Fuzzy Prime Ideals of ADL's

Chigurupalli Santhi Sundar Raj, Natnael Teshale Amare and Uppasetti Madana Swamy

***Abstract*—In this paper the concept of prime L-fuzzy ideals and L-fuzzy prime ideals of an ADL A with truth values in a complete lattice L satisfying the infinite meet distributive law are introduced. All prime L-fuzzy ideals of a given ADL A are determined by establishing a one-to- one correspondence between prime L-fuzzy ideals of an ADL A and the pairs (P,α), where P is a prime ideal of A and α is a prime element in L. Also, here minimal prime L-fuzzy ideals and L-fuzzy minimal prime ideals of an ADL A are introduced and characterized.**

***Index Terms*—Almost Distributive Lattice (ADL), complete lattice, L-fuzzy minimal prime ideal L-fuzzy prime ideal, minimal prime L-fuzzy ideal, prime L-fuzzy ideal.**

I. Introduction

A fuzzy subset of a set X is a function from X into $I=[0,1]$, as in [1]. J.A. Goguen [2] explored, generalized and continued the work of L.A. Zadeh and realized that the unit interval $[0,1]$ is not sufficient to take the truth values of general fuzzy statements. Wang-Jing Liu [3] introduced the notion of a fuzzy ideal of a ring in the case when $L=[0,1]$ of real numbers and T.K. Mukherjee and M.K. Sen [4] introduced the notion of a fuzzy prime ideal and continued the study of fuzzy ideals. U.M. Swamy and K.L.N. Swamy [5] introduced the concept of fuzzy prime ideal of a ring with truth values in a complete lattice satisfying the infinite meet distributive law.

The concept of prime ideal of an Almost Distributive Lattice was introduced by U.M. Swamy and G.C. Rao, in 1981 [6]. U.M. Swamy, Ch. Santhi Sundar Raj and Natnael Teshale A [7] have introduced the notion of L-fuzzy ideals of an ADL with the truth values in a complete lattice L satisfying the infinite meet distributive law.

In this paper, we introduce and study prime L-fuzzy ideals and L-fuzzy prime ideals of an ADL A, where L is a complete lattice satisfying the infinite meet distributive law. Also, in this paper we introduce minimal prime L-fuzzy ideals and L-fuzzy minimal prime ideals of an ADL A.

II. Preliminaries

First we give necessary definitions and results mostly taken from [6] and [7] which will be used in the later text.

Definition 2.1: An algebra $A=(A,\wedge,\vee,0)$ of type $(2,2,0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a,b and $c\in A$.

1) $0\wedge a=0$
2) $a\vee 0=a$
3) $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$
4) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$
5) $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$
6) $(a\vee b)\wedge b=b$.

Any bounded below distributive lattice is an ADL, where 0 is the smallest element. Any nonempty set X can be made into an ADL by fixing an arbitrarily chosen element 0 in X and by defining the binary operations $\wedge$ and $\vee$ on X by

$$a\wedge b=\begin{cases}0, & \text{if } a=0\\ b, & \text{if } a\neq 0\end{cases} \quad\text{and}\quad a\vee b=\begin{cases}b, & \text{if } a=0\\ a, & \text{if } a\neq 0.\end{cases}$$

This ADL $(X,\wedge,\vee,0)$ is called a discrete ADL.

Definition 2.2: Let $A=(A,\wedge,\vee,0)$ be an ADL. For any a and $b\in A$, define $a\leq b$ if $a=a\wedge b$ ($\Leftrightarrow a\vee b=b$). Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem 2.3: The following hold for any a,b and c in an ADL A.

