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# PERFORATE AND IMPERFORATE CURRENCY BANDS:

## EXCHANGE RATE MANAGEMENT AND THE TERM STRUCTURE

## OF INTEREST RATE DIFFERENTIALS

Paolo A. Pesenti

Yale University

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## Abstract

This paper provides a simple analytical characterization of an
exchange rate regime which is consistent with the main stylized facts of
current currency band institutional schemes, namely the reduction of
exchange rate volatility, the concentration of the exchange rate around
its central parity and the coexistence of marginal and intramarginal
interventions. For particular policy rules, even an informal (so called
perforate) target zone which allows the exchange rate to be above (below)
any upper (lower) limit with non zero probability can make the exchange
rate less responsive to shocks in fundamentals than a formal band defended
by interventions at the boundaries. The implications for the dynamic
behavior of interest rates, including a closed form solution for the term
structure of interest rate differentials as a function of the current spot
exchange rate, are derived and analyzed in detail.

Since the pathbreaking contribution by Krugman (1988), followed among others by Miller and Weller (1988 and 1989), Froot and Obstfeld (1989 a and b), Flood and Garber (1989), Bertola and Caballero (1989 and 1990), Svensson (1989 and 1990), Lewis (1990) and Buiter and Pesenti (1990), special attention has been devoted in international financial theory to the properties of exchange rate dynamics within a target zone. The sensible intuition of the *standard* model is that exchange rates under a target zone regime are less responsive to shocks in fundamentals than exchange rates under free float, provided that the intervention rules of the Central Bank are common knowledge. The narrower the target zone, the lower the degree of sensitivity of exchange rates to fundamental shocks. This result holds even if the defense of the target zone is not perfectly credible. Variations on this theme have analyzed alternative policy rules for defending the band, the effects of expected realignments of the central parity (Bertola–Caballero (1989 and 1990)), the relation between exchange rates and the term structure of interest rates (Svensson (1989 and 1990)), the optimality and the sustainability of these regimes (Avesani (1990), Delgado and Dumas (1990)) and the presence of rational speculative *bubbles* (Buiter and Pesenti (1990)).

However, the standard model seems unable to explain some of the stylized facts that characterize exchange rate behavior under the current institutional arrangements, and in particular the European Monetary System (EMS). In these regimes the exchange rate is more frequently observed in the neighborhood of the central parity rather than in the neighborhood of the upper or lower limits of the band as predicted by the standard target zone theory. In other words, the empirical distribution of the exchange rate concentrates probability mass around the central parity rather than being bimodal at the boundaries, as is the case for the asymptotic distribution derived in the target zone model. Moreover, in the literature the formalization of the intramarginal mechanism of defense does not seem to provide a satisfactory stylization of a realistic policy rule (Flood and Garber (1989)); for

instance, intramarginal open market operations and/or foreign market interventions are assumed to take place only *after* the exchange rate has already reached (and retreated from) one of its boundaries. Finally, the model does not analyze the *coexistence* of marginal *and* intramarginal interventions in an exchange rate band, a well known characteristic of the EMS during the 80's[1].

The model introduced in this paper represents a generalization of the standard target zone theory consistent with the stylized facts above. Section 1 describes a model in which the exchange rate central parity is defended but the exchange rate can *perforate* any exogenously given ceiling or floor with non zero probability. The probability of this event is a function of the policy followed by the Central Bank, so that this scenario characterizes an *informal* target zone. The general model of an *imperforate* target zone presented in section 2 provides a theoretical refinement which includes the *perforate* band model and the standard theory as particular cases. Sections 3 and 4 analyze the relation between exchange rates and interest rates in a perforate zone and provide empirically testable implications of the model, including a closed form solution for the term structure of interest rate differentials as a function of the exchange rate.

## 1) A perforate target zone model

I define as *perforate* a target zone if the (asymptotic) probability for the exchange rate being above the upper boundary or below the lower boundary is not zero and depends on the parameters of the policy rule followed by the Central Bank.

A simple model of a perforate target zone can be sketched as follows. Define as s(t)

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[1] See for instance Giavazzi—Giovannini (1989), ch.2 and 4. A notable departure from the standard model is provided by Lewis (1990) who considers a setup in which the authorities intervene intramarginally by temporarily stopping the movement of the fundamentals, with increasing probability the farther fundamentals are from the target level. These stochastic rules imply a form of mean—reversion to fundamentals.

the logarithm of the spot exchange rate, expressed as the price of one unit of foreign currency in terms of domestic currency. f(t) denotes the fundamental variables relevant for determining s(t) according to a given theoretical structural model. The fundamental stochastic process is described by

$$(1) \qquad\qquad df = \hat{\eta}dt + \sigma d\omega$$

where $\hat{\eta}$ represents the instantaneous drift, $\omega$ the standard Wiener process and $\sigma$ the instantaneous standard deviation of the fundamental.

The Central Bank affects the dynamics of the fundamental, and consequently the exchange rate, through the choice of the value of the drift; no other control variable is available. The opportunity set of the Central Bank is given by two values of $\hat{\eta}$: $\eta_L < 0$ and $\eta_H > 0$. For simplicity, it is assumed that $\eta \equiv \eta_H = -\eta_L$. The analysis can be easily generalized to the case $\eta_H \neq -\eta_L$. Intuitively, the choice of $\eta_L$ implies the adoption of a *contractionary* monetary policy and the choice of $\eta_H$ implies the adoption of an *expansionary* monetary policy. The infinitesimal $\hat{\eta}$ dt measures the instantaneous intervention of the Central Bank on the foreign exchange market, and the cumulative value of the drift $\int_0^t \hat{\eta} \, d\tau$ represents the difference between the log of the stock of money supply at time t and the log of the stock of money at time 0.

The policy rule followed by the Central Bank can be characterized as a two–valued *bang–bang control* of brownian drift[2]. The positive drift $\eta_H = \eta$ is chosen when the fundamental f is less or equal to a predetermined threshold $\delta$ and the negative drift $\eta_L = -\eta$ is chosen when f $> \delta$. Without loss of generality we normalize the threshold value choosing $\delta = 0$. The heuristic interpretation of this rule is that the Central Bank *pushes* the economy as hard as possible to the right when the process f finds itself to the left of 0,

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[2] See Shreve (1981).

and vice versa for f to the right of 0. It can be shown that this is the optimal policy rule followed by the Central Bank when the objective function is given by $J(f,\hat{\eta})$ defined as

$$(2) \qquad J(f,\hat{\eta}) = E_f \int_0^\infty e^{-\gamma t} [f(\tau)]^2 d\tau$$

where the discounting factor $\gamma$ is positive and the control variable $\hat{\eta}(f)$ can be chosen within a bounded range $[\eta_L, \eta_H]$[3] (Beneš–Shepp–Witsenhausen (1980), Karatzas–Shreve (1988), ch.6)[4].

The exchange rate process is given in equation (3) or (3a)

$$(3) \qquad s(t)\, dt = f(t)\, dt + \alpha^{-1} E_t ds(t)$$

$$(3\ a) \qquad E_t\, ds/dt = \alpha(s - f)$$

---

[3] An alternative but qualitatively analogous intervention policy could be specified as follows: the positive drift is chosen when $f \leq \delta_1$, the negative drift is chosen when $f > \delta_2$ and a zero drift is chosen when $\delta_1 < f \leq \delta_2$, where $\delta_1$ and $\delta_2$ are predetermined thresholds. It can be shown that this policy is optimal if the objective function in equation (2) takes into account the "running costs" on the control equal to $|\hat{\eta}|$, that is if $J(f,\hat{\eta}) = E_f \int_0^\infty e^{-\gamma t}[f(\tau)^2 + |\hat{\eta}|]\, d\tau$ (Benes–Karatzas (1981)). Again, qualitatively analogous is the case in which the absolute value of the drift is an increasing function of the distance between f and the constant $\delta$ (this specification could be analyzed by modeling the regulated fundamental as an Ornstein–Uhlenbeck process).

[4] The general solution of this stochastic optimal control problem when $\sigma = 1$ is

$$\hat{\eta}(f) = \begin{cases} \eta_H & \text{if } f \leq \delta \\ \eta_L & \text{if } f > \delta \end{cases}$$

where $\delta \equiv \dfrac{1}{\sqrt{\eta_H^2 + 2\gamma} + \eta_H} - \dfrac{1}{\sqrt{\eta_L^2 + 2\gamma} - \eta_L}$, so that $\delta = 0$ if $\eta_H = -\eta_L$.

where $E_t$ is the mathematical expectation operator conditional on the information available at time t to the private sector and the controller, and $\alpha$ a constant positive parameter. Both s(t) and f(t) are assumed to be observable at time t; the assumption of observability of f can be relaxed, as we discuss below. The structure of the model (including the policy rule) is common knowledge. The law of motion (3) implies that the current exchange rate is a loglinear function of the current value of the fundamental and the expected rate of depreciation. This typical *forward looking* equation can be easily derived from the family of monetary models, with the parameter $\alpha^{-1}$ representing the semielasticity of money demand to the interest rate. More complex structural models based on the presence of *feedback* from the exchange rate to the fundamental can be modeled following Miller–Weller (1988 and 1989).

The forward saddlepoint solution to eq. (3) expresses the exchange rate as a function of current and expected discounted future values of the fundamental, namely

$$(4) \qquad s(t) = \alpha \int_t^{\infty} e^{-\alpha(\tau - t)} E_t f(\tau) d\tau$$

It is useful to express equation (4) in state space rather than time series representation. Consider the solution to eq. (3) within the class of functions s which depend on the current value of the fundamental only, or

$$(5) \qquad s(t) = g(f(t)).$$

where g is a continuous twice differentiable function in f. If the drift of the fundamental process were $\eta_H = \eta$ for *any* value of f, the trajectory (5) in the space $f - s$ would look

like a 45 degree line with intercept $\eta/\alpha$ as in figure 1. In fact, by eq. (1) with $\hat\eta = \eta$ we obtain

$$(6) \qquad E_t \, f(\tau) = f(t) + \eta(\tau{-}t) \ \text{ for } \ \tau \geq t.$$

Substituting (6) into (4) yields the solution described above. Analogously, if the drift of the fundamental process were $\eta_L = -\eta$ for *any* value of f, the trajectory would look like a 45 degree line with intercept $-\eta/\alpha$.

