of all such operators by $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$, with the t_n -norm

$$\|T\|_{VN, (r; r_1, \dots, r_n; s)} = \inf \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_r \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w, s'} \prod_{k=1}^n \left\| (\phi_i^k)_{i=1}^\infty \right\|_{w, r'_k},$$

where the infimum is taken over all representations of T as in (4). This t_n -normed space is a complete metrizable topological vector space.

Remarks 2.2. (a) By choosing $s' = \infty$ in Definition 2.1, we obtain virtually $(r; r_1, \dots, r_n)$ -nuclear n -linear operators presented in Definition 1.1.

(b) We have $\mathcal{N}_{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y) \subset \mathcal{L}_{VN}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$ and

$$\|T\| \leq \|T\|_{VN, (r; r_1, \dots, r_n; s)} \leq N_{(r; r_1, \dots, r_n; s)}(T),$$

for every T is in $\mathcal{N}_{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$.

By definition every T in $\mathcal{L}_f(X_1, \dots, X_n; Y)$ has a finite representation

$$T = \sum_{j \in \mathbb{N}_m^n} \lambda_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n b_j. \tag{5}$$

It is clear that we have a t_n -norm on $\mathcal{L}_f(X_1, \dots, X_n; Y)$ defined by

$$\|T\|_{VN_f, (r; r_1, \dots, r_n; s)} = \inf \left\| (\lambda_j)_{j \in \mathbb{N}_m^n} \right\|_r \left\| (b_j)_{j \in \mathbb{N}_m^n} \right\|_{w, s'} \prod_{k=1}^n \left\| (\phi_i^k)_{i=1}^m \right\|_{w, r'_k},$$

where the infimum is taken over all finite representations of T as in (5).

The next result collects some elementary facts about virtually $(r; r_1, \dots, r_n; s)$ -nuclear n -linear operators.

PROPOSITION 2.3. (i) The vector space $\mathcal{L}_f(X_1, \dots, X_n; Y)$ of the continuous n -linear operators of finite type is dense in $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$.

(ii) Ideal property: If E_1, \dots, E_n , and Y_0 are Banach spaces and $T \in \mathcal{L}(X_1, \dots, X_n; Y)$, $S_k \in \mathcal{L}(E_k, X_k)$, $k = 1, \dots, n$, and $R \in \mathcal{L}(Y, Y_0)$ with T virtually $(r; r_1, \dots, r_n; s)$ -nuclear, then $R \circ T \circ (S_1, \dots, S_n)$ is virtually $(r; r_1, \dots, r_n; s)$ -nuclear and

$$\|R \circ T \circ (S_1, \dots, S_n)\|_{VN, (r; r_1, \dots, r_n; s)} \leq \|R\| \|T\|_{VN, (r; r_1, \dots, r_n; s)} \prod_{k=1}^n \|S_k\|.$$

(iii) $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is virtually $(r; r_1, \dots, r_n; s)$ -nuclear if and only there are bounded linear operators $A_k \in \mathcal{L}(X_k; l_{r'_k})$, $k = 1, \dots, n$, $B \in \mathcal{L}(l_1(\mathbb{N}^n); Y)$ and $(\lambda_j)_{j \in \mathbb{N}^n} \in l_r(\mathbb{N}^n)$, if $r < \infty$ (or $(\lambda_j)_{j \in \mathbb{N}^n} \in c_0(\mathbb{N}^n)$, if $r = +\infty$), such that

$$T = B \circ \mathcal{D}_{(\lambda_j)_{j \in \mathbb{N}^n}} \circ (A_1, \dots, A_n),$$

