ANALYTIC MAPS OF PARABOLIC AND ELLIPTIC TYPE WITH TRIVIAL CENTRALISERS

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ABSTRACT. We prove that for a dense set of irrational numbers α, the analytic centraliser of the map $e^{2\pi i\alpha}z + z^2$ near 0 is trivial. We also prove that some analytic circle diffeomorphisms in the Arnold family, with irrational rotation numbers, have trivial centralisers. These provide the first examples of such maps with trivial centralisers.

1. INTRODUCTION

For $\alpha \in \mathbb{R}$, let $\mathcal{H}^\omega_\alpha$ denote the set of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ of the form

$$h(z) = e^{2\pi i\alpha}z + O(z^2),$$

defined near 0. We also consider the class $\mathcal{C}^\omega_\alpha$ of orientation preserving analytic diffeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$ with rotation number α. Let $\mathcal{H}^\omega = \cup_{\alpha\in\mathbb{R}}\mathcal{H}^\omega_\alpha$ and $\mathcal{C}^\omega = \cup_{\alpha\in\mathbb{R}}\mathcal{C}^\omega_\alpha$.

The analytic *centraliser* of an element $h \in \mathcal{H}^\omega_\alpha$, denoted by $\mathrm{Cent}(h)$, is the set of elements of $\mathcal{H}^\omega$ which commute with h near 0. From dynamical point of view, any element of $\mathrm{Cent}(h)$ is a conformal symmetry of the dynamics of h, that is, the conformal change of coordinates g which conjugate h to itself, $g^{-1} \circ h \circ g = h$. Evidently, $\mathrm{Cent}(h)$ forms a group, where the action is the composition of the elements. For every $k \in \mathbb{Z}$, a suitable restriction of the k-fold composition $h^{\circ k}$ is defined near 0 and belongs to $\mathrm{Cent}(h)$. If the only elements of $\mathrm{Cent}(h)$ are of the form $h^{\circ k}$ for some $k \in \mathbb{Z}$, it is said that h has a *trivial centraliser*. In the same fashion, for $h \in \mathcal{C}^\omega$, the collection $\mathrm{Cent}(h)$ of elements of $\mathcal{C}^\omega$ which commute with h enjoys the same features.

Theorem 1.1. *There is a dense set of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $\mathrm{Cent}(e^{2\pi i\alpha}z + z^2)$ is trivial.*

The above theorem is proved using a successive perturbation argument and the following statement for parabolic maps which we prove in this paper.

Theorem 1.2. *For every $p/q \in \mathbb{Q}$, $\mathrm{Cent}(e^{2\pi i\alpha}z + z^2)$ is trivial.*

The main idea we employ to prove the above theorems also allows us to deal with analytic circle diffeomorphisms in the Arnold family,

$$S_{a,b}(x) = x + a + b\sin(2\pi x),$$

for $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$.

Theorem 1.3. *For every $b \in (0, 1/(2\pi))$ there is $a \in \mathbb{R}$ such that $\mathrm{Cent}(S_{a,b})$ is trivial and the rotation number of $S_{a,b}$ belongs to $\mathbb{R} \setminus \mathbb{Q}$.*

Indeed, we prove that for each fixed $b \in (0, 1/(2\pi))$, the set of rotation numbers of the maps $S_{a,b}$ which have an irrational rotation number and a trivial centraliser is dense in $\mathbb{R}$. The above theorem is obtained from a successive perturbation argument and the analogue of Theorem 1.2 for maps $S_{a,b}$ with a parabolic cycle.

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The main tool used to deal with parabolic maps is Ecalle cylinders and horn maps, first studied and applied by Douady-Hubbard [DH84] and Lavaures [Lav89].

To our knowledge, Theorems 1.1 and 1.3 provide the first examples in $\mathcal{H}^\omega$ and $\mathcal{C}^\omega$ with irrational rotation numbers and trivial analytic centralisers. Below we briefly explain how these results fit in the frame of the dynamics of such analytic diffeomorphisms.

When an element $h \in \mathcal{H}^\omega_\alpha$, for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, is locally conformally conjugate to its linear part near 0, $\mathrm{Cent}(h)$ is a large set. That is, if $\phi^{-1} \circ h \circ \phi(w) = e^{2\pi i \alpha} w$ near 0, for some $\phi \in \mathcal{H}^\omega$, then for any $\mu \in \mathbb{C} \setminus \{0\}$, h commutes with the map $z \mapsto \phi(\mu\phi^{-1}(z))$. Indeed, here, $\mathrm{Cent}(h)$ is isomorphic to $\mathbb{C} \setminus \{0\}$. The problem of understanding $\mathrm{Cent}(h)$ precedes the problem of local conjugation of h to its linear part. That is because, the space of solutions for the conjugation problem are the right-cosets of $\mathrm{Cent}(h)$. In this spirit, the size of $\mathrm{Cent}(h)$ may be thought of a measure of linearisability of h near 0. The same argument applies to analytic circle diffeomorphisms.

For $h \in \mathcal{H}^\omega$, $\mathrm{Cent}(h)$ projects onto a subgroup of $\mathbb{R}/\mathbb{Z}$ through $g \mapsto \log g'(0)/(2\pi i)$. Similarly, for $h \in \mathcal{C}^\omega$, one maps $g \in \mathrm{Cent}(h)$ to its rotation number. Let $\mathcal{G}(h) \subset \mathbb{R}/\mathbb{Z}$ denote the image of this projection.

By remarkable results of Siegel and Herman [Sie42, Her79] there is a full-measure set $\mathscr{C} \subset \mathbb{R} \setminus \mathbb{Q}$ such that for every $\alpha \in \mathscr{C}$, any $h \in \mathcal{H}^\omega_\alpha \cup \mathcal{C}^\omega_\alpha$ is analytically linearisable. But, for genetic choice of α, there are $h \in \mathcal{H}^\omega_\alpha$ and $h \in \mathcal{C}^\omega_\alpha$ which are not linearisable [Cre38, Arn61]. We note that if f and g commute, and one of them is linearisable at 0, then the other one must also be linearisable through the same map. This implies that if $h \in \mathcal{H}^\omega_\alpha \cup \mathcal{C}^\omega_\alpha$ is not linearisable, then $\mathcal{G}(h) \subseteq (\mathbb{R} \setminus \mathscr{C})/\mathbb{Z}$. However, by a profound result of Moser [Mos90], $\mathcal{G}(h)$ may not be any subgroup of that set. That is because there is an arithmetic restriction on the rotations of commuting non-linearisable maps. The optimal size of $\mathcal{G}(h)$, for nonlinearisable h in $\mathcal{H}^\omega_\alpha$ and $\mathcal{C}^\omega_\alpha$, remains open. This complication is due to the rich structure of the local dynamics of such maps near 0, see [PM95, Che17] and the references therein. However, a complete solution for smooth circle diffeomorphisms is presented in [FK09].

