Asymmetric unimodal maps with non-universal period-doubling scaling laws

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Abstract

We consider a family of strongly-asymmetric unimodal maps $\{f_t\}_{t\in[0,1]}$ of the form $f_t = t\cdot f$ where $f\colon [0,1]\to[0,1]$ is unimodal, $f(0)=f(1)=0$, $f(c)=1$ is of the form and

$$f(x) = \begin{cases} 1 - K_-|x-c| + o(|x-c|) & \text{for } x < c, \\ 1 - K_+|x-c|^\beta + o(|x-c|^\beta) & \text{for } x > c, \end{cases}$$

where we assume that $\beta > 1$. We show that such a family contains a Feigenbaum-Coullet-Tresser 2^∞ map, and develop a renormalization theory for these maps. The scalings of the renormalization intervals of the 2^∞ map turn out to be super-exponential and non-universal (i.e. to depend on the map) and the scaling-law is different for odd and even steps of the renormalization. The conjugacy between the attracting Cantor sets of two such maps is smooth if and only if some invariant is satisfied. We also show that the Feigenbaum-Coullet-Tresser map does not have wandering intervals, but surprisingly we were only able to prove this using our rather detailed scaling results.

Contents

1 Introduction

The theory of one-dimensional dynamics is rather well developed. Up till now the starting point of this theory has always been some 'real bounds' argument which implies absence of wandering intervals and certain non-linearity bounds for high iterates of the map. This then makes it possible to obtain, under suitable conditions, the existence of invariant measures and a renormalization theory culminating in universality properties. The latter means that two maps which are topologically conjugate, are in fact Hölder, quasi-symmetrically or even smoothly conjugate. In this paper, we will consider a setting where it seems one is only able to obtain the absence of wandering intervals after developing a rather intricate set of bounds, essentially amounting to a renormalization theory.

The question whether two maps which are combinatorially the same, are in fact topologically conjugate hinges on absence of wandering intervals. The first results in this direction were obtained for circle diffeomorphisms in the 1920's by Denjoy, for critical circle maps by Yoccoz [56] and for circle maps with plateaus by [36]. For interval maps there are results, in increasing generality, by Misiurewicz [45], Guckenheimer [15], de Melo-van Strien [41], Block-Lyubich [4], Lyubich [30], de Melo, Martens, van Strien, [37] and Vargas-van Strien [52]. On the other hand, interval exchange transformations can have wandering intervals, see e.g. [34]. There are some rather important cases whether nothing is known about the absence of wandering intervals. For example the case of smooth circle homeomorphisms with at least two singularities with one of the form $\pm|x-c_1|^\alpha$, $\alpha < 1$ and the other of the form $\pm|x-c_2|^\beta$ with $\beta > 1$ is wide open. The situation which we will consider in this paper is that of a unimodal interval map with a strong asymmetry, for example

$$f(x) = \begin{cases} f(c) - K_-|x-c|^\alpha & \text{for } x < c \\ f(c) + K_+|x-c|^\beta & \text{for } x > c \end{cases}$$

where $1 \le \alpha < \beta$. As mentioned, absence of wandering intervals is not known in this setting.

In this paper we will obtain absence of wandering intervals for such maps in the 'least expanding' case, namely when the map is infinitely renormalizable of the Feigenbaum-Coullet-Tresser type. To do this we develop a renormalization theory, which then provides enough information to show absence of wandering intervals.

Renormalisation and rigidity results were proved previously for circle diffeomorphisms with Diophantine conditions on the rotation number [16, 57]. For circle maps with discontinuities of the derivative (break type singularities) there are quite a few results, see e.g. [17, 18, 1, 19, 20]. For circle maps with plateaus there is for example [38]. For smooth homeomorphism of the circle with a critical point, there are results by [10, 11, 21, 55, 2]. For infinitely renormalizable unimodal interval maps there is a rich history, starting with the conjectures of Feigenbaum and Coullet-Tresser. Rigorous proofs were finally provided by [53, 43, 44, 3], see also [12, 49, 50, 51]. Note that for interval maps smooth rigidity is not possible, so the natural context there is quasi-symmetric rigidity. This was proved in increasing generality in [13, 31, 24, 7], see also [5, 7, 24, 29]. For Lorenz maps there is another very interesting phenomenon: in this case the renormalization operator can have several (degenerate) fixed points even when the left and right critical exponent at the discontinuity is the same. This can happen even for bounded combinatorics, and return maps can degenerate [40, 54].

Absence of wandering interval for our class of maps implies that the maps we consider are all topologically conjugate to the well-known Feigenbaum-Coullet-Tresser map.

One of the main challenges in dealing with these maps is that there are no Koebe space extensions. More precisely, 'real bounds' coming from diffeomorphic extensions of some first entry maps definitely do NOT hold. As far as we know this is the first type of unimodal map for which such bounds are known not to exist.

Let us now summarise the results in this paper.

- Although the period doubling diagram, see Figure 1, looks qualitatively the same as for the quadratic family, there are important differences: when n is odd, the periodic orbit of period 2^n doubles its period when it contain the critical point rather than when its multiplier is -1.

- In spite of the absence of Koebe space, we are able to obtain very precise scaling laws. Here the scaling laws are rather different than for the usual 'symmetric' Feigenbaum-Coullet-Tresser case where the scalings are geometric and universal (the rates only depend on the order of the critical point) and so there if we denote by $[a_k, b_k]$ the k-th renormalization interval we have

$$|b_{k+1} - a_{k+1}| \sim \alpha |b_k - a_k|$$

  for some $0 < \alpha < 1$ which does not depend on which unimodal map one takes (provided its critical point is quadratic). In our setting, the scalings of their lengths are quite different for even and odd steps, namely

$$\begin{aligned} |b_{2k+2} - a_{2k+2}| &\sim \beta^{\frac{-2}{\beta-1}} K_0^{\frac{1}{\beta-1}} \lambda^{-2} |b_{2k+1} - a_{2k+1}|^2 \\ |b_{2k+1} - a_{2k+1}| &\sim \lambda |b_{2k} - a_{2k}| \end{aligned}$$

  where λ is the root of $\lambda^\beta + \lambda - 1 = 0$ and $K_0 = K_+/K_-$. Moreover, there exists $\Theta > 0$ so that

$$|b_{2k} - a_{2k}| \sim \beta^{\frac{2}{\beta-1}} K_0^{\frac{-1}{\beta-1}} \exp(-2^k \Theta). \tag{1}$$

- In the usual Feigenbaum-Coullet-Tresser 2^∞ case, two maps with quadratic critical points are necessarily differentiably conjugate along the closure of the forward iterates of the critical point. This phenomenon is usually referred to as *universality*. Here this universality no longer holds: two maps $f, \tilde{f}$ are Lipschitz (and even differentiably conjugate) if and only if

$$\beta = \tilde{\beta}, \Theta = \tilde{\Theta}.$$

This means that this case is rather more similar to [33, 38] where there are also necessary and sufficient conditions for these maps to be differentiably conjugate at the turning point.

- One of the consequences of this fact is that f and its renormalizations are not Lipschitz conjugate even at the critical point c.
- In the 'symmetric' case the n-th renormalization of the function converges to some analytic function with unknown closed formula. Here we obtain a degenerate limit, but whose form is entirely explicit.
- The 2^∞ maps we consider do not have wandering intervals.

Open questions.

Before stating our results rigorously, let us discuss questions and possible directions for further research.

Super-exponential scaling when $\beta > \alpha > 1$. In this paper we always assumed that the left critical order α of our map is equal to 1. We believe that the super-exponential scaling of the points a_n and b_n that we have shown here, also holds when $1 < \alpha < \beta$ and also for more general combinatorics.

Absence of wild attractors when $\beta > \alpha > 1$. It is well-known that in the 'symmetric' case, the so-called Fibonacci map has a wild attractor provided the order of the critical point is large. Inspired by our belief that one has super-exponential scaling, we believe that such attractors do not exist when $\beta > \alpha > 1$, even if these numbers are arbitrarily large.

Absence of wandering intervals. In this paper we only proved absence of wandering intervals for the 2^∞ combinatorics and when $\beta \geq \alpha = 1$. We believe one has absence of wandering intervals without these assumptions. In fact, we tried and failed to prove this result in the case that $\beta > \alpha > 1$.

Monotonicity of bifurcations. Notice numerical simulations suggest that the bifurcations from the family f_t from equation (4) are monotone: no periodic orbit seems to disappear when t increases. When instead we consider the family

$$f_t(x) = \begin{cases} t - 1 - t|x|^\alpha & \text{when } x < 0, \\ t - 1 - tx^\beta & \text{when } x \geq 0. \end{cases} \tag{2}$$

with $\alpha, \beta > 1$ large, then there are partial results towards monotonicity in [27] see also [28]. Only when $\alpha = \beta$ is an even integer it is known that one has monotonicity. This was proved using complex methods by Sullivan, Thurston, Tsujii, Milnor, Douady, for references see [27].

More precise rigidity results. Consider continuous degree one circle maps, which are smooth local diffeomorphisms outside a single plateau and with x^β behaviour at the boundary points of this plateau. In earlier papers [36] it was shown that such maps have no wandering intervals, and in [47] it was shown that one has super-exponential decay of scales when $\beta \in (1, 2)$ when the rotation number is golden mean. In [38], Martens and Palmisano show that there exist invariants for Lipschitz, differentiable and $C^{1+\epsilon}$ conjugacy. For related results see [6]. A similar obstruction to differentiable conjugacy also appears in [33].

Parameter scaling. Consider the family f_t defined in (4) and let t_n be the parameter where the turning point 0 has period 2^n for f_{t_n} and let t_* be so that f_{t_*} has 2^∞ dynamics. Computer experiments suggest that the parameters t_n scale also super-exponentially. We are hopeful that we will be able to elaborate the methods in this paper to prove the following

Conjecture 1 (Non-universality of parameter bifurcations)**.**

$$|t_{n+2} - t_*| \sim \kappa |t_n - t_*|^2 \tag{3}$$

where κ depends non-trivially on the two parameters β, Θ associate to the family f_t and so is not a universal parameter, where Θ is defined through equation (1).

So we conjecture that, in our setting, the parameter scaling is super-exponential and non-universal. This is in contrast to the universality results for generic smooth families of unimodal maps with a quadratic critical point (where the genericity assumption is that the family is assumed to be transversal to the stable manifold of the renormalization operator) where one has the parameter scaling

$$|t_{n+2} - t_*| \sim \lambda |t_n - t_*|$$

where λ is universal and so does not depend on the family.

Renormalisation theory in the smooth setting. The renormalization theory we develop here is done by obtaining large bounds. This is quite different from the renormalization theory obtained for real analytic unimodal maps, [53, 43, 44, 32, 3, 12], see also [50, 35, 14]. It would be interesting to tie these approaches together.

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2 The setting of this paper

Consider the class $\mathcal{A}_{\alpha,\beta}$ of continuous unimodal maps $f : [a_0, b_0] \to [a_0, b_0]$ where $a_0 < 0 < b_0$ and with the following properties:

1 $f(a_0) = f(b_0) = a_0$ and outside the turning point $c := 0$ the map f is C^3 and has Schwarzian derivative $Sf \leq 0$. The authors believe that the results in this paper also hold without the $Sf \leq 0$ assumption.

2 $c = 0$ is the unique extremal value of f and $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$.

3 Near the critical point $c = 0$ the map f behaves as $f(x) \approx -K_-|x|^\alpha + f(0)$ for $x < 0$ and $|x|$ small and $f(x) \approx -K_+x^\beta + f(0)$ for small positive values of x. The constants should satisfy $K_- > 0$, $K_+ > 0$ and $\beta > \alpha \geq 1$.

We say that $f \in \mathcal{A}_{\alpha,\beta}(2^\infty)$ if in addition

4 The map f has 2^∞ combinatorics, i.e. f is an infinitely renormalizable *Feigenbaum-Coullet-Tresser* period doubling map.

Almost everywhere in the paper we shall assume that $\alpha = 1$, in this case we will denote $\mathcal{A}_{1,\beta}$ just by $\mathcal{A}$.

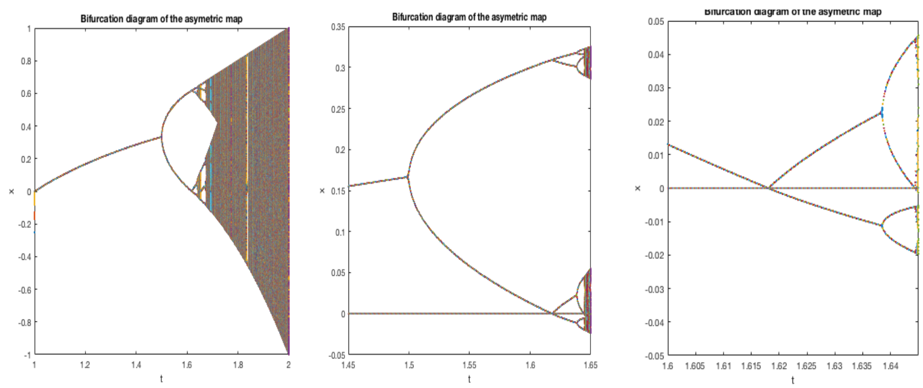

Figure 1: The bifurcation diagram of the family of asymmetric maps $\{f_t\}_{t\in[1,2]}$, defined in (4) together with two zoomed-in versions with the position of the critical point $x = 0$ marked. Note that the doubling bifurcation from period 2^n to period 2^{n+1} when n is odd is not the classical one; in the current asymmetric case the period doubles precisely when 0 is periodic (rather than when the multiplier is equal to -1), as is explained in Theorem 11. The parameter scalings also appears to be rather different than that for the quadratic family.

As will be shown in Theorem 11 in Subsection 6, there exist many maps within the class $\mathcal{A}(2^\infty)$. For example, there exists $t_* \in (1, 2)$ so that $f_{t_*} \in \mathcal{A}(2^\infty)$ where $f_t \colon [-1, 1] \to [-1, 1]$, $t \in [1, 2]$ is defined by

$$f_t(x) = \begin{cases} t(1 + x) \ - 1 & \text{when } x < 0, \\ t(1 - x^\beta) - 1 & \text{when } x \geq 0. \end{cases} \tag{4}$$

As we will see in Subsection 6 this family f_t undergoes unusual period doubling bifurcations, see Figure 1.

Since the power laws of f at both sides of 0 are different, most proofs from the theory of one-dimensional dynamics do not apply. The stumbling block appears already when trying to recover real bounds. As these form the cornerstone for everything else, this is the first issue to overcome. In the 'symmetric' unimodal case the standard proof relies on the simple but powerful smallest interval argument, see Lemma 3. In the symmetric case this argument gives space on both sides of some interval, and in the asymmetric case only on one side, which prevents Koebe like distortion results. It turns out that this is not just a technical issue as the most basic real bounds do not hold. Indeed, the first entry map from the critical value into a periodic renormalization interval around the critical point does NOT have

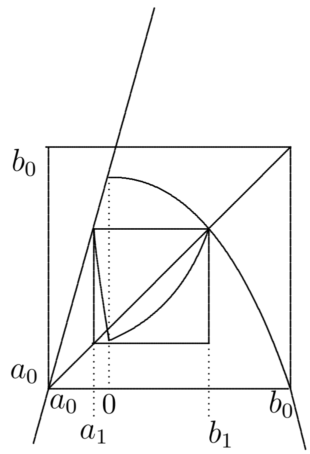

Figure 2: f together with its renormalization and its semi-extension.

a diffeomorphic extension with Koebe space, see for example Theorem 2. Moreover, entirely new scaling phenomena appear as a result of this asymmetry.

The purpose of this paper is to overcome this gap in the literature by obtaining results on real bounds, scaling laws and absence of wandering intervals. Although we believe that the results described in this paper go through for all maps in $\mathcal{A}_{\alpha,\beta}$ with $1 \leq \alpha < \beta$, we were only able to do this under the assumption that $\alpha = 1$. In particular, we will develop a renormalization theory in this setting and show that the scaling laws for such maps are universal, but entirely different from those of smooth maps with non-flat critical points.