(1) $a\wedge 0=0=0\wedge a$ and $a\vee 0=a=0\vee a$
(2) $a\wedge a=a=a\vee a$
(3) $a\wedge b\leq b\leq b\vee a$
(4) $a\wedge b=a\Leftrightarrow a\vee b=b$
(5) $a\wedge b=b\Leftrightarrow a\vee b=a$
(6) $(a\wedge b)\wedge c=a\wedge(b\wedge c)$ (i.e., $\wedge$ is associative)
(7) $a\vee(b\vee a)=a\vee b$
(8) $a\leq b\Rightarrow a\wedge b=a=b\wedge a$ $\big(\Leftrightarrow a\vee b=b=b\vee a\big)$
(9) $(a\wedge b)\wedge c=(b\wedge a)\wedge c$
(10) $(a\vee b)\wedge c=(b\vee a)\wedge c$
(11) $a\wedge b=b\wedge a\Leftrightarrow a\vee b=b\vee a$
(12) $a\wedge b=\inf\{a,b\}\Leftrightarrow a\wedge b=b\wedge a\Leftrightarrow a\vee b=\sup\{a,b\}$.

An element $m\in A$ is said to be maximal if, for any $x\in A$, $m\leq x$ implies $m=x$. It can be easily observed that m is maximal if and only if $m\wedge x=x$ for all $x\in A$.

Definition 2.4: Let I be a non empty subset of an ADL A. Then I is called an ideal of A if $a,b\in I\Rightarrow a\vee b\in I$ and $a\wedge x\in I$ for all $x\in A$.
As a consequence, for any ideal I of A, $x\wedge a\in I$ for all $a\in I$ and $x\in A$. For any $S\subseteq A$, the smallest ideal of A containing S is called the ideal generated by S in A and is denoted by $(S]$. It is known that

$$(S]=\Big\{\big(\bigvee_{i=1}^{n}x_i\big)\wedge a\mid n\geq 0, x_i\in S \textit{ and } a\in A\Big\}.$$

when $S=\{x\}$, we write $(x]$ for $(\{x\}]$. Note that $(x]=\{x\wedge a\mid a\in A\}$.

Definition 2.5: An L-fuzzy subset λ of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law. If L is the unit interval $[0,1]$ of real numbers, then these are the usual fuzzy subsets of X.

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Ch. Santhi Sundar Raj and A. Natnael Teshale are with the Department of Engineering Mathematics, Andhra University, Visakhapatnam- 530003, A.P., India. E-mail: santhisundarraj@yahoo.com
A. Natnael Teshale is with the Department of Mathematics, University of Gondar, Ethiopia. E-mail: yenatnaelau@yahoo.com
U.M. Swamy is with the Department of Mathematics, Andhra University, Visakhapatnam- 530003, A.P., India. E-mail: umswamy@yahoo.com

For any $\alpha \in L$, the set $\lambda_\alpha = \{x \in X : \alpha \leq \lambda(x)\}$ is called the α-cut of λ.

Definition 2.6: An L -fuzzy subset λ of A is said to be an L -fuzzy ideal of A, if $\lambda(0) = 1$ and $\lambda(x \vee y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in A$.

Lemma 2.7: Let λ be an L-fuzzy ideal of A, S a non-empty subset of A and $x, y \in A$. Then we have the following.

(1) $x \wedge y = y$ and $y \wedge x = x \Longrightarrow \lambda(x) = \lambda(y)$
(2) $\lambda(x \wedge y) = \lambda(y \wedge x)$
(3) $x \in (S] \Longrightarrow \lambda(x) \geq \bigwedge_{i=1}^n \lambda(a_i)$ for some $a_1, a_2, \ldots, a_n \in S$
(4) $x \in (y] \Longrightarrow \lambda(x) \geq \lambda(y)$
(5) If m is a maximal element in A then $\lambda(m) \leq \lambda(x)$, for all x
(6) $\lambda(m) = \lambda(n)$ for all maximal elements m and n in A.

Theorem 2.8: The set of all L-fuzzy ideals of A is a complete distributive lattice, in which the supremum $\bigvee_{i \in \Delta} \lambda_i$ and infimum $\bigwedge_{i \in \Delta} \lambda_i$ of any family $\{\lambda_i : i \in \Delta\}$ of L-fuzzy ideals of A are given by

$$\Big(\bigvee_{i \in \Delta} \lambda_i\Big)(x) = \bigvee \Big\{ \bigwedge_{a \in F} \Big(\bigvee_{i \in \Delta} \lambda_i(a)\Big) : x \in (F], F \subset\subset A \Big\}$$

and $$\Big(\bigwedge_{i \in \Delta} \lambda_i\Big)(x) = \bigwedge_{i \in \Delta} \lambda_i(x)$$

III. Prime L-Fuzzy Ideals

Let us recall from [6] that a proper ideal P of an ADL A is said to be prime if for any $x, y \in A$, $x \wedge y \in P$ implies that $x \in P$ or $y \in P$; (equivalently, for any ideals I and J of A, $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.)