The "combination" of these two trajectories is *not* the solution to equation (4) when f is regulated as described above. If this were the case, at the point $f = 0$ the exchange rate would discretely jump from the value $\eta/\alpha$ to the value $-\eta/\alpha$ in response to an anticipated intervention (the change in drift). This would imply the presence of a foreseeable arbitrage opportunity and a fortiori the absence of equilibrium in the foreign exchange market.

The correct solution is found as follows. First, apply Ito's Lemma to eq. (5). This yields

$$(7) \qquad ds = g'(f) \, df + \ g''(f)/2 \, df^2 = g'(f)[\hat\eta \, dt + \sigma \, d\omega] + g''(f)/2 \ \sigma^2 \, dt.$$

An implication of eq. (7) is that the standard deviation of the instantaneous rate of change of the exchange rate is a function of f, namely $\sigma g'(f)$. Taking the conditional expectation of eq. (7) and comparing this expression with the law of motion (3) we obtain

$$(8) \qquad E_t \, ds = g' \, E_t \hat\eta \, dt + g''/2 \ \sigma^2 \, dt = \alpha(s{-}f) \, dt.$$

The second order differential equation (8) can be easily solved by recalling that for $f > 0$ it is $E_t \, \hat\eta = -\eta$, and for $f \leq 0$ it is $E_t \, \hat\eta = \eta$. The general closed form solution is given by

$$s = f - \frac{\eta}{\alpha} + A_1 e^{\lambda_1 f} + A_2 e^{\lambda_2 f} \quad \text{for } f > 0$$

(9)

$$s = f + \frac{\eta}{\alpha} + A_3 e^{\lambda_3 f} + A_4 e^{\lambda_4 f} \quad \text{for } f \leq 0$$

where $\lambda_{1,2} = \sigma^{-2} [-\eta_L \pm \sqrt{\eta_L^2 + 2\alpha\sigma^2}]$, $\lambda_{3,4} = \sigma^{-2} [-\eta_H \pm \sqrt{\eta_H^2 + 2\alpha\sigma^2}]$. Note that $\lambda_1 = -\lambda_4$ is positive, while $\lambda_2 = -\lambda_3$ is negative.

The boundary conditions that define the constants $A_1 - A_4$ are chosen as follows. First, as discussed above no expected discrete jump of the exchange rate can take place at $f = 0$. This value matching (no arbitrage) condition implies

(10 a)
$$g(0^+) = g(0)$$

where $g(0^+) \equiv \lim_{f \to 0, \, f > 0} g(f)$. In terms of eq. (9), we obtain

(10 b)
$$-\eta/\alpha + A_1 + A_2 = \eta/\alpha + A_3 + A_4$$

Second, the log of the exchange rate is assumed to be 0 when the fundamental is 0 as well. In other words

(11 a)
$$g(0) = 0$$

(11 b)
$$-\frac{\eta}{\alpha} + A_1 + A_2 = 0$$

Conditions (11) simply identify the origin in the space f—s without loss of generality. It is always possible to obtain conditions (11) by defining appropriately the units of measurement.

Finally, the trajectory must asymptotically reach the two 45 degree lines $-\eta/\alpha + f$ and $\eta/\alpha + f$ for $f \to +\infty$ and $f \to -\infty$ respectively. The intuition for this is that the difference between the free float trajectory and the equilibrium trajectory reflects the anticipation of the policy change. If the economy is *far away* from the origin, the event of the change in drift is perceived as very remote in time, and its effects on the *expectation bias* are small[5]. These considerations lead to the conditions $(15-16)$ below:

(15) $$\lim_{f \to \infty} g(f) - (-\eta/\alpha + f) = A_1 e^{\lambda_1 f} + A_2 e^{\lambda_2 f} = 0$$

(16) $$\lim_{f \to -\infty} g(f) - (\eta/\alpha + f) = A_3 e^{\lambda_3 f} + A_4 e^{\lambda_4 f} = 0$$

Since $\lambda_1 > 0$ and $\lambda_2 < 0$, any value of $A_1$ different from zero would be incompatible with condition (15); analogously, since $\lambda_3 > 0$ and $\lambda_4 < 0$ any value of $A_4$ different from 0 would not satisfy condition (16).

To summarize, the appropriate boundary conditions imply $A_1 = A_4 = 0$ and $A_2 = -A_3 = \frac{\eta}{\alpha}$, so that the final solution (9) under conditions (10–11–15–16) is given by

(17)
$$s = f - \frac{\eta}{\alpha} + \frac{\eta}{\alpha} e^{\lambda_2 f} \qquad \text{for } f > 0$$

$$s = f + \frac{\eta}{\alpha} - \frac{\eta}{\alpha} e^{\lambda_3 f} \qquad \text{for } f \leq 0.$$

The graph of $s$ defined by eq. (17) is shown in figure 1. It is convex for $f$ positive and concave for $f$ negative[6]. It is easy to check that the first derivative $g'$ is always

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[5] This condition is equivalent to the assumption that no *intrinsic bubbles* can arise. See Froot—Obstfeld (1989 c) and Buiter—Pesenti (1990).

[6] It can be easily verified that at the origin a discontinuity of the second derivative occurs. The derivation of the equilibrium trajectory provided in eq. (7) is not affected by this result, since even in the presence of discontinuities in $g''$ the basic Ito's formula is equivalent to the Tanaka's formula. See Harrison (1985), p.70.

continuous, positive and less than 1. In the negative range of f we have $\frac{ds}{df} = 1 - \frac{\eta}{\alpha} \lambda_3 e^{\lambda_3 f}$.
For $f \to -\infty$, the first derivative is 1. At f=0, the first derivative is $1 - \frac{\eta}{\alpha} \lambda_3 < 1$.

Substituting the expression for $\lambda_3$ we obtain $\left.\frac{ds}{df}\right|_{f=0} = \frac{\alpha\sigma^2 + \eta^2 - \eta\sqrt{\eta^2 + 2\alpha\sigma^2}}{\alpha\sigma^2}$

and it is straightforward to verify that the numerator is positive. Given the concavity of s
for negative f, it must be the case that $0 < ds/df \leq 1$ when $f \leq 0$. Analogous reasoning
shows that $0 < ds/df \leq 1$ even for $f > 0$.

Note that in the absence of *bang–bang control* of the drift the equilibrium trajectory
would look like a 45 degree line with $\frac{ds}{df} = 1$. This latter case corresponds to a free float
regime for the exchange rate. Since $g'(f)$ measures the elasticity of the exchange rate to
the fundamental, it has been shown that the exchange rate within a perforate target zone is
less responsive to fundamental shocks than the freely floating exchange rate[7]. Analogously,
the (instantaneous) volatility of the exchange rate within a perforate zone, measured by
$\sigma g'(f)$ is always less than the (instantaneous) volatility of the freely floating exchange rate
given by $\sigma$.

It is important to note that the monotonicity of s with respect to f implies that
the Central Bank does not need to observe the fundamental shocks *directly*. The contingent
choice of $\hat{\eta}$ can be simply based on the current observed value of the exchange rate.

The asymptotic (steady state) distribution[8] for the fundamental has density
function $\pi(f)$ defined as

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[7] Bertola–Caballero (1989) consider a setup based on the presence of stochastic realignments of the
central parity. For particular values of the parameters the model is able to generate a backward
S–shaped trajectory as in the perforate target zone, but for these values the elasticity of the exchange
rate with respect to the fundamental is greater than 1.

[8] See for instance Malliaris–Brock (1982), pp.106–108.

$$(18) \qquad \pi(f) = \frac{\eta}{\sigma^2} e^{\frac{2\hat{\eta}}{\sigma^2} f} = \begin{cases} \eta\sigma^{-2}\exp[2\eta\sigma^{-2}f] & \text{for } f \le 0 \\ \eta\sigma^{-2}\exp[-2\eta\sigma^{-2}f] & \text{for } f > 0 \end{cases}$$

The distribution (18) is the combination of two (positive and negative) exponential functions truncated at 0 and normalized to have $\int_{-\infty}^{\infty} \pi(f)\, df = 1$. Since the exchange rate is a monotonic increasing function in $f$, the asymptotic density function of $s$, defined as $\pi^s(s)$, is proportional to $\pi/g'$ or

$$(19) \qquad \pi^s(s) \propto \frac{\pi[g^{-1}(s)]}{g'[g^{-1}(s)]}$$

The density function $\pi^s(s)$ is convex, unimodal with mean and median equal to 0. It is shown in figure 2 for different values of $\eta$, with $\eta_1 < \eta_2 < \eta_3$.

We want now to show that the solution above characterizes a perforate target zone as defined at the beginning of the section. In other words, we want to show that for any given upper and lower exchange rate target $s_H$ and $s_L$, with $s_L < 0 < s_H$, the asymptotic probability for the exchange rate being outside the range $[s_L, s_H]$ is a decreasing function of the *strength* of the policy rule measured by $\eta$. Given $s_H$ and $s_L$ we can find the values of the fundamental $f_H \equiv g^{-1}(s_H) > 0$ and $f_L \equiv g^{-1}(s_L) < 0$ at which the exchange rate reaches the target values, for $\eta$ given. Moreover, it can be shown that $\frac{df_H}{d\eta}\Big|_{s=s_H} > 0$ and $\frac{df_L}{d\eta}\Big|_{s=s_L} < 0$. We can evaluate now the probability $P$ defined as

$$(20) \qquad \begin{aligned} P &\equiv \Pr\{s > s_H, s < s_L\} = \Pr\{f > f_H, f < f_L\} = \\ &= 1 + \Pi(f_L) - \Pi(f_H) \end{aligned}$$

where $\Pi(f)$ is the cumulative distribution function with density function $\pi(f)$. Simple algebra shows that

(20 a) $$P = 0.5 \left[ \exp(2\eta f_L / \sigma^2) + \exp(-2\eta f_H / \sigma^2) \right]$$

(20 b) $$\frac{\partial P}{\partial \eta} = \sigma^{-2} \left[ f_L \exp(2\eta f_L / \sigma^2) - f_H \exp(-2\eta f_H / \sigma^2) \right] < 0$$

(20 c) $$\frac{\partial P}{\partial f_H} = 0.5 \left[ -2\eta / \sigma^2 \exp(-2\eta f_H / \sigma^2) \right] < 0$$

(20 d) $$\frac{\partial P}{\partial f_L} = 0.5 \left[ 2\eta / \sigma^2 \exp(2\eta f_L / \sigma^2) \right] > 0.$$

(20 e) $$\frac{dP}{d\eta} = \frac{\partial P}{\partial \eta} + \frac{\partial P}{\partial f_H} \frac{df_H}{d\eta} + \frac{\partial P}{\partial f_L} \frac{df_L}{d\eta} < 0$$

Eq. (20 a–e) imply that the larger the policy parameter $\eta$ and the larger the size of the *informal* target zone, the smaller the probability for the exchange rate being outside the band. When $\eta$ goes to infinite, this probability goes to zero.

