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Hepheastus Account on Troyanski's Polyhedral War*

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Abstract: We survey several open problems in the theory of isomorphically polyhedral Banach spaces related to Troyanski's research.

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*There shone the image of the master-mind:
There earth, there heaven, there ocean he design'd
The unwearied sun, the moon completely round
The starry lights that heaven's high convex crown'd*

This paper is based on a talk delivered by the first author in the Geometry of Banach spaces conference in honor of S. Troyanski. We have focused in aspects of his recent work on isomorphically polyhedral spaces. The introductory texts belong to Chapter XVIII of the Iliad, in the translation of Alexander Pope.

1. ISOMORPHICALLY POLYHEDRAL BANACH SPACES

*Two cities radiant on the shield appear,
The image one of peace, and one of war.
Here sacred pomp and genial feast delight,
And solemn dance, and hymeneal rite;*

A Banach space is said to be polyhedral if the unit ball of every finite dimensional subspace is the closed convex hull of a finite number of points. Polyhedrality is a geometrical notion: c_0 is polyhedral while c is not. The

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isomorphic notion associated with polyhedrality is: A Banach space is said to be *isomorphically polyhedral* if it admits a polyhedral renorming.

Probably the simplest examples of isomorphically polyhedral spaces are provided by the spaces of continuous functions on some compact ordinal space $C(\alpha)$; and its subspaces, such as Schreier space [45] or further variations of it, like spaces generated by a compact family of subsets of $\{0, 1\}^{\mathbb{N}}$ (see [2, 9]).

Fonf [22, 23, 24, 25] obtained most basic results on polyhedral spaces, among them, that isomorphically polyhedral spaces are c_0 -saturated (a result about which Diestel says [20] “its proof is too clever, by half”. The result is important, on one side because c_0 -saturated are difficult objects in Banach space theory; but also because after Fonf’s result what people wanting to construct exotic c_0 -saturated spaces actually do is to construct polyhedral spaces.

2. BOUNDARIES

*Another part (a prospect differing far)
Glow'd with refulgent arms, and horrid war.
Two mighty hosts a leaguer' d town embrace,
And one would pillage, one would burn the place*

A boundary for a Banach space X is a set \mathfrak{B} of the unit sphere of X^* with the property that, for every $x \in X$, there is $f \in \mathfrak{B}$ such that $|f(x)| = \|x\|$.

A basic corner in the theory is that polyhedral spaces have small boundaries [24, 26], although a while can be discussed about the meaning of “small”. A beautiful closing result in the theory is the one that fixes “small” as “of the same cardinal as the density character of the space”. Precisely

THEOREM. *A polyhedral Banach space admits a boundary with the same cardinality as the density character of the space.*

In particular, separable isomorphically polyhedral spaces admit a countable boundary. Actually, any Banach space admitting a countable boundary (f_n) must be (separable and) isomorphically polyhedral since the renorming

$$r(x) = \lim(1 + \varepsilon_n)|f_n(x)|$$

with $\lim \varepsilon_n = 0$ is polyhedral. There are further possibilities in this “small” regard: In [5] Arias de Reyna introduces the notion of small set in a metric

space as follows: a set A is *small* if there exists a sequence of balls $B(x_n, r_n)$ with $\lim_{n \rightarrow \infty} r_n = 0$ such that for each $m \in \mathbb{N}$, one has:

$$A \subset \bigcup_{n \geq m} B(x_n, r_n).$$

It can be proved [14]:

LEMMA 2.1. *A separable Banach space is isomorphically polyhedral if and only if it can be renormed so that it has a small boundary.*

Of course, polyhedral spaces may admit renormings with other boundaries: An example of Fonf and Veselý [30] shows a renorming of c_0 admitting an uncountable small set of extreme points. An example of Livny [41] provides a renorming of c_0 admitting a non-small set of extreme points. Also axiomatic counts: Arias de Reyna and A. Durán show in [5] that under Martin's axiom a set of cardinal \aleph_1 is small. Therefore, assuming MA, if a separable Banach space has a boundary of cardinal \aleph_1 it is isomorphically polyhedral.