3 Statement of results

Existence of infinitely renormalizable maps. When $\alpha = 1$ then the usual proof concerning full families breaks down. Nevertheless we have the following theorem, showing that every family such as the one defined in (4) contains a map in $\mathcal{A}(2^\infty)$.

Theorem 1. For the family defined in (4) there exists a parameter t_* so that $f_{t_*} \in \mathcal{A}(2^\infty)$.

In fact, the proof of this theorem will show that any family similar to (4) (not necessarily with $\alpha = 1$) is *full* in the sense that for each parameter t there exists t_* so that f_{t_*} has the same kneading invariant as $Q_t(x) = tx(1-x)$.

Some notation. As usual, let $[a_k, b_k], k = 0, 1, ...$, be the sequence of renormalization intervals of f. This sequence is constructed in the following way. Let b_1 be a fixed point of f with negative multiplier and a_1 be its preimage. Then

$c_2 := f^2(0) \in [a_1, b_1]$. Notice that $a_0 < a_1 < 0 < b_1 < b_0$. Since the map f is assumed to be of Feigenbaum-Coullet-Tresser 2^∞ type, $f^2|[a_1, b_1]$ is again unimodal; it decreases on $[a_1, 0]$ and increases on $[0, b_1]$. The branch $f^2|[a_1, 0]$ has a fixed point which we will denote by a_2 and b_2 will denote its preimage by $f^2|[0, b_1]$. Using again that f is a 2^∞ map, $f^4|[a_2, b_2]$ is unimodal, and we can continue this process indefinitely and obtain a sequence of points $a_k < 0 < b_k$ and unimodal maps $f^{2^k} \colon [a_k, b_k] \to [a_k, b_k]$.

We say that the interval T is a τ-scaled neighbourhood of $J \subset T$ if both components of $T \setminus J$ have at least size $\tau \cdot |J|$. We shall also use the notations

$$u_k \sim v_k \quad \Longleftrightarrow \quad \tfrac{u_k}{v_k} \to 1 \text{ as } k \to \infty$$

$$u_k \approx v_k \quad \Longleftrightarrow \quad 0 < \liminf \tfrac{u_k}{v_k} \leq \limsup \tfrac{u_k}{v_k} < \infty \text{ as } k \to \infty.$$

Given two intervals $U, V \subset \mathbb{R}$ we define $[U, V]$ to be the smallest interval containing both.

No diffeomorphic extensions The main source of difficulties lies in the following theorem, which shows the difference with the 'symmetric' case:

Theorem 2. For every $\tau > 0$ there exists $k_0 \geq 0$ so that if $T \ni f(0)$ is the maximal interval on which $f^{2^k-1}|T$ is diffeomorphic, then $f^{2^k-1}(T)$ does **not** contain a τ-scaled neighbourhood of $[a_k, b_k]$ for any $k \geq k_0$.

Semi-extensions. To overcome this issue, we will introduce the notion of semi-extension. Since $\alpha = 1$, the derivative of f near the critical point of the left branch of f is non-zero and we can extend this branch smoothly (C^3) and monotonically to $f_1 : [a_0, \epsilon_0] \to \mathbb{R}$ in such a way that $\epsilon_0 > 0$, $f_1|[a_0, 0] = f$, the derivative of f_1 is strictly positive, and the Schwarzian derivative of f_1 is ≤ 0. For consistency, the right branch of f will be denoted by f_2, i.e. $f_2 = f|[0, b_0]$.

Definition (Semi-extensions)**.** Let J be an interval and $f^n|J$ be monotone. Then $F : T \to \mathbb{R}$ is called *monotonic semi-extension* of $f^n|J$ if

- $J \subset T$ and $F|J = f^n|J$;
- $F = f_{i_1} \circ \cdots \circ f_{i_n}$, where $i_k \in \{1, 2\}$ for $k = 1, ..., n$.

We will call such an extension *maximal* if T is the maximal interval satisfying the above properties.

Big bounds for the first entry maps to $[a_k, b_k]$ when k is even. It turns out that these semi-extensions are surprisingly useful since the branch f_1 is essentially linear near 0. Indeed, the semi-extension of the first entry map from an interval $J \ni f(0)$ to $[a_k, b_k]$ becomes almost linear for $k \to \infty$ and even.

Theorem 3 ('Big Bounds')**.** Let $f^{2^k-1}\colon J \to [a_k, b_k]$ be the first entry map of $J \ni f(0)$ into $[a_k, b_k]$ and let $F_k\colon T_k \to \mathbb{R}$ be the maximal monotonic semi-extension of $f^{2^k-1}\colon J \to [a_k, b_k]$. Take $\tau_k > 0$ be maximal so that $F_k(T_k)$ is τ_k-scaled neighbourhood of $[a_k, b_k]$. Then

- $\lim \tau_{2k-1} = \lambda$ where $\lambda \in (0,1)$ is the root of the equation $\lambda^\beta + \lambda = 1$.
- $\tau_{2k} \approx b_{2k}^{-1/2}$ grows super-exponentially with k. In fact, $\log \tau_{2k}$ grows exponentially, see also equation (9) below.

Remark 1. As we will show in Theorem 9 and Section 11, this theorem does not hold when we drop the assumption that $J \ni f(0)$. This will complicate for example the proof of Theorem 10 (on absence of wandering intervals).

Scaling laws. From this theorem we will obtain that the geometry of the ω-limit set is quite different from the one found in smooth unimodal maps with 2^∞ combinatorics. In the next theorem we describe this scaling. By definition $f(a_k) = f(b_k)$ and therefore

$$a_k \sim -K_0 b_k^\beta, \text{ where } K_0 = K_+/K_-. \tag{5}$$

Thus the scaling properties of the renormalization intervals can be described just by the scaling properties of b_k.

Theorem 4 (Scaling laws)**.** The following scaling properties hold for b_k:

- For large even values of k one has

$$\begin{array}{rcl} b_{k+1} & \sim & \lambda b_k \\ c_{2^k} & \sim & b_k, \end{array} \tag{6}$$

  where as before $\lambda \in (0,1)$ is the root of the equation $\lambda^\beta + \lambda = 1$.

- For large odd values of k one has

$$\begin{array}{rcl} b_{k+1} & \sim & \beta^{\frac{-2}{\beta-1}} K_0^{\frac{1}{\beta-1}} \lambda^{-2} b_k^2 \\ c_{2^k} & \sim & -\beta^{-\frac{\beta+1}{\beta-1}} K_0^{\frac{\beta}{\beta-1}} \lambda^{-\beta-1} b_k^{\beta+1} \end{array} \tag{7}$$

- The length of the renormalization intervals decays super-exponentially fast: there exists $\Theta > 0$ so that

$$\log\left(\frac{1}{b_{2k}}\right) \sim \log\left(\frac{1}{|b_{2k} - a_{2k}|}\right) \sim \Theta \cdot 2^k. \tag{8}$$

More precisely,

$$1/b_{2k} \sim \beta^{\frac{-2}{\beta-1}} K_0^{\frac{1}{\beta-1}} \exp(2^k \Theta). \tag{9}$$

In (6) the convergence is super-exponentially: b_{k+1}/b_k converges to λ super-exponentially fast.

The parameter Θ can be arbitrarily large. The parameter Θ is determined by the asymptotic behaviour of $1/b_{2k}$. In the next theorem we show that Θ indeed varies within the space $\mathcal{A}(2^\infty)$:

Corollary 1. For each $\Theta_0 > 0$ there exists a map $f \in \mathcal{A}_{1,\beta}(2^\infty)$ so that $\Theta(f) > \Theta_0$.

Proof. From formula (9) it follows immediately that $\Theta(R^2(f)) = 2 \cdot \Theta(f)$. $\square$

Renormalisation limits. The above scaling laws make it possible to compute the renormalization map R^k for k even with quite a lot of accuracy:

Theorem 5 (Renormalization limits of R^k)**.** For k even we have

$$f^{2^k}(x) = \begin{cases} c_{2^k} - s_k|x| + O(b_k^{\frac{3}{2}}) & \text{when } x \in [a_k, 0] \\ c_{2^k} - t_k x^\beta + O(b_k^{\frac{3}{2}}) & \text{when } x \in [0, b_k] \end{cases} \tag{10}$$

where

$$s_k \sim \frac{b_k^{1-\beta}}{K_0} \text{ and } t_k \sim b_k^{1-\beta}. \tag{11}$$

As usual we can state the renormalization results by rescaling the intervals to a fixed interval. So let $R^k f$ denote the k-th renormalization of f. In other words, let $l_k : [0,1] \to [a_k, b_k]$ be the linear map such that $l(0) = a_k$ and $l(1) = b_k$ and define $R^k f := l_k^{-1} \circ f^{2^k} \circ l_k$. Let $\hat{c}_k$ denote the the critical point of $R^k f$. From (5) it is clear that $\hat{c}_k \to 0$ as $k \to \infty$. Therefore, the left branch of $R^k f$ gets more and more degenerate and disappears in the limit.

Theorem 6. The right branch of the renormalizations of f converge super exponentially fast in the C^1 norm to

$$\lim_{k\to\infty}(R^{2k}f)|[\hat{c}_k,1] = 1 - x^\beta$$

$$\lim_{k\to\infty}(R^{2k+1}f)|[\hat{c}_k,1] = x^\beta.$$

Let $m_k : [-1,0] \to [0,\hat{c}_k]$ be the linear orientation preserving maps mapping the boundary to the boundary. Then in the C^1 norm

$$\lim_{k\to\infty}(R^{2k}f)\circ m_{2k} = x+1$$

$$\lim_{k\to\infty}(R^{2k+1}f)\circ m_{2k+1} = -\lambda^{\beta^2-1}(x+\lambda^{-\beta})^\beta + \lambda^{-1}.$$

Here the convergence is super exponentially fast as well and $\lambda \in (0,1)$ is the root of $\lambda^\beta + \lambda = 1$ as before.

It is easy to see that $\lambda^\beta + \lambda = 1$ implies that $-\lambda^{\beta^2-1}(x+\lambda^{-\beta})^\beta + \lambda^{-1}$ is equal to 1 when $x = -1$ and equal to 0 when $x = 0$. Note that the asymptotic expression for the left branch of $R^{2k+1}f$ is an explicit but non-trivial expression.

Remark 2. One can prove also convergence in the C^N norm in the above theorem if f is a smooth function outside of zero. If the map f is only assumed to have finite smoothness this can be done as in [22] or following the approach in [5]. If f is real analytic (on each side of 0) then this can be done by complex tools: then $f^{2^k} = E_k \circ f$ where E_k extends holomorphically to a diffeomorphism whose range is $B(0,\tau_k|b_k|)$. Using the Koebe Lemma (in the complex case) we then obtain that, for k even, $DE_k = DE_k(c_1) + o(k)$ and $D^iE_k = o_i(k)$ for each $i \geq 2$. The speeds of convergence can be obtained from Koebe and from the speed of τ_k.

Metric invariants and universality. Theorem 4 implies that two maps $f, \tilde{f} \in \mathcal{A}(2^\infty)$ are not necessarily differentiably conjugate on their postcritical sets. In fact, there are necessary and sufficient conditions which are needed for universality:

Theorem 7 (Complete invariants for C^1 universality)**.** Take two maps $f \in \mathcal{A}_{1,\beta}(2^\infty)$ and $\tilde{f} \in \mathcal{A}_{1,\tilde{\beta}}(2^\infty)$, with as before $\beta, \tilde{\beta} > 1$. Then there exists a homeomorphism h which is a conjugacy between the postcritical sets of $f, \tilde{f}$ and

1. h is Hölder at 0;

2. h is Lipschitz at $0 \iff h$ is differentiable at $0 \iff \Theta = \tilde{\Theta}$ and $\beta = \tilde{\beta}$.

Here Θ is defined through equation (8) in Theorem 4.

Moreover, let $\Lambda = \overline{\cup_n f^n(0)}$ be the attracting Cantor set and $\tilde{\Lambda}$ be the corresponding set for $\tilde{f}$. Then $\Theta = \tilde{\Theta}$ and $\beta = \tilde{\beta}$ implies that the conjugacy $h\colon \Lambda \to \tilde{\Lambda}$ is differentiable in the sense that the following limit exists

$$\lim_{y \in \Lambda, y \to x} \frac{h(y) - h(x)}{y - x} \neq 0$$

and depends continuously on $x \in \Lambda$.

Corollary 2. f and $R^2(f)$ are not Lipschitz conjugate.

Proof. This follows from the previous theorem and Corollary 1. □

Hausdorff dimension of the Attracting Cantor set. As in the symmetric case the closure of the orbit of the critical point of $f \in \mathcal{A}(2^\infty)$ is a Cantor set which we denote as $\Lambda(f)$.

Theorem 8. The Hausdorff dimension of the Cantor set $\Lambda(f)$, where $f \in \mathcal{A}(2^\infty)$, is zero.

Absence of Koebe space.

Theorem 9 (Absence of Koebe space)**.** For each $\tau > 0$ there exists x and k so that the maximal semi-extension of the first entry map of f from x into $[a_k, b_k]$ does **not** contain a τ-scaled neighbourhood of $[a_k, b_k]$.

Absence of wandering intervals. As usually, one says that W is. a wandering interval if all iterates of W are disjoint and if W is not in the basin of a periodic attractor. Existing proofs for absence of wandering intervals do not go through. Indeed, we used an argument which is quite different from anything we have seen in the literature showing that

Theorem 10. No map $f \in \mathcal{A}_{1,\beta}(2^\infty)$ has wandering intervals.

4 Some background material

In the proofs below we will need the well-known *Koebe Theorem*.

Lemma 1 (Koebe Lemma)**.** Let $g\colon T \to g(T)$ be a C^3 diffeomorphism with $Sg < 0$. Assume that $J \subset T$ is an interval so that $g(T)$ contains a τ-scaled neighbourhood of $g(J)$, i.e. $g(T) \supset (1+\tau)g(J)$. Then for all $x, y \in J$,

$$\frac{\tau^2}{(1+\tau)^2} \leq \frac{Dg(x)}{Dg(y)} \leq \frac{(1+\tau)^2}{\tau^2}$$

and

$$\frac{\tau}{1+\tau}\frac{|g(J)|}{|J|} \leq |Dg(y)| \leq \frac{1+\tau}{\tau}\frac{|g(J)|}{|J|}.$$

Proof. See the proof of Theorem IV.1.2 in [42]. □

Integrating the last inequalities immediately gives:

Lemma 2 (Corollary of Koebe)**.** Let g be as in the previous lemma and let $L\colon J \to g(J)$ be the affine surjective map with the same orientation as g. Then for all $x \in J$,

$$Lx - \frac{1}{1+\tau}|g(J)| \leq g(x) \leq Lx + \frac{1}{\tau}|g(J)|, \quad |\frac{Dg(x)}{DL(x)} - 1| \leq \frac{1}{\tau}.$$

5 Unusual bifurcations of families of maps with strong asymmetries

In this section we will consider the local bifurcation of families of maps g_t with strong asymmetries. For simplicity, take $\beta > 1, A > 1$ and let us consider a concrete example:

$$g_t(x) = \begin{cases} A|x| + t & \text{for } x \leq 0 \\ x^\beta + t & \text{for } x \geq 0. \end{cases}$$

For $t > 0$ this maps has an attracting fixed point, whereas for *any* $t < 0$ near 0 this has a repelling fixed point $p(t)$ and an attracting periodic orbit $\{q_1(t), q_2(t)\}$ with period 2 with $q_1(t) < p(t) < 0 < q_2(t)$, see the left panel of Figure 3. So periodic doubling occurs precisely when 0 is a fixed point of g_t . We will call this an asymmetric period doubling bifurcation.

Note that if we take a map with the opposite orientation, say $\hat{g}_t(x) = -g_t(x)$, then the attracting fixed point disappears as soon as $t < 0$ (so this is the analogue of the saddle-node bifurcation).

In the next section we will consider the analogue of the periodic doubling phenomena for a family of maps f_t in $\mathcal{A}_{1,\beta}$. During this parameter window only period doubling occurs. The usual period doubling occurs when an attracting periodic orbit of period 2^{2n} becomes repelling and creates an attracting periodic orbit of period 2^{2n+1} (when the multiplier is equal to -1). On the other hand, the asymmetric periodic doubling occurs when an attracting periodic orbit of period 2^{2n+1} looses stability as it goes through the turning point 0.