## 2) An imperforate target zone model

I define as *imperforate* a two–sided target zone if the (asymptotic) probability for the exchange rate being above some upper boundary and below some finite lower boundary is zero. Consider a target zone with central parity $0$, upper boundary $s_H$ and lower boundary $s_L$, with $s_L < 0 < s_H$. The boundaries are defended by infinitesimal reflecting interventions as in Krugman (1988), so that the fundamental stochastic process is now

(1') $$df = \hat{\eta} dt + \sigma d\omega - dI_H + dI_L$$

where the regulators $I_H$ and $I_L$ are two right continuous increasing functions in $f$. The upper regulator $I_H$ increases only when $s(f(t))$ reaches $s_H$, and $I_L$ increases only when $s(f(t))$ reaches $s_L$. The interpretation of eq. (1') is that the Central Bank controls the

fundamental drift (that is, intervenes intramarginally) in order to reduce the probability for the exchange rate being at the boundaries. If and when the exchange rate hits the upper or lower limit, a marginal intervention occurs maintaining the exchange rate within the band.

Define as $f_H$ the value of the fundamental at which $s = s_H$ and $f_L$ the value of $f$ at which $s = s_L$. Obviously, in an imperforate target zone reflecting interventions keep the fundamental within the range $[f_L, f_H]$. Since $f_L$ and $f_H$ are endogenously determined, we need now six rather than four boundary conditions as before. The solution to eq. (3) is still eq. (9) with boundary conditions (10) and (11), but conditions (15) and (16) are replaced now by the following four equations :

$$(15') \qquad s_H = f_H - \frac{\eta}{\alpha} + A_1 e^{\lambda_1 f_H} + A_2 e^{\lambda_2 f_H}$$

$$(16') \qquad s_L = f_L + \frac{\eta}{\alpha} + A_3 e^{\lambda_3 f_L} + A_4 e^{\lambda_4 f_L}$$

$$(20) \qquad 1 + \lambda_1 A_1 e^{\lambda_1 f_H} + \lambda_2 A_2 e^{\lambda_2 f_H} = 0$$

$$(21) \qquad 1 + \lambda_3 A_3 e^{\lambda_3 f_L} + \lambda_4 A_4 e^{\lambda_4 f_L} = 0.$$

Equations (15') and (16') guarantee that the exchange rate reaches its zenith when the fundamental reaches $f_H$ and $s$ reaches its nadir when $f = f_L$. Equations (20) and (21) (usually and improperly called *smooth pasting conditions*) imply that at the boundaries the first derivative $g'$ is zero. The economic intuition for these latter requirements is that at the upper (lower) boundary of a credible band the exchange rate is *known* to appreciate (depreciate) without uncertainty. Since the conditional standard deviation of the rate of depreciation is $\sigma g'$, it must be the case that at the boundaries

$g'(f_H) = g'(f_L) = 0.$

The trajectory (9) under conditions (10–11–15'–16'–20–21) is shown in figure 3. It can be thought of as two S–shaped curves joining at the origin. For $\eta = 0$ we recover the same results of the standard theory and the trajectory looks like a S–shaped curve; this implies that the asymptotic distribution of the exchange rate concentrates probability mass on the edges of the band[9]. For $s_L \rightarrow -\infty$ and $s_H \rightarrow \infty$ we recover the results of the perforate zone presented in the previous section.

Obviously, for $\eta \neq 0$ an imperforate target zone as described above is always more stabilizing than a standard target zone defended by marginal interventions only or a perforate informal target zone defended by intramarginal interventions (changes in drift) only. It is interesting to note that for relatively large values of $\eta$, even a *perforate* target zone can be more stabilizing than a currency band defended by marginal interventions[10]. In other words, an informal target zone can in some cases be more effective than a formal one if the latter is defended only at the boundaries. At any rate, an informal zone is *always* less effective than an imperforate band when both marginal and intramarginal interventions are adopted.

The asymptotic density function of the fundamental in the presence of an imperforate target zone $\pi(f)$ is now defined as

(18')
$$\pi(f) = \frac{\mu}{\sigma^2} e^{\frac{2\eta}{\sigma^2}\hat{f}} = \left\{ \begin{array}{ll} \mu\sigma^{-2}\exp[2\eta\sigma^{-2}f] & \text{for } f_L \leq f \leq 0 \\ \mu\sigma^{-2}\exp[-2\eta\sigma^{-2}f] & \text{for } 0 < f \leq f_H \end{array} \right\}$$

where $\mu \equiv 2\eta/[2 - \exp(2\eta f_L/\sigma^2) - \exp(-2\eta f_H/\sigma^2)$. As before, this is the combination of

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[9] See Svensson (1989).

[10] Analytically, this result is obtained by comparing $g'(f)$ in eq. (17) with $g'(f)$ obtained by solving the system (9–10–11–15'–16'–20–21) when $\eta = 0$, keeping constant the values of $\sigma$, $\alpha$, $s_H$ and $s_L$.

two truncated exponentials. Obviously, in the imperforate case the support $[f_L, f_H]$ is bounded.

By eq. (19), the asymptotic distribution of the exchange rate is now trimodal at the central parity and at the edges (see figure 4). For $\eta = 0$ the density function is *U–shaped* as in the standard theory. For $\eta \to +\infty$ the probability mass is concentrated on the central parity and the distribution degenerates. In the intermediate cases more probability mass is distributed around the origin than in the U–shaped distribution of the standard theory.


## 3) Interest rate differential and exchange rate in a perforate band


If uncovered interest parity occurs and agents are risk neutral, the instantaneous interest rate differential is equal to the expected rate of depreciation by no arbitrage. In other words, defining as i the instantaneous domestic interest rate, $i^*$ the instantaneous foreign interest rate and $\xi \equiv i - i^*$ the instantaneous interest rate differential, it is

(22)
$$\xi(t) \equiv i(t) - i^*(t) = E_t \frac{ds}{dt} = \alpha \, (s(t) - f(t)) =$$

$$= \begin{cases} \eta \, (e^{-\lambda f} - 1) < 0 & \text{if } f > 0 \\ \eta \, ( 1 - e^{\lambda f}) \geq 0 & \text{if } f \leq 0 \end{cases}$$

where $\lambda \equiv \lambda_3 = -\lambda_2$. Note that the perforate band for the exchange rate induces an *imperforate* band for the interest rate differential such that $-\eta < \xi < \eta$. The relation between $\xi$ and $f$ is shown in figure 5. It is negatively sloped, concave for f<0 and convex for f>0.

We can now consider the stochastic process for $\xi$. Since

(23) $$d\xi = \alpha(ds - df) = \alpha[(\xi - \hat{\eta})\, dt + |(g'-1)|\, \sigma\, d\omega]$$

we can express the instantaneous variation of the interest rate differential as a function of the interest rate differential itself rather than as a function of the fundamental. In fact, equation (23) leads to

(24a) $$d\xi = \alpha(\xi + \eta)\, dt + \lambda\sigma(\xi + \eta)\, d\omega \quad \text{if} -\eta < \xi < 0$$

(24b) $$d\xi = \alpha(\xi - \eta)\, dt - \lambda\sigma(\xi - \eta)\, d\omega \quad \text{if} \ 0 \le \xi < \eta$$

The equations above implies that the a perforate zone for the exchange rate stabilizes the interest rate differential around its "parity", that is 0 in the symmetric case $\eta_L = -\eta_H$ we are considering. In other words, when the domestic interest rate is higher than or equal to the foreign one, the interest rate differential is instantaneously expected to decrease, and when the foreign interest rate is higher than the domestic one the interest rate differential is expected to increase. Intuitively, intramarginal interventions are expected to affect both fundamental and exchange rate in the same direction, but the exchange rate is less responsive to interventions than the fundamental. Since the interest rate differential is proportional to the difference between exchange rate and fundamental, expected depreciation (equal to interest rate differential) and exchange rate (or fundamental) move in opposite directions.

It is very convenient now to define the exchange rate in terms of the (potentially observable) interest rate differential rather than in terms of the (likely unobservable) fundamental. From equation (22) it is $\xi = \alpha(s-f)$, or $s = \frac{\xi}{\alpha} + f$. For positive values of the fundamental we have $\xi = \eta(e^{-\lambda f} - 1)$ or, inverting, $f = -\frac{1}{\lambda}\log\left(\frac{\xi}{\eta} + 1\right)$. For negative values of the fundamental we have $\xi = \eta(1 - e^{\lambda f})$ or $f = \frac{1}{\lambda}\log\left(1 - \frac{\xi}{\eta}\right)$, so that

(25a) $$s = \frac{\xi}{\alpha} - \frac{1}{\lambda}\log\left(\frac{\xi}{\eta} + 1\right) \qquad \text{for } \xi < 0$$

(25b)
$$s = \frac{\xi}{\alpha} + \frac{1}{\lambda}\log(1 - \frac{\xi}{\eta}) \qquad \text{for } \xi \geq 0$$

The relation between exchange rate and interest rate differential is represented in figure 6. It is negatively sloped, convex for negative $\xi$ and concave for positive $\xi$. In the standard target zone model the analogous relation between s and $\xi$ is negatively sloped as well, but the pattern of concavity is reversed.