where $\mathcal{D}_{(\lambda_j)_{j \in \mathbb{N}^n}} : l_{r'_1} \times \dots \times l_{r'_n} \rightarrow l_1(\mathbb{N}^n)$ defined by $\mathcal{D}_{(\lambda_j)_{j \in \mathbb{N}^n}}((\xi_{j_1}^1)_{j_1=1}^\infty, \dots, (\xi_{j_n}^n)_{j_n=1}^\infty) = (\lambda_j \xi_{j_1}^1 \dots \xi_{j_n}^n)_{j \in \mathbb{N}^n}$ for $(\xi_{j_1}^1)_{j_1=1}^\infty \in l_{r'_1}$, is a virtually $(r; r_1, \dots, r_n; s)$ -nuclear with

$$\|\mathcal{D}_{(\lambda_j)_{j \in \mathbb{N}^n}}\|_{VN, (r; r_1, \dots, r_n; s)} = \|(\lambda_j)_{j \in \mathbb{N}^n}\|_r.$$

In this case

$$\|T\|_{VN, (r; r_1, \dots, r_n; s)} = \inf \|B\| \|(\lambda_j)_{j \in \mathbb{N}^n}\|_r \prod_{k=1}^n \|A_k\|,$$

where the infimum is taken over all such factorizations.

3. DUALITY

The natural question is to find out when we have

$$\|T\|_{VN, (r; r_1, \dots, r_n; s)} = \|T\|_{VN_f, (r; r_1, \dots, r_n; s)},$$

for each $T \in \mathcal{L}_f(X_1, \dots, X_n; Y)$.

Of course we have

$$\|T\|_{VN,(r;r_1,\dots,r_n;s)} \leq \|T\|_{VN_f,(r;r_1,\dots,r_n;s)}.$$

Below we will see that the reverse implication holds to be true for some certain Banach spaces X_k 's ($k = 1, \dots, n$). We start with finite dimensional spaces X_k 's. The following theorem can be proved as in [9, Proposition 4.6].

THEOREM 3.1. *If the spaces X_k ($k = 1, \dots, n$) are finite dimensional vector spaces, then*

$$\|T\|_{VN_f,(r;r_1,\dots,r_n;s)} \leq \|T\|_{VN,(r;r_1,\dots,r_n;s)},$$

for every $T \in \mathcal{L}_f(X_1, \dots, X_n; Y)$.

As in [9, Proposition 4.8], we get the following, which extends Theorem 3.1 to infinite dimensional Banach spaces with the λ -bounded approximation property (λ -BAP, for short).

PROPOSITION 3.2. *If the spaces X_k^* 's ($k = 1, \dots, n$) have the λ_k -BAP, then*

$$\|T\|_{VN,(r;r_1,\dots,r_n;s)} \geq \|T\|_{VN_f,(r;r_1,\dots,r_n;s)},$$

for all $T \in \mathcal{L}_f(X_1, \dots, X_n; Y)$.

Proof. We consider $T_k \in \mathcal{L}(X_k; \mathcal{L}(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n; Y))$, defined by

$$T_k(x^k)(x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^n) = T(x^1, \dots, x^{k-1}, x^k, x^{k+1}, \dots, x^n),$$

for $x^k \in X_k$, $k = 1, \dots, n$.

Since X_k^* has the λ_k -bounded approximation property for some $\lambda_k > 0$, given $\epsilon > 0$, we can find $S_k \in \mathcal{L}_f(D_k, X_k)$, such that $T_k = T_k \circ S_k$ and $\|S_k\| \leq (1 + \epsilon)\lambda_k$. Hence, for all $x^k \in X_k$, for $k = 1, \dots, n$, we have

$$T(x^1, \dots, x^{k-1}, S_k(x^k), x^{k+1}, \dots, x^n) = T(x^1, \dots, x^{k-1}, x^k, x^{k+1}, \dots, x^n).$$

Now, we can write

$$T(x^1, \dots, x^n) = T \circ (S_1, \dots, S_n)(x^1, \dots, x^n), \quad \forall x^k \in X_k, \quad k = 1, \dots, n.$$