6 The existence of a 2^∞ map within the space of one-sided linear unimodal maps and a full family result

This section is the only one in this paper where we consider maps in $\mathcal{A}_{\alpha,\beta}$ where we allow $\alpha \geq 1$. In fact, in the proof we assume $\alpha = 1$, because when $\alpha > 1$ the proof is simpler.

We say that a non degenerate interval I is *restrictive* of period $d > 0$ of a unimodal map f if it contains the critical point of f, the interiors of $I, f(I), \dots, f^{d-1}(I)$ are disjoint and $f^d(I) \subset I$, $f^d(\partial I) \subset \partial I$. If a map f has a restrictive interval I of period d is called *renormalizable* and $f^d|I$ is called a renormalization of f. Note that any renormalization of a unimodal map is unimodal.

The maps in class $\mathcal{A}_{\alpha,\beta}(2^\infty)$ we defined are all infinitely renormalizable, moreover all the restrictive intervals $I_1 \supset I_2 \cdots \supset I_n \cdots$ are of periods $2, 2^2, \dots, 2^n, \dots$.

The following theorem implies Theorem 1:

Theorem 11. Consider a family $f_t\colon [a_0, b_0]$, $t \in [0,1]$ in $\mathcal{A}_{\alpha,\beta}$ with $1 \leq \alpha < \beta$ so that $t \mapsto f_t|[a_0, 0] \in C^1$ and $t \mapsto f_t|[0, b_0] \in C^1$ are continuous and so that f_0 has a unique attracting fixed point and so that f_1 is surjective. Then there exist two sequences of parameters $u_1 < u_2 < \cdots < v_2 < v_1$ such that

- for $t \in (u_n, v_n]$ the map f_t is 2^n renormalizable, more precisely, there exists a non degenerate restrictive interval $I_{n,t}$ of period 2 of the map $f_t^{2^{n-1}}|I_{n-1,t}$ continuously depending on the parameter $t \in (u_n, v_n]$ (here we set $I_{0,t} = [a_0, b_0]$);

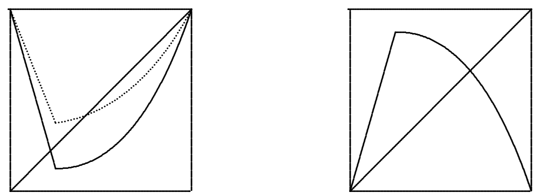

Figure 3: $f^{2^n}|I_{n,t}$ for n odd (on the left) and n even (on the right). When $n \geq 2$ is even then $I_{n,t} \to \{0\}$ as $t \downarrow u_n$ and for $t \in (u_n, v_n)$ the only fixed point of $f_t^{2^n}$ in the interior of $I_{n,t}$ lies to the right of 0.

- when n is even then $f_{u_n}^{2^{n-1}}(0) = 0$ and $\lim_{t\downarrow u_n} I_{n,t} = \{0\}$, while for n is odd f_{u_n} has a parabolic periodic orbit of period 2^{n-1} with multiplier -1 and and $\lim_{t\downarrow u_n} I_{n,t}$ is non-degenerate;
- $f_{v_n}^{2^n}(I_{n,v_n}) = I_{n,v_n}$, that is $f_{v_n}^{2^n}|I_{n,v_n}$ is surjective.

Clearly, $f_t \in \mathcal{A}_{\alpha,\beta}$ for any $t \in \cap_n(u_n, v_n)$.

Note that $\cap_n(u_n, v_n) \neq \emptyset$ because the intervals (u_n, v_n) are properly nested. In particular, the family (4) (with $\beta > 1$) contains a map in the class $\mathcal{A}_{\alpha,\beta}(2^\infty)$.

Proof. The proof we will give of this theorem is almost the same as a proof based on a bifurcation analysis for smooth unimodal maps and will use the following two properties:

(1) whenever f_t has an attracting periodic orbit then 0 is in the immediate basin of this attractor. This holds since f has negative Schwarzian derivative, and therefore the immediate basin of a periodic attractor contains a turning point of an iterate of f and hence 0 is also in the immediate basin of this periodic attractor.

(2) whenever 0 is a (topologically) *attracting* periodic point of f_{t_0} of period n then f_t has a periodic attractor of period n or period $2n$ for each t near t_0. Note that within this class of maps it is no longer true that if 0 is periodic then it is also attracting (it can be repelling on one side when $\alpha = 1$).

Analysing what bifurcations occur in the family f_t analogous to the period doubling bifurcations which occur in the quadratic family, we will prove inductively that there exists a nested sequence of maximal parameter intervals described by the theorem.

Slightly abusing notation we set $u_0 = 0$, $v_0 = 1$ and $I_{0,t} = [a_0, b_0]$. Clearly all the properties stated in the theorem are satisfied except one claiming that the critical point is fixed by f_0. This does not affect the proof which follows. So assume by induction that such parameter interval $[u_n, v_n]$ exists for some integer n. There are two possibilities.

(i) n is even. In this case for each $t \in [u_n, v_n]$, $f_t^{2^n}|I_{n,t}$ is of type $+-$ and $\alpha\beta$, i.e., orientation preserving (resp. reversing) to the left (right) of 0 and the order of the critical point is of order α to the left of 0 and of order β to the right of 0. We know that $f_{v_n}^{2^n}|I_{n,v_n} = I_{n,v_n}$, therefore there exists an orientation reversing fixed point $p_n > 0$ of $f_{v_n}^{2^n}|I_{n,v_n}$. Note that this fixed point is repelling because the orbit of the critical point of $f_{v_n}^{2^n}$ belongs to the boundary of I_{n,v_n}. Since the multiplier of p_n is not equal to one this fixed point persists when we change a parameter in a neighbourhood of v_n, that is there is a continuous function $p_{n,t}$ defined for t in some interval $W_n\ niv_n$ such that t $f_t^{2^n}(p_{n,t}) = p_{n,t}$ and $p_{n,v_n} = p_n$. We will assume that W_n is the maximal interval where such a function can be defined. Let $u_{n+1} < v_n$ be maximal such that $Df_{u_{n+1}}^{2^n}(p_{n,u_{n+1}}) = -1$, that is $p_{n,u_{n+1}}$ becomes a parabolic periodic point of f with multiplier -1. Such a point u_{n+1} exists and $u_{n+1} > u_n$ because the multiplier of $p_{n,t}$ varies continuously with the parameter $t \in W_n \cap (u_n, v_n]$, since $Df_t^{2^n}(p_{n,t}) < -1$ for $t = v_n$ and since for any t we have $\lim_{x\downarrow 0} Df_t^{2^n}(x) = 0$ while $f_{u_n}^{2^{n-1}}(0) = 0$.

For $t \in [u_{n+1}, v_n]$ let $\hat{p}_{n,t} < 0$ denote a preimage of $p_{n,t}$ under $f_t^{2^n}|I_{n,t}$ and let $I_{n+1,t} = [\hat{p}_{n,t}, p_{n,t}]$. Since f has negative Schwarzian derivative it follows that $p_{n,u_{n+1}}$ is a parabolic periodic point of $f_{u_{n+1}}$ and that the critical point belongs to the basing of attraction of $p_{n,u_{n+1}}$. This in turn implies that $f_{u_{n+1}}^{2^{n+1}}(I_{n+1,u_{n+1}}) \subset I_{n+1,u_{n+1}}$, i.e., $I_{n+1,u_{n+1}}$ is a restrictive interval of $f_{u_{n+1}}^{2^n}$ of period 2. Note that if t is slightly larger than u_{n+1}, the interval $I_{n+1,t}$ is still a restrictive interval of period 2 of the corresponding map. We know that $f_{v_n}^{2^n}(0)$ belongs to the boundary of I_{n,v_n} and therefore $f_{v_n}^{2^{n+1}}(0) \notin I_{n+1,v_n}$. Define v_{n+1} to be infimum of all parameters $t > u_{n+1}$ such that $f_{v_{n+1}}^{2^{n+1}}(0) \notin I_{n+1,v_n}$, thus $f_{v_{n+1}}^{2^{n+1}}(0)$ belong to the boundary of $I_{n+1,v_{n+1}}$. It must be the left boundary point (that is $f_{v_{n+1}}^{2^{n+1}}(0) = \hat{p}_{n,v_{n+1}}$) because otherwise the condition $Df_t^{2^n}(p_{n,t}) \leq -1$ for $t \in [u_{n+1}, v_n]$ would be broken.

It is easy to see that the constructed points u_{n+1}, v_{n+1} and the intervals $I_{n+1,t}$ satisfy all the induction assumptions. Note that in this case the intervals $I_{n+1,t}$ are non degenerate for all $t \in [u_{n+1}, v_{n+1}]$.

(ii) n is odd. In this case $f_{u_n}^{2^n}|I_n$ is of type $-+$ and $\alpha\beta$. The construction will be very similar to the case of even n with some modifications relating to the asymmetric period doubling bifurcation.

Arguments similar to the case when n is even show that there exists a maximal $u_{n+1} < v_n$ such that $f^{2^n}_{u_{n+1}}(0) = 0$. Then for all $t \in [u_{n+1}, v_n]$ there exists an orientation reversing fixed point $p_{n,t} \in I_{n,t}$ of $f_t^{2^n}$. Note that $p_{n,t}$ is negative (i.e. it is to the left of the critical point). Define $\hat{p}_{n,t} > 0$ to be a preimage of $p_{n,t}$ under $f_t^{2^n}|I_{n,t}$ and let $I_{n+1,t} = [\hat{p}_{n,t}, p_{n,t}]$ for all $t \in [u_{n+1}, v_n]$ as before. Note that $p_{n,u_{n+1}} = \hat{p}_{n,u_{n+1}} = 0$ and the interval $I_{n+1,u_{n+1}}$ degenerates to the critical point. For all other values of the parameters the intervals $I_{n+1,t}$ are non degenerate. In Section 5 it was explained that for values of parameters t slightly larger than u_{n+1} the interval $I_{n+1,t}$ is a restrictive interval of period 2 of the map $f_t^{2^n}$. As before define $v_{n+1} > u_{n+1}$ to be maximal such that $I_{n+1,t}$ is a restrictive interval of period 2 of the map $f_t^{2^n}$ for all $t \in (u_{n+1}, v_{n+1})$ and note that $v_{n+1} < v_n$.

□

In fact, we have

Theorem 12. Any family $\{f_t\}$ as in Theorem 11 is a full family in the following sense. Take a quadratic interval map Q without periodic attractors. Then there exists a parameter t so that f_t combinatorially equivalent to Q.

Proof. The proof of this theorem can be deduced from the previous proof as this shows that the kneading invariant of a family of maps f_t in $\mathcal{A}_{\alpha,\beta}$ bifurcates in the same way as in the quadratic family (but no assertion about monotonicity of the bifurcations can be deduced from this), see also [46]. Another way to prove this is by modifying the proof following the Thurston mapping approach from [41, Theorem II.IV.1]. □

7 The smallest interval argument

The usual smallest interval argument in the current setting gives a weaker statement than in the 'symmetric' case:

Lemma 3. There exists $\tau > 1$ so that the following holds. Consider $I = [a_n, b_n]$ and choose $x \notin I$. Assume that there exists $k > 0$ (minimal) so that $f^k(x) \subset I$. Then there exists an interval $T \ni x$ so that $f^k|T$ is a diffeomorphism and $f^k(T) \supset [\tau a_n, \tau b_n]$.

Proof. For completeness let us include the proof of this lemma. By maximality of T and since $f^i(x) \notin I$ for all $i = 0, \ldots, k-1$ there exist integers $0 < i_0, i_1 < 2^n$ so that $f^k(T) \supset [f^{i_0}(I), f^{i_1}(I)]$ where $f^{i_0}(I)$ and $f^{i_1}(I)$ are to the left respectively to the right of I. So it suffices to show that $[f^{i_0}(I), f^{i_1}(I)] \supset [\tau a_n, \tau b_n]$ for some universal choice of $\tau > 0$.

Write $I_i = f^i(I)$ and let $3 \leq m \leq 2^n$ be so that I_m is the smallest of the intervals $I_3, \ldots, I_{2^n}$. Let K_m be the smallest interval containing the left and right neighbours of I_m from the collection $I_1, \ldots, I_{2^n}$ (such neighbouring intervals exist because $, \geq 3$). It follows that K_m contains a τ_0-scaled neighbourhood of I_m where $\tau_0 > 0$ is independent on n (here we use that I_1, I_2 are not much smaller than I_3). Let $K_1 \supset I_1$ be the maximal interval on which $f^{i_0-1}|K_1$ is a diffeomorphism with $f^{i_0-1}(K_1) \subset K_m$. By maximality, $f^{i_0-1}(K_1) = K_m$. By Koebe it follows that K_1 contains a τ_1-scaled neighbourhood of I_1. Hence $K_0 := f^{-1}(K_1)$ contains $[\tau_1' a_n, \tau_1'' b_n']$ where $\tau_1' = \tau_1^{1/\alpha}$ and $\tau_1'' = \tau_1^{1/\beta}$. Note that because $|a_n| << b_n$, this latter interval is no longer a definite interval around $[a_n, b_n]$. Note also that by the choice of K_m the interval K_0 is contained in any interval of the form $[f^{i_0}(I), f^{i_1}(I)]$ where $f^{i_0}(I)$ and $f^{i_1}(I)$ are to the left respectively to the right of I. □

8 Proof of the first part of Theorem 3: big bounds

Since $\alpha = 1$, we can consider a semi-extension of f of the 'linear' branch and use the following strategy. First, using the standard smallest interval argument we have already shown that there exists a definite space to the right of the renormalization intervals. Next we will show that either there is definite space to the left of the renormalization interval for the semi-extension or this space is at least as big as the space on the previous level. Considering several scenarios, this will imply that there is some definite space on both sides of the renormalization intervals. Having space on both sides of the renormalization intervals we can repeat the argument used to obtain it and get as much space as one may want. From this the rest follows.

8.1 Using semi-extensions

Let $f^{2^k-1}\colon J_k \to [a_k, b_k]$ be the branch of the first entry map to $[a_k, b_k]$ for which $c_1 := f(0) \in J_k$. Note that this is a surjective diffeomorphism. Let $\hat{T}_k \supset J_k$ be the maximal interval so that $f^{2^k-1}|\hat{T}_k$ is a diffeomorphism and let $[\hat{A}_k, \hat{B}_k] := f^{2^k-1}(\hat{T}_k)$ where $\hat{A}_k < \hat{B}_k$. Note that $f^{2^k-1}|\hat{T}_k$ is orientation preserving (reversing) when k is even (odd). We also define an interval $[A_k, B_k] \supset [\hat{A}_k, \hat{B}_k]$, with $A_k < B_k$, associated to the semi-extension as follows. Let $E_k : T_k \to [A_k, B_k]$

be the maximal monotone surjective *semi-extension* of $f^{2^k-1}\colon J_k \to [a_k, b_k]$ such that $A_k \le a_k < 0 < b_k \le B_k$. (In principle this extension depends on the choice of the extension $f_1\colon [0,\epsilon) \to \mathbb{R}$ of $f\colon [a_0, 0] \to \mathbb{R}$.)

Let $[a_k', e_k] = f_1^{-1}(T_k)$, $a_k' < a_k < 0 < e_k$, and therefore $E_k \circ f_1 : [a_k', e_k] \to [A_k, B_k]$ is the maximal monotone surjective semi-extension of $f^{2^k} : [a_k, 0] \to [a_k, b_k]$. Also, define the point $b_k' > b_k$ as the right boundary point of the interval $f_2^{-1}(T_k)$. Furthermore, define $d_k \in [0, e_k]$ such that $E_k \circ f_1(d_k) = b_k$ for even values of k. When k is odd the point d_k is not defined.