In [Her79, Yoc95, Yoc02], Herman and Yoccoz carry out a ground breaking study of the centraliser and conjugation problem for circle diffeomorphisms and germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$. In particular, Herman proves the existence of C^∞ circle diffeomorphisms with irrational rotation number having uncountably many C^∞ symmetries, and Yoccoz proves the existence of C^∞ circle diffeomorphisms with irrational rotation numbers and trivial centralisers. Perez-Marco in [PM95] elaborated a construction of Yoccoz to build elements $h \in \mathcal{H}^\omega$ and $h \in \mathcal{C}^\omega$, with irrational rotation number, such that $\mathcal{G}(h)$ is uncountable. His construction provides remarkable examples where $\mathcal{G}(h)$ contains infinitely many elements of finite order. In this paper we close the problem of the existence of maps in $\mathcal{H}^\omega$ and $\mathcal{C}^\omega$ with irrational rotation number and trivial centraliser. In light of the above discussions, our result shows that quadratic polynomials and the Arnold family provide the least linearisable elements in $\mathcal{H}^\omega$ and $\mathcal{C}^\omega$, respectively. This is consistent with the spirit of Yoccoz's argument in [Yoc95], that is, if some $e^{2\pi i \alpha} z + z^2$ is linearisable, then any element of $\mathcal{H}^\omega_\alpha$ is linearisable.

It is worth noting that the commutation problem for rational functions of the Riemann sphere was already studied by Fatou and Julia in 1920's [Jul22, Fat23] using iteration methods. A complete classification of such pairs was successfully obtain by Ritt [Rit23], using topological and analytic methods, and was reproved by Eremenko [Ere89] using modern iteration techniques. If iterates of g and h are not identical, modulo conjugation, they are either power maps, Chebyshev polynomials, or Lattès maps. The global commutation problem for entire functions of the complex plane still remains open, although substantial progress has been made so far, see for instance [GI59, Bak62, Lan99, Ng01, BRS16]. The global commutation problem on higher dimensional complex spaces has been widely studied using iteration methods in recent years, see [DS02, DS04, Kau18] and the references therein. For an extensive discussion on the centraliser and conjugation problems in low-dimensions one may refer to [Kop70] and the more recent survey article [OR10].

2. PARABOLIC CASE

Fix an arbitrary rational number $p/q \in \mathbb{Q}$ with $(p,q) = 1$. Also fix an arbitrary g in $\mathrm{Cent}(Q_{p/q})$.

The map $F = Q_{p/q}^{\circ q}$ has a parabolic fixed point at 0 with multiplier $+1$, and there are q attracting directions. It follows that the parabolic fixed point of F at 0 has multiplicity $q+1$. That is, the Taylor series expansion of F near 0 is of the form

$$F(z) = Q_{p/q}^{\circ q}(z) = z + \sum_{k=q+1}^{2^q} a_k z^k, \tag{1}$$

with $a_{q+1} \neq 0$.

Lemma 2.1. *We have $g'(0)^q = 1$.*

Proof. Let $g(z) = \sum_{k=1}^{\infty} b_k z^k$ denote the Taylor series expansion of g about 0. First we show that $b_1 \neq 0$. Assume, for a contradiction, that $n \geq 2$ is the smallest positive integer with $b_n \neq 0$. Note that $F \circ g = g \circ F$ near 0. By identifying the coefficient of z^{n+q} in the Taylor series expansion of $F \circ g$ and $g \circ F$ we conclude that $b_{n+q} + n b_n a_{q+1} = b_{n+q}$. Since $a_{q+1} \neq 0$, that gives us $b_n = 0$, which contradicts the choice of n.

Now we identify the coefficients of z^{q+1} in the power series expansions of $F \circ g$ and $g \circ F$, and obtain $b_{q+1} + b_1^{q+1} a_{q+1} = b_{q+1} + b_1 a_{q+1}$. This implies that $(b_1^{q+1} - b_1) a_{q+1} = 0$. Since $a_{q+1} \neq 0$ and $b_1 \neq 0$, we must have $b_1^q = 1$. □

By Lemma 2.1, there is an integer j with $0 \leq j \leq q-1$ such that $(Q_{p/q}^{\circ j} \circ g)'(0) = 1$. Consider the holomorphic map

$$G(z) = Q_{p/q}^{\circ j} \circ g, \tag{2}$$

which is defined near 0 and commutes with F.

Lemma 2.2. *The multiplicity of G at 0 is $q+1$. That is, $G(z) = z + \sum_{i=q+1}^{\infty} b_i z^i$, with $b_{q+1} \neq 0$.*

Proof. Assume that $G(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \dots$ is a convergent Taylor series with $b_{n+1} \neq 0$. Observe that

$$\begin{aligned} F \circ G(z) &= \big(z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \dots\big) \\ &\quad + a_{q+1} \big(z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \dots\big)^{q+1} \\ &\quad \vdots \\ &\quad + a_{q+j} \big(z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \dots\big)^{q+j} \\ &\quad \vdots \\ &= \big(z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \dots\big) \\ &\quad + a_{q+1} \big(z^{q+1} + b_{n+1}(q+1) z^{q+n+1} + \dots\big) \\ &\quad \vdots \\ &\quad + a_{q+j} \big(z^{q+j} + b_{n+1}(q+j) z^{q+n+j} + \dots\big) \\ &\quad \vdots \end{aligned}$$

The coefficient of z^{q+n+1} in the above expansion is

$$b_{q+n+1} + a_{q+1} b_{n+1}(q+1) + a_{q+n+1}.$$

Similarly, the coefficient of z^{n+q+1} in the expansion of $G \circ F$ is

$$a_{q+n+1} + b_{n+1} a_{q+1}(n+1) + b_{q+n+1}.$$

Since $F \circ G = G \circ F$ near 0, the above values must be identical. Using $a_{q+1} \neq 0$ and $b_{n+1} \neq 0$, we conclude that $q = n$. □

We shall use the theory of Leau-Fatou flower, Fatou coordinates, and horn maps to exploit the local dynamics of F near 0. One may refer to [Mil06] and [Dou94] for the basic definitions and constructions we present below, although conventions may be different.

For $s > 0$, define the open sets

$$\Omega^s_{att} = \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta > s - |\operatorname{Im}\zeta|\}, \quad \Omega^s_{rep} = \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < -s + |\operatorname{Im}\zeta|\}.$$

Also, consider the map $I : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$,

$$I(z) = \frac{-1}{q a_{q+1} z^q}.$$

For $s > 0$ there are holomorphic and injective branches of I^{-1} defined on Ω^s_{att} and Ω^s_{rep}.