If J_k denotes the natural injection from $S_k(D_k)$ into X_k , we can write $S_k = J_k \circ \tilde{S}_k$ ($\tilde{S}_k \in \mathcal{L}_f(D_k, S_k(D_k))$), with $\|\tilde{S}_k\| = \|S_k\|$. Therefore we can

say that $T \circ (J_1, \dots, J_n) \in \mathcal{L}_f((S_1(D_1), \dots, S_n(D_n)); Y)$. By Theorem 3.1 and Proposition 2.3 (ii) we have

$$\begin{aligned} \|T\|_{VN_f, (r; r_1, \dots, r_n; s)} &= \|T \circ (S_1, \dots, S_n)\|_{VN_f, (r; r_1, \dots, r_n; s)} \\ &\leq \|T\|_{VN, (r; r_1, \dots, r_n; s)} \prod_{k=1}^n \|S_k\| \\ &\leq \|T\|_{VN, (r; r_1, \dots, r_n; s)} (1 + \epsilon)^n \prod_{k=1}^n \lambda_k. \end{aligned}$$

This implies that

$$\|T\|_{VN_f, (r; r_1, \dots, r_n; s)} \leq \left(\prod_{k=1}^n \lambda_k \right) \|T\|_{VN, (r; r_1, \dots, r_n; s)}.$$

For each $\epsilon > 0$, we choose a representation

$$T = \sum_{j \in \mathbb{N}^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j$$

such that

$$\left\| (\sigma_j)_{j \in \mathbb{N}^n} \right\|_r \left\| (y_j)_{j \in \mathbb{N}^n} \right\|_{w, s'} \prod_{k=1}^n \left\| (\phi_i^k)_{i=1}^\infty \right\|_{w, r'_k} \leq (1 + \epsilon) \|T\|_{VN, (r; r_1, \dots, r_n; s)}.$$

We can find $m \in \mathbb{N}$ such that

$$\left(\prod_{k=1}^n \lambda_k \right) \left\| \sum_{j \in \mathbb{N}^n / \mathbb{N}_m^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j \right\|_{VN_f, (r; r_1, \dots, r_n; s)} \leq \epsilon \|T\|_{VN, (r; r_1, \dots, r_n; s)}.$$

We use the triangular inequality for t_n -norms in order to write

$$\begin{aligned} \left(\|T\|_{VN_f, (r; r_1, \dots, r_n; s)} \right)^{t_n} &\leq \left(\left\| \sum_{j \in \mathbb{N}_m^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j \right\|_{VN_f, (r; r_1, \dots, r_n; s)} \right)^{t_n} \\ &\quad + \left(\left\| \sum_{j \in \mathbb{N}^n / \mathbb{N}_m^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j \right\|_{VN_f, (r; r_1, \dots, r_n; s)} \right)^{t_n} \end{aligned}$$

$$\begin{aligned} &\leq (1 + \epsilon)^{t_n} \left(\|T\|_{VN,(r;r_1,\dots,r_n;s)} \right)^{t_n} \\ &\quad + \left(\prod_{k=1}^n \lambda_k \right)^{t_n} \left(\left\| \sum_{j \in \mathbb{N}^n / \mathbb{N}_m^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j \right\|_{VN,(r;r_1,\dots,r_n;s)} \right)^{t_n} \\ &\leq [(1 + \epsilon)^{t_n} + \epsilon^{t_n}] \left(\|T\|_{VN,(r;r_1,\dots,r_n;s)} \right)^{t_n}. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary we have

$$\|T\|_{VN_f,(r;r_1,\dots,r_n;s)} \leq \|T\|_{VN,(r;r_1,\dots,r_n;s)},$$

and this proves the theorem. \blacksquare

For Banach spaces with λ -bounded approximation property, Proposition 3.2 can be seen as a generalization of a result obtained by B. Cerna [4, Lemma 2.1].

Now, we also give another generalization of [4, Lemma 2.1].

