1969

Radiative Corrections to Low Energy Theorems

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RADIATIVE CORRECTIONS TO LOW ENERGY THEOREMS

A thesis submitted to the Faculty of The Rockefeller University
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

by

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Approved for publication
Sept. 24, 1969
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Rockefeller University

May, 1969

The Rockefeller University
New York, New York

ACKNOWLEDGEMENTS

I would like to express my gratitude to my thesis advisor, Professor A. Pais, for his guidance and encouragement. I am grateful to Professor M. A. B. Bég, with whom I also had many valuable discussions. I would further like to acknowledge helpful conversations with Professor H. Pagels and Dr. K. Y. Lin.

ABSTRACT

A general method is presented for evaluating, in a model-independent way, the soft virtual photon radiative corrections to an arbitrary hadronic process. It is shown that all the results concerning infrared divergences obtained within the theory of quantum electrodynamics of the electron-photon system are, in fact, exact in strong interactions. The problem of radiative corrections to low-energy theorems is the primary concern of this investigation. The threshold contributions of intermediate soft photon states are non-analytic in the photon-frequency (ω). While the procedure is general enough to permit, in principle, the calculation of the leading terms (as $\omega \to o$) of these radiative corrections to all orders in e, in this paper only the leading e^2 radiative corrections are computed explicitly: they are of the order $\ln \omega$ for the bremsstrahlung (as first noted by Soloviev); of the order $\omega \ln \omega$ for pion-photo-production. Accordingly, in the presence of radiative corrections, there are no longer any low-energy theorems for the $O(\omega^o)$ bremsstrahlung and $O(\omega)$ pion-photo-production amplitudes. The e^4 Compton amplitude $O(\omega^2 \ln\omega)$ is computed and shown to be independent of the target spin. A review of the usual low energy theorems, which have been proven to the lowest order in e, also will be presented.

CONTENTS

CHAPTER I

INTRODUCTION

In our study of the elementary particles one of the most important orientations is provided by the classification of their interactions into four basic types, that is:

Strong Interactions	1
Electromagnetic Interactions	10^{-2}
Weak Interactions	10^{-12}
Gravitational Interactions	10^{-40}

The dimensionless number on the right hand side characterizes the strength of these forces. This division allows us to untangle complicated phenomena as the manifestations of each of these four types of interactions.

It is, however, important to keep in mind that although we divide the interactions into distinctive classes, in reality they are mixed together in a "non-linear" way--there is no physical process belonging purely to one type. In fact, to describe any one reaction exactly we will, in principle, have to take all the forces into account. Such an exact description is clearly impractical, and influences due to the forces of weak strength can be neglected. Accordingly, gravitational effects are almost never included in our discussions of the strong, electromagnetic, or weak interactions of the elementary particles. On the other hand, the influences of strong interactions in a primarily non-strong reaction are, in general, not negligible. Consider the case of Compton scattering by a proton target. Even though this is primarily an electromagnetic effect (i. e. this reaction will not take place in an ideal world where electromagnetic interactions are "turned off"), yet no quantitative description (in this case not even a meaningful approximation) is possible without taking into account strong interactions. Also, the electromagnetic properties of a simple electron are influenced by strong interactions in so far as hadrons can also occur as virtual intermediate states. Accordingly these non-linear influences complicate considerably our description of the elementary particles.

In the absence of a comprehensive strong interaction theory we will have to rely heavily on general symmetry principles and simple phenomenology. In this way we can deduce a number of structure-independent statements about physical processes, that is, statements that are true independent of the dynamic details.

In this thesis we will discuss a number of low energy theorems for processes involving photons. Thye are structure-independent statements that relate the scattering amplitude in the low frequency (ω) limit to the corresponding amplitude without the photon. These theorems are derived by requiring that the principle of relativity (Lorentz invariance) be satisfied, that photons have zero rest mass, and that the amplitudes have some simple analyticity properties in the region of $\omega = 0$.

It is well known that low energy theorems, valid to lowest order in the electromagnetic coupling but to all orders in the strong interactions, have been proved for Compton scattering, bremsstrahlung and pion photoproduction. The purpose of this thesis is to examine the validity of these theorems in the presence of higher order radiative corrections. In the low frequency limit, the leading terms in the radiative corrections are shown to be structure independent, i.e. there also exist low energy theorems for radiative corrections themselves, valid to all orders in the strong interactions.

In the next chapter we will present a review of the usual derivations of the low energy theorems which are valid only to the lowest order in e. In Chapter III radiative corrections to these theorems will then be calculated.

CHAPTER II

LOWEST ORDER IN e LOW ENERGY THEOREMS

1. Single-Photon Cases

Consider the processes $\alpha \to \beta + \gamma$ where α and β are arbitrary hadron states and γ is a photon of four momentum $k_\lambda \equiv (\vec{k}, i\omega)$ and polarization e_λ. The S-matrix for the process is

$$_{out}\langle \beta\gamma | \alpha \rangle_{in} = -i(2\pi)^4 \delta(P_\alpha - P_\beta - k)(2\omega)^{-\frac{3}{2}}(2\pi)^{-\frac{1}{2}} N_\alpha N_\beta e^*_\lambda M_\lambda \qquad (2.1)$$

where P_α, P_β, N_α and N_β are respectively the total four momenta and normalization factors[1] of the particles in the α and β states. The amplitude M_λ, which is a vector in the polarization space, is related to the matrix element of the electromagnetic current operator (J_λ) by

$$N_\alpha N_\beta M_\lambda = {}_{out}\langle \beta | J_\lambda | \alpha \rangle_{in} \qquad (2.2)$$

Electromagnetic current conservation

$$\partial_\lambda J_\lambda = 0 \qquad (2.3)$$

then implies that

$$k_\lambda M_\lambda = 0 \tag{2.4}$$

Clearly this is just the requirement of gauge invariance.[2]

We are interested in the behavior of the amplitude in the limit $k \to 0$. To derive the low energy theorems we separate the amplitude into two parts;

$$M_\lambda = S_\lambda + R_\lambda \tag{2.5}$$

where S_λ is the part of the amplitude that is singular at $k = 0$; R_λ is the remainder and is analytic at $k = 0$. Furthermore, the separation is made in a gauge invariant way, i.e. S_λ and R_λ are both gauge invariant amplitudes:[3]

$$k_\lambda S_\lambda \equiv 0 \tag{2.6}$$

$$k_\lambda R_\lambda = 0 \tag{2.7}$$

Differentiating on both sides of Eq. (2.7) with respect to k_λ, we have

$$R_\lambda = -k_\mu \left[\frac{\partial R_\mu}{\partial k_\lambda} \right] \tag{2.8}$$

Since $\lim_{k \to 0} \left[\frac{\partial R_\mu}{\partial k_\lambda} \right]$ exists (by construction) we can conclude that R_λ must be of the order k. Consequently,

$$M_\lambda = S_\lambda + O(\omega) \qquad (2.9)$$

Thus the existence of low energy theorems rests on the possibility of computing the gauge invariant singular amplitude S_λ in a structure-independent way. S_λ is in general very complicated, since the amplitude has poles and branch points at $k = 0$. In this chapter we will first demonstrate (as is well known) that the pole terms can be calculated exactly from the on-shell amplitude ${}_{out}\langle \beta | \alpha \rangle_{in}$; in the next chapter it will be shown, as our main result, that the threshold contributions of the cuts can also be computed in a structure-independent way from ${}_{out}\langle \beta | \alpha \rangle_{in}$. The pole terms correspond to the contributions of the virtual single-particle state and the threshold factors of the cuts are the contributions of the single-particle-plus-soft (zero frequency) photons intermediate states. In the usual derivations of low energy theorems these soft photon cuts are always ignored, hence restricting the validity of the results to the lowest order in e. In this chapter we will present a review of these usual derivations (i.e. we will calculate only the poles on S_λ); in the next chapter computation of the radiative corrections to the low energy theorems will be discussed in detail.

a. Pole Terms

The renormalized amplitude M_λ for the process may be represented by a set of irreducible diagrams with renormalized propagator and renormalized vertices. We can divide the diagrams into two categories according to where the photon line is attached: (1) the photon is attached to an internal line (internal insertion diagrams), (2) the photon line is attached to an external line (external insertion diagrams). In the limit of $k \to 0$ only the diagrams in category (2) contain k^{-1} factors. To see that this is the case, consider the external insertion diagram depicted in Fig. 1:

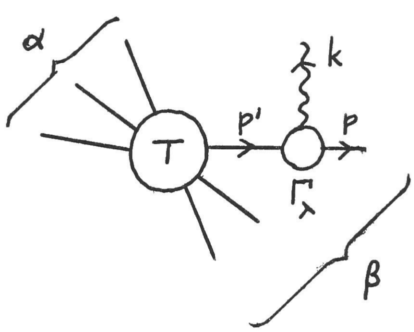


Fig. 1. Single photon external insertion diagram.

Let p be the momentum of the particle in β on to which the photon is attached. Fig. 1 represents a contribution to the amplitude of:

$$\Gamma_\lambda(p, p')\, D(p')\, T(\beta', \alpha) \qquad (2.10)$$

with $p' = p + k$. $\Gamma_\lambda(p, p')$ is the renormalized electromagnetic vertex function. The variables denote the momenta of final and initial lines. The initial leg is off its mass-shell (by 2pk). $D(p')$ is the renormalized propagator for the intermediate state of momentum p'. $T(\beta', \alpha)$ is the amplitude for $\alpha \to \beta$ with one of the final β particles off mass-shell (by 2pk). In the limit of $k \to 0$, the intermediate single particle approaches a free particle, thus $D(p') \to D^{O}(p')$, with $D^{O}(p') \sim [(p+k)^2 + m^2]^{-1} = (2pk)^{-1}$; showing that there is indeed a k^{-1} singularity. Furthermore, (and this is crucial for the structure-independent calculation) we can set the matrices T and Γ_λ on their mass-shell values (denoted by T^{O} and Γ_λ^{O}) since they are residues of the pole. In other words, off-shell contributions are non-singular. The singular part of Eq. (2.10) is then

$$\Gamma_\lambda^{O}(p, p')\, D^{O}(p')\, T^{O}(\beta', \alpha) \qquad (2.11)$$

We can summarize the foregoing discussion by giving a set of rules for computing S_λ when cuts are ignored. The general procedure is as follows: in the external insertion diagrams, for each internal line, which is an external line before insertion, write a free propagator with a pole at the physical mass, place all the amplitudes[4] (or vertices) in the numerator on their mass shell. This is then the entire singular part of the amplitude at $k = 0$. If the pole term is not already gauge invariant, an analytic term is added (so no extra singular part is introduced in this process) to make a gauge invariant S_λ.[5]

We shall illustrate the use of this general rule by working out a number of simple examples.

b. Bremsstrahlung

For $\alpha \to \beta + \gamma$, with $\alpha \to \beta$ being some physically allowed transition, we have a bremsstrahlung process.

We will consider the simplest case when the basic process is the elastic scattering between a spin zero neutral particle (r) and a spin zero charged particle (p) : $r_1 + p_1 \to r_2 + p_2$. Fig. 2 shows the external insertion diagrams.

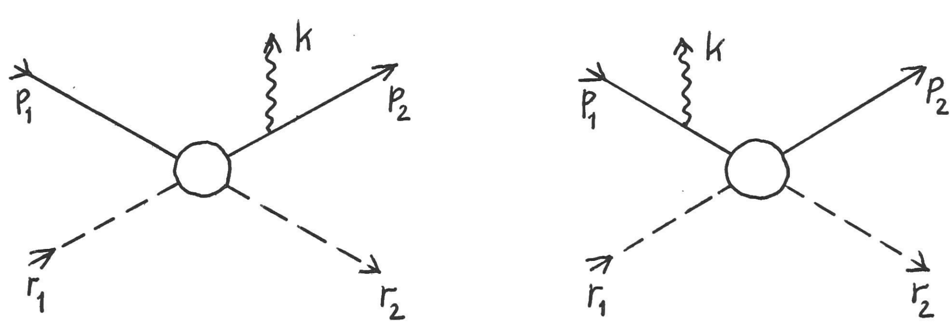


Fig. 2 Photon emission from external charged particle lines in bremsstrahlung.

The amplitude $T(\nu, t)$ for $r_1 + p_1 \to r_2 + p_2$ is specified by two variables: total energy $\nu \equiv p_1 r_1 + p_2 r_2$ and momentum-transfer $t \equiv (r_1 - r_2)^2$. The on-shell matrix elements of the current are of course known.

Applying the general rule we can immediately write down singular terms of the bremsstrahlung amplitude:

$$S'_\lambda = e \frac{(2p_2 + k)_\lambda}{(p_2+k)^2 + m^2} T(\nu + kr_2, t) + e\, T(\nu - kr_1, t) \frac{(2p_1 - k)_\lambda}{(p_1 - k)^2 + m^2}$$

$$= e \frac{p_{2\lambda}}{p_2 k} [T(\nu, t) + kr_2 \frac{\partial T}{\partial \nu}] - e \frac{p_{1\lambda}}{p_1 k} [T(\nu, t) - kr_1 \frac{\partial T}{\partial \nu}] + O(\omega)$$

(2.12)

We note that S'_λ is not gauge invariant by itself,

$$k_\lambda S'_\lambda = e(kr_2 + kr_1) \frac{\partial T}{\partial \nu} + O(\omega^2)$$

We thus obtain the gauge invariant singular term,

$$S_\lambda = S'_\lambda - e(r_2 + r_1)_\lambda \frac{\partial T}{\partial \nu} + O(\omega)$$

$$= e \left[\frac{p_{2\lambda}}{p_2 k} - \frac{p_{1\lambda}}{p_1 k} \right] T(\nu, t) \qquad (2.13)$$

$$+ e \left[\frac{r_2 k}{p_2 k} p_{2\lambda} - r_{2\lambda} + \frac{r_1 k}{p_1 k} p_{1\lambda} - r_{1\lambda} \right] \frac{\partial T}{\partial \nu} + O(\omega)$$

Substituting this expression for S_λ in Eq. (2.9) we have the low energy theorem for the bremsstrahlung first derived by Low in 1958.[6] Of course, the theorem for the ω^{-1} amplitude was known long before the work of Low. It has simple classical interpretation and corresponds to the radiation by a charged particle that undergoes an essentially instantaneous change of velocity. However, Low has shown, as in Eq. (2.13), that to the lowest order in e, not only the leading ω^{-1} term, but also the next order ω^{0} term is structure independent.

c. Pion Photoproduction

For $\alpha \to \beta + \gamma$, with $\alpha \to \beta$ being a pion-nucleon vertex, we have the simple, but physically interesting case of pion photoproduction: $k(e_\lambda) + p_1 \to q(a) + p_2$. The pion has four momentum q and isospin index a, and the nucleon has initial (final) momenta p_1 (p_2).

We are interested in the limit of $k \to 0$, i.e. the low energy limit in the no-recoil approximation. The external insertions of the photon onto the pion-nucleon vertex are just the Born approximation diagrams shown in Fig. 3.

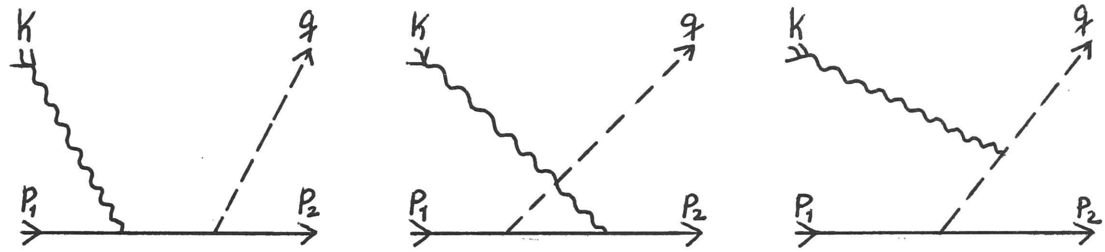


Fig. 3. Photon absorption by external particle lines in pion photoproduction.

The Born amplitude is:

$$S_\lambda^{a'} = \frac{eg}{2}\,\bar{u}(p_2)\Big\{ \tau^a\gamma_5\,[\,i(p_1+k)\cdot\gamma+m]^{-1}\Big[(1+\tau^3)\,\gamma_\lambda-\Big(\frac{\kappa^S+\tau^3\kappa^V}{2\,m}\Big)\sigma_{\lambda\mu}k_\mu\Big]$$

$$+\Big[(1+\tau^3)\,\gamma_\lambda-\Big(\frac{\kappa^S+\tau^3\kappa^V}{2m}\Big)\sigma_{\lambda\mu}k_\mu\Big]\,[\,i(p_2-k)\cdot\gamma-m]^{-1}\,\gamma_5\,\tau^a$$

$$+\,2\,\frac{(2q-k)_\lambda}{(q-k)^2+\mu^2}\,\gamma_5\,e^{ab3}\,\tau^b\Big\}u(p_1) \qquad (2.14)$$

where g is the pion nucleon coupling constant; κ^S and κ^V are respectively the isoscalar and isovector part of the nucleon anomalous magnetic moments measured in units of $\frac{e}{2m}$: $\kappa^S = \kappa$ (proton) + κ(neutron), $\kappa^V = \kappa$(proton) - κ(neutron). τ's are Pauli matrices, m is the nucleon mass and μ is the pion mass.