Consider two complex numbers v_{att} and v_{rep} such that

$$q a_{q+1} v_{att}^q = -1, \quad v_{rep} = e^{-\pi i/q} v_{att}.$$

Evidently, $I(v_{att}) = +1$ and $I(v_{rep}) = -1$. For $s > 0$, there is an injective and holomorphic branch of I^{-1} defined on Ω^s_{att} such that $I^{-1}(\Omega^s_{att})$ contains εv_{att}, for sufficiently small $\varepsilon > 0$. Similarly, there is an injective branch of I^{-1} defined on Ω^s_{rep} such that $I^{-1}(\Omega^s_{rep})$ contains εv_{rep}, for sufficiently small $\varepsilon > 0$. From now on, we shall fix these choices of inverse branches for I^{-1} on Ω^s_{att} and Ω^s_{rep}. This is independent of $s > 0$.

Let

$$\begin{aligned} W_{att} &= \{z \in \mathbb{C}\setminus\{0\} \mid |\arg(z/v_{att})| \le \pi/q\}, \\ W_{rep} &= \{z \in \mathbb{C}\setminus\{0\} \mid |\arg(z/v_{rep})| \le \pi/q\}, \\ W'_{att} &= \{z \in \mathbb{C}\setminus\{0\} \mid |\arg(z/v_{att})| \le \pi/q - \pi/(4q)\}, \\ W'_{rep} &= \{z \in \mathbb{C}\setminus\{0\} \mid |\arg(z/v_{rep})| \le \pi/q - \pi/(4q)\}, \end{aligned}$$

where arg denotes a branch of argument with values in $[-\pi, +\pi]$.

Let U be a Jordan neighbourhood of 0 such that G is defined on U and both G and F are injective on U. Since $F'(0) = 1$ and $G'(0) = 1$, there is $\delta > 0$ such that $B(0,\delta) \subset U$ and

$$\begin{aligned} F(W'_{att} \cap B(0,\delta)) \subset W_{att}, \quad F(W'_{rep} \cap B(0,\delta)) \subset W_{rep}, \\ G(W'_{att} \cap B(0,\delta)) \subset W_{att}, \quad G(W'_{rep} \cap B(0,\delta)) \subset W_{rep}. \end{aligned} \tag{3}$$

We may choose $r > 0$ such that

$$I^{-1}(\Omega^r_{att}) \subset W'_{att} \cap B(0,\delta), \quad I^{-1}(\Omega^r_{rep}) \subset W'_{rep} \cap B(0,\delta). \tag{4}$$

Now we may lift $F : W'_{att} \cap B(0,\delta) \to W_{att}$ and $F : W'_{rep} \cap B(0,\delta) \to W_{rep}$ via the change of coordinate $I(z) = \zeta$ to define injective holomorphic maps

$$\tilde{F}_{att} : \Omega^r_{att} \to \mathbb{C}, \quad \text{and} \quad \tilde{F}_{rep} : \Omega^r_{rep} \to \mathbb{C}.$$

Straightforward calculations show that $\tilde{F}$ is of the form

$$\tilde{F}_{att}(\zeta) = \zeta + 1 + O(1/|\zeta|^{1/q}), \quad \tilde{F}_{rep}(\zeta) = \zeta + 1 + O(1/|\zeta|^{1/q}),$$

as $|\zeta| \to +\infty$. There is $s > 0$ such that,

$$\begin{aligned} |\tilde{F}_{att}(\zeta) - (\zeta+1)| \le 1/4, \quad \forall \zeta \in \Omega^s_{att}, \\ |\tilde{F}_{rep}(\zeta) - (\zeta+1)| \le 1/4, \quad \forall \zeta \in \Omega^s_{rep}. \end{aligned}$$

There are injective holomorphic maps

$$\Phi_{att} : \Omega^s_{att} \to \mathbb{C}, \quad \Phi_{rep} : \Omega^s_{rep} \to \mathbb{C},$$

such that

$$\Phi_{att} \circ \tilde{F}_{att} = \Phi_{att} + 1, \quad \text{on } \Omega^s_{att},$$

$$\Phi_{rep} \circ \tilde{F}_{rep} = \Phi_{rep} + 1, \quad \text{on } \tilde{F}^{-1}_{rep}(\Omega^s_{rep}).$$

It is known that

$$|\Phi_{att}(\zeta)/\zeta - 1| \to 0, \quad \text{as } \operatorname{Re}\zeta \to +\infty, \tag{5}$$

$$|\Phi_{rep}(\zeta)/\zeta - 1| \to 0, \quad \text{as } \operatorname{Re}\zeta \to -\infty. \tag{6}$$

Let us define

$$\mathcal{P}^s_{att} = I^{-1}(\Omega^s_{att}), \quad \mathcal{P}^s_{rep} = I^{-1}(\Omega^s_{rep}).$$

Then, the injective holomorphic maps

$$\phi_{att} = \Phi_{att} \circ I : \mathcal{P}^s_{att} \to \mathbb{C}, \quad \phi_{rep} = \Phi_{rep} \circ I : \mathcal{P}^s_{rep} \to \mathbb{C},$$

satisfy

$$\begin{aligned} \phi_{att} \circ F &= \phi_{att} + 1, \quad \text{on } \mathcal{P}^s_{att}, \\ \phi_{rep} \circ F &= \phi_{rep} + 1, \quad \text{on } F^{-1}(\mathcal{P}^s_{rep}). \end{aligned} \tag{7}$$

The map ϕ_{att} is an *attracting Fatou coordinate* for F, and ϕ_{rep} is a *repelling Fatou coordinate* for F.

Let

$$\mu = b_{q+1}/a_{q+1}.$$

Lemma 2.3. *There is $t \geq 0$ such that*

(i) *$G(z) = \phi^{-1}_{att} \circ T_\mu \circ \phi_{att}(z)$, for all $z \in \mathcal{P}^t_{att}$,*
(ii) *$G(z) = \phi^{-1}_{rep} \circ T_\mu \circ \phi_{rep}(z)$, for all $z \in \mathcal{P}^t_{rep}$.*

Proof. By Equations (3) and (4), we may lift $G : W'_{att} \cap B(0,\delta) \to W_{att}$ via the change of coordinate $I(z) = \zeta$ to define an injective holomorphic map $\tilde{G}_{att} : \Omega^r_{att} \to \mathbb{C}$. We note that $\tilde{G}_{att}$ is of the form

$$\tilde{G}_{att}(\zeta) = \zeta + \frac{b_{q+1}}{a_{q+1}} + O\left(\frac{1}{|\zeta|^{1/q}}\right), \text{ as } |\zeta| \to +\infty.$$