As in the standard target zone model, there always exist a trade off between the instantaneous volatility of the interest rate differential and the exchange rate. Expressing all variables as a function of the interest rate differential (and considering only positive values of $\xi$ for simplicity), the instantaneous volatilities are $\sigma^S = (1 - \frac{\lambda}{\alpha}(\xi+\eta))\sigma$ and $\sigma^\xi = \lambda(\xi+\eta)\sigma$, so that $\sigma^S + \frac{1}{\alpha}\sigma^\xi = \sigma$. Since $\sigma$ is constant, a negative relation between $\sigma^S$ and $\sigma^\xi$ occurs. The volatility of the exchange rate reaches its peak and the volatility of the interest rate differential is zero only asymptotically when $|s| \to +\infty$ and $|\xi| = \eta$. The volatility of the exchange rate is minimized (but is not zero) and the volatility of the interest rate differential reaches its maximum when the exchange rate reaches its central parity: at $s = \xi = 0$ in fact $\sigma^S = (1 - \frac{\lambda\eta}{\alpha})\sigma$ and $\sigma^\xi = \lambda\eta\sigma$.

## 4) The term structure of interest rate differentials in a perforate band

In the previous sections we have analyzed the instantaneous and asymptotic properties of exchange rates. The complete characterization of the stylized perforate zone regime requires now the analysis of the "intermediate" cases defined in relation to a finite time period. In particular, we want to analyze the term structure of interest rate differentials induced by the presence of a mean reverting exchange rate in a perforate band.

We define as T the maturity of a zero coupon (pure discount) bond and as $\tau \equiv T - t$ the time to maturity. The interest rate differential for a given time to maturity $\tau$ is assumed to be function only of $\tau$ and the current exchange rate. For simplicity of

notation we pose $t = 0$, so that the *current* level of the exchange rate is denoted by $s_0$. The assumption of unbiased expectation hypothesis leads to the following relation

(26)
$$\xi(\tau, s_0) = \frac{E_{s_0} s(\tau) - s_0}{\tau}$$

Equation (26) is the finite time equivalent of equation (22). For $\tau \to 0$ we recover the familiar instantaneous uncovered interest parity analyzed in section 3 above, with $\xi(0, s_0) = \alpha(s_0 - g^{-1}(s_0))$ where $g^{-1}(s_0) = f_0$ and $f_0$ denotes the *current* level of the fundamental.

In order to determine the interest rate differential $\xi(\tau, s_0)$ we need to evaluate the expectation of the exchange rate at time $T - t$ conditional on the current level of s. In general the computation of $E_{s_0} s(\tau)$ is not straightforward. Svensson (1990) considers the strategies of solution (analytical and numerical) for the standard target zone case with infinitesimal reflecting interventions at the boundaries. In the lucky case of a perforate band, it is possible to provide a relatively simple analytical characterization in closed form.

The starting point is given by the computation of the transition density function for the fundamental in the presence of bang–bang control of brownian drift, defined as $\pi_\tau[f, f_0]$ such that

(27)
$$\pi_\tau[f, f_0] \, df = E_{f_0}[F(\tau) \in df]$$

where $F(\tau)$ is the stochastic variable with Ito differential as in eq. (1). In other words, $\pi_\tau[f, f_0]$ denotes the probability for the fundamental being equal to the value f at time $\tau$ contingent on the current fundamental being equal to $f_0$. Starting from the results by Shreve (1981) and Karatzas–Shreve (1988, p.441), after straightforward algebra and rescaling it can be shown that

(28a)
$$\pi_\tau[f,f_0] = \frac{1}{\sigma\sqrt{\tau}}\,\varphi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right] + \frac{\eta}{\sigma^2}\,e^{-2\eta\sigma^{-2}f}\left[1 - \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right]\right]$$

if $f_0 \geq 0,\ f > 0$

(28b)
$$\pi_\tau[f,f_0] = \frac{e^{2\eta\sigma^{-2}f_0}}{\sigma\sqrt{\tau}}\,\varphi\left[\frac{f_0 - f + \eta\tau}{\sigma\sqrt{\tau}}\right] + \frac{\eta}{\sigma^2}\,e^{2\eta\sigma^{-2}f}\left[1 - \Phi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right]\right]$$

if $f_0 \geq 0,\ f \leq 0$

(28c)
$$\pi_\tau[f,f_0,\eta] = \pi_\tau[-f, -f_0, -\eta]$$

where $\varphi(x) \equiv \dfrac{1}{\sqrt{2\pi}}\,e^{-(x^2/2)}$ and $\Phi(x) \equiv \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-(y^2/2)}\,dy$ respectively denote the

density function and the cumulative function of a standard normal random variable (in Appendix A it is shown that eq. (28) defines a density function). Given the symmetry of Brownian motion, we can assume $f_0 \geq 0$ without loss of generality: for $f_0 < 0$ it is possible to compute $\pi_\tau[f,f_0]$ according to (28c). Note that for $\tau \to \infty$ we recover the asymptotic distribution (independent of the current value of the fundamental) analyzed in section 1 (eq. 18).

Since $f_0 = g^{-1}(s_0)$, the expected value of the exchange rate at time $\tau$ given its current value $s_0$ is given by

(29)
$$E_{s_0} s(\tau) = \int_{-\infty}^{\infty} g(f)\,\pi_\tau[f, g^{-1}(s_0)]\,df = \int_{-\infty}^{\infty} g(f)\,\pi_\tau[f, f_0]\,df =$$

$$= (f_0 - \eta\tau - \frac{\eta}{\alpha})\,\Phi\left[\frac{f_0 - \eta\tau}{\sigma\sqrt{\tau}}\right] + (f_0 + \eta\tau + \frac{\eta}{\alpha})\exp\left[\frac{2\eta f_0}{\sigma^2}\right]\left[1 - \Phi\left[\frac{f_0 + \eta\tau}{\sigma\sqrt{\tau}}\right]\right] +$$

$$+\frac{\eta}{\alpha}\exp\left[\lambda(\frac{\sigma^2\lambda\tau}{2}+\eta\tau-f_0)\right]\left[1-\Phi\left[\frac{\eta\tau-f_0+\sigma^2\lambda\tau}{\sigma\sqrt{\tau}}\right]\right]-$$

$$-\frac{\eta}{\alpha}\exp\left[\frac{2\eta f_0}{\sigma^2}+\lambda(\frac{\sigma^2\lambda\tau}{2}+\eta\tau+f_0)\right]\left[1-\Phi\left[\frac{\eta\tau+f_0+\sigma^2\lambda\tau}{\sigma\sqrt{\tau}}\right]\right].$$

Algebraic details of the computation of eq. (29) are provided in Appendix B.

The expected value of the exchange rate as a function of time to maturity is represented in Figure 7 for alternative (positive) values of $s_0$. If the exchange rate "starts off" positive, it is expected to decrease monotonically in the term and rapidly approach its steady state mean 0. If the current level of s is 0, the exchange rate is not expected to change. The higher the current value of the exchange rate, the higher the expected value of s at any future time $\tau$. Figure 8 shows the term structure of interest rate differentials according to eq.(26). A positive current level of the (log of) exchange rate (and fundamental) implies a negative current level of the instantaneous interest rate differential denoted by $\xi(0,s_0)$ in Figure 8. The interest rate differential increases monotonically in the term and reaches asymptotically its steady state mean 0. These results are qualitatively similar to the findings by Svensson (1990) for the standard target zone model defended only by marginal interventions. In Figure 8 three term structures are plotted, corresponding to alternative values of the policy parameter $\eta$, with $\eta_1<\eta_2<\eta_3$. The higher the value of the drift, the faster $\xi(\tau,s_0)$ approaches its steady state mean.

Figure 9 shows the interest rate differential as a function of the fundamental (only positive values of f are considered) for given values of $\tau$. For a very short term $\tau_1$ the graph of $\xi(\tau,f)$ is identical to Figure 5. For longer terms $\tau_2$ and $\tau_3$ the graph becomes flatter and concave in the neighborhood of the origin but steeper and convex for relatively large values of the fundamental. As a net result, the graph of $\xi(\tau_3,f)$ for instance is relatively linear despite of the change in concavity while the graph of $\xi(\tau_1,f)$ is highly non

linear. Figure 10 shows the (empirically testable) relation between the term structure of interest rate differentials on the x–axis and the log of the exchange rate (again, only positive values of s are considered) on the y–axis. As in Figure 9, we find that for short terms the relation is highly non linear ($\tau_1$ in Figure 10) and the graph is identical to Figure 6. For longer terms $\tau_2$ and $\tau_3$ the graphs become closer and closer to the vertical axis (partially "losing" non–linearity, as shown for instance by the graph for $\tau_3$). At any rate, for relatively high values of s the interest rate differential, independent of time to maturity, tends to reach its lower boundary $-\eta$.

These results are intuitive given the setup of the perforate zone regime. The long term interest rate differential is relatively close to the steady state mean 0 independent of the current value of the fundamental and exchange rate, since intramarginal interventions are asymptotically expected to stabilize fundamental and exchange rate around their parities. At the same time, if the current value of the exchange rate (and fundamental) happens to be relatively large, the behavior of the exchange rate in a perforate zone regime is similar to the behavior in free float, so that the interest rate differential is close to its lower (free float) limit $-\eta$ independent of the term to maturity.

## 5) Conclusions

The model presented in this paper has provided a simple analytical characterization of an exchange rate regime which is consistent with the main stylized facts of the EMS and other currency band schemes, namely the reduction of exchange rate volatility, the concentration of the exchange rate around its central parity and the coexistence of marginal and intramarginal interventions. In general, the model of a *perforate* currency band as described above seems to provide an extremely promising setup for analyzing the properties of international asset prices and returns, in the presence of institutional schemes of exchange rate management equidistant from the textbook cases of free float and fixed

exchange rate regimes. This paper has analyzed the relation between exchange rates and interest rates in a perforate zone and provided empirically testable implications of the model, including a closed form solution for the term structure of interest rate differentials as a function of the spot exchange rate.

The setup here introduced can be considered as an intermediate step toward a theoretical *explanation* of the stylized facts themselves. Starting from the realistic a *priori* that the Central Bank is concerned with reducing the volatility of the exchange rate (an objective function implicit in the entire literature on target zones), an *imperforate* target zone as defined above is always more effective than a target zone defended by marginal interventions only. Moreover, in some cases even an informal (*perforate*) target zone which allows for the exchange rate being above (below) any upper (lower) target level with non zero probability can make the exchange rate less responsive to shocks in fundamentals than a formal zone defended by interventions at the boundaries.