Since $k_\lambda\, S_\lambda^{a'} = 0$, the Born approximation to the pion photoproduction is the S_λ and by Eq. (2.9) is the total amplitude up to (but not including) terms of order ω . In the rest frame of target nucleon with transverse photon[7], the low energy theorem takes on a simple form:

$$\vec{e} \cdot \vec{M} = i \frac{eg}{4m} \vec{\sigma} \cdot \vec{\epsilon} \, [\, \tau^{a}, \tau^{3}] + O(\omega) + O(\mu) \qquad (2.15)$$

This theorem was first derived by Kroll and Ruderman[8] and is often used to define the pion-nucleon coupling constant.

<u>d. Low-Energy for One Soft Pion (PCAC)</u>

We have seen how the inputs of Lorentz invariance, current conservation (gauge invariance) and simple analyticity properties of the amplitude lead to low energy theorems for processes involving a low frequency photon. There is another class of low energy theorems which relate the matrix element for any strong process with the matrix element for the corresponding process in which an additional zero mass, zero energy pion (soft pion) is emitted or absorbed. The difference in inputs for these soft pion theorems is that instead of current conservation we have "partial conservation" of axial current (PCAC). The purpose of our digression is simply to point out the similarities (and the differences) in the techniques used in deriving all the low energy theorems. Detailed discussion of these important soft pion theorems can be found in a number of excellent review articles.[9]

PCAC states that

$$\partial_\lambda A_\lambda = c\, \phi_\pi , \qquad (2.16)$$

with $c = \dfrac{-i\sqrt{2}\, m\mu^2 g_A}{g}$, where A_λ is the strangeness-conserving weak axial vector current; ϕ_π is the renormalized field operator which creates the π^+; g_A is the β-decay axial vector coupling constant. By partial conservation of the axial current we mean that for an idealized world where pions have zero rest mass, this current will be strictly conserved and this idealization is assumed to be a "good" approximation to reality since pion mass is relatively small on the scales of hadrons.

Using Eq. (2.16) we can immediately obtain the following relations among the matrix elements[10] (just as the gauge invariance requirement of the amplitude in Eq. (2.4) follows directly from the divergenceless condition of the electromagnetic current in Eq. (2.3)):

$$i q_\lambda \; {}_{out}\langle\beta| A_\lambda |\alpha\rangle_{in} = c \; {}_{out}\langle\beta|\phi_\pi|\alpha\rangle_{in} \qquad (2.17)$$

${}_{out}\langle\beta|\phi_\pi|\alpha\rangle_{in}$ in the right hand side is the amplitude for the process

$\alpha \rightarrow \beta + \pi$ (the pion carries momentum $q = P_\alpha - P_\beta$). In the limit of $q \rightarrow 0$, on the left hand side only the q^{-1} pole terms in the amplitude ${}_{out}\langle \beta | A_\lambda | \alpha \rangle_{in}$ can contribute. The pole terms can be computed through the external insertion (of axial currents) diagrams from the physical amplitude of ${}_{out}\langle \beta | \alpha \rangle_{in}$. This then gives us the soft pion theorems.

We will consider one such soft pion theorem as applied to the pion nucleon scattering (i. e. $\alpha \rightarrow \beta$ is the pion nucleon vertex). Nucleons have momenta p_1, p_2 and pions have momenta q_1, q_2 and isospin indices a, b. The amplitude has the following kinematical structure:

$$\bar{u}(p_2) \left\{ T_{\pi N}^{(+)} \delta^{ab} + T_{\pi N}^{(-)} \tfrac{1}{2} [\tau^a, \tau^b] \right\} u(p_1)$$

with

$$T_{\pi N}^{(\pm)} = A_{\pi N}^{(\pm)} + i\, q_i \cdot \gamma B_{\pi N}^{(\pm)}$$

The energy and momentum-transfer variables are usually chosen to be:

$$\nu = -q_1 (p_1 + p_2)/2m$$

$$\nu_B = -q_1 q_2 / 2m$$

Since the theorems refer to the off-shell values of the amplitude, we will explicitly display the dependence on q_1^2 and q_2^2 in the amplitude: $A_{\pi N}^{(\pm)}(\nu, \nu_B, q_1^2, q_2^2)$; $B_{\pi N}^{(\pm)}(\nu, \nu_B, q_1^2, q_2^2)$.

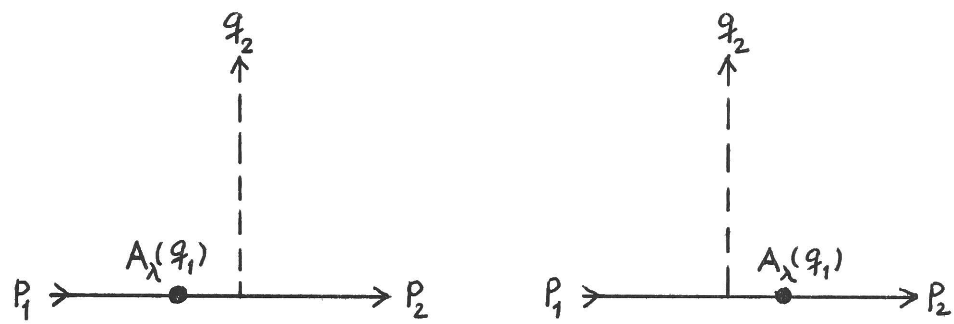


Fig. 4. Axial current external insertion diagrams.

From the external insertion graphs shown in Fig. 4, we obtain the pole terms for the amplitude $_{out}\langle p_2, q_2(b) \,|\, A_\lambda^a \,|\, p_1 \rangle_{in}$

$$g\, g_A \bar{u}(p_2) \Big\{ \tau^b \gamma_5 \, [\, i(p_1+q_1)\cdot\gamma + m \,]^{-1} \tau^a \gamma_\lambda \gamma_5 + \tau^a \gamma_\lambda \gamma_5 \, [\, i(p_1 - q_2)\cdot\gamma + m \,]^{-1} \tau^b \gamma_5 \Big\} u(p_1) \qquad (2.18)$$

We then obtain, through Eq. (2.17), the value of the amplitude in the limit $q_1 \to 0$ (i. e. one of the pions is off-shell).

$$A_{\pi N}^{(+)}(0,0,0,-\mu^2) = \frac{g^2}{m} \; ; \; A_{\pi N}^{(-)}(0,0,0,-\mu^2) = 0 \qquad (2.19)$$

This is Adler's consistency condition for the pion nucleon scattering amplitude.[11]

<u>Remark 1.</u> The β decay axial vector coupling g_A never appears in any final results derived using Eq. (2.16). It is always canceled on both sides: it appears on the right hand side in the PCAC constant c, and on the left hand side it appears in the external insertion graphs as the coupling between the axial vector current and the one particle state. However, it survives in the cases where more than one soft photon appears, as will be discussed in Section 2.c.

<u>Remark 2.</u> The analogy of the soft pion with the photon case is of course not complete. The pion theorems can only yield values of the amplitudes at unphysical points while theorems for low frequency photons give physical amplitudes. Furthermore, the low energy theorems emerge differently from the divergence conditions of Eq. (2.4) and Eq. (2.16). The theorem for the photon is really a low energy theorem for current, while the pion comes into the picture as divergence of the current. A more analogous case will be the low energy theorem for axial current.[12]

2. Many-Photon Cases

The techniques used in deriving low energy theorems in Section 1 of this chapter may be readily extended to cases in which several low frequency photons are absorbed or emitted.

Let us consider the two photon case: $\alpha \rightarrow \beta + \gamma_1 + \gamma_2$. The S-matrix for the process is

$$_{out}\langle\beta\,\gamma_1\,\gamma_2\,|\,\alpha\rangle_{in} = -i(2\pi)^4\,\delta(P_\alpha - P_\beta - k_1 - k_2)\,(2\pi)^{-3}(4\omega_1\omega_2)^{-\frac{1}{2}} N_\alpha N_\beta e^*_{1\lambda} e^*_{2\mu} M_{\lambda\mu}$$

with obvious notations. Gauge invariance then implies[13]

$$k_{1\lambda} M_{\lambda\mu} = 0$$
$$k_{2\mu} M_{\lambda\mu} = 0 \qquad (2.20)$$

To prove low energy theorems, we again separate the amplitude into two parts:

$$M_{\lambda\mu} = S_{\lambda\mu} + R_{\lambda\mu} \qquad (2.21)$$

$S_{\lambda\mu}$ is the gauge invariant singular part and $R_{\lambda\mu}$ is the remainder. From the divergenceless conditions of

$$k_{1\lambda} R_{\lambda\mu} = 0$$
$$k_{2\mu} R_{\lambda\mu} = 0 \qquad (2.22)$$

we obtain by appropriate differentiations

$$R_{\lambda\mu} = k_{1_\nu}\ k_{2_\sigma}\ \left[\frac{\partial^2\ R_{\nu\sigma}}{\partial k_{1_\lambda}\ \partial k_{2_\mu}}\right] \tag{2.23}$$

Hence we have the low energy theorem:[14]

$$M_{\lambda\mu} = S_{\lambda\mu} + O\ (\ \omega_1\ \omega_2) \tag{2.24}$$

We note that now $S_{\lambda\mu}$ contains double pole terms $(\omega_1\omega_2)^{-1}$ coming from diagrams in which both photons are attached to external lines (see Fig. 5). These are the leading terms in the limit of $\omega_1, \omega_2 \rightarrow 0$.

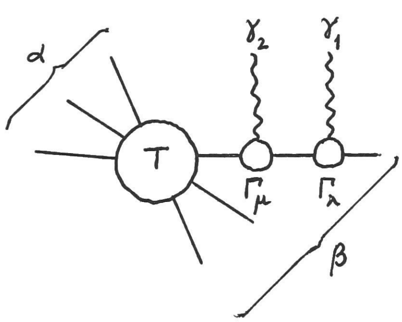
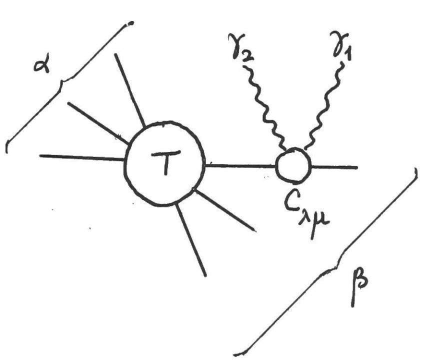


Fig. 5. Double pole diagram.

There are also single pole diagrams (Fig. 6).



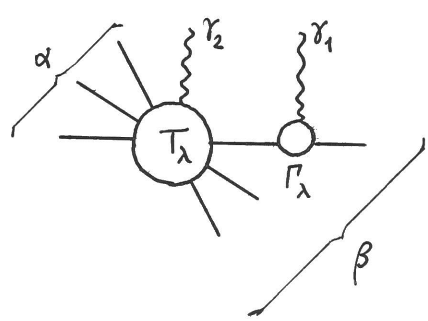


Fig. 6. Single pole diagrams.

The rules derived in Section 1 may be extended in an obvious way to compute $S_{\lambda\mu}$ from the on-shell "elastic" amplitude $T(\alpha \to \beta)$ and single photon amplitude $T_\lambda(\alpha \to \beta + \gamma)$.[15]

a. Compton Scattering

Compton scattering is the simplest two-photon process $\alpha + \gamma_1 \to \beta + \gamma_2$: α and β are merely single particle states. Kinematics are displayed in Fig. 7.

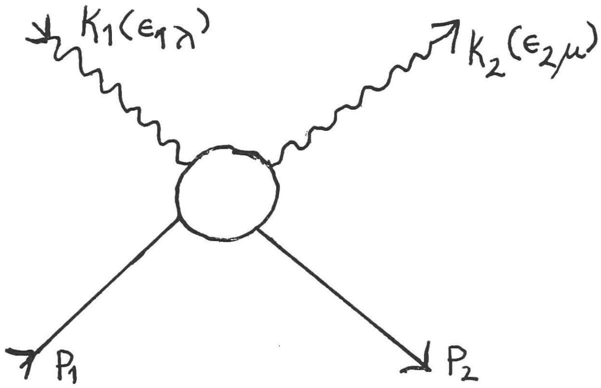


Fig. 7. Compton scattering.

The energies of the initial and final photons are not independent:

$$p_1 (k_1 - k_2) = k_1 \, k_2 \tag{2.25}$$

This is the Compton formula, which, in laboratory systems, reads

$$\omega_1 - \omega_2 = \frac{\omega_1 \omega_2}{m} (1 - \cos \theta) \tag{2.26}$$

θ being the laboratory scattering angle. Whenever the difference

ω_1 -ω_2, which is quadratic in photon frequencies, can be neglected, we will simply refer to both ω_1 and ω_2 as ω. The low energy theorem in Eq. (2.24) reads for the Compton scattering amplitude $M_{\lambda\mu}$ as

$$M_{\lambda\mu} = S_{\lambda\mu} + O(\omega^2) \qquad (2.27)$$

$S_{\lambda\mu}$ can be computed from the simple external insertion diagrams (the pole diagrams) as shown in Fig. 8.

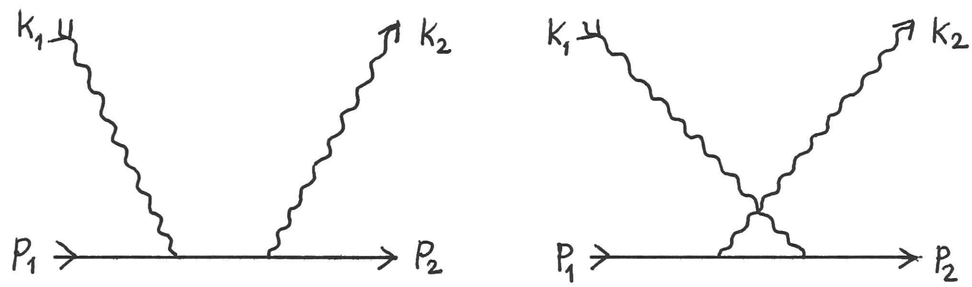


Fig. 8. Pole diagrams in Compton scattering.

Again, an analytic term may be added to the pole terms to obtain the gauge invariant $S_{\lambda\mu}$. We will often depict such a factor by a "seagull diagram" (which may not in general coincide with the perturbation seagull diagrams in spin zero electrodynamics).

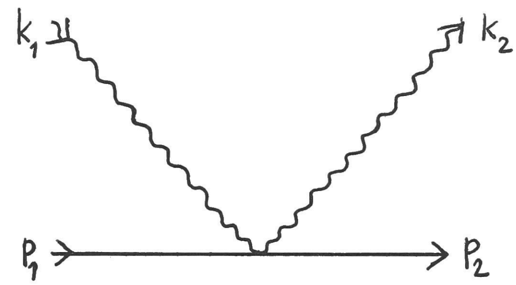


Fig. 9. Seagull diagram in Compton scattering.

Clearly $S_{\lambda\mu}$ may be computed from our knowledge of the on-shell electromagnetic vertices alone. We shall first demonstrate this by working out the familiar results for spin zero and spin one half targets.

For the spin zero target, th external insertion diagrams in Fig. 8 contribute a pole term of

$$S'_{\lambda\mu} = -e^2 \left[\frac{(2p_1+k_1)_\lambda (2p_2+k_2)_\mu}{(p_1+k_1)^2 + m^2} + \frac{(2p_1-k_2)_\mu (2p_2-k_1)_\lambda}{(p_1-k_2)^2 + m^2} \right] \tag{2.28}$$

It is well known that an extra (seagull) factor is needed to make the pole terms gauge invariant: $S_{\lambda\mu} = S'_{\lambda\mu} + 2e^2 \delta_{\lambda\mu}$. In the laboratory system with transverse photon only the seagull term survives. The low energy theorem reads

$$e_{1_i} M_{ij} e^*_{2_j} = 2e^2 \vec{e}_1 \cdot \vec{e}_2^{\,*} + O(\omega^2) \tag{2.29}$$

where $i(j) = 1, 2, 3$.

For the spin one half target, the pole terms are

$$S'_{\lambda\mu} = e^2\bar{u}(p_2)\Big\{[\gamma_\mu + (\frac{\kappa}{2m})\sigma_{\mu\nu}k_{2\nu}][i(p_1+k_1)\cdot\gamma + m]^{-1}[\gamma_\lambda - (\frac{\kappa}{2m})\sigma_{\lambda\nu}k_{1\nu}]$$

$$+[\gamma_\lambda - (\frac{\kappa}{2m})\sigma_{\lambda\nu}k_{1\nu}][i(p_1-k_2)\cdot\gamma + m]^{-1}[\gamma_\mu + (\frac{\kappa}{2m})\sigma_{\mu\nu}k_{2\nu}]\Big\}u(p_1)$$

(2.30)

where κ is the anomalous magnetic moment of the target. $S'_{\lambda\mu}$ is already gauge invariant by itself. Therefore $S_{\lambda\mu} = S'_{\lambda\mu}$.

Remark. Since only on-shell vertices are involved there is some freedom as to how to parameterize the vertex functions: for example, in Eq. (2.30) we have used

$$\langle p'|J_\lambda|p\rangle = -eN_pN_{p'}\bar{u}(p')[\gamma_\lambda - (\frac{\kappa}{2m})\sigma_{\lambda\mu}(p'-p)_\mu]u(p) \quad \text{for} \quad (p'-p)^2 = 0$$

(2.31)

which can be rewritten (using the Dirac equation) as

$$\langle p'|J_\lambda|p\rangle = ieN_pN_{p'}u(p')[(p'+p)_\lambda - i(\frac{1+\kappa}{2m})\sigma_{\lambda\mu}(p'-p)_\mu]u(p)$$

(2.32)

Had we used the second form, the pole term $S'_{\lambda\mu}$ would not be gauge invariant so seagull terms had to be added. The low energy theorem itself is of course not altered.

