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GROWTH WITH FLUCTUATIONS

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Notes: Center Discussion Papers are preliminary materials circulated to stimulate discussion and critical comments.

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ABSTRACT

In this paper we integrate fluctuations in the intensity and factor bias of innovation with a model of long run growth. We introduce two conceptual tools for the derivation of comparative dynamic theorems and implement these tools under specific assumptions on the behavior of the model. We conclude with a discussion of the integration of long run and short run macroeconomic phenomena.

INTRODUCTION:

Since the seminal contribution of Robert Solow [Solow (1956)], the neoclassical growth model has become a tradition. In his Nobel lecture in 1987, Prof. Solow called for an integration of long run growth theory with the short run theory of business cycles--a special type of instability, often found in the real world, where the durations of alternating prosperity and recession periods last considerably longer than the (quarterly) observable "random errors" [Solow (1988), page 311]. The need for integration of "long-term GNP growth and short-term cyclical fluctuations" has been noted even more recently by Assar Lindbeck [Lindbeck (1990), page 3]. Both Solow and Lindbeck call for an interpretation of business cycles as short run deviations from long run equilibrium. There appears to be a consensus that short run deviations ultimately stem from imperfections in the labor and/or financial markets. The long run real (non-monetary) model presented in this paper shows that business cycles can occur even when both these markets are functioning perfectly as long as the rhythm of the unfolding technological advance is not smooth.

In historical perspective, technology change has been the basic growth promotion force since the arrival of the modern growth epoch [Kuznets (1956)]. It has, moreover, long been recognized [Schumpeter (1939)] that innovation, associated with the exploration of the mysterious frontier of science and technology, occurs with an unpredictable rhythm. Fluctuations in the intensity and factor bias of innovations account for the business cycles dealt with in this paper. Our basic purpose is to integrate these fluctuations with long run growth.

We present a model of "growth with fluctuations" in which the instability of investment (dK/dt) and capital accumulation ($i=\eta_K$), induced by the irregularity of innovations, is propagated in a system of endogenous variables $[E(t)]$ that describe cyclical fluctuations along growth paths $[P^\Delta(t)]$. (The notation $\eta_x=(dx/dt)/x$ stands for the growth rate of $x(t)$. All time series are defined over a sufficiently long, but finite, time interval $t \in [0, T]$; \bar{x} will be referred to as the initial value of $x(t)$.)

Theorems dealing with business cycles require definitions of "prosperity," "recession" and "turning point" which are properties of the endogenous variables $E(t)$. In our paper, these notions are defined in terms of capital acceleration or deceleration as follows:

Definition: At any point in time $E(t)$ is in one of the following states:

- 0.1 a) State of prosperity: $\Omega^+ = \{t | \eta_i(t) > 0\}$
- b) State of recession: $\Omega^- = \{t | \eta_i(t) < 0\}$
- c) State of zero acceleration (or turning point): $\Omega^0 = \{t | \eta_i(t) = 0\}$

For a typical $i(t, \bar{i})$ path in Diagram 1b, the alternation of subphases of prosperity and recession are marked off by the turning points where $i(t)$ takes on a local maximum (downturn) or minimum (upturn) value. A business cycle is formed by consecutive phases of prosperity and recession. Thus, in our paper, the business cycle is formally defined in terms of the direction of change of $i(t)$. In the real world, Ω^0 is a set of isolated points such that a growing economy is in a state of prosperity or recession most of the time.

The main objective of this paper is the derivation of comparative dynamic theorems which show the impact of the fluctuations of the exogenous variables $X(t)$ on the cyclical and trendal behavior of all components of the endogenous $E(t)$ individually and/or collectively. For this purpose, we shall introduce the "natural rate" (Theorem Two) and the "potential value" (Theorem Three) which together constitute the major conceptual tools of advanced here. The "natural rate" is useful for understanding the impact of $X(t)$ on an individual endogenous variable $e(t)$ because it acts as a moving target towards

which an endogenous variable will always move. The "potential value" is useful for analyzing the behavior of all $e(t) \in E(t)$ as a system because it acts as a boundary for $E(t)$.

Section I describes our model and discusses dynamic determinism. Section II introduces comparative dynamics through the "natural rate" and "potential value" concepts. The general model of Section I contains many meaningful submodels when special assumptions are postulated for the exogenous variables. The conceptual tools of Section II will be applied to these submodels in Section III. In the concluding Section IV we suggest the direction of future research to integrate long run and short run macroeconomic phenomena.

Section I: THE GENERAL MODEL

In this section we introduce the model by formally defining the endogenous and exogenous variables relative to the production function. We then discuss the meaning of a "solution" to the model. Finally, we prove that the endogenous variables are dynamically determined when the exogenous variables and the initial rapidity of capital accumulation (\bar{i}) are specified.

We shall postulate a dynamic neoclassical production function $Q=f(K,L,t)^1$ with constant returns to scale (CRS). All essential variables of the model may be defined relative to the production function as follows:

1.1 a) Static Production Concepts:

- i) $w=f_L(K,L,t)$; $\pi=f_K(K,L,t)$ (wage and "interest rate"²)
- ii) $p=Q/L=f(K,L,t)/L$ (labor productivity)
 $d=Q/K=f(K,L,t)/K$ (capital productivity)
- iii) $\epsilon_{LL}=-\frac{\partial f_L}{\partial L}L/f_L$; $\epsilon_{KK}=-\frac{\partial f_K}{\partial K}K/f_K$ (laws of diminishing returns)
 $\epsilon=\epsilon_{KK} + \epsilon_{LL}$ ("inelasticity" of substitution)
- iv) $K^*=K/L$ (capital per head)

b) Technological Growth Promotion Forces:

- i) $J = (\partial Q / \partial t) / Q = f_t(K, L, t) / f(K, L, t)$ (innovation intensity)
- ii) $\phi_L = w/p$; $\phi_K = \pi/d$; $\phi_L + \phi_K = 1$ (factor shares)
- iii) $\phi_L = \epsilon_{KK} / \epsilon$; $\phi_K = \epsilon_{LL} / \epsilon$ (indices of non-substitutability)

c) Savings Pushed Growth Promotion Forces:

- i) $dK/dt = sQ$ (savings pushed investment)
- ii) $i = \eta_K = sd$ (Harrod-Domar equation)
- iii) $c = (1-s)p$ (per capita consumption)

d) Demographic Growth Promotion Forces:

$$\eta_L = r \quad (\text{population growth rate})$$

e) Rapidity and Welfare Indicators Growth Profile:

- i) Rapidity indicators: $(i, q) = (\eta_K, \eta_Q)$ (capital and GNP growth rates)
- ii) Welfare indicators: $(\eta_p, \eta_w, \eta_c, \eta_{K^*})$ (growth rate of p, w, c, K^*)
- iii) Growth Profile: $E = \{(d, \pi), (i, q), (\eta_p, \eta_w, \eta_c, \eta_{K^*})\}$

The static production concepts involve the familiar wage and interest rate, factor productivities, and per capita wealth, as well as ϵ which describes the severity of the law of diminishing returns. Our paper recognizes three basic types of exogenous "growth promotion forces" traceable to demography (i.e. the population growth rate $r = \eta_L$ in 1.1d), austerity (i.e. the saving rate s in 1.1c) and technology (1.1b).

In the technology category, we include two crucial dimensions of innovations which are identifiable from $Q = f(K, L, t)$: the intensity of innovation J (1.1bi) and the labor share ϕ_L (1.1bii) (a proxy for factor bias in the Hicksian sense, see 1.9 below). It may be shown (see Appendix) that under CRS, the factor shares ϕ_L and ϕ_K measure the fraction of the "inelasticity of substitution" (ϵ) accounted for by the other factor of production respectively (see 1.1biii). For example, an empirical measure of $\phi_L = 2/3$ would imply that, during the modern epoch, a lion's share (66%) of the

factor non-substitutability (ϵ) is accounted for by the law of diminishing returns to capital (ϵ_{KK}). An increase in the labor share is a proxy for increasing capital rigidity--a technological phenomenon.

The growth profile of endogenous variables (vector E in 1.1eiii) has eight components that define an "equilibrium system" of factor prices and quantities. Its components are classified into three subsets: capital efficiency indicators (i.e. the level of average (d) and marginal (π) capital productivity, where π also represents the interest rate), rapidity indicators (i.e. the GNP and capital growth rates), and welfare indicators (i.e. the growth rates of p, w, c, K^*). The value of the vector E through time represents a growth and fluctuations profile to be determined endogenously.

This choice of endogenous variables is perfectly natural in view of the long run constancy of the levels of d and π and the positive growth rates of $K, Q, p, w, c,$ and K^* since the arrival of the modern epoch [Kaldor (1963)]. Of the components in E, the capital growth rate $i = n_K$ takes on the most strategic role (see 0.1). Once $i(t)$ is determined endogenously, it is, in turn, propagated to determine the system of the other seven variables over time (see Theorem One (1.11)).

For the model presented above to be logically consistent we must define a "solution path." The exogenous rules of innovation over time ($J = [\partial f(K, L, t) / \partial t] / f(K, L, t) = \alpha(t)$ and $\phi_L = [\partial f(K, L, t) L / \partial L] / f(K, L, t) = \beta(t)$ in 1.2a) are not postulated as globally valid, but are valid only along a particular "solution path" as will be formally defined below.³

Let the rules of growth ($X(t) = \chi(t)$), the initial rapidity of capital accumulation (\bar{i}), and the initial "size" (\bar{K} and \bar{L}) be postulated as follows:

$$1.2 \text{ a) } X = (J, \phi_L, r, s) = [\alpha(t), \beta(t), \rho(t), \sigma(t)] = \chi(t)$$

$$\alpha(t) \geq 0, \quad 0 < \beta(t) < 1, \quad \rho(t) > 0, \quad 0 < \sigma(t) < 1 \quad \text{for } t \in [0, T]$$

$$\text{b) } i(0) = \bar{i} \quad (\text{take-off speed})$$

$$\text{c) } [K(0), L(0)] = (\bar{K}, \bar{L}) > 0 \quad (\text{initial "size"})$$

Except for the indicated inequalities, all seven parameter values (i.e. the four time series and three constants) can be specified arbitrarily. This is a generalization of the original Solow model which effectively ruled out fluctuations.

Equation 1.2 implies that the population growth path $L^\Delta(t)$ is exogenous (see 1.4b below), while the capital growth path $K^\Delta(t)$ is endogenous (see 1.4c below). The time path of $i^\Delta(t)$, given the take-off speed \bar{i} , will be denoted by

$$1.3) i^\Delta(t) = i^\Delta(t; \bar{i}) \text{ with } i^\Delta(0; \bar{i}) = \bar{i}$$

Once $i^\Delta(t)$ is determined, we can construct the factor endowment path $P^\Delta(t)$ to be referred to as a solution path for $i^\Delta(t)$.

$$1.4) a) P^\Delta(t) = [K^\Delta(t), L^\Delta(t)] \quad t \in [0, T] \text{ where}$$

$$b) L^\Delta(t) = \bar{L}m(t) \text{ where } m(t) = \exp\left[\int \rho(r) dr\right] \quad (\rho(r) \text{ in 1.2a, } \bar{L} \text{ in 1.2c)}$$

$$c) K^\Delta(t) = \bar{K}n(t) \text{ where } n(t) = \exp\left[\int i^\Delta(r) dr\right] \quad (\text{for } \bar{K} \text{ in 1.2c) satisfying}$$

$$d) \eta_L^\Delta(t) = \rho(t), \quad \eta_K^\Delta(t) = i^\Delta(t), \quad L^\Delta(0) = \bar{L}, \text{ and } K^\Delta(0) = \bar{K} \text{ by construction.}$$

As an example, a factor endowment path $P^\Delta(t)$ is represented by a curve defined parametrically in the positive quadrant of the input space along which growth takes place (Diagram 1a). Notice that the solution path satisfies the conditions in 1.4d by construction.