A complementary interpretation can be sketched as follows. The stabilizing properties of a target zone defended by marginal interventions derive from the expectations induced by the announcement of such interventions. Before the exchange rate hits one of its boundaries, private agents do not observe any signal of the willingness of the Central Bank to defend the zone, so that the reduction in exchange rate volatility exclusively relies on the degree of credibility of the policy rule. In the framework analyzed in this paper, instead, intramarginal interventions give unequivocal observable signs of the intentions of the Central Bank at any moment in time, strengthening the credibility of the intervention rule and its effectiveness. If common beliefs of private agents do not exclude St. Thomas–like skepticism regarding the future actions of the Central Bank, the tangibility of continuous intramarginal interventions may become a necessary condition for guaranteeing the success of any scheme of exchange rate management.

## APPENDIX A: Equation (28) as a density function

We want to show that $\int\limits_{-\infty}^{\infty} \pi_{_T}[f, f_0]\, df = 1$ choosing $f_0 > 0$ without loss of generality. This integral can be thought of as the sum of the following four components:

(A1)
$$\int\limits_0^{\infty} \frac{1}{\sigma\sqrt{\tau}}\, \varphi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right]\, df = \Phi\left[\frac{f_0 - \eta\tau}{\sigma\sqrt{\tau}}\right]$$

(A2)
$$\int\limits_{-\infty}^{0} e^{2\eta\sigma^{-2}f_0}\, \frac{1}{\sigma\sqrt{\tau}}\, \varphi\left[\frac{f_0 - f + \eta\tau}{\sigma\sqrt{\tau}}\right]\, df = e^{2\eta\sigma^{-2}f_0}\left(1 - \Phi\left[\frac{f_0 + \eta\tau}{\sigma\sqrt{\tau}}\right]\right)$$

(A3)
$$\int\limits_0^{\infty} \frac{\eta}{\sigma^2}\, e^{-2\eta\sigma^{-2}f}\left[1 - \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right]\right]\, df =$$
$$= \frac{1}{2}\left[1 - \Phi\left[\frac{f_0 - \eta\tau}{\sigma\sqrt{\tau}}\right] - e^{2\eta\sigma^{-2}f_0}\left(1 - \Phi\left[\frac{f_0 + \eta\tau}{\sigma\sqrt{\tau}}\right]\right)\right]$$

(A4)
$$\int\limits_{-\infty}^{0} \frac{\eta}{\sigma^2}\, e^{2\eta\sigma^{-2}f}\left[1 - \Phi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right]\right]\, df =$$
$$= \frac{1}{2}\left[1 - \Phi\left[\frac{f_0 - \eta\tau}{\sigma\sqrt{\tau}}\right] - e^{2\eta\sigma^{-2}f_0}\left(1 - \Phi\left[\frac{f_0 + \eta\tau}{\sigma\sqrt{\tau}}\right]\right)\right]$$

The computation of (A1) and (A2) is trivial. Moreover it is easy to show that equations (A3) and (A4) are identical simply by transforming the variable of integration: After posing $f \equiv -q$ expression (A4) becomes $-\int\limits_{\infty}^{0} \frac{\eta}{\sigma^2} e^{-2\eta\sigma^{-2}q}[1 - \Phi\left[\frac{f_0 + q - \eta\tau}{\sigma\sqrt{\tau}}\right]]\, dq$ which is equal to eq. (A3).

The right hand side of eq. (A3) can be derived by integrating by parts as follows:

$$\int_0^\infty \frac{\eta}{\sigma^2} e^{-2\eta\sigma^{-2}f} \left[1 - \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right]\right] df =$$

$$\int_0^\infty \frac{\eta}{\sigma^2} e^{-2\eta\sigma^{-2}f} df - \int_0^\infty \frac{\eta}{\sigma^2} e^{-2\eta\sigma^{-2}f} \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right] df =$$

$$-\frac{1}{2}e^{-2\eta\sigma^{-2}f}\Big|_0^\infty - \left[-\frac{1}{2}e^{-2\eta\sigma^{-2}f} \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right]\Big|_0^\infty + \right.$$

$$\left. + \int_0^\infty \frac{1}{2} e^{-2\eta\sigma^{-2}f} \frac{1}{\sigma\sqrt{\tau}} \varphi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right] df\right] =$$

$$= \frac{1}{2} - \frac{1}{2}\Phi\left[\frac{f_0 - \eta\tau}{\sigma\sqrt{\tau}}\right] - \frac{1}{2}\int_0^\infty \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left[\frac{-(f_0+f+\eta\tau)^2 + 4f_0\eta\tau}{2\sigma^2\tau}\right] df =$$

$$\frac{1}{2} - \frac{1}{2}\Phi\left[\frac{f_0 - \eta\tau}{\sigma\sqrt{\tau}}\right] - \frac{1}{2}e^{2\eta\sigma^{-2}f_0} \Phi\left[\frac{f_0 + f + \eta\tau}{\sigma\sqrt{\tau}}\right]\Big|_0^\infty \qquad \text{Q.E.D.}$$

Finally, it can be easily checked that the sum of the right hand sides of (A1) to (A4) is equal to 1.

## APPENDIX B: Computation of $E_t \, s(\tau)$

Equation (29) can be thought of as the sum of the following twelve expressions:

(B1)
$$-\frac{\eta}{\alpha} \int_0^\infty \frac{1}{\sigma\sqrt{\tau}} \, \varphi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right] \, df$$

(B2)
$$\frac{\eta}{\alpha} \int_{-\infty}^0 e^{2\eta\sigma^{-2}f_0} \frac{1}{\sigma\sqrt{\tau}} \, \varphi\left[\frac{f_0 - f + \eta\tau}{\sigma\sqrt{\tau}}\right] \, df$$

(B3)
$$-\frac{\eta}{\alpha} \int_0^\infty \frac{\eta}{\sigma^2} e^{-2\eta\sigma^{-2}f} [1 - \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right]] \, df$$

(B4)
$$\frac{\eta}{\alpha} \int_{-\infty}^0 \frac{\eta}{\sigma^2} e^{2\eta\sigma^{-2}f} [1 - \Phi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right]] \, df$$

(B5)
$$\frac{\eta}{\alpha} \int_0^\infty e^{-\lambda f} \frac{1}{\sigma\sqrt{\tau}} \, \varphi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right] \, df$$

(B6)
$$-\frac{\eta}{\alpha} \int_{-\infty}^0 e^{2\eta\sigma^{-2}f_0} e^{\lambda f} \frac{1}{\sigma\sqrt{\tau}} \, \varphi\left[\frac{f_0 - f + \eta\tau}{\sigma\sqrt{\tau}}\right] \, df$$

(B7)
$$\frac{\eta}{\alpha} \int_0^\infty \frac{\eta}{\sigma^2} e^{-\lambda f} e^{-2\eta\sigma^{-2}f} [1 - \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right]] \, df$$

(B8)
$$-\frac{\eta}{\alpha} \int_{-\infty}^0 \frac{\eta}{\sigma^2} e^{\lambda f} e^{2\eta\sigma^{-2}f} [1 - \Phi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right]] \, df$$

(B9)
$$\int\limits_{0}^{\infty} f \frac{1}{\sigma\sqrt{\tau}} \varphi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right] df$$

(B10)
$$\int\limits_{-\infty}^{0} f\, e^{2\eta\sigma^{-2}f_0} \frac{1}{\sigma\sqrt{\tau}} \varphi\left[\frac{f_0 - f + \eta\tau}{\sigma\sqrt{\tau}}\right] df$$

(B11)
$$\int\limits_{0}^{\infty} f \frac{\eta}{\sigma^2} e^{-2\eta\sigma^{-2}f} [1 - \Phi\left[\frac{f_0 + f - \eta\tau}{\sigma\sqrt{\tau}}\right]] df$$

(B12)
$$\int\limits_{-\infty}^{0} f \frac{\eta}{\sigma^2} e^{2\eta\sigma^{-2}f} [1 - \Phi\left[\frac{f_0 - f - \eta\tau}{\sigma\sqrt{\tau}}\right]] df$$

The computation of (B1 − B4) is immediate given the results of Appendix A. The sum of (B1) to (B4) gives

$$-\frac{\eta}{\alpha} \Phi\left[\frac{f_0 - \eta\tau}{\sigma\sqrt{\tau}}\right] + \frac{\eta}{\alpha} e^{2\eta\sigma^{-2}f_0} (1 - \Phi\left[\frac{f_0 + \eta\tau}{\sigma\sqrt{\tau}}\right]).$$

Notice now that the sum of expressions (B7) and (B8) is 0, and analogously the sum of expressions (B11) and (B12) is 0. These results can be easily checked by posing $f \equiv -q$, so that expression (B8) becomes

$$+\frac{\eta}{\alpha} \int\limits_{\infty}^{0} \frac{\eta}{\sigma^2} e^{-\lambda q} e^{-2\eta\sigma^{-2}q} [1 - \Phi\left[\frac{f_0 + q - \eta\tau}{\sigma\sqrt{\tau}}\right]] dq$$

and expression (B12) becomes

$$\int\limits_{\infty}^{0} q \, \frac{\eta}{\sigma^2} \, e^{-2\eta\sigma^{-2}q} \, [1 - \Phi\left[\frac{f_0 + q - \eta\tau}{\sigma\sqrt{\tau}}\right]] \, dq.$$

In order to compute expression (B5), first rewrite (B5) as

$$\frac{\eta}{\sigma\alpha\sqrt{2\pi\tau}} \int\limits_{0}^{\infty} \exp\left[-\lambda f - \frac{(f_0 - f - \eta\tau)^2}{2\sigma^2\tau}\right] df = \frac{\eta}{\sigma\alpha\sqrt{2\pi\tau}} \int\limits_{0}^{\infty} e^{-(af^2 + 2bf + c)} df$$

where  $a \equiv \dfrac{1}{2\sigma^2\tau}$, $b \equiv \dfrac{\eta\tau - f_0 + \sigma^2\lambda\tau}{2\sigma^2\tau}$  and  $c \equiv \dfrac{(f_0 - \eta\tau)^2}{2\sigma^2\tau}$.

