

Yale University

EliScholar – A Digital Platform for Scholarly Publishing at Yale

Discussion Papers

Economic Growth Center

12-1-1990

Lecture Notes on Economic Growth: Five Prototype Models of Endogenous Growth, Volume II

Xavier Sala-i-Martin

Follow this and additional works at: <https://elischolar.library.yale.edu/egcenter-discussion-paper-series>

Recommended Citation

Sala-i-Martin, Xavier, "Lecture Notes on Economic Growth: Five Prototype Models of Endogenous Growth, Volume II" (1990). *Discussion Papers*. 630.

<https://elischolar.library.yale.edu/egcenter-discussion-paper-series/630>

This Discussion Paper is brought to you for free and open access by the Economic Growth Center at EliScholar – A Digital Platform for Scholarly Publishing at Yale. It has been accepted for inclusion in Discussion Papers by an authorized administrator of EliScholar – A Digital Platform for Scholarly Publishing at Yale. For more information, please contact elischolar@yale.edu.

ECONOMIC GROWTH CENTER

YALE UNIVERSITY

Box 1987, Yale Station
New Haven, Connecticut 06520

CENTER DISCUSSION PAPER NO. 622

LECTURE NOTES ON ECONOMIC GROWTH:
FIVE PROTOTYPE MODELS OF ENDOGENOUS GROWTH
VOLUME II

Xavier Sala-i-Martin

Yale University

December 1990

Notes: Center Discussion Papers are preliminary materials circulated to stimulate discussion and critical comments.

These notes were developed while I was assisting Robert J. Barro to teach the first year graduate macroeconomics class at Harvard in 1988-89 and 1989-90. The two vintages of first year students suffered the first draft of these notes and made all kinds of useful comments/amendments. Thank you kids. I am especially grateful to Susan Guthrie for carefully reading them and helping me think about section 1. I also received comments from Serge Marquié and (Kid) Casey Mulligan from Harvard, and Julie Lee from Yale.

This paper is the revised version of the first draft, November 1988.

See Discussion Paper No. 621 for Volume I of Lecture Notes on Economic Growth.

LECTURE NOTES ON ECONOMIC GROWTH:
FIVE PROTOTYPE MODELS OF ENDOGENOUS GROWTH

VOLUME II

Abstract

This paper explores the five simplest models of endogenous growth. We start with the AK model (Rebelo (1990)) and argue that all endogenous growth models can be viewed as variations or microfoundations of it. We then examine the Barro (1990) model of government spending and growth. Next we look at the Arrow-Sheshinski-Romer model of learning by doing and externalities. The Lucas (1988) model of human capital accumulation is then considered. Finally, we present a simple model of R&D and growth.

KEY WORDS: Economic Growth, Increasing Returns, Externality, Endogenous Growth

"A view of Economic Growth that depends so heavily on an exogenous variable, let alone one so difficult to measure as the quantity of knowledge, is hardly intellectually satisfactory. From a quantitative, empirical point of view, we are left with time as an explanatory variable. Now trend projections, however necessary they may be in practice, are basically a confession of ignorance, and, what is worse from a practical viewpoint, are not policy variables" (Arrow (1962), p.155).

INTRODUCTION

In Section 1 of the first part of the notes we saw that the key to endogenous growth was the inexistence of diminishing returns to the inputs that can be accumulated. This implies that the "return to investment" (RI) in all these types of models ends up being a constant A^*

$$(1) r = A^*$$

Endogenous Growth models combine this return to investment with the usual return to consumption schedule, which was derived from a constant intertemporal elasticity of substitution (IES) utility function

$$(2) r = \rho + \sigma\gamma$$

which states that the return to consumption (RC) is a premium on the discount rate. The premium is larger the larger the economy is expected to grow (larger γ) and the more willing people are to smooth consumption (large σ). In steady state the growth rate γ is constant so equations 1 and 2 can be pictured as in Figure 1. Notice that the crossing point determines the steady state growth rate of the economy. In order to interpret this figure, it will be useful to compare it with Figure 2, which represents the Neoclassical model with exogenous productivity growth. The return to

consumption is the same as in Figure 1 and the return to investment is a vertical line at $\gamma=g$, where g is the exogenous productivity growth rate.

Figure 2 says that changes in the parameters that affect the savings rate such as the discount rate ρ and the coefficient of IES σ (these changes are represented by shifts and twists of the Return to Consumption line) affect the steady state interest rate but not the steady state growth rate. The reason is that the long run growth rate in these models is exogenously determined at g . Notice, on the other hand, that the same shifts in σ and ρ in Figure 1 imply changes in the long growth rate of the economy. Changes in the return to investment A^* also have effects on the long run growth rate.

Almost all the endogenous growth literature is concerned with the parameter A^{*1} . If we find the determinants of A^* and how policy affects them, we will know what determines long run economic growth. In this second part of the notes we will start with the simplest model of endogenous growth where the production function is assumed to be linear in the only input, capital. This simple model is very important since all the other endogenous growth models can be thought of as extensions or microfoundations of the basic linear one.

In section (5) we will explore the Barro model of government spending, distortionary taxes and growth. In section 6 we will show a model of learning by doing where the return to investment is kept constant by agents that constantly improve technology (learn) as they work (do). In section 7 we explore a model of human capital accumulation where people become more productive as they invest in their human capital (study). In

1

Paradoxically, almost no work has been done in trying to understand the determinants of the discount rate and the elasticity of intertemporal substitution. Notice that if we knew why some countries are more impatient or more willing to intertemporally substitute consumption than others, we would know what determines long run growth given the return to investment A^* . The parameters in the utility function, however, have always been taken as given and, therefore, not subject to policy actions. One exception is the work on fertility choice by Barro and Becker (1988) and others, where discount rates are linked to income through the willingness and ability to raise children.

the final Section we present the simplest model of R&D, where growth is kept alive through the constant introduction of new varieties of capital goods.

(4) CONVEX ENDOGENOUS GROWTH MODELS: REBELO (1990)

(a) The Model.

The simplest possible endogenous growth model is the so called "AK" model developed by Rebelo (1990). The production function is assumed to be linear in the only input, capital. Hence, the production function is both constant returns to scale and constant returns to capital.

$$(4.1) Y = F(K,L) = AK.$$

where A is an exogenous constant and K is aggregate capital broadly defined (so not only it includes physical capital but it may also include human capital as well as stock of knowledge and maybe other types of capital such as financial capital, etc.). Assume for simplicity that the population does not grow at all ($n=0$) and that the depreciation rate is zero (none of the results depend on these two assumptions). The utility function is the usual constant Intertemporal Elasticity of Substitution.

$$(4.2) U(0) = \int_0^{\infty} e^{-rt} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] dt$$

As we did in the last section, we will start thinking about a model of household production and then we will show that the market economy will yield the same solution. In the household production model, the dynamic capital accumulation constraint is

$$(4.3) \dot{k} = Ak - c$$

where use of (4.1) has been made. Households maximize (4.2) subject to

(4.3). The Hamiltonian is the usual:

$$(4.4) \quad H(\cdot) = e^{-\rho t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] + \nu \left[Ak - c \right]$$

And the FOC are:

$$(4.5) \quad e^{-\rho t} c^{-\sigma} = \nu$$

$$(4.6) \quad \dot{\nu} = -\nu A$$

$$(4.7) \quad TVC$$

Take logs and derivatives of (4.5) to get $\dot{\nu}/\nu = -\rho - \sigma\gamma$ where, again $\gamma = \dot{c}/c$ is the balanced growth rate of per capita consumption. By substituting this in (4.6) we can get the growth rate as a function of the "first principles" parameters:

$$(4.8) \quad \dot{c}/c = \gamma = (A - \rho)/\sigma$$

We can rewrite (4.8) as

$$(4.9) \quad \gamma\sigma + \rho = A.$$

Again, the left hand side is the return to consumption and the right hand side is the return to investment. The return to consumption depends on the discount rate (maybe because people like their children but they like themselves better) and it depends on the growth rate for smoothing reasons: if $\sigma > 0$, people like to smooth consumption. If consumption is growing people want to smooth their consumption paths by bringing some future consumption to the present. The return to investment is simply A (there are no adjustment costs or diminishing returns to capital so the return is independent to the growth rate or the capital stock). To find the steady state growth rate of per capita capital, divide both sides of the

dynamic constraint by the capital labor ratio and call $\dot{k}/k = \gamma'$:

$$(4.10) \quad \dot{k}/k = \gamma' = A - c/k \rightarrow \gamma' - A = -\delta = -c/k$$

Taking logs and derivatives of both sides of (4.10), and given that γ is constant in the steady state, we get

$$(4.11) \quad \dot{c}/c = \gamma = \dot{k}/k = \gamma'$$

That is, in steady state capital and consumption grow at the SAME constant rate γ . Finally, by taking logs and derivatives of the production function (2.1) we see that output will also grow at the same rate γ .