Let $\psi = \psi(K, L, t)$ be an arbitrary function of K , L and t . We can define its path value along the solution path by the following time series:

$$1.5) \psi^\Delta = \psi^\Delta(t) = \psi\left[K^\Delta(t), L^\Delta(t), t\right] \text{ for } [K^\Delta(t), L^\Delta(t)] \text{ defined in 1.4bc.}$$

Notice that the path value ψ^Δ is derived by restricting the domain of ψ to $P^\Delta(t)$. Thus, in the determination of the path value, much information provided by ψ "globally" is both redundant and irrelevant.

The endogenous vector E (1.1eiii) and the exogenous vector X (1.2a) both have path values E^Δ and X^Δ along the solution path $P^\Delta(t)$. "Solution" of our model (indeed any growth model where these assumptions are made) obviously involves the simultaneous determination of both a growth path $P^\Delta(t)$ [which initiates from (\bar{K}, \bar{L})] and a neoclassical production function $Q=f(K, L, t)$ that satisfy the specified rules of innovation according to the following definition:

Definition: A time series $i^\Delta(t)$ is a solution to 1.2 if $i^\Delta(0)=\bar{i}$ and if there exists a neoclassical production function $f(K, L, t)$ such that the path value X^Δ along the solution path $P^\Delta(t)$ (1.4) satisfies the condition

$$1.7 \text{ a) } X^\Delta(t) = \chi(t) \text{ (for } \chi(t) \text{ in 1.2a).}$$

The endogenous variable $E(t)$ propagated by $i^\Delta(t)$ is the path value E^Δ of E along the solution path $P^\Delta(t)$.

$$\text{b) } E(t) = E^\Delta(t) \text{ (for } P^\Delta(t) \text{ in 1.4)}$$

According to the definition, all that is required of $f(K, L, t)$ is that it provide the non-redundant information along a narrow band of open neighborhoods covering $P^\Delta(t)$ (see Diagram 1a). The following lemma assures the existence of a solution path.

Lemma One: For the set of parameter values $(\bar{K}, \bar{L}, \bar{i}, \alpha(t), \beta(t), \rho(t), \sigma(t))$ the factor endowment path,

$$1.8 \text{ a) } L^\Delta = \bar{L} \exp\left[\int \rho(t) dt\right]$$

$$\text{b) } K^\Delta = \bar{K} \exp\left[\int i^\Delta(t, \bar{i}) dt\right],$$

constructed from the solution $i^\Delta(t, \bar{i})$ of the differential equation in 1.14a, is a solution path when the neoclassical production function is constructed as

$$\text{c) } Q = k \exp[m(t)] K^{(1-\beta(t))} L^{\beta(t)} \text{ where}$$

$$\text{d) } m(t) = \int_0^t \{\alpha(z) + \beta'(z) \ln[K^\Delta(z)/L^\Delta(z)]\} dz$$

$$\text{e) } k = \bar{i} (\bar{K}/\bar{L})^{\bar{\beta}} / \bar{\sigma}$$

for $K^\Delta(t)/L^\Delta(t)$ defined in 1.8ab.

Proof: see Appendix.

This lemma is illustrated in the input space of Diagram 1a. For each initial size (e.g. $P^i(0)$ $i=1,2,3\dots$ represented by a superscript) and for each take-off speed ($\bar{i}, \bar{i}', \bar{i}'' \dots$ represented by a subscript), there is at

least one solution path $P^\Delta(t)$. Each initial size $P^i(0)$ may be thought of as a "knot" of a "bundle of fibers" (i.e. the individual growth paths) that take on certain shapes in the K-L space, depending on the take-off speed (\bar{i}). Along each "fiber" the growth profile $E(t)$ is determined (1.7b). Note that in each bundle, points of different fibers on the same vertical line occur at the same time because $L(t)$ is exogenous.

For a dynamic production function $Q=f(K,L,t)$, the labor using bias B_L in the Hicksian sense is defined as

$$1.9) B_L = H_L - J \text{ where } H_L = (\partial f_L / \partial t) / f_L \quad [\text{see Fei and Ranis (1964)}]$$

From 1.8c, it can be readily shown that the rate of increase of β is the labor using bias in the Hicksian sense.

$$1.10) B_L = \eta_\beta(t)$$

Notice that 1.8c is a neoclassical production function of the Cobb-Douglas type. As is shown in Diagram 1a, at time t' the total output $Q^\Delta(t')$, defined for the input point $P^\Delta(t') = [K^\Delta(t'), L^\Delta(t')]$ on the solution path, depends upon the cumulative results of all innovative activities before t' . Obviously, the state of the arts at t' is always a "heritage of history" as the economy moves along a particular expansion path where innovative knowledge of a specific intensity and bias accumulates.

The term $m(t)$ (in 1.8d) represents the cumulative result of the intensity α and factor bias η_β of innovation. [Note that factor bias $[\beta'(t)]$ will have a different impact on output expansion depending on the capital-labor ratio $[\ln(K^\Delta(z)/L^\Delta(z))]$ at the time of the innovation.] Innovation in the real world is obviously a historical accumulation of knowledge (i.e. intensity and factor bias) that can only take place along a specific factor endowment path through which a dynamic production function is cumulatively built up. Therefore, our modelling of innovation seems to be quite natural and realistic.

Having defined a "solution" to the model, the next step is to show that the endogenous growth profile (1.1eiii) is determined when $\chi(t)$ and \bar{i} are given. The variables in the growth profile can be written as functions of the capital growth rate and the exogenous variables as follows:

- 1.11 a) $d = i/s$ (Harrod-Domar equation)
- b) $\pi = f_K = \phi_K d = \phi_K i/s$ (by 1.11a)
- c) $q = \phi_L r + \phi_K i + J$ (Solow growth equation)
- d) $\eta_p = q - r = \phi_L r + \phi_K i + J - r = J + \phi_K(i - r)$ (by $p=Q/L$ and 1.11c)
- e) $\eta_w = \eta_{\phi_L} + \eta_p = \eta_{\phi_L} + \phi_K(i - r) + J$ (by $w=\phi_L p$ and 1.11d)
- f) $\eta_c = \eta_p + \eta_{1-s} = J + \phi_K(i - r) + \eta_{1-s}$ [by $c = p(1-s)$ and 1.11d]
- g) $\eta_{K^*} = i - r$ (by $K^* = K/L$)

Using 1.11, $E(t) = \{(d, \pi), (i, q), (\eta_p, \eta_w, \eta_c, \eta_{K^*})\}$ can be written in vector notation as a transformation of $i(t)$:

- 1.12 a) $E(t) = A(t)i(t) + B(t)$ where
- b) $A(t) = \{(1/s, \phi_K/s), (1, \phi_K), (\phi_K, \phi_K, \phi_K, 1)\} > 0$
- c) $B(t) = \{(0, 0), (0, \phi_L G), (\phi_L G - r, \eta_{\phi_L} + \phi_L G - r, \phi_L G - r + \eta_{1-s}, -r)\}$
- d) $G = U + r$ and $U = J/\phi_L$

We shall refer to G as a "growth promotion factor" that has demographic (r) and technological (U) components. Since $A(t)$ and $B(t)$ are determined by the exogenous $X(t)$, these equations imply that the time paths of all indicators in $E(t)$ are determined endogenously when $i(t)$ is determined as the solution to the following differential equation in i :

$$1.13) \quad \eta_i = \phi_L(G + \eta_s/\phi_L - i)$$

$$\text{Proof: } i = sd \rightarrow \eta_i = \eta_d + \eta_s = \eta_Q - i + \eta_s = J + \phi_K i + \phi_L r - i + \eta_s \text{ (by 1.11c)} = J + \phi_L(r - i) + \eta_s = \phi_L(J/\phi_L + r - i) + \eta_s$$

Q.E.D.

Note that 1.13 is valid globally. When restricted to the factor endowment path $P^\Delta(t)$ (1.4), we have the differential equation in 1.14a, which implies

the solution in 1.14b. Equation 1.14c follows from 1.12. Thus, we have shown dynamic determinism in our model as is stated formally in the following theorem:

Theorem One: For a specific $\chi(t)$, every solution $i^\Delta(t)$ satisfies the differential equation:

1.14 a) i) $\eta_i^\Delta = \beta(t)[G^\Delta + \eta_\sigma(t)/\beta(t) - i^\Delta]$ where

ii) $G^\Delta = \alpha(t)/\beta(t) + \rho(t)$

and hence $i^\Delta(t)$ is uniquely determined

b) $i^\Delta(t) = i^\Delta(t, \bar{i})$ where $i^\Delta(0) = \bar{i}$

when the take-off speed \bar{i} is specified. The endogenous $E(t)$ is a linear transformation of $i^\Delta(t; \bar{i})$

c) i) $E(t) = A(t)i^\Delta(t, \bar{i}) + B(t)$ where

ii) $A(t) = \{[1/\sigma, (1-\beta)/\sigma], [1, 1-\beta], [1-\beta, 1-\beta, 1-\beta, 1]\} > 0$

iii) $B(t) = [(0, 0), (0, \beta G^\Delta), (\omega^\Delta, \eta_\beta^\Delta + \omega^\Delta, \omega^\Delta + \eta_{1-\sigma}^\Delta, -\rho)]$

iv) $\omega^\Delta = \beta G^\Delta - \rho$

where the values of $B(t)$ and the strictly positive $A(t)$ are determined by the values at time t of

d) $\chi^0(t) = [\alpha(t), \beta(t), \rho(t), \sigma(t), \eta_\beta(t), \eta_\sigma(t)]$

which is derived from the exogenous $\chi(t)$.

Thus the capital accumulation path $i(t)$ is determined first in the causal order and is then propagated by the linear transformation in 1.14c to determine $E(t)$. Note that $\chi^0(t)$ contains the growth rate of labor using bias η_β and the rate of increase of savings η_σ which are essential growth relevant characteristics deduced from the exogenous $\chi(t)$.

In our paper, the dynamic determination of $E(t)$ has certain properties which can be proved formally. It is intuitively clear that, in a neoclassical world with CRS, the initial size is irrelevant for the determination of $E(t)$. Indeed, the following corollary of Theorem One states that "fibers" in different "bundles" will have the same $E(t)$ when the take-off speed is the same.

Corollary One: The endogenous $E(t)$ is uniquely determined by the take-off speed

$$1.15) E^\Delta(t; \bar{i}, \bar{K}, \bar{L}) = E^\Delta(t; \bar{i}, \bar{K}', \bar{L}')$$

independently of the initial size, i.e. $(\bar{K}, \bar{L}) \neq (\bar{K}', \bar{L}')$.

Thus the initial size of the economy is irrelevant for a theory of growth with fluctuations which concentrates on $E(t)$. Corollary One implies that all "fibers" with the same \bar{i} are merely a rigid spatial translation of each other (when K and L are measured on a logarithmic scale, see 1.4bc). The economy will always move on a particular path (fiber) with a particular size. However, as an implication of this corollary, the path need not be specified explicitly.

The corollary implies, moreover, that in each "bundle" of Diagram 1a precisely one fiber corresponds to a specific take-off speed (\bar{i}). The set of growth profiles are \bar{i} -decisive as stated in the following corollary.

Corollary Two: The set $\Sigma = \{E(t)\}$ is \bar{i} -decisive as every component of $E(t)$ is indexed by \bar{i}

1.16 a) $\Sigma = \{E(t)\} = \{E^\Delta(t; \bar{i})\}$ where $E^\Delta(t)$ has eight ordered components;

b) for any pair $\{E^\Delta(t; \bar{i}), E^\Delta(t; \bar{i}')\} \in \Sigma$ we have

$$i^\Delta(t; \bar{i}) > i^\Delta(t; \bar{i}') \text{ for all } t \in [0, T] \text{ if } \bar{i} > \bar{i}'$$

i.e. the capital growth rate is perpetually higher at any t when the take-off speed is higher.