Gleit and McGuigan introduce in [33] the so-called property (*): A boundary \mathfrak{B} is said to have property (*) if every weak*-accumulation point f verifies $|f(x)| < 1$ for every point $\|x\| = 1$. One has:

PROPOSITION 1. *A Banach space admitting a boundary with property (*) is polyhedral.*

Proof. Let \mathfrak{B} a boundary with property (*) for a Banach space X , and let F be a finite-dimensional subspace of X . Pick (x_n) a dense subset of the unit sphere of F and for each n some element $f_n \in \mathfrak{B}$ so that $f_n(x_n) = 1$. If the set (f_n) is infinite, it must admit a weak*-accumulation point f . Therefore some subsequence (f_m) is norm convergent on F to f . And some subsequence (x_k) of (x_m) is norm convergent to a certain $x \in F$. So $f(x) = 1$ against what property (*) says. Thus, the set (f_n) is finite. Since it is a boundary for F by compactness, X is polyhedral. ■

On the other hand, although this is not easy to prove, separable polyhedral spaces always admit a boundary with property (*): the one formed by the set of w*-strongly extreme points, namely, those norm one f for which there exists $x_0 \in X$ so that $f(x_0) = 1$ and whenever (f_n) is a sequence for which $\lim f_n(x_0) = 1$ then $\lim \|f_n - f\| = 0$. Moreover, Fonf [25] and Veselý [48] show it is a minimal boundary. What happens for nonseparable spaces is a different matter. Actually, in [27, p.452] it is explicitly posed the problem:

PROBLEM 1. *Does every polyhedral Banach space admit a boundary with property (*)?*

Let ω_1 be the first uncountable ordinal and let \aleph_1 be its cardinal. Let E be a Banach space with density character \aleph_1 . The space E can be written as $E = \bigcup_{\alpha < \omega_1} E_\alpha$ in which each E_α is separable and for $\alpha < \beta$ one has $E_\alpha \subset E_\beta$. A renorming process for the subspaces E_α will be called *continuous* if for each $x \in E$, whenever $\alpha = \lim \alpha_n$ then one has $r_\alpha(x) = \lim r_{\alpha_n}(x)$ when this makes sense. One has the following partial result:

PROPOSITION 2. *Let E be a Banach space with density character \aleph_1 . If the decomposition $E = \bigcup_{\alpha < \omega_1} E_\alpha$ admits a continuous polyhedral renorming process then E admits a polyhedral renorming the set of whose weak*-strongly extreme points has property (*).*

Proof. Let r_α be a polyhedral C -renorming of E_α having a countable boundary with property (*). Let \mathcal{U} be a free ultrafilter refining the order filter and set the C -renorming of E given by

$$r(x) = \lim_{\mathcal{U}} r_\alpha(x).$$

The continuity hypothesis and the fact that every continuous function $\omega_1 \rightarrow \mathbb{R}$ is eventually constant means that for each x one eventually has $r(x) = r_\alpha(x)$ for large α . Therefore, the same happens by compactness on any finite dimensional subspace. It is therefore obvious that r is a polyhedral renorming of E .

Let now \mathbb{S}_E be the set of all weak*-strongly exposed points of $[E, r]$. To each $f \in \mathbb{S}_E$ we can associate a norm one point $p(f) \in E$ with the property described in the definition.

LEMMA 2.2. *Let E be a Banach space and let X be a subspace. Let $f \in \mathbb{S}_E$ and assume that $p(f) \in X$. Then $f|_X \in \mathbb{S}_X$.*

Proof. Let f be a weak*-strongly extreme point of E . If $p(f) \in X$ is such that $\|p(f)\| = 1$ and $f(p(f)) = 1$ then $\|f|_X\| = 1$ and $f|_X(p(f)) = 1$. Let $f_n \in X^*$ be a sequence of functionals such that $\lim f_n(p(f)) = 1$. Take F_n some Hahn-Banach extension of f_n to E ; one has $\lim F_n(p(f)) = 1$, hence

$$0 \leq \lim \|f_n - f|_F\| \leq \lim \|F_n - f\| = 0. \quad \blacksquare$$

All points in \mathbb{S}_X are restrictions of elements of \mathbb{S}_E : indeed, the restrictions of elements of \mathbb{S}_E form a boundary for X , so it contains \mathbb{S}_E . And each $f \in \mathbb{S}_E$ such that $p(f) \in X$ is such that $f|_X$ cannot be the restriction of any other $g \in E^*$: Otherwise, since $g(p(f)) = 1$ it follows that $\|g - f\| = 0$.

Since E admits a boundary with cardinality \aleph_1 , by [25, 48], the boundary \mathbb{S}_E also has cardinality \aleph_1 , as well as the set $P = \{p(f) : f \in \mathbb{S}_E\}$. Write P as the union of a chain of countable subsets P_α -indexed by the first uncountable ordinal ω_1 - so that $P = \cup_{\alpha < \omega_1} P_\alpha$. And thus there is no loss of generality assuming that $p(f_\alpha) \in E_\alpha$.