In the laboratory system the low energy Compton amplitude for spin one half target is

$$e_{1_i} M_{ij}\, e_{2_j}^* = \frac{e^2}{m} \vec{e}_1 \cdot \vec{e}_2^* + 2i\,\omega \left(\frac{e}{2m}\right)^2 (1+2\kappa)\, \vec{\sigma} \cdot \vec{e}_1 \times \vec{e}_2^*$$

$$- 2i\omega \left(\frac{e}{2m}\right)^2 (1+\kappa)^2 \vec{\sigma}\, (\vec{e}_1 \times \hat{k}_1) \times (\vec{e}_2^* \times \hat{k}_2)$$

$$- 2i\omega \left(\frac{e}{2m}\right)^2 (1+\kappa) \left[\vec{\sigma} (\vec{e}_1 \times \hat{k}_1)(\hat{k}_1 \vec{e}_2^*) - \vec{\sigma} (\vec{e}_2^* \times \hat{k}_2)(\hat{k}_2 \vec{e}_1) \right]$$

$$+ O(\omega^2) \qquad (2.33)$$

Pauli spinors are understood and not displayed. In the forward direction this complicated expression reduces to

$$M(\omega, \theta = 0) = \frac{e^2}{m} \vec{e}_1 \cdot \vec{e}_2^* - 2i\omega\, \kappa^2 \left(\frac{e}{2m}\right)^2 \vec{\sigma} \cdot \vec{e}_1 \times \vec{e}_2^* + O(\omega^2)$$

(2.34)

The zeroth order term is identical to the classical Thomson amplitude. It is first derived as a low energy theorem by Thirring in 1950.[16] The linear order theorem (which is proportional to the magnetic moment) was the result obtained by Low[17] and, independently, by Gell-Mann and Goldberger[18] in 1954.

Physically it is not surprising that the amplitudes should, in the low frequency limit, take on some classic features, since the long-wavelength photons cannot sense the effects coming from the (quantum mechanical) virtual states which are essentially of the short range nature. This is also the physical reason why the external insertion diagrams play such a prominant role in our consideration of the low energy theorems.

It is well known that in classical electrodynamics, the electric charge may be measured in two ways: by measuring the Coulomb force on the particle placed as a test charge in a static field or, alternatively, by measuring the Thomson cross section for scattering of light by the (unbound) particle. The low energy theorem in Eq.(2.33) states that the above correspondence between these two measurements (as generalized to include magnetic moments) remains to be true in an exact quantum field theory. As we have seen in the derivation of Eq. (2.33), the electromagnetic couplings (charge and magentic moment) enter into the Compton amplitude through the expression for the on-shell matrix element of electromagnetic current operator, $< p' | J_\lambda | p >$ of Eq. (2.31). Thus it remains for us to show that $< p' | J_\lambda | p >$ indeed gives a direct expression of the electromagnetic couplings measured in a weak, slowly varying external electromagnetic field: A_λ^{ext} . This is not difficult to do: the amplitude for such an interaction is

$$<p' | H_I | p > = \int <p' | J_\lambda | p > A_\lambda^{ext}(x) \, d^3x \qquad (2.35)$$

which in the non-relativistic limit reduces to

$$< p'|H_I|p> = \int e^{-i\vec{q}\cdot\vec{x}} [e A_0^{ext}(\vec{x}) - (1+\kappa)\frac{e}{2m}\vec{\sigma}\cdot\vec{H}^{ext}(\vec{x})] d^3x \qquad (2.36)$$

where $\vec{q} = \vec{p}' - \vec{p}$ and $\vec{H}^{ext}(\vec{x}) = \vec{\nabla} \times \vec{A}^{ext}(\vec{x})$. This is the basis for the physical interpretation of electromagnetic vertex function of Eq. (2.31). Accordingly, we can use Eq. (2.33) as our empirical definitions of charge and magnetic moment.

Remark. In the usual discussion of low energy theorems the P and T non-invariant effects are neglected, though they may be included. Including them involves more complicated expressions for the current matrix elements corresponding to such terms as the electric dipole in the currents. The general procedure for obtaining the low energy theorems is not altered. We will then find that the linear order Compton amplitude is determined not only by the magnetic moment but also by the electric dipole moment of the target particle.[19]

Compton scattering by target of arbitrary spin has been studied by Pais.[20] Since the detailed calculations are quite involved, we will only give an outline of the problem. There are two aspects in this problem:

(i) to show that the zeroth and linear order theorems derived for the case of spin zero and one half, hold for targets of arbitrary spin $S \geq 1$; related to this problem is the proof of a new low energy theorem stating that the double spin flip amplitude $\{\vec{S}\vec{e}_1, \vec{S}\vec{e}_2\}$ vanishes in the zero frequency limit;

(ii) to show that there exist low energy theorems for all $2S+1$ intrinsic multipole moments.

Since the general relation Eq. (2.27) holds irrespective of target spin, the task of (i) is to show that, given the electromagnetic vertex function for particles of high spin, the low frequency limit of the pole term $S_{\lambda\mu}$ is of the same form for any spin and there are no extra terms such as $\{\vec{S}\vec{e}_1, \vec{S}\vec{e}_2\}$ in the zeroth order amplitude. The general structure of the pole terms are again the same for all spins, thus we should concentrate on the difference of the vertex functions $\Gamma_\lambda(p', p)$ for different spins. For low energy theorems, the only relevant part of the matrix element is that of $\Gamma_\lambda(p', p)$ with $p' \rightarrow p$. We have already discussed the physical interpretation of these quantities for spin one half <u>via</u> the consideration of scattering in a slowly varying, weak external electromagnetic field. For such interactions involving systems of higher spin it is well known that all we have to do is to <u>add</u> more terms corresponding to <u>higher</u> order multiple-moments-interactions. In the momentun-space they correspond to higher order terms in $(p'-p)^2$; thus ultimately they contribute higher powers of ω in the pole terms $S_{\lambda\mu}$. In fact the minimal power of ω with which 2^L-pole moment will enter into the low energy amplitude is ω^L. Accordingly, starting with the definition of multipole moments we can show that there are no extra terms in $O(\omega^o)$ and $O(\omega)$ amplitude and the low energy theorem for target of spin S is just Eq. (2.33) with $\frac{\vec{\sigma}}{2}$ replaced by $\vec{S}$.

This brings us natrually to the task (ii). Since the Compton amplitude containing 2^L-pole moments with $L \geqslant 2$ is of the order $O(\omega^2)$, the question natrually arises: can higher multipole moments be defined empirically, as in the case of the charge and the magnetic moment, via low energy Compton scattering? This question is answered in the positive by Pais[20] and also by Bardakci and Pagels.[21] In particular, they have written down a low energy theorem of the quadrupole moment. Their work is motivated by the investigation of Singh,[22] who first proved a different type of low energy theorem to order ω^2.

While the entire ω^2 term in the Compton amplitude cannot be determined in a structure-independent way, it is still possible that low energy theorems exist for some parts of the ω^2 amplitude. We will state briefly the basic ideas underlying the derivations of these higher order theorems. Let us re-examine Eq. (2.27), which is derived without any specific knowledge as to the kinematic structure of the amplitude. The amplitude $M_{\lambda\mu}$ may be expanded into a linearly independent tensor basis

$$M_{\lambda\mu} = \tilde{a}\, A_{\lambda\mu} + \tilde{b}\, B_{\lambda\mu} + \ldots \qquad (2.37)$$

(there may be more spin indices $M_{\lambda\mu,\alpha\beta\cdots}$, but we will not exhibit them here). Where $\tilde{a}, \tilde{b}, \ldots$ are scalar amplitudes and $A_{\lambda\mu}$, $B_{\lambda\mu}\ldots$ are tensors constructed out from momenta $k_1, k_2 \ldots$ Eq. (2.27) states that it is possible for us to find the basis so that all $A_{\lambda\mu}, B_{\lambda\mu}\ldots$ are $O(\omega^2)$ and the amplitudes $\tilde{a}, \tilde{b}, \ldots$ are free of kinematical singularities (i.e. the only singular part of $\tilde{a}, \tilde{b}, \ldots$ are those dynamic pole terms and thus the non-pole terms are necessarily of the order ω^2). But once an explicit tensor base is constructed, it is not hard to convince oneself that some of the base $A_{\lambda\mu}, B_{\lambda\mu}\ldots$ are of orders higher than ω^2. Suppose $A_{\lambda\mu}$ is $O(\omega^3)$ and $\tilde{a}$ is free of kinematical singularities, then we have a higher order low energy theorem:

$$\tilde{a}\, A_{\lambda\mu} = [\tilde{a}\, A_{\lambda\mu}]_{\text{pole}} + O(\omega^3) \qquad (2.38)$$

Accordingly we will have to construct a linearly independent tensor basis with maximum powers of ω (i.e. to remove kinematical zeroes) but still compatible with the requirement that the amplitude be free of kinematical singularities. In general this is a very difficult task, and to do this, we will have to make full use of the gauge invariance requirements.

The existence proof of the $2S+1$ multipole moment low energy theorems involves then showing that certain amplitude, which is of the appropriate order in ω, is free of kinematical singularity and the pole term contribution to that amplitude contains the relevant multipole moment. More concretely, we make an expansion of the amplitude in the photon frequency and the scattering angle cos θ (the multipole expansion), the proof of the theorem then involves showing that the correct terms in this double-expansion get contributions only from the singular parts.

b. Theorems for Non-Commuting Currents

So far we have considered the consequences derived from divergence conditions of one current or several identical currents. Now we will extend out treatment to cases involving the low energy limit of several mutually non-commuting currents. As we shall see, the commutator of these currents will play a role in the low energy theorem. Here we will only consider the case of conserved currents, but the procedure can easily be generalized to include currents having non-vanishing divergences.

Bég [23] has treated the case of Compton scattering of "charged photons" by a nucleon: $p_1 + k_1(a) \to p_2 + k_2(b)$. The electromagnetic current has an isoscalar part (Y_λ) and a part which is the third component of an isovector (V_λ):

$$J_\lambda = e(Y_\lambda + V^3_\lambda) \qquad (2.39)$$

A "charged photon" is then a fictitious particle coupled to $V^1_\lambda \pm iV^2_\lambda$ components of the same isovector current. Or, simply, the photon now has an extra isospin label a(b) = 1, 2, 3, coupled to the current

$$\langle p' | V_\lambda^a | p \rangle = -N_{p'} N_p \bar{u}(p') \frac{\tau^a}{2} [F_1^V(q^2) \gamma_\lambda - \frac{\kappa^V}{2m} F_2^V(q^2) \sigma_{\lambda\mu} q_\mu] u(p)$$

$$q \equiv p' - p \; ; \; F_1^V(0) = F_2^V(0) = 1 \qquad (2.40)$$

and

$$\partial_\lambda V_\lambda^a = 0 \qquad (2.41)$$

The corresponding amplitude $M^{ab}_{\lambda\mu}$ is

$$N_{p'} N_p M^{ab}_{\lambda\mu} = \div i \int d x e^{ik_1 x} \langle p_2 | [T(V_\mu^b(0) V_\lambda^a(x)) + i\rho^{ba}_{\mu\lambda} \delta(x)] | p_1 \rangle$$

(2.42)

where $\rho^{ba}_{\mu\lambda}$ (the seagull term) precisely compensates for the noncovariant nature of the T-product.[24]

In the one current case the divergence condition on the current leads immediately to a similar divergence condition on the amplitude, Eq. (2.3) and Eq. (2.4). When more than one current is involved, as here, the divergence condition on the amplitude will also depend on the commutator of the currents. This is brought about by the following identity:

$$\partial_\lambda [T(V_\mu^b(0) V_\lambda^a(x))] = T(V_\mu^b(0) \partial_\lambda V_\lambda^a(x)) + \delta(x_0) [V_\mu^b(0), V_0(x)] \qquad (2.43)$$

The delta function comes from differentiation of the θ function in the T-product. Thus, Eq. (2.41), Eq. (2.42) and Eq. (2.43) imply:

$$k_{1\lambda} M^{ab}_{\lambda\mu} = -N^{-1}_{p_1} N^{-1}_{p_2} \left\{ \int dx\, e^{ik_1 x} \delta(x_0) < p_2 | \, [V^b_\mu(0), V^a_0(x)] \, | p_1 > \right.$$

$$\left. - \int dx\, e^{ik_1 x} < p_2 | \, i \partial_\lambda [\rho^{ba}_{\lambda\mu} \delta(x)] \, | p_1 > \right\} \qquad (2.44)$$

The commutator is well known:

$$\delta(x_0)[V^b_\mu(0), V^a_0(x)] = \delta(x)\, i\, e\, e^{bac} V^c_\mu + \text{Schwinger term.} \qquad (2.45)$$

The Schwinger term is proportional to the gradient of a delta function and is structure dependent. Thus, from Eq. (2.44), it would appear that the Schwinger terms may prevent the derivation of structure-independent low energy theorems. However, this is merely an apparent difficulty. This may be seen from the case of ordinary Compton scattering considered in the last subsection. There we had two identical currents; their commutator vanishes apart from the Schwinger term, i.e.

$$\delta(x_0)\, [J_\mu(0), J_0(x)] = \text{Schwinger term.} \qquad (2.46)$$

On the other hand, we know from gauge invariance that the physical Compton amplitude satisfies the condition in Eq. (2.20): the divergence vanishes regardless of the value of the Schwinger term. Therefore, the Schwinger term must have been canceled by the divergence of the seagull term. In fact, for the case of electromagnetic currents, we know explicitly how this cancellation comes about, and it has been conjectured that this cancellation holds for all "weak" currents. Thus, with this conjecture, as far as the derivation of divergence conditions for the amplitudes are concerned, we may consistently ignore the "seagulls" and Schwinger terms and still obtain the correct answer.[25]

Consequently, Eq. (2.44) reads

$$k_{1\lambda} \, M^{ab}_{\lambda\mu} = i \, N^{-1}_{p_1} \, N^{-1}_{p_2} \, e^{abc} < p_2 | \, V^c_\mu \, | \, p_1 > \qquad (2.47a)$$

Similarly we have

$$k_{2\mu} \, M^{ab}_{\lambda\mu} = i \, N^{-1}_{p_1} \, N^{-1}_{p_2} \, e^{abc} < p_2 | \, V^c_\lambda \, | \, p_1 > \qquad (2.47b)$$

The low energy theorem follows from the identification of the singular part $S^{ab}_{\lambda\mu}$ of the amplitude:

$$M^{ab}_{\lambda\mu} = S^{ab}_{\lambda\mu} + R^{ab}_{\lambda\mu} \qquad (2.48)$$

The separation is made such that

$$k_{1_\lambda} S^{ab}_{\lambda\mu} = k_{2_\nu} S^{ab}_{\mu\nu} = i N^{-1}_{p_1} N^{-1}_{p_2} e^{abc} < p_2 | V^c_\mu | p_1 > \qquad (2.49)$$

or

$$k_{1_\lambda} R^{ab}_{\lambda\mu} = k_{2_\mu} R^{ab}_{\lambda\mu} = 0 \qquad (2.50)$$

The last equation implies that $R^{ab}_{\lambda\mu} = O(\omega^2)$. Thus we have

$$M^{ab}_{\lambda\mu} = S^{ab}_{\lambda\mu} + O(\omega^2) \qquad (2.51)$$

The pole term $S^{ab}_{\lambda\mu}$ may be computed in exactly the same fashion as in the ordinary Compton case. We then have the theorems for the forward amplitude derived by Bég

$$e^2 e_{1_i} M^{ab}_{ij} e^*_{2_j} = \delta^{ab} \left[\frac{e^2}{4m} \vec{e}_1 \cdot \vec{e}_2^{\,*} - \frac{i\omega}{2} \left(\frac{e^2}{2m}\right) \kappa^{V^2} \vec{\sigma} \cdot \vec{e}_1 \times \vec{e}_2^{\,*} \right]$$

$$+ \frac{1}{2} [\tau^a, \tau^b] \left\{ e^2\omega \left[\frac{(1+\kappa^V)^2}{8m^2} - \left(\frac{\partial G^V_E(q^2)}{\partial q^2}\right)_{q^2=0} - \frac{1}{8m^2} \right] \vec{e}_1 \cdot \vec{e}_2^{\,*} \right.$$

$$\left. - i e^2 \frac{(1+\kappa^V)}{4m} \vec{\sigma} \cdot \vec{e}_1 \times \vec{e}_2^{\,*} \right\} + O(\omega^2) \qquad (2.52)$$

Where $G^V_E(q^2)$ (electric isovector Sachs form factor) is related to the form factors defined in Eq. (2.40) by

$$G^V_E(q^2) = F^V_1(q^2) + \frac{\kappa^V q^2}{4m^2} F^V_2(q^2)$$

Adler and Dothan[12] have derived a number of low energy theorems for the process of radiative μ capture : $\mu + p \rightarrow n + \nu_\mu + \gamma$. This is a complicated problem since one of the two currents involved has non-vanishing divergence.

c. Low-Energy Theorems for Many Soft Pions (PCAC and Current Algebra)

In the last subsection we have already seen that current commutators play a role in the derivation of low energy theorems involving several non-commuting currents. Similar situation happens for low energy theorems involving more than one soft pion.