By properties of the error function  erf(x)  (see for instance Abramowitz M. — Stegun I. (eds.) (1964, 1972[10]) — **Handbook of Mathematical Functions**, National Bureau of Standards, eq. 7.4.32 p. 303), it is

$$\int\limits_{0}^{\infty} e^{-(ax^2 + 2bx + c)} \, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp(\frac{b^2 - ac}{a}) \, erf\left[\frac{ax + b}{\sqrt{a}}\right]\Bigg|_{0}^{\infty}$$

where  $erf(x) = 2\Phi(x\sqrt{2}) - 1$  for  $x \geq 0$ (**Handbook**, cit., eq. 26.2.29 p.934).

In our case expression (B5) becomes

$$\frac{\eta}{2\sigma\alpha\sqrt{2a\tau}} \exp(\frac{b^2 - ac}{a}) \, [2\,\Phi\left[\frac{af + b}{\sqrt{a}}\sqrt{2}\right] - 1]\Bigg|_{0}^{\infty} = \frac{\eta}{\sigma\alpha\sqrt{2a\tau}} \exp(\frac{b^2 - ac}{a}) \, [1 - \Phi\left[\frac{\sqrt{2}\,b}{\sqrt{a}}\right]]$$

and after substituting for the values of  a, b and c  expression (B5) is equal to

$$\frac{\eta}{\alpha} \exp[\lambda(\frac{\sigma^2\lambda\tau}{2} + \eta\tau - f_0)] \, (1 - \Phi\left[\frac{\eta\tau - f_0 + \sigma^2\lambda\tau}{\sigma\sqrt{\tau}}\right]).$$

Following the previous scheme, it is possible to compute expression (B6) as

$$-\frac{\eta}{\alpha}\exp[\frac{2\eta f_0}{\sigma^2}+\lambda(\frac{\sigma^2\lambda\tau}{2}+\eta\tau+f_0)]\;(1-\Phi\left[\frac{\eta\tau+f_0+\sigma^2\lambda\tau}{\sigma\sqrt{\tau}}\right]).$$

In order to evaluate expression (B9), consider the following equation:

$$\frac{d}{df}\left[\sigma\sqrt{\tau}\;\varphi\left[\frac{f_0-f-\eta\tau}{\sigma\sqrt{\tau}}\right]\right]=\frac{d}{df}\frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}\exp\left[\frac{-(f_0-f-\eta\tau)^2}{2\sigma^2\tau}\right]=(f_0-f-\eta\tau)\frac{1}{\sigma\sqrt{\tau}}\;\varphi\left[\frac{f_0-f-\eta\tau}{\sigma\sqrt{\tau}}\right].$$

Integrating the equation above and rearranging, expression (B9) becomes equal to

$$(f_0-\eta\tau)\;\Phi\left[\frac{f_0-\eta\tau}{\sigma\sqrt{\tau}}\right]+\sigma\sqrt{\tau}\;\varphi\left[\frac{f_0-\eta\tau}{\sigma\sqrt{\tau}}\right].$$

Analogously, it can be checked that expression (B10) becomes

$$e^{2\eta\sigma^{-2}f_0}\left[(f_0+\eta\tau)\left[1-\Phi\left[\frac{f_0+\eta\tau}{\sigma\sqrt{\tau}}\right]\right]-\sigma\sqrt{\tau}\;\varphi\left[\frac{f_0+\eta\tau}{\sigma\sqrt{\tau}}\right]\right].$$

Finally, note that the expression

$$+\sigma\sqrt{\tau}\left[\varphi\left[\frac{f_0-\eta\tau}{\sigma\sqrt{\tau}}\right]-\exp\left[\frac{2\eta f_0}{\sigma^2}\right]\varphi\left[\frac{f_0+\eta\tau}{\sigma\sqrt{\tau}}\right]\right]$$

is equal to zero. The sum of expressions (B1) to (B12) gives the right hand side of eq. (29).