The following definition is analogous to that of a prime ideal of A. Here after A stands for an ADL with a maximal element. An L-fuzzy ideal λ of A is called proper if $\lambda(x) \neq 1$ for some $x \in A$.

Definition 3.1: A proper L-fuzzy ideal λ of A is called a prime L-fuzzy ideal if for any L-fuzzy ideals ν and μ of A, $\nu \wedge \mu \leq \lambda$ implies either $\nu \leq \lambda$ or $\mu \leq \lambda$.
An element $x \neq 1$ in L is called prime if for any $a, b \in L$ $a \wedge b \leq x$ implies either $a \leq x$ or $b \leq x$.

Now, we determine all prime L-fuzzy ideals of A by establishing a correspondence between prime L-fuzzy ideals and pairs (I, α), where I is a prime ideal of A and α is a prime element in L. First, we recall from [7] that for any ideal I of A and $\alpha \in L$, the L-fuzzy ideal α_I of A defined by

$$\alpha_I(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{if } x \notin I. \end{cases}$$

and that α_I is called the α-level L-fuzzy ideal correspondence to I.

Theorem 3.2: Let I be an ideal of an ADL A and $\alpha \in L$. Then α_I is a prime L-fuzzy ideal of A if and only if I is a prime ideal of A and α is a prime element in L.

Proof: Suppose that α_I is a prime L-fuzzy ideal of A. Since α_I is proper, $\alpha_I(x) \neq 1$, for some $x \in A$. Therefore $x \notin I$ and hence $I \subsetneqq A$. If J and K are ideals of A such that $J \cap K \subseteq I$. Then $\alpha_J \wedge \alpha_K = \alpha_{J \wedge K} \leq \alpha_I$ and hence $\alpha_J \leq \alpha_I$ or $\alpha_K \leq \alpha_I$, so that $J \subseteq I$ or $K \subseteq I$. Therefore, I is a prime ideal of A. Also, for any $\gamma, \beta \in L$,

$$\begin{aligned} \gamma \wedge \beta \leq \alpha &\Rightarrow (\gamma \wedge \beta)_I \leq \alpha_I \\ &\Rightarrow \gamma_I \wedge \beta_I \leq \alpha_I \\ &\Rightarrow \gamma_I \leq \alpha_I \text{ or } \beta_I \leq \alpha_I \\ &\Rightarrow \gamma \leq \alpha \text{ or } \beta \leq \alpha. \end{aligned}$$

Therefore, α is a prime element in L.
Conversely, suppose that I is a prime ideal of A and α is a prime element in L. Since I is proper and $\alpha \neq 1, \alpha_I$ is clearly a proper L-fuzzy ideal of A. Let λ and μ be any L-fuzzy ideals of A such that $\lambda \nleq \alpha_I$ and $\mu \nleq \alpha_I$. Then there exists $x, y \in A$ such that $\lambda(x) \nleq \alpha_I(x)$ and $\mu(y) \nleq \alpha_I(y)$. This implies that $\alpha_I(x) = \alpha = \alpha_I(y)$ (otherwise, $\alpha_I(x) = 1 \geq \lambda(x)$ and $\alpha_I(y) = 1 \geq \mu(y)$) and hence $x \notin I$ and $y \notin I$. Since I is a prime ideal, $x \wedge y \notin I$. Also, since α is prime and $\lambda(x) \nleq \alpha$ and $\mu(y) \nleq \alpha$, we have that $\lambda(x) \wedge \mu(y) \nleq \alpha$.
Now, $(\lambda \wedge \mu)(x \wedge y) = \lambda(x \wedge y) \wedge \mu(x \wedge y) \geq \lambda(x) \wedge \mu(y)$ (since λ and μ are antitones) and hence $(\lambda \wedge \mu)(x \wedge y) \nleq \alpha = \alpha_I(x \wedge y)$ so that, $(\lambda \wedge \mu) \nleq \alpha_I$. Hence, α_I is a prime L-fuzzy ideal of A. ■

Theorem 3.3: A proper L-fuzzy ideal λ of A is prime if and only if the following are satisfied.