In particular, if $|\zeta|$ is large enough, $|\tilde{G}_{att}(\zeta) - (\zeta + \mu)| \leq 1$. This implies that there is $t > s$ such that

$$\tilde{G}_{att}(\Omega^t_{att}) \subset \Omega^s_{att}.$$

Let

$$V = \Phi_{att}(\Omega^t_{att}).$$

Note that since $\tilde{F}_{att}(\Omega^t_{att}) \subset \Omega^t_{att}$, $V + 1 \subset V$. By Equation (5), if $\operatorname{Re}\zeta$ is large enough, $|\Phi_{att}(\zeta) - \zeta| \leq |\zeta|/3$. This implies that

$$V/\mathbb{Z} = \mathbb{C}/\mathbb{Z}.$$

Consider the injective holomorphic map

$$\hat{G}_{att} = \Phi_{att} \circ \tilde{G}_{att} \circ \Phi^{-1}_{att} : V \to \mathbb{C}.$$

Since F commutes with G near 0, $\tilde{F}_{att}$ commutes with $\tilde{G}_{att}$ on the common domain of definition Ω^t_{att}. Therefore, for $w \in V$, we have

$$\begin{aligned}\hat{G}_{att} \circ T_1(w) &= \Phi_{att} \circ \tilde{G}_{att} \circ \Phi_{att}^{-1} \circ T_1(w)\\ &= \Phi_{att} \circ \tilde{G}_{att} \circ \tilde{F}_{att} \circ \Phi_{att}^{-1}(w)\\ &= \Phi_{att} \circ \tilde{F}_{att} \circ \tilde{G}_{att} \circ \Phi_{att}^{-1}(w)\\ &= T_1 \circ \Phi_{att} \circ \tilde{G}_{att} \circ \Phi_{att}^{-1}(w) = T_1 \circ \hat{G}_{att}(w).\end{aligned}$$

Since $V/\mathbb{Z} = \mathbb{C}/\mathbb{Z}$, the above relation implies that $\hat{G}_{att}$ induces a well-defined injective holomorphic map from $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C}/\mathbb{Z}$. Thus, $\hat{G}_{att}$ is a translation on $V/\mathbb{Z}$, and hence, $\hat{G}_{att}$ is a translation on V, say T_τ. However, since $\Phi'_{att}(\zeta) \to +1$, as $\operatorname{Re}\zeta \to +\infty$, and $\tilde{G}_{att}(\zeta)$ is asymptotically a translation by μ near $+\infty$, we must have $\tau = \mu$. That is, $\hat{G}_{att} = T_\mu$.

For $z \in \mathcal{P}^t_{att}$, we have

$$\begin{aligned}\phi_{att}^{-1} \circ T_\mu \circ \phi_{att} &= I^{-1} \circ \Phi_{att}^{-1} \circ T_\mu \circ \Phi_{att} \circ I\\ &= I^{-1} \circ \Phi_{att}^{-1} \circ \hat{G}_{att} \circ \Phi_{att} \circ I = I^{-1} \circ \tilde{G}_{att} \circ I = G.\end{aligned}$$

Part (ii): As in the previous part, we may lift $G : W'_{rep} \cap B(0, \delta) \to W_{rep}$ to obtain an injective holomorphic map $\tilde{G}_{rep} : \Omega^r_{rep} \to \mathbb{C}$ of the form $\tilde{G}_{rep} = \zeta + \mu + o(1)$, as $|\zeta| \to +\infty$. Then, one may repeat the argument in part (i) with $\tilde{F}_{rep}$ and Φ_{rep}. □

Let B denote the set of $z \in \mathbb{C}$ such that $F^{\circ n}(z) \to 0$, as $n \to +\infty$. Evidently, $\mathcal{P}^s_{att}$ is contained in B. Let B_1 denote the connected component of B which contains $\mathcal{P}^s_{att}$. (That is, B_1 is the immediate basin of attraction of 0 in the direction of v_{att}.) For every $z \in B_1$, there is $k \in \mathbb{N}$ with $F^{\circ k}(z) \in \mathcal{P}^s_{att}$. By the maximum principle, B_1 is a simply connected subset of $\mathbb{C}$. We may employ the functional relation in Equation 7, to extend $\phi_{att} : \mathcal{P}^s_{att} \to \mathbb{C}$ to a holomorphic map

$$\phi_{att} : B_1 \to \mathbb{C},$$

such that $\phi_{att} \circ F = \phi_{att} + 1$ over all of B_1.

Consider the trip

$$\Pi = \{w \in \mathbb{C} \mid -t - |\mu| - 1 < \operatorname{Re} w < -t\} \subset \Omega^t_{rep}.$$

By the estimate in (6), if $w \in \Pi$ with $\operatorname{Im} w$ sufficiently large, $\Phi^{-1}_{rep}(w) \in \Omega^s_{att}$, and hence $\phi_{rep}(w) \in B_1$. On the other hand, for some $w \in \Pi$, $\phi_{rep}(w)$ does not belong to B_1. Otherwise, a neighbourhood of 0 lies in B_1, which is not possible since 0 belongs to the Julia set of F.

Let Π' denote the connected component of the set $\{w \in \Pi \mid \phi^{-1}_{rep}(w) \in B_1\}$ which contains the top end of Π. We may consider the map

$$h = \phi_{att} \circ \phi^{-1}_{rep} : \Pi' \to \mathbb{C}.$$

This is a *horn map* of F. By the functional equations for ϕ_{att} and ϕ_{rep}, we must have $h(\zeta+1) = h(\zeta)+1$, whenever both side of the equation are defined. Thus, h induces a holomorphic map

$$H : \operatorname{Dom} H \to \mathbb{C},$$

on a punctured neighbourhood of 0 so that $H \circ e^{2\pi i \zeta} = e^{2\pi i h(\zeta)}$. By the estimates in (5) and (6), $\operatorname{Im} h(\zeta) \to +\infty$, as $\operatorname{Im}\zeta \to +\infty$. This implies that H has a removable singularity at 0. That is $\operatorname{Dom} H$ contains a neighbourhood of 0. [1]

Lemma 2.4. *The map H has infinitely many critical points, all mapped to the same value.*

[1] The map H is only defined modulo pre-composition and post-composition by linear maps of the form $w \mapsto \lambda w$. This is due to the freedom in the choice of ϕ_{att} and ϕ_{rep} up to post-compositions with translations. However, we are not concerned with those choices here.

Proof. Let c_1 denoted the unique critical point of F within B_1. The map ϕ_{att} has a simple critical point at c_1. It follows from Equation (7) that any $z \in B_1$ which is mapped to c_1 under some iterate of F is a critical point of ϕ_{att}. The closure of the set of such points is equal to the boundary of B_1.