Since E_k is orientation preserving (reversing) when k is even (odd), the following holds:

- for even values of k

$$\begin{aligned} A_k &= E_k \circ f_1(a_k') = E_k \circ f_2(b_k'), \\ B_k &= E_k \circ f_1(e_k) \end{aligned}$$

- and for odd k

$$\begin{aligned} B_k &= E_k \circ f_1(a_k') = E_k \circ f_2(b_k'), \\ A_k &= E_k \circ f_1(e_k). \end{aligned}$$

As we will show in Lemma 4, $B_k = \hat{B}_k$ but in general $A_k \neq \hat{A}_k$.

Let us list a number of more or less obvious relations between the points we defined. For example, assertion (4) and (5) show that if some metric properties hold for the non-dynamically defined points b_k' and e_k then the semi-extension from one level can be used to obtain a semi-extension of the next level.

Lemma 4. Let $k \ge 2$ be an even integer. Then

1. $B_{k+1} = B_{k+2} = \hat{B}_{k+1} = \hat{B}_{k+2} = c_{2^k}$;
2. $e_{k+2} < d_k$;
3. $\hat{A}_k = \hat{A}_{k+1} = c_{2^{k-1}}$;
4. if $b_k' < B_k$, then $e_{k+1} < e_k$ and $A_{k+1} = A_k$.
5. if $e_{k+1} < b_{k+1}$, then $b_{k+2}' < b_{k+1}$ and $A_{k+2} = A_{k+1}$.

Proof. Since $f^{2^k}[a_{k+1}, b_{k+1}] \subset [0, b_k]$, we have $E_{k+1} = E_k \circ f_2 \circ E_k | T_{k+1}$, where E_k is orientation preserving and f_2 is orientation reversing. Since the diffeomorphic range of E_k is $[\hat{A}_k, \hat{B}_k] \supset [a_k, b_k] \ni 0$ and $E_k \circ f_2$ maps $(0, b_k]$ diffeomorphically onto $[a_k, c_{2^k})$, it follows that $B_{k+1} = \hat{B}_{k+1} = E_k \circ f_2(0) = c_{2^k}$

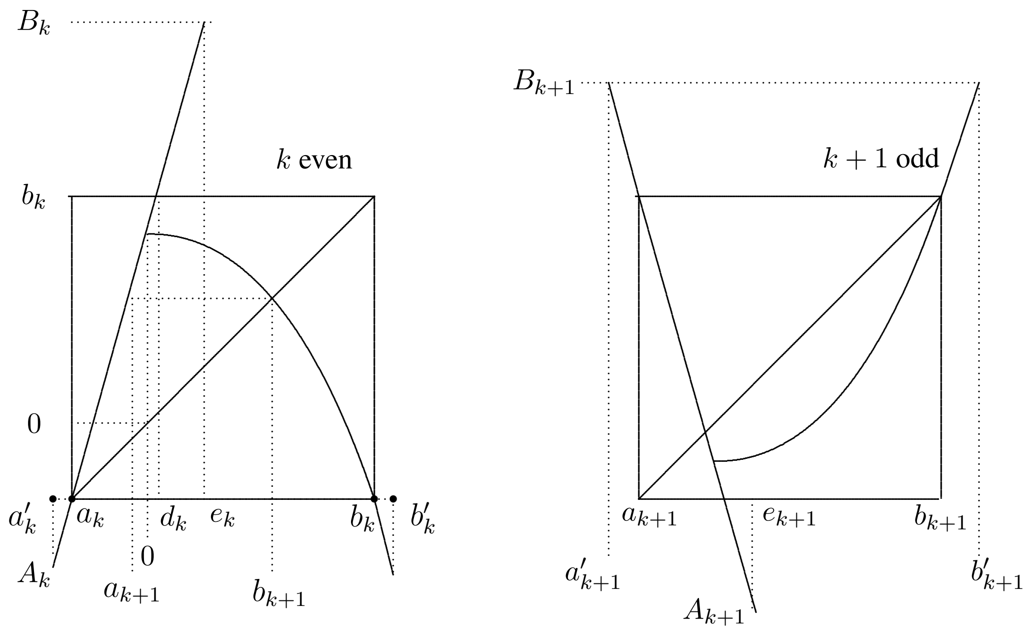

Figure 4: $f^{2^k}|I_k$ and $f^{2^{k+1}}|I_{k+1}$ when k is even and their semi-extensions. Note that the points d_k, e_k, a'_k, b'_k are defined using the semi-extension rather than dynamically.

and $A_{k+1} \leq \hat{A}_{k+1} \leq a_k$. Taking a'_{k+1} to be the point in (a_k, a_{k+1}) for which $f^{2^k}(a'_{k+1}) = E_k \circ f_1(a'_{k+1}) = 0$ one has $f^{2^{k+1}}(a'_{k+1}) = E_{k+1} \circ f_1(a'_{k+1}) = E_k \circ f_2 \circ E_k \circ f_1(a'_{k+1}) = E_k \circ f_2 \circ E_k(0) = B_{k+1}$.

Similarly, since $f^{2^{k+1}}[a_{k+2}, b_{k+2}] \subset [a_{k+1}, 0]$, $E_{k+2} = E_{k+1} \circ f_1 \circ E_{k+1}|T_{k+2}$ where E_{k+1} is orientation reversing and f_1 is orientation preserving. Since $a_k < a'_{k+1} < a_{k+1} < c_{2^{k+1}} = E_{k+1}(c_1) < 0$, $E_{k+1} \circ f_1(a'_{k+1}) = B_{k+1} = \hat{B}_{k+1}$ and since the diffeomorphic range of E_{k+1} is $[\hat{A}_{k+1}, \hat{B}_{k+1}) \supset [a_k, c_{2^k}) \supset (a'_{k+1}, 0)$ it follows that $B_{k+2} = \hat{B}_{k+2} = B_{k+1} = \hat{B}_{k+1} = c_{2^k}$ and $\hat{A}_{k+2} = c_{2^{k+1}}$, proving in particular statement (1).

By definition $E_{k+2} \circ f_1(e_{k+2}) = B_{k+2}$. Since $E_{k+1} \circ f_1(a'_{k+1}) = B_{k+1} = B_{k+2}$ and $E_{k+2} = E_{k+1} \circ f_1 \circ E_{k+1}|T_{k+2}$ we have that $E_{k+1} \circ f_1(e_{k+2}) = a'_{k+1}$. Since $a'_{k+1} \in (a_k, a_{k+1})$, $E_{k+1} \circ f_1(d_k) = E_k \circ f_2 \circ E_k \circ f_1(d_k) = E_k \circ f_2(b_k) = a_k$ and E_{k+1} is orientation reversing, it follows that $e_{k+2} < d_k$, proving statement (2).

Statement (3) follows as in statement (1).

To prove statement (4), assume $b'_k < B_k$. Then E_k has range $[A_k, B_k] \supset [A_k, b'_k]$. Note that the left endpoint of the domain of E_k is $f_2(b'_k)$ and $E_k \circ f_2(b'_k) =$

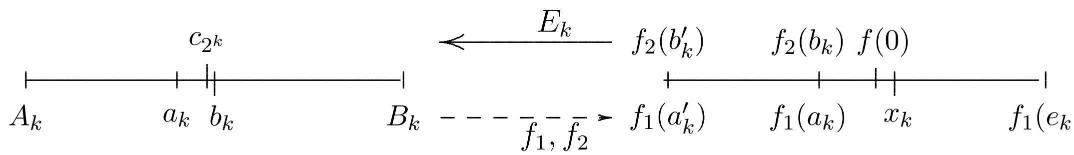

Figure 5: When k is even, $E_{k+1} = E_k \circ f_2 \circ E_k$ and E_k is orientation preserving. Here $E_k(x_k) = b_k$. It is not clear where b'_k and a'_k are in relation to B_k and A_k.

A_k. Since $E_{k+1} = E_k \circ f_2 \circ E_k$ it follows that the range of E_{k+1} is equal to $[A_k, B_{k+1}]$ and so $A_{k+1} = A_k$. Moreover, $A_k = A_{k+1} = E_{k+1} \circ f_1(e_{k+1}) = E_k \circ f_2 \circ E_k \circ f_1(e_{k+1})$ and $E_k \circ f_2(b'_k) = A_k$. Since E_{k+1} and f_1, f_2 are all injective, $b'_k = E_k \circ f_1(e_{k+1})$. Therefore, and since $B_k = E_k \circ f_1(e_k)$ and f_1, E_k are increasing, $b'_k < B_k$ implies that $e_{k+1} < e_k$.

Finally, to prove statement (5), note that $E_{k+1}|[f(a_{k+1}), f(0))$ maps diffeomorphically onto $(c_{2^{k+1}}, b_{k+1}]$ and if $e_{k+1} < b_{k+1}$ then this last interval contains $(c_{2^{k+1}}, e_{k+1}]$. Since $E_{k+1} \circ f_1$ maps the latter interval diffeomorphically onto $[A_{k+1}, c_{2^{k+2}})$ and since $E_{k+2} = E_{k+1} \circ f_1 \circ E_{k+1}|T_{k+2}$ it follows that $A_{k+2} = A_{k+1}$ and $b'_{k+2} = f^{2^{k+1}}|[0, b_{k+1}](e_{k+1}) < b_{k+1}$. □

Lemma 5. There exists $C > 0$ so that for all k even

$$d_k \leq C b_{k+1}^{\beta-1} b_k. \tag{12}$$

Proof. For k even, b_{k+1} is a repelling fixed point of f^{2^k}, so $|Df^{2^k}(b_{k+1})| > 1$. When k is large this implies that

$$b_{k+1}^{\beta-1}|DE_k(f(b_{k+1}))| \approx |Df^{2^k}(b_{k+1})| > 1.$$

Since $|Df^{2^k}(a_{k+1})| \approx |a_{k+1}|^{\alpha-1}|DE_k(f(a_{k+1}))|$ and $f(a_{k+1}) = f(b_{k+1})$ it follows that

$$Df^{2^k}(a_{k+1}) > C_.|a_{k+1}|^{\alpha-1} b_{k+1}^{1-\beta} \text{ and } |DE_k(f(b_{k+1}))| > C_. b_{k+1}^{1-\beta}. \tag{13}$$

Diffeomorphic branches of maps with negative Schwarzian derivative expand cross-ratios. Applying this fact to the diffeomorphism $E_k \circ f_1 \colon [a_{k+1}, e_k] \to [b_{k+1}, B_k]$ and the four points $a_{k+1}, a^+_{k+1}, d_k, e_k$ (which map to $b_{k+1}, b^+_{k+1}, b_k, B_k$) (where we take $a^+_{k+1} = a_{k+1} + h$ with $h > 0$ close to 0 and b^+_{k+1} the image of this point) we obtain the inequality

$$\frac{(e_k - a^+_{k+1})(d_k - a_{k+1})}{(a_{k+1} - a^+_{k+1})(e_k - d_k)} \leq \frac{(B_k - b^+_{k+1})(b_k - b_{k+1})}{(b_{k+1} - b^+_{k+1})(B_k - b_k)}.$$

Taking $h \downarrow 0$, we get

$$\begin{aligned} d_k &< d_k - a_{k+1} \leq \frac{(B_k - b_{k+1})}{(B_k - b_k)}(b_k - b_{k+1})\frac{(e_k - d_k)}{(e_k - a_{k+1})}\frac{1}{Df^{2^k}(a_{k+1})} \\ &\leq C b_{k+1}^{\beta - 1} b_k. \end{aligned} \tag{14}$$

Here we use that the first factor in the long expression is bounded from above by Lemma 3, the second by b_k, the third factor by 1 and in the final factor we use the bound from (13). $\square$

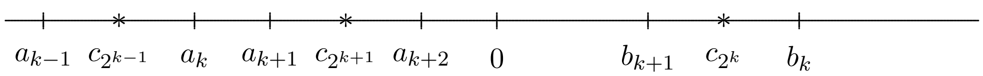

Figure 6: The ordering of several dynamically relevant point; here k is even.

Lemma 6. There exists a constant $C > 0$ such that for large even values of k,

$$|A_{k+2}| > \min(Cb_{k+1}, \frac{1}{2}|a_k|).$$

Proof. Note that $E_{k+2} = E_{k+1} \circ f_1 \circ E_{k+1}|T_{k+2}$ and that E_{k+1} maps $\hat{J}_{k+1} \ni f(0)$ diffeomorphically onto $[\hat{A}_{k+1}, \hat{B}_{k+1}] = [c_{2^{k-1}}, c_{2^k}] \supset [a_k, c_{2^k}]$.

If $d_k \leq c_{2^k}$ then the last interval contains $[a_k, d_k]$. Moreover, $E_k \circ f_1$ maps $[a_k, d_k]$ diffeomorphically to $[a_k, b_k] \supset [0, b_k]$ and the latter interval is mapped diffeomorphically by f^{2^k} to $[a_k, c_{2^k}]$. Since $E_{k+1} = f^{2^k} \circ E_k \circ f_1|T_{k+1}$, it follows that $A_{k+2} \leq a_k$ and since both numbers are negative we get $|A_{k+2}| \geq |a_k|$.

If $d_k > c_{2^k}$ then the same consideration shows that $A_{k+2} = E_{k+1} \circ f_1(c_{2^k})$. If $|A_{k+2}| > \frac{1}{2}|a_k|$ or $|A_{k+2}| > \frac{1}{2}b_{k+1}$ there is nothing to prove. So in the remainder of the proof of this lemma assume that $|A_{k+2}| \leq \frac{1}{2}|a_k|$ and $|A_{k+2}| \leq \frac{1}{2}b_{k+1}$. The interval $[A_{k+2}, a_{k+2}]$ is well-inside the interval $[a_k, c_{2^k}]$ as $c_{2^k} > b_{k+1} > 2|A_{k+2}|$ and $|a_k| \geq 2|A_{k+2}|$. Moreover, $[\hat{A}_{k+1}, \hat{B}_{k+1}] = [c_{2^{k-1}}, c_{2^k}]$ is the diffeomorphic range of $E_{k+1}|\hat{J}_{k+1}$, $[c_{2^{k-1}}, c_{2^k}] \supset [a_k, c_{2^k}]$ and $[f(a_{k+2}), f_1(c_{2^k})] \subset \hat{J}_{k+1}$. So $[A_{k+2}, a_{k+2}] = E_{k+1}[f(a_{k+2}), f_1(c_{2^k})]$ is well-inside the diffeomorphic range of $E_{k+1}|\hat{J}_{k+1}$ and so the distortion of E_{k+1} restricted to $[f(a_{k+2}), f_1(c_{2^k})]$ is bounded.

It follows that the distortion of $E_{k+1} \circ f_1|_{[a_{k+2}, c_{2^k}]}$ is bounded. Since the derivative of $f^{2^{k+1}}$ at its fixed point a_{k+2} is larger than one, this implies that $|D(E_{k+1} \circ f_1)(x)| > C_5$ for all $x \in [a_{k+2}, c_{2^k}]$. Since $a_{k+2} < 0 < b_{k+1} < c_{2^k}$, E_{k+1} is orientation reversing and $E_{k+1} \circ f_1(0) = c_{2^{k+1}} < 0$,

$$|A_{k+2}| = |E_{k+1} \circ f_1(c_{2^k})| > |E_{k+1} \circ f_1(b_{k+1})| > C_5 b_{k+1}. \qquad \square$$

Lemma 7. There exists $C > 0$ such that the following holds. Let k be a sufficiently large even integer. Then either

- $|A_k| > Cb_{k+1}$ or
- $e_k < b_{k+1}$.

Proof. Suppose $e_k \geq b_{k+1}$. Then due to Lemma 4(2) and inequality (12) we know that for k large and even,

$$b_{k+1} \leq e_k < d_{k-2} < C_4 b_{k-1}^{\beta-1} b_{k-2} < b_{k-2}^{\beta}. \tag{15}$$

From Lemma 6 we know that either $|A_k| > Cb_{k-1}$ or $|A_k| > \frac{1}{2}|a_{k-2}|$. In the first case we have nothing to do because $b_{k+1} < b_{k-1}$. In the second case it follows from (15) that $|A_k| > \frac{1}{2}|a_{k-2}| > Cb_{k-2}^{\beta} > C_6 b_{k+1}$. □

Lemma 8. For any $C > 0$ there exist $0 < \lambda_1 < \lambda_2 < 1$ such that the following holds. Let k be large even integer and $|A_k| > Cb_{k+1}$. Then

$$|Df^{2^k}|[b_{k+1}, b_k]| > \lambda_1 \ , \tag{16}$$

$$\lambda_1 b_k < b_{k+1} < \lambda_2 b_k. \tag{17}$$

Proof. Consider two cases.