(b) Transitional Dynamics

We just showed that, in the steady state, consumption, capital and output grow at the same constant rate. Equation 4.8 tells us that consumption will always grow at a constant rate given by $\sigma^{-1}(A-\rho)$ so consumption is always in steady state. Given this, equation (4.10) says that IF CAPITAL GROWTH is constant, then all variables grow at the same rate. Can we say that capital always grows at a constant rate?. To answer this question let us start by taking the budget constraint (4.3) and integrate it between 0 and T (pre multiply both sides by the integrating factor e^{-At} and take into account that c grows at a constant rate $(A-\rho)/\sigma$ so $c_t = c_0 e^{(A-\rho)t/\sigma}$).

$$(4.12) \quad \int_0^T (\dot{k}_t - Ak_t) e^{-At} dt = -c_0 \int_0^T e^{(A-\rho-\sigma A)t/\sigma} dt$$

the solution of which is

$$(4.14) \quad k_T = \left[k_0 - c_0 \sigma / (\rho - (1-\sigma)A) \right] e^{AT} + \left[c_0 \sigma / (\rho - (1-\sigma)A) \right] e^{(A-\rho)T/\sigma}$$

or

$$(4.15) \quad k_T = \alpha e^{AT} + \beta e^{(A-\rho)T/\sigma}$$

where $\alpha = \left[k_0 - c_0 \sigma / (\rho - (1-\sigma)A) \right]$ and $\beta = \left[c_0 \sigma / (\rho - (1-\sigma)A) \right]$. We can now put k_t in the transversality condition and let T go to infinity

$$\begin{aligned} (4.16) \quad k_T u'(c_T) e^{-\rho T} &= (\alpha e^{AT} + \beta e^{(A-\rho)T/\sigma}) c_T^{-\sigma} e^{-\rho T} \\ &= (\alpha e^{AT} + \beta e^{(A-\rho)T/\sigma}) c_0^{-\sigma} e^{-\sigma(A-\rho)T/\sigma} e^{-\rho T} \\ &= (\alpha e^{AT} + \beta e^{(A-\rho)T/\sigma}) c_0^{-\sigma} e^{-AT} \\ &= (\alpha + \beta e^{(A(1-\sigma)-\rho)T/\sigma}) c_T^{-\sigma} e^{-\rho T} \end{aligned}$$

For the limit of this expression to be zero as T goes to infinity we need two things:

(a) $A(1-\sigma) - \rho < 0$ which we know is satisfied (this is the "bounded utility condition" that we imposed at the outset)

and (b) $\alpha = 0$

But if $\alpha = 0$, k_T can be rewritten as

$$(4.17) \quad k_T = \beta e^{(A-\rho)T/\sigma}$$

which is equivalent to say (just take logs and derivatives of both sides) that $\dot{k}_t/k_t = (A-\rho)/\sigma$ at all times. Hence, capital also grows at a constant rate all the time so there are no transitional dynamics in this model.

(c) Savings, Growth and, Convergence.

Finally, it is interesting to analyze what this economy predicts about the interaction between the savings and the growth rates. Let us write the savings rate

$$(4.18) \quad \text{savings rate} = s/y = \dot{k}/y = (\dot{k}/k)(k/y) = \gamma(1/A) = (1-\rho/A)/\sigma$$

The growth rate of a country depends on its saving rate² and on how productive its technology is ($\gamma=(s/y)A$). Determinants of the saving rate are ρ and σ . The more patient a society is (low ρ), the larger the saving and growth rates. The more willing to substitute intertemporally (low σ), also the larger the saving and growth rates. What determines A remains unexplained and it will be the subject of the next few models.

This is the first model that does not predict convergence. Suppose that countries have the same parameters (A, σ, ρ) but for some reason, they differ in their initial $k(0)$. Since they will all grow at the same constant rate γ , the poor countries will always be poorer in levels. Suppose that countries differ also in their productivity parameters ($A_i \neq A_j$ for $i \neq j$). This implies that "low growth" countries will remain "low growth countries" forever, independently of initial income or product (this contrasts with the neoclassical result where poorer countries tend to grow faster to their steady state level of income). An alternative way to see this is to use the linearization around steady state developed in section 2 and to let the capital share, α , go to one. Notice that in this case $\mu=0$ so the "negative" eigenvalue is $\lambda_1 = \rho - n = (\rho - n) = 0$. The convergence equation (2.19) says that the coefficient on initial income predicted by this model is exactly zero. Again, this convergence implication has been used by a heterogeneity of authors to test validity of the neoclassical.

(d) The "market" model.

In solving the model the way we did in Section (a), we implicitly assumed that households do the production at home. Alternatively we could

2

If population and depreciation rates were not set equal to zero we would have that $s=S/y=(k+(\delta+n)k)/k=(k/k+(\delta+n))(k/y)=(\gamma+(\delta+n))/A$ which implies that $\gamma=sA-(\delta+n)$. This is the growth rate we found in the introductory section of the first part of these notes when we were dealing with a constant savings rate.

have modeled households maximizing utility subject to a financial constraint of the form

$$(4.12) \quad \dot{b}_t = rb_t - c_t$$

where b is financial wealth and r is the return to financial wealth. Financial wealth is made out of physical capital plus bonds. Because it is assumed that the economy is closed, the net supply of bonds is zero so, at the aggregate level b is equal to k . The first order condition of this problem is the usual

$$(4.14) \quad r = \rho + \sigma\gamma$$

which can be interpreted as the return to consumption (RC). In steady state (that is when γ is constant), this relation is an upward sloping line in the r, γ space. Firms, on the other hand, are assumed to produce output with the linear technology (4.1). They also take the interest rate as given and choose inputs and outputs so as to maximize profits. The first order conditions require the equalization of interest rate and marginal product of capital

$$(4.15) \quad r = A$$

which, in the $r-\gamma$ space is represented by a flat line at A . Notice that the combination of (4.14) and (4.15), which is depicted in Figure 1, yields the same steady state we found for the household production model of Section (a). It just remains to be shown that the growth rates of capital and consumption are the same. We can do that by substituting (4.15) in (4.12) to get

$$(4.16) \quad \dot{k} = Ak - c$$

we can divide both sides by k , realize that, in steady state \dot{k}/k is a

constant. Put all constant in one side of the equation, take logs and derivatives of both sides and conclude that the growth rate of consumption is the same as the growth rate of capital. Hence, the solution to the market model is the same as the solution to the household production model.

(5) THE BARRO (1990) MODEL OF PUBLIC SPENDING

(a) The Model of Household Production.

This is a growth model that tries to link growth to fiscal variables. If we think Rebelo's k as representing a BROAD MEASURE OF CAPITAL we could read this model as a particular version of Rebelo's.

Barro assumes that some inputs are publicly provided private goods. It is hard to think what these goods really are in actual economies. Two natural extensions of the Barro (1990) model are developed in Barro and Sala-i-Martin (1990). The first one considers pure non rival public good in the Samuelson (1954) sense. This model, however, ends up having the unappealing implication that economies with large population will grow faster. The second variation considers public goods subject to congestion (such as highways, airports or courts of law). This second model is the most realistic of the three and it does not have the scale effect that the pure public goods model has. In these notes, however, we will show how to solve the original Barro model, the other two being straightforward extensions of it.

The key assumption is that the production function is CR to government spending (g)³ and capital (k) together but it is DR to k and g separately. In its Cobb-Douglas specification the aggregate production is

$$(5.1) \quad y = f(k, g) = Ak^{(1-\alpha)}g^\alpha$$

3

In a slight abuse of notation we are denoting government spending by g . Thus, g does NOT correspond to exogenous productivity growth as it did in previous sections.