On the other hand, by Equation (7), those critical points are mapped to $\phi_{att}(c_1)$, $\phi_{att}(c_1) - 1$, $\phi_{att}(c_1) - 2, \dots$. Since ϕ_{rep}^{-1} is conformal on $\Pi' \subset \Omega_{rep}^t$, we conclude that the only critical values of h are at $\phi_{att}(c_1), \phi_{att}(c_1) - 1, \phi_{att}(c_1) - 2, \dots$. All those points project to the same value in $\mathbb{C}/\mathbb{Z}$. □

Lemma 2.5. *The map H commutes with $\xi \mapsto e^{2\pi i\mu}\xi$ near 0.*

Proof. By Lemma 2.3, $G = \phi_{att}^{-1} \circ T_\mu \circ \phi_{att}$ on $\mathcal{P}_{att}^t$, and $G = \phi_{rep}^{-1} \circ T_\mu \circ \phi_{rep}$ on $\mathcal{P}_{rep}^t$. Thus,

$$\phi_{att}^{-1} \circ T_\mu \circ \phi_{att} = \phi_{rep}^{-1} \circ T_\mu \circ \phi_{rep},$$

at any point in $\mathcal{P}_{att}^t \cap \mathcal{P}_{rep}^t$ where both sides of the equation are defined. Equivalently,

$$T_\mu \circ \phi_{att} \circ \phi_{rep}^{-1} = \phi_{att} \circ \phi_{rep}^{-1} \circ T_\mu,$$

whenever both sides of the equation are defined. We note that $T_\mu^{-1}(\Pi') \cap \Pi'$ is a non-empty open set, where both sides of the above equation are defined. This implies that the horn map h commutes with T_μ. Hence, H commutes with the map $\xi \mapsto e^{2\pi i\mu}\xi$. □

Lemma 2.6. *We have $\mu \in \mathbb{Z}$.*

Proof. First note that $\operatorname{Dom} H$ is invariant under multiplication by $e^{2\pi i\mu}$. That is, on the set $e^{2\pi i\mu} \cdot \operatorname{Dom} H$ we may define H as $\xi \mapsto e^{2\pi i\mu} H(e^{-2\pi i\mu}\xi)$. This matches H on $(e^{2\pi i\mu} \cdot \operatorname{Dom} H) \cap \operatorname{Dom} H$.

Let c denote a critical point of H. Differentiating $H(e^{2\pi i\mu}\xi) = e^{2\pi i\mu}H(\xi)$ at c, we note that $e^{2\pi i\mu}c$ is a critical point of H. However, $H(e^{2\pi i\mu}c) = e^{2\pi i\mu}H(c)$ is a critical value of H. By Lemma 2.4, we must have $H(c) = e^{2\pi i\mu}H(c)$, which using $H(c) \neq 0$, we conclude that $\mu \in \mathbb{Z}$. □

Proof of Theorem 1.2. By Lemma 2.3, $G = \phi_{att}^{-1} \circ T_\mu \circ \phi_{att}$ on $\mathcal{P}_{att}^t$, and by Lemma 2.6, μ is an integer. Thus, on $\mathcal{P}_{att}^t$,

$$G = \phi_{att}^{-1} \circ T_1^{\circ\mu} \circ \phi_{att} = (\phi_{att}^{-1} \circ T_1 \circ \phi_{att}) \circ (\phi_{att}^{-1} \circ T_1 \circ \phi_{att}) \circ \cdots \circ (\phi_{att}^{-1} \circ T_1 \circ \phi_{att}) = F^{\circ\mu}.$$

As $\mathcal{P}_{att}^t$ is a non-empty open set, we must have $G = F^{\circ\mu}$ on a neighbourhood of 0.

Looking back at definitions (1) and (2), we conclude that $(Q_{p/q}^{\circ q})^{\circ\mu} = Q_{p/q}^{\circ j} \circ g$, on a neighbourhood of 0, for some $0 \le j \le q-1$. Thus, $g = Q_{p/q}^{\circ(q\mu - j)}$ near 0. □

3. Elliptic case

Let $g(z) = \sum_{k=1}^\infty g_k z^k \in \operatorname{Cent}(Q_\alpha)$. It is easy to see that $|g_1| = 1$. Let us say that g is *r-good*, if $|g_k| \le r^{1-k}$ for all $k \ge 1$. Note that if g is r-good, then it is defined and holomorphic on the disk $|z| < r$.

Lemma 3.1. *For every $p/q \in \mathbb{Q}$ and every $r > 0$, $Q_{p/q}^{\circ k}$ is r-good for only finitely many values of $k \in \mathbb{Z}$.*

Proof. As $Q_{p/q}$ has a parabolic fixed point at 0, the family of iterates $\{Q_{p/q}^{\circ k}\}_{k\ge 0}$ and $\{Q_{p/q}^{\circ -k}\}_{k \ge 0}$ have no uniformly convergent subsequence on any neighbourhood of 0. □

We let

$$K(p/q, r) = \left\{ k \in \mathbb{Z} \, ; \, Q_{p/q}^{\circ k} \text{ is } r\text{-good} \right\}.$$

By the above lemma, $K(p/q, r)$ is a finite set.

Lemma 3.2. *For every $p/q \in \mathbb{Q}$ and every $r > 0$, there exists $\delta(p/q, r) > 0$ such that for every $p'/q' \in \mathbb{Q}$ with $|p'/q' - p/q| \le \delta(p/q, r)$ we have $K(p'/q', r) \subseteq K(p/q, r)$.*

Proof. By the compactness of the set of r-good holomorphic maps, there is $N(r)$ such that any r-good map has less than $N(r)$ critical points is the disk $|z| < r/2$.

As L tends to $+\infty$, the set of the critical points of $Q_{p/q}^{\circ L}$ increases, and accumulates on 0. Let $L \in \mathbb{N}$ be such that $Q_{p/q}^{\circ L}$ has at least $N(r)$ critical points in the open disk $|z| < r/2$. If p'/q' is close enough to p/q, then $Q_{p'/q'}^{\circ L}$ has at least $N(R)$ critical points in the open disk $|z| < r/2$. For $l \geq L$, $Q_{p'/q'}^{\circ l}$ has at least all those critical points, so it is not r-good.

Let $M \in \mathbb{N}$ be such that $Q_{p/q}^{\circ -M}$, and hence $Q_{p/q}^{\circ -m}$ for any $m \geq M$, does not extend to the open disk $|z| < r$. Then, the same is true for p'/q' close to p/q.