Let now f be a weak*-accumulation point of a family $F \subset \mathbb{S}_E$ and let $x \in E$ be so that $r(x) = 1$. Again, no loss of generality assuming that $r_\alpha(x) = 1$ for all α . Pick a sequence $f_{1,n} \subset F$ such that $|f_{1,n}(x) - f(x)| < 2^{-n}$ and then E_{α_1} such that $p(f_{1,n}) \in E_{\alpha_1}$ for all n . Pick a countable subset $f_{2,n}$ of F so that $f|_{E_{\alpha_1}}$ is a weak*-accumulation point of $f_{1,n}|_{E_{\alpha_1}}$. Find E_{α_2} containing all points $p(f_{2,n})$ and iterate the construction. Set $\beta = \sup \alpha_n$ so that $E_\beta = \overline{\cup_n E_{\alpha_n}}$. If, after relabelling, we set $(f_n) = \cup_{n,k} f_{k,n}$ then $f_n|_{E_\beta}$ is a weak*-accumulation point of $f_n|_{E_\beta}$ and since $p(f_n) \in E_\beta$, $f_n|_{E_\beta} \in \mathbb{S}_{E_\beta}$ and thus, by property (*), $f(x) < 1$. ■

Which polyhedral spaces have a boundary with property (*)? In [27] two new classes have been identified. Let X be a Banach space and let M be a nonempty set; an operator $T : X \rightarrow c_0(S_{X^*} \times M)$ is said to be a Talagrand operator if, for any $x \in X$ there is a pair $(f, m) \in S_{X^*} \times M$ with $f(x) = \|x\|$ and $T(x)(f, m) \neq 0$.

THEOREM 1. [27, Prop. 7] *If a Banach space admits a Talagrand operator then it also admits, for every $\varepsilon > 0$, a $(1 + \varepsilon)$ -equivalent polyhedral renorming with property (*).*

Proof. Let X be a Banach space admitting a Talagrand operator $T : X \rightarrow c_0(S_{X^*} \times M)$. For every $f \in X^*$ put $A_f = \{T^* \delta_{(f,m)} : m \in M\}$ and then form the set $A = \cup_{\|f\|=1} A_f$. The only weak*-limit point of A is 0. Set

$$B = \bigcup_{\|f\|=1} f \pm A_f$$

and introduce in X the renorming $|x| = \sup_{g \in B} g(x)$. This is a $(1 + \varepsilon)$ -renorming of X since $(1 + \varepsilon)^{-1}|x| \leq \|x\| < |x|$. The norm $|\cdot|$ has a B as a boundary with property (*): let g be a weak*-limit point of B ; since the

only weak*-limit point of A is 0, then $\|g\|^* \leq 1$. Thus, if $|x| \leq 1$ one has $g(x) \leq \|x\| < |x| = 1$. ■

The $1+\varepsilon$ could be important in the nonseparable setting, until the nonseparable analogue for the following beautiful result of Deville, Fonf and Hajek [18] be obtained: Every norm in a separable isomorphically polyhedral space admits a $(1+\varepsilon)$ -equivalent polyhedral norm. Another interesting new polyhedral renorming appears in [27, Thm. 10]:

THEOREM 2. *If K is a countable height compact space then $C(K)$ is isomorphically polyhedral.*

Proof. Write $K = \bigcup_n (K^n \setminus K^{n+1})$ where K^n is the n^{th} -derived set of K , and fix $\varepsilon > 0$. Set $I_t = \{n \in \mathbb{N} : t \in K^n \setminus K^{n+1}\}$ then the function

$$\psi(t) = 1 + \varepsilon \sum_{n \in I_t} 2^{-n}$$

and finally a norm on $C(K)$:

$$\|x\| = \sup_{t \in K} \{\psi(t)|f(t)|\}.$$

One has $\|\cdot\|_\infty \leq \|\cdot\| \leq (1+\varepsilon)\|\cdot\|_\infty$ and the set $\mathcal{B} = \{\pm\psi(t)\delta_t : t \in K\}$ is a boundary for $\|\cdot\|$. Let us show it has property (*): If f is a weak*-accumulation point of \mathcal{B} then $f = \alpha\delta_t$ for some $t \in K$ and $\alpha \in \mathbb{R}$. Let us show that $|\alpha| < \psi(t)$, which is what one needs to get $\|f\| < 1$.