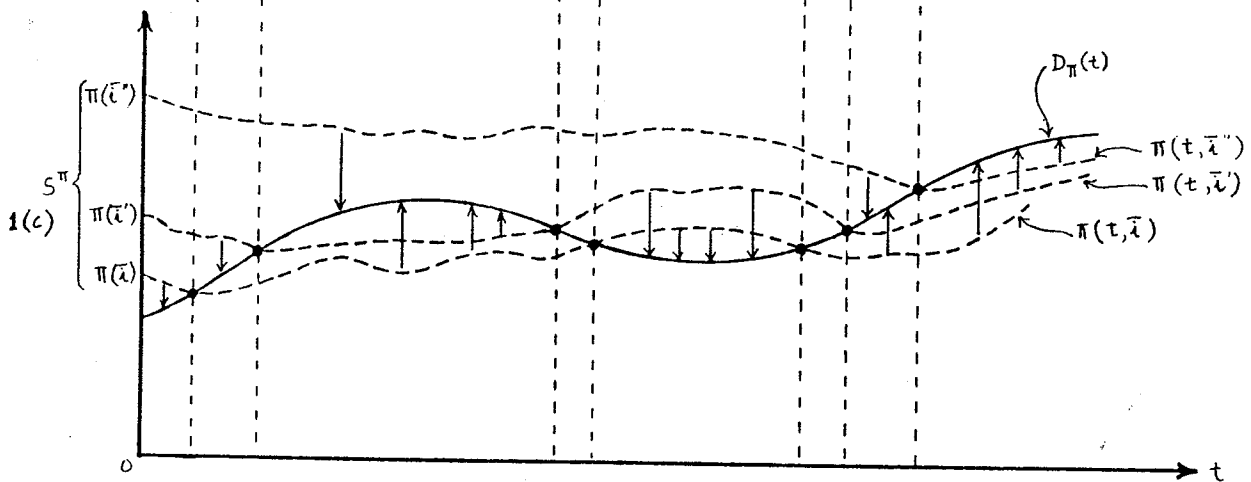
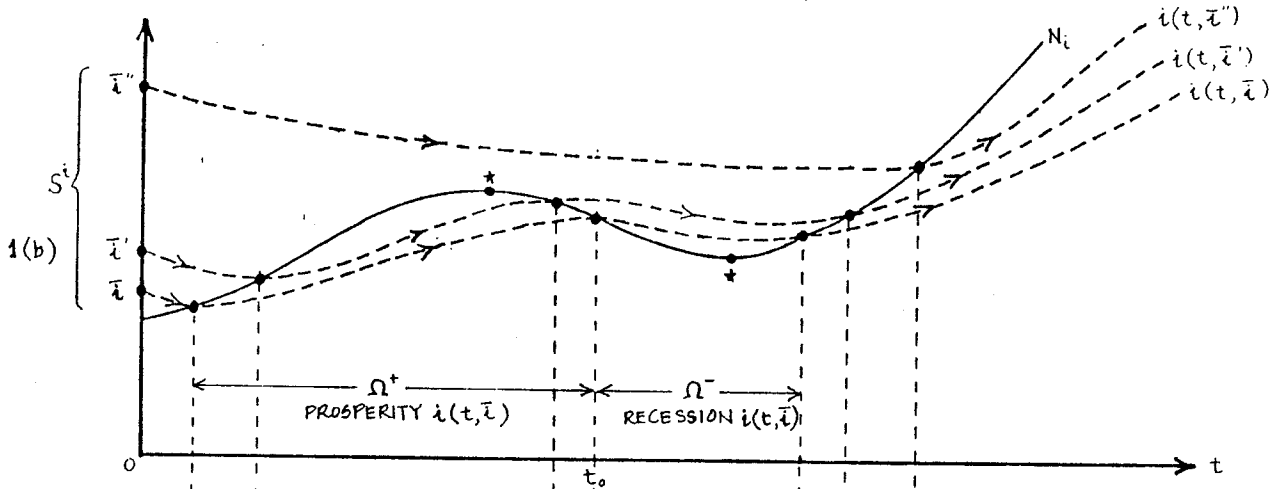
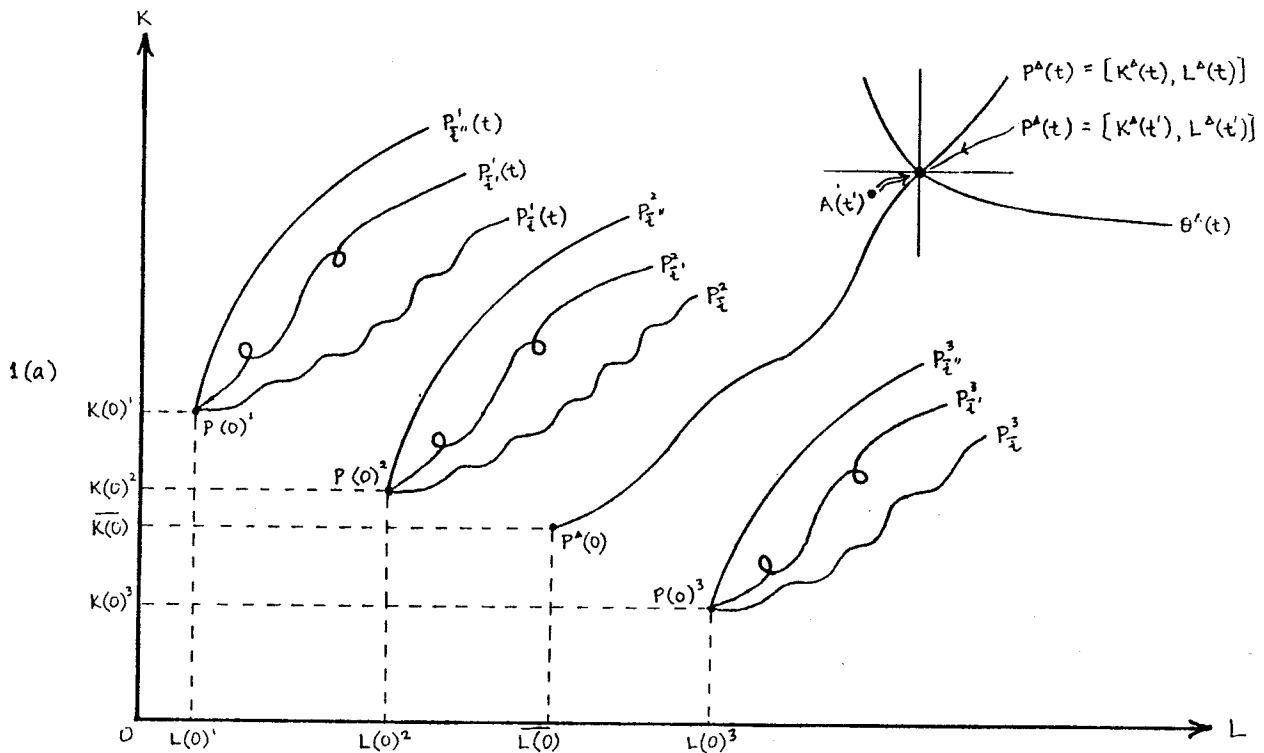
Every $E(t)$ is indexed by \bar{i} which determines a unique solution path along which $E^\Delta(t) = E(t)$. Thus, in a neoclassical world, a theory of growth and fluctuation is intrinsically size-neutral (Corollary One) and \bar{i} -decisive (Corollary Two). In such a world, the elementary building blocks that we will be working with in this paper are the following set of indicator paths:

1.17 a) $\{(s^d, s^\pi), (s^i, s^q), (s^{\eta p}, s^{\eta w}, s^{\eta c}, s^{\eta k^*})\}$

b) $S^e = \{e(t, \bar{i}) \text{ for all real } \bar{i}\}$ for $e \in E$

c) $\Sigma = \Pi S^e$ [product over $e \in E(t)$]

DIAGRAM 1



Notice that every indicator in every S^e is indexed by a unique take-off speed. The set of interest paths S^π (paths of capital growth rate S^i) are represented by the dotted curves in Diagram 1c (1b). Two i -paths (Diagram 1b) never cross each other [i.e. $i(t, \bar{i})$ dominates $i(t, \bar{i}')$ over time when the take-off speed is higher ($\bar{i} > \bar{i}'$)]. The set Σ is a subset of the Cartesian product space in 1.17c.

Note that only the current information of $\chi^0(t)$ is required to construct $A(t)$ and $B(t)$ at time t for the propagation of $i^\Delta(t)$ in 1.14c. This is an important property which will be defined formally in 2.3. The following corollary states that the set Σ (i.e. the endogenous growth profiles in 1.16a) are linearly ordered by domination according to \bar{i} (where the inequality sign stands for vector domination in 1.18a).

Corollary Three: In Σ , $E(t; \bar{i})$ dominates $E(t; \bar{i}')$ over time when the take-off speed is higher, i.e.

1.18 a) $E^\Delta(t; \bar{i}) > E^\Delta(t; \bar{i}')$ for all $t \in [0, T]$ if $\bar{i} > \bar{i}'$

which implies that in S^e , $e(t; \bar{i})$ dominates $e(t; \bar{i}')$ over time when the take-off speed is higher, i.e.

b) $e(t; \bar{i}) > e(t; \bar{i}')$ for all $t \in [0, T]$ if $\bar{i} > \bar{i}'$

Proof: follows from 1.16c and $A(t) > 0$ in 1.14c. Q.E.D.

In Diagram 1c, the dotted interest paths in S^π do not cross as a higher curve corresponds to a higher take-off speed. If any component of $E(t)$ is higher than the corresponding component of $E'(t)$ at any time, then each component of $E(t)$ is higher than the corresponding component of $E'(t)$ all the time.

In summary, the theory of growth with fluctuations presented in this paper is size-neutral (Corollary One) and \bar{i} -decisive (Corollary Two), with the growth profiles linearly ordered by domination according to \bar{i} (Corollary Three).

This first section has described the model, defining the exogenous $X(t)$ and endogenous $E(t)$ variables. The logical consistency of the model has been

shown through the definition of a solution (1.7) and the proof its existence (Lemma One). We have also shown that the endogenous variables are dynamically determined when $\chi(t)$ and \bar{i} are specified (Theorem One). In the next section we will introduce two conceptual tools appropriate for comparative dynamic analysis in this model.

Section II: CONCEPTUAL TOOLS

It is our intent to present comparative dynamic theorems for the analysis of the impact of fluctuations of the exogenous $\chi(t)$ on the cyclical and trendal behavior of the components of the endogenous $E(t)$. For this purpose, we develop two conceptual tools of analysis. The "natural rate" acts as a moving target towards which an endogenous variable will always move. The "potential value" acts as a ceiling or floor for the entire system of endogenous variables. In this section we will formally present the "natural rate" and the "threshold value" which will be used for comparative dynamic analysis in Section III.

To begin with, a natural rate N_e is a time series defined for a particular $e(t) \in E(t)$ which helps us to predict the behavior of all e in S^e . We can axiomatically define a natural rate N_e for every indicator $e \in E$, according to the following definition:

Definition: For each $e \in E$, a natural rate N_e of e is a time series satisfying

$$2.1) \eta_e \begin{matrix} \geq \\ < \end{matrix} 0 \longleftrightarrow N_e \begin{matrix} \geq \\ < \end{matrix} e$$

for all $e \in S^e$, where the arrow means "if and only if" applied to $>$, $=$ and $<$ separately.