PROPOSITION 3.3. *Let $T : X_1 \times \dots \times X_n \longrightarrow L_s(\Omega, \mu)$ be defined by*

$$T(x^1, \dots, x^n) = \sum_{j \in \mathbb{N}_m^n} \lambda_j \phi_{j_1}^1(x^1) \dots \phi_{j_n}^n(x^n) b_j,$$

where $\frac{1}{s} = \frac{1}{r'_1} + \dots + \frac{1}{r'_n}$. Then, $\|T\|_{VN_f,(\infty;r_1,\dots,r_n;s)} = \|T\|_{VN,(\infty;r_1,\dots,r_n;s)} = \|T\|$.

Proof. It is clear that for $\frac{1}{s} = \frac{1}{r'_1} + \dots + \frac{1}{r'_n}$, we have

$$\|T\| \leq \|T\|_{VN,(\infty;r_1,\dots,r_n;s)} \leq \|T\|_{VN_f,(\infty;r_1,\dots,r_n;s)}.$$

Moreover,

$$\|T\| \|x^1\| \dots \|x^n\| \geq \left(\int_{\Omega} \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j \phi_{j_1}^1(x^1) \dots \phi_{j_n}^n(x^n) b_j(t) \right|^s d\mu(t) \right)^{1/s} \quad (6)$$

Since $\phi_{j_i}^i$ is surjective there exists $\xi_i \in X_i$ such that $\phi_{j_i}^i(\xi_i) = M_i/2^{j_i/r'_i}$, where

$$M_i = \sup_{\|x^i\|_{X_i} \leq 1} \left(\sum_{j_i=1}^m |\langle \phi_{j_i}^i, x^i \rangle|^{r'_i} \right)^{1/r'_i},$$

We will show that $\|\xi_i\| \leq 1$ and $M_i < +\infty$ for $i = 1, \dots, n$. From the definition of M_i for a fixed i and for $\epsilon > 0$ we have

$$M_i \|\xi_i\| < (1 + \epsilon) \left(\sum_{j_i=1}^m M_i^{r'_i} / 2^{j_i} \right)^{1/r'_i},$$

which implies that

$$\|\xi_i\| < (1 + \epsilon), \text{ for all } \epsilon > 0.$$

So, considering $\|\xi_i\| < 1$ in equation (6) we have

$$\|T\| \geq \left(\int_{\Omega} \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j M_1 / 2^{j_1/r'_1} \dots M_n / 2^{j_n/r'_n} b_j(t) \right|^s d\mu(t) \right)^{1/s},$$

if $k = \max \{j_1, \dots, j_n\}$ we get

$$\|T\| \geq \left(\int_{\Omega} \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j \frac{b_j(t)}{2^{k/s}} \right|^s d\mu(t) \right)^{1/s} \prod_{i=1}^n M_i. \tag{7}$$

Let $z(t) = \sum_{j \in \mathbb{N}_m^n} \lambda_j \frac{b_j(t)}{2^{k/s}}$, then for all $s \geq 1$ we have

$$|\langle \varphi, z \rangle| = \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j \left\langle \varphi, \frac{b_j}{2^{k/s}} \right\rangle \right| \leq \|\varphi\| \|z\|. \tag{8}$$

By renumbering multi-finite indices $j \in \mathbb{N}_m^n$, we can rewrite this finite sum as

$$z(t) = \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}}.$$

In addition, let $M = span_{k \in \{1, \dots, f(m,n)\} - k_0} \left\{ \frac{b_k}{2^{k/s}} \right\}$ where k_0 is a fixed number belongs to $\{1, \dots, f(m,n)\}$, and $f(m,n) \in \mathbb{N}$. Moreover, as a consequence of the Hahn-Banach theorem there exists φ such that $\|\varphi\| = \frac{1}{d}$, $\langle \varphi, x \rangle = 0$ for all $x \in M$ and $\left\langle \varphi, \frac{b_{k_0}}{2^{k_0/s}} \right\rangle = 1$, where $d = \inf_{x \in M} \left\| x - \frac{b_{k_0}}{2^{k_0/s}} \right\|$ and further one can choose λ_{k_0} such that

$$|\lambda_{k_0}| = \max_{k=1, \dots, f(m,n)} |\lambda_k| = \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty; \text{ where } j = (j_1, \dots, j_n).$$