Just as in Eq. (2.43) in the last subsection, the basic identity is as follows:

$$\left(\frac{\partial}{\partial x_\lambda} \frac{\partial}{\partial y_\mu} \cdots \right) T(A^a_\lambda(x)\, A^b_\mu(y) \cdots)$$

$$= T\,(\partial_\lambda A^a_\lambda\,(x),\ \partial_\mu\, A^b_\mu\,(y) \cdots) + \text{equal time commutators} \tag{2.53}$$

PCAC, $\partial_\lambda A^a_\lambda = c\,\phi^a_\pi$, then relates the first term on the right hand side to a multi-pion amplitude. In the soft pion limit only the structure-independent pole terms of axial current matrix element and the commutator factor will contribute. The general case is clearly rather complex. Accordingly we will concentrate on one special example: the pion nucleon scattering in the limit when both pions are soft.

Consider the two amplitudes: the pion nucleon amplitude (kinematics are the same as in Section 1d),

$$N_{p_1} N_{p_2} T^{ab}_{\pi N} = -i(q_1^2+\mu^2)(q_2^2+\mu^2)\int dx\, e^{-iq_2 x} < p_2 | \, T(\phi^b_\pi(x)\, \phi^a_\pi(0)) \, | p_1 > \tag{2.54}$$

and the corresponding amplitude for axial currents,

$$N_{p_1} N_{p_2} M^{ab}_{\lambda\mu} = -i \int dx\, e^{-iq_2 x} < p_2 | \, T(A^b_\mu(x)\, A^a_\lambda(0)) \, | p_1 > \tag{2.55}$$

It follows from the identity in Eq. (2.53) and from integration by parts that

$$\begin{aligned} N_{p_1} N_{p_2} q_{1\lambda} q_{2\mu} M^{ab}_{\lambda\mu} = & -i \int dx\, e^{-iq_2 x} < p_2 | \, T(\partial_\mu A^b_\mu(x), \partial_\lambda A^a_\lambda(0)) \, | p_1 > \\ & - q_{1\lambda} \int dx\, e^{-iq_2 x} \delta(x_0) < p_2 | [A^b_0(x), A^a_\lambda(0)] \, | p_1 > \\ & + i \int dx\, e^{-q_2 x} \delta(x_0) < p_2 | [\partial_\mu A^b_\mu(x), A^a_0(0)] \, | p_1 > \end{aligned} \tag{2.56}$$

Using PCAC and our knowledge of the commutators:

$$\delta(x_0)[A_0^b(x), A_\lambda^a(0)] = i\, e^{bac} V_\lambda^c(x)\delta(x)$$

$$\delta(x_0)[\partial_\mu A_\mu^b(x), A_0^a(0)] = \text{symmetric in } a, b. \qquad (2.57)$$

We may convert Eq. (2.56) into a constraint on the amplitudes:

$$q_{1\lambda} q_{2\mu} M_{\lambda\mu}^{[a,b]} = \frac{c^2}{(q_1^2+\mu^2)(q_2^2+\mu^2)} T_{\pi N}^{[a,b]} + i N_{p_1}^{-1} N_{p_2}^{-1} q_{1\lambda} e^{abc} <p_2 | V_\lambda^c(0) | p_1>$$

$$(2.58)$$

The superscript [a, b] indicates the isospin-antisymmetric part of the amplitude. In the limit of $q_1, q_2 \to 0$, only the pole terms in $M_{\lambda\mu}^{ab}$ contribute. We will also take out pole terms in $T_{\pi N}^{[a,b]}$. In this way Eq. (2.58) yields the well known result of Adler and Weisberger.[26] In terms of the amplitudes and variables defined in Section 1d, the low energy theorem reads

$$\lim_{\nu \to 0} \frac{T_{\pi N}^{(-)}(\nu, 0, 0, 0)}{\nu} = \frac{g^2}{2m^2}\left(1 - \frac{1}{g_A^2}\right) \qquad (2.59)$$

where $T_{\pi N}^{(-)}(\nu, 0, 0, 0) = A_{\pi N}^{(-)}(\nu, 0, 0, 0) + \nu B_{\pi N}^{(-)}(\nu, 0, 0, 0)$. Because the derivation involves a non-linear relation for the axial currents Eq. (2.57), g_A appears in the final result (see Remark Sec. 1d).

d. Sum Rules

Low energy theorems, when supplemented with "no-subtraction" assumption in a dispersion relation, lead to the sum rules, which relate experimental quantities and thus can be tested in the laboratory. In fact, a number of low energy theorems discussed in previous sections were first expressed in terms of sum rules in their original derivations. We will simply proceed to give several examples.

Consider the forward Compton-scattering amplitude

$$f(\omega) = f_1(\omega)\, \vec{e}_1 \cdot \vec{e}_2^{\,*} + i\, f_2(\omega)\, \vec{\sigma} \cdot \vec{e}_1 \times \vec{e}_2^{\,*} \qquad (2.60)$$

The amplitudes are normalized such that the optical theorem reads

$$\mathrm{Im}\, f_1(\omega) = \frac{\omega}{4\pi}\, \frac{\sigma_P(\omega) + \sigma_A(\omega)}{2} \equiv \frac{\omega}{4\pi}\, \sigma_{tot}(\omega) \qquad (2.61)$$

$$\mathrm{Im}\, f_2(\omega) = \frac{\omega}{4\pi}\, \frac{\sigma_P(\omega) - \sigma_A(\omega)}{2} \qquad (2.62)$$

where σ_P is the total cross section for circularly polarized photons with their helicity parallel to the target spin; σ_A, antiparallel. The dispersion relation for these two amplitudes may be written with the assumption of no-subtraction (i.e. the relevant integral converges) as

$$\mathrm{Re}\, f_1(\omega) = \frac{2}{\pi} P \int_0^{\infty} \frac{\omega' d\omega' \, \mathrm{Im}\, f_1(\omega')}{\omega'^2 - \omega^2} \tag{2.63}$$

$$\mathrm{Re}\, f_2(\omega) = \frac{2\omega}{\pi} P \int_0^{\infty} \frac{d\omega' \, \mathrm{Im}\, f_2(\omega')}{\omega'(\omega'^2 - \omega^2)} \tag{2.64}$$

Using the low energy theorem of Eq. (2.34), noting $M(\omega) = -4\pi f(\omega)$, and the optical theorems of Eq. (2.61) and Eq. (2.62), the above relation immediately yields the following sum rules:

$$-\frac{e^2}{m} = \frac{2}{\pi} \int d\omega' \sigma_{tot}(\omega') \tag{2.65}$$

and

$$\frac{e^2 \kappa^2}{m^2} = \frac{2}{\pi} \int \frac{d\omega'}{\omega'} [\sigma_P(\omega') - \sigma_A(\omega')] \tag{2.66}$$

Eq. (2.65) is an obvious contradiction since the right hand side is positive definite. We conclude that for this particular amplitude the no-subtraction assumption ($f_1(\omega) \to 0$ as $\omega \to \infty$) must be false. On the other hand, if one analyzes the contributions to the integral in Eq. (2.66) for proton Compton scattering ($\omega' < 1$ BeV) on the basis of both theoretical assumptions and experimental data on photoproduction from hydrogen, one finds both sides of the sum rules agree to better than ten percent.

Next we consider the forward pion nucleon scattering. Taking the limit of $\nu \to 0$ and $\mu \to 0$ on both sides of the unsubtracted dispersion relation (with the same notation as in Section 1d.)

$$\frac{T^{(-)}_{\pi N}(\nu, \frac{\mu^2}{2m}, -\mu^2, -\mu^2)}{\nu} = \text{pole term} + \frac{1}{\pi} \int_{\mu}^{\infty} \frac{d\nu' \, q' [\sigma^{\pi^- P}(\nu') - \sigma^{\pi^+ P}(\nu')]}{\nu'^2 - \nu^2} \tag{2.67}$$

with $q' = (\nu'^2 - \mu^2)^{\frac{1}{2}}$,

we have, using the soft pion theorem of Eq. (2.59) and noting that the pole term vanishes in the limit of $\nu_B = 0$, the Adler-Weisberger sum rule:[26]

$$\frac{1}{g_A^2} = 1 + \frac{2m^2}{\pi g^2} \int_0^{\infty} \frac{d\nu'}{\nu'} [\sigma_0^{\pi^- p}(\nu') - \sigma_0^{\pi^+ p}(\nu')] \tag{2.68}$$

where $\sigma_0^{\pi^- p}$ and $\sigma_0^{\pi^+ p}$ are respectively the total cross section of zero mass π^- and π^+ scattering on a proton target.

3. Alternative Methods of Derivation

a. Low's Non-Covariant Method

So far our discussion of the low energy theorems has been carried out in a manifestly covariant framework. While this approach has some advantages of showing, in a concise manner, the general content of the theorems, frequently it is not the simplest method in actual computations because of its host of auxiliary constraints. The non-covariant (physical covariance is of course not lost) "Low equation"[28] is often the preferred technique for considering such processes as Compton scattering and pion photoproduction.

Consider Compton scattering on a target of arbitrary spin. The tensorial amplitude $M_{\lambda\mu}$ used in Section 2.a. is related to the current operators by

$$N_{p_1} N_{p_2} M_{\lambda\mu} = -i \int dx \, e^{-ik_2 x} < p_2 | T(J_\mu(x) J_\lambda(0)) | p_1 > \quad (2.69)$$

In addition, there may be seagull terms; however, they do not contribute to $\lambda = 0$, $\mu = 0$ components of the amplitude, and they are not singular in the limit of $\omega \to 0$.[24] To display the singular structure of the amplitude we expand the T-product in terms of the θ-functions, inserting complete set of states, the space-time integrals can be performed to give the "Low equation."

$$M_{\lambda\mu} = N_{p_1}^{-1} N_{p_2}^{-1} (2\pi)^3 \sum_n \Big[\frac{<\vec{p}_2 | J_\mu(0) | \vec{n}> <\vec{n} | J_\lambda(0) | \vec{p}_1 >}{\omega_1 + E(\vec{p}_1) - E_n} \Big|_{\vec{n} = \vec{p}_1 + \vec{k}_1}$$

$$+ \frac{< \vec{p}_2 | J_\lambda(0) | \vec{n}> <\vec{n} | J_\mu(0) | \vec{p}_1 >}{-\omega_2 + E(\vec{p}_1) - E_n} \Big|_{\vec{n} = \vec{p}_1 - \vec{k}_2} \Big]$$

(2.70)

with $E(\vec{p}) = (\vec{p}^2 + m^2)^{\frac{1}{2}}$. $|\vec{n}>$ denotes a general (on-shell) intermediate state with total three momentum $\vec{n}$ and total energy E_n. The singular parts of the amplitudes simply correspond to terms in the sum having denominators that may vanish in the $\omega \to 0$ limit.

We will explicitly separate out the structure-independent, single particle contribution as $U_{\lambda\mu}$ (the "unexcited" term) which has denominators proportional to ω in the limit of $\omega \to 0$

$$M_{\lambda\mu} = U_{\lambda\mu} + E_{\lambda\mu} \tag{2.71}$$

In the transverse gauge, the physical amplitude is M_{ij}. We will express the unknown E_{ij} amplitude in terms of the known U_{ij}, U_{00} and the simpler object E_{00} <u>via</u> the gauge invariance condition, which is implied by Eq. (2.20),

$$k_{1\,i}\, M_{ij}\, k_{2\,j} = \omega_1\, \omega_2\, M_{00} \tag{2.72}$$

or

$$k_{1\,i} E_{ij} k_{2\,j} = \omega_1\, \omega_2\, U_{00} - k_{1\,i} U_{ij} k_{2\,j} + \omega_1 \omega_2\, E_{00} \tag{2.73}$$

Since the seagull term does not contribute to M_{00}, we have from Eq. (2.69) and Eq. (2.70) the following representation for E_{00},

$$E_{00} = N_{p_1}^{-1} N_{p_2}^{-1} (2\pi)^3 \sum_n{}' \Big[\frac{\langle \vec{p}_2 | J_0 | \vec{n} \rangle \langle \vec{n} | J_0 | \vec{p}_1 \rangle}{\omega_1 + E(p_1) - E_n} \Big|_{\vec{n} = \vec{p}_1 + \vec{k}_1}$$

$$+ \frac{\langle \vec{p}_2 | J_0 | \vec{n} \rangle \langle \vec{n} | J_0 | \vec{p}_1 \rangle}{-\omega_2 + E(\vec{p}_1) - E_n} \Big|_{\vec{n} = \vec{p}_1 - \vec{k}_2} \Big] \qquad (2.74)$$

where the prime over the summation sign indicates that the one particle state is excluded from the sum. We note that Eq. (2.74) may be written, using the continuity equation, Eq. (2.3), as

$$E_{00} = N_{p_1}^{-1} N_{p_2}^{-1} (2\pi)^3 \sum{}' \Big[\frac{k_{2_j} \langle \vec{p}_2 | J_j | \vec{n} \rangle \langle \vec{n} | J_i | \vec{p}_1 \rangle k_{1_i}}{(E(\vec{p}_2) - E_n)(\omega_1 + E(\vec{p}_1) - E_n)(E_n - E(\vec{p}_1))} \Big|_{\vec{n} = \vec{p}_1 + \vec{k}_1}$$

$$+ \frac{k_{1_i} \langle \vec{p}_2 | J_i | \vec{n} \rangle \langle \vec{n} | J_j | \vec{p}_1 \rangle k_{2_j}}{(E(\vec{p}_2) - E_n)(-\omega_2 + E(\vec{p}_1) - E_n)(E_n - E(\vec{p}_1))} \Big|_{\vec{n} = \vec{p}_1 - \vec{k}_2} \Big]$$

$$\equiv k_{1_i} k_{2_j} [\Lambda_{ij}(\omega_2, \vec{k}_2; \omega_1, \vec{k}_1) + \Lambda_{ji}(-\omega_1, -\vec{k}_1; -\omega_2, -\vec{k}_2)] \qquad (2.75)$$

Since we are excluding the (zero mass) photons from our intermediate state $|\vec{n}\rangle$, the denominators will never vanish in the limit of $\omega \to 0$. Accordingly, Λ_{ij} (a three tensor) is free of singularities.[29] From this we can immediately conclude that E_{00} is of the order of ω^2. Evidently, this statement is enough for the derivation of the usual low energy theorems, Eq. (2.33), since Eq. (2.73) indicates that E_{ij}, hence M_{ij}, can be computed accurately to the order ω. For the derivation of the higher order in ω theorems, we must construct suitable tensor basis for E_{ij} and Λ_{ij} to the appropriate order in ω. Pais has given a complete procedure for such a construction.[20] A low energy theorem results if one can show that, for the particular amplitude in question, E_{00} plays no role (to the order in which we are interested) and, hence, only the structure-independent $U_{\lambda\mu}$ term will contribute.

b. S-Matrix Method

A recent development is the derivation of the Compton low energy theorems using only the properties of the physical helicity amplitudes for the process.[30] The basic idea is of course still the same: We want to identify the amplitude that is free of kinematical singularities (and zeroes) then the corresponding dynamic pole term, which are calculable, will necessarily give the leading contributions--hence the low energy theorems.

One of the interesting features of the S-matrix approach is that gauge invariance, which plays such an important role in identifying these kinematical singularity- and zero- free amplitudes in our field theoretical derivations, is never used explicitly. This is possible because a photon is described in the helicity formalism as having two helicities (in every Lorentz frame) and it is known that this description is equivalent to the statement of gauge invariance. It is thus satisfying to see that low energy theorems may be derived directly from their physical basis: Lorentz invariance, masslessness of the photon[31] and simple analyticity properties of the amplitude.

We will only give an outline of this approach. Kinematics is still the same as (Fig. 7) previous discussions of the Compton scattering. We shall use the following covariant variables:

$$s = -(p_1 + k_1)^2$$
$$t = -(k_1 - k_2)^2$$
$$u = -(p_1 - k_2)^2$$

and

$$s + t + u = 2m^2 \qquad (2.76)$$

We note that $s - m^2 = O(\omega)$, $u - m^2 = O(\omega)$ and $t = O(\omega^2)$ in the limit of $\omega \to 0$ with fixed angle. For the direct channel process, where s is the square of the total center of mass energy: $p_1(a) + k_1(b) \to p_2(c) + k_2(d)$. The variables in the parentheses are the helicities of the particles. The s-channel helicity amplitude $M^{s}_{cd;ab}(s,t)$ is assumed to be free of kinematical singularities in the variable t, and it in general contains kinematic zeroes.

Consider the scattering in the center of mass system with scattering angle θ_s, (the angle between initial and final photons). We note that the projection of the total angular momentum of the initial (final) state is $a-b \equiv \lambda (c-d \equiv \mu)$ along the respective directions of motion. It then follows from angular momentum conservation that helicity amplitudes vanish (kinematical zeroes) for $\theta_s = 0 \ (\pi)$ unless $\lambda = \mu \ (\lambda = -\mu)$. Formally this property of the helicity amplitudes comes out from the standard expansion:[32]

$$M^{s}_{cd;ab}(s,t) = \sum_{J} (2J+1) M^{s(J)}_{cd;ab}(s) \, d^{J}_{\lambda\mu}(\theta_t) \qquad (2.77)$$

when we use the fact that the rotational matrices $d^{J}_{\lambda\mu}(\theta_s)$ may be written as $(\sin \frac{\theta_s}{2})^{|\lambda-\mu|} (\cos \frac{\theta_t}{2})^{|\lambda+\mu|}$ time a (Jacobi) polynomial in $\cos \theta_s$. Accordingly we can define a new amplitude

$$\overline{M}^{s}_{cd;ab}(s,t) = M^{s}_{cd;ab}(s,t) / (\sin \frac{\theta_s}{2})^{|\lambda-\mu|} (\cos \frac{\theta_s}{2})^{|\lambda+\mu|} \qquad (2.78)$$

which is now free of kinematical zeroes and still free of kinematical singularity in t.