For example, N_π is the natural rate of interest and N_i is the natural rate of the capital growth rate. In each case, N_e acts like a moving target for e . This can be explained with the aid of Diagram 1b where the solid curve N_i is a natural rate of i . A dotted time path $i(t)$ lies below (above) N_i during prosperity (recession). The market rate i always "chases after" N_i as a moving target. The turning points of $i(t)$ always fall on N_i because an upturn

(downturn) of $i(t)$ can only occur when N_i is rising (falling). Thus an upturn (or downturn) of $i(t)$ will always follow that of the N_i (indicated by a *) with a time lag. Hence, the space of $i(t)$ is partitioned into two regions by N_i : $i(t)$ will fall (rise) when it lies above (below) N_i . A similar economic interpretation can be given to N_i , N_π , and N_d as follows:

2.2) At any time t' , the economy is in a state of prosperity [recession] if and only if

$$a) i(t') < N_i(t') \quad [i(t') > N_i(t')].$$

The market rate of interest will increase [decrease] if and only if

$$b) \pi(t') < N_\pi(t') \quad [\pi(t') > N_\pi(t')].$$

The capital efficiency will increase [decrease] if and only if

$$c) d(t') < N_d(t') \quad [d(t') > N_d(t')].$$

The natural rates are determined by the exogenous parameters $\chi(t)$ (see Theorem Two, below). The following definition of "concurrent determination" applies to the determination of the natural rates:

Definition: A time series $\theta(t)$ is determined concurrently by $\chi^0(t)$ if there exists a function F^θ of real arguments such that, for each t' ,

$$2.3) \theta(t') = F^\theta[\alpha(t'), \beta(t'), \rho(t'), \sigma(t'), \eta_\beta(t'), \eta_\sigma(t')] = F^\theta[\chi^0(t')] \\ \text{for } \chi^0 \text{ in 1.14d.}$$

Notice that $i(t; \bar{i})$ (1.14b) is not determined by $\chi(t)$ concurrently because $i(t'; \bar{i})$ is affected, cumulatively, by all values of $i(t)$ before t' . We have the following theorem that states the existence and the uniqueness of the triplet of (axiomatically defined) natural rates N_i , N_π , and N_d :

Theorem Two: There exist natural rates N_i , N_π and N_d implied by

$$2.4) a) \eta_i = \phi_L(N_i - i) \text{ where } N_i = G + \eta_s/\phi_L$$

$$b) \eta_d = s\phi_L(N_d - d) \text{ where } N_d = G/s$$

$$c) \eta_\pi = (s\phi_L/\phi_K)(N_\pi - \pi) \text{ where } N_\pi = \phi_K G/s - \eta\phi_L/s$$

since, for $e=i, d, \pi$.

$$d) \eta_e \begin{matrix} > \\ \approx \\ < \end{matrix} 0 \longleftrightarrow N_e \begin{matrix} > \\ \approx \\ < \end{matrix} e$$

because ϕ_L , ϕ_K , and s are strictly positive.

Proof: a) follows from 1.13

b) follows from $i=sd$

c) follows from substituting $d=\pi/\phi_K$ and $\eta_d=\eta_\pi - \eta\phi_K$ into b, using $(\phi_K/\phi_L)\eta\phi_K = -\eta\phi_L$.

The uniqueness of the natural rates is easy to prove. Q.E.D.

The triplet of natural rates (interpreted in 2.3) is reproduced in column two of Table One. Notice that these natural rates, written as functions of K, L, t , become time series when their path values are derived by substitution of $\chi(t)$. From now on, we will interpret the natural rates (and the potential values in the next section) as path values. (The notation for χ (1.2a) and χ^0 (1.14d) will be dispensed with in Table One.)

Table One: Natural Rates, Potential Values and Stationary Values

	I	II	III
	<u>POTENTIAL VALUES</u>	<u>NATURAL RATES</u>	<u>STATIONARY VALUES</u>
Capital Productivity Indicators:			
d	$D_d = (G + \eta_s/\phi_L)/s$	$N_d = G/s$	$P_d = \bar{G}/s$
π	$D_\pi = \phi_K(G + \eta_s/\phi_L)/s$	$N_\pi = \phi_K G/s - \eta\phi_L/s$	$P_\pi = \bar{G}\phi_K/s$
Rapidness Indicators:			
i	$D_i = G + \eta_s/\phi_L$	$N_i = G + \eta_s/\phi_L$	$P_i = \bar{G}$
q	$D_q = G + (\phi_K/\phi_L)\eta_s$		$P_q = \bar{G}$
Welfare Indicators:			
η_p	$D_{\eta_p} = U + (\phi_K/\phi_L)\eta_s$		$P_{\eta_p} = \bar{U}$
η_w	$D_{\eta_w} = U + (\phi_K/\phi_L)\eta_s + \eta\phi_L$		$P_{\eta_w} = \bar{U}$
η_c	$D_{\eta_c} = U + [(\phi_K/\phi_L) - s/(1-s)]\eta_s$		$P_{\eta_c} = \bar{U}$
η_{K^*}	$D_{\eta_{K^*}} = U + \eta_s/\phi_L$		$P_{\eta_{K^*}} = \bar{U}$

where $G = J/\phi_L + r$ and $U = J/\phi_L$

Notice that, in all cases, the path values of N_d , N_π and N_i (2.4) are determined concurrently by $\chi^0(t)$ (1.14d). The concurrent determination of the natural rates by $\chi^0(t)$ is essential for comparative dynamic theorems which focus on the impact of a change in any component of $X(t)$ on any $N_e(t)$ (and inferentially on the direction of change of the market rate $e(t)$). An increase in G through time (as caused by an increase of J , r and/or a decrease of capital rigidity $\phi_L = \epsilon_{KK}/\epsilon$) may be thought of as an enhanced "prosperity orientation" of the exogenous growth promotion factor G that will lead to higher values of N_i , N_π , and N_d directly and will cause i , π , and d to increase, with time lags, indirectly.

There are two distinct meanings attached to an "austerity orientation:" a high value of s or an increasing value of s (i.e. $\eta_s > 0$). Other parameters held constant, the following theorem concerns the impact of an "austerity orientation."

2.5 a) N_i will be raised by a high rate of increase of s (high $\eta_s > 0$).

b) N_d will be depressed by a high level of s .

Equation 2.5b states that a high saving rate will decrease capital efficiency because of the law of diminishing returns to capital. However, it is a high rate of increase of s which will lead to "prosperity" (2.5a, see Section III, below). The natural rate N_π is important because it acts as a moving target for π and can therefore be used in theorems of this sort to analyze the impact of $\chi(t)$ on $\pi(t)$.

In economic doctrine, the theory of the real rate of interest π has always been a controversial issue. In the dynamic loanable funds theory advanced here, the real interest theory is an integral part of an equilibrium system $E(t)$ of growth with fluctuations. Note that all parameter values affect $N_\pi(t)$ which is "chased" as a target by $\pi(t)$. At any point in time, other things held constant, we see that the natural rate of interest ($N_\pi = \phi_K G/s - \eta \phi_L/s$) will be higher when

2.6 a) G is higher (growth promotion has a prosperity orientation)

b) $\phi_L = \epsilon_{KK} / \epsilon$ is lower (law of diminishing returns to K is less pronounced)

c) $\eta\phi_L < 0$ (innovation carries a capital using bias in the Hicksian sense)

d) s is low (under the assumption $G > \eta\phi_L / (1 - \phi_L)$ the capital

using bias of technological change does not overwhelm the positive term G .)

Note that 2.6d represents the characteristic position of the loanable funds theory which states that the real rate of interest will be suppressed by a high saving capacity--provided innovations are not biased too much in a capital using direction (i.e. a positive $\eta\phi_L$ is not too large) at the time of prosperity. Because the market rate of interest π moves toward N_π as a target, we expect the impacts discussed in 2.6 to happen to π with a time lag. (Possible exceptions to this rule occur in the presence of short run monetary factors (see concluding section)).

While the natural rate is useful for comparative dynamic analysis of the effects of fluctuations of $\chi(t)$ on $e(t)$, the potential value can be used to analyze the behavior of $E(t)$ as a set of variables. For the set of growth profiles Σ , we can define a potential profile $D(t)$, axiomatically, as follows:

Definition: A vector of time series $D(t) = \{(D_d, D_\pi), (D_i, D_q), (D_{\eta_p}, D_{\eta_w}, D_{\eta_c}, D_{\eta_{K^*}})\}$ is a potential profile of Σ if at any time t' and for all $E(t') \in \Sigma$,

2.7 a) $E(t')$ in prosperity -----> $E(t') < D(t')$ [i.e. $D(t')$ dominates $E(t')$]

b) $E(t')$ in recession -----> $E(t') > D(t')$ [i.e. $E(t')$ dominates $D(t')$]

c) $E(t')$ at turning point -----> $E(t') = D(t')$

For every $e \in E$, D_e will be referred to as a potential value of e .

From the definition of potential value, it follows that the potential capital growth rate $D_i(t)$ is the same as the natural capital growth rate $N_i(t)$.

2.8) $D_i = N_i$

The concept of potential values can be explained with the aid of the solid curve in Diagram 1c which represents the potential interest rate $D_{\pi}(t)$. During a time of prosperity (depression), the market rate of interest lies below (above) the potential interest curve. The market rate equals the potential rate only when $E(t)$ reaches a turning point (0.1c). The potential profile $D(t)$ describes the "cyclical thresholds" of all components of $E(t)$.

In Diagram 1bc, at time t_0 , the natural rate takes on a value $N_i(t_0)$. Since the set S^i is linearly ordered by domination (Corollary Three), there is precisely one take-off speed \bar{i}_0 for which $i^{\Delta}(t; \bar{i}_0)$ will reach a turning point at t_0 [i.e. $i^{\Delta}(t_0; \bar{i}_0) = N_i(t_0)$]. Let the potential value $D(t)$ at the time t_0 be defined as

2.9 a) $D(t_0) = E(t_0)$ where the $i(t)$ component of $E(t_0)$ satisfies

$$b) i(t_0; \bar{i}_0) = N_i(t_0)$$

Thus $D(t)$ is unambiguously defined for all $t \in [0, T]$. The construction of $D(t)$ in 2.9 implies that $D(t)$ is the growth profile $E(t)$ when the capital component of $E(t)$ takes on the natural value at t . This, in conjunction with Corollary Three (i.e. the linear ranking of all $E(t) \in \Sigma$), in turn implies Theorem Three:

Theorem Three: The potential value $D(t)$ is a linear transformation of the natural rate $N_i(t)$

$$2.10) D(t) = A(t)N_i(t) + B(t)$$

where $A(t)$ and $B(t)$ are defined in 1.12c.

Notice that $D(t)$ is determined concurrently by $\chi(t)$ because $A(t)$, $B(t)$ and $N_i(t)$ all have this property. The potential values of all components of $D(t)$ are indicated in column I of Table One. These values are obtained by a routine calculation of the linear transformations in 2.10.

The potential value D_e links the behavior of every component of $E(t)$ to the key component $i(t)$ cyclically through $N_i(t)$. We shall show that this linkage implies a particular thesis of business cycles as propagated by exogenous forces. The business cycle is formed of coherent fluctuations of the system of macroeconomic variables $E(t)$.