Taking into account these last relations in equation (8) we can get,

$$\|z\| \geq |\lambda_{k_0}| d. \tag{9}$$

Since $x = \sum_{k=1, k \neq k_0}^{f(m,n)} \frac{-b_k}{2^{k/s}} \in M$, then for a given $\epsilon > 0$, we have

$$(1 + \epsilon) d > \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\|.$$

Therefore, from (9) we get

$$(1 + \epsilon) \|z\| > \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\|. \tag{10}$$

We know that

$$\left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'} = \sup_{\|\psi\|_s \leq 1} \left(\sum_{j \in \mathbb{N}^n} |\psi(b_j)|^{s'} \right)^{1/s'} = \sup_{a \in B_{l_s}^{f(m,n)}} \left\| \sum_{k=1}^{f(m,n)} a_k b_k \right\|,$$

and since $a_k = \frac{1}{2^{k/s}}$ for $k = 1, \dots, f(m, n)$, given $\tilde{\epsilon} > 0$ we have

$$(1 + \tilde{\epsilon}) \left\| \sum_{k=1}^{f(m,n)} \frac{b_k}{2^{k/s}} \right\| \geq \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'}.$$

From the last relation and the equation (10) we obtain

$$(1 + \epsilon) (1 + \tilde{\epsilon}) \|z\| > \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'} \text{ for all } \epsilon \text{ and } \tilde{\epsilon} > 0. \tag{11}$$

Therefore, from the relations (7) and (11) we get

$$\begin{aligned} \|T\| &\geq \left\| (\lambda_j)_{j \in \mathbb{N}^n} \right\|_\infty \left\| (b_j)_{j \in \mathbb{N}^n} \right\|_{w,s'} \prod_{i=1}^n M_i \\ &\geq \|T\|_{VN_f, (\infty; r_1, \dots, r_n; s)}. \end{aligned}$$

■

We will prove a new link between the topological dual of virtually $(r; r_1, \dots, r_n; s)$ -nuclear n -linear operators and multiple $(r'; r'_1, \dots, r'_n; s')$ -summing operators. The proof of the next theorem is similar to the proof of Theorem 7.3.1 in [10]. We included the detailed proof here for completeness.

THEOREM 3.4. *If the spaces X_k^* 's ($k = 1, \dots, n$) have the λ_k -BAP, then the topological dual of $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$ is isomorphic isometrically to $\mathcal{L}_{mas}^{(r'; r'_1, \dots, r'_n; s')}(X_1^*, \dots, X_n^*; Y^*)$, for $r, r_k \in [1, +\infty[$, $k = 1, \dots, n$ through the mapping \mathcal{B} define by*

$$\mathcal{B}(\Psi)(\phi^1, \dots, \phi^n)(b) = \Psi(\phi^1 \times \dots \times \phi^n b),$$

for all $b \in Y$, $\phi^k \in X_k^*$, $k = 1, \dots, n$ and $\Psi \in (\mathcal{L}_{VN}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y))^*$.

Proof. It is easy to see that the correspondence

$$\Psi \in (\mathcal{L}_{VN}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y))^* \longrightarrow \mathcal{B}(\Psi) \in \mathcal{L}_{mas}^{(r'; r'_1, \dots, r'_n; s')}(X_1^*, \dots, X_n^*; Y^*)$$

defined by

$$\mathcal{B}(\Psi)(\phi^1, \dots, \phi^n)(b) = \Psi(\phi^1 \times \dots \times \phi^n b), \quad \phi^k \in X_k^*, \quad k = 1, \dots, n \text{ and } b \in Y,$$

is linear and injective. To show the surjectivity let $T \in \mathcal{L}_{mas}^{(r'; r'_1, \dots, r'_n; s')}(X_1^*, \dots, X_n^*; Y^*)$ and consider the linear functional Ψ_T on the space $(\mathcal{L}_f(X_1, \dots, X_n; Y), \|\cdot\|_{VN_f, (r; r_1, \dots, r_n; s)})$ given by