Finally, if $k \notin K(p/q, r)$ and $-M \leq k \leq L$, $Q_{p'/q'}^{\circ k}$ may not be r-good if p'/q' is too close to p/q, because otherwise one could take limits to conclude that $Q_{p/q}^{\circ k}$ is r-good. □

Lemma 3.3. *For every $p/q \in \mathbb{Q}$, every $r > 0$, and every $\epsilon > 0$, there exists $\kappa(p/q, r, \epsilon) > 0$ which satisfies the following. For every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ with $|\alpha - p/q| \leq \kappa(p/q, r, \epsilon)$, and every $g(z) = e^{2\pi i\beta}z + O(z^2)$ which commutes with Q_α and is r-good, there exists $k \in K(p/q, r)$ such that $|\beta - kp/q| < \epsilon \mod \mathbb{Z}$.*

Proof. If the result does not hold, we may take a sequence $\alpha_n \to p/q$ and r-good maps $g_n(z) = e^{2\pi i\beta_n}z + O(z^2)$ which commute with Q_{α_n}. By the compactness of the set of r-good maps, we may choose a convergent subsequence of the g_n converging to a limit g which is r-good and commutes with $Q_{p/q}$. Then, g will not be of the form $Q_{p/q}^{\circ k}$ for some $k \in K(p/q, r)$. This contradicts Theorem 1.2 and Lemma 3.1. □

Lemma 3.4. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, if a holomorphic germ of the form $g(z) = e^{2\pi ik\alpha}z + O(z^2)$, for some $k \in \mathbb{Z}$, commutes with Q_α, then $g = Q_\alpha^{\circ k}$ near 0.*

Proof. By considering $Q_\alpha^{\circ -k} \circ g$ instead, we may assume that $k = 0$. Then, by an inductive argument, one may show that the coefficients of the Taylor series expansion of g, except the first term, must be 0. That is, $g(z) = z$. □

proof of Theorem 1.1. Start with any rational number p_1/q_1. We inductively define a strictly increasing sequence of rational numbers p_n/q_n, for $n \geq 1$, so that for all $1 \leq l \leq j < n$ we have

$$|p_n/q_n - p_j/q_j| < \delta(p_j/q_j, 1/j), \tag{8}$$

$$|p_n/q_n - p_j/q_j| < \kappa(p_j/q_j, 1/l, 1/j), \tag{9}$$

$$|p_n/q_n - p_j/q_j| < 1/q_j^2. \tag{10}$$

Let $\alpha = \lim_{n\to\infty} p_n/q_n$. Since the sequence p_n/q_n is strictly increasing, it follows from Equation (10) that $q_n \to \infty$, as $n \to \infty$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Taking limit as $n \to \infty$ in Equation (9), we note that $|\alpha - p_j/q_j| \leq \kappa(p_j/q_j, 1/l, 1/j)$, for every $1 \leq l \leq j$.

Assume that $g(z) = e^{2\pi i\beta}z + O(z^2)$ is a germ of a holomorphic map which commutes with Q_α. There is $l \geq 1$ such that g is $1/l$-good.

By Equation (8) and Lemma 3.2, we obtain $K(p_j/q_j, 1/l) \subseteq K(p_l/q_l, 1/l)$, for $1 \leq l \leq j$.

By Lemma 3.3, for every $j \geq l$, there exists $k \in \mathbb{Z}$ with $k \in K(p_j/q_j, 1/l) \subseteq K(p_l/q_l, 1/l)$ such that $|\beta - kp_j/q_j| < 1/j \mod \mathbb{Z}$. Taking limits of the latter inequality, as $j \to \infty$, we obtain $\beta = k\alpha$, for some k in the same range. Combining with Lemma 3.4, we conclude that $g = Q_\alpha^{\circ k}$ near 0. □

4. Circle maps

We shall employ techniques from complex dynamics to study the analytic symmetries of the maps $S_{a,b}$. So we consider the complexified family of maps $S_{a,b}(z) = z + a + b\sin(2\pi z)$, for $z \in \mathbb{C}$, but real values of a and b. Using the projection $z \mapsto e^{2\pi iz}$ from $\mathbb{C}$ to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $S_{a,b}$ induces the holomorphic map

$$f_{a,b}(w) = e^{2\pi ia} w e^{\pi b(w-1/w)}$$

from $\mathbb{C}^*$ to $\mathbb{C}^*$. Evidently, $f_{a,b}$ preserves the unit circle $\mathbb{T} = \{w \in\subset \mathbb{C} \;;\; |z| = 1\}$, and for $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$, $f_{a,b}$ is a diffeomorphism of $\mathbb{T}$. Below we always assume that $a \in \mathbb{R}$ and $b \in (0, 1/(2\pi))$.

Theorem 4.1. *Assume that $f_{a,b}$ has a parabolic cycle on $\mathbb{T}$, for some a and b. Then,* $\mathrm{Cent}(f_{a,b})$ *is trivial.*

Let us fix an arbitrary $f_{a,b}$ which has a parabolic cycle on $\mathbb{T}$, say $\{w_i\}_{i=1}^n$, of period $n \geq 1$. By relabelling if necessary, we may assume that $f_{a,b}(w_i) = w_{i+1}$, with the subscripts calculated modulo n. Consider the map

$$F_{a,b} = f_{a,b}^{\circ n} : \mathbb{C}^* \to \mathbb{C}^*.$$

Each w_i is a parabolic fixed point of $F_{a,b}$ with multiplier $+1$. For $1 \leq i \leq n$, let $U_i \subset \mathbb{C}^*$ denote the immediate basin of attraction of w_i for the iterates of $F_{a,b}$. That is, U_i is the union of the connected components of the basin of attraction of w_i which contain w_i on their boundary. The following lemma is a special case of a more general result by Geyer [Gey01, thm 4.4].

Lemma 4.2. *For every $1 \leq i \leq n$, U_i consists of a single connected component, which is invariant under τ, and contains precisely two distinct critical points of $F_{a,b}$. Moreover, $\cup_{i=1}^n \overline{U_i} = \mathbb{T}$.*

Proof. The critical points of $f_{a,b}$ are the solutions of the equation $f'_{a,b}(w) = e^{2\pi ia} e^{\pi b(w-1/w)}(1+\pi b(w+1/w)) = 0$. Evidently, if w is a solution of this equation, then $\overline{w}$, $1/w$ and $1/\overline{w}$ are all solutions of the equation. Thus, $w = \overline{w}$, and hence, the distinct solution of the equation are of the form c_1 and $c_2 = \tau(c_1)$, for some $c_1 \in (-1, 0)$.

Since $F_{a,b}$ is τ-symmetric, it follows that $\tau(U_i) = U_i$, for $1 \leq i \leq n$. Moreover, since $F_{a,b}(w_i) = w_i$, every connected component of each U_i is invariant under $F_{a,b}$. By a classical result of Fatou, see [Mil06], every connected component of each U_i contains at least one critical point of $F_{a,b}$. On the other hand, the critical points of $F_{a,b}$ are the pre-images of the critical points of $f_{a,b}$. Since $f_{a,b}(U_i) = U_{i+1}$, it follows that there is j with $1 \leq j \leq n$, such that U_j contains the critical points c_1 and c_2. Moreover, c_1 and c_2 are the only critical points of $F_{a,b}$ inside U_j. Then, the critical values of $f_{a,b}$ belong to U_{j+1}, which is distinct from U_j.