Case 1: $|a_k| < \frac{1}{2}Cb_{k+1}$. Then $|b_{k+1} - a_k| < (1 + \frac{1}{2}C)b_{k+1}$. At the same time $|A_k - a_k| > \frac{1}{2}Cb_{k+1}$ and we see that $|A_k - a_k| > C_7|b_{k+1} - a_k|$ for some $C_7 > 0$ which depends only on C.

Case 2: $|a_k| \geq \frac{1}{2}Cb_{k+1}$. Then $|b_{k+1} - a_k| \leq (1 + \frac{2}{C})a_k$. According to Lemma 3, $|A_k| > K|a_k|$ for some universal $K > 1$, therefore $|A_k - a_k| > (K-1)|a_k|$ and we again get $|A_k - a_k| > C_8|b_{k+1} - a_k|$ for some $C_8 > 0$ which depends only on C and K.

From this and Lemma 3, we get that the range of the map $E_k \colon [f(b_{k+1}), f(b_k)] \to [a_k, b_{k+1}]$ can be diffeomorphically semi-extended to a C_9-scaled neighbourhood of the interval $[a_k, b_{k+1}]$, and therefore the distortion of the map $E_k|[f(b_{k+1}), f(b_k)]$ is bounded.

On the interval $[b_{k+1}, b_k]$ the absolute value of Df is increasing, hence

$$|Df^{2^k}(x)| = |DE_k(f(x))||Df(x)| > C_{10}|Df^{2^k}(b_{k+1})|$$

for all $x \in [b_{k+1}, b_k]$ and some constant $C_{10} > 0$ which depends only on C. Since b_{k+1} is a repelling fixed point of f^{2^k}, we get $|Df^{2^k}(b_{k+1})| > 1$ and $|Df^{2^k}| > C_{10}$ on $[b_{k+1}, b_k]$. This implies the existence of $\lambda_1 > 0$ as in equations (16) and (17).

To prove the existence of $\lambda_2 < 1$ in (17), note that by Lemma 3 and Koebe that E_k has bounded distortion on the range $[b_k/2, b_k]$. Moreover, f_2 has bounded distortion on $[b_k/2, b_k]$. By contradiction assume that $b_{k+1}/b_k \approx 1$. Then there exists a point $x \in [b_{k+1}, b_k]$ for which $(E_k \circ f_2)(x) \in [b_k/2, b_{k+1}]$ and $|D(E_k \circ f_2)(x)|$ is large. But since $(E_k \circ f_2)(y) \in [b_{k+1}, b_k]$ for all $y \in [b_k/2, b_{k+1}]$, it follows that $|D(E_k \circ f_2)(y)|$ is also large for all such y. But this contradicts that $(E_k \circ f_2)$ maps $[b_k/2, b_{k+1}]$ into $[b_{k+1}, b_k]$. Thus the existence of $\lambda_2 < 1$ follows. □

8.2 Getting some space some of the time

Now we are ready to combine the results from the previous subsection.

Lemma 9. There exists a constant $C > 0$ and an infinite sequence of even integers $k_1 < k_2 < \dots$ such that

$$|A_{k_i}| > Cb_{k_i},$$

and therefore, the distortion of the maps $E_{k_i}|J_{k_i}$ is universally bounded.

Proof. It follows from Lemma 7 that either there exist infinitely many even integers k_i such that $|A_{k_i}| > Cb_{k_i+1}$ or there exists an even integer k_0 such that $e_k < b_{k+1}$ for all even $k \geq k_0$.

In the first case we are done because of Lemmas 3 and 8, so suppose that we are in the second case. Since $0 < e_{k+1} \leq e_k$, Lemma 4(5) implies $b'_{k+2} < b_{k+1}$ and $A_{k+2} = A_{k+1}$ for all even $k \geq k_0$. Notice that $b_{k+1} < c_{2^k} = B_{k+2}$, and therefore Lemma 4(1) gives $b'_{k+2} < B_{k+2}$. Then from Lemma 4(4) it follows that $A_{k+3} = A_{k+2}$. So, we see that $A_k = A_{k_0+1}$ for all $k > k_0$ and since $b_k \to 0$ we get $|A_k| > b_k$ for all k large enough.

The boundedness of the distortion of the maps $E_{k_i}|J_{k_i}$ follows from Lemma 3 and from $|A_{k_i}| > Cb_{k_i}$. □

8.3 The proof of the first part of Theorem 3: getting huge space all the time

Lemma 10. For every constant $C > 0$ there exists a constant $\tau_* > 0$ such that the following holds. Let k be a large even integer and $|A_k| > Cb_k$. Then

$$b_{k+2} < \tau_* b_k^{2-1/\beta}, \tag{18}$$

$$b_k - c_{2^k} < \tau_* b_k^{\beta}, \tag{19}$$

$$d_k < \tau_* b_k^{2\beta-1}. \tag{20}$$

Proof. Due to Lemma 3 we always have some space to the right of the renormalization interval, and since we assumed that $|A_k| > Cb_k$, therefore the distortion of the map $E_k|J_k$ is bounded by a constant depending only on C. The map $E_{k+1}|J_{k+1}$ can be decomposed as $E_{k+1}|J_{k+1} = E_k|J_k \circ f|[b_{k+1}, b_k] \circ E_k|J_{k+1}$. Due to Lemma 8 we know that $b_{k+1} > \lambda_1 b_k$, and hence, the distortion of the map $f|[b_{k+1}, b_k]$ is bounded. Thus, the distortion of $E_{k+1}|J_{k+1}$ is bounded as a composition of three maps of bounded distortion. Then the distortion of the map $f^{2^{k+1}}|[a_{k+1}, 0]$ is bounded again. Combining this with $f^{2^{k+1}}(a_{k+1}) = b_{k+1}$ and $f^{2^{k+1}}(0) = c_{2^{k+1}} \in [a_{k+1}, a_{k+2}]$ we get

$$Df^{2^{k+1}}|[a_{k+1}, 0] > C_{11} b_{k+1}/|a_{k+1}|. \tag{21}$$

This implies the following estimate on the position of a_{k+2} and, therefore, of b_{k+2}:

$$\begin{array}{lclcl} |a_{k+2}| & < & \frac{|a_{k+1}|^2}{C_{11} b_{k+1}} & < & C_{12} b_k^{2\beta-1}, \\ |b_{k+2}| & & < & & C_{13} b_k^{2-1/\beta}, \end{array} \tag{22}$$

for some universal constants $C_{12} > 0$ and $C_{13} > 0$.

Since k is even we know that $c_{2^k} \in [b_{k+1}, b_k]$ and $c_{2^{k+1}} \in [a_{k+1}, a_{k+2}]$ and so in particular $f^{2^k}[c_{2^k}, b_k] \subset [a_k, 0]$. Due to Lemma 8 the derivative of $f^{2^k}|[b_{k+1}, b_k]$ is bounded away from zero, hence

$$|b_k - c_{2^k}| < \lambda_1^{-1} |a_k| < C_{14} b_k^{\beta} \ll b_k \tag{23}$$

for some universal constant C_{14}. Combining this with equation (21), and since $f^{2^k}[0, d_k] = [c_{2^k}, b_k]$, this gives us a much better estimate for d_k (compared to inequality (5)):

$$d_k < C_{11}^{-1} |b_k - c_{2^k}| \cdot |a_{k+1}|/b_{k+1} < C_{15} b_k^{\beta} |a_{k+1}|/b_{k+1} < C_{15} b_k^{2\beta-1} \tag{24}$$

for some $C_{15} > 0$. □

Lemma 11. For every constant $C_0 > 0$ there exists a constant $\tau_* > 0$ such that the following holds. Let k be a large even integer, C be a constant greater that C_0, and $|A_k| > Cb_k$, $B_k > (1+C)b_k$. Then

$$|A_{k+2}| > \tau_* \min(C, b_k^{1-\beta})b_k.$$

Proof. Set

$$\begin{aligned} \tilde{A}_k &= -\tfrac{1}{2}Cb_k \\ \tilde{B}_k &= (1+\tfrac{1}{2}C)b_k. \end{aligned} \tag{25}$$

Let $\tilde{e}_k, \tilde{b}_k$ be points such that $E_k \circ f_1(\tilde{e}_k) = \tilde{B}_k$ and $E_k \circ f_2(\tilde{b}_k) = \tilde{A}_k$. Arguing as before we see that the distortions of maps $E_k \circ f_1|[a_k, \tilde{e}_k]$ and $E_k \circ f_2|[b_{k+1}, \tilde{b}_k]$ are bounded by some constant depending on C_0. Therefore, for all $x \in [a_k, \tilde{e}_k]$,

$$\begin{aligned} D(E_k \circ f_1)(x) &> C\frac{b_k - a_k}{d_k - a_k} \\ &> C_{17} b_k^{1-\beta}. \end{aligned} \tag{26}$$

In the same way we get the estimate on the derivative of the other branch:

$$D(E_k \circ f_2)(x) > C_{18}$$

for all $x \in [b_{k+1}, \tilde{b}_k]$. Now consider the following cases.

Case 1.a. Assume that $\tilde{e}_k < b_{k+1}$ and $\tilde{B}_k > \tilde{b}_k$. Then, arguing as in Lemma 4(4,5) we obtain that $|A_{k+2}| > |\tilde{A}_k|$ and we are done in this case.

Case 1.b. Now suppose $\tilde{e}_k < b_{k+1}$ and $\tilde{B}_k \leq \tilde{b}_k$. Then

$$\begin{aligned} |E_{k+1} \circ f_1([d_k, \tilde{e}_k])| &> C_{18}|\tilde{B}_k - b_k| \\ &= \frac{1}{2}C_{18}Cb_k. \end{aligned} \tag{27}$$

Using an argument similar to prove Lemma 2(4) we get $|A_{k+2}| > \frac{1}{2}C_{18}Cb_k$ and this case is also done.

Case 2: $\tilde{e}_k > b_{k+1}$. From the derivative estimate we know

$$\begin{aligned} E_k \circ f_1([d_k, b_{k+1}]) &> C_{17} b_k^{1-\beta}|b_{k+1} - d_k| \\ &> C_{19} b_k^{2-\beta}. \end{aligned} \tag{28}$$

Here we used inequalities (17) and (20).

We finish by considering two subcases as in Case 1. If $E_k \circ f_1(b_{k+1}) > \tilde{b}_k$, then as before $|A_{k+2}| > |\tilde{A}_k|$. Otherwise,

$$|A_{k+2}| > C_{18}C_{19}b_k^{2-\beta}.$$ □

The following lemma completes the proof of the first part of the 'Big Bounds' Theorem 3. The actual bounds for the space that are claimed in that theorem will be only obtained in the improved bounds from Lemma 13.

Lemma 12 (Koebe Space for the semi-extension)**.** There exists $\hat{\lambda} > 0$ so that as k even and $k \to \infty$,

$$\frac{|b_{k+2} - a_{k+2}|}{|a_{k+2} - A_{k+2}|} = O(b_k^{1-1/\beta}), \frac{|b_{k+2} - a_{k+2}|}{|B_{k+2} - b_{k+2}|} = O(b_k^{1-1/\beta}) \tag{29}$$

and

$$\frac{|b_{k+1} - a_{k+1}|}{|a_{k+1} - A_{k+1}|} = O(b_{k-2}^{1-1/\beta}), \frac{|b_{k+1} - a_{k+1}|}{|B_{k+1} - b_{k+1}|} \geq \hat{\lambda}. \tag{30}$$

In particular, the range of the map $E_k|J_k$ can be monotonically semi-extended to a τ_k scaled neighbourhood of $[a_k, b_k]$ where $\tau_k \approx O(b_{k-2}^{1-1/\beta})$ for k even and $\tau_k \approx 1$ for k odd.

Moreover, $O(b_k^{1-1/\beta})$ converges super-exponentially to zero: $\log(b_k)$ converges exponentially to zero.

Proof. This lemma is a consequence of the previous two lemmas. Let k be a large (even) integer from the sequence given by Lemma 9. Then, from Lemmas 10 and 11 it follows that

$$\begin{aligned} |A_{k+2}| &> C_{20} b_k^{\frac{1}{\beta}-1} b_{k+2}, \\ |B_{k+2}| &> C_{20} b_k^{\frac{1}{\beta}-1} b_{k+2}, \end{aligned} \tag{31}$$

for some universal constant $C_{20} > 0$. Since $\beta > 1$ we see that if k is large enough, we get huge improvement on the relative size of extension interval $[A_{k+2}, B_{k+2}]$ compared to the renormalization interval $[a_{k+2}, b_{k+2}]$. From this point the argument can be applied inductively and (29) follows.

Lemma 8 gives $|a_{k+1} - b_{k+1}| \approx |a_k - b_k|$. By the proof of Lemma 4(4) either $A_{k+1} = A_k$ (if $b'_k < B_k$) or $A_{k+1} = E_k \circ f_2(B_k)$ (if $B_k \leq b'_k$). In the former case we use (29) and get $\dfrac{|b_{k+1} - a_{k+1}|}{|a_{k+1} - A_{k+1}|} \approx \dfrac{|b_k - a_k|}{|a_k - A_k|} = O(b_{k-2}^{1-1/\beta})$. So let us check what happens when $B_k \leq b'_k$. Using (31) we obtain (*) $\dfrac{|f(0) - f_2(b_k)|}{|f(0) - f(B_k)|} \approx b_k^\beta / B_k^\beta = O(b_{k-2}^{\beta-1})$. On the other hand, the expression in (29) and Koebe imply $\dfrac{|x - f_2(b_k)|}{|f_2(b_k) - f_2(b'_k)|} = O(b_{k-2}^{1-1/\beta})$ where x is so that $E_k(x) = b_k$, see Figure 5. Since

$c_{2^k} \sim b_k$ we have $|x - f(a_k| \approx |f(a_k) - f(0)|$ this implies (**) $\dfrac{|f(0) - f_2(b_k)|}{|f_2(0) - f_2(b'_k)|} = O(b_{k-2}^{1-1/\beta})$. Since $b_{k-2}^{1-1/\beta} >> b_{k-2}^{\beta-1}$ and comparing (*) and (**) we can conclude that either $B_k > b'_k$ or (by Koebe) $E_k \circ f_2(B_k)| \geq (1/2)|A_k|$. In either case (30) holds.

Since $B_{k+1} = c_{2^k} \sim b_k$, we have by (17) that there exist universal constants $0 < \lambda'_1 < \lambda'_2 < 1$ so that $\dfrac{|b_{k+1} - a_{k+1}|}{|B_{k+1} - b_{k+1}|} \sim \dfrac{|b_{k+1}|}{|b_k - b_{k+1}|} \in (\lambda'_1, \lambda'_2)$. Which proves the second expression in (30) and that this expression cannot be improved.