(1) λ is two valued
(2) $\lambda(m)$ is a prime element in L, for any maximal element m in A
(3) λ_1 is a prime ideal of A.

Proof: Suppose that λ is a prime L-fuzzy ideal of A.
(1): Suppose λ assumes more than two values. Then there exists $x, y \in A$ and $\alpha \neq \beta \in L - \{1\}$ such that $\lambda(x) = \alpha$, $\lambda(y) = \beta$ and $\lambda(0) = 1$. Now, define L-fuzzy subsets ν and μ of A as follows:

$$\nu(z) = \begin{cases} 1 & \text{if } z \in (x] \\ 0 & \text{if } z \notin (x] \end{cases} \quad \text{and} \quad \mu(z) = \begin{cases} 1 & \text{if } z = 0 \\ \alpha & \text{if } z \neq 0. \end{cases}$$

Then, clearly $\nu = 0_{(x]}$ and $\mu = \alpha_{(0]}$ and hence ν and μ are L-fuzzy ideals. Also,for
$z = 0 \Rightarrow \big(\nu \wedge \mu\big)(0) = \nu(0) \wedge \mu(0) = 1 \wedge 1 = 1 = \lambda(0)$.
$0 \neq z \in (x] \Rightarrow \nu(z) \wedge \mu(z) = 1 \wedge \alpha = \alpha = \lambda(x) \leq \lambda(z)$ (since λ is an antitone and $z \wedge x \leq x$, we have
$\lambda(x) \leq \lambda(z \wedge x) = \lambda(x \wedge z) = \lambda(z)$)
and $z \notin (x] \Rightarrow \nu(z) \wedge \mu(z) = 0 \wedge \alpha = 0 \leq \lambda(z)$. Therefore, $\nu \wedge \mu \leq \lambda$. Since λ is prime, $\nu \leq \lambda$ or $\mu \leq \lambda$. But $\nu \nleq \lambda$(since $\nu(x) = 1, \lambda(x) = \alpha$ and $1 \neq \alpha$).
Therefore, $\mu \leq \lambda$. In particular, $\mu(y) \leq \lambda(y) \neq \lambda(0)$, we get that $y \neq 0$ and $\alpha = \mu(y) = \beta$, which is a contradiction.
(2): Let m be a maximal element in A. Since λ is proper, $\lambda(x) \neq 1$, for some $x \in A$ and hence $\lambda(m) \neq \lambda(0) = 1$
$\big(\ \lambda(m) = 1 \Rightarrow \lambda(m \vee x) = 1 \Rightarrow \lambda(m) \wedge \lambda(x) = 1 \Rightarrow \lambda(x) = 1.$
$\big)$
Let α and $\beta \in L$ such that $\alpha \wedge \beta \leq \lambda(m)$. Define ν and μ of A as:

$$\nu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \beta & \text{if } x \neq 0. \end{cases}$$

Then, it can be easily proved that ν and μ are L-fuzzy ideals of A and $\nu \wedge \mu \leq \lambda$. Since λ is prime, $\nu \leq \lambda$ or $\mu \leq \lambda$, inparticular, $\nu(m) \leq \lambda(m)$ or $\mu(m) \leq \lambda(m)$. Therefore, $\alpha \leq$

$\lambda(m)$ or $\beta \leq \lambda(m)$ and hence $\lambda(m)$ is prime.
(3): Let $I = \{x \in A : \lambda(x) = 1\}$. Clearly, I is a proper ideal of A, since λ is proper. Let α be the other value of λ. Then

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{if } x \notin I \end{cases}$$

and hence $\lambda = \alpha_I$. By theorem 3.2, I is prime.
Conversely suppose that λ is an L-fuzzy ideal of A satisfying the conditions (1),(2) and (3). By (1), there exists $\alpha(\neq 1) \in L$ such that $\lambda(x) = \alpha$, for each $x \in A - \{0\}$. Then for any maximal element m of A, $\lambda(m) = \alpha$. By (2), α is prime. Let $I = \{x \in A : \lambda(x) = 1\}$. Then I is a prime ideal of A (by (3)). Therefore, $\lambda = \alpha_I$ and hence λ is a prime L-fuzzy ideal of A (by theorem 3.2). ■

The results 3.2 and 3.3 yield the following.