As usual, we will assume that individuals choose a consumption path so as to maximize the traditional CES utility function subject to the dynamic constraint. We assume that every individual is a very small part of the society so each of them takes public spending as given. For simplicity, we will assume that the government has to balance its budget at all times (no public debt is permitted) and that the only public source of income is an income tax. The program, therefore becomes:

$$(5.2) \text{ MAX } U(0) = \int_0^{\infty} e^{-\rho t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] dt$$

$$(5.3) \text{ Subject to } \dot{k} = (1-\tau)Ak^{1-\alpha}g^\alpha - c \text{ and } k(0)$$

$$(5.4) \text{ where } g = \tau y = \tau Ak^{1-\alpha}g^\alpha$$

where τ is the constant average and marginal income tax rate. Notice that for the first time we find a model that could potentially yield a non-optimal competitive equilibrium. That is, the "private" equilibrium will have to be solved assuming that individuals take g as given. Individual choices, however, affect everybody's output. "When I increase my production, I increase public income (because the government maintains a fixed ratio of public expenditure to output or, equivalently, a fixed income tax) which, in turn, increases productivity for everybody". This externality could potentially be a source of non optimality in the sense that if the model is solved by a planner, he will take the externality into account and yield a solution that may not be the same as the competitive equilibrium one.

(b) Equilibrium

Since individuals take government expenditures as given to them, the problem they face is a concave one (constant returns to the inputs they can choose, namely labor and private capital). Hence, there will be a set of prices that support the competitive equilibrium. To find such an equilibrium, we have to optimize taking g as given. The Hamiltonian,

therefore is:

$$(5.5) H() = e^{-\rho t} \left[\frac{c^{1-\sigma} - 1}{1-\sigma} \right] + \nu \left[(1-\tau) A k^{(1-\alpha)} g^\alpha - c \right]$$

The FOC of the program are:

$$(5.6) e^{-\rho t} c^{-\sigma} = \nu$$

$$(5.7) \dot{\nu} = -\nu \left[(1-\tau) A (1-\alpha) k^{-\alpha} g^\alpha \right] = -\nu \left[(1-\tau) A (1-\alpha) (g/k)^\alpha \right]$$

$$(5.8) \text{TVC}$$

By taking logs and derivatives of (5.6) and substituting in (5.7) we will get the usual growth condition that states that the balanced growth rate is proportional to the difference between the MPK and the discount rate, ρ .

$$(5.9) \dot{c}/c \equiv \gamma = \sigma^{-1} \left[(1-\tau) A (1-\alpha) (g/k)^\alpha - \rho \right]$$

We can now manipulate the government budget constraint (5.4) to get the size of the government $\tau = g/y = g / (A k^{(1-\alpha)} g^\alpha) = (g/k)^{(1-\alpha)} A^{-1}$. Let's substitute for (g/k) to get:

$$(5.10) g/k = (\tau A)^{1/(1-\alpha)}$$

Plug (5.10) in (5.9) to get the growth rate as a function of the parameters τ , ρ , σ , A and α .

$$(5.11) \gamma = \sigma^{-1} \left[A^* - \rho \right]$$

where $A^* = (1-\alpha) A^{1/(1-\alpha)} (1-\tau)^\alpha / (1-\alpha)$. As usual, by dividing the dynamic constraint by k , taking logs and derivatives of both sides we will see that the growth rate of capital is the same as the growth rate of consumption ($\dot{k}/k = \dot{c}/c = \gamma$). Suppose now that a benevolent government tried to maximize the

growth rate of the economy taking into account that people behave competitively⁴. In other words, what is the τ that maximizes γ in (5.11)? Just take derivatives, set them equal to zero and get that the optimal τ is $\tau^* = \alpha$. We can calculate the saving rate in the usual way:

$$(5.12) \quad s/y = \dot{k}/y = (\dot{k}/k)(k/y) = \gamma A^{-1/(1-\alpha)} \tau^{-\alpha/(1-\alpha)}$$

Finally, the usual condition for $U(0)$ to be bounded (take limits of the term inside the integral when t tends to infinity and let them go to zero):

$$(5.14) \quad \lim_{t \rightarrow \infty} e^{-\rho t} c_t^{1-\sigma} - e^{-\rho t} = \lim_{t \rightarrow \infty} e^{-\rho t} c(0) e^{\gamma(1-\sigma)t}$$

The condition $\rho > \gamma(1-\sigma)$ ensures that this limit is zero and, therefore, that $U(0)$ is bounded.

(c) The Command Economy.

The command economy solution will take into account the fact that private output affects public income and (through the production function) other people's marginal product of capital. In other words, to solve for the command economy we have to substitute the public budget constraint into the Hamiltonian and take the FOC from it. The new Hamiltonian is:

$$(5.15) \quad H() = e^{-\rho t} \left[\frac{c^{1-\sigma} - 1}{1-\sigma} \right] + \nu \left[(1-\tau)kA^{1/(1-\alpha)} \tau^{\alpha/(1-\alpha)} - c \right]$$

4

It is not clear that this is what a benevolent dictator would like to do. It would seem more reasonable to assume that he wants to maximize the utility (not the growth rate) of the representative consumer. For the Cobb Douglas production and CES utility assumptions assumed here, however, the two are the same (see Barro (1990)).

The FOC are:

$$(5.16) \quad e^{-\rho t} c_t^{-\sigma} = \nu$$

$$(5.17) \quad \dot{\nu} = -\nu \left[(1-\tau)A^{1/(1-\alpha)} \tau^{\alpha/(1-\alpha)} - \rho \right]$$

By substituting in the usual way we will get the "command growth rate", γ^c .

$$(5.17) \quad \gamma^c = \sigma^{-1} \left[(1-\tau)A^{1/(1-\alpha)} \tau^{\alpha/(1-\alpha)} - \rho \right]$$

Since $0 < \alpha < 1$, It is clear that the competitive growth rate (5.11) is smaller than the command rate for all values of τ . A decentralized economy involves too little growth. The reason is that it also involves too little savings⁵. Notice, finally, that the growth rate is maximized at $\tau^* = \alpha$, the same result as in the competitive equilibrium.

Growth in this model is achieved through the government action: when private individuals decide to save one unit of consumption and purchase a unit of capital with it, the government is forced (because he wants to maintain a constant τ) to provide one more unit of public input. This avoids diminishing returns capital so individuals keep investing forever at constant rates, which is the ultimate source of growth.

(d) Market Equilibrium

As we did in Section 4, we have implicitly assumed that households in this economy produce their own output. As before, the same results would obtain if they operated in a perfectly competitive market. Again, the first order conditions from the consumer side can be represented by equation (4.14). In steady state, this can be depicted as an upward sloping line in Figure 3. Firms, once again, maximize profits subject to the constraint that

5

$$s = S/y = \dot{k}/y = (k/k)(k/y) = \gamma^c A^{-1/(1-\alpha)} \tau^{-\alpha/(1-\alpha)}$$

net output is given by

$$(5.18) \quad (1-\tau)Ak^{(1-\alpha)}g^\alpha$$

The first order condition will entail the equalization of marginal product to the real interest rate. After substituting in the government budget constraint this return to investment is

$$(5.19) \quad r = \alpha(1-\tau)A^{1/\alpha} \tau^{(1-\alpha)/\alpha}$$

which corresponds to the horizontal line called RI_{private} in Figure 3. The intersection of the two lines yields the growth rate in (5.11). It remains to be shown that the growth rate of capital and consumption are the same. We can verify that by plugging (5.19) in the household's budget constraint (4.3) to get

$$(5.20) \quad \dot{k} = (1-\tau)\alpha A^{1/\alpha} \tau^{(1-\alpha)/\alpha} k - c$$

We can divide (5.20) by k . Realize that in the steady state \dot{k}/k is constant, put all the constants in (5.20) in one side and take logs and derivatives to find that $\dot{c}/c = \dot{k}/k$. Hence, the market solution is equivalent to the household production solution.