Pick $n \in \mathbb{N}$ such that $t \in K^n \setminus K^{n+1}$. Since this set is discrete, one can find an open set V so that $\overline{K^n \setminus K^{n+1}} \cap V = \{t\}$. Now, if $s \in V$ and $s \neq t$ then $n \in I_t \setminus I_s$. We set $J = I_t \cup \{i \in \mathbb{N} : i > n+1\}$ and consider the open set

$$U = K \setminus \bigcup \left\{ \overline{K^n \setminus K^{n+1}} : i \in \mathbb{N} \setminus J \right\}.$$

By the definition of J , it is clear that $t \in U$ and $I_s \subset J$ whenever $s \in U$. Moreover, the way we choose V gives that $n \in (U \cap V) \setminus \{t\}$ and we have:

$$I_s \setminus I_t \subset J \setminus I_t \subset \{i \in \mathbb{N} : i > n+1\}.$$

Thus, for such an s we obtain the following inequality

$$\psi(t) - \psi(s) = \varepsilon \sum_{i \in I_t \setminus I_s} 2^{-i} - \varepsilon \sum_{i \in I_s \setminus I_t} 2^{-i} \geq 2^{-n} - \varepsilon \sum_{i > n+1} 2^{-i} = \varepsilon 2^{-(n+1)} > 0.$$

Thus, one can find a net $\{s_\lambda\} \subset (U \cap V) \setminus \{t\}$ such that $\lim s_\lambda = t$ and $\lim \psi(s_\lambda) = |\alpha|$. From this, it follows that $\psi(t) - |\alpha| \geq \varepsilon 2^{-(n+1)} > 0$. ■

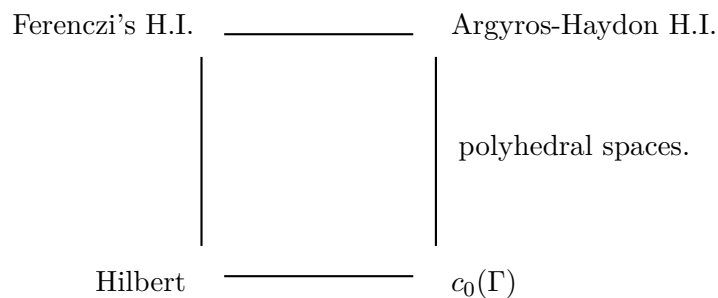
There is a connection between both results, although it is difficult to prove [27, Thm. 10]: *If K is a tree then $C(K)$ is isomorphically polyhedral if and only if it admits a Talagrand operator.* A comment in [27, p. 455] remarks that Theorems 1 and 2 have different targets:

- All $C[0, \alpha]$ spaces admit Talagrand operators $T : C[0, \alpha] \rightarrow c_0(\alpha)$ given by $Tf(\gamma) = f(\gamma) - f(\gamma + 1)$ when $\gamma < \alpha$ and $Tf(\alpha) = 0$. For α large enough they have not countable height.
- The Ciesielski-Pol space is the space of continuous functions on a compact space \mathcal{CP} having height three and with the remarkable property that no injective operator $C(\mathcal{CP}) \rightarrow c_0(\Gamma)$ exists. Since Talagrand operators are injective (remark in [27, p.455]), the space $C(\mathcal{CP})$ admits no Talagrand operator.

3. TWISTING ISOMORPHICALLY POLYHEDRAL SPACES

*A field deep furrow’d next the god design’d,
The third time labour’d by the sweating hind;
The shining shares full many ploughmen guide,
And turn their crooked yokes on every side.*

The currently known theory suggests that there are two general settings that help to twist a space: One of them is “unconditionality properties”, and the other is “the belonging to an interpolation scale”. Actually, let us represent the first setting in the x -axis and the second in the y -axis with the agreement that 0 means that the property expressed is at its best and 1 at its worst. For instance, ℓ_2 will be placed at $(0, 0)$: it has unconditional basis (unconditionality at its top) and is in the middle of all reasonable scales — Köthe-spaces, ℓ_∞ -moduli, L_∞ -moduli, $B(H)$ -moduli...— i.e., “belonging to a scale” at its top. We would get a picture such as:



Hilbert spaces, and in general Banach spaces with an unconditional basis, can be twisted, except in the case of c_0 because of Sobczyk's theorem, using the Kalton-Peck method, which means using the correspondence between twisted sums and quasi-linear maps and then a quasi-linear map of the type

$$\mathcal{K}_\phi(x) = x\phi\left(\log \frac{|x|}{\|x\|}\right)$$

where ϕ is a certain Lipschitz map. It is no strange that Kalton-Peck maps can be shown to produce nontrivial a twisting only when the space does not contain c_0 . On the other hand, Kalton-Peck maps on ℓ_p -spaces can also be produced via complex interpolation. And it is in this way that complex interpolation can do the twist even when no unconditional structure is present.