By the maximum principle, every connected component of each U_i is a simply connected region. Since the critical values of $f_{a,b}$ belong to U_{j+1}, any other U_i does not contain any critical values of $f_{a,b}$. These imply that for $1 \leq l \leq n-1$ there is a conformal branch of $f_{a,b}^{\circ -l}$ from U_j to U_{j-l}. Therefore, each U_i contains exactly two critical points of $F_{a,b}$.

Every connected component of each U_i is invariant under $F_{a,b}$ and τ, and contains at least one critical point of $F_{a,b}$. Therefore, the number of the critical points in U_i is two times the number of the connected components of U_i. Since U_i contains exactly two critical points of $F_{a,b}$, U_i consists of a single connected component containing both critical points.

Since each U_i has a single connected component, each w_i has a single attraction vector and a single repulsion vectors. As $\mathbb{T}$ is invariant, the attraction and repulsion vectors are the tangent vectors to $\mathbb{T}$ at w_i. Fix an arbitrary i and consider an arc of $\mathbb{T}$ cut off by w_i and w_{i+1} which does not contain any other w_l. This arc is invariant under $F_{a,b}$, and the orbit of any point on this arc must converge to either w_i or w_{i+1}. Otherwise, there will be another fixed point of $F_{a,b}$ on this arc which is distinct from w_i and w_{i+1}, and is either attracting or parabolic. This is a contradiction since such a cycle requires its own critical points distinct from the grand orbit of c_1 and c_2. □

By relabelling the points w_i, and U_i accordingly, we may assume that U_1 contains the critical points c_1 and c_2 of $f_{a,b}$.

Since there is only one attracting direction for $F_{a,b}$ at w_1, it follows that the multiplicity of the parabolic fixed point at w_1 is equal to $+2$. As in the previous section, there are attracting and repelling Fatou coordinates

$$\phi_{att} : \mathcal{P}_{att} \to \mathbb{C}, \qquad \phi_{rep} : \mathcal{P}_{rep} \to \mathbb{C},$$

satisfying the functional equations

$$\phi_{att} \circ F_{a,b} = \phi_{att} + 1, \qquad \phi_{rep} \circ F_{a,b} = \phi_{rep} + 1,$$

with $\phi_{att}(\mathcal{P}_{att}) = \Omega^s_{att}$ and $\phi_{rep}(\mathcal{P}_{rep}) = \Omega^s_{rep}$ for some $s > 0$, $F^{\circ j}_{a,b}$ converges to w_1 uniformly on compact subsets of $\mathcal{P}_{att}$ as $j \to +\infty$, and $F^{\circ j}_{a,b}$ converges to w_1 uniformly on compact subsets of $\mathcal{P}_{rep}$ as $j \to -\infty$. The attracting coordinate may be extended to a holomorphic map $\phi_{att} : U_1 \to \mathbb{C}$ using the above functional equation.

The map

$$h = \phi_{att} \circ \phi^{-1}_{rep}$$

has a maximal domain of definition, which is $\phi^{-1}_{rep}(U_1) + \mathbb{Z}$. This induces a holomorphic map H defined on a neighbourhood of 0, with $H(0) = 0$.

Lemma 4.3. *The horn map H has infinitely many critical points, which are mapped to critical values v_1 and v_2 satisfying $\arg v_1 = \arg v_2$.*

Proof. Any pre-image of c_1 and c_2 under $F_{a,b}$ within U_1 is a critical point of ϕ_{att}. The set of the accumulation points of those pre-images is equal to the boundary of U_1 (which is contained in the Julia set of $F_{a,b}$). By the functional equation for ϕ_{att}, ϕ_{att} maps those critical points into the set $\phi_{att}(c_1) + \mathbb{Z}$ or $\phi_{att}(c_2) + \mathbb{Z}$. On the other hand, ϕ_{rep} is conformal on Ω^s_{rep}. This implies that the only critical value of h are contained in $(\phi_{att}(c_1) + \mathbb{Z}) \cup (\phi_{att}(c_2) + \mathbb{Z})$.

Since $F_{a,b}$ is τ-symmetric, both ϕ_{att} and ϕ_{rep} are τ-symmetric. That is, $\phi_{att} \circ \tau = \overline{\phi_{att}}$ and $\phi_{rep} \circ \tau = \overline{\phi_{rep}}$. This is due the uniqueness of a Fatou-coordinate up to translation by a constant. Combining with the above paragraph, we conclude that $\overline{\phi_{att}(c_1)} = \phi_{att}(c_2)$, and hence the critical values of H have the same argument. □

Proof of Theorem 4.1. The proof already starts at the beginning of this section. Fix an arbitrary $f_{a,b}$ with a parabolic cycle $\{w_i\}^n_{i=1}$ of period n. Let us also fix an arbitrary $g \in \mathrm{Cent}(f_{a,b})$. The commutation implies that $g(w_1)$ is a periodic point of period n for $f_{a,b}$. By Lemma 4.2, $f_{a,b}$ has a unique periodic cycle, which is $\{w_i\}^n_{i=1}$. Therefore, there is an integer $k \geq 1$ such that $f^{\circ k}_{a,b} \circ g(w_1) = w_1$. Let us define the analytic map

$$G = f^{\circ k}_{a,b} \circ g : \mathbb{T} \to \mathbb{T}.$$

As $F_{a,b}$ commutes with G, $F_{a,b}(w_1) = w_1$, $F'_{a,b}(w_1) = 1$ we may repeat Lemma2.1 to conclude that $G'(w_1) = 1$. On the other hand, since the multiplicity of $F_{a,b}$ at w_1 is equal to $+2$, we may repeat Lemma 2.2 to conclude that the multiplicity of G at w_1 is also equal to $+2$. That is, G is of the form

$$G(w) = G(w_1) + (w - w_1) + b_2(w - w_1)^2 + \dots,$$

near 0, with $b_2 \neq 0$. As in the previous section, we must have $G = \phi^{-1}_{att} \circ T_\mu \circ \phi_{att}$ on $\mathcal{P}_{att}$ and $G = \phi^{-1}_{rep} \circ T_\mu \circ \phi_{rep}$ on $\mathcal{P}_{rep}$, where $\mu = 2b_2/F''_{a,b}(0)$. Repeating Lemma 2.5, we conclude that H must commute with the rotation $\xi \mapsto e^{2\pi i \mu}\xi$ near 0. Now, as in the proof of Lemma 2.6, we use Lemma 4.3 instead of Lemma 2.5, to say that if c is a critical point of H, then we must have $\arg H(c) = \arg(e^{2\pi i \mu} H(c))$. This implies that $\mathrm{Re}\, \mu \in \mathbb{Z}$. On the other hand, if $\mathrm{Im}\, \mu \neq 0$, since the domain of definition of H is invariant under $\xi \mapsto e^{2\pi i \mu}$, we conclude that H is defined over all of $\mathbb{C}$. But this is a contraction since H has infinitely many critical points in a bounded region of the plane. Therefore, $\mu \in \mathbb{Z}$, and hence $G = F^{\circ \mu}_{a,b}$. This completes the proof of Theorem 4.1 □