The planner, on the other hand, would take into consideration the government budget constraint before calculating first order conditions. That is it would maximize profits subject to the constraint that net output is given by

$$(5.20) \quad (1-\tau)A^{1/\alpha} \tau^{(1-\alpha)/\alpha} k$$

The equalization of real return to the marginal product of capital yields

$$(5.21) \quad r = (1-\tau)A^{1/\alpha} \tau^{(1-\alpha)/\alpha}$$

which is larger than the r in (5.19) since $\alpha < 1$. This social rate of return is pictured as a horizontal line called RI_{planner} in Figure 3. Notice that the intersection between (5.21) and (4.14) in Figure 3 yields a superior steady state growth rate for the planner economy. This is, again, because the planner takes into account the fact that when firms raise output they raise government revenue and, given the public budget constraint, raise productive public spending and everybody else's productivity. Since competitive firms do not take into account such an externality (ie their perceived return is smaller than the planner) they underinvest so the competitive growth rate is lower than optimal.

(6) LEARNING BY DOING, EXTERNALITIES AND INCREASING RETURNS.

(a) The Model.

In the paper that started the literature on endogenous economic growth, Romer (1986) follows Arrow (1962) and Sheshinski (1967) in solving the $\beta=1$ problem by postulating increasing returns to scale at the economy wide level but CRS at the firm level. That is, in order to support the equilibrium with a set of competitive prices he needs to assume that the Increasing Returns are external to the firm. This externality, however, will yield non optimal equilibria.

Arrow argues that the acquisition of knowledge (learning) is related to experience. He cites examples from the airframe industry where there is strong evidence of the interplay between experience and increasing productivity. He argues that a good measure of increase in experience is investment because "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning takes place with continuous new stimuli" (p. 157). It follows that an index of experience is cumulative investment or capital stock. More formally, let the production function for firm i be a function of its capital stock, its

capital labor corrected by the state of knowledge at time t , $A(t)$

$$(6.1) Y_{it} = F(K_{it}, A(t)L_{it})$$

and let experience be a function of the past investments of ALL the firms in the economy which, under the assumption of no depreciation is equal to the aggregate capital stock $\kappa(t)$

$$G(t) = \int_{-\infty}^t I(v)dv = \kappa(t)$$

Based on the experience of the airframe industry Arrow further assumes that the relation between experience and the state of knowledge is

$$A(t) = G(t)^\eta$$

where $\eta < 1$. It follows that the individual production function can be rewritten as

$$Y_i = F(K_i, L_i, \kappa) = K_i^\beta L_i^{1-\beta} \kappa^\eta$$

This production is CR in K_i and L_i holding κ fixed and IRS if we consider the three "inputs" at the same time. We will assume that the number of firms is a large constant number M . Since M is large, every firm will take the aggregate stock of capital as given even though $\kappa = \sum_{i=1}^M k_i = Mk$. This, again, will give rise to an externality that will make the competitive equilibrium non optimal in the sense that a command economy would achieve a larger growth rate in the steady state and a larger utility. By aggregating across firms, the aggregate production function is

$$(6.1)' Y = F(K, L, \kappa) = K^\beta L^{1-\beta} \kappa^\eta$$

where $K = Mk_i$ and $L = ML_i$. It is convenient to work in per capita terms so

let's divide both sides of (6.1)' by L to get

$$(6.2) y = k^\beta \kappa^\eta$$

where $k=K/L$ and $y=Y/L$. Households maximize a typical CES utility function subject to the dynamic constraint (assume again no population growth⁶):

$$(6.3) \dot{k} = k^\beta \kappa^\eta - c$$

(b) Competitive Equilibrium

Households in a competitive economy will take the aggregate stock of capital as given. To solve the program we have to set up the familiar Hamiltonian

$$(6.4) H() = e^{-\rho t} \left[\frac{c^{1-\sigma} - 1}{1-\sigma} \right] + \nu \left[k^\beta \kappa^\eta - c \right]$$

The First Order Conditions are:

$$(6.5) e^{-\rho t} c_t^{-\sigma} = \nu$$

$$(6.6) \dot{\nu} = -\nu \left[\beta k^{-(1-\beta)} \kappa^\eta \right]$$

$$(6.7) TVC$$

Equilibrium in the capital market requires that total capital be equal to the sum of individual capital stocks: $\kappa = Lk$. By using this condition, taking logs and derivatives of (6.5) and plugging in equation

6

As shown in Section 1, this assumption is crucial for this particular type of model.

(6.6) we will, once again, get the growth rate for consumption:

$$(6.8) \dot{c}/c \equiv \gamma = \sigma^{-1} \left[\beta k^{-(1-\eta-\beta)} L^\eta - \rho \right]$$

which states that the growth rate of consumption is proportional to the difference between the marginal product of capital and the individual discount rate. We can divide both sides of the dynamic constraint (6.3) by k and then take logs and derivatives to show that the capital stock will grow, in the steady state, at the same rate as consumption. Notice that equation (6.8) has the implication that countries with a lot of population experience large growth. This "Scale Effect" is certainly counterfactual (see Backus, Kehoe and Kehoe (1990) for an empirical study of scale effects). This scale effect is due to the assumption that the externality is captured by the aggregate capital stock. If the externality was captured by the average capital stock instead, the growth rate would be $\gamma = (\beta - \rho) / \sigma$, which is independent of total labor supply.

Is this model capable of generating positive steady state growth ($\gamma > 0$)? Suppose first that $\beta + \eta < 1$. If this is the case, the model is exactly equal to the Ramsey-Cass-Koopmans model (only that the relevant capital share is not β but $\beta + \eta$). The steady state implies $\gamma = 0$. This gives an important insight⁷: IRS by themselves are not enough to generate persistent growth!. What we need is VERY increasing returns. That is, we need η to be large enough so as to satisfy $\beta + \eta = 1$. In this case, the model looks very much like Rebelo's (in fact, let's define A^* as being equal to βL^η and condition (6.9) in this model exactly matches condition (4.8) in the Rebelo one)⁸. The

7

This is the familiar result that in order to have sustainable growth, we need CRS in all inputs that can be accumulated!

8

The difference between the two models is that the private and social marginal products of capital are different for the Romer model but not for the Rebelo one.

steady state rate of growth will be

$$(6.9) \quad \gamma = \sigma^{-1}(A^* - \rho)$$

where $A^* = \beta L^\eta$. Romer shows how a technology that exhibits IRS of the form $\beta + \eta > 1$ can generate increasing growth rates (as opposed to decreasing rates when $\beta + \eta < 1$ or constant rates when $\beta + \eta = 1$). We will not deal with that case in here but we can mention that it corresponds to the explosive growth case depicted in figure 3 where there are IRS in the inputs that can be accumulated.

A planner confronted with a production function of the form (6.1) would take into account that when a firm invests, it increases the stock of knowledge from which ALL other firms in the economy may benefit. Hence, when calculating first order conditions, it would take derivatives with respect to all capital (including the part that is external to the firm) and find a growth rate of the form

$$(6.10) \quad \gamma_{\text{planner}} = \sigma^{-1}((\beta + \eta)k^{-(1-\beta-\eta)}L^\eta - \rho)$$

which is larger than the competitive one. In other words, competitive household producers would achieve a lower than optimal growth rate because they fail to internalize the knowledge spill over in production. This leads them to underinvest and, therefore, undergrow.

An interesting extension of this model to the open economy is provided by Young (1989). He sets up a two country world with one developed (the North) and one less developed (the South) countries. There are two goods, high-technology and low-technology. When trade between the two regions occurs, the North specializes in high-technology and the south does the opposite (like in a comparative advantage model). Since the production of high technology is assumed to lead to more rapid learning by doing, the effect of free trade is to increase growth in the North but decrease it in the South.

(e) Market equilibrium

Again we can show that the household production model just shown corresponds to an equilibrium in which households and firms interact through competitive markets. As usual the main first order condition for private households is equation (4.14) which, in steady state, corresponds to the upward sloping line called RC in Figure 4. Firms maximize profits subject to the production function

$$(6.11) \quad y = k^\beta k^\eta L^\eta$$

and taking k^η as given. The first order conditions entail the equalization of the private marginal product of capital to the real interest rate

$$(6.12) \quad r = \beta k^{\beta+\eta-1} L^\eta$$

which corresponds to the horizontal line RI_{private} in Figure 4. The intersection of RC and RI_{private} yields the competitive equilibrium steady state growth rate. Notice, in turn, that a planner would take into account that investment in firm i has an effect on the aggregate stock of capital so the social rate of return is

$$(6.14) \quad r = (\beta+\eta)k^{\beta+\eta-1} L^\eta$$

which corresponds to RI_{planner} in Figure 4. Notice that as long as there is a positive externality ($\eta > 0$), the competitive return and therefore the competitive growth rate is smaller than that of the planner.