Theorem 3.4: Let λ be an L-fuzzy subset of A. Then λ is a prime L-fuzzy ideal of A if and only if there exists a prime ideal P of A and a prime element α in L such that $\lambda = \alpha_P$.

IV. L-Fuzzy Prime Ideals

In this section, we introduce the notion of an $L-$fuzzy prime ideal which is weaker than that of a prime L-fuzzy ideal.

Definition 4.1: A proper L-fuzzy ideal λ of A is called an L-fuzzy prime ideal of A if for any $x, y \in A$,
$\lambda(x \wedge y) = \lambda(x)$ or $\lambda(y)$.

The following theorem gives a characterization of an L-fuzzy prime ideal.

Theorem 4.2: Let λ be a proper L-fuzzy ideal of A. Then the following are equivalent to each other.

(1) for each $\alpha \in L$, $\lambda_\alpha = A$ or λ_α is a prime ideal of A
(2) λ is an L-fuzzy prime ideal of A
(3) for any $x, y \in A$, $\lambda(x \wedge y) \leq \lambda(x) \vee \lambda(y)$ and either $\lambda(x) \leq \lambda(y)$ or $\lambda(y) \leq \lambda(x)$.

Proof: $(1) \Rightarrow (2)$: Let $x, y \in A$ and $\alpha = \lambda(x \wedge y)$. Then $x \wedge y \in \lambda_\alpha$ and hence $x \in \lambda_\alpha$ or $y \in \lambda_\alpha$.

$$x \in \lambda_\alpha \Rightarrow \lambda(x \wedge y) = \alpha \leq \lambda(x) \leq \lambda(x \wedge y)$$
$$\Rightarrow \lambda(x \wedge y) = \lambda(x)$$

Similarly, $y \in \lambda_\alpha \Rightarrow \lambda(x \wedge y) = \lambda(y)$.
$(2) \Rightarrow (3)$: Let $x, y \in A$. Then, $\lambda(x \wedge y) = \lambda(x)$ or $\lambda(y)$.
$\lambda(x \wedge y) = \lambda(x) \Rightarrow \lambda(x \wedge y) = \lambda(x) \leq \lambda(x) \vee \lambda(y)$ and $\lambda(y) \leq \lambda(x \wedge y) = \lambda(x)$.
Similarly, $\lambda(x \wedge y) = \lambda(y) \Rightarrow \lambda(x \wedge y) \leq \lambda(x) \vee \lambda(y)$ and $\lambda(x) \leq \lambda(y)$.
$(3) \Rightarrow (1)$: Let $\alpha \in L$ be fixed. If $\lambda_\alpha \neq A$, then λ_α is a proper ideal of A. Also, for any $x, y \in A$,

$$x \wedge y \in \lambda_\alpha \Rightarrow \alpha \leq \lambda(x \wedge y) \leq \lambda(x) \vee \lambda(y) = \lambda(x) \text{ or } \lambda(y)$$
$$\Rightarrow \alpha \leq \lambda(x) \text{ or } \alpha \leq \lambda(y)$$
$$\Rightarrow x \in \lambda_\alpha \text{ or } y \in \lambda_\alpha$$

Therefore, λ_α is prime. ■

Theorem 4.3: A prime L-fuzzy ideal of A is an L-fuzzy prime ideal of A.