The final statement follows from inequality (18). □

9 Proof of Theorems 3-7: scaling laws, renormalization limits and universality

A first error bound for the map f^{2^k} on $[a_k, b_k]$ when k is even. Let k be even and x_k be so that $E_k(x_k) = b_k$, see Figure 5. Then $E_k \colon [f(a_k), x_k] \to [a_k, b_k]$ is the first entry map and τ_k be the Koebe space of $E_k|[f(a_k), x_k]$. Let L_k be the affine map which agrees with E_k on the boundary points of $[f(a_k), f(0)]$. By the Corollary of Koebe, Lemma 2, we obtain

$$E_k(x) = L_k x + O(b_k/\tau_k) \text{ and } DE_k(x) = DL_k(1 + O(1/\tau_k)) \text{ for all } x \in [f(a_k), f(0)] \tag{32}$$

By the previous lemma $\tau_k \approx b_{k-2}^{1/\beta-1} \to \infty$. In particular it follows that $O(b_k/\tau_k) = o(b_k)$. Obviously $DL_k \approx b_k/|a_k| \approx b_k^{1-\beta}$. Hence

$$E_k(x) = L_k x + o(b_k) \text{ and } DE_k(x) \sim DL_k, \tag{33}$$

for all $x \in [f(a_k), f(0)]$. Later on, we will improve the error bound in this expression. Hence

$$f^{2^k}(x) = \begin{cases} c_{2^k} - s_k|x| + o(b_k) & \text{when } x \in [a_k, 0], \\ c_{2^k} - t_k x^\beta + o(b_k) & \text{when } x \in [0, b_k], \end{cases} \tag{34}$$

where $s_k > 0$ is so that $c_{2^k} - s_k|a_k| + o(|b_k|) = -|a_k|$ and $t_k > 0$ is so that $c_{2^k} - t_k b_k^\beta + o(|b_k|) = -|a_k|$. By (19) we have $c_{2^k} = b_k + O(b_k^\beta) \sim b_k$ and since $a_k \sim -K_0 b_k^\beta$, this implies

$$s_k \sim \frac{b_k^{1-\beta}}{K_0} \text{ and } t_k \sim b_k^{1-\beta}. \tag{35}$$

Equation (33) also gives

$$Df^{2^k}(x) \sim \begin{cases} s_k & \text{when } x \in [a_k, 0), \\ -t_k \beta x^{\beta-1} & \text{when } x \in (0, b_k]. \end{cases} \tag{36}$$

For simplicity we will write

$$f_{l,k} := f^{2^k}|[a_k, 0] \text{ and } f_{r,k} := f^{2^k}|[0, b_k].$$

To avoid an overload of notation we usually write

$$f_l = f_{l,k} \text{ and } f_r = f_{r,k}$$

if it clear from the context which k is used.

The scaling law from b_k to b_{k+1} when k is even. Write $b_{k+1} = \lambda_k b_k$. Then (34) implies

$$c_{2^k} - t_k \lambda_k^\beta b_k^\beta + o(b_k) = f^{2^k}(b_{k+1}) = b_{k+1} = \lambda_k b_k. \tag{37}$$

By (19)

$$c_{2^k} = b_k + O(b_k^\beta)$$

and combining this with (35) and (37) implies

$$1 - \lambda_k^\beta + o(1) = \lambda_k.$$

So taking $\lambda \in (0, 1)$ be the root of $1 - \lambda^\beta = \lambda$ this gives $\lambda_k = \lambda + o(1)$ and

$$b_{k+1} = \lambda b_k + o(b_k).$$

Later on we will improve on this statement, see (55).

The approximate scaling law from b_k to b_{k+2} when k is even. Fix some $\delta > 0$ and let C_k be so that $c_{2^{k+1}} = -C_k b_k^\delta$. Below we will determine δ and C_k. Note that

$$a_{k+1} < c_{2^{k+1}} < 0 < c_{2^{k+2}} < b_{k+2} < b_{k+1} < c_{3 \cdot 2^k} < c_{2^k} < b_k.$$

Then using (35) and (36)

$$c_{2^k} - c_{3 \cdot 2^k} = f^{2^k}(0) - f^{2^k}(c_{2^{k+1}}) = f_l(0) - f_l(c_{2^{k+1}}) \sim \frac{C_k}{K_0} b_k^\delta b_k^{1-\beta}. \tag{38}$$

Since f_r has bounded distortion and bounded derivative on $[b_{k+1}, b_k]$ this implies

$$c_{2^{k+2}} - c_{2^{k+1}} = f_r \circ f_l(c_{2^{k+1}}) - f_r(c_{2^k}) = f_r(c_{3.2^k}) - f_r(c_{2^k}) \approx C_k b_k^\delta b_k^{1-\beta}. \quad (39)$$

In fact,

$$|c_{2^k} - c_{3.2^k}| \approx |c_{2^{k+2}} - c_{2^{k+1}}| < |b_{k+2} - a_{k+1}| < o(b_k) \quad (40)$$

where $\approx$ follows from the fact that Df_r is bounded from above and below on $[b_{k+1}, b_k]$, where the first $<$ follows from the ordering of the points and where $< o(b_k)$ follows from equation (18) and $|a_{k+1}| \approx b_{k+1}^\beta$. Combining this with $c_{2^k} \sim b_k$, equations (36) and (35) give $f_r'(b_k) \sim -\beta$ and $f_r'(x) \sim -\beta$ for all $x \in [c_{3 \cdot 2^k}, c_{2^k}]$. Hence (39) in fact improves to

$$c_{2^{k+2}} - c_{2^{k+1}} \sim \frac{\beta C_k}{K_0} b_k^\delta b_k^{1-\beta}. \quad (41)$$

Since $|c_{2^{k+1}}| = C_k b_k^\delta << \dfrac{\beta C_k}{K_0} b_k^\delta b_k^{1-\beta}$ and using that $b_{k+2} \sim c_{2^{k+2}}$, equation (41) gives

$$b_{k+2} \sim c_{2^{k+2}} \sim \frac{\beta C_k}{K_0} b_k^\delta b_k^{1-\beta} \text{ and } a_{k+2} \sim -K_0 [\frac{\beta C_k}{K_0} b_k^\delta b_k^{1-\beta}]^\beta. \quad (42)$$

Next note that $f^{2^{k+1}}(a_{k+2}) = f_r \circ f_l(a_{k+2})$. Using that $f_l|[a_k, 0]$ has derivative everywhere $\sim \frac{1}{K_0} b_k^{1-\beta}$ and equation (18) we have that $|a_{k+2}| \le K_0 |b_{k+2}|^\beta < C|b_k|^{2\beta-1}$ and therefore equation (42) implies

$$f_l(a_{k+2}) - f_l(0) \le C b_k^{2\beta-1} b_k^{1-\beta} = C b_k^\beta.$$

Therefore $f_l(a_{k+2}) \sim b_k$ and so equation (36) implies

$$f_r'(x) \sim -\beta \text{ for all } x \in [f_l(a_{k+2}), b_k]. \quad (43)$$

Since, by (42),

$$f_l(a_{k+2}) - f_l(0) \sim \frac{b_k^{1-\beta}}{K_0} K_0 [\frac{\beta C_k}{K_0} b_k^\delta b_k^{1-\beta}]^\beta = \left[\frac{\beta C_k}{K_0}\right]^\beta b_k^{\beta\delta+1-\beta^2}.$$

Hence (43) implies

$$f^{2^{k+1}}(a_{k+2}) - c_{2^{k+1}} = f_r \circ f_l(a_{k+2}) - f_r(f_l(0)) \sim \beta \left[\frac{\beta C_k}{K_0}\right]^\beta b_k^{\beta\delta+1-\beta^2}. \quad (44)$$

By (42), $f^{2^{k+1}}(a_{k+2}) = a_{k+2} \approx -C_k^\beta [b_k^\delta b_k^{1-\beta}]^\beta = -C_k^\beta b_k^{\beta\delta+\beta-\beta^2}$ is orders smaller than the right hand side of (44), and thus it follows that

$$c_{2^{k+1}} \sim -\beta \left[\frac{\beta C_k}{K_0}\right]^\beta b_k^{\beta\delta+1-\beta^2}. \tag{45}$$

Using $c_{2^{k+1}} = -C_k b_k^\delta$ we obtain as a natural choice

$$\delta = \beta\delta + 1 - \beta^2 \text{ which gives } \delta = \beta + 1 \tag{46}$$

and

$$C_k \sim \beta \left[\frac{\beta C_k}{K_0}\right]^\beta \text{ and therefore } C_k \sim \left[\frac{K_0^\beta}{\beta^{\beta+1}}\right]^{1/(\beta-1)}. \tag{47}$$

Hence from (42), $b_{k+2} \sim c_{2^{k+2}}$ and $c_{2^{k+1}} = -C_k b_k^\delta$ we obtain

$$b_{k+2} \sim \frac{\beta}{K_0} \left[\frac{K_0^\beta}{\beta^{\beta+1}}\right]^{1/(\beta-1)} b_k^2 = \beta^{-2/(\beta-1)} K_0^{1/(\beta-1)} b_k^2 \tag{48}$$

and

$$c_{2^{k+1}} \sim -\left[\frac{K_0^\beta}{\beta^{\beta+1}}\right]^{1/(\beta-1)} b_k^{\beta+1}. \tag{49}$$

Since $b_{k+1} \sim \lambda b_k$ this gives

$$b_{k+2} \sim \beta^{\frac{-2}{\beta-1}} K_0^{\frac{1}{\beta-1}} \lambda^{-2} b_{k+1}^2 \tag{50}$$

and

$$c_{2^{k+1}} \sim -\beta^{-\frac{\beta+1}{\beta-1}} K_0^{\frac{\beta}{\beta-1}} \lambda^{-\beta-1} b_{k+1}^{\beta+1} \tag{51}$$

The usual Koebe space does not hold and the proof of Theorem 2 Let $T \ni f(0)$ be the maximal interval on which $f^{2^k-1}|T$ is diffeomorphic. Then by Lemma 4 we have that $f^{2^k-1} = [\hat{A}_k, \hat{B}_k] \supset [a_k, b_k]$ where

$$\hat{A}_k = c_{2^{k-1}}, \hat{B}_k = c_{2^{k-2}} \text{ when } k \text{ is even}$$

$$\hat{A}_k = c_{2^{k-2}}, \hat{B}_k = c_{2^{k-1}} \text{ when } k \text{ is odd.}$$

When k is even then

$$\hat{A}_k = c_{2^{k-1}} \approx b_{k-1}^{\beta+1} \approx b_k^{(\beta+1)/2} = o(b_k)$$

and when k is odd then

$$\hat{A}_k = c_{2^{k-2}} \approx b_{k-2}^{\beta+1} \approx b_k^{(\beta+1)/2} = o(b_k).$$

So in either case there exists no $\tau > 0$ so that $[\hat{A}_k, \hat{B}_k]$ is a τ-scaled neighbourhood of $[a_k, b_k]$ for k large. In other words, there is no Koebe space (on the left) for the diffeomorphic extension of the first entry map into $[a_k, b_k]$.

Improved Koebe Space for the semi-extension and the proof of Theorem 3 (Big Bounds). We can now prove Theorem 3 and an improved version of Lemma 12:

Lemma 13 (Improved Koebe Space)**.** The range of the map $E_k|J_k$ can be monotonically semi-extended to a τ_k scaled neighbourhood of $[a_k, b_k]$ where $\tau_k \approx b_{k-2}/b_k \approx b_k^{-1/2}$ when k is even and $\tau_k \approx 1$ for k odd.

Proof. The map $E_k|J_k$ can be monotonically semi-extended onto $[A_k, B_k]$. As we saw in Lemmas 11 and 12 we have $|A_k| \geq b_{k-2}$ for k even. By Lemma 4 and the previous bounds, we have for k even $B_k = c_{2^{k-2}} \approx b_{k-2}$. It follows from this and (48) that $\tau_k \approx b_{k-2}/b_k \approx b_k^{-1/2}$. Note that for k odd, $B_k = b_{k-1}$ and so $\tau_k = b_k/B_k = b_k/b_{k-1} \to \lambda$ as $k \to \infty$ and k odd. □

Proof of Theorems 5 and 6 (Renormalization limits of R^k): Given the previous lemma, we obtain that the Koebe space is of the order $\tau_k \approx b_k^{-\frac{1}{2}}$. It follows that $O(b_k/\tau_k) = O(b_k^{\frac{3}{2}})$ and so (32) gives

$$f^{2^k}(x) = \begin{cases} c_{2^k} - s_k|x| + O(b_k^{\frac{3}{2}}) & \text{when } x \in [a_k, 0] \\ c_{2^k} - t_k x^\beta + O(b_k^{\frac{3}{2}}) & \text{when } x \in [0, b_k] \end{cases} \tag{52}$$

with

$$s_k \sim \frac{b_k^{1-\beta}}{K_0} \text{ and } t_k \sim b_k^{1-\beta}. \tag{53}$$

The proof of Theorem 6 follows the above and an explicit calculation. For example,

$$\lim_{k\to\infty} (R^{2k+1}f) \circ m_{2k+1}$$

is composition of the asymptotically linear left branch of $R^{2k}f$ and of the part of the right branch of $R^{2k}f$ corresponding to $[b_{k+1}, c_{2^k}]$ where $c_{2^k} \sim b_k$.

Improved scaling law from b_k to b_{k+1} when k is even. Arguing as in (37) and below we have

$$c_{2^k} - t_k\lambda_k^\beta b_k^\beta = \lambda_k b_k + O(b_k^{\frac{3}{2}}) \tag{54}$$

and therefore

$$b_k - \lambda_k^\beta b_k + O(b_k^\beta) = \lambda_k b_k + O(b_k^{\frac{3}{2}})$$

This means

$$b_k - \lambda_k^\beta b_k = \lambda_k b_k + O(b_k^{\frac{3}{2}}) + O(b_k^\beta)$$

and so

$$\lambda_k = \lambda + O(b_k^{\frac{1}{2}}) + O(b_k^{\beta-1}) \tag{55}$$

where as before $\lambda \in (0,1)$ is the root of $1-\lambda^\beta = \lambda$. In the same way, we obtain that the $\sim$ expressions in this Section 9 are in fact equalities with a multiplicative error of the form $1+O(b_k^\epsilon)$ for some $\epsilon > 0$.

One can similarly also obtain exponential convergence for the constants in the scaling for b_{k+1} to b_{k+2}.

The growth rate of $\log b_k$ and the completion of the proof of Theorem 4. Let $\mu_k = \log(1/b_{2k})$. As we saw $\mu_k \to \infty$. Let us give a sharper estimate here. According to (48) $\mu_{k+1} = 2\mu_k + D_k$ for all $k \geq 0$ where

$$D_k \sim D := \log(\beta^{\frac{2}{\beta-1}} K_0^{\frac{-1}{\beta-1}}). \tag{56}$$

It follows that $\mu_k/2^k = (\mu_0 + D_{k-1}/2^k + \cdots + D_0/2)$ and therefore there exists $\Theta > 0$ so that $\frac{\mu_k}{2^k} \to \Theta$. Moreover,

$$\Theta - \mu_k/2^k = \sum_{i\geq k} D_i/2^{i+1} = \sum_{i\geq k} D/2^{i+1} + \sum_{i\geq k}(D_i - D)/2^{i+1} = D/2^k + o(1)/2^k.$$

Hence

$$\log(1/b_{2k+1}) \sim \log(1/b_{2k}) = \mu_k = 2^k\Theta - D + o(1) \tag{57}$$

and so using (56)

$$1/b_{2k} = \beta^{-\frac{2}{\beta-1}} K_0^{\frac{1}{\beta-1}} \exp(2^k\Theta + o(1)). \tag{58}$$

Necessary and sufficient invariants for $h\colon \{c_{2^k}\}_{k\geq 0} \to \{\tilde{c}_{2^k}\}_{k\geq 0}$ to be Lipschitz. Assume that $h\colon \{c_{2^k}\}_{k\geq 0} \to \{\tilde{c}_{2^k}\}_{k\geq 0}$ is a conjugacy between f and $\tilde{f}$ and is Lipschitz at 0. This implies

$$\tilde{c}_{2^{2k}} \approx c_{2^{2k}}, \tilde{c}_{2^{2k+1}} \approx c_{2^{2k+1}}. \tag{59}$$

Since $b_{2k+1} \sim \lambda b_{2k}$, $c_{2^{2k}} \sim b_{2k}$ where $\lambda \in (0,1)$ is the root of the equation $\lambda^\beta + \lambda = 1$, (59) implies

$$\tilde{b}_{2k} \approx b_{2k} \text{ and } \tilde{\lambda}^{-1}\tilde{b}_{2k+1} \approx \lambda^{-1} b_{2k+1} \tag{60}$$

By Theorem 4 and (59) we also have

$$-\tilde{\beta}^{-\frac{\tilde{\beta}+1}{\tilde{\beta}-1}} \tilde{K}_0^{\frac{\tilde{\beta}}{\tilde{\beta}-1}} \tilde{\lambda}^{-\tilde{\beta}-1} \tilde{b}_{2k+1}^{\tilde{\beta}+1} \sim \tilde{c}_{2^{2k+1}} \approx c_{2^{2k+1}} \approx -\beta^{-\frac{\beta+1}{\beta-1}} K_0^{\frac{\beta}{\beta-1}} \lambda^{-\beta-1} b_{2k+1}^{\beta+1}. \tag{61}$$

This, the 2nd expression in (60) and $b_{2k+1} \to 0$ imply that

$$\beta = \tilde{\beta} \text{ and therefore } \lambda = \tilde{\lambda}. \tag{62}$$

Finally (58) and (59) imply that

$$1 \approx \tilde{c}_{2^k}/c_{2^k} \sim \tilde{b}_{2k}/b_{2k} = \left[\frac{K_0}{\tilde{K}_0}\right]^{\frac{-1}{\beta-1}} \exp(2^k(\Theta - \tilde{\Theta}) + o(1)) \tag{63}$$

Hence

$$\Theta = \tilde{\Theta}. \tag{64}$$

Thus we have shown that the existence of a Lipschitz conjugacy implies

$$\beta = \tilde{\beta} \text{ and } \Theta = \tilde{\Theta}. \tag{65}$$

That h is Lipschitz conjugate when (65) follows from the above equations.