To close the model, it remains to be shown that the capital stock grows at the same rate as consumption. This can be done by substituting (6.12) into the household budget constraint (4.3) to get

$$(6.15) \quad \dot{k} = \beta L^\eta - c$$

We can divide both sides by k , note that in steady state \dot{k}/k is

constant, and put all the constants in the same side. Take logs and derivatives and conclude that $\dot{c}/c = \dot{k}/k$.

(f) The Relation Between Increasing Returns and Endogenous Growth.

Although some people relate endogenous growth to increasing returns we are now in a position to say the Increasing Returns ARE NEITHER NECESSARY NOR SUFFICIENT TO GENERATE ENDOGENOUS GROWTH. We saw that they were not necessary in the Rebelo (1990) and Barro (1990) models of Constant Returns to Scale. In the other hand we just saw that they are not sufficient since, the Romer model where $\eta < 1 - \beta$, failed to generate endogenous growth.

(7) HUMAN CAPITAL ACCUMULATION: LUCAS (1988)

(a) The Model.

The first model in Lucas (1988)⁹ argues that we can have CRS in inputs that can be accumulated by arguing that ALL inputs can be accumulated. Hence, we do not need government externalities (Barro 1990) or private capital externalities (Romer 1986). To this end, he introduces human capital instead of plain "number of physical bodies" in the production function. As opposed to the "exogenous" productivity model of section 2, human capital here can change through investment (individuals will choose how much time they invest in their studies). Hence, we can accumulate all inputs of the production function. If we postulate a CRS production function, we will have a version of the Rebelo model in which the broad measure of capital includes human and physical capital. All we need to generate growth is to have the incentive to invest in human capital be

9

We will not talk about the second one which deals with acquired comparative advantage.

nondecreasing in human capital. That is we need to postulate a production function of human capital which is constant returns to human capital so its marginal product (which determines the incentive to spend time studying) is constant.

Let u be the fraction of non-leisure time individuals spend working (producing output Y), h be a measure of the average quality of workers and L be the number of bodies so uhL is the total effective labor used to produce Y . The production function, therefore is something like:

$$(7.1) Y = AK^\beta [uhL]^{(1-\beta)}$$

The term uhL is often called human capital. This production function exhibits constant returns to physical and human capital since doubling K and uhL doubles final output¹⁰. Notice that if we think of $K^\beta [uhL]^{(1-\beta)}$ as being a broad measure of capital (capital is capital is capital no matter whether it is human or it is physical), we are back to the Rebelo model (again, this is provided that the incentive to study does not decrease over time so we end up not accumulating any human capital). The production function in (7.1) would be enough to generate endogenous growth. Yet Lucas postulates an externality in human capital to reflect the fact that people are more productive when they are around clever people. If we let h_a^ψ be the average human capital of the labor force, the production function becomes

$$(7.1)' Y = AK^\beta [uhL]^{(1-\beta)} h_a^\psi$$

where h_a^ψ represents the externality from average human capital. This externality increases the degree of homogeneity of the production function

10

The production function exhibits sharply increasing returns since the doubling of K , h and L more than doubles output. That is, this production function is homogeneous of degree $2-\beta$ in K , h and L . Notice that $2-\beta$ is larger than one as long as $\beta < 1$.

to $2+\psi-\beta > 2-\beta > 1$. Yet as we just mentioned, this externality is not essential for endogenous growth but Lucas assumes it in order to get some other results on population movements¹¹. Individuals choose a stream of consumption so as to maximize the standard intertemporal utility function subject to the capital accumulation constraint:

$$(7.2) \quad \dot{K} = AK^\beta [uhL]^{1-\beta} h_a^\psi - c$$

To complete the model we need to specify how individuals accumulate knowledge. Of course they do it by studying! We can write this semi universal truth in a differential equation format¹²:

$$(7.3) \quad \dot{h} = \phi h(1-u)$$

Under this particular functional form, there are constant returns to scale in the production of human capital. That is, the growth rate of knowledge (\dot{h}/h) is proportional to the time spent in studying ($1-u$). The constant of proportionality is some "studying productivity" parameter, ϕ .

11

The implications for migration depend on whether the externality comes from aggregate or average human capital. There are arguments in favor of both types of externalities: we could say that people go to lunch with the person they happen to find in the corridor every day so what matters is the quality of the average person they happen to find. In this case the externality would come from average human capital. Notice that this specification implies that adding a person with lower than average education lowers everybody's productivity. An alternative specification would be that people benefit from everybody around them, no matter what the quality of that person is. This would imply an externality from aggregate, not average, human capital. Although providing microfoundations to the Lucas production function seems a reasonable and interesting exercise, I will not pursue this line of research here and I will just assume that the externality comes from average human capital.

12

I say semi universal rather than universal because people may learn when they work as we saw in section 6.

The assumption of non-diminishing returns in the "production of knowledge technology" is crucial. We will see that it is this sector that drives the economy to a sustained positive growth rate. To keep things simple let's assume again that L is constant and let's normalize it to one.

(b) Market Solution.

Individuals choose a stream of consumption c_t and the proportion of time they want to spend working (u) as opposed to studying ($1-u$) subject to the two constraints (7.2) and (7.3). They take h_a as given. The Hamiltonian is:

$$(7.4) H() = e^{-\rho t} \left[\frac{c^{1-\sigma} - 1}{1-\sigma} \right] + \nu \left[AK^\beta [uh]^{(1-\beta)} h_a^\psi - c \right] + \lambda \left[h\phi(1-u) \right]$$

The four FOC (wrt C, u, K and h respectively) are:

$$(7.5) e^{-\rho t} c^{-\sigma} = \nu$$

$$(7.6) \nu \left[AK^\beta h^{(1-\beta)} (1-\beta) u^{-\beta} h_a^\psi \right] - \lambda h \phi = 0$$

$$(7.7) \dot{\nu} = -\nu \left[\beta AK^{(\beta-1)} (uh)^{(1-\beta)} h_a^\psi \right]$$

$$(7.8) \dot{\lambda} = -\nu \left[(1-\beta) AK^\beta u^{(1-\beta)} h^{-\beta} h_a^\psi \right] - \lambda \left[\phi(1-u) \right]$$

As a consistency condition we require that

$$(7.9) h_a = h.$$

We can start, as usual by taking logs and derivatives of (7.5) and using (7.7) and (7.9) to get:

$$(7.10) \dot{c}/c \equiv \gamma = \sigma^{-1} \left[\beta AK^{-(1-\beta)} u^{(1-\beta)} h^{(1+\psi-\beta)} - \rho \right]$$

By dividing the dynamic constraint of physical capital

accumulation by K we will find that

$$(7.11) \quad \dot{K}/K \equiv \gamma_k = AK^{-(1-\beta)} u^{(1-\beta)} h^{(1+\psi-\beta)} - c/K$$

Now realize that the first part of the second term is (from eq (7.10)) equal to $(\gamma\sigma+\rho)/\beta$. Let's put all the constants on the Right Hand Side, take logs and derivatives of both sides to get that $\dot{c}/c = \gamma = \dot{k}/k = \gamma_k$. So capital and consumption grow at the same rate γ . Now we have one more growth rate to go: the growth rate of human capital ($\dot{h}/h \equiv \gamma_h$). Take eq (7.10), put all the constants on the Left Hand Side to get:

$$(7.12) \quad (\gamma\sigma+\rho)/A\beta = K^{-(1-\beta)} u^{(1-\beta)} h^{(1+\psi-\beta)}$$

Let's take logs and derivatives of both sides to get

$$(7.14) \quad 0 = -(1-\beta)\dot{K}/K + (1+\psi-\beta)\dot{h}/h$$

which implies

$$(7.14)' \quad \dot{h}/h \equiv \gamma_h = \gamma(1-\beta)/(1+\psi-\beta)$$

proportion (growth rate of h is smaller if there is an externality, $\psi > 0$). In the absence of an externality ($\psi=0$), the two growth rates are the same. Now we have to find the value of either γ or γ_h as a function of the parameters of the model. We can start with (7.6):

$$(7.6)' \quad \nu/\lambda = \phi / \left[A(1-\beta)K^\beta u^{-\beta} h^{(\psi-\beta)} \right]$$

we can again take logs and derivatives of both sides and get

$$(7.15) \quad \dot{\nu}/\nu + \beta \dot{k}/k + (\psi-\beta)\dot{h}/h = \dot{\lambda}/\lambda \quad \rightarrow \quad \dot{\nu}/\nu + \beta\gamma + (\psi-\beta)\gamma_h = \dot{\lambda}/\lambda$$