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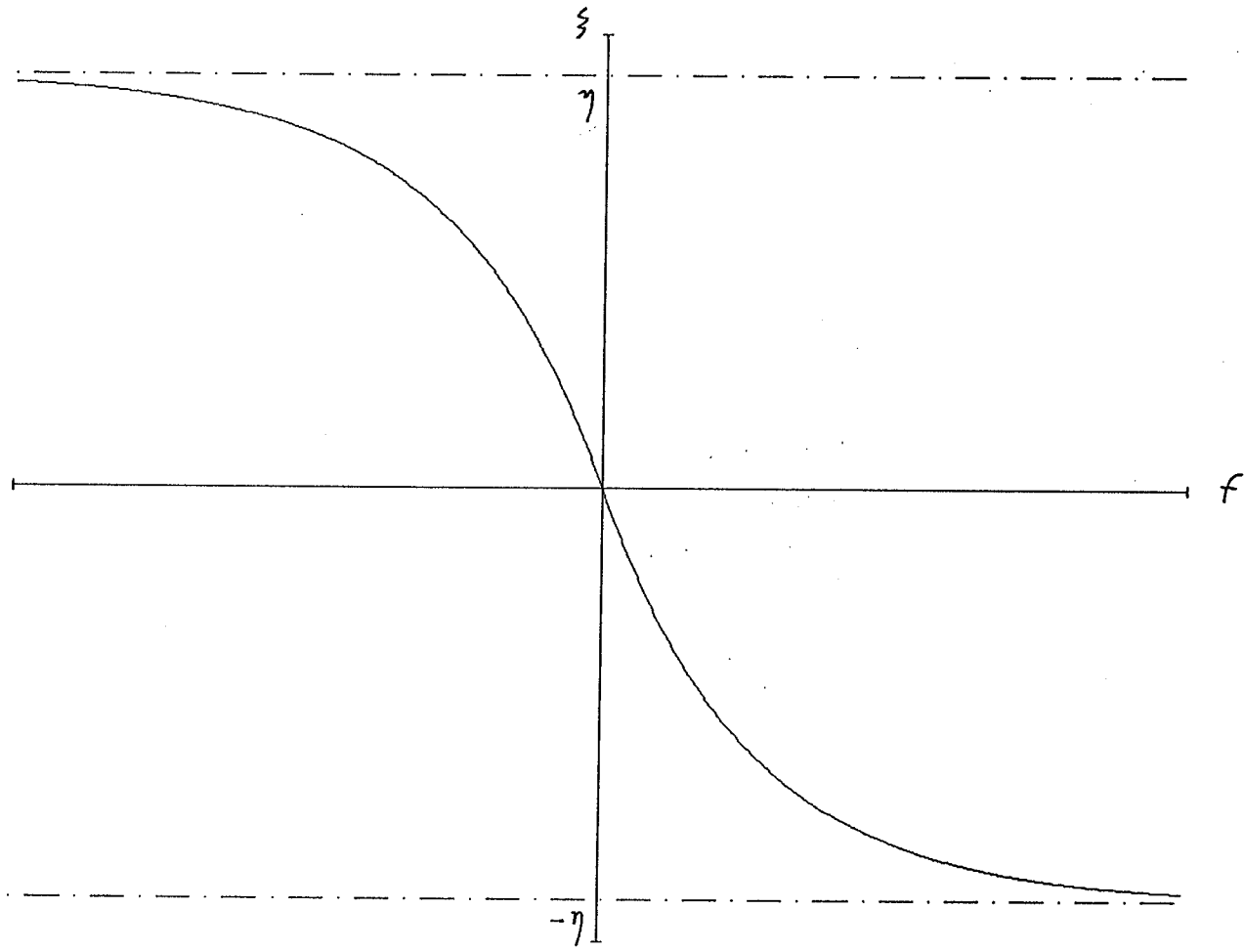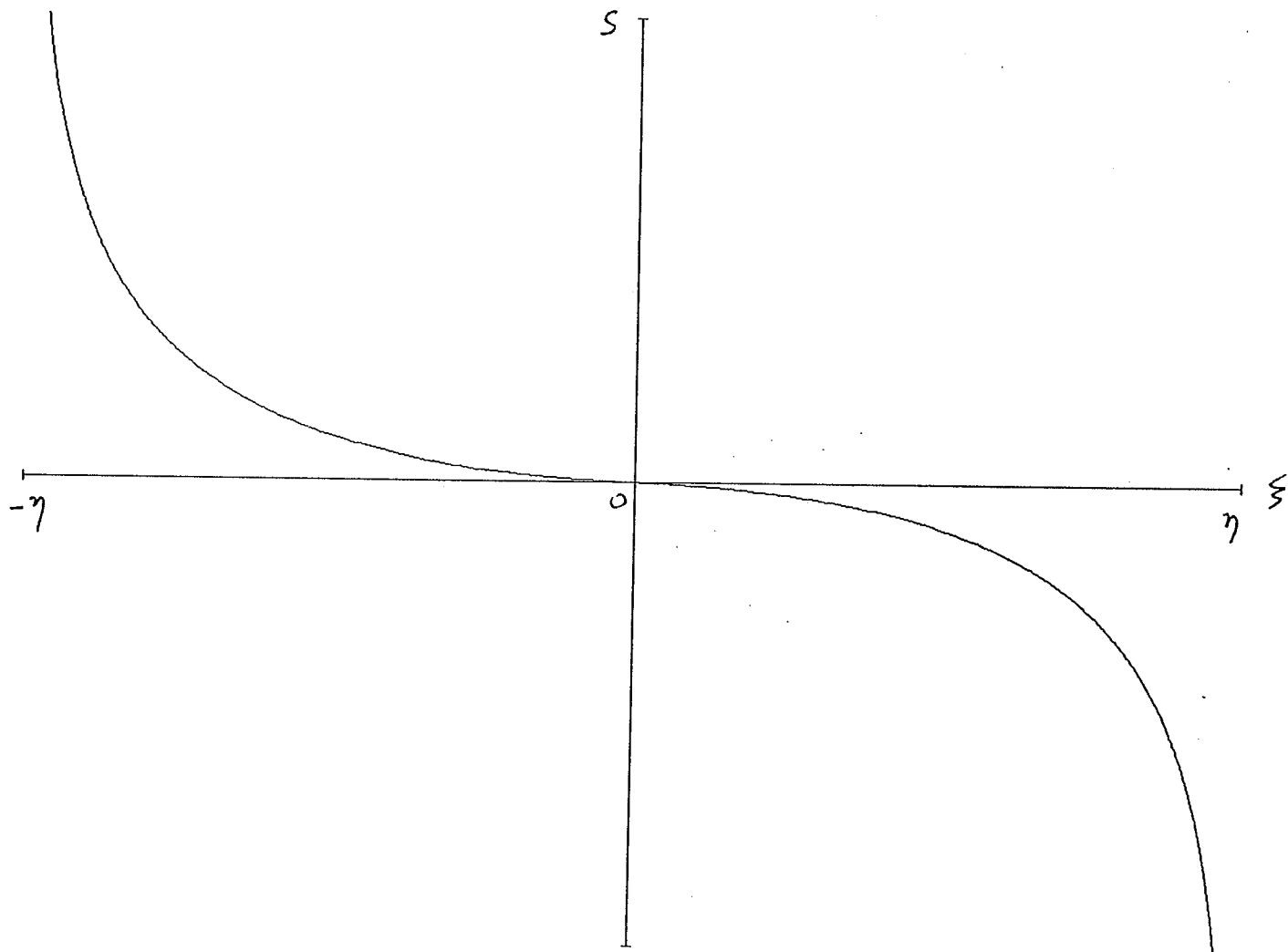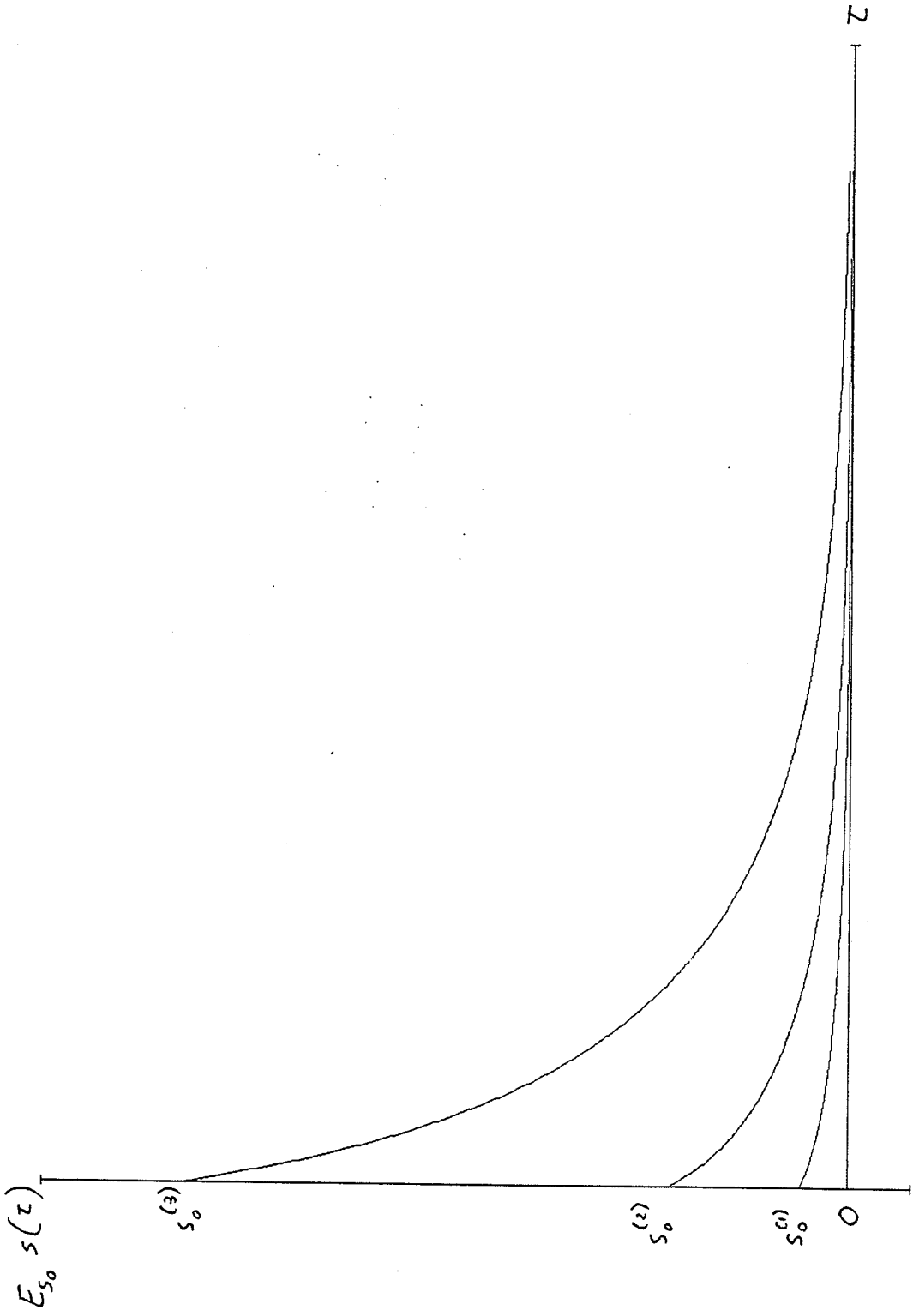
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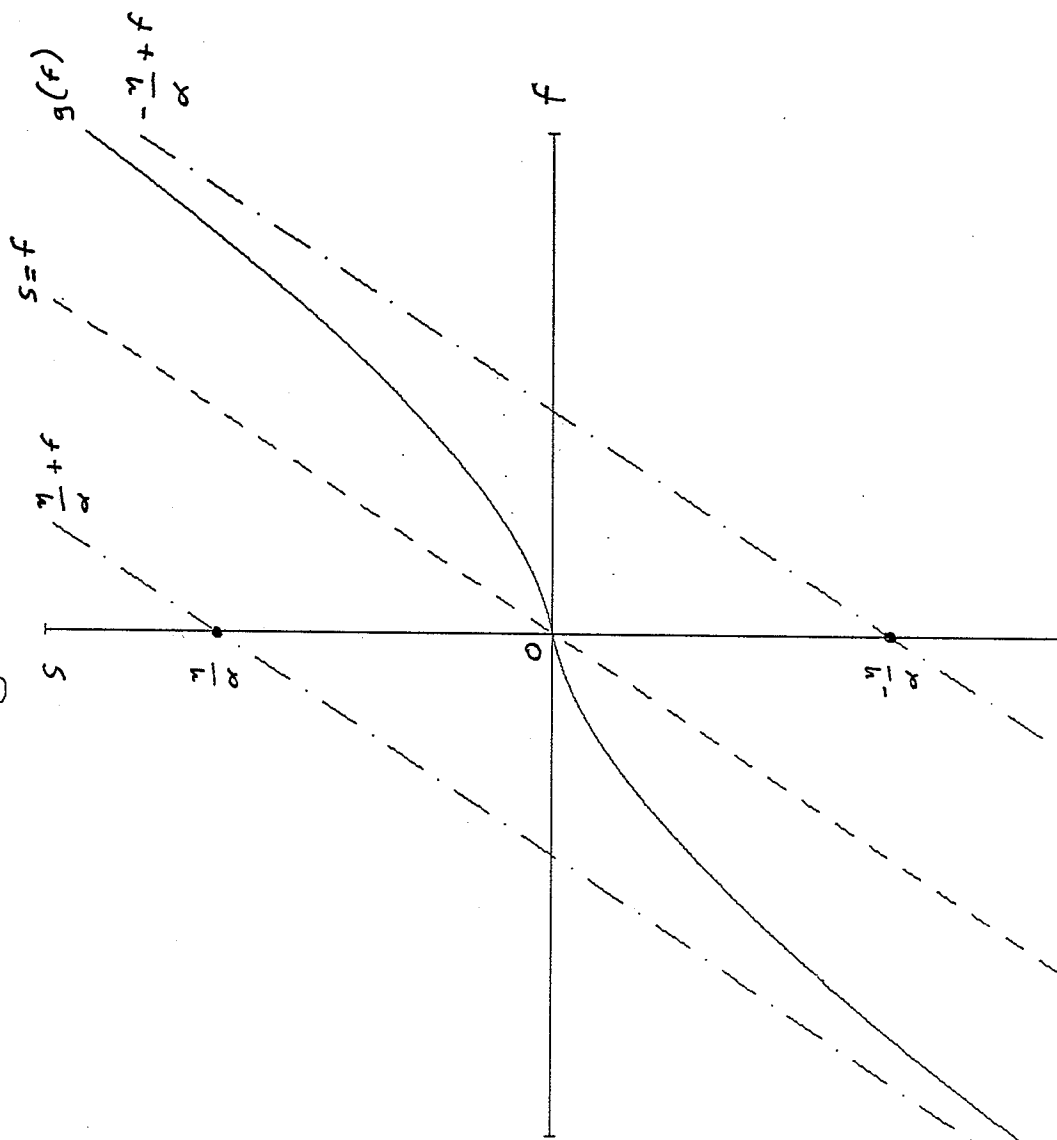
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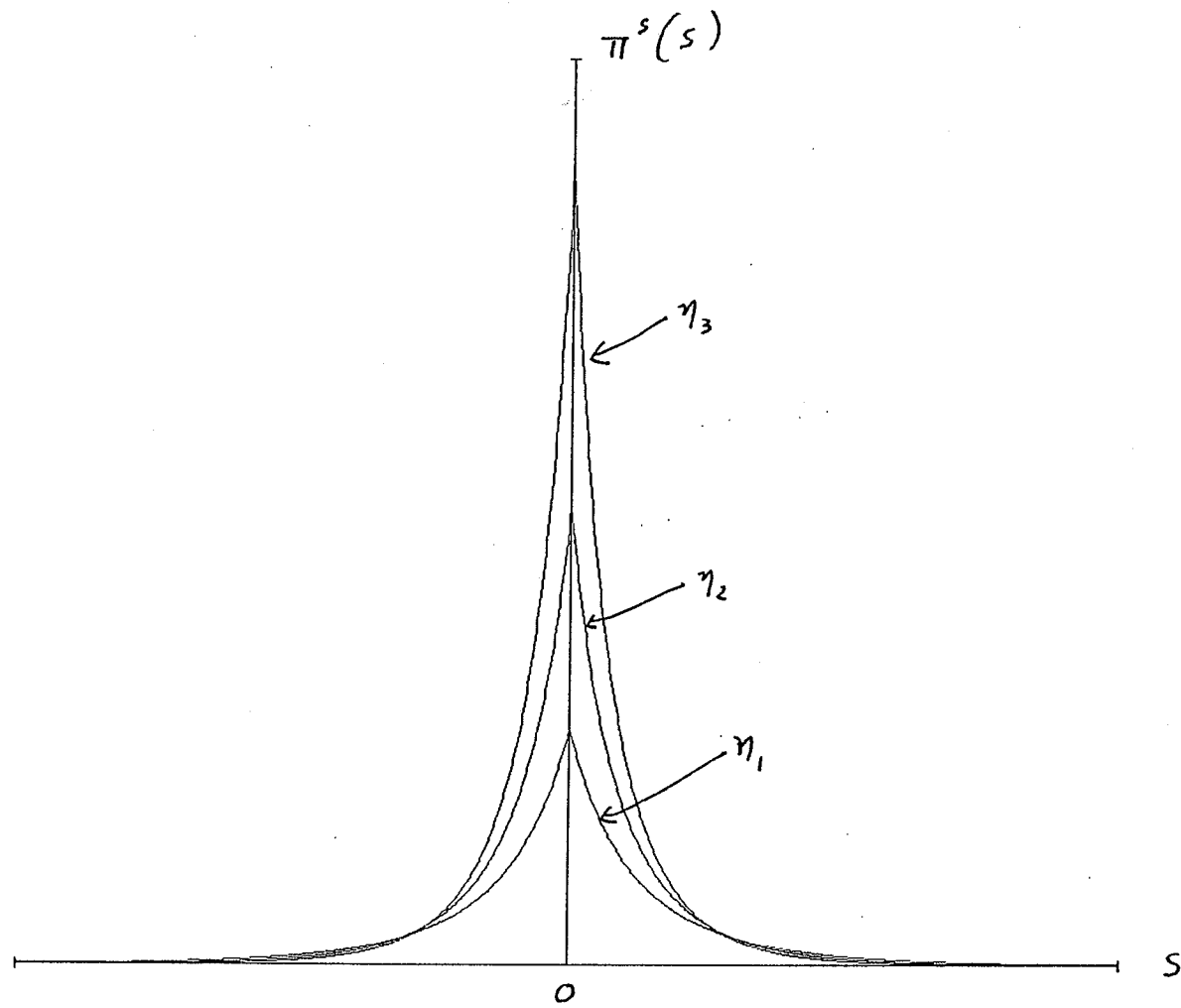