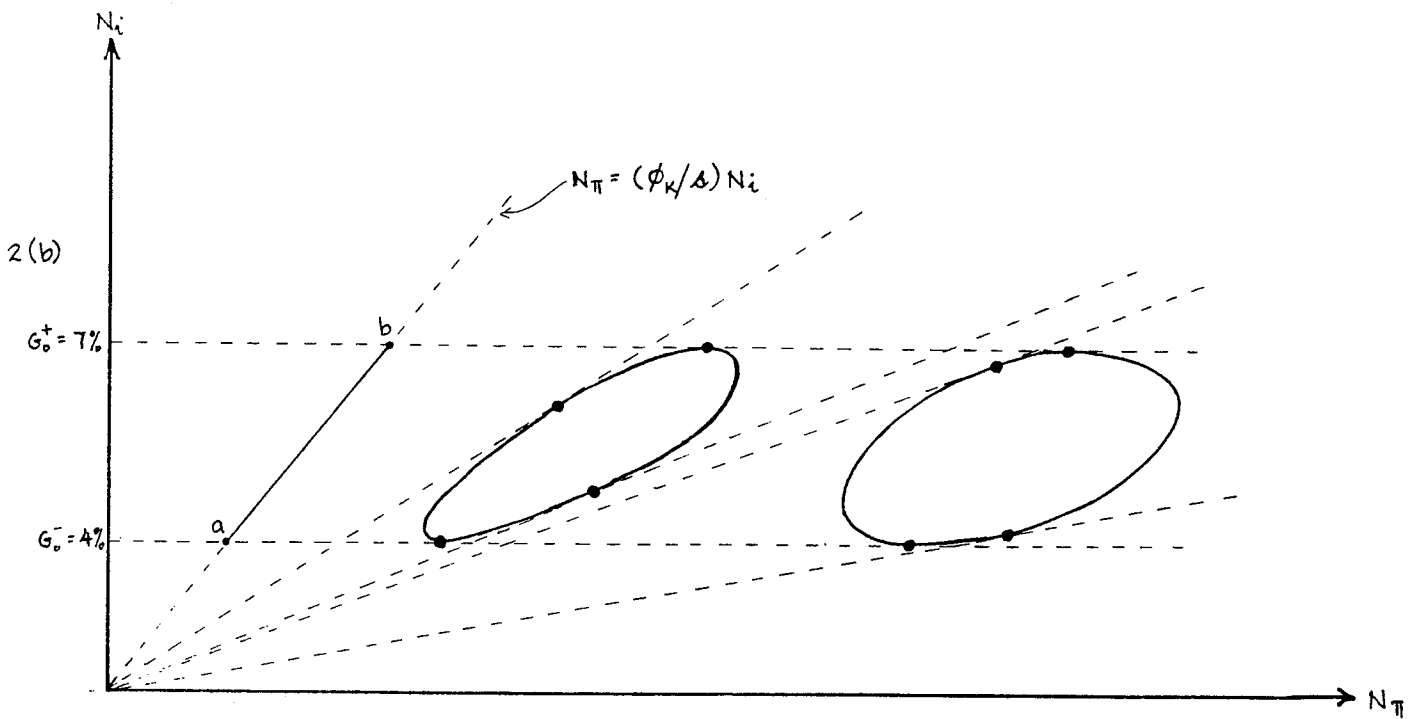
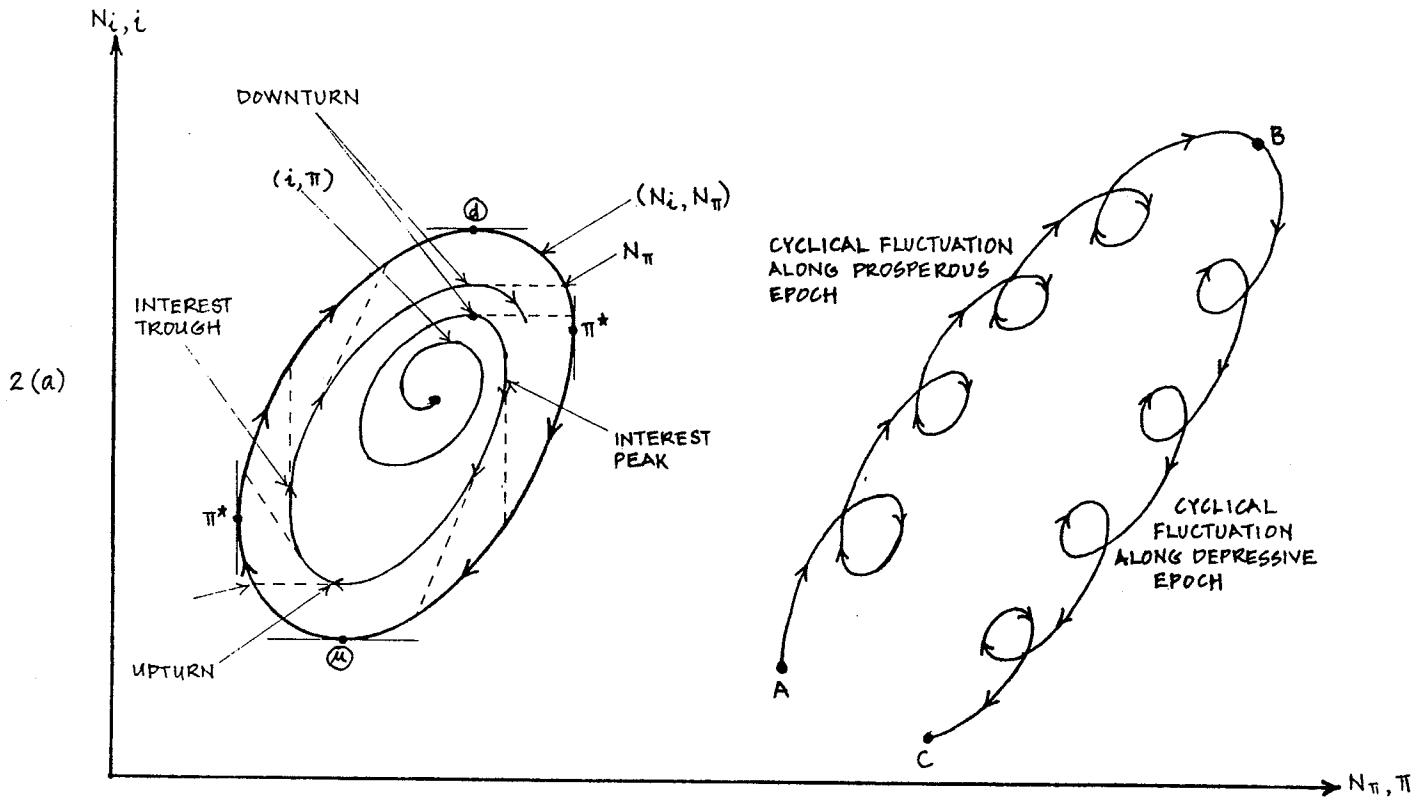
Notice that $D(t)$ is parametrically defined in the eight-dimension indicator space when t increases from 0 to T . The definition of $D(t)$ implies that prosperity (depression) will always occur when the potential profile $D(t)$ dominates (is dominated by) the observable market profile $E(t)$. Any variable in $E(t)$ takes on its potential value if and only if all variables take on their respective potential values. Therefore, we may consider the exogenously and concurrently determined $D(t)$ as a causal system that generates growth and fluctuations in $E(t)$ such that $e(t)$ will always move to "realize the potential" $D(t)$.

The relation between the natural rates N_e and the potential rates D_e may be explained with the aid of Diagram 2 in which the loop stands for the parametrically defined locus of points (N_i, N_π) , while the spiral inside the loop stands for the points (i, π) as t increases from 0 to T . As an analogy, imagine that a moth which flies perpetually around the loop clockwise is being chased by a bird which always flies toward the moth along the straight dotted lines.

The vertical location of the spiral (bird) contains phases of prosperity and recession marked off by the upturns or downturns of $i(t)$ [always lagging behind that of the moth, indicated by u and d]. The cyclic behavior of the interest rate (represented by the horizontal position of the spiral) follows a pattern that takes on a trough π_* (peak π^*) value during prosperity (depression) as it chases after the natural rate of interest. The market rate of interest achieves its potential value ($\pi = D_\pi$) at the turning point of the business cycle when the bird reaches its maximum or minimum height on the spiral.

In the above example, the loop represents the causation for a case of "pure fluctuation." The right-hand side of the diagram shows the causation for the case of a prosperous growth epoch (from A to B) to be followed by a depressive growth epoch (from B to C). In all cases, a unique profile of growth and fluctuations $E(t)$ in eight-dimensional space is determined when a

DIAGRAM 2



particular take-off speed \bar{i} is specified in 1.2a to determine an initial $E(0)$.

In the history of macroeconomics, few concepts can rival the importance of the Wicksellian natural rate of interest N_π (see further discussion in the concluding section). The basic notion of N_π is the restoration of equilibrium when the market rate deviates from N_π , an intrinsically dynamic idea. Notice that the abstract targeting property of N_π is shared by N_i and N_d . In fact, N_i actually has a higher causal order of determination than N_π or N_d (see order of proof 2.4). This is hardly a surprise because the interest rate π is only a price that "accommodates" the process of real capital accumulation during which $i = \eta_K$ always chases after its own target N_i . The concept of the natural rate of interest N_π owes its very existence to N_i .

The conceptual tools N_e and D_e presented in this section will be used in the following sections for comparative dynamic analysis under special assumptions regarding the behavior of the exogenous variables.

Section III: COMPARATIVE DYNAMIC THEOREMS UNDER SPECIAL ASSUMPTIONS ON $X(t)$

In the preceding sections, $X = (J, \phi_L, r, s)$ can be postulated arbitrarily to fit empirical reality. For analytical purposes, special convergent assumptions will be postulated for the components of X in this section.

We will use the notation $H(e_1, e_2, \dots, e_n)$ to denote the fact that the set e_i fluctuates harmonically through time, i.e.

$$3.1) \eta_{e_i} \begin{matrix} \geq \\ < \end{matrix} 0 \longleftrightarrow \eta_{e_j} \begin{matrix} \geq \\ < \end{matrix} 0$$

We have defined the natural rates only for the triplet (i, π, d) , while a potential value D_e is defined for all $e \in E$ (Section II). The following lemma follows trivially from the definitions of N_e and D_e .

$$3.2) N_e = D_e \text{ if and only if } H(i, e) \text{ for all } t \in [0, T]$$

An indicator e that moves harmonically with i may be referred to as procyclic because it increases (decreases) during prosperity (recession). Equation 3.2 states that a natural rate N_e can be defined (and is equal to the potential value D_e) for all procyclic variables. Referring to the linear transformation in 1.12, the following statement is obvious:

3.3) A sufficient condition for $H(i,e)$ [i.e. for e to be procyclic] is that the e -th component of $A(t)$ and $B(t)$ (in 1.12) are constant.

The general model contains many submodels when special assumptions of the following type are postulated for a component $X(t)$:

3.4 a) $x(t) \rightarrow \bar{x}$ (convergence)

b) $x(t) \rightarrow \bar{x}$ and $\eta_x \rightarrow 0$ (strong convergence)

c) $x(t) = \bar{x}$ (constancy)

These assumptions, though unrealistic, are useful analytically for the study of comparative stationary states defined by a convergent $E(t)$ as follows:

3.5) $E(t) \rightarrow \bar{E}$ where \bar{E} is a constant vector.

The stationary state of this model implies constancy of (d,π) , while all other indicators can grow at positive constant rates which is consistent with the long run "stylized facts" recognized by Kaldor.

The original Solow model began a tradition which stresses the notion of the stationary state for analytical purposes. A necessary condition for the existence of the stationary state is the convergence of the natural rates N_e for all $e \in E$. Theorem Four establishes a set of sufficient conditions for the convergence of N_i and the existence of stationary states:

Theorem Four: When $X(t)$ converges to $\bar{X} = (\bar{J}, \bar{\phi}_L, \bar{r}, \bar{s})$ and when s strongly converges ($\eta_s \rightarrow 0$), then N_i converges, i.e.

3.6 a) $N_i \rightarrow \bar{P}_i = \bar{J} / \bar{\phi}_L + \bar{r}$ (stationary value of i).

If, in addition, ϕ_L converges strongly, then $E(t)$ converges to a stationary state

$$b) E(t) \text{ -----} \rightarrow \bar{E} = \{(\bar{P}_d, \bar{P}_\pi), (\bar{P}_i, \bar{P}_q), (\bar{P}_{\eta_p}, \bar{P}_{\eta_w}, \bar{P}_{\eta_c}, \bar{P}_{\eta_{k^*}})\}$$

where \bar{P}_e are indicated in column three of Table One for all $e \in E$.

Proof: 3.6b follows from 3.6a. The proof of 3.6a is available from the authors.

Notice that a stationary state, as determined by rules of growth $X(t)$, is a property of Σ , independent of the take-off speed. The theorem is useful for the derivation of comparative stationary state theorems as will be discussed below.

In macroeconomics, the essential endogenous variables are the triplet (i, q, π) . For simplification, it is empirically reasonable to assume that the pair (r, ϕ_L) is constant. We then have the following corollary:

Corollary Four: When the population growth rate r and the labor share ϕ_L are constant, the profiles of (N_i, N_π) and (D_i, D_π, D_q) become

$$3.7 \text{ a) } D_i = N_i = G^0 + \eta_s / \bar{\phi}_L \quad (\phi_L \approx .66) \quad (\text{capital accumulation rate})$$

$$b) N_\pi = \bar{\phi}_K G^0 / s; \quad D_\pi = \bar{\phi}_K N_i / s \quad (\bar{\phi}_K \approx .33) \quad (\text{interest rate})$$

$$c) D_q = G^0 + (\bar{\phi}_K / \bar{\phi}_L) \eta_s \quad (\text{GNP growth rate) where}$$

$$d) G^0 = J / \bar{\phi}_L + \bar{r} \text{ with } H(G^0, J)$$

which are determined by the time profile of $(J, s, \eta_s, 1/s)$ when the values $\bar{\phi}_L$ and \bar{r} are given.