We know $\dot{\nu}/\nu$ from equation (7.7)

$$(7.16) \quad \dot{\nu}/\nu = -(\gamma\sigma + \rho)$$

To find the value of $\dot{\lambda}/\lambda$ let's divide both sides of equation (7.8) by λ , plug $\dot{\nu}/\nu$ from equation (7.6)' to get:

$$(7.17) \quad \dot{\lambda}/\lambda = -\phi$$

That is, the shadow price of human capital decreases at a constant rate ϕ (recall that ϕ is the productivity parameter of the "production of knowledge" technology). We can substitute $\gamma_h = \gamma(1-\beta)/(1+\psi-\beta)$ plus equations (7.16) and (7.17) in (7.15) to get that

$$(7.18) \quad \gamma_h = \left[(\phi - \rho)(1 - \beta) \right] / \left[\sigma(1 + \psi - \beta) - \psi \right]$$

Notice that if there is no externality, the growth rates are $\gamma = \gamma_h = (\phi - \rho)/\sigma$. It is interesting to note that this is the growth rate Rebelo gets in his CRS model but with ϕ rather than A . The sector that really drives the economy is the production of human capital.

In this model, unlike Rebelo's, the economy is not always in the steady state balanced growth path. It has some complicated transitional dynamics which we will not try to derive mathematically. Although Lucas conjectures about how this transition looks like, very little is known about it.

(c) The Command Economy Solution.

To solve for the command solution, we have to internalize the externality: we have to solve taking into account the fact that h_a is equal to h . Since the procedure is the same as in the Barro model I will not do it here. Let me just say that the solution you should get is the following

"efficient" growth rate for the the human capital sector:

$$(7.24) \quad \gamma_h = \sigma^{-1} \left[\phi - (1-\beta)\rho / (1+\psi-\beta) \right]$$

Notice that, as one should have expected, in the absence of externality, the growth rate is the same one we got for the market solution. That is, in the absence of externality, the competitive equilibrium gives the optimal incentive to invest in education and, therefore, the optimal growth rate. When the externality is positive, in the other hand, the "efficient" growth rate is always larger than the market rate. That is, the market economy does not grow enough. The reason is that the private return to study is lower than the social one, so in a market economy people will not invest in human capital as much as would be socially optimal.

An interesting extension of this model to the open economy is provided by Stokey (1990). She constructs a model where different qualities of goods are produced by people with different human capital stocks. She finds that free trade may be bad for poor countries because it may discourage uses this framework to analyze the impact of opening the economy to trade and finds that opening the economy may be bad for growth in poor countries as individuals are discouraged from investing in human capital.

(8) R&D MODELS OF GROWTH: (1988)

(a) The Model

There is a heterogeneity of growth models that emphasizes R&D as an important engine of economic growth. We can think of R&D as contributing to growth in at least two ways. First it allows to introduce new types of capital goods which may or may not be more productive than the existing ones. Output is a function of all existing varieties or qualities of capital goods. If it exhibits "constant returns to the number of varieties or qualities" (we will define later what we mean by that) we will get endogenous growth even when there are diminishing returns to each type of

capital. This approach has been taken by Romer (1987) and Barro and Sala-i-Martin (1990) among others.

The second contribution of R&D to economic growth is that it may have some spillovers on the aggregate stock of knowledge: as scientists spend time thinking about the development of new products or techniques, they increase the stock of knowledge. A larger stock of knowledge, in turn, reduces the costs of R&D. Hence, under some conditions the existence of spillovers from R&D activities will generate a "Constant Returns to Investing to R&D" which keeps firms investing constant amounts of resources in R&D and increasing the stock of knowledge at a constant rate. Since general knowledge reduces the cost of producing manufacturing goods, the amount of manufacturing production will also be growing at a constant rate over time. As we just mentioned, what is needed in order to generate endogenous growth is the incentive to do R&D not to decrease over time. Because what drives growth is the fact that the Stock of Knowledge is growing as a side product of R&D, it does not really matter why firms do R&D in the first place. Thus there are models where firms develop new varieties of consumption goods (Grossman and Helpman (1989,c) or new varieties of varieties of production goods (Grossman and Helpman (1989, a and b) and where the quality of new good is the same as all the others. And on the other hand there are models where firms try to increase the quality of a constant number of varieties goods (either consumption or investment goods) (Aghion and Howitt (1989) or Grossman (1989), and Grossman and Helpman (1989 d,e)). The four type of models will yield exactly the same results.

These R&D models have been used by Grossman and Helpman (in a variety of papers) to analyze the open economy implications of endogenous growth models. Trade of goods has implication for growth because it implies international transmission of knowledge. They also use this framework to develop models where there is a race between first world countries trying to create new products and third world countries trying to imitate them.

An interesting finding of these line of research is the endogenous growth can be generated through the accumulation of knowledge alone. In particular, no investment in physical capital is needed. This is an interesting finding despite the fact that the data show that investment in

physical capital is highly correlated with GNP growth. In order to generate such a correlation these models have to include some physical capital whose accumulation responds to growth rather than the other way around.

In this section we will explore the simplest version of an R&D model (taken from Barro and Sala-i-Martin (1990)) where growth arises from the assumption that the production function exhibits constant returns to varieties according to the following production function:

$$(8.1) \quad Y_t = A \sum_{i=1}^{N_t} x_i^{1-\alpha}$$

where x_i 's are intermediate inputs and A is some technological parameter (that could be related to fiscal policy or other things). In words, output is produced with a set of N_t inputs x_i (the amount of inputs available N_t has a time subscript indicating that it can, and will, change over time). We can think of x_i as being different types of capital goods, which we will call "useful" capital goods¹³. Spence (1976) and Dixit and Stiglitz (1977) first modeled utility as depending on a variety of consumption goods in a formulation similar to (8.1). Ethier (1983) reinterpreted the Spence-Dixit-Stiglitz utility function in terms of production of a single output using a variety of inputs, which is the approach taken here.

Imagine that the useful capital goods are produced from some kind of "raw" capital. Raw capital foregone consumption. Instead of directly nailing this raw capital to the floor, we must first transform it into useful capital. Suppose that we have a certain amount K_t of raw capital. This aggregate raw quantity has to be divided among all different types of useful capital goods. The assumed production function has the property that if we divide the total available raw capital into N varieties of useful

13

It is very easy to introduce inputs that cannot be accumulated such as labor or land. If we call this input L, the production function

will be $y = AL^{(1-\alpha)} \left(\sum_{i=1}^{N_t} x_i^\alpha \right)$

capital we get LESS than if we divide it into $N+1$ varieties, and that in turn is less than the output we get with $N+2$ and so on¹⁴. This means that, given a certain amount of aggregate raw K , we can produce an infinite amount of output by simply dividing it into an infinite amount of varieties, each of which is infinitesimally small¹⁵. To prevent that from happening (and therefore to make the economy meaningfully scarce) we need to argue that, at any given moment in time the amount of varieties is limited. We do it by assuming that in order to transform raw into useful capital we need to pay a fixed R&D cost. After paying the fix cost, we can transform raw into useful capital at a constant marginal cost. The fixed research cost is modeled differently by different people. Some papers assume that it is in terms of output and some others assume that is in terms of labor (human capital). The distortionary effects are going to be different according to how this research cost is modeled, yet the main conclusions are similar across models.

Of course this fixed cost structure will limit the amount of available varieties for a given level of K_t . But it also means that if we want this useful capital to be provided through a market mechanism, it cannot be a competitive one. Thus, we will assume monopolistic behavior.

Summarizing, these types of models will have the following three

14

Suppose we have a fixed amount of capital and we divide it equally among N varieties. The implied output is (set $L=1$ for simplicity) $Y_1 = N(K/N)^\alpha$. Suppose instead that we divide total K into $N+1$ varieties, output now is $Y_2 = (N+1)(K/(N+1))^\alpha$. Of course $Y_2 - Y_1 = K^\alpha [N^{(1-\alpha)} - (N+1)^{(1-\alpha)}] > 0$.