Proof: Let λ be a prime L-fuzzy ideal of A. Then $\lambda = \alpha_I$ for some prime ideal P of A and α a prime element in L. Since $\alpha < 1$, λ is a proper.
Let $x, y \in A$. Then

$$x \wedge y \in I \Rightarrow \lambda(x \wedge y) = 1 \text{ and } x \in I \text{ or } y \in I$$
$$\Rightarrow \lambda(x \wedge y) = 1 = \lambda(x) \text{ or } \lambda(y)$$
$$\text{and } x \wedge y \notin I \Rightarrow x \notin I \text{ and } y \notin I$$
$$\Rightarrow \lambda(x \wedge y) = \alpha = \lambda(x) = \lambda(y)$$

Therefore, λ is an L-fuzzy prime ideal of A. ■

The converse of the above theorem is not true; for consider the given example below.

Example 4.4: Let $A = \{0, a, b, c\}$, $L = \{0, t, 1\}$ with $0 < t < 1$ and let $\vee$ and $\wedge$ be binary operations on A defined by

∨	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

∧	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then, $(A, \wedge, \vee, 0)$ is an ADL. Now define $\lambda : A \to L$ by $\lambda(0) = 1$,
$\lambda(a) = \lambda(b) = 0$ and $\lambda(c) = t$. Therefore, $\lambda_0 = A$, $\lambda_t = \{0, c\}$ and $\lambda_1 = \{0\}$ are prime ideals of A. Therefore, λ is an L-fuzzy prime ideal of A, while λ is not a prime L-fuzzy ideal of A, since λ is not exactly two valued.

Finally, in this section we slightly generalize α-level fuzzy ideals of A and identify general prime ideals of A with L-fuzzy prime ideals of A.

Theorem 4.5: Let I a proper ideal of A and $\alpha, \beta \in L$. Let $\langle \alpha, \beta \rangle_I$ be an $L-$fuzzy subset of A defined by

$$\langle \alpha, \beta \rangle_I(x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } 0 \neq x \in I \\ \beta & \text{if } x \notin I. \end{cases}$$

Then,

(1) $\langle \alpha, \beta \rangle_I$ is an L-fuzzy ideal of A if and only if $\beta \leq \alpha$ and, in this case $\langle \alpha, \beta \rangle_I$ is proper if and only if $\beta < 1$.
(2) I is a prime ideal of A if and only if χ_I is an L-fuzzy prime ideal of A
(3) Suppose that 0 be a prime element in L. Then, I is a prime ideal of A if and only if $\langle \alpha, \beta \rangle_I$ is an L-fuzzy prime ideal of A for all $1 \neq \beta \leq \alpha$ in L.

Proof: (1) and (2) are striaght forward and simple verifications.
(3): Suppose that I is a prime ideal of A and $1 \neq \beta \leq \alpha$ in L. Let $x, y \in I$. Then,

$$x \wedge y = 0 \Rightarrow x = 0 \text{ or } y = 0$$
$$\Rightarrow \langle \alpha, \beta \rangle_I(x \wedge y) = 1 = \langle \alpha, \beta \rangle_I(x) \text{ or } \langle \alpha, \beta \rangle_I(y)$$
$$0 \neq x \wedge y \in I \Rightarrow 0 \neq x \in I \text{ or } 0 \neq y \in I$$
$$\Rightarrow \langle \alpha, \beta \rangle_I(x \wedge y) = \alpha = \langle \alpha, \beta \rangle_I(x) \text{ or } \langle \alpha, \beta \rangle_I(y)$$
$$\text{and } x \wedge y \notin I \Rightarrow x \notin I \text{ and } y \notin I$$
$$\Rightarrow \langle \alpha, \beta \rangle_I(x \wedge y) = \beta = \langle \alpha, \beta \rangle_I(x) = \langle \alpha, \beta \rangle_I(y).$$

Converse follows from the fact that $\chi_I = \langle 1, 0 \rangle_I$. ■

V. MINIMAL PRIME L-FUZZY IDEALS

Let us recall from [**?**] that a prime ideal P an ADL of A containing an ideal I is said to be a minimal prime ideal belonging to I if there is no prime ideal of A containing I and properly contained in P.

Definition 5.1: Let λ be a prime L-fuzzy ideal of A. Then λ is said to be minimal if λ is a minimal member in the set of all prime L-fuzzy ideals of A under the point-wise partial ordering. A minimal prime L-fuzzy ideal belonging to $\chi_{\{0\}}$ is simply called a minimal prime L-fuzzy ideal.
In this section, we characterize all minimal prime L-fuzzy ideals of A in terms of minimal prime ideals of A and minimal prime elements of L.