Corollary Four will be used in the analysis of four submodels which make special assumptions on the behavior of the exogenous variables, including the constancy of \bar{r} and $\bar{\phi}_L$. The assumptions and the implied harmonic motion of these four submodels are as follows.

$$3.8) \text{ M}^1: \text{ Assuming } \eta_s = \eta_r = \eta_J = J = 0 \text{ implies } H(i, d, \eta_{k^*}) \text{ ("Solow" Model)}$$

$$\text{M}^2: \text{ Assuming } \eta_s = \eta_r = \eta_{\phi_L} = \eta_J = 0, \bar{J} > 0 \text{ implies } H(\text{all components of } E) \text{ ("Kaldor" Model)}$$

$$\text{M}^3: \text{ Assuming } \eta_s = \eta_r = \eta_{\phi_K} = 0 \text{ implies } H(i, d, \pi) \text{ (US Growth Model)}$$

$$\text{M}^4: \text{ Assuming } \eta_J = \eta_{\phi_L} = \eta_r = 0 \text{ implies } H(i, q) \text{ (Austerity Model)}$$

In M^3 (M^4) the intensity of innovation J (saving rate s) is allowed to vary freely, while the other parameters are held constant in order to show the impact of changes in J (s) on $E(t)$. The model M^2 shows the impact of a

positive innovation intensity $J > 0$ as compared with the case $J = 0$ in M^1 . With the aid of 3.3 and 1.12, one can readily verify that the assumptions on $X(t)$ postulated for the above submodels imply the harmonic sets of procyclic variables as indicated. The natural rates of the procyclic variables, which can be derived from the potential values of Table One (by 3.2), will be used in the analysis of the submodels.

The remainder of Section III is devoted to the analysis of the submodels individually and is organized as follows:

- A) Growth Without Fluctuations (M^1 and M^2)
- B) Growth With Fluctuations in Epochs of Prosperity and Recession (M^3)
- C) The Impact of Austerity on Growth and the Interest Rate (M^4)

A) Growth Without Fluctuations (M^1 and M^2)

The models M^2 and M^1 (see 3.8) will be discussed together in this section because they share the common properties of the constant $(\bar{r}, \bar{s}, \bar{J})$ that rule out all meaningful fluctuations. Due to this simplicity, we will concentrate on two theoretical issues: the behavior of $E(t)$ as a system and the behavior of the real rate of interest in a dynamic loanable funds theory context.

In M^2 , by Theorem Four, $i(t)$ converges to the constant natural rate of capital growth:

$$3.9) \quad i \rightarrow \bar{N}_i = \bar{J} / \bar{\phi}_L + \bar{r}$$

This, in turn, implies two types of transitional behavior depending on the take-off speed \bar{i} .

$$3.10) \quad a) \quad \bar{i} > \bar{N}_i: \quad i \rightarrow \bar{N}_i \text{ with } \eta_i < 0 \text{ (capital shallowing growth)}$$

$$b) \quad \bar{i} < \bar{N}_i: \quad i \rightarrow \bar{N}_i \text{ with } \eta_i > 0 \text{ (capital deepening growth)}$$

Theorem Three implies that all components $E(t)$ are procyclic as they converge to the stationary values shown in the last column of Table I. Notice that as a system these natural rates satisfy the following inequality:

$$3.11) 0 < \bar{U} = \bar{J}/\bar{\phi}_L < \bar{G} = \bar{J}/\bar{\phi}_L + r < \bar{G}\bar{\phi}_K/\bar{s} < \bar{G}/\bar{s}$$

(assuming $s < \phi_K$)

The fact that all these rates are positive implies the stylized facts of Kaldor in the long run: the level of the interest rate ($\bar{\pi} = \bar{G}/\bar{\phi}_K/\bar{s}$) and of the capital-output ratio ($1/d = \bar{s}/\bar{G}$) are constant, while GNP, capital ($q = i = \bar{G}$) and the welfare indicators ($\eta_p = \eta_w = \eta_c = \eta_{K^*} = \bar{U} = \bar{J}/\bar{\phi}_L$) are growing at constant positive rates. The model shows the overwhelming importance of innovation intensity ($J > 0$) in that austerity s is irrelevant to both growth rapidity (q, i at \bar{G}) and the rate of welfare gains (at \bar{U}) [see Solow(1988), page 308]--supporting the Kuznets thesis that the primary growth promotion force in the modern economy is technology.

The model shows that the market rate of interest will always move toward the natural rate $\bar{G}\bar{\phi}_K/\bar{s}$ in the long run. The loanable funds theory suggests that the market rate is not merely a matter of "liquidity preference" as it is affected by all growth promotion forces. It will be raised by high innovation intensity J , high capital using bias ϕ_K , and high population growth rate r , all of which contribute to a higher marginal product of capital and investment demand. The interest rate is lowered by a higher savings rate s that increases the supply of loanable funds. (Thus the comparative dynamic theorems of 2.6 can be simplified in the setting of comparative stationary state analysis.)

The model M^1 is called the "Solow" model because of the assumptions $\eta_s = \eta_r = J = 0$. However, the model is very different from the original Solow model because of the tolerance of arbitrary fluctuations of ϕ_L . The harmonic relation $H(i, d, \eta_{K^*})$ implies the following natural rates:

$$3.12) \text{ a) Constant Natural Rates: } (N_d, N_i, N_{\eta_{K^*}}) = (D_d, D_i, D_{\eta_{K^*}}) = (\bar{r}/\bar{s}, \bar{r}, 0)$$

$$\text{ b) Variable Natural Rate: } N_{\pi} = (\phi_K \bar{r}/\bar{s} - \eta_{\phi_L}/\bar{s})$$

Thus while (i, d, η_K^*) move toward their natural rates in 3.12a, the other variables in $E(t)$ can fluctuate perpetually with the fluctuation of ϕ_K . The economy will not reach a long run stationary state. The absence of innovation activities ($J=0$) is a major weakness of M^1 which will be remedied in the next submodel.

B) Growth With Fluctuations in Epochs of Prosperity and Recession (M^3)

The model M^3 allows us to analyze growth with fluctuations induced by technology change because innovation intensity $J(t)$ is permitted to vary arbitrarily. This model is particularly useful for the analysis of $E(t)$ when fluctuations of $J(t)$ are superimposed on a long run increasing or decreasing trend.

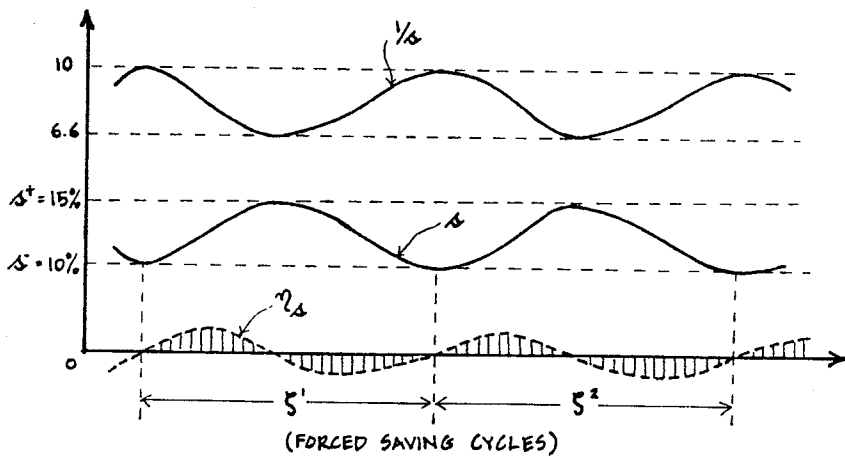
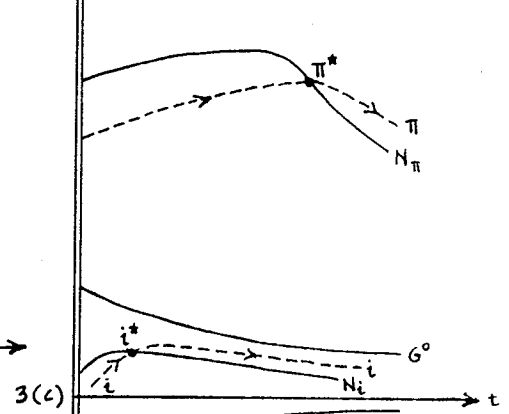
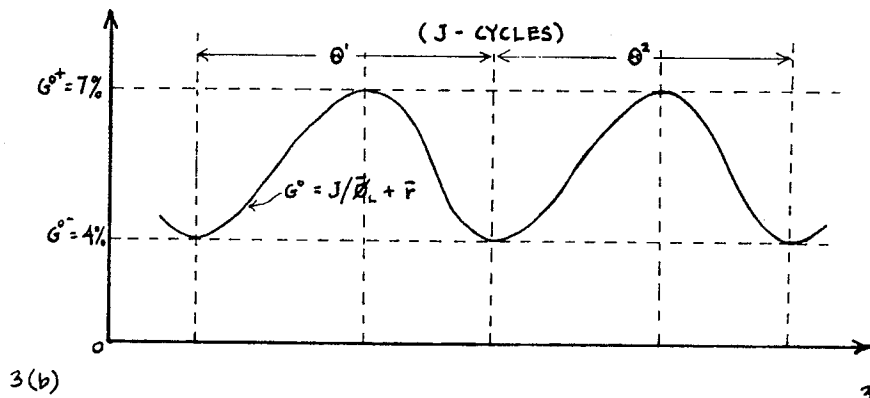
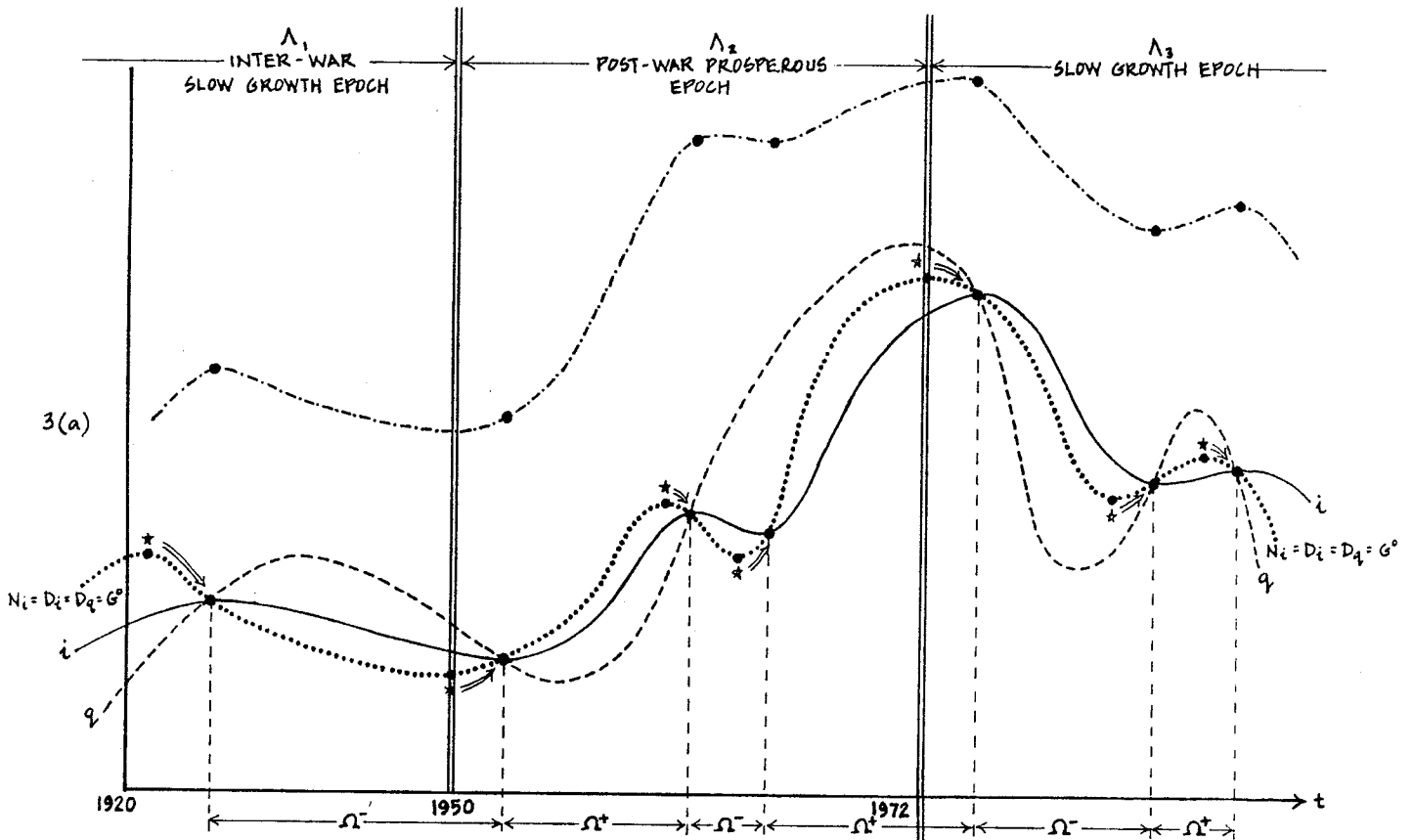
The interwar period Λ_1 (1920-1939) has been the most depressed U.S. growth epoch in recent times, while the postwar period Λ_2 (1950-1972) constituted a period of unprecedented prosperity. This postwar prosperity was brought about, in part, by a sequence of major technological changes constituting a virtual "second industrial revolution" which appears, however, to have ground to a halt in the most recent slow growth epoch Λ_3 (1973-1990). The three growth epochs are modeled by the postulation of a fluctuating time profile for $J(t)$ along an increasing (decreasing) trend in Λ_2 (Λ_1, Λ_3). Diagram 3 illustrates the profile of growth of the U.S. economy (1920-1990).