In exactly the same manner we can consider the crossed channel process where t is the square of the total center of mass energy with center of mass scattering angle θ_t: $k_2(d')+k_1(b') \rightarrow p_2(c')+\bar{p}_1(a')$. The amplitude $(\lambda' \equiv d'-b', \mu' \equiv c'-a')$

$$\bar{M}^{t}_{c'a';d'b'}(s,t) = M^{t}_{c'a';a'b'} / (\sin \frac{\theta_t}{2})^{|\lambda'-\mu'|} (\cos \frac{\theta_t}{2})^{|\lambda'+\mu'|} \tag{2.79}$$

is then free of kinematical singularities in the variable s.

Finally, using the crossing properties relating s- and t-channel amplitudes, one can construct amplitudes which are free of kinematical singularities and zeroes in <u>both</u> s and t. In general the crossing matrix is very complicated. We will only consider the spin zero case, where there are two independent transitions in each channel:

$$\begin{aligned} M^{s}_{01;01} &= (\cos \frac{\theta_s}{2})^2 \bar{M}^{s}_{01;01} \\ M^{s}_{01;0-1} &= (\sin \frac{\theta_s}{2})^2 \bar{M}^{s}_{01;0-1} \end{aligned} \tag{2.80}$$

and

$$\begin{aligned} M^{t}_{11;00} &= \bar{M}^{t}_{11;00} \\ M^{t}_{1-1;00} &= (\sin \frac{\theta_t}{2})^2 \bar{M}^{t}_{1-1;00} \end{aligned} \tag{2.81}$$

The crossing relation is

$$M^{s}_{01;01} = -M^{t}_{1-1;00}$$
$$M^{s}_{01;0-1} = M^{t}_{11;00} \qquad (2.82)$$

In terms of the $\overline{M}$ amplitude, they are

$$\frac{\overline{M}^{s}_{01;01}}{(s-m^2)^2} = \frac{\overline{M}^{t}_{1-1;00}}{t(t-4m^2)}$$
$$\frac{s\overline{M}^{s}_{01;0-1}}{(s-m^2)^2} = -\frac{\overline{M}^{t}_{11;00}}{t} \qquad (2.83)$$

Where we have used the kinematical relations

$$\sin\frac{\theta_s}{2} = (-st)^{\frac{1}{2}}/(s-m^2)$$
$$\cos\frac{\theta_s}{2} = [(s-m^2)+st]^{\frac{1}{2}}/(s-m^2)$$
$$\cos\theta_t = (s-u)/[t(t-4m^2)]^{\frac{1}{2}} \qquad (2.84)$$

$\overline{M}^{s(t)}$ is free of singularities in t(s) respectively, therefore $\frac{\overline{M}^{s}_{01;01}}{(s-m^2)^2}$ and $\frac{s\,\overline{M}^{s}_{01;0-1}}{(s-m^2)^2}$ must be free of kinematical singularities both in t and in s.

Going back to the original helicity amplitudes,

$$M^s_{01;01} = \left[\frac{\overline{M}^s_{01;01}}{(s-m^2)^2} \right] (s-m^2)^2 \left(\cos\frac{\theta_s}{2}\right)^2$$

$$M^s_{01;0-1} = \left[\frac{s\overline{M}^s_{01;0-1}}{(s-m^2)^2} \right] \frac{1}{s} (s-m^2)^2 \left(\sin\frac{\theta_s}{2}\right)^2 \qquad (2.85)$$

we observe that they are both of the order ω^2, except for the dynamical pole terms. Consequently,

$$M^s_{cd;ab} = [M^s_{cd;ab}]_{pole} + O(\omega^2) \qquad (2.86)$$

This is the same result derived previously in Eq. (2.27).

As we can see, the practical usefulness this S-matrix approach is rather debatable, particularly in view of the fact that Eq. (2.86) is proved only for the special case of spin zero target (for particles with spin it is far more complicated). In fact, with the helicity approach there is no way to make a general spin-independent statement; we simply have to work it out for each target spin. Furthermore, it is not clear how this technique may be applied to processes involving more than four particle lines (e.g. bremsstrahlung).

CHAPTER III

RADIATIVE CORRECTIONS TO LOW ENERGY THEOREMS

1. Outline of the Problem

In the last chapter we saw how low energy theorems may be derived for processes involving photons, based on the inputs of Lorentz invariance and of masslessness of the photon. We have also seen that the feature which distinguishes these theorems from all other low energy theorems is that they yield amplitudes in the physical region and involve no extrapolation from unphysical points. This is also because the photon has zero rest mass. On the other hand, the massless nature of the photon complicates the analyticity properties of the amplitude. In the last chapter we assumed that the only singular part at $\omega = 0$ was the one containing the single particle poles, and we ignored the fact that the photons in intermediate states brought about cuts (in fact an infinite number of them), extending down to the point $\omega = 0$. Accordingly, to obtain low energy theorems that are correct to higher orders in electromagnetism, we must be able to calculate the threshold contributions of these cuts.

Consider again the pole diagram of Fig. 1. But now we will add a photon γ' to the intermediate single particle state, showing it in Fig. 10.

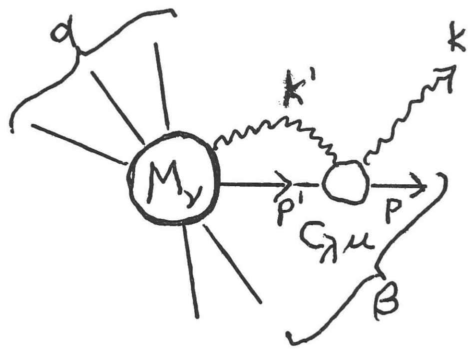


Fig. 10. Singular contributions by the state of the particle plus a soft photon.

Fig. 10 (with the same kinematics as in Fig. 1) represents a contribution to the amplitude of:

$$-\int \frac{dk'}{(2\pi)^4} D_{\mu\nu}(k')C_{\lambda\mu}(kp;k'p')D(p')M_\nu(k'\beta';\alpha) \qquad (3.1)$$

with $p' = p+k-k'$. $D_{\mu\nu}(k')$ is the renormalized propagator for the virtual photon with momentum k', and polarization indices ν, μ; $D(p')$ is the renormalized propagator for the particle of momentum p'; $C_{\lambda\mu}(kp;k'p')$ is the off-shell amplitude for the (Compton) process $p'+k'(\mu)\to p+k(\lambda)$; $M_\nu(k',\beta';\alpha)$ is the off-shell amplitude for the (bremsstrahlung) process $\alpha\to\beta'+k'(\nu)$.

In the limit of $k, k'\to 0$, both intermediate particles approach free particles:

$$D_\nu(k') \to \frac{\delta_\nu}{i(k'^2-i\epsilon)} \qquad (3.2)$$

$$D(p') \to \frac{1}{i(p'^2+m^2-i\epsilon)} = \frac{1}{i[2p(k-k')+(k-k')^2]} \qquad (3.3)$$

Eq. (3.3) shows that the threshold contribution of the dk' integration will contain a logarithmic singularity coming from the integration over the $(p'^2+m^2)^{-1}$ pole.

As we are only interested in this singular part, a number of simplifications can be made in Eq. (3.1). (i) At the threshold, the virtual photon γ' is really on its mass shell (i.e. $k' \to 0$ implies $k'^2 \to 0$). Thus, for the purpose of computing the singular part, the following substitution can formally be made:[34]

$$\int \frac{dk'}{(2\pi)^4} \frac{1}{i(k'^2-ie)} \xrightarrow[(k'_\lambda \to 0)]{} \int \frac{d\vec{k}'}{(2\pi)^3} \frac{1}{2\omega'} \tag{3.4}$$

with $\omega' = |\vec{k}'|$ in the integrand. (ii) Since the logarithms are brought about by the integration over the $(p'^2+m^2)^{-1}$ pole, we are allowed to set $p'^2+m^2=0$ in the numerator which is the residue of the pole. Accordingly, the singular part of Eq. (3.1) is

$$-\frac{1}{(2\pi)^3} \int_0 \frac{d\vec{k}'}{2\omega'} (p'^2+m^2)^{-1} \left[C^0_{\lambda\mu}(kp;k'p') \, M^0_\mu(k',\beta';\alpha) \right]; \; p' \equiv p+k-k' \tag{3.5}$$

where $C^0_{\lambda\mu}$ and M^0_μ are the on-shell soft photon amplitudes and they are lowest order in e. This is the counterpart of the result for the single particle pole shown in Eq. (2.11).

The purpose of this hueristic derivation [33] is to show that the threshold contribution of the intermediate particle-plus-soft-photon state is indeed non-analytic in the photon frequency, and that these terms may be computed in a structure independent way (since only on-shell quantities are involved).

It is clear that the validity of the usual low-energy theorems to higher orders in e will depend on what orders of photon frequencies these non-analytic threshold terms are. For example, the existence of a $\omega \ln\omega$ term will then limit the original series expansion in powers of ω to no higher than ω^0, and so on. At the end of Section 3.a., a more detailed discussion will be given of the question "what does one mean by 'validity' of the low-energy theorems to higher orders in e?"

Of course, infrared divergences are associated with the virtual soft photon radiative corrections. It is clear that here two expansions are made of the scattering amplitudes: one in the electric charge (e) and another in the photon frequency (ω). Soft photons complicate both these expansions: for the expansion in e we have the infrared divergences; for the expansion ω we have the non-analytic terms, for example, $\omega \ln \omega$. Thus, the zero-mass property of the photon leads to infinities in the coefficients of both power-series expansions. Both these types of complications will appear in the discussion of higher order radiative corrections to low-energy theorems.

In the next part of this section we will give an outline of the general procedure for evaluating the radiative corrections due to soft virtual photons in an arbitrary hadronic process.

a. General Strategy for Evaluation

The e^2 radiative correction amplitude $T(2)$ for the hadronic process $\alpha \to \beta$ can be represented as

$$T(2) = \frac{1}{2!} \int \frac{dk'}{(2\pi)^4} \frac{\delta_{\lambda\mu}}{i(k'^2 - ie)} M_{\lambda\mu} \tag{3.6}$$

$M_{\lambda\mu}$ is the amplitude for the process $\alpha + \gamma' \to \beta + \gamma'$, γ' being virtual photons, with polarization indices λ and μ respectively.

Regardless of whether the photons are on- or off-shell, the low-energy theorem (to the lowest order in e) holds, and it informs us (as discussed in Section II-2) that the leading terms in $M_{\mu\lambda}$ for $k' \to 0$ come from diagrams in which the two virtual photon-lines are attached to the external charged particles. For the special cases (which we will consider in Sections 3 and 4 of this chapter) of soft photon radiative corrections to processes which themselves have soft photon(s) in their initial and final states, the relevant $M_{\lambda\mu}$ is then given by diagrams where all the soft photons, real and virtual, are emitted or absorbed from the external charged lines. Fig. 13 is such an example.

Thus our strategy for evaluating the leading term in soft photon radiative corrections is the following: we will first cut the virtual photon line in two, then attach all the soft photon lines of the resultant amplitude onto the external lines. Radiative corrections can then be obtained through the appropriate threshold integration *via* Eq. (3.6).

b. Principal Results

Using the general procedure as outlined above, we shall demonstrate in Section 2 that the results, obtained with respect to infrared divergence within the theory of quantum electrodynamics, are not modified when strong interactions are included to all orders.[35]

In Section 3 the soft photon radiative correction will be evaluated when the hadronic process also involves a low-frequency photon, $\alpha \to \beta + \gamma$. Besides giving the expected infrared divergent factors, the soft virtual photons also bring about non-analytic terms in the ω expansion: in Subsection 3a for the bremsstrahlung ($\alpha \to \beta$ being some physically allowed process) we have a term proportional to $\ln\omega$; while the ω^{o} term, for which Low has proved the low-energy theorem to order e, Eq. (2.13), is shown to be structure-dependent in the presence of radiative corrections. The low-energy theorem for the $e^3 \ln \omega$ bremsstrahlung amplitude obtained here agrees with the results derived by Soloviev.[36] In Subsection 3b for pion-photoproduction ($\alpha \to \beta$ being the pion-nucleon vertex) we have a term proportional to $\omega \ln\omega$. It is interesting to note that our result shows that the radiative correction to the ω term, for which Fubini _et at._[37] have proved a PCAC low-energy theorem is structure dependent.

In Section 4 our method is extended to include Compton scattering. The leading threshold contribution $O(e^4 \omega^2 \ln\omega)$ is shown to be completely determined by the charge of the target particle. This result has been previously derived _via_ dispersion relations for the forward differential cross section by Gerasimov and Soloviev,[38] then for the amplitude by Roy and Singh[39] and, independently, by this author.[40] It is rederived here without extraneous high energy assumptions. The simplicity of the actual computation allows us to see immediately that the results for the spin-zero case actually hold for targets of arbitrary spin.[41]

2. Infrared Divergences

The physical basis of infrared divergences was elucidated long ago starting with the work of Bloch and Nordsieck.[42] They treated a simplified model in which a fixed classical-current density interacts with a quantized electromagnetic field. The virtue of this model is that it can be solved (without recourse to the usual perturbation method) and thus leads to insight into the difficulties brought about by photons in the long wave-length region. The principal conclusion is the following: when a charged particle is scattered, it always radiates an infinite number of soft photons (with finite total energy). The cross section of any process involving charged particles and a definite number of photons, for which the perturbation expansion in the fine structure constant is not valid, is an unphysical quantity. What is measured in reality is the probability for scattering with an energy loss (due to radiation) of less than ΔE, which is the energy resolution of the experimental set-up. A perturbation expansion for this physically observable quantity is, however, permissible; the infrared divergence in the radiative correction, and the corresponding divergence in the cross section for photon emissions, cancel order by order.

This cancellation of the infrared divergence in the cross sections has been demonstrated explicitly for quantum electrodynamics of the electron-photon system. The proof, as given for instance by Jauch and Rohrlich,[43] depends on some special properties of quantum electrodynamics: Consequently, we must inquire whether the cancellation remains true for the case of hadrons, if strong interactions are included to all orders. We will show that this is indeed the case, as is to be anticipated.

Consider the process $\alpha \rightarrow \beta$ where α and β are some arbitrary hadron states involving charged particles. It is well known that the infrared divergent part of the cross section $d\sigma$ for $\alpha \rightarrow \beta + \gamma$ can be factored out and is proportional to the $d\sigma$ for $\alpha \rightarrow \beta$. This is just the usual ω^{-1} bremsstrahlung low-frequency theorem. Our main interest is therefore the factorization of infrared divergent radiative corrections to $\alpha \rightarrow \beta$ due to virtual photons.

For definitness, consider the case where $\alpha(\beta)$ consists of one charged particle p_1 (p_2) and one neutral particle r_1 (r_2).

According to the general method of evaluating the soft photon radiative corrections as outlined in the last section, we first consider the process $p_1 + r_1 + \gamma' \rightarrow p_2 + r_2 + \gamma'$ with the virtual photons γ' attached to the external charged lines, p_1 and p_2, as shown in Fig. 11.

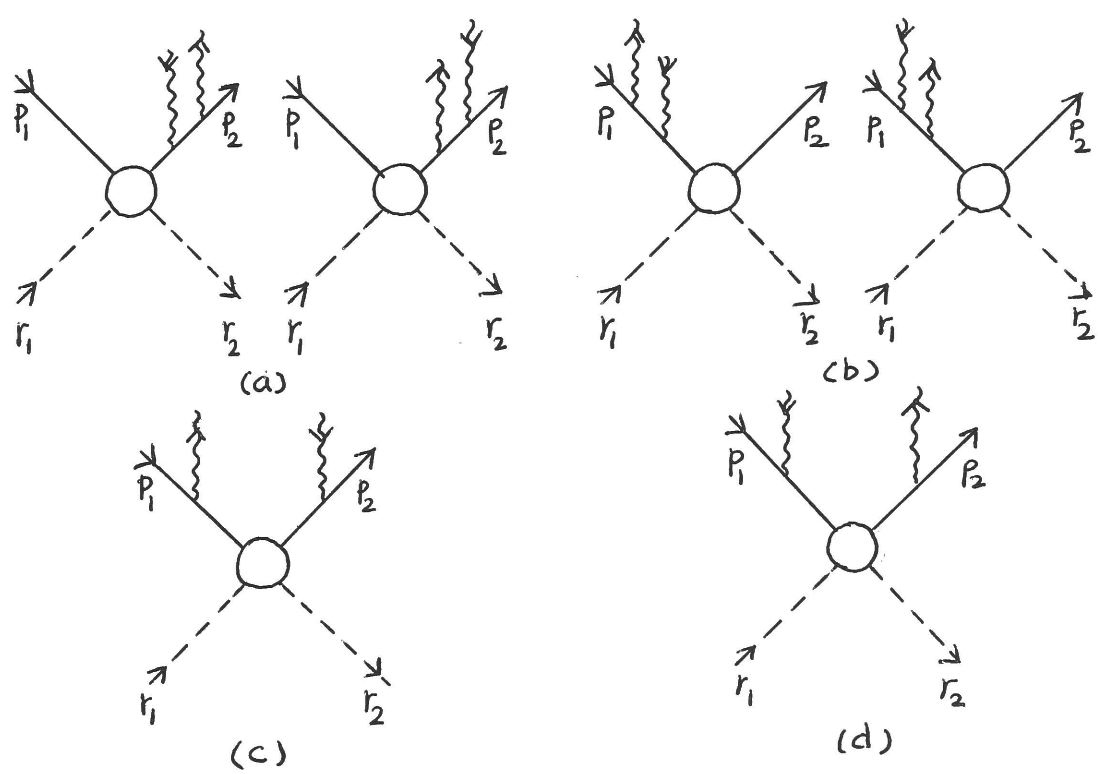


Fig. 11. The double pole terms in scattering involving two low frequency photons.