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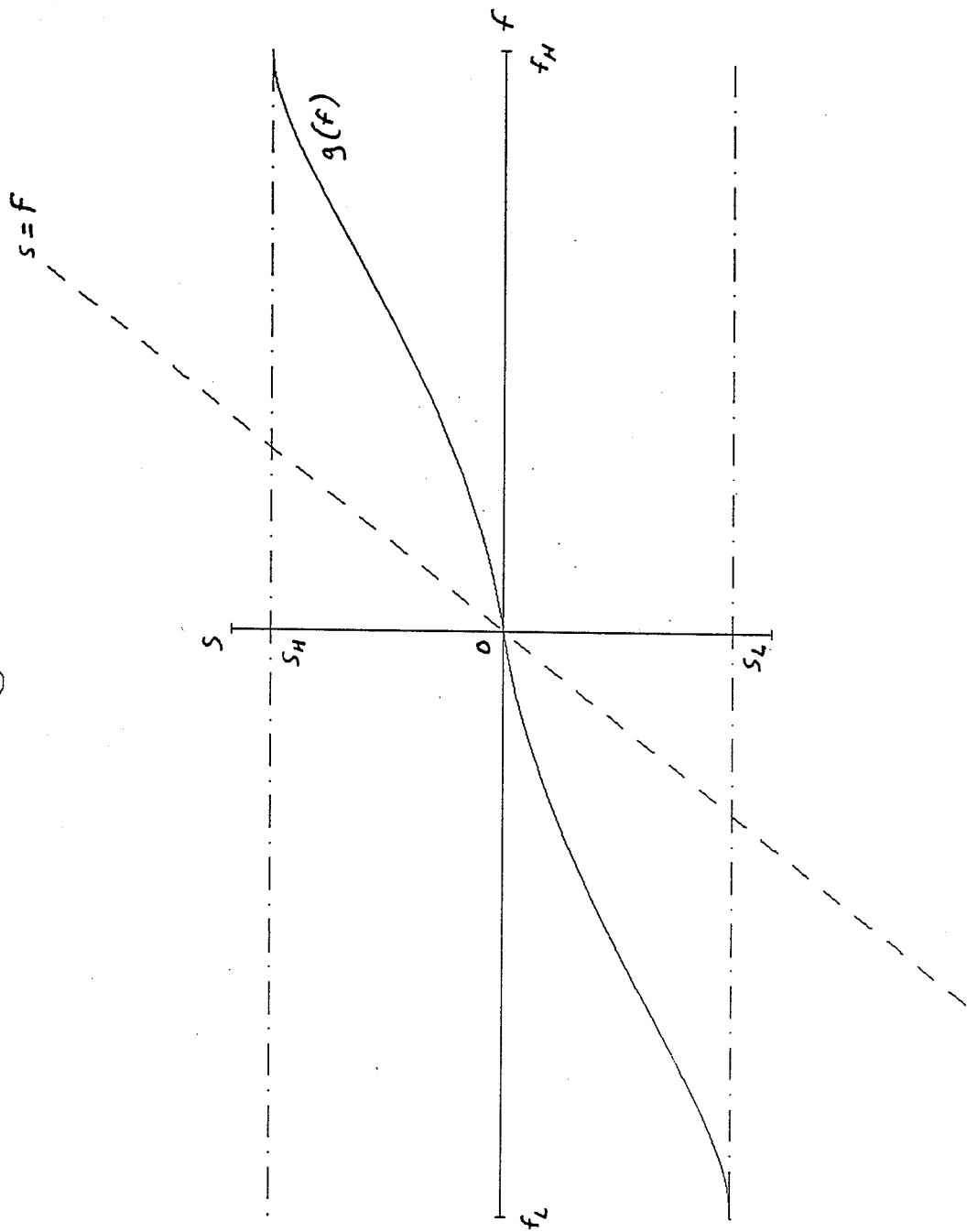
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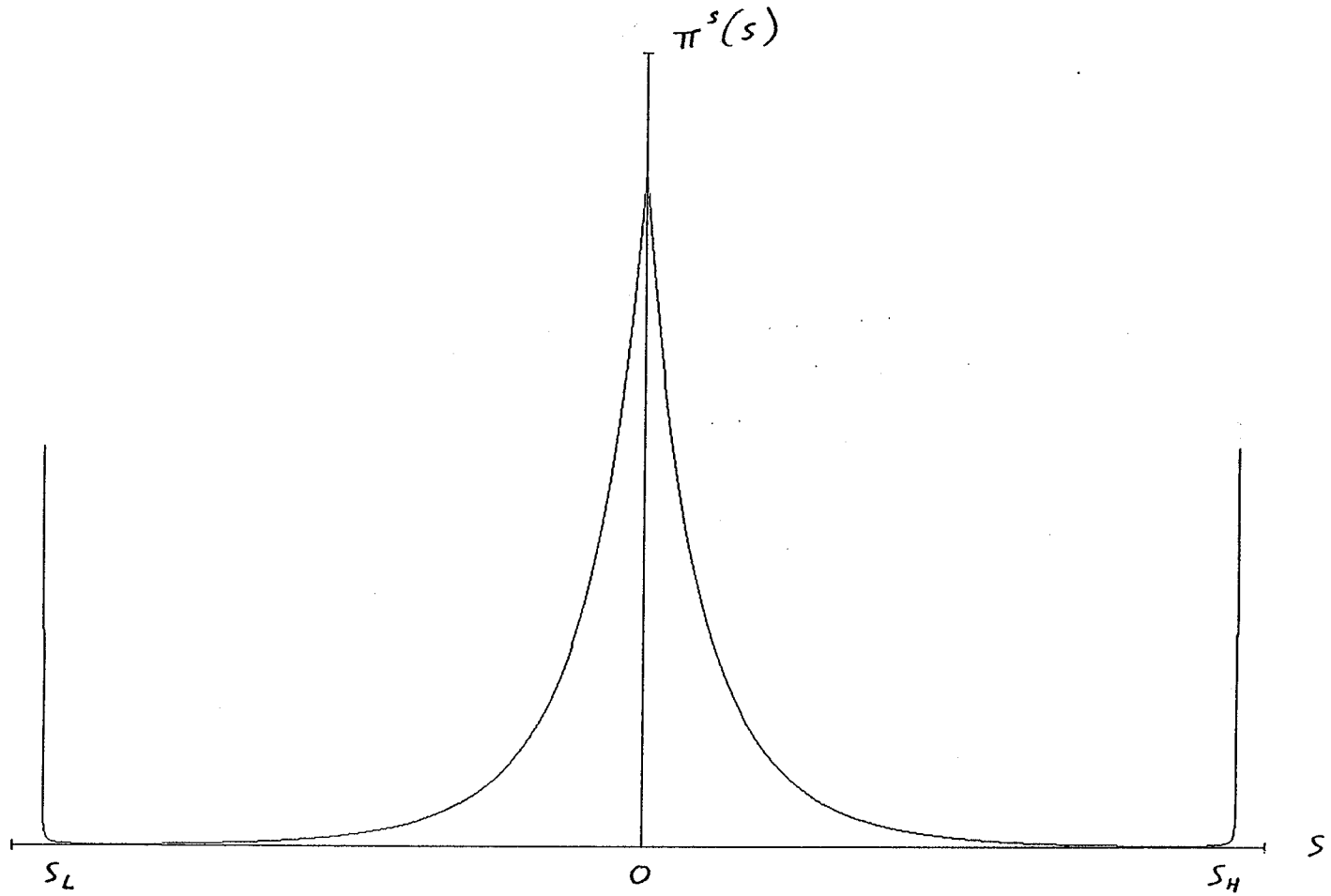
# Figure 5

# Figure 6

Figure 7

Figure 8

Figure 9

Figure 10



$s$

$\xi(z_3, s)$

$\xi(z_2, s)$

$\xi(z_1, s)$

$\xi(z, s)$

$l_-$