In the picture above, polyhedral spaces will have coordinates $(1, y)$ (unconditionality still can work but they will not be at the center of interpolation scales). In other words, they are difficult to twist and no general method works. It makes sense to pose the problem:

PROBLEM 2. *Let X be a polyhedral space. Does exist a nontrivial exact sequence*

$$0 \longrightarrow X \longrightarrow \spadesuit \longrightarrow X \longrightarrow 0?$$

Recall that a property \mathcal{P} is said to be a 3-space property if whenever Y and X/Y have \mathcal{P} then also X has \mathcal{P} .

The Ciesielski-Pol space $C(\mathcal{C}\mathcal{P})$ mentioned at the end of the previous section can be represented as a twisted sum of two $c_0(\Gamma)$ spaces. Therefore:

PROPOSITION 3. *Admitting a Talagrand operator is not a 3-space property.*

Which, of course, suggests the 3-space problem for isomorphic polyhedrality:

PROBLEM 3. *Is "to be isomorphically polyhedral" a 3-space property?*

The first author believes the answer is negative. Since no current method is known to twist polyhedral spaces, let us briefly review the known examples and constructions of twisted sums of (isomorphically) polyhedral spaces and exact sequences involving isomorphically polyhedral spaces.

3.1. JOHNSON-LINDENSTRAUSS SEQUENCES. Let $\mathcal{M} = \{M_\alpha : \alpha \in \Gamma\}$ be an uncountable almost-disjoint family –which means that $M_\alpha \cap M_\beta$ is finite for different α, β – by infinite subsets of \mathbb{N} . Form the subspace $X_{\mathcal{M}} \subset \ell_\infty$ formed by the closure of the linear span in ℓ_∞ of the characteristic functions $\{1_n : n \in \mathbb{N}\}$ and $\{1_{M_\alpha} : \alpha \in \Gamma\}$. Since the images of $\{1_{M_\alpha}\}_{\alpha \in \Gamma}$ in ℓ_∞/c_0 generate a copy of $c_0(\mathfrak{c})$, one gets the nontrivial exact sequence

$$0 \longrightarrow c_0 \longrightarrow X_{\mathcal{M}} \longrightarrow c_0(|\mathcal{M}|) \longrightarrow 0. \quad (1)$$

The space $X_{\mathcal{M}}$ is easily shown to be isomorphic to a $C(\Delta_{\mathcal{M}})$ space, where $\Delta_{\mathcal{M}}$ denotes the compact having as isolated points the nodes of the dyadic tree, which accumulate at the branches, which accumulate at some infinity point. Hence $\Delta_{\mathcal{M}}$ is a finite height compact with its third derived set is empty. Thus, the sequence 3 has in fact the form

$$0 \longrightarrow c_0 \longrightarrow C(\Delta_{\mathcal{M}}) \longrightarrow c_0(|\mathcal{M}|) \longrightarrow 0. \quad (2)$$

There exist almost disjoint families of \mathbb{N} of cardinal the continuum: for instance, enumerate the nodes of the dyadic tree, let Γ be the set of branches and for each $\alpha \in \Gamma$ let M_α be the set of numbers assigned to the nodes in the branch α . The existence of such families was first observed by Sierpinski [47]. An elegant proof due to Whitley [49] of the fact that (the canonical copy of) c_0 is uncomplemented in ℓ_∞ relies on the following fact: Whitley observes that the kernel of every functional of c_0^\perp contains all but perhaps a countable quantity of members of the family. Thus, if $\ell_\infty = c_0 \oplus Z$ and $P : \ell_\infty \rightarrow \ell_\infty$ is a projection onto Z then P comes defined by a sequence (f_n) of functionals of c_0^\perp . But the kernel of an operator $(f_n) : \ell_\infty \rightarrow \ell_\infty$ cannot be c_0 since it is $\bigcap_{n \in \mathbb{N}} \ker f_n$, and thus it must contain uncountably many members of the family, which necessarily makes it nonseparable. In an entirely analogous form, c_0 is equally uncomplemented in the subspace $X_{\mathcal{M}}$ of ℓ_∞ .