Fix an arbitrary $b \in (0, 1/(2\pi))$. By a general theorem of Poincaré, $f_{a,b}$ has a period point on $\mathbb{T}$ if and only if its rotation number $\rho(f_{a,b}) \in \mathbb{Q}$. Moreover, by classical results, the map $a \mapsto \rho(f_{a,b})$ is an increasing function of $a \in (0,1)$. It is locally strictly increasing at irrational values, that is, if $\rho(f_{a,b}) \in \mathbb{R} \setminus \mathbb{Q}$ for some a, then for $a' \in (0,1)$ with $a' > a$, $\rho(f_{a',b}) > \rho(f_{a,b})$. However, at rational values, the map is constant on a closed interval. [2]

Given $r > 1$, we say that an analytic homeomorphism $g : \mathbb{T} \to \mathbb{T}$ is r-good, if g is holomorphic on the annulus $1/r < |z| < r$ and maps that annulus to the annulus $1/2 < |z| < 2$. Evidently, every analytic homeomorphism of T is r-good for some $r > 1$. Moreover, by Schwarz-Pick lemma, for every $r > 1$, the class of r-good analytic homeomorphisms of $\mathbb{T}$ forms a compact class of maps.

Let us consider the sets

$$P = \{(a, b) \in (0, 1) \times (0, 1/(2\pi))\ ;\ f_{a,b} \text{ has a parabolic cycle on } \mathbb{T}\}.$$

and for each $b \in (0, 1/(2\pi))$,

$$P_b = \{a \in (0, 1)\ ;\ (a, b) \in P\}.$$

Lemma 4.4. *For every $(a, b) \in P$, $f_{a,b}^{\circ k}$ is r-good for only finitely many values of k.*

Proof. The proof of Lemma 3.1 may be repeated here to show this statement. □

For $(a, b) \in P$, we define

$$K'(a, b, r) = \left\{k \in \mathbb{Z}\ ;\ f_{a,b}^{\circ k} \text{ is } r\text{-good}\right\}.$$

Lemma 4.5. *For every $(a, b) \in P$ and every $r > 0$, there exists $\delta'(a, b, r) > 0$ such that for every $a' \in P_b$ with $|a' - a| \le \delta'(a, b, r)$ we have*

$$K'(a', b, r) \subseteq K'(a, b, r).$$

Proof. This is the same as the proof of Lemma 3.2. □

Lemma 4.6. *For every $(a, b) \in P$, every $r > 0$, and every $\epsilon > 0$, there exists $\kappa'(a, b, r, \epsilon) > 0$ which satisfies the following. For every $a' \in P_b$ with $|a' - a| \le \kappa'(a, b, r, \epsilon)$ and $\rho(f_{a',b}) \in \mathbb{R} \setminus \mathbb{Q}$, and every r-good map g which commutes with $f_{a',b}$, there exists $k \in K'(a, b, r)$ such that $|\rho(g) - k\rho(f_{a,b})| < \epsilon$* mod $\mathbb{Z}$.

Proof. The proof is identical to the one for Lemma 3.3. Here one uses the continuity of the map $x \mapsto \rho(f_{x,b})$, for $x \in \mathbb{R}$. □

Lemma 4.7. *Assume that $\rho(f_{a,b}) \in \mathbb{R} \setminus \mathbb{Q}$. If $g : \mathbb{T} \to \mathbb{T}$ is an analytic map which commutes with $f_{a,b}$ and $\rho(g) = k\rho(f)$ for some $k \in \mathbb{Z}$, then $g = f^{\circ k}$ on $\mathbb{T}$.*

Proof. By considering $f_{a,b}^{\circ -k} \circ g$ instead, we may assume that $\rho(g) = 0$. By Poincaré's theorem, g has a fixed point, and then by the commutation of $f_{a,b}$ and g, any iterate of that fixed point by $f_{a,b}$ must be a fixed point of g. Since the orbit of any point in $\mathbb{T}$ by $f_{a,b}$ is dense on $\mathbb{T}$, g has a dense set of fixed points. Thus, g is the identity map on $\mathbb{T}$. □

Proof of Theorem 1.3. The proof is similar to the one for Theorem 1.1, using Theorem 4.1 instead of Theorem 1.2.

Fix an arbitrary $b \in (0, 1/(2\pi))$, and start with an arbitrary $a \in P_b$. We inductively define an strictly increasing sequence of parameters $a_n \in P_b$, for $n \ge 1$, so that for all $1 \le l \le j < n$ we have

$$|a_n - a_j| < \delta'(a_j, b, 1/j), \tag{11}$$

$$|a_n - a_j| < \kappa'(a_j, b, 1/l, 1/j), \tag{12}$$

[2] The set of a and b where $\rho(f_{a,b})$ is a rational number has non-empty interior, and is known as Arnold tongues. One may refer to [Arn61], [Fag99] for basic features of those loci, and the global dynamics of the complexified standard family.

$$|\rho(f_{a_n,b}) - \rho(f_{a_j,b})| < 1/q_j^2, \tag{13}$$

where $p_j/q_j = \rho(f_{a_j,b}) \in \mathbb{Q}$ and $(p_j, q_j) = 1$.

Let $a = \lim_{n\to\infty} a_n$. Since the sequence a_n is strictly increasing, the sequence p_n/q_n must be increasing with at most two consecutive terms identical. It follows from Equation (13) that $q_n \to \infty$, as $n \to \infty$, and $\rho(f_{a,b}) \in \mathbb{R} \setminus \mathbb{Q}$.

Taking limit as $n \to \infty$ in Equation (12), we note that $|a - a_j| \leq \kappa'(a_j, b, 1/l, 1/j)$, for every $1 \leq l \leq j$.

Assume that g is a orientation preserving analytic homeomorphism of $\mathbb{T}$ which commutes with $f_{a,b}$. There is $l \geq 1$ such that g is $1/l$-good.

By Equation (8) and Lemma 3.2, we obtain $K'(a_j, b, 1/l) \subseteq K'(a_l, b, 1/l)$, for $1 \leq l \leq j$.

By Lemma 3.3, for every $j \geq l$, there exists $k \in \mathbb{Z}$ with $k \in K'(a_j, b, 1/l) \subseteq K'(a_l, b, 1/l)$ such that $|\rho(g) - kp_j/q_j| < 1/j \mod \mathbb{Z}$. Taking limits of the latter inequality, as $j \to \infty$, we obtain $\rho(g) = k\rho(f_{a,b})$, for some k in the same range. Combining with Lemma 3.4, we conclude that $g = f_{a,b}^{\circ k}$ on $\mathbb{T}$. □

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