$$\Psi_T(S) = \sum_{j \in \mathbb{N}_m^n} \lambda_j T(\varphi_{j_1}^1, \dots, \varphi_{j_n}^n)(b_j)$$

for every $S \in \mathcal{L}_f(X_1, \dots, X_n; Y)$ with a finite representation of the form

$$S = \sum_{j \in \mathbb{N}_m^n} \lambda_j \varphi_{j_1}^1 \times \dots \times \varphi_{j_n}^n b_j.$$

Hence, by Hölder's inequality and Definition 1.3 it follows that

$$\begin{aligned} |\Psi_T(S)| &\leq \left\| (\lambda_j)_{j \in \mathbb{N}_m^n} \right\|_r \left\| (T(\varphi_{j_1}^1, \dots, \varphi_{j_n}^n)(b_j))_{j \in \mathbb{N}_m^n} \right\|_{r'} \\ &\leq \|T\|_{mas(r'; r'_1, \dots, r'_n; s')} \left\| (\lambda_j)_{j \in \mathbb{N}_m^n} \right\|_r \prod_{k=1}^n \left\| (\varphi_i^k)_{i=1}^m \right\|_{w, r'_k} \left\| (b_j)_{j \in \mathbb{N}_m^n} \right\|_{w, s'}. \end{aligned}$$

This shows that

$$|\Psi_T(S)| \leq \|T\|_{\text{mas}(r';r'_1, \dots, r'_n; s')} \|S\|_{V_{N_f}(r; r_1, \dots, r_n; s)},$$

for all $S \in \mathcal{L}_f(X_1, \dots, X_n; Y)$.

Since on $\mathcal{L}_f(X_1, \dots, X_n; Y)$, under our hypothesis for X_1, \dots, X_n , we have

$$\|\cdot\|_{V_{N_f}(r; r_1, \dots, r_n; s)} = \|\cdot\|_{V_{N}(r; r_1, \dots, r_n; s)},$$

we conclude that Ψ_T is continuous on $\mathcal{L}_f(X_1, \dots, X_n; Y)$ for $\|\cdot\|_{V_{N}(r; r_1, \dots, r_n; s)}$ and

$$\|\Psi_T\| \leq \|T\|_{\text{mas}(r';r'_1, \dots, r'_n; s')}.$$

By Proposition 2.3 (i), $\mathcal{L}_f(X_1, \dots, X_n; Y)$ is dense in $\mathcal{L}_{V_N}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$. Hence we can extend Ψ_T to a continuous functional $\tilde{\Psi}_T$ on $\mathcal{L}_{V_N}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y)$ in a unique way, with

$$\|\tilde{\Psi}_T\| \leq \|T\|_{\text{mas}(r';r'_1, \dots, r'_n; s')}.$$

Finally we note that $\mathcal{B}(\tilde{\Psi}_T) = T$.

To show the reverse inequality let $\Psi \in (\mathcal{L}_{V_N}^{(r; r_1, \dots, r_n; s)}(X_1, \dots, X_n; Y))^*$ and consider the corresponding n -linear mapping $\mathcal{B}(\Psi) \in \mathcal{L}(X_1^*, \dots, X_n^*; Y^*)$, defined by $\mathcal{B}(\Psi)(\phi^1, \dots, \phi^n)(b) = \Psi(\phi^1 \times \dots \times \phi^n b)$, for $\phi^k \in X_k^*$, $k = 1, \dots, n$ and $b \in Y$. Let us consider $n \in \mathbb{N}$ and $\varphi_{j_k}^k \in X_k^*$, for $k = 1, \dots, n$, and $(b_j)_{j \in \mathbb{N}_m^n} \in l_{s'}^w(Y; \mathbb{N}_m^n)$. There is $(\lambda_j)_{j \in \mathbb{N}_m^n} \in l_r(\mathbb{N}_m^n)$ such that $\left\| (\lambda_j)_{j \in \mathbb{N}_m^n} \right\|_r = 1$ and

$$\left\| \left(\mathcal{B}(\Psi)(\varphi_{j_1}^1, \dots, \varphi_{j_n}^n)(b_j) \right)_{j \in \mathbb{N}_m^n} \right\|_{r'} = \sum_{j \in \mathbb{N}_m^n} \lambda_j \left| \mathcal{B}(\Psi)(\varphi_{j_1}^1, \dots, \varphi_{j_n}^n)(b_j) \right|$$