Koszmider [36, Question 5] raises the question of whether all the $C(\Delta_{\mathcal{M}})$ -spaces generated by an almost disjoint family \mathcal{M} must be isomorphic. Marciszewski and Pol answer in [44] the question by showing that there exist $2^{\mathfrak{c}}$ almost disjoint families \mathcal{M} generating non-isomorphic $C(\Delta_{\mathcal{M}})$ -spaces. Since the compact $\Delta_{\mathcal{M}}$ are of finite height, one has

PROPOSITION 4. *The spaces $C(\Delta_{\mathcal{M}})$ are isomorphically polyhedral.*

Similar constructions to the previous ones can be carried over for other cardinals. One needs a generalized version of Sierpinski's almost-disjoint uncountable family, due to Sierpinski and Tarski [37]: given an infinite set I

there exists a family \mathcal{M} of infinite subsets of I with $|\mathcal{M}| > |I|$ and $M \cap N$ finite for every $M \neq N$ in \mathcal{M} . In this way one would obtain nontrivial exact sequences

$$0 \longrightarrow c_0(\Gamma) \longrightarrow X \longrightarrow c_0(\Gamma) \longrightarrow 0.$$

3.2. THE CIESIELSKI-POL SPACE. It is an exact sequence

$$0 \longrightarrow c_0(\mathfrak{c}) \longrightarrow C(\mathcal{CP}) \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

in which the twisted sum space $C(\mathcal{CP})$ not only is not WCG: it does not admit any injective operator into $c_0(I)$. Since the Ciesielski-Pol compact \mathcal{CP} has finite height, one has

PROPOSITION 5. *The space $C(\mathcal{CP})$ is isomorphically polyhedral.*

3.3. WEAKLY COMPACTLY GENERATED TWISTINGS OF $c_0(\Gamma)$. The Johnson-Lindenstrauss spaces $C(\Delta_{\mathcal{M}})$ that provide nontrivial exact sequences

$$0 \longrightarrow c_0 \longrightarrow C(\Delta_{\mathcal{M}}) \longrightarrow c_0(|\mathcal{M}|) \longrightarrow 0, \quad (3)$$

are not weakly compactly generated (WCG) since c_0 is complemented inside every WCG space. In [3] it is obtained a nontrivial exact sequence

$$0 \longrightarrow c_0(\aleph) \longrightarrow C(\mathcal{ACGJM}) \longrightarrow c_0(\aleph) \longrightarrow 0$$

in which \mathcal{ACGJM} is an Eberlein compact, and thus $C(\mathcal{ACGJM})$ is WCG. Under GCH one can choose $\aleph = \aleph_\omega$ (and this is the smallest cardinal allowing a WCG nontrivial twisted sum of $c_0(\aleph)$). In this same line, Marciszewski shows in [43] that the space $C(\mathcal{BM})$ obtained by Bell and Marciszewski in [7] (here BM is an Eberlein compact of weight \mathfrak{c} and height 3 that cannot be embedded into any space formed by all the characteristic functions of subsets of X of cardinality lesser than or equal to n) is actually a nontrivial twisted sum of two $c_0(\Gamma)$. The Bell-Marciszewski compact \mathcal{BM} has finite height, and thus:

PROPOSITION 6. *The space $C(\mathcal{BM})$ is isomorphically polyhedral.*

Returning to the space $C(\mathcal{ACGJM})$, this actually is a c_0 -sum of spaces $C(K_n)$ spaces so that each of the compacta K_n has finite height. Since:

LEMMA 3.1. [34] *If X is isomorphically polyhedral then $c_0(I, X)$ is isomorphically polyhedral.*

Proof. It is enough to assume that X is polyhedral and show that also $c_0(I, X)$ is polyhedral. The proof mimics [34, Lemma 1]: let Z be a finite dimensional subspace of $c_0(I, X)$. There is a finite set of indices $F = \{i_1, \dots, i_n\}$ such that $\|P_j(z)\| \leq \frac{1}{2}\|z\|$ for all $z \in Z$ and all $j \notin F$. Hence $\|z\| = \max\{\|P_{i_k}(z)\| : 1 \leq k \leq n\}$ and since X is polyhedral the unit ball of Z , namely $\bigcap_{1 \leq k \leq n} \{z \in Z : \|P_{i_k}(z)\| \leq 1\}$ is a polytope. ■