So WITH THE SAME AMOUNT OF AGGREGATE CAPITAL, the final output is larger the more we divide it among different varieties. We could think of K as being the total amount of talent or human capital devoted to work. We could transform all this talent into one input or we could divide it into different inputs (or activities). The more activities we have (given the total amount of talent) the more output we get. This, somehow, captures Adam Smith's idea of increasing return due to division of labor or specialization.

15

So there are diminishing returns to each variety but there are constant returns to the number of varieties, N .

kinds of agents:

(1) The Producers of the final (consumption) good use labor and all available varieties of useful capital. The production function is (8.1). They rent capital each good i at rate R_i . Their optimizing behavior will generate input demand functions of the form $x_i = R_i^{-\epsilon} \psi$ where R_i is the rental rate of good i , ϵ is an elasticity depending on the parameters of the model and ψ represents other parameters.

(2) Producers of useful capital goods x_i . They use Raw capital and produce useful capital. They pay an R&D fee equal to β and, after this, they will be able to produce and rent unlimited amounts of x_i at a constant marginal cost, ϑ . Because of this "fixed cost" technology, their behavior will not be competitive but, instead, monopolistic. They will choose the rental rate R_i so as to maximize profits subject to the demand functions for their useful capital goods ($x_i = R_i^{-\epsilon} \psi$). We will assume that everybody in the economy can invest in R&D and develop a new variety of investment goods. The existence of free entry will drive profits to zero at every moment in time.¹⁶ This zero profit condition will imply that the quantity of each variety of useful capital is fixed and, therefore, that any increase in the demand for useful capital will be satisfied through increases in the amount of varieties rather than increases in the quantities of the existing ones.

(3) Consumers who receive income Y and decide how much to consume (C) and save (K) each period. Their savings are flows of raw capital that can be used by the firms. Consumers can trade units of raw capital today (which is the same good as consumption) for units tomorrow at the real interest rate r_t . Since we are neglecting labor, their only source of income is the income from lending raw capital ($Y_t = r_t K_t$) or bonds, which will have to yield the same return given that there is no uncertainty. The utility function for consumer will be assumed to be CIES. The model will be solved in three steps.

16

This also means that we can neglect profits on the income side of the consumer budget constraint.

(a.1) Producers of Final Goods.

As mentioned earlier, they rent each of the N_t varieties of useful capital x_i at rate R_i and they combine them according to (8.1) to produce the only consumer good, Y_t which they sell at unit price. They maximize the present value of all future cash flows:

$$(8.2) \int_0^{\infty} e^{-rt} \left[Y_t - \sum_{i=1}^{N_t} p_i I_{it} \right] dt$$

subject to (8.1). The straightforward first order conditions yield the following demand function for good i

$$(8.2) A(1-\alpha)x_i^{-\alpha} = R_i$$

The demand function in (8.2) has a constant elasticity equal to α .

(a. 2) Producers of Useful Capital Goods.

As mentioned earlier, the technology to produce useful capital out of raw capital requires a fixed R&D cost, β (measured in units of output¹⁷) which allows them to develop a new variety which can then be produced at a constant marginal cost, ϕ and rented at rate R_i . Entrepreneurs in this sector choose the rental rate so as to maximize profits taking the demand

Romer (1990) and Grossman and Helpman (1990) assume that the research technology uses labor only. In this case, changes in real wages have an effect on the fixed R&D cost.

for their output, x_i , as given by (8.2).

$$(8.3) \text{ MAX}_{R_i} \int_0^{\infty} e^{-rt} \left[R_i x_i(t) - \vartheta I_i(t) \right] dt - \vartheta x_i(0) - \beta$$

subject to $I_i(t) = \dot{x}_i(t)$ and to (8.2). Notice that (8.3) includes the rental income from zero to infinite discounted at the real interest rate r , the future marginal costs of producing increasing quantities of x_i ($I_i(t)$ is the increase in production of x_i), the marginal cost of producing the initial quantity $x_i(0)$ and the fixed R&D cost β . We can set up the Hamiltonian for this problem

$$(8.4) H = e^{-rt} \left[R_i x_i(t) - \vartheta I_i(t) \right] - \vartheta x_i(0) - \beta + q_i(t) I_i(t)$$

where $q_i(t)$ is the dynamic multiplier. The first order conditions entail

$$(8.5) e^{-rt} \vartheta = q_i(t)$$

$$(8.6) -\dot{q}_i(t) = e^{-rt} A(1-\alpha)^2 x_i^{-\alpha}$$

Notice that, after taking logs and derivatives, equation (8.5) can be transformed into

$$(8.7) \dot{q} = -qr$$

We can now substitute (8.7) into (8.6) to get

$$(8.8) r = A(1-\alpha)^2 x_i^{-\alpha} / \vartheta$$

which is a relation between x_i and the real interest rate. It can be

rewritten as

$$(8.9) \quad x_i(t) = \left[A(1-\alpha)^2 / r\theta \right]^{(1/\alpha)}$$

which is independent of time and i . This implies that the quantities of all goods will be the same and that they will be constant over time. Hence, $I_i(t)$ is zero for all t 's. The optimal rental rate can be found by substituting x_i in (8.2)

$$(8.10) \quad R_i = r\theta / (1-\alpha)$$

This rental rate can be interpreted as follows: the asset value of a firm that invests in R&D and discovers a new variety is the present value of all future rental incomes, R_i/r . Equation (8.10) says that the price of such an asset is a constant markup over the marginal cost (this result comes from the constant elasticity demand functions in (8.2)). Notice that R_i is independent of i so all goods will have the same market rental rate.

The free entry in the R&D business condition implies that the cost of investing in R&D will equal the present value of all future gains. That is

$$(8.11) \quad \int_0^{\infty} e^{-rt} \left[R_i x_i(t) - \theta I_i(t) \right] dt - \theta x_i(0) = \beta$$

Since $x_i(t)$ equals $x_i(0)$ for all periods and I is zero, this present value condition implies

$$(8.12) \quad R_i x_i / r = \theta x_i + \beta$$

We can use the rental rate (8.10) and (8.12) to find a relation between x_i and the parameters of the model

$$(8.14) \quad x_i = \beta(1-\alpha) / \alpha$$

We can now combine (8.14) and (8.9) to get rid of x_i

$$(8.15) \quad r = A(1-\alpha)^{2-\alpha} \alpha^2 / (\beta^\alpha \vartheta) \equiv A^*$$

which is constant relation between the real interest rate and the parameters of the model. This corresponds to the flat return to investment line (RI) in Figure 1.

(a.3) Consumers

To close the model we need to find the Return to Consumption schedule. Of course we will find it by allowing individuals to maximize the typical CIES infinite horizon utility function

$$(8.16) \quad \text{MAX} \quad U(0) = \int_0^{\infty} e^{-(\rho-n)t} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right] dt$$

Subject to the budget constraint.

$$(8.17) \quad rK_t = c_t + \dot{K}_t$$

where $K_t = \sum_{i=1}^N x_i = Nx$, where x is the constant stock of each and everyone of the varieties of capital goods x_i . Equation (8.17) says that raw capital is just foregone consumption (in the same units). Thus, $\dot{K} > 0$ means that some resources are allocated to the increase in the number of varieties ($\dot{N} > 0$), that is, investment in R&D or to the increase in the quantity of existing varieties ($\dot{x} > 0$). Individuals receive income from lending their units of raw capital at the current interest rate r (there is no labor income or profits since the free entry condition implies zero profits at all times). The first order conditions are the usual ones which can be combined to yield

$$(8.17) \quad c/c = (r-\rho)/\sigma$$

This completes the description of the model. We can use equations (8.17) and (8.16) to get the growth rate of the economy:

$$(8.17)' \quad \dot{c}/c = (A^* - \rho)/\sigma$$

Notice that growth is constant because the return to saving (the interest rate r) is constant (A^*) as the economy grows so the incentive to save never vanishes (just as in the simple Rebelo model).

Because the equilibrium interest rate is constant, we can divide the consumer budget constraint by K_t , take logs and derivatives of both sides and get that, in steady state, consumption and capital grow at the same rate ($\dot{c}/c = \dot{K}/K$). Finally, taking logs and derivatives of $K = Nx$, we see that N_t also grows at the same rate as K (since $\dot{x}/x = 0$)

$$(8.19) \quad \dot{c}_t/c_t = \dot{K}_t/K_t = \dot{N}_t/N_t$$

In words, all capital accumulation takes place in the form of new varieties rather than in deepening the old ones.