Now we can choose α_j , $|\alpha_j| = 1$, $j \in \mathbb{N}_m^n$ such that

$$\begin{aligned} \sum_{j \in \mathbb{N}_m^n} \lambda_j \left| \mathcal{B}(\Psi)(\varphi_{j_1}^1, \dots, \varphi_{j_n}^n)(b_j) \right| &= \sum_{j \in \mathbb{N}_m^n} \lambda_j \alpha_j \mathcal{B}(\Psi)(\varphi_{j_1}^1, \dots, \varphi_{j_n}^n)(b_j) \\ &= \Psi \left(\sum_{j \in \mathbb{N}_m^n} \lambda_j \alpha_j \varphi_{j_1}^1 \times \dots \times \varphi_{j_n}^n b_j \right) = (*). \end{aligned}$$

By the continuity of Ψ and the Hölder's inequality we have

$$\begin{aligned} (*) &\leq \|\Psi\| \left\| (\lambda_{j_k} \alpha_{j_k})_{j_k \in \mathbb{N}_m} \right\|_r \prod_{k=1}^n \left\| (\varphi_{j_k}^k)_{j \in \mathbb{N}_m^n} \right\|_{w, r'_k} \left\| (b_j)_{j \in \mathbb{N}_m^n} \right\|_{w, s'} \\ &= \|\Psi\| \prod_{k=1}^n \left\| (\varphi_{j_k}^k)_{j \in \mathbb{N}_m^n} \right\|_{w, r'_k} \left\| (b_j)_{j \in \mathbb{N}_m^n} \right\|_{w, s'}. \end{aligned}$$

This shows that $\mathcal{B}(\Psi) \in \mathcal{L}_{mas}^{(r'; r'_1, \dots, r'_n; s')} (X_1^*, \dots, X_n^*; Y^*)$ and

$$\|\mathcal{B}(\Psi)\|_{mas(r'; r'_1, \dots, r'_n; s')} \leq \|\Psi\|.$$

■

If we replace \mathbb{N}^n by \mathbb{N} and s' by ∞ in Theorem 3.4, we obtain the following known cases.

COROLLARY 3.5. *If the spaces X_k^* 's ($k = 1, \dots, n$) have the λ_k - bounded approximation property, then*

(i) *The topological dual of $\mathcal{N}_{(r; r_1, \dots, r_n; s)} (X_1, \dots, X_n; Y)$ is isometrically isomorphic to $\mathcal{L}_{as, (r'; r'_1, \dots, r'_n; s')} (X_1^*, \dots, X_n^*; Y^*)$, for r, r_k and $s \in [1, +\infty]$, $k = 1, \dots, n$ through the mapping $\mathcal{B}(\Psi)$ given as follows:*

$$\mathcal{B}(\Psi) (\phi^1, \dots, \phi^n) (b) := \Psi (\phi^1 \times \dots \times \phi^n b),$$

where Ψ is in the topological dual of $\mathcal{N}_{(r; r_1, \dots, r_n; s)} (X_1, \dots, X_n; Y)$, $\phi^k \in X_k^*$, $k = 1, \dots, n$ and $b \in Y$.

(ii) *The topological dual of $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n)} (X_1, \dots, X_n; Y)$ is isometrically isomorphic to $\mathcal{L}_{mas}^{(r'; r'_1, \dots, r'_n)} (X_1^*, \dots, X_n^*; Y^*)$.*

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