It turns out that

PROPOSITION 7. *The space $C(\mathcal{ACSGJM})$ is isomorphically polyhedral.*

Apart from the general 3-space problem for isomorphic polyhedrality, the following problem is also open:

PROBLEM 4. *Is every twisted sum of two $c_0(\Gamma)$ isomorphically polyhedral?*

Observe that if twisted sums of $c_0(I)$ admit a continuous polyhedral renorming process then the answer would be yes. It is also unknown if it does there exist an exact sequence

$$0 \longrightarrow c_0(I) \longrightarrow X \longrightarrow c_0(J) \longrightarrow 0$$

in which X is not isomorphic to a $C(K)$ -space. To be a $C(K)$ -space is not a 3-space property [8, 11, 6] (see also below). Regarding this problem, it was shown in [15] that every twisted sum of $c_0(\Gamma)$ and a space with property (V) has property (V).

3.4. TWISTED SUMS WITH $C(\omega^\omega)$. In [8] it is shown not only that “to be a $C(K)$ space” is not a 3-space property, but even the existence of exact sequences

$$0 \longrightarrow C(\omega^\omega) \longrightarrow \Omega \xrightarrow{q} c_0 \longrightarrow 0$$

in which the quotient map q is strictly singular. This fact makes Ω fail Pełczyński's property (V). Recall that a Banach space is called a Lindenstrauss space if its dual is isometric to some $L_1(\mu)$ -space. Lindenstrauss spaces share with $C(K)$ -spaces Pełczyński's property (V); so, the space Ω is not isomorphic to a Lindenstrauss space. Of course it is an \mathcal{L}_∞ -space since this is a 3-space property.

3.5. GASPARIS SEQUENCES. Gasparis shows in [31, 32] the following surprising result:

PROPOSITION 8. *For every Banach space Z with a shrinking unconditional basis satisfying an upper p -estimate for some $p > 1$ there exists an exact sequence*

$$0 \longrightarrow \ker q \longrightarrow \mathcal{G}(Z) \longrightarrow Z \longrightarrow 0$$

in which $\mathcal{G}(Z)$ is isomorphically polyhedral and has an unconditional basis.

The result applies, in particular, to spaces Z such as: reflexive Banach spaces with an unconditional basis and non-trivial type, Tsirelson's original space or spaces such as $\ell_p(c_0)$ for $1 < p < +\infty$. In the particular case $Z = \ell_p$, $1 < p < +\infty$, the space (ℓ_p) is not a subspace of any $C(\alpha)$ -space for α a countable ordinal, and is not a quotient of any separable \mathcal{L}_∞ -space.

Related previous examples were Leung's example [42] of a c_0 -saturated Banach space with an unconditional basis and a quotient isomorphic to ℓ_2 (see also [4, Thm.8.8]) and Alspach's example [1] of an ℓ_1 -predual, hence polyhedral space \mathcal{A} , that is a quotient of $C(\omega^\omega)$ but not a subspace of any $C(\alpha)$ with α countable.

4. ISOMORPHICALLY POLYHEDRAL \mathcal{L}_∞ -SPACES

*Another field rose high with waving grain;
With bended sickles stand the reaper train.
Next, ripe in yellow gold, a vineyard shines,
Bent with the ponderous harvest of its vines*

Lazar [38] completes Lindenstrauss results [39] by showing that a Lindenstrauss space X is polyhedral if and only if every compact operator $\tau : Y \rightarrow X$ admits norm-preserving compact extensions to any superspace. Does a similar extension result hold for isomorphically Lindenstrauss and isomorphically polyhedral spaces? In fact, the connection between the properties of being an \mathcal{L}_∞ -space, isomorphically polyhedral and isomorphically Lindenstrauss have not yet been clarified. More precisely:

1. There are polyhedral spaces which are not \mathcal{L}_∞ : Schreier space is a subspace of $C(\omega^\omega)$ not of \mathcal{L}_∞ -type; or else, any subspace of c_0 other than c_0 (subspaces of $c_0(\Gamma)$ are \mathcal{L}_∞ -spaces if and only if isomorphic to $c_0(I)$)