The growth rates implied by the optimal or command solution to this model are smaller than the ones in a competitive setup. The reason is that the producers of useful capital goods charge a monopoly rent which is higher than the competitive one. This implies that the private return to investment falls short of the social return and hence, the steady state growth of the decentralized is smaller than the socially optimal rate. Pareto optimal solutions can be achieved if the government raises the private incentive to invest, which in this model can be achieved by subsidizing the purchase of goods (at rate α) or by subsidizing the income on capital (at rate $\alpha/(1-\alpha)$). In this sense, the results are similar to the ones we found in the public spending, the learning by doing or the human capital models with externality.

References

Aghion, P. and P. Howitt, "A Model of Growth through Creative Destruction", mimeo university of Western Ontario, May 1989.

Arrow, K. J., "The Economic Implications of Learning by Doing", *Review of Economic Studies*, June 1962.

Backus, D., P. Kehoe, and T. Kehoe, "In search of scale effects in trade and growth", mimeo University of Minnesota, May 1990.

Barro, R. (1974), "Are Government Bonds Net Wealth?", *Journal of Political Economy* 82: 1095-1117, November/December.

Barro, R. (1989,a), "A Cross-country Study of Growth, Saving, and Government", NBER w.p. no. 2855, February 1989.

Barro, R. (1989,b), "Economic Growth in a Cross Section of Countries", NBER w.p. no. 3120, September 1989.

Barro, R. J. (1989,c), "Macroeconomics", 3rd. edition. *John Wiley & Sons*, New York.

Barro, R. J. (1990), "Government Spending in a Simple Model of Endogenous Growth", *Journal of Political Economy*, October 1990.

Barro, R. J., and G. Becker, "Fertility Choice in a Model of Economic Growth", *Econometrica*, March 1989.

Barro, R. J. and Xavier Sala-i-Martin, "Public Finance in Models of Economic Growth", NBER wp # 3362, May 1990 A.

Barro, R.J and X. Sala-i-Martin, "Economic Growth and Convergence Across the United States", NBER wp # 3419, August 1990 B.

Baumol (1986), "Productivity Growth, Convergence and Welfare: What the Long Run Data Show", *American Economic Review*, 76 (December):1072-1085.

Becker, G., Murphy, K. and Tamura (1990), "Human Capital, Fertility, and Economic Growth", *Journal of Political Economy*, October 1990.

Blanchard, O. J. and S. Fischer, "Lectures on Macroeconomics", M.I.T. press, 1989.

Cass, D. (1965), "Optimum Growth in an Aggregative Model of Capital Accumulation", *Review of Economic Studies*, 32, July 1965, 233-240.

Delong, B. (1988), "Productivity Growth, Convergence, and Welfare: Comment", *American Economic Review*, 78, December 1988, 1138-1154.

Dixit and Stiglitz (1977), "Monopolistic Competition and Optimum Product Diversity", *American Economic Review*, 67, June 1977, 297-308.

Domar, E. D., "Capital Expansion, Rate of Growth, and Employment", *Econometrica*, April 1946.

Edwards, S. (1989), "Openness, Outward Orientation, Trade Liberalization and Economic Performance in Developing Countries", mimeo, University of California at Los Angeles, March 1989.

Ethier, W. J. (1982), "National and International Returns to Scale in the Modern Theory of International Trade", *American Economic Review*, 72, June 1982, 389-405.

Gale, D. and Sutherland, W. R. (1968), "Analysis of a One Good Model of Economic Development", in Dantzig, G. B. and Veinott, A. F. eds, "*Mathematics of the Decision Sciences*", Part II. American Mathematical Society, Providence, RI. 1968

Greenwald, B., M. Salinger and J. Stiglitz, "Imperfect Capital Markets and Productivity Growth", mimeo April 1990.

Greenwood, J. and Jovanovic, B. (1990), "Financial Development, Growth and the Distribution of Income", *Journal of Political Economy*, October 1990.

Greenwood, M. J. (1975), "Research on Internal Migration in the United States: A Survey", *Journal of Economic Literature* vol. XIII, #2, 397-433. June 1975.

Grossman, G., "Explaining Japan's Innovation and Trade: A Model of Quality Competition and Dynamic Comparative Advantage", NBER wp# 3194, December 1989.

Grossman, G. and Helpman, E. (1989a), "Comparative Advantage and Long Run growth", NBER WP # 2809, January 1989.

Grossman, G. and Helpman, E. (1989b), "Growth and Welfare in a Small Open Economy", NBER WP #2970, July 1989.

Grossman, G. and Helpman, E. (1989c), "Product Development and International Trade", *Journal of Political Economy*, 97, December 1989, 1261-1283.

Grossman, G. and Helpman, E. (1989d), "Endogenous Product Cycles", NBER wp# 2913, March 1989.

Grossman, G. and Helpman, E. (1989e), "Quality Ladders and Product Cycles", NBER wp#3201, December 1989.

Harrod, R. F., "Review of Joan Robinson's *Essays in the Theory of Employment*", *The Economic Journal* vol 47, 1937.

Harrod, R.F., "An Essay in Dynamic Theory", *The Economic Journal*, March 1939.

Helpman, E. (1988), "Growth, Technical Progress and Trade", NBER WP #2592.

Hicks, J. R., "The Theory of Wages", Macmillan, London.

Jones, L and Manuelli, R. (1990), "A model of optimal equilibrium growth", *Journal of Political Economy*, October 1990.

King, R. G. and S. T. Rebelo, (1989), "Transitional Dynamics and Economic Growth in the Neoclassical Model", NBER WP # 3185, November 1989.

King, R. and Rebelo, S. (1990), "Public Policy and Economic Growth: Developing Neoclassical Implications", *Journal of Political Economy*, October 1990.

Koopmans, T. C., "On the Concept of Optimal Growth", in "*The Econometric Approach to Development Planning*", North Holland, Amsterdam., 1965.

Kurtz, M., (1968), "The General Instability of a Class of Competitive Growth Processes", *Review of Economic Studies*, march 1968, p. 155.

Lucas, R. E. (1988), "On the mechanics of economic development", *Journal of Monetary Economics*, 22, p 3-42, June 1988

Mankiw, N.G., D. Romer, D., and D. N. Weil, (1990), "A Contribution to the Empirics of Economic Growth", Mimeo Harvard University.

Murphy, K., A. Shleifer, and R. Vishny, "The Allocation of Talent: Implications for Growth", mimeo University of Chicago, May 1990.

Phelps, E., "The Golden Rule of Capital Accumulation: A Fable for Growthmen", AER 1961.

Phelps, E. S., "The New View of Investment: A Neoclassical Analysis", Q.J.E., November 1962.

Phelps, E. (1966), "Golden Rules of Economic Growth", Norton, New York, 1966.

Ramsey, F. P. (1928), "A mathematical theory of saving", *Economic Journal*, 38, p. 543-59.

Rebelo, S. "Long Run Policy Analysis and Long Run Growth", NBER WP, April 1990.

Robinson, J., "The Classification of Inventions", *Review of Economic Studies*, February 1938.

Robinson, J., "The Production Function and the Theory of Savings", *Review of Economic Studies*, 1954.

Romer, P., (1986), "Increasing returns and long run growth", *Journal of Political Economy*, 94, p. 1002-1037.

Romer, P. (1987), "Growth Based on Increasing Returns Due to Specialization", *American Economic Review*, May 1987.

Romer, P. (1989,a), "Capital Accumulation in the Theory of Long-Run Growth", in Robert Barro editor, "*Modern Business Cycle Theory*", Harvard University Press, Cambridge, MA. 1989.

Romer, P. (1989,b), "Human Capital and Growth: Theory and Evidence", mimeo The University of Chicago, April 1989.

Romer, P. (1990), "Endogenous Technological Change", *Journal of Political Economy*, October 1990.

Sala-i-Martin, X., "On Growth and States", Unpublished PhD dissertation, Harvard University, May 1990.

Samuelson, P. A., (1954) "The Pure Theory of Public Expenditures", *Review of Economic & Statistics*, 36, November 1954, 387-389.

Sheshinski, E., "