As usual, by a minimal prime element of L we mean a minimal element in the poset of all prime elements of L.

Now we have the following:

Theorem 5.2: Let λ be an L-fuzzy ideal of A. Then λ is a minimal prime L-fuzzy ideal of A if and only if $\lambda = \alpha_I$, for some minimal prime ideal I of A and a minimal prime element α in L.

Proof: Suppose that $\lambda = \alpha_I$ for some minimal prime ideal I of A and minimal prime element in L. Then by theorem 3.4, λ is prime L-fuzzy ideal of A. Let μ be a prime L-fuzzy ideal of A and $\mu \leq \lambda$. Then by theorem 3.4, $\mu = \beta_J$ for some prime ideal J of A and a prime element β in L. Therefore, $\beta_J \leq \alpha_I$. This implies that, $\beta \leq \alpha$ and $J \subseteq I$. By using the minimality of I and α, we get that $\beta = \alpha$ and $J = I$. Therefore, $\mu = \lambda$ and hence λ is a minimal prime L-fuzzy ideal of A.
Conversely suppose that λ is a minimal prime L-fuzzy ideal of A. Then by theorem 3.4, there exists a prime ideal I of A and a prime element α in L such that
$\lambda = \alpha_I$. Let J be a prime ideal of A such that $J \subseteq I$. Then $\alpha_J \leq \alpha_I$, by the minimality of λ, $\alpha_J = \alpha_I$. Therefore, $J = I$ and hence I is minimal prime ideal of A. Let β be a prime element in L and $\beta \leq \alpha$. Then $\beta_I \leq \alpha_I$. This implies, $\beta_I = \alpha_I$ and hence $\beta = \alpha$. Thus α is a minimal prime element in L. ■

If the smallest element 0 in L is prime, then 0 will be the only minimal prime element in L. Note that $\chi_P = 0_P$, for any ideal P of A.

The following is a simple verification.

Theorem 5.3: Let 0 be a prime element in L. Then an L-fuzzy ideal λ of A is a minimal prime L-fuzzy ideal of A if and only if $\lambda = \chi_P$, for some minimal prime ideal P of A. More over, $P \mapsto \chi_P$ is a bijection of the set of minimal prime ideals of A onto the set of minimal prime L-fuzzy ideals of A.

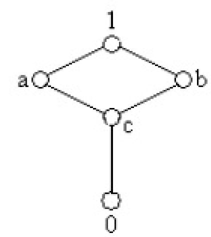
VI. L-FUZZY MINIMAL PRIME IDEALS

By an L-fuzzy minimal prime ideal of A we mean, as usual, a minimal element in the set of all L-fuzzy prime ideals of A under the point-wise partial ordering. In this section, we characterize all L-fuzzy minimal prime ideals of A in terms of their α-cuts.

Theorem 6.1: (1) If λ is an L-fuzzy prime ideal of A, then $\lambda_1 = \{x \in A : \lambda(x) = 1\}$ is a prime ideal of A.
(2) Let λ be an L-fuzzy prime ideal of A. If λ is an L-fuzzy minimal prime ideal of A, then λ_1 is a minimal prime ideal of A.

Proof: (1) Let λ be an L-fuzzy prime ideal of A. Then λ_1 is a proper ideal of A since λ is proper. Let $x, y \in A$. Then,
$x \wedge y \in \lambda_1 \Rightarrow \lambda(x \wedge y) = 1$
$\Rightarrow 1 = \lambda(x \wedge y) = \lambda(x)$ or $\lambda(y)$ (by 4.1)
$\Rightarrow x \in \lambda_1$ or $y \in \lambda_1$.
Thus, λ_1 is a prime ideal of A.

The converse is not true. For, consider the lattice $A = \{0, a, b, c, 1\}$ represented by the Hasse diagram is given below.