Necessary and sufficient invariants for $h\colon \{c_{2^k}\}_{k\geq 0} \to \{\tilde{c}_{2^k}\}_{k\geq 0}$ to be differentiable at 0. By the previous paragraph, (65) are necessary conditions for h to be differentiable at 0. Let us show that these conditions are also sufficient. So assume that (65) holds. This and (58) imply

$$\frac{\tilde{c}_{2^{2k}}}{c_{2^{2k}}} \sim \frac{\tilde{b}_{2k}}{b_{2k}} \sim \frac{\beta^{\frac{-2}{\beta-1}} K_0^{\frac{1}{\beta-1}}}{\tilde{\beta}^{\frac{-2}{\tilde{\beta}-1}} \tilde{K}_0^{\frac{1}{\beta-1}}} \exp(2^k(\Theta - \tilde{\Theta}) + o(1)) \sim \left(\frac{K_0}{\tilde{K}_0}\right)^{\frac{1}{\beta-1}} := \rho. \tag{66}$$

By Theorem 4, $\tilde\beta = \beta$, $\tilde\lambda = \lambda$ and $b_{2k+1} \sim \lambda b_{2k}$, $\tilde b_{2k+1} \sim \tilde\lambda b_{2k}$ and the previous expression (and $\rho := [K_0/\tilde K_0]^{\frac{1}{\beta-1}}$) we get

$$\begin{aligned}\frac{\tilde c_{2^{2k+1}}}{c_{2^{2k+1}}} &\sim \frac{-\tilde\beta^{-\frac{\tilde\beta+1}{\tilde\beta-1}}\tilde K_0^{\frac{\tilde\beta}{\tilde\beta-1}}\tilde\lambda^{-\tilde\beta-1}\tilde b_{2k+1}^{\tilde\beta+1}}{-\beta^{-\frac{\beta+1}{\beta-1}}K_0^{\frac{\beta}{\beta-1}}\lambda^{-\beta-1}b_{2k+1}^{\beta+1}} = \left[\frac{\tilde K_0}{K_0}\right]^{\frac{\beta}{\beta-1}}\left[\frac{\tilde b_{2k+1}}{b_{2k+1}}\right]^{\beta+1} \sim \\ &\sim \left[\frac{\tilde K_0}{K_0}\right]^{\frac{\beta}{\beta-1}}\left[\frac{\tilde b_{2k}}{b_{2k}}\right]^{\beta+1} \sim \left[\frac{\tilde K_0}{K_0}\right]^{\frac{\beta}{\beta-1}}\rho^{\beta+1} = \rho^{-\beta}\rho^{\beta+1} = \rho.\end{aligned} \tag{67}$$

Another ratio. Even though we shall not use this, let us calculate another ratio. Writing as before $c_{2^{2k+1}} = -C_{2k}b_{2k}^\delta$ we have according to (46) and (47) we have $\delta = \beta + 1$ and $C_{2k} \sim \left[\dfrac{K_0^\beta}{\beta^{\beta+1}}\right]^{1/(\beta-1)}$. So according to (38) we have

$$c_{2^{2k}} - c_{3\cdot 2^{2k}} \sim \frac{C_{2k}}{K_0}b_{2k}^2 \sim \frac{K_0^{1/(\beta-1)}}{\beta^{(\beta+1)/(\beta-1)}}b_{2k}^2 \tag{68}$$

So assuming that (65) holds we have using (66)

$$\frac{\tilde c_{2^{2k}} - \tilde c_{3\cdot 2^{2k}}}{c_{2^{2k}} - c_{3\cdot 2^{2k}}} \sim \frac{\tilde K_0^{1/(\beta-1)}}{K_0^{1/(\beta-1)}}\frac{\tilde b_{2k}^2}{b_{2k}^2} \sim \frac{\tilde K_0^{1/(\beta-1)}}{K_0^{1/(\beta-1)}}\rho^2 = \rho$$

The invariants (65) are sufficient for the conjugacy $h\colon \Lambda \to \tilde\Lambda$ to be differentiable at 0, where Λ is the attracting Cantor set $\overline{\cup_{n\geq 0} f^n(0)}$. Regardless whether or not (65) holds, there exists a topological conjugacy $h\colon \Lambda \to \tilde\Lambda$ between f and $\tilde f$; in fact, in the next section we will show that $f, \tilde f$ do not wandering intervals, and then we will also know that there exists a topological conjugacy h on the entire space. Let us show now that the conjugacy $h\colon \Lambda \to \tilde\Lambda$ is necessarily differentiable on Λ when (65) is satisfied.

To do this, note that when k is even that $\Lambda \cap [a_k, b_k]$ is contained in the union of following intervals U_k, V_k, W_k, X_k where $U_k = [x_k, c_{4\cdot 2^k}]$ where $x_k < 0$ is chosen so that $f(x_k) = f(c_{4\cdot 2^k})$ and let $U_k^- = [x_k, 0]$, $U_k^+ = [0, c_{4\cdot 2^k}]$, $V_k = f_l(U_k^-)$, $W_k = f_r(V_k)$ and $X_k = f_l(W_k)$. For simplicity also define $R_k := [X_k, V_k]$, $L_k = [W_k, U_k]$ and $(U_k, X_k) := [c_{4\cdot 2^k}, c_{3\cdot 2^k}]$.

Lemma 14.

$$\liminf \frac{|W_k|}{|L_k|} > 0. \tag{69}$$

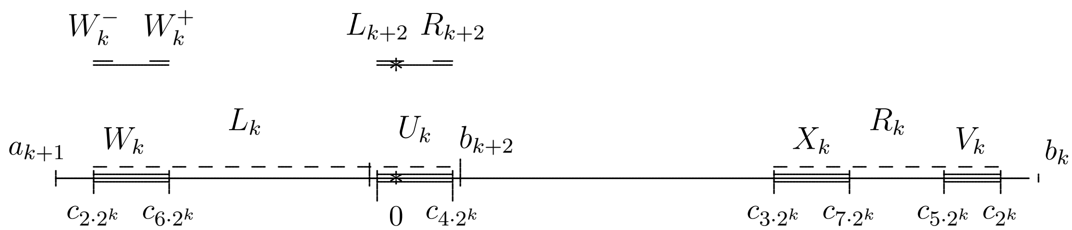

Figure 7: These four intervals contain the postcritical set in $[a_k, b_k]$. We will pull back the analogue of the dashed intervals for level $k+2$ inside W_k.

and

$$\frac{|R_k|}{|(U_k, X_k)|} \to 0 \text{ and } \frac{|L_k|}{|(U_k, X_k)|} \to 0 \text{ as } k \to \infty. \tag{70}$$

Proof. Note that $|U_k^-| = |x_k| \approx |c_{4\cdot 2^k}|^\beta \sim b_{k+2}^\beta \approx b_k^{2\beta}$,

$$|V_k| = |c_{2^k} - c_{5\cdot 2^k}| = |f_l(U_k^-)| \approx s_k |U_k^-| \approx b_k^{1-\beta} b_k^{2\beta} = b_k^{1+\beta}$$

and by (43),

$$|W_k| = |f_r(V_k)| \approx \beta b_k^{1+\beta} \approx |c_{2^{k+1}} - 0|$$

where in the last $\approx$ we used (51). This implies that the size of W_k is comparable to its distance to 0; in other words for any two points $u_k, v_k \in W_k$ we merely have $u_k \approx v_k$, showing (69). To prove (70), note that

$$|U_k| \sim |U_k^+| = |c_{2^{k+2}}| \sim b_{k+2} \approx b_k^2$$

and therefore

$$|L_k| = |[W_k, U_k]| \approx b_k^{1+\beta} + b_k^2 \approx b_k^2.$$

Similarly, by (38) and $\delta = 1 + \beta$ we have

$$|R_k| = |[X_k, V_k]| = |c_{2^k} - c_{3\cdot 2^k}| \approx b_k^2. \tag{71}$$

These two statements imply $|(U_k, X_k)| \sim |[0, c_{2^k}]| \sim b_k$ and therefore (70). $\square$

It follows from (71) that when $u_k \in R_k$ arbitrarily then $u_k \sim b_k$ as $k \to \infty$ and therefore we will be able to use R_k instead of the intervals X_k and V_k. Equation (69) will require us to choose much smaller intervals inside W_k.

Lemma 15. Let W_k^- and W_k^+ in W_k which are mapped by $f_r \circ f_l$ onto R_{k+2} resp. L_{k+2}, where we take W_k^- is to the left of W_k^+. Then

$$\frac{|W_k^-|}{|W_k|}, \frac{|W_k^+|}{|W_k|} \to 0. \tag{72}$$

Note that

$$\Lambda \cap [a_k, b_k] \subset W_k^- \cup W_k^+ \cup U_k \cup X_k \cup V_k. \tag{73}$$

Proof. Since (70) also holds for $k+2$ replaced by k, there exists four intervals in U_k (with two in L_{k+2} and two in R_{k+2}) so that the gap between L_{k+2} and R_{k+2} is huge compared to the size of these two intervals. Now consider the orientation reversing map $f_r \circ f_l \colon W_k \to U_k$. Since this map has bounded distortion (72) holds. □

Note that for each $x \in \Lambda \cap [a_k, b_k]$ either $x \in [a_{k+2}, b_{k+2}]$ or x is contained in one of the sets X_k, V_k, W_k^+ or W_k^-. Moreover, as we have shown, if $u_k, v_k \in Q_k$ and $u_k \to 0$ where Q_k is either $R_k = [X_k, V_k]$, W_k^+ or W_k^- then $u_k \sim v_k$.

It remains to obtain asymptotic expressions for at least one point in each these intervals. Let us start with W_k^+. This interval contains a point z_k so that $f_r \circ f_l(z_k) = 0$. It follows that

$$|c_{2^{k+1}} - 0| = |f_r(f_l(0)) - f_r(f_l(z_k))| \sim \beta|f_l(0) - f_l(z_k)| \sim \beta|z_k|s_k$$

Since $s_k \sim \dfrac{b_k^{1-\beta}}{K_0}$ and $c_{2^{k+1}} \sim -\left[\dfrac{K_0^\beta}{\beta^{\beta+1}}\right]^{1/(\beta-1)} b_k^{\beta+1}$ it follows that

$$z_k \sim -\frac{1}{\beta}\left[\frac{K_0^\beta}{\beta^{\beta+1}}\right]^{1/(\beta-1)} b_k^{\beta+1}\frac{K_0}{b_k^{1-\beta}} = -\left[\frac{K_0^{2\beta-1}}{\beta^{2\beta}}\right]^{1/(\beta-1)} b_k^{2\beta}. \tag{74}$$

Similarly, $c_{2^{k+1}} \in W_k^-$ and according to (51)

$$c_{2^{k+1}} \sim -\left[\frac{K_0^\beta}{\beta^{\beta+1}}\right]^{1/(\beta-1)} b_k^{\beta+1}. \tag{75}$$

Finally, $c_{3\cdot 2^k}, c_{2^k} \in R_k$, by (40)

$$c_{3\cdot 2^k} \sim c_{2^k} \sim b_k. \tag{76}$$

Let us now take the homeomorphism h between Λ and $\tilde{\Lambda}$ defined so that $h(f^n(0)) = \tilde{f}^n(0)$ and show that h is differentiable at 0, provided that $\beta = \tilde{\beta}$, $\Theta = \tilde{\Theta}$ and $K_0 = \tilde{K}_0$. Because of these assumptions, equation (58) gives that for $k \to \infty$ even,

$$\frac{\tilde{b}_k}{b_k} \to \rho := \left[\frac{K_0}{\tilde{K}_0}\right]^{\frac{1}{\beta-1}} = 1. \tag{77}$$

Let $u_k \in \Lambda$ and take $\tilde{u}_k = h(u_k)$. By renumbering if necessary we may assume that $u_k \in W_k^- \cup W_k^+ \cup X_k \cup V_k$. From (75) follows that for $u_k \in W_k^-$, $\tilde{u}_k \in \tilde{W}_k^-$,

$$\tilde{u}_k/u_k \to [\tilde{K}_0/K_0]^{(2\beta-1)/(\beta-1)}(\tilde{b}_k/b_k)^{2\beta} \sim \rho^{1-2\beta}\rho^{2\beta} = \rho.$$

From (74), $u_k \in W_k^+$, $\tilde{u}_k \in \tilde{W}_k^+$,

$$\tilde{u}_k/u_k \to [\tilde{K}_0/K_0]^{\beta/(\beta-1)}(\tilde{b}_k/b_k)^{\beta+1} \sim \rho^{-\beta}\rho^{1+\beta} = \rho.$$

Finally from (76) we have $\tilde{u}_k/u_k \to \rho$ for $u_k \in X_k \cup V_k$ and $\tilde{u}_k \in \tilde{X}_k \cup \tilde{V}_k$. It follows that $h\colon \Lambda \to \tilde{\Lambda}$ is differentiable at 0.

The invariants (65) are sufficient for the conjugacy $h\colon \Lambda \to \tilde{\Lambda}$ to be differentiable along Λ, where $\Lambda = \overline{\cup_{n\geq 0} f^n(0)}$. Let $\Delta_{k,0} = [a_k, b_k]$, $\Delta_{k,i} = f^i(\Delta_k^0)$, $i = 1, \ldots, 2^k - 1$ and $\Delta_k = \cup_{0\leq i\leq 2^k-1}\Delta_{k,i}$. Note that $\Lambda = \cap_k \Delta_k$. Moreover, let $\tilde{\Delta}_{k,i}, \tilde{\Delta}_k$ be the corresponding the sets for $\tilde{f}$. As in [42, Section VI.9], define $\Omega = \{0,1\}^{\mathbb{N}}$ and a continuous map $\phi\colon \Omega \to \Lambda$ defined by associating to $\omega \in \Omega = \{0,1\}^{\mathbb{N}}$ the point $\cap_k \Delta^{j(k,\omega)}$ where $j(k,\omega) = \sum_{i=0}^{k-1} \omega(i)2^j$. Denote the interval $\Delta_{k,j(k,\omega)}$ by $[\omega(0), \ldots, \omega(k-1)]_k$ and let $[\omega(0), \ldots, \omega(k-1)]_{k,\sim}$ be the corresponding interval for $\tilde{f}$. Because f has the period doubling combinatorics,

$$[\omega(0), \ldots, \omega(k-1)]_k \subset [\omega(0), \ldots, \omega(k-2)]_{k-1}.$$

Let Ω^* be the dual Cantor set consisting of all left infinite words

$$\{\omega = (\ldots, \omega(k), \ldots, \omega(1), \omega(0)), \omega(i) \in \{0,1\}\}$$

with the product topology. From the scaling law (58) we obtain that

$$\frac{[0, \ldots, 0, 0, 0]_{k+2}}{[0, \ldots, 0, 0]_k} = (1 + \epsilon_k)\exp(2^k(\Theta - 4\Theta)).$$

From the calculation in (56)- (58) it follows that $\prod_{n\geq k}(1+\epsilon_n)$ goes to one as $k\to\infty$. (In fact, one can show that ϵ_n tends exponentially fast to zero.) From the above consideration we also have that for $j_1, j_2 \in \{0,1\}$

$$\frac{[0,\dots,0,j_1,j_2]_{k+2}}{[0,\dots,0,0]_k} = (1+\epsilon_k)\kappa(\beta,j_1,j_2)\exp(-2^k\Psi(\Theta,\beta,j_1,j_2))$$

where $\kappa(\beta,j_1,j_2) > 0$ and $\Psi(\Theta,\beta,j_1,j_2)$ are constants which can be computed explicitly as above (and which only depend on β,Θ,j_1,j_2). Using the fact that the Koebe space of the semi-extension of the first entry map from Δ_k^i into $\Delta_{k,2^k}\subset\Delta_{k,0}$ tends exponentially fast to infinity, and therefore the non-linearity of the first entry map tends exponentially fast to zero, we obtain

$$\frac{[\omega(k+1),\dots,\omega(2),j_1,j_2]_{k+2}}{[\omega(k+1),\dots,\omega(2)]_k} = (1+\epsilon_k)\kappa(\beta,j_1,j_2)\exp(-2^k\Psi(\Theta,\beta,j_1,j_2)).$$

Hence, as in [42, Proof of Theorems VI.9.3 and VI.9.1], using the property that $\prod_{n\geq k}(1+\epsilon_n)$ converges to 1 as $k\to\infty$ and assuming that (65) holds we obtain that for each sequence $\omega\in\Omega^*$

$$\frac{[\omega(k-1),\dots,\omega(0)]_{k,\sim}}{[\omega(k-1),\dots,\omega(0)]_k}$$

converges and the value of the limit depends continuously on $\omega\in\Omega^*$. From this it follows that the conjugacy is differentiable along Λ.