We treat the two photons as distinguishable, and the overcounting is compensated by the appropriate combinatorial factors. For graphs in (a) and (b) in Fig. 11 we shall, for the moment, imagine that the two photons have different momenta, k_1' and k_2', so that the contributions from these two graphs will be unambiguous.[44,45] For $k_1' \to 0$ and $k_2' \to 0$, we have from (a),

$$a_{\mu\lambda} = + e^2 \frac{p_{2\mu} p_{2\lambda}}{p_2 (k_2' - k_1')} \left[\frac{1}{p_2 k_2'} - \frac{1}{p_2 k_1'} \right] T(0) \qquad (3.7)$$

where $T(0)$ is the amplitude to the lowest order in e for $p_1 + r_1 \to p_2 + r_2$. Setting $k' = k_1' = k_2'$, we have

$$a_{\mu\lambda} = - e^2 \frac{p_{2\mu} \, p_{2\lambda}}{(p_2 k')(p_2 k')} T(0) \qquad (3.8)$$

For the graph (c) we obtain,

$$c_{\mu\lambda} = + e^2 \frac{p_{2\mu} p_{2\lambda}}{(p_2 k')(p_1 k')} T(0) \qquad (3.9)$$

Substituting $p_1 \leftrightarrow p_2$ in $a_{\mu\lambda}$ and $c_{\mu\lambda}$ results in $b_{\mu\lambda}$ and $d_{\mu\lambda}$. The whole amplitude is then

$$M_{\mu\lambda} = -e^2 \left(\frac{p_{2\mu}}{p_2 k'} - \frac{p_{1\mu}}{p_1 k'} \right) \left(\frac{p_{2\lambda}}{p_2 k'} - \frac{p_{1\lambda}}{p_1 k'} \right) T(0) \qquad (3.10)$$

We have for the e^2 radiative correction

$$T(2) = -\frac{e^2}{2} \int \frac{d\vec{k}'}{(2\pi)^3} \frac{1}{2\omega'} \left(\frac{p_2}{p_2 k'} - \frac{p_1}{p_1 k'} \right)^2 T(0) \qquad (3.11)$$

The interference term with $T(0)$ gives the e^2 cross section, which just cancels the corresponding cross section for the bremsstrahlung of a soft photon, $r_1 + p_1 \rightarrow r_2 + p_2 + \gamma$. It is clear that the above result can easily be extended to include cases of $\alpha \rightarrow \beta$ with more complicated charge configurations.

For later reference we note that the infrared divergent factor in Eq. (3.11) can be computed to be:

$$T(2) = b(t) \int \frac{d\omega'}{\omega'} T(0) \qquad (3.12)$$

where $t = -(p_2 - p_1)^2$. For the equal mass case $(p_1^2 = p_2^2)$, b takes on a simple form,

$$b(t) = \frac{\alpha}{\pi} \left[1 + \frac{t - 2m^2}{D^{\frac{1}{2}}} \ln \left| \frac{t - D^{\frac{1}{2}}}{t + D^{\frac{1}{2}}} \right| \right] \qquad (3.13)$$

$D = t^2 - 4tm^2$ and for $t \rightarrow 0$, we have

$$b(t) = \frac{\alpha}{3\pi} \left(\frac{t}{m^2} \right) + O(t^2) \tag{3.14}$$

It should be noted that the validity of our result as stated in Eqs. (3.11) or (3.12) rests on the generality of our procedure in applying the low-energy theorem to the amplitude $\alpha + \gamma' \rightarrow \beta + \gamma'$ in the integrand. The results should be true in any local field theory.

3. Single-Photon Cases

We now consider the slightly more complicated problem of soft virtual photon radiative correction to the process $\alpha \rightarrow \beta + \gamma$, the emitted photon also being soft ($k_\lambda \rightarrow 0$). Here we expect to obtain not only the infrared divergent terms but also the non-analytic threshold factors ($\sim \ln \omega$) of the intermediate soft photon state. Similarly, as in the last section, we first consider the process of $\alpha + \gamma' \rightarrow \beta + \gamma + \gamma'$. All the photons being soft, the low-energy theorem informs us that the leading contribution comes from those graphs in which all three photon lines, real and virtual, are attached to the external charged particles in the basic process $\alpha \to \beta$.

a. Bremsstrahlung

We will first consider the radiative corrections to the theorem derived in the Subsection 1. b. of Chapter II.

By the above argument it is then sufficient for our calculation of the soft virtual photon radiative corrections to consider the set of diagrams in Fig. 12. We have included "seagull" diagrams so that the total contribution in the soft photon limit will be gauge invariant.

It is clear that (a) and (b) in Fig. 12 contribute only to the infrared divergent terms:

$$(a+b)_\lambda = - \frac{e^3}{(2\pi)^3} \int \frac{d\vec{k}'}{2\omega'} \left[\frac{p_{2\mu} \, p_{2\mu}}{2(p_2 k')^2} + \frac{p_{1\mu} p_{1\mu}}{2(p_1 k')^2} \right] \frac{p_{2\lambda}}{p_2 k} \, T(0) \tag{3.15}$$

where $T(0)$ denotes the zeroth order in e, on-shell amplitude for the process $r_1 + p_1 \to r_2 + p_2$ without photon emission.

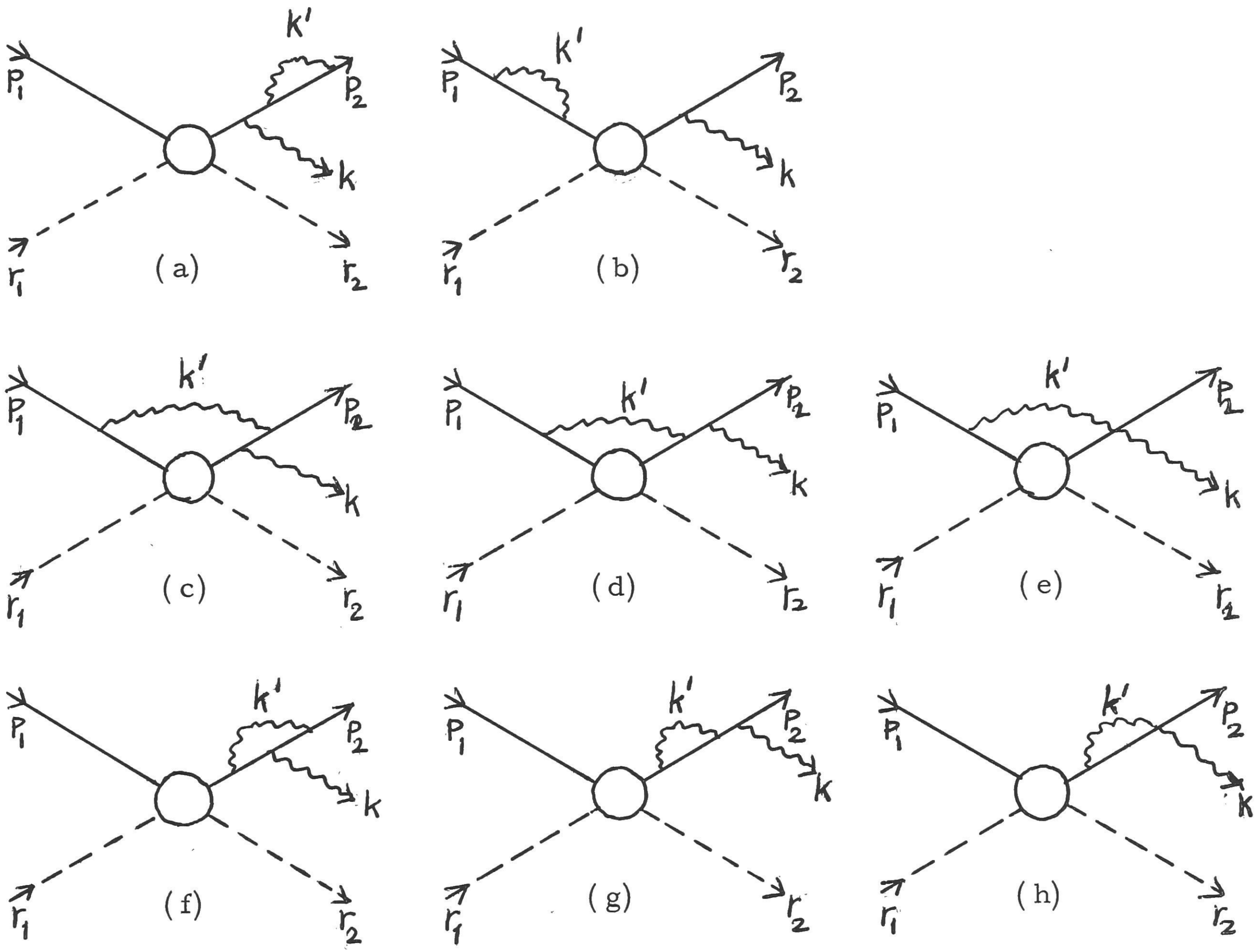


Fig. 12. Soft photon radiative correction to bremsstrahlung. There are eight more crossed diagrams corresponding to the photon k being emitted from the initial charged line. They are related to the above diagrams by substitutions:

$$p_1 \leftrightarrow p_2 \qquad \text{and} \qquad k_\lambda \leftrightarrow -k_\lambda .$$

The contribution from (c) to (h) can be written in simple form,

$$(c + d + \dots h)_\lambda = - \frac{e^3}{(2\pi)^3} \int \frac{d\vec{k}'}{2\ \omega'} \frac{1}{p'^2 + m^2} \times$$

$$\times\ 2 \left[\frac{p_{2\mu}\, p'_\lambda}{p_2\, k'} - \frac{p_{2\lambda}\, p'_\mu}{p_2\, k} + \delta_{\lambda\mu} \right] \left[\frac{p'_\mu}{p_2\, k} - \frac{p_{1\mu}}{p_1\, k'} \right] \quad T(0) \tag{3.16}$$

where $p' = p_2 + k - k'$. The $\ln\omega$ terms come about because the propagator $(p'^2 + m^2)^{-1}$ diverges as $k \to 0$ and $k' \to 0$. Before proceeding with the computation of these terms, we will first separate out the infrared divergent factor coming from graph (c) which is proportional to $(p_2 k')^{-1} (p_1 k')^{-1}$. This can be accomplished easily by noting

$$\frac{1}{p'^2 + m^2} \times \frac{1}{p_2\, k'} = \left(\frac{1}{p'^2 + m^2} + \frac{1}{2\, p_2\, k'} \right) \frac{1}{p_2 k - k'k}$$

Dropping terms that contribute neither to the divergent factor nor to the $\ln\omega$ terms, we have

$$(c+d+\ldots h)_\lambda = -\frac{1}{(2\pi)^3}\int \frac{d\vec{k}'}{2\omega'}\left\{ -e^2 \frac{p_{1\mu}p_{2\mu}}{(p_1k')(p_2k')}\frac{p_{2\lambda}}{p_2k}T(0)\right.$$

$$\left. + \frac{2e^2}{p'^2+m^2}\left[\frac{p_{2\mu}p'_\lambda}{p'k} - \frac{p_{2\lambda}p'_\mu}{p_2k} + \delta_{\lambda\mu}\right]\left[\frac{p'_\mu}{p_2k} - \frac{p_{1\mu}}{p_1k'}\right] e\,T(0)\right\}$$

(3.17)

The first term combined with $(a+b)_\lambda$ in Eq. (3.15) gives the expected infrared divergent factors, which are proportional to the lowest order bremsstrahlung amplitude.

In the following discussion we will concentrate on the $\ln\omega$ factors coming from the second term in Eq. (3.17). Since these logarithms are brought about by the integration over the $(p'^2+m^2)^{-1}$ pole terms, we are allowed to set $p'^2 + m^2 = 0$ in the numerator, which is the residue of the pole. With this observation Eq. (3.17) gives just the result which we have derived heuristically in Eq. (3.5). Keeping only the leading terms in the integrand, $O(\omega^0)$ for $C^0_{\lambda\mu}$; $O(\omega^{-1})$ for M^0_μ we have

$$(c'+d+\ldots h)_\lambda = \frac{e^3T(0)}{(2\pi)^3}\int \frac{d\vec{k}'}{2\omega'}\frac{1}{p_2(k-k')}\left[(k'_\lambda - p_{2\lambda}\frac{k'k}{p_2k})\left(\frac{p_2}{p_2k}\right)^2\right.$$

$$\left. + \frac{1}{p_1k'}(p_{1\lambda} - p_{2\lambda}\frac{p_1k}{p_2k}) - \frac{1}{p_1k'}(k'_\lambda - p_{2\lambda}\frac{k'k}{p_2k})\frac{p_1p_2}{p_2k}\right]$$

(3.18)

The prime over the amplitude c_λ indicates that the infrared divergent factor has been extracted. The first term vanishes after the $\vec{dk'}$ integration, since $\int \frac{\vec{dk'}}{2\omega'} \frac{k'_\lambda}{p_2(k_2 - k')}$ must be proportional to $p_{2\lambda}$, therefore we have only two types of integrals to calculate:

$$\int \frac{\vec{dk'}}{2\,\omega'} \frac{1}{p_2(k-k')} \frac{1}{p_1 k'} \equiv A \ln | p_2 k | \qquad (3.19)$$

and

$$\int \frac{\vec{dk'}}{2\,\omega'} \frac{1}{p_2(k-k')} \frac{k'_\lambda}{p_1 k'} \equiv (B\, p_{1\lambda} + C\, p_{2\lambda}) \ln | p_2 k | \qquad (3.20)$$

giving

$$(c' + d + \ldots h)_\lambda = \frac{e^3\, T(0)}{(2\pi)^3} \left(p_{1\lambda} - p_{2\lambda} \frac{p_1 k}{p_2 k}\right) \left(A - B \frac{p_1 p_2}{p_2 k}\right) \ln | p_2 k | \qquad (3.21)$$

A and B can be evaluated easily by going first to the rest frame of p_2:

$$A = -4\,\pi D^{-\frac{1}{2}} \ln \left| \frac{t - D^{\frac{1}{2}}}{t + D^{\frac{1}{2}}} \right| \qquad (3.22)$$

$$B = 4\pi\, (p_2 k)\, D^{-1} \left(2 - 4\, p_1 p_2 D^{-\frac{1}{2}} \ln \left| \frac{t - D^{\frac{1}{2}}}{t + D^{\frac{1}{2}}} \right| \right) \qquad (3.23)$$

where t and D are defined as in Section 2. It is then straightforward to check that

$$\frac{e^2}{(2\pi)^3}\left(A - B\,\frac{p_1 p_2}{p_2\, k}\right) = -2b'(t) \qquad (3.24)$$

b' being the derivative with respect to the variable t of the function b defined in Eq. (3.12). Adding the contribution from the crossed term and noting the relation, $\ln|p_1 k| = \ln|p_2 k| + O(k^0)$, we obtain for the soft photon radiative correction to the bremsstrahlung amplitude:

$$M_\lambda(3) = e\left[\frac{p_{2\lambda}}{p_2\, k} - \frac{p_{1\lambda}}{p_1\, k}\right] T(0)\left[\, b(t) \int \frac{d\omega'}{\omega'} - 2b'(t)(p_2 - p_1)k \ln|p_1 k|\,\right]$$

$$+\, O(k^0) \qquad (3.25)$$

This is the result first obtained by Soloviev. It agrees with the low energy limit of the exact relativistic calculation in perturbation theory by Fomin[46] for the process of bremsstrahlung by an electron scattered in Coulomb field.

When the velocities of the charged particles are non-relativistic and $p_1^2 = p_2^2 = -m^2$, $t \approx -(\vec{p}_2 - \vec{p}_1)^2 \equiv -\vec{\Delta}^2$. Hence the angular factor $b'(t)$ can be expanded and Eq. (3.25) is reduced to the following simple form:

$$\vec{M}(3) = \frac{-e^3}{12\pi^2 \omega\, m^3}\, \vec{\Delta}\, [\, \vec{\Delta}^2 \ln \omega_{min} - 2\vec{\Delta}\cdot\vec{k} \ln \omega]\, T(0) \qquad (3.25\ \mathrm{N.R})$$

For $p_1^2 \neq p_2^2$ (e.g. $\pi^- + p \rightarrow \pi^o + n + \gamma$) there is no simplification in the non-relativistic limit.

With this concrete expression for the low-energy behavior of the e^3 bremsstrahlung amplitude, we can now give a more detailed discussion of the question: "what does one mean by the 'validity' of the low-energy theorem to higher orders in electromagnetism?"

The theorem proved by Low states that not only the leading ω^{-1} term, but also the next order ω^o term, in the bremsstrahlung (M_λ) can be computed from the corresponding scattering amplitude without photon emission (T), see Eq. (2.13). The proof is carried out only in the lowest order in e: $M_\lambda = M_\lambda(1)$ and $T = T(0)$. (The number in the brackets denotes the order in e.) If the theorem were valid to the n^{th} order in α, then the relation in Eq. (2.13) would still hold, with M_λ and T taken to the corresponding orders: $M_\lambda = M_\lambda(1) + \ldots M_\lambda(2n+1)$ and $T = T(0) + \ldots T(2n)$. The radiative corrections to M_λ and T naturally include infrared divergent factors. This is precisely what we have in Eq. (3.25): the pole term is multiplied by an infrared divergent factor. The corresponding divergent factors on both sides are identical, as they necessarily must be. Thus we say that the infrared divergent radiative corrections to the low-energy theorems are really of a trivial kind. They do not change the form of the low-energy theorems derived in the lowest order. To be sure, due to the existence of these infrared factors, ΔE will have to be introduced in the cross sections. But this is a general feature of any physical measurement and is not limited in any way to the low-energy behavior of scattering.

We now see that it is the non-analytic factor ($\sim \omega^n \ln \omega$) which is crucial to the validity of the higher order theorems. It invalidates the original expansion in ω. A more concrete way of viewing the problem is to note that as in the case of bremsstrahlung the existence of threshold terms $\ln \omega$ indicates that the non-threshold contributions are of the order ω^o, its precise value being cut-off dependent. Accordingly, there is no longer any low-energy theorem for the ω^o term in the e^3 amplitude, since soft photons introduce structure-dependent terms of this order.

b. Pion Photoproduction

To the lowest order in e, we have shown that the leading ω^{o} term in the pion photoproduction amplitude is completely determined by the electric charge and the pion-nucleon coupling constant g, (the Kroll-Ruderman Theorem). To calculate the radiative corrections we can follow the procedure used in Sub-section a. To order e^3 the pole term is still the same as in Eq. (2.15) with g being the e^2 radiative correction to the pion-nucleon coupling constant including the expected infrared divergent factors. As for the non-analytic threshold terms, we can simply make use of the general expression in Eq. (3.5), with M_{μ}^{o} being the lowest order pion photoproduction amplitude. Since the initial and final momenta of the nucleon are constrained in this scattering, the calculation is considerably simplified when performed in the laboratory system and with the transverse photon gauge. Evaluating the integrand by the Thomson theorem for the Compton amplitude and by Eq. (2.15), we have

$$\vec{e}\,\vec{M}^{a}(3) = \frac{-1}{(2\pi)^3}\left(i\,\frac{e^3 g}{2\,m^2}\right)\int_0 \frac{\omega' d\omega'}{2}\int d\Omega' \sum_{e'} (\vec{\sigma}\,\vec{e}\,')(\vec{e}\,'\,\vec{e}) \times$$

$$\times\left\{\frac{\frac{1}{4}[\tau^a, \tau^3]\,(1+\tau^3)}{\omega' - \omega} + \frac{\frac{1}{4}(1+\tau^3)[\tau^a, \tau^3]}{\omega' + \omega}\right\} \quad (3.26)$$

which yields the threshold contribution:

$$[-i\,\frac{eg}{2m}\,\vec{\sigma}\,\vec{e}\,]\,\frac{2\alpha}{3\pi}\,\left(\frac{\omega}{m}\right)\,\ln\omega\;\tau^a(1-\delta_{3a}) + O(\omega) + O(\mu) \quad (3.27)$$

The $\omega \ln \omega$ term comes in two isospin channels. The term proportional to τ^a is one of the amplitudes for which Fubini, et al. have given a PCAC low-energy theorem to order ω. Our result shows that this term is structure-dependent when radiative corrections are included, for the same reason as was discussed in Subsection a for the ω^o term in bremsstrahlung amplitudes. The independent isospin amplitude $\tau^a\delta_{3a}$ comes about because the intermediate photon has an isovector part.

4. Compton Scattering

We begin the calculation by considering the amplitude for the process $p_1 + k_1\,(e_{1_\lambda}) + k'(e'_\nu) \rightarrow p_2 + k_2\,(e_{2_\mu}) + k'\,(e'_\rho)$, which also serves to define the kinematics of the problem . The target (p) is allowed to have arbitrary spin. As we shall see, $O(e^4\omega^2 \ln\omega)$ Compton low-energy theorem to be derived in this section is spin-independent. When all photons have small energy, the leading terms are associated with diagrams which can be obtained by making all possible permutations of the four photon lines in Fig. 13

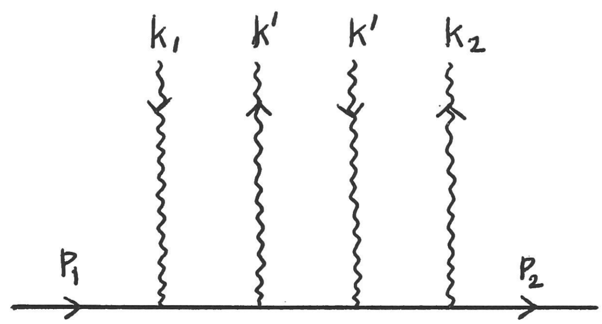


Fig. 13. External line emission diagram for the process, $p_1 + k_1 + k' \rightarrow p_2 + k_2 + k'$.

We then tie the k' lines together, and integrate (with the integration range: $0 < |\vec{k}'| < \delta, \delta \rightarrow 0$) to obtain the soft virtual photon radiative correction to the low-energy Compton scattering amplitude (see Fig. 14, again we have included the "seagull" graphs, g to n).

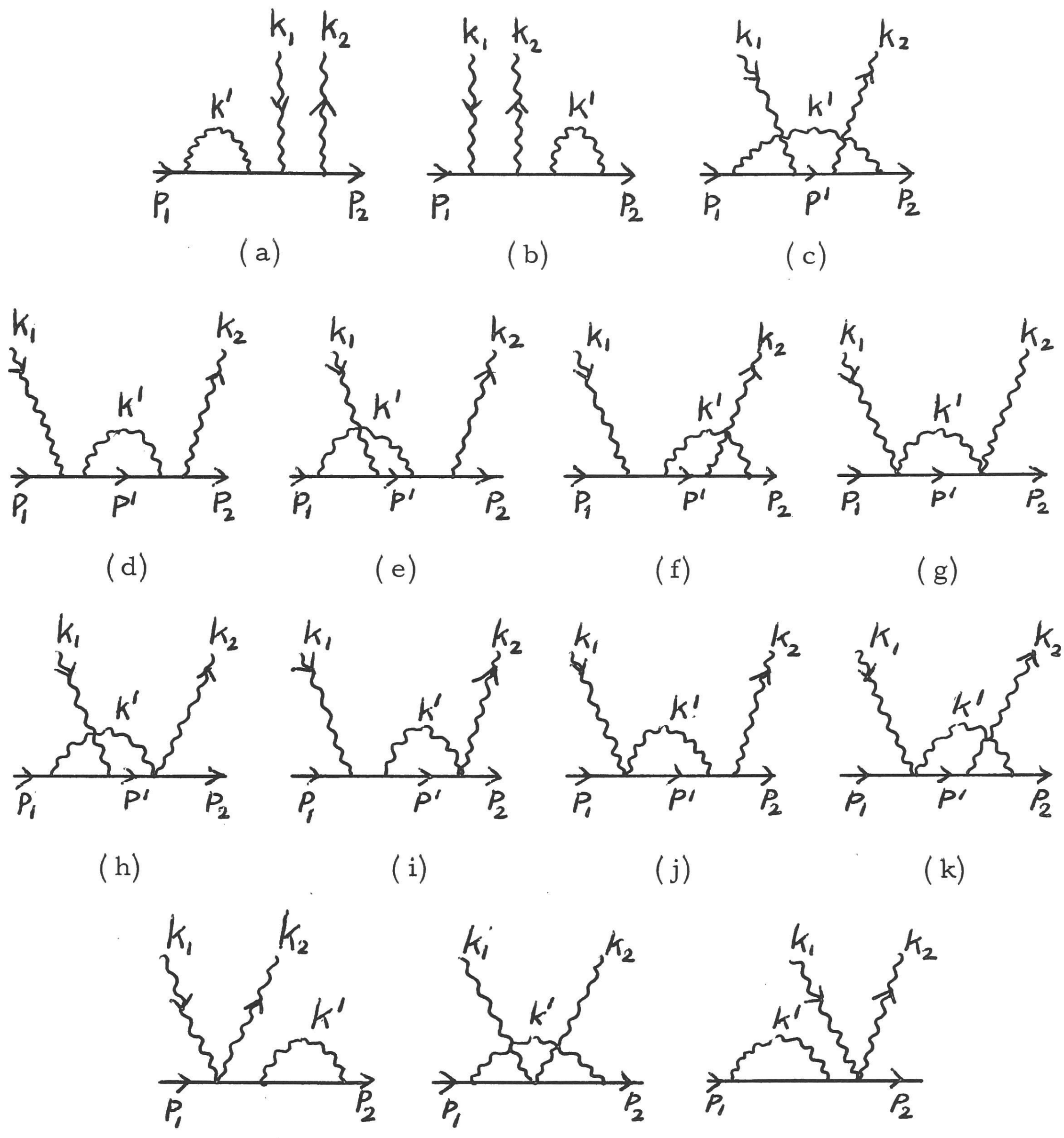


Fig. 14. Soft photon radiative correction to Compton scattering. There are eleven more crossed diagrams corresponding to an exchange of k_1 and k_2 in diagrams (a) to (k). The contributions of the crossed diagrams can be obtained by the substitution $k_{1\lambda} \leftrightarrow -k_{2\lambda}$.

Just as for pion photoproduction, the actual computation is greatly simiplified when it is performed in a special Lorentz frame $(\vec{p}_1 = 0)$ with a particular gauge (the Coulomb gauge).

It is clear that graphs (l) (m) (n), and (a) (b) (c) with their cross graphs, give rise to the infrared divergence. By Eq. (3.12) and (3.14) [the momentum transfer $t = -2\omega_1 \omega_2 \times (1 - \cos\theta)$ vanishes in the soft photon limit].

$$- \frac{2\alpha}{3\pi} \frac{\omega^2}{m^2} (1-\cos\theta) \int \frac{d\omega'}{\omega'} C^{(2)} \qquad (3.28)$$

$C^{(2)}$ is the lowest order Compton amplitude.

We will now concentrate on the evaluation, in graphs (c) to (k), of the singular threshold contributions from intermediate soft photon states. For this, we can make direct use of Eq. (3.5)

$$C^{(4)'} = - \frac{e_{1\lambda} e_{2\mu}}{(2\pi)^3} \int \frac{d\vec{k}'}{2\omega'} \left(\sum_{e'} e'_\nu e'_\rho \right) \frac{C^{(2)}_{\mu\nu}(k_2, p_2; k', p') C^{(2)}_{\rho\lambda}(k', p'; p_1, k_1)}{p'^2 + m^2} \qquad (3.29)$$

where $p' = p_1 + k_1 - k'$. The numerator is the product of two physical e^2 Compton amplitudes. To the order we are interested in, we have

$$e'_\rho \, C^{(2)}_{\rho\lambda} \, e_{1\lambda} = 2 e^2 \, \vec{e}_1 \, \vec{e}' + \omega S [\vec{e}_1 , \vec{e}'] + O(\omega^2) \qquad (3.30)$$

and

$$e_{2\mu} \, C^{(2)}_{\mu\nu} \, e'_\nu = 2 e^2 \Big\{ \vec{e}_2 \, \vec{e}' + (\vec{k}_1 \, \vec{e}')(\vec{k}_1 \, \vec{e}_2) / m\omega'$$

$$- (\vec{k}_1 - \vec{k}') \cdot \vec{e}_2 (\vec{k}_1 - \vec{k}_2) \cdot \vec{e}' / m \omega_2 \Big\} + \omega S[\vec{e}_2 , \vec{e}'] + O(\omega^2) \qquad (3.31)$$

where $\omega S \; \vec{e}, \vec{e}'$ stands for possible spin-dependent terms. $S[\vec{e}, \vec{e}']$ by itself is of order ω^o and is odd under crossing. As we shall see, none of these spin-dependent factors will contribute to the final results with respect to the radiative correction $O(\omega^2 \ln\omega)$.

Thus, in the laboratory system, with transverse gauge, Eq. (3.29) reads

$$C^{(4)\,'} = \frac{4 e^4}{(2\pi)^3} \int \frac{d\vec{k}'}{2\omega'} \sum_{e'} \Big\{ \frac{1}{2m[\omega_1 - \omega' - (\omega_1 \omega' - \vec{k}_1 \vec{k}') / m]} \, x$$

$$x \, [(\vec{e}_1 \vec{e}') (\vec{e}_2 \vec{e}') - (\vec{e}_1 \vec{e}') (\vec{k}_1 - \vec{k}') \cdot e_2 (\vec{k}_1 - \vec{k}_2) \, . e' / m \omega_2$$

$$+ (\vec{e}_1 \, \vec{e}') (\vec{k}_1 \vec{e}') (\vec{k}_1 \vec{e}_2) / m \, \omega' + \omega \, (\vec{e}_1 \, \vec{e}') \, S[\vec{e}_2 , \vec{e}']$$

$$+ \omega \, S[\vec{e}_1 , \vec{e}'] \, (\vec{e}_2 \vec{e}') + O(\omega^2)] \Big\} \qquad (3.32)$$

The only term we have to be careful with is the one proportional to $(\vec{e}_1 \vec{e}')(\vec{e}_2\vec{e}')$; here we must keep the recoil term proportional to $(k_1\ k')$ in the denominator. Concentrating on the threshold contribution of this term, we have

$$\frac{e^4}{(2\pi)^3} \frac{1}{m} \sum_{e'} \int_0 \frac{\omega' d\omega' d\Omega'}{\omega_1 - \omega'} (\vec{e}_1 \vec{e}')(\vec{e}_2 e')\left[1 + \frac{\omega\omega' - \vec{k}_1 \vec{k}_1}{m(\omega_1 - \omega')} + \ldots\right]$$

$$= \frac{e^4}{(2\pi)^3} \frac{1}{m} \frac{8\pi}{3} (\vec{e}_1 \vec{e}_2)\left[\omega_1 \ln\omega - 2\frac{\omega_1^2}{m} \ln\ \omega_1\right] + O(\omega^2) \qquad (3.33)$$

For the second and the third terms in Eq. (3.32) the recoil factors can be dropped; after a simple computation, the result is

$$\frac{e^4}{3\pi^2} \frac{(\vec{e}_2 \vec{k}_1)(\vec{e}_1 \vec{k}_2)}{m^2} \ln\ \omega_1 \qquad (3.34)$$

The spin-dependent terms will give a term of the order $\omega_1^2 \ln\omega_1$. When the contribution from the crossed graphs is added, the leading $\omega_1 \ln \omega_1$ term and the spin dependent terms are both canceled by the crossed terms. By Eqs. (3.28) (3.33) and (2.26) $C^{(4)}$ is given to be,

$$C^{(4)} = -\frac{e^4}{3\pi^2}\Big\{\Big[\Big(\frac{\omega^2}{m}\Big)^2 (3+\cos\theta)(\vec{e}_1\,\vec{e}_2) - \frac{2(\vec{e}_1\,\vec{k}_2)(\vec{e}_2\,\vec{k}_1)}{m^2}\Big]\ln\omega$$

$$+ \Big(\frac{\omega}{m}\Big)^2 (1-\cos\theta)\,(\vec{e}_1\,\vec{e}_2) \int \frac{d\omega'}{\omega'}\Big\} + O(\omega^2) \qquad (3.35)$$

The cross product of the e^4 amplitude and the Thomson amplitude $2\,e^2\,\vec{e}_1\,\vec{e}_2$ gives rise to the leading e^6 differential cross section,

$$d\,\sigma(6) = d\,\Omega\,\frac{2\alpha}{3\pi}\Big(\frac{\alpha}{m}\Big)^2\Big(\frac{\omega}{m}\Big)^2\Big[(\cos^3\theta - 3\cos^2\theta - 3\cos\theta - 3)\ln\omega$$

$$-(1-\cos\theta)(1+\cos^2\theta)\int\frac{d\omega'}{\omega'}\Big] + O(\omega^2) \qquad (3.36)$$

This result agrees with the low-energy limits of the exact relativistic results calculated in perturbation theory by Corinaldesi and Jost for spin-zero targets, and by Brown and Feynman for electron targets.[47]

With our approach the spin-independent nature of the results in Eqs. (3.35) and (3.36) are understood[48] and furthermore they are shown to be exact in strong interactions.

Remark 1. One may ask why the e^2 radiative correction invalidates the ω^0 bremsstrahlung low-energy theorem, but not the ω Compton theorem. The reason lies essentially in the special crossing properties of the Compton scattering. In bremsstrahlung, the momenta of the initial and final charged particles are not correlated in the soft photon limit; on the other hand, in Compton scattering we have the kinematic relation: $k_1(p_1 - p_2) = k_1 k_2$. Accordingly, the leading $\omega \ln\omega$ term [see Eq. (3.33)] is cancelled when the corresponding contribution from the crossed graphs is added. It can be understood in a similar way that the $\omega \ln \omega$ amplitude vanishes for the π^0 photo-production.

Remark 2. Recently a new low energy theorem is proved by Dr. K. Y. Lin for the e^4 Compton scattering: the entire $\omega^3 \ln\omega$ amplitude is determined by the charge and magnetic moment of the target particle.[49]

5. Discussion

We will now discuss very briefly the problem presented by higher order radiative corrections and examine the possibilities of deriving low energy theorems that are not only exact in strong interactions but also in electromagnetism. In particular it would be interesting to investigate the validity of the important Compton theorems to all orders in e, to see whether it is possible to define exact electromagnetic multipole moments via low energy Compton scattering.

While we have only presented an explicit computation of the e^2 radiative corrections, the techniques used here allow, in principle, the calculation of higher order correction terms. The general procedure would be to cut all virtual photon lines in two and then attach the resultant soft photon lines, real or virtual, to the external particle legs in all possible permutations. Fig. 15 represents such a graph for bremsstrahlung. To compute these pole diagrams, we would follow the now-familar rule: for each explicitly displayed internal line write a free propagator with a pole at the physical mass and for the amplitudes and vertices use only on-shell quantities.

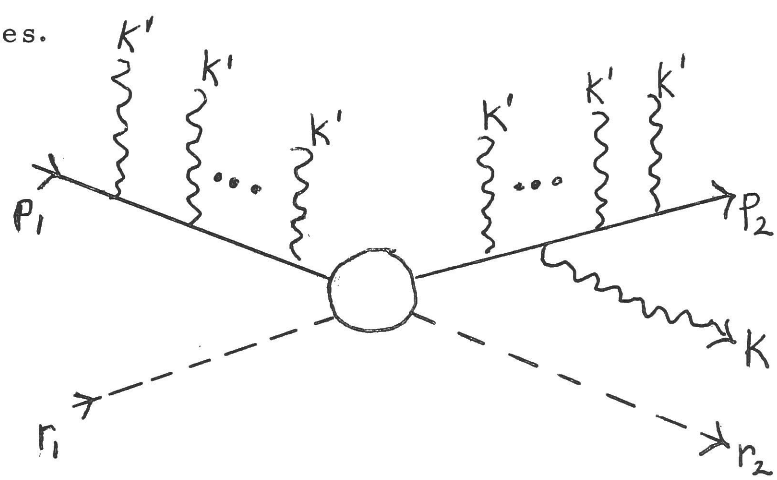


Fig. 15. Pole diagram in bremsstrahlung.

To obtain radiative corrections we would join the virtual lines and extract the appropriate threshold factors in the integrations (using Eq. (3.4)). In other words, the soft photon radiative corrections come from diagrams as exemplified by Fig. 16. We note that it is not necessary, for the purpose of computing the leading terms in soft photon radiative corrections, to include graphs with loops; this is, of course, a great simplification.

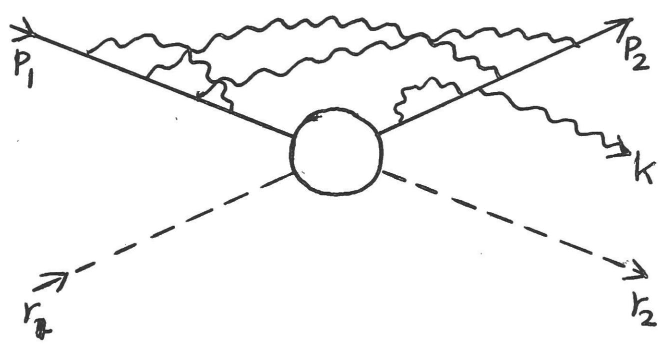


Fig. 16. Soft photon radiative correction to bremsstrahlung.

Naturally it is reasonable that only external lines are involved in the calculation of these soft photon radiative corrections. Physically this corresponds to the fact that only the features that are external to the major scattering event are of importance in the low energy limit. This is also the underlying reason for the structure independnet nature of the calculation.

It is well known that the infrared divergence may be summed to all orders in e into an exponential factor: $M = e^{B}\tilde{M}$ where $\tilde{M}$ is free of infrared divergences and B is the multiplicative factor calculated in the e^2 order. The non-analytic term (which is in $\tilde{M}$) in which we are interested, is essentially the next leading term. Although the calculation will evidently be a formidable combinatorial prbblem (since we have to join virtual lines in all possible orders), it is conceivable that, with the procedure outlined here, that the leading non-analytic terms may be summed to all orders, thus obtaining low energy theorems that are exact in e.

APPENDIX

Eq. (3.5) offers us a general procedure for evaluating the non-analytic threshold factors. Here we will derive it via the non-covariant Low equation. We shall, for definiteness, consider the radiative corrections to the process $p_0 + r_0 \rightarrow p + r + k$ discussed in Sections II 1b and III 3a. The vectorial amplitude (M_λ) is related to the matrix element of the current operators by:

$$M_\lambda(r, p, k; r_0, p_0) = -N_p^{-1} N_{p_0}^{-1} \int dx\, dy$$
$$e^{-ir\,x + i\,r_0\,y} < \vec{p} \,|\, T(j(x)\ J_\lambda(0)\ j(y))\,|\, \vec{p}_0 > \tag{A.1}$$

J_λ is the electromagnetic current operator and j's are the source currents for the initial and final neutral particles. There may be, in addition, equal-time commutator factors, which precisely compensate the non-covariant nature of the T-product. Since we are only interested in the singular threshold contributions of soft photon intermediate states, we can ignore these factors as they are not singular in the limit $\omega \rightarrow 0$.

First, we write out the time-ordered product:

$$\begin{aligned} T(j(x)\ J_\lambda(0)\, j(y)) = \ & \theta(x_0)\,\theta(-y_0)\, j(x) J_\lambda(0)\, j(y) \\ & + \theta(y_0)\,\theta(-x_0)\, j(y)\, J_\lambda(0)\, j(x) \\ & + \theta(-x_0)\,\theta(x_0 - y_0)\, J_\lambda(0)\, j(x)\, j(y) \\ & + \theta(-y_0)\,\theta(y_0 - x_0)\, J_\lambda(0)\, j(y)\, j(x) \\ & + \theta(x_0 - y_0)\,\theta(y_0)\, j(x)\, j(y)\, J_\lambda(0) \\ & + \theta(y_0 - x_0)\,\theta(x_0)\, j(y)\, j(x)\, J_\lambda(0) \end{aligned} \tag{A. 2}$$

It is not difficult to check that only the last four terms contain amplitudes having energy denominators that may vanish in the zero-frequency limit. Consider the contribution coming from the third term in Eq. (A. 2):

$$\sum_n \int dx\, dy\, e^{-ir\, x + ir_0 y}\, \theta(-x_0)\,\theta(x_0 - y_0)\, \langle \vec{p} \,|\, J_\lambda(0) \,|\, \vec{n} \rangle \langle \vec{n} | j(x) j(y) \,|\, \vec{p}_0 \rangle \tag{A. 3}$$

where $|\vec{n}\rangle$ denotes a general on-(mass)-shell intermediate state with total three-momentum $\vec{n}$ and energy E_n. By translational invariance and appropriate change of space-time variables, we have

$$\sum_n \int dx\, dy\, e^{-i(n-p-k)y+i(n-p_0-r_0)x}$$

$$\theta(-y_0)\,\theta(x_0)\langle\vec{p}|\,J_\lambda(0)\,|\vec{n}\rangle\langle\vec{n}|\,j(x)\,j(0)\,|\vec{p}_0\rangle$$

$$= \sum_n (2\pi)^3\,\delta(\vec{n}-\vec{p}-\vec{k})\,\frac{i\langle\vec{p}|J_\lambda(0)|\vec{n}\rangle\int dx\, e^{i(n-p_0-r_1)x}\,\theta(x_0)\langle\vec{n}|j(x)\,j(0)|\vec{p}_0\rangle}{\omega+E-E_n}$$

(A. 4)

The fourth term in Eq. (A. 2) may be written in a similar way, and when it is combined with Eq. (A. 4) the result is:

$$-(2\pi)^3 N_p^{-1} N_{p'}^{-1} \sum_n \delta(\vec{n}-\vec{p}-\vec{k})\,\frac{\langle\vec{p}|J_\lambda|\vec{n}\rangle\langle\vec{n},\vec{r}|M|\vec{p}_0,\vec{r}_0\rangle}{E_n-E-\omega} \qquad (A.\,5)$$

where

$$\langle\vec{n},\vec{r}|\,M\,|\vec{p}_0,\vec{r}_0\rangle \equiv -i\int dx\, e^{-irx}\big[\,\theta(-x_0)\langle\vec{n}|\,j(0)\,j(x)|\,\vec{p}_0\rangle$$

$$+\,\theta(x_0)\langle\vec{n}|\,j(x)\,j(0)\,|\,\vec{p}_0\rangle\, e^{i(E_n-E-\omega)x_0}\,\big]$$

For singular terms, the off-shell effects in the numerator may be neglected. Eq. (3. 5) then corresponds to the contribution by the state of a particle plus a soft photon, $\vec{n} = \vec{p}'+\vec{k}'$ and $E_n = E' + \omega'$ with $E' = (\vec{p}'^2+m^2)^{\frac{1}{2}}$, to the sum in Eq. (A. 5).

We note the following correspondences:

$$\sum_n \rightarrow \int d\vec{p}\,' \, d\vec{k}\,' \sum_{e'}$$

$$2E'(E' + \omega' - E - \omega) \rightarrow p'^2 + m^2$$

$$\langle \vec{p} \,|\, J_\lambda \,|\, \vec{p}\,', \vec{k}\,' (\vec{e}\,') \rangle \rightarrow (2\pi)^{-\frac{3}{2}} (2\omega')^{-\frac{1}{2}} N_p N_{p'} e'_\mu C^o_{\lambda\mu}(k, p; k', p')$$

$$\langle \vec{r}, \vec{p}\,', \vec{k}\,'(\vec{e}\,') \,|\, M \,|\, \vec{p}_0 \, \vec{r}_0 \rangle \rightarrow (2\pi)^{-\frac{3}{2}} (2\omega')^{-\frac{1}{2}} N_{p_0} N_{p'} e'_\nu M^o_\nu(k', r, p'; r_0, p_0)$$

$$\sum_{e'} e'_\mu \, e'_\nu \rightarrow \delta_{\mu\nu}$$

The contributions corresponding to the fifth and sixth terms in Eq. (A. 2) are just the crossed graphs.

FOOTNOTES

1. We use the metric: $p_\lambda = (\vec{p}, i\, p_0)$ and the normalization factors:

$$N_p = \begin{cases} \dfrac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\dfrac{1}{2\,p_0}} & \text{for bosons} \\ \dfrac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\dfrac{m}{p_0}} & \text{for fermions.} \end{cases}$$

2. Gauge invariance states that the amplitude $e_\lambda M_\lambda$ is not changed under the substitution: $e_\lambda \rightarrow e_\lambda + \alpha k_\lambda$.

3. Since we are only interested in the low energy behavior of the amplitude (to an accuracy of $O(\omega)$), it will be adequate for our purpose to write

$$k_\lambda S_\lambda = O(\omega^2)$$

$$k_\lambda R_\lambda = O(\omega^2)$$

instead of Eq. (2.6) and (2.7).

4. All the "graphs" we will draw in this paper have complete vertices, i.e. each graph will correspond to set of Feynman diagrams reflecting that fact that we are taking strong interactions to all orders. Accordingly, we will have various form factors.

5. To be more precise, let S'_λ be the pole term calculated directly from the external insertion diagram: $M_\lambda = S'_\lambda + R'_\lambda$. A part is then taken out from R'_λ, $S_\lambda = S'_\lambda + (R'_\lambda - R_\lambda)$ so that $k_\lambda S_\lambda = 0$. This can always be done because S'_λ includes all the singular contributions to the amplitude and because $k_\lambda M_\lambda = 0$.

6. F. E. Low, Phys. Rev. 110, 974 (1958).

7. Transverse gauge: $\vec{e}_\lambda = (e, 0)$ and $\vec{e}\,\vec{k} = 0$.

8. N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 233 (1954).

9. See, for example, S. L. Adler and R. F. Dashen, Current Algebras (W. A. Benjamin, Inc., New York, 1968).

10. This, however, is not true in the presence of an external electromagnetic or weak perturbation. The divergence condition on the amplitude will then depend on the commutator of the axial current and the perturbative Lagrangian. An example of such a low energy theorem for one soft pion is the theorem for the pion photoproduction derived by S. Fubini, G. Furlan and C. Rossetti, Nuovo Cimento 40, 1171 (1965).

11. S. L. Adler, Phys. Rev. 139, B1638 (1965).

12. Such low energy theorems for non-conserved currents have been discussed by S. L. Adler and Y. Dothan, Phys. Rev.

151, 1267 (1966).

13. Instead of using current conservation we invoke directly the requirement of gauge invariance (see footnote 2), since we do not wish to become involved in the problem of Schwinger terms, which will be discussed in Subsection 2b.

14. The generalization to the case of n soft photons is

$$M_{\lambda_1\lambda_2 \cdots \lambda_n} = S_{\lambda_1\lambda_2 \cdots \lambda_n} + O(\omega_1\omega_2 \ldots \omega_n) .$$

15. For an explicit computation of S_λ in the two-photon bremsstrahlung case, see L. S. Borwn and R. L. Goble, Phys. Rev. 173, 1505 (1968).

16. W. Thirring, Phil. Mag. 41, 1193 (1950).

17. F. E. Low, Phys. Rev. 96, 1428 (1954).

18. M. Gell-Mann and M. L. Goldberger, Phys. Rev. 96, 1433 (1954).

19. D. J. Almond, University of Oxford Preprint (1968).

20. A. Pais, Phys. Rev. Letters 19, 544 (1967), Nuovo Cimento 53, 433 (1968).

21. K. Bardakci and H. Pagels, Phys. Rev. 166, 1783 (1968).

22. V. Singh, Phys . Rev. Letters 19, 730 (1967).

23. M. A. B. Beg, Phys. Rev. Letter 17, 333 (1966).

24. See, for example, K. Johnson, Nucl. Phys. 25, 431 (1961).

25. For a detailed discussion of this conjecture, see S. L. Adler and R. F. Dashen (ref. 9).

26. S. L. Adler, Phys. Rev. 140, B 736 (1965); W. I. Weisberger, Phys. Rev. 143, 1302 (1966).

27. S. D. Drell and A. C. Hearn, Phys. Rev. Letters 16, 908 (1966). S. B. Gerasimov, Yadern, Fiz. 2, 598 (1965) [English translation Soviet J. Nucl. Phys. 2, 430 (1966)].

28. F. E. Low, Phys. Rev. 97, 1392 (1955) and reference 17.

29 V. Singh, ref. 22; and alternative derivation without making and expansion into infinite sum is given by J. S. Bell, Nuovo Cimento 52 A, 935 (1967).

30. H. D. I. Abarbanel and M. L. Goldberger, Phys. Rev. 165, 1594 (1968).

31. In the helicity formalism, that a photon has two helicity states is a direct consequences of masslessness and Lorentz invariance.

32. M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).

33. An alternative derivation *via* the Low equation is presented in the Appendix.

34. For an evaluation of similar types of integrals, see, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964)p. 173.

35. Similar results have been proved independently in renormalizable theories by S. R. Choudhury (private conversation).

36. L. D. Soloviev, Nucl. Phys. 64 , 657 (1965).

37. S. Fubini, G. Furlan and C. Rossetti, Nuovo Cimento 40, 1171, (1965). Also see footnote 10.

38. S. B. Gerasimov and L. D. Soloviev, Nucl. Phys. 74, 589 (1965).

39. S. M. Roy and V. Singh, Phys. Rev. Letters 21, 861 (1968).

40. T. P. Cheng, Phys. Rev. 176, 1674 (1968).

41. This result is known independently to K. Y. Lin, (private conversation).

42. F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

43. J.M. Jauch and F. Rohrlich, Helv. Phys. Acta. **27**, 613 (1954), and *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), Chapter 16.

44. This is just a convenient device to avoid involvement in any discussion of the renormalization problem which would only obscure the simple kinematical properties being presented here, (see footnote 45). The contribution of (a) as given in Eqs. (3.7) and (3.8) is identical to the result obtained by an appropriate differentiation of (the integrand of) the electromagnetic self-energy diagram "$\Sigma(p)$". After the virtual photon integration, it gives the infrared divergent factors of Z_2 (the wave function renormalization constant).

45. In quantum electrodynamics of the electron-photon system, infrared divergences are usually discussed together with the renormalization problems of the theory, since some of the renormalization constants are themselves infrared divergent. Thus graphs where the virtual photon line is not attached at both ends to external charged particles (hence non-infrared according to the criteria as stated above) often become infrared divergent after renormalization. Accordingly, the following question has been raised by Dr. S.R. Choudhury in

a private discussion: for a general scattering process, does renormalization introduce extra infrared divergent terms in the physical amplitudes ? that is, above and beyond those coming from graphs where the virtual photon line is attached at both ends to external charged particles. We note that this problem arises from the fact that the renormalization constants are usually defined at points involving particles on their mass-shell. If we had chosen different points (as we have the freedom to do) where all the charged particles in question are off-shell, no extra infrared divergences would be produced to begin with, thus eliminating the issue of the introduction and cancellation of such spurious divergences. In other words, renormalization can only shuffle infrared divergences from one part of the amplitude to another. No new factors are introduced in the full amplitude. Consequently, as far as soft photon radiative corrections are concerned, we can consistently ignore the problem of renormalization and still get the correct results.

46. P.I. Fomin, Zh. Eksperium. i. Teor. Fiz. 35, 707 (1958) [English translation: Soviet Physics JETP 35, 491 (1959)].

47. F. Corinaldesi and R. Jost, Helv. Phys. Acta 21, 183 (1948), L.M. Brown and R.P. Feynman, Phys. Rev. 85, 231 (1952). See also W. Heitler and S.T. Ma, Phil. Mag. 11, 651 (1949).

48. That in the perturbation calculation the results of the e^6 Compton cross section for spin 1/2 should be identical in the low-frequency limit to that for spin 0 target was pointed out as a puzzle by W. Heitler [The Quantum Theory of Radiation, 3rd edition (Oxford, 1954) p. 332] .

49. K.Y. Lin, Rockefeller University preprint, May, 1969.

End