Define $\lambda : A \to [0,1]$ by $\lambda(0) = 1$, $\lambda(c) = 0.75$, $\lambda(b) = 0.5$ and $\lambda(a) = \lambda(1) = 0$. Then, $\lambda_1 = \{0\}$ which is a prime ideal of A, while , λ is not an L-fuzzy prime ideal of A, since $\lambda(a \wedge b) = \lambda(c) = 0.75 \neq \lambda(a)$ and $\lambda(b)$.
(2) Suppose that λ is an L-fuzzy minimal prime ideal of A. Let Q be a prime ideal of A and $Q \subset \lambda_1$. Then χ_Q is an L-fuzzy prime ideal of A and $\chi_Q \lneqq \lambda$. This implies that λ is not an L-fuzzy minimal prime ideal of A, which is a contradiction. Thus λ_1 is a minimal prime ideal of A.

The converse is not true; for in the above example, if $\lambda(0) = 1$ and $\lambda(x) = 0.5$ for all $x \neq 0$, then it can be easily checked that λ is an L-fuzzy prime ideal of A and $\lambda_\alpha = A$ if $0 \leq \alpha \leq 0.5$ and $\lambda_\alpha = \{0\}$ if $0.5 < \alpha \leq 1$. In particular, λ_1 is a minimal prime ideal of A. But, λ is not an L-fuzzy minimal prime ideal of A, since if we define $\mu(0) = 1$ and $\mu(x) = 0.25$ for all $x \neq 0$, then μ is an L-fuzzy prime ideal of A and $\mu \lneqq \lambda$. ■

The following theorem is a characterization of L-fuzzy minimal prime ideals of A.

Theorem 6.2: Let λ be an L-fuzzy prime ideal of A and 0 be a prime element in L. Then λ is an L-fuzzy minimal prime ideal of A if and only if λ_α is a minimal prime ideal of A, for all $\alpha \in L$.

Proof: Suppose λ is an L-fuzzy minimal prime ideal of A and λ_α is not minimal prime ideal of A, for some $0 < \alpha < 1$ in L. Then there exists a prime ideal Q of A such that $Q \subset \lambda_\alpha$. Define $\mu : A \to L$ by

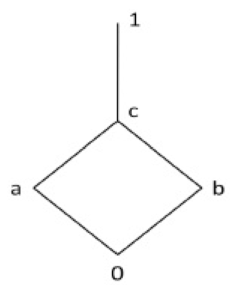
$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } 0 \neq x \in Q \\ 0 & \text{if } x \notin Q. \end{cases}$$

Then, clearly $\mu = (\alpha, 0)_Q$ and hence μ is an L-fuzzy prime ideal of A (by theorem 4.5 (3)). Also, $\mu \leq \lambda$. Since $Q \subset \lambda_\alpha$, there exists $y \in \lambda_\alpha$ such that $y \notin Q$. Therefore, $\mu(y) = 0 <$

$\alpha \leq \lambda(y)$. Therefore, $\mu \lneq \lambda$, which is a contradiction. Thus for each $\alpha \in L$, λ_α is a minimal prime ideal of A.
Conversely, suppose for each $\alpha \in L$, λ_α is a minimal prime ideal of A. Let μ be an L-fuzzy prime ideal of A such that $\mu \leq \lambda$. Then for each $\alpha \in L$, $\mu_\alpha \subseteq \lambda_\alpha$. By the minimality of λ_α, we have $\mu_\alpha = \lambda_\alpha$ and hence $\mu = \lambda$. Therefore λ is an L-fuzzy minimal prime ideal of A. ■

Remark 6.3: If λ is an L-fuzzy minimal prime ideal of A, the each α-cut of λ need not be minimal prime ideal of A.

For, consider the example given in the following. Let $A = \{0, a, b, c, 1\}$ be the lattice represented by the Hasse diagram is given below.



Define $\lambda : A \to [0,1]$ by $\lambda(0) = \lambda(a) = 1$, $\lambda(b) = \lambda(c) = 0.5$ and $\lambda(1) = 0$. It can be easily verified that, λ is an L-fuzzy prime ideal of A and for any $t \in [0, 0.5]$, $\lambda_t = \{0, a, b, c\}$ is a prime ideal of A but not minimal.

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