2. There are Lindenstrauss spaces not polyhedral: $C[0, 1]$.
3. A result of Fonf [22] asserts that preduals of ℓ_1 are isomorphically polyhedral.
4. The result fails for $\ell_1(\Gamma)$ (thanks are due to V. Fonf for this information): Kunen's compact \mathcal{K} provides, under CH, a scattered, non metrizable, one point compactification of a non-Lindelöf locally compact space K_0 having all its finite powers K_0^n hereditarily separable. The outcome of this is that the corresponding $C(\mathcal{K})$ space has the extraordinary property that every uncountable set of elements contains one that belongs to the closure of the convex hull of the others. And this property was used by Jiménez and Moreno [35] to show that every equivalent renorming of $C(\mathcal{K})$ has only a countable number of weak*-strongly exposed points. Thus, no equivalent renorming can be polyhedral. At the same time $C(\mathcal{K})^* = \ell_1(\Gamma)$ as every scattered compact does. [21].
5. Fonf asked once whether isomorphically polyhedral \mathcal{L}_∞ -spaces are isomorphically Lindenstrauss.
6. Is it possible to embed an isomorphically polyhedral space as a subspace of an isomorphically polyhedral \mathcal{L}_∞ -space?

5. POLYHEDRALITY IN PIECES

*Next this, the eye the art of Hephaestus leads
Deep through fair forests, and a length of meads,
A figured dance succeeds; such once was seen
In lofty Knossos for the Cretan queen*

Any finite dimensional space admits, for every $\varepsilon > 0$ a $(1 + \varepsilon)$ -equivalent polyhedral renorming. So one cannot expect to get a polyhedral norm out from renormings of the finite dimensional pieces. Nevertheless, things change if one asks some compatibility condition, as we will show now. Let $[A, \|\cdot\|]$ be a finite dimensional Banach space. We define $n[A, \|\cdot\|]$ as the minimum of those N , if they exist, so that $[A, \|\cdot\|]$ is isometric to a subspace of ℓ_∞^N . Let in what follows denote with ϕ_A an into isometry $\phi_A : [A, \|\cdot\|] \rightarrow \ell_\infty^{n(A)}$.

PROPOSITION 9. *A Banach space E is λ -isomorphically polyhedral if and only if each finite dimensional subspace F admits a λ -equivalent polyhedral renorming r_F so that the following compatibility assumption is satisfied:*

$$\forall A, B \quad \exists C : A + B \subset C \quad \begin{cases} n([A, r_{C|A}]) = n([A, r_A]); \\ n([B, r_{C|B}]) = n([B, r_B]). \end{cases}$$

Proof. Let $FIN(E)$ denote the space of all finite dimensional subspaces of X and for each $F \in FIN(E)$ let r_F be the λ -equivalent polyhedral renorming satisfying the compatibility assumption. Set the order

$$A \leq B \iff \begin{cases} A \subset B; \\ n([A, r_{B|A}]) = n([A, r_A]). \end{cases}$$

The compatibility condition is there to guarantee that the ordering is filtering; i.e., that for all A, B there is some C so that $A \leq C$ and $B \leq C$. After that, the sets $W_A = \{F : A \leq F\}$ form a filter base. Let thus \mathcal{U} be a free ultrafilter refining that base. The renorming:

$$r(x) = \lim_{\mathcal{U}(F)} r_F(x)$$

is polyhedral: Let A be a finite dimensional subspace; since $W_A \in \mathcal{U}$ it makes sense to define a map $\phi : [A, r] \rightarrow \ell_\infty^{n(A)}$ as $\phi(a) = \lim_{\mathcal{U}(F)} \phi_F(a)$. The map ϕ is obviously linear and

$$\phi(a) = \lim_{\mathcal{U}(F)} \phi_F(a) = \lim_{\mathcal{U}(F)} r_F(a) = r(a)$$

shows it is an into isometry. ■

One could ask whether compatibility assumptions could be removed forcing the hypothesis to assume polyhedral renorming of all separable subspaces. Precisely: *Assume that X is a Banach space all whose separable subspace are λ -isomorphically polyhedral. Must X be isomorphically polyhedral?* The answer is no again because of Kunen compact \mathcal{K} (and again we thank Fonf for addressing us to this space). We already know that no equivalent renorming of $C(\mathcal{K})$ can be polyhedral. On the other hand, a metrizable scattered compact is a countable ordinal space, so each separable subspace of $C(\mathcal{K})$ is actually a subspace of some $C(\alpha)$ space for countable ordinal α , hence isomorphically polyhedral. Alternatively, $C(\mathcal{K})$ has a countable boundary, and so does every separable subspace; which, therefore, must be polyhedral.

*Thus the broad shield complete the artist crown'd
With his last hand, and pour'd the ocean round:
In living silver seem'd the waves to roll,
And beat the buckler's verge, and bound the whole.*

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