10 The Hausdorff dimension of the attracting Cantor set is zero

Recall that for every $k>0$ and $i=0,\dots,2^k-1$ we have defined $\Delta_{k,i} := f^i([a_k,b_k])$.

Let us make a few observations on locations of certain intervals Δ inside their parents. In what follows k is assumed to be even. First, observe that the both intervals $\Delta_{k+2,2^k}$ and $\Delta_{k+2,3\cdot 2^k}$ belong to $[c_{3\cdot 2^k}, c_{2^k}]$. Secondly, $\Delta_{k+2,2\cdot 2^k}\subset[c_{2\cdot 2^k}, c_{4\cdot 2^k}]$. Also note that all 4 mentioned intervals belong to $\Delta_{k,0}$.

Using formulas (38), (39) and (48) we see that $|\Delta| < C|\Delta_{k,2^k}|^2$ for $\Delta = \Delta_{k+2,2^k}, \Delta_{k+2,2\cdot 2^k}, \Delta_{k+2,3\cdot 2^k}, \Delta_{k+2,4\cdot 2^k}$, where C is some universal constant.

Fix some integer $1\leq i\leq 2^k-1$. The distortion of the map $f^{2^k-i}:\Delta_{k,i}\to\Delta_{k,0}$ is asymptotically small due to Theorem 3 and Lemma 1 (k is still assumed even). We know that $f^{2^k-i}(\Delta_{k,i}) = [a_k, c_{2^k}]$ and this interval is very close to $\Delta_{k,0} := [a_k, b_k]$ due to formula (6). Hence, if $\Delta \subset \Delta_{k,i}$ is one of four intervals of the form $\Delta_{k+2,m}$, then $|\Delta| < C|\Delta_{k,0}||\Delta_{k,i}|$, where C is another universal constant. This estimate implies that for any $\gamma > 0$ there exists k_0 (depending on f) such that if $k > k_0$ and k is even, $|\Delta|^\gamma < \frac{1}{4}|\Delta_{k,i}|^\gamma$. Therefore,

$$\sum_{i=0}^{4\cdot 2^k-1} |\Delta_{k+2,i}|^\gamma < \sum_{i=0}^{2^k-1} |\Delta_{k,i}|^\gamma.$$

Thus we have shown that the Hausdorff dimension of Λ is zero.

11 Absence of any Koebe space for general first entry maps

Define R_k to be the first return map to $[a_k, b_k]$.

Theorem 13 (Theorem 9 - Absence of Koebe space)**.** For each $\tau > 0$ there exists x and k so that the maximal semi-extension of the first entry map from x into $[a_k, b_k]$ does **not** contain a τ-scaled neighbourhood of $[a_k, b_k]$.

Proof. Assume that $x \in I$ and n is so that $y = f^n(x)$ is a first entry to $[a_{2i-1}, b_{2i-1}]$ and that in fact $y \in [b_{2i}, b_{2i-1}]$. Moreover, assume that $y' = R_{2i-1}(y) \in [a_{2i}, b_{2i}]$. Write $y' = f^m(x)$ so y' is a first entry of x into $[a_{2i}, b_{2i}]$ under f^m. Since $f^m = R_{2i-1} \circ f^n$, the maximal diffeomorphic extension (or even semi-extension) of f^m is at most that of R_{2i-1}. The diffeomorphic range of the latter map is $[c_{2^{2i-1}}, B_{2i-1}]$. By Theorem 4 we have $c_{2^{2i-1}} \approx -b_{2i-1}^{\beta+1}$.

The length of $[a_{2i}, b_{2i}]$ is $\sim b_{2i} \approx b_{2i+1} \approx b_{2i-1}^2$, and since $\beta > 1$, therefore the space $[c_{2^{2i-1}}, a_{2i}]$ is minute compared to the size of the interval $[a_{2i}, b_{2i}]$ when i large. It follows that when i is large, there exists no $\tau > 1$ so that the range of the extension $[c_{2^{2i-1}}, B_{2i-1}]$ contains a τ-scaled neighbourhood of $[a_{2i}, b_{2i}]$. In fact, the range of the extension is also not a τ-scaled neighbourhood of $[a_{2i+1}, b_{2i+1}]$ for the same reason. □

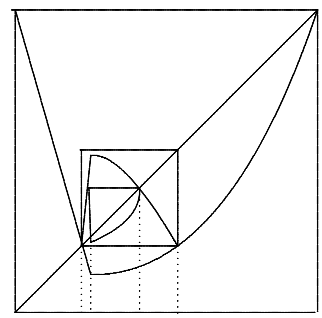

Figure 8: The return maps R_j to $[a_j, b_j]$ for $j = 2i-1, 2i, 2i+1$.

12 Proof of Theorem 10: absence of wandering intervals

Lemma 16 (The orbit of a potential wandering interval)**.** If f has a wandering interval W, then

1. $W_k := f^k(W)$ accumulates onto 0, so for some sequence of k_j's tending to infinity $W_{k_j} \to 0$;
2. there exists i_0 so that if $W_k \subset [a_{2i_0-1}, b_{2i_0-1}]$ for some k then $W_k \subset \bigcup_{i \geq i_0} [b_{2i}, b_{2i-1}]$;
3. if $W_k \subset [b_{2i}, b_{2i-1}]$ then $W_k \subset [b_{2i}, \eta_i b_{2i-1}]$ where $\eta_i \to 0$ as $i \to \infty$.

Proof. Assume by contradiction that W is a maximal wandering interval for f. The sequence of intervals $W_i := f^i(W)$ must accumulate to 0 for some subsequence $i_j \to \infty$. Indeed, otherwise we can modify the map in a small neighbourhood of 0 to obtain a C^2 map whose orbit of W is the same as that of f. But then a theorem of Mañé, see [41][Theorem III.5.1] gives a contradiction. It follows that $W_i \not\ni 0$ for all $i \geq 0$. So for any k there exists a *minimal* $n(k) \geq 0$ so that $W_{n(k)} \subset I_k = [a_k, b_k]$ where $n(k) \to \infty$ as $k \to \infty$. Since all iterates of W are disjoint, $W_i \cap \{a_k, b_k\} = \emptyset$ for all $i \geq 0, k \geq 0$.

By minimality of $n(k)$, $W_i \cap [a_k, b_k] = \emptyset$ for all $i < n(k)$. Hence if we take $T_k \supset W$ to be the maximal interval so that $f^{n(k)}|T_k$ is a diffeomorphism then by Lemma 3 there exists $\tau > 1$ so that $f^{n(k)}(T_k)$ contains $[\tau a_k, \tau b_k]$.

(1) Let us first show that $W_{n(k)}$ lies to the right of 0 for all k large. Indeed, assume by contradiction that there exists infinitely many k's so that $W_{n(k)} \subset [a_k, 0]$.

For each such k, $f^{n(k)}(T_k) \supset [\tau a_k, \tau b_k]$ is a scaled-neighbourhood of $W_{n(k)}$. By Koebe it follows that T_k also contains a τ'-scaled neighbourhood of W where $\tau' > 0$ is the same for infinitely many k's. This shows that there exists an interval $W' \supset W$ which strictly contains W on which all iterates of f are diffeomorphic, contradicting the maximality of W.

(2) Let us now show that there exists k_0 so that if $k \geq k_0$ is even then $W_{n(k)}$ cannot be contained in $[b_{k+1}, b_k]$. Indeed, when k is even then by Theorem 4, $[\tau a_k, \tau b_k]$ is a scaled neighbourhood of $[b_{k+1}, b_k]$ and so as in the previous case we obtain a contradiction.

From (1) and (2) it follows that for all k large, $W_{n(k)}$ is contained in $\bigcup_i [b_{2i}, b_{2i-1}]$. Similarly to (2), we have that if $W_{n(k)}$ is contained in $[b_{2i}, b_{2i-1}]$ then in fact it is contained in $[b_{2i}, \eta b_{2i-1}]$ where $\eta \in (0,1)$ is small when i is large. Here we use that $W_{n(k)}$ must be contained in a fundamental domain of the fixed point b_{2i-1} of R_{2i-1}. $\square$

As above let $n(k) \geq 0$ be minimal so that $W_{n(k)} \subset I_k = [a_k, b_k]$. From the previous lemma it follows that $W_{n(k)}$ is contained in $[b_{2i}, b_{2i-1}]$ for some $2i-1 \geq k$ and therefore $n(2i-1) = n(k)$. The first return map R_{2i-1} to $[a_{2i-1}, b_{2i-1}]$ is drawn in Figure 8 and satisfies $R_{2i-1}(x) < x$ for $x \in [0, b_{2i-1}]$. It follows that there exists $m_k \geq 1$ so that

$$R_{2i-1}^j(W_{n(k)}) \subset [b_{2i}, b_{2i-1}] \text{ for all } 0 \leq j < m_k \tag{78}$$

and then for some $i' > i$,

$$R_{2i-1}^{m_k}(W_{n(k)}) \subset [b_{2i'}, b_{2i'-1}]. \tag{79}$$

In other words, the next first entry into $[a_{2i}, b_{2i}]$ is in fact into $[b_{2i'}, b_{2i'-1}]$ and in particular $n(2i-1) < n(2i) = \cdots = n(2i'-1)$.

Lemma 17. f does not wandering intervals.

Proof. Let us write $R_{2i-1} = \phi_{2i-1}(x^\beta)$ on $[0, b_{2i-1}]$ where ϕ_{2i-1} is an orientation preserving diffeomorphism. For convenience we will write ϕ rather than ϕ_{2i-1}. Let us first obtain an estimate for ϕ. It follows from Lemma 12 and part (3) of Lemma 16 $|\phi'(x)/\phi'(\hat{x}) - 1| \leq \epsilon$ for all $x, \hat{x} \in [b_{2i}^\beta, \eta b_{2i-1}^\beta]$ where $\epsilon > 0$ is small when η is small and i is large. It follows that there exists $\gamma > 0$ so that

$$-\gamma\epsilon \leq \phi'(x) - \gamma \leq \gamma\epsilon. \tag{80}$$

Since $\phi(0) = c_{2^{2i-1}} < 0$ it follows that

$$\phi(0) + (1-\epsilon)\gamma x \leq \phi(x) \leq \phi(0) + (1+\epsilon)\gamma x \leq (1+\epsilon)\gamma x. \qquad (81)$$

Note that $|c_{2^{2i-1}}| \approx |b_{2i-1}^{\beta+1}| << |b_{2i-1}|$ and therefore $R_{2i-1}(b_{2i-1}) = b_{2i-1}$ implies that $\gamma \approx b_{2i-1}^{1-\beta}$.

From (57) we have $\log(1/b_{2i-1}) \approx 2^i$, $\log(1/b_{2i}) \approx 2^{i+1}$, and therefore $\log(\log(1/b_{2i-1})) \approx i\log 2 + O(1)$, $\log(\log(1/b_{2i})) \approx (i+1)\log 2 + O(1)$ and so the length of the intervals $[b_{2i}, b_{2i-1}]$ is bounded in double logarithmic coordinates.

Let us show that R_{2i-1} is expanding in double logarithmic coordinates. So define $l_2(x) = \log(\log(1/x))$ where we assume $x \in [b_{2i}, \eta b_{2i-1}]$. Then $Dl_2(x) = \dfrac{-1}{x\log(1/x)}$ and $x = l_2^{-1}(y) = e^{-e^y}$. Moreover,

$$D(l_2 \circ R_{2i-1} \circ l_2^{-1})(y) = D(l_2 \circ \phi \circ f \circ l_2^{-1})(y) = \frac{\phi'(e^{-\beta e^y})(\beta e^y)e^{-\beta e^y}}{\phi(e^{-\beta e^y})\log(1/\phi(e^{-\beta e^y}))}$$

Since $x = l_2^{-1}(y) = e^{-e^y}$, $\log x = -e^y$ and $\log(1/x^\beta) = \beta e^y$ this is equal to

$$\frac{\phi'(x^\beta)x^\beta\log(1/x^\beta)}{\phi(x^\beta)\log(1/\phi(x^\beta))} \geq (1-\epsilon)\gamma\frac{x^\beta\log(1/x^\beta)}{\phi(x^\beta)\log(1/\phi(x^\beta))}$$

where in the inequality we used (80). Since $t \mapsto t\log(1/t)$ is increasing for $t > 0$ small and because of (81) the latter expression is bounded below by

$$\geq (1-\epsilon)\gamma\frac{x^\beta\log(1/x^\beta)}{(1+\epsilon)\gamma x^\beta\log(1/((1+\epsilon)\gamma x^\beta))} = \frac{(1-\epsilon)}{(1+\epsilon)}\frac{\log(1/x^\beta)}{\log(1/((1+\epsilon)\gamma x^\beta))}.$$

Since $\gamma \approx b_{2i-1}^{1-\beta}$, there exists $C_0 > 0$ so that this is bounded below by

$$\geq \frac{1-\epsilon}{1+\epsilon}\ \frac{\log(1/x^\beta)}{\log(1/x^\beta) + (1-\beta)\log(1/b_{2i-1}) + \log(C_0)}.$$

Since the latter expression is increasing in x for $x \in [0, b_{2i-1}]$ and since $x \in [b_{2i}, b_{2i-1}]$ this is bounded from below by

$$\frac{1-\epsilon}{1+\epsilon}\ \frac{\beta\log(1/b_{2i})}{\beta\log(1/b_{2i}) + (1-\beta)\log(1/b_{2i-1}) + \log(C_0)}.$$

Since $b_{2i} \approx b_{2i-1}^2$ this is bounded from below by

$$\frac{1-\epsilon}{1+\epsilon}\ \frac{2\beta\log(1/b_{2i-1}) + \log(C_0'')}{2\beta\log(1/b_{2i-1}) + (1-\beta)\log(1/b_{2i-1}) + \log(C_0')} \geq \frac{2\beta}{1+\beta} - o(\epsilon) > 1$$

provided i is large and $\epsilon > 0$ is small. It follows that in double-logarithmic coordinates R_{2i-1} is expanding on $[b_{2i}, \eta b_{2i-1}]$.

It follows that if W is a wandering interval above, then in double-logarithmic coordinates the iterates described in (78) and (79) increase each step in length by a factor $(\beta+1)/2$. So their length tends to infinity. But this violates that all iterates are contained in $\cup_{i \geq i_0}[b_{2i}, b_{2i-1}]$ because, as we saw, in double-logarathmic coordinates the length of the intervals $[b_{2i}, b_{2i-1}]$ is uniformly bounded from above. $\square$

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