We shall now concentrate on the behavior of the triplet (q, i, π) as in Corollary Four. The constancy of s (i.e. $\eta_s=0$ in 3.7) implies

- 3.13 a) $N_i = D_i = D_q = G^0 \equiv J/\bar{\phi}_L + \bar{r} \geq 0$
 b) $N_\pi = D_\pi = \bar{\phi}_K G^0/s \geq 0$ implying
 c) $H(N_i, N_\pi, G^0, D_q, J)$ (harmonic relation in $X(t)$)
 d) $H(i, \pi)$ (harmonic relation in $E(t)$, see 1.11b)

Because of the harmony $H(G^0, J)$ in the causal system, Diagram 3 need not illustrate both J and G^0 , but shows only G^0 , the solid growth promotion curve. When $\eta_J=0$, the local extreme values of G^0 are indicated by a "*" Since i

DIAGRAM 3



chases after G^0 as a moving target, the "*" points predict the turning points of i that will occur with a time lag (illustrated by the double arrows). [This will always occur provided \bar{i} does not deviate drastically from G^0 , as will be explained in the next section.] Prosperity Ω^+ and recession Ω^- are marked off below the horizontal axis in Diagram 3a. The superimposition of Ω^i on Λ^j conveys the sense of the modeling of growth with fluctuations in our paper.

The natural interest rate N_π (not shown in Diagram 3) satisfies $H(N_i, N_\pi)$. The market rate of interest π is procyclic $H(i, \pi)$ as it chases after N_π . The model shows the real (non-monetary) interest rate fluctuating along an increasing (decreasing) trend in Ω^+ (Ω^-). The GNP growth rate q (Diagram 3a) fluctuates around the potential value $D_q = N_i$. Thus q increases (decreases) in the trendal sense during Ω^+ (Ω^-). In a perpetually fluctuating world, q as well as all other components of $E(t)$ will catch up with their potential values $D_e(t)$ periodically at the turning points--even though they may not be procyclic all the time.

The epochal variations of the real rate of interest in the real world are not adequately explained by the short run "liquidity theory." For instance, it is very unlikely that the low real rate of interest (3-4%) in the 1950's gave way to much higher rates (7-8%) in Λ_3 because "liquidity preference" increased or the quantity of real money declined. Our analysis suggests that the long run increase of π is a product of a prosperous growth epoch. The mainline liquidity theory is thus a short run theory of deviations of the market rate from the long run real rate (which is, itself, a deviation from the natural rate N_i) due to "imperfections" in the financial market. Liquidity theory needs to be integrated with long run growth theory (see Section IV, below).

In M^3 , the prediction $H(i, \pi)$ (e.g. their simultaneous trendal decline during Λ_3) is contradicted by data--which is hardly a surprise. A more satisfactory theory can be constructed when s is postulated to decline during Λ_3 , which is consistent with the empirical realities of the U.S. Quite

possibly the decline of s during Λ_3 is due to a continuation of the habit of "consumption orientation" ($\eta_s < 0$) formed during the prosperous epoch Λ_2 .

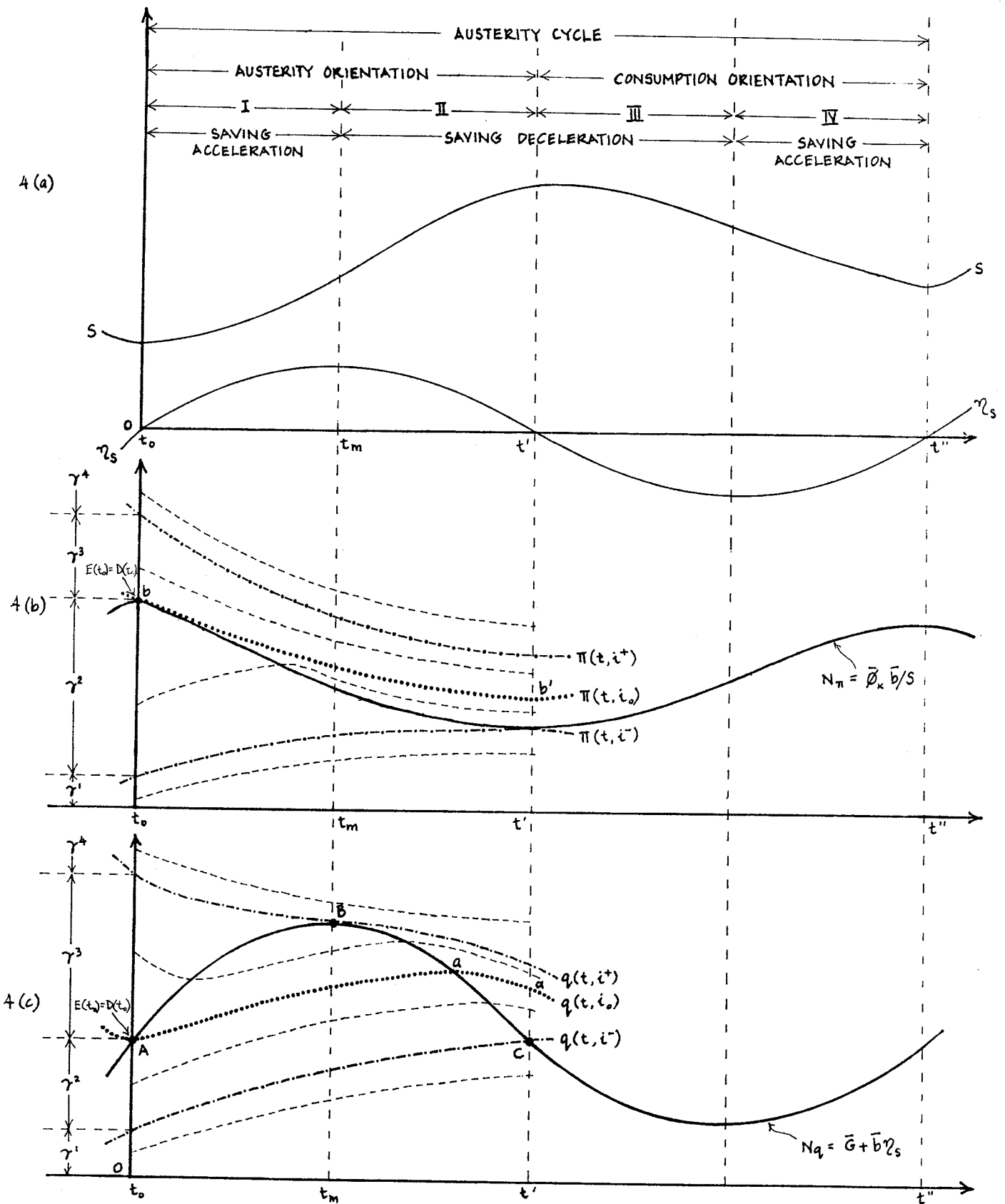
The declining patterns of G^0 and s have opposite effects on $N_{\pi} = \bar{\phi}_K G^0 / \bar{s}$ (3.7b) as illustrated below Λ_3 in Diagram 3c. We can show (see Section IV, below) that the downturn of the real interest rate (i.e. at the point π^* in Diagram 3c) tends to follow the downturn of i (i.e. i^* in Diagram 3c) with a time lag. This, in essence, is the unconditional prediction of our model for the direction of change of the U.S. real rate of interest in the years ahead. According to the loanable funds theory of our model, the decline in π is the inevitable consequence of the decline in G^0 when the decline in s ceases.

The above analysis suggests that the modeling of the cyclical and trendal relationship of (J, s) holds the key to a real theory of growth with fluctuations. Once the time profiles of the natural rates (N_i, N_{π}, N_q) and the potential values (D_i, D_{π}, D_q) are established by Corollary Four (3.7), the market behavior of (i, π, q) (e.g. the rapidity of their chasing of N_e) is only a technical matter of theoretical prediction and/or mechanical simulation.

C) The Impact of Austerity on Growth and the Interest Rate (M^4)

A crucial doctrinal controversy between Keynesian and loanable funds theories is the denial in The General Theory of the beneficial impact of austerity (proxied by a higher s and/or η_s) for growth promotion (i.e. raising q) and for the suppression of the interest rate (i.e. lowering π). The model M^4 can be used to examine the impact on (i, q, π) when s goes through an "austerity cycle" between t_0 and t' , as shown in Diagram 4a. The cycle has four quarters, with different meanings of "austerity orientation" (defined in terms of the direction of change of s or η_s) as indicated. The two meanings of austerity have different operational significance: s affects π and η_s affects i . We shall concentrate our analysis on the first two quarters where s increases monotonically (from t_0 to t') reaching a turning point (at t_m) where η_s takes on a maximum value.

DIAGRAM 4



In this model, the procyclic movement of q ($H(i,q)$) (3.8) implies that we need only consider the movement of (q,π) , the natural rates of which are given by 3.14 (see 3.7):

$$3.14 \text{ a) } N_q = D_q = \bar{G} + \eta_s \bar{b} \text{ where } \bar{G} = \bar{J}/\bar{\phi}_L + \bar{r}, \quad \bar{b} = \bar{\phi}_K/\bar{\phi}_L$$

$$\text{b) i) } N_\pi = \bar{\phi}_K \bar{G}/s$$

$$\text{ii) } N_\pi = D_\pi \text{ when } \eta_s = 0 \text{ (by } D_\pi = N_\pi + \bar{b}\eta_s/s)$$

Thus we have

$$3.15 \text{ a) } H(N_q, \eta_s) \text{ (harmonic movement between } N_q \text{ and } \eta_s)$$

$$\text{b) } H(N_\pi, -s) \text{ (counter-harmonic movement between } N_\pi \text{ and } s)$$

The natural interest rate N_π (GNP growth rate N_q) as shown in Diagram 4b (4c) moves counter-harmonically (harmonically) with s (η_s). In terms of the natural rate N_π , we see that the characteristic position of the loanable funds school $H(N_\pi, -s)$ is sustained. From 3.14a, we see that the first (second) quarter corresponds to a period of increasing (decreasing) N_q when s accelerates, $\eta_{\eta_s} > 0$ (decelerates, $\eta_{\eta_s} < 0$). In M^4 the "business cycle" is induced by savings.

The set of interest rate paths S^π (GNP growth paths S^q) is represented by the dotted curves in Diagram 4b (4c). In Diagram 4c, there are three special paths in S^q that pass through the points A, B and C:

$$3.16 \text{ a) } q(t_m, i^+) = N_q(t_m) \quad [q(t, i^+) \text{ passes through B}]$$

$$\text{b) } q(t_0, i_0) = N_q(t_0) \quad [q(t, i_0) \text{ passes through A}]$$

$$\text{c) } q(t_0, i^-) = N_q(t') \quad [q(t, i^-) \text{ passes through C}]$$

These paths determine three take-off speeds (i^-, i_0, i^+) that satisfy the following inequalities because N_q is inverse U-shaped:

$$3.17) \quad i^- < i_0 < i^+$$

The triplet of real numbers (i^-, i_0, i^+) marks off four line segments ($\gamma^1, \gamma^2, \gamma^3$, and γ^4) that represent a classification of all take-off speeds \bar{i} .

Corollary Three implies that all $E(t) \in \Sigma$ are classified accordingly (see

Diagram 4bc). When s expands in the first two quarters, we have the following behavior patterns of (q, π) depending upon the take-off speed.

- 3.18 a) $\gamma^1 = (\bar{i} | \bar{i} < i^-)$: monotonic increase of π and q
- b) $\gamma^2 = (\bar{i} | i^- < \bar{i} < i_0)$: π reaches a maximum before t_m
 q reaches a maximum after t_m
- c) $\gamma^3 = (\bar{i} | i_0 < \bar{i} < i^+)$: π decreases monotonically
 q reaches a minimum (maximum) before (after) t_m
- d) $\gamma^4 = (\bar{i} | i^+ < \bar{i})$: monotonic decrease of π and q

To interpret these results, notice that at the beginning point t_0 , there is a unique boundary case $i[t, i(t_0)]$ for which $E(t)$ is "at par with the potential profile" [i.e. $E(t_0) = D(t_0)$]. At t_0 , all other $E(t)$ either dominate or are dominated by $D(t)$ for historical reasons. For the boundary case, we see the sustained beneficial interest suppression effect (i.e. the declining curve bb' in Diagram 4b) and the exhaustion of the growth promotion effect after t_m when the saving rate ceases to accelerate (i.e. the inverse-U shaped curve Aaa' in Diagram 4c). Thus, in the absence of historical deviance, a higher saving rate will depress the real rate of interest, but it takes savings acceleration to promote growth. In Diagram 4c, recession will begin (at a) with a time lag (behind B) when the saving rate ceases to accelerate.

When at t_0 the economy is disturbed in either direction by historical forces, as is generally the case, the causal relations indicated for the boundary case can be disturbed temporarily for the moderate deviation cases (3.18bc) and can be disturbed beyond recognition for the severe deviation cases (3.18ad). Therefore, the controversy between the Keynesian and loanable funds theories cannot be settled by short run data even in the absence of monetary disturbances. Due to the effects of historical deviations, one cannot refute a dynamically formulated loanable funds theory of interest by the absence of an instantaneous correlation between s and π . The validity of the loanable funds theory remains a long run phenomenon.

Section IV: DIRECTIONS FOR FUTURE RESEARCH

As early as the 1950's, James Tobin, a leading Keynesian, attempted an integration of long run growth with short run instability by stressing the demand for monetary assets and the supply of labor [Tobin(1955)]. Thirty years later, Robert Solow has called for a return to this attempt to integrate long run theory and short run macroeconomics [Solow (1988)]:

..a theory of equilibrium growth badly needed- and still needs- a theory of deviations from the equilibrium path....
 ...(by dealing with) the implications of real wage rigidity and the possibility of liquidity trap (page 309). ...(This) really involves the integration of short run and long run macroeconomics, of growth theory and business cycle theory.....a problem that has still not been solved (page 310).

The need for an integration of long and short run theories in macroeconomics has been noted even more recently by Assar Lindbeck (1990):

Long-term GNP growth and short-term 'cyclical' fluctuations ..., the dichotomy between capacity growth and capacity utilization, (should be regarded as) part of one and the same process.... At the present time, economists are not able to integrate long-term growth and short-term cyclical instability in a unified and realistic framework....A fundamental analytical issue in short-term macroeconomics is to choose between market clearing and non-market clearing specifications of the labor market.... It is necessary to model unemployment as a disequilibrium phenomenon in the labor market. (page 3-4)

The key message of these common pleas is that short run macroeconomics (i.e. the effective demand theory of Keynes) should be interpreted as deviations from long run equilibrium magnitudes determined in the growth process. There appears to be a consensus that short run deviations ultimately stem from imperfections in the labor and/or financial markets that disturb the neoclassical "Euler Theorem" representative of the symmetric principle of income distribution in the long run. Notice that the long run growth model represents a real theory for a barter economy, while the short run theory is a monetary phenomenon. Therefore, the integration of short run deviations with long run growth calls for a monetary reformulation of the real model [see Tobin (1965)].

Our long run real (non-monetary) model shows that business cycles can occur even when both factor markets are functioning perfectly as long as the rhythm of the unfolding technological frontier is not smooth. Factor market imperfections caused by monetary disturbances and/or "systematic confusion" [Lindbeck (1990), page 4] are contributory, rather than primary, causes of cyclical fluctuations.

In Diagram 1a, with unemployment equilibrium at t' , the actual employment point $A(t')$ is a deviation from the factor endowment point $P^\Delta(t')$ which dominates $A(t')$, i.e.

$$4.1) A(t') \leq P^\Delta(t')$$

as indicated by the double arrow pointing northeast. A short run employment theory (i.e. on the extent of the deviation in 4.1 which represents unemployment and/or under-capacity utilization) cannot be equated with business cycles because it is conceivable to have full employment growth with cyclical fluctuations.

It follows that the integration of long and short run macroeconomics should proceed in two steps. The first step is a theory of full-employment growth with fluctuations that explains price inflation and the money rate of interest. In such a model $E(t)$ is viewed as a deviation from natural monetary rates which reflect not only all the long run real growth variables $X(t)$, but also the short run "liquidity" demand for money. Such a monetary theory of growth with fluctuations implies an acknowledgement of our intellectual debt to Wicksell, recognized by Solow [Solow (1988), page 309]. The second step is a theory of factor price distortions possibly causing unemployment and under-utilization of capacity as "deviations" (see 4.1). We will comment on these two steps in what follows (in reverse order).

With respect to the second step, the formulation of an unemployment theory, our model shows that what is being determined in the labor market in the long run is always a growth rate of real wages η_w . In an imperfect

labor market, the rate of growth of money wages η_w' may deviate from η_w . For example, an upward deviation $\eta_w' > \eta_w$ leads to cost-push inflation and unemployment in the Keynesian tradition. If an upward deviation is more pronounced during prosperity, when laborers are less sensitive to job security, then a monetary version of our model provides an explanation of a "Phillips curve-type" relationship between the inflation rate and $i = \eta_K$ (rather than the employment level) which can be statistically verified. (The short run significance of N_{η_w} and/or D_{η_w} remains to be explored.)

With respect to the first step, full-employment growth, notice that business cycles are formed of certain features in the fluctuations of the components of $E(t)$. Any business cycle theory worthy of the name should be able to "forecast" these features from those postulated for the exogenous variables $X(t)$. Making use of the theoretical simplifications (on $\bar{r}, \bar{\phi}_L$) introduced in Corollary Four (4.4), we may suggest way to integrate monetary theory with the long run model of this paper by means of natural monetary rates.

In the upper deck of Diagram 3b, the fluctuation of $G^0 = J/\bar{\phi}_L + \bar{r}$ (with an amplitude of, say, $G^{0-} = 4\%$ and $G^{0+} = 7\%$) is shown to exhibit recurring innovation cycles θ^i ($i=1,2$), with turning points lagging behind those of the saving cycles ζ^i ($i=1,2$) (with an amplitude between $s^- = 10\%$ and $s^+ = 15\%$ in the lower deck of Diagram 3b). This pattern of time lag is based on the interpretation of ζ^i as the effective investment (or forced saving) rate that has a particular pattern of cyclic behavior: shortly before the arrival of an upturn in J (i.e. when revival is "in the air"), there is always monetary expansion in the banking system to allow the entrepreneurs to acquire a higher percentage of real GNP for effective investment [Friedman and Schwartz (1963)].

Since the time profiles of η_s and $1/s$ are determined (as shown by the dotted curves in the lower deck of Diagram 3b), Corollary Four allows us to deduce the time path of (N_i, N_π) as illustrated (in Diagram 2b) by the following cases where (N_i, N_π) is represented by either

- 4.2 a) a segment of the radial line ab , if s is constant,
 b) a counter-clockwise loop, if the upturn of ζ^i leads to that of θ^i , or
 c) a wider clockwise loop, if the amplitude of fluctuations of ζ^i is wide
 and the upturn of ζ^i lags behind that of θ^i .

All these cases represent "causal systems" of pure business cycles in that the market rate (i, π) ("bird") will chase after the natural monetary rates (N_i, N_π) ("moth") with a theoretically predictable locus when the initial take-off speed is specified (see Diagram 2ab).

When s is interpreted as the effective investment rate rather than as voluntary savings by private households, the model can be integrated with fiscal operations and international trade as well as monetary expansion. The extensions of our model of dynamic growth with fluctuations in those directions, however, lies beyond the scope of the present paper.

APPENDIX:

In this appendix we will prove 1.1biii and Lemma One.

To prove 1.1biii, differentiate the Euler Theorem partially with respect to t :

- A.1 a) $-Lf_{LL} = Kf_{KL}$ or $\epsilon_{LL} = -Lf_{LL}/f_L = Kf_{KL}/f_L$ and similarly
 b) $\epsilon_{KK} = Lf_{KL}/f_K$
 c) $\epsilon_{KK}/\epsilon_{LL} = \phi_L/\phi_K$ and $1 + \epsilon_{KK}/\epsilon_{LL} = 1 + \phi_L/\phi_K = 1/\phi_K$; thus
 d) $\epsilon/\epsilon_{LL} = (\epsilon_{LL} + \epsilon_{KK})/\epsilon_{LL} = 1/\phi_K$ Q.E.D.

To complete the proof of Lemma One we have to show that, according to 1.7a:

- A.2 a) $J = (\partial Q/\partial t)/Q = \alpha(t)$ along $P^\Delta(t)$
 b) $\phi_L = \beta(t)$ along $P^\Delta(t)$
 c) $i^\Delta(t, \bar{I}) = d^\Delta \sigma(t)$ along $P^\Delta(t)$

Note that A.2b follows from the Cobb-Douglas specification in 1.8c. To prove

A.2a we differentiate the $\ln Q$ (1.8c) partially with respect to t :

$$A.3) J = \alpha(t) + \beta'(t) \ln K^\Delta(t)/L^\Delta(t) - \beta'(t) [\ln K^\Delta(t)/L^\Delta(t)] = \alpha(t)$$

To prove A.2c, note that the Harrod-Domar equation holds initially and dynamically.

- A.4 a) $\bar{i} = d(0)\sigma(0)$
 b) $\eta_{i^\Delta}(t, \bar{i}) = \eta_{d^\Delta} + \eta_\sigma(t)$

A.4b follows from the fact that $i^\Delta(t, \bar{i})$ in 1.8b is a solution of the differential equation 1.14a in the text.

Acknowledgement. The authors acknowledge the collaboration of Prof. Gustav Ranis at an early stage of this project.

Endnotes:

¹ $f_K > 0$, $f_L > 0$, $\lim(f_K) \rightarrow 0$ as $K \rightarrow 0$, and $\lim(f_L) \rightarrow 0$ as $L \rightarrow 0$

² π is technically the one period return on investment in terms of Q , which is positively correlated with the interest rate. We will refer to π as the "interest rate" in the remainder of this paper.

³ We acknowledge benefitting from discussion with Prof. T.N. Srinivasan which resulted in the formal definition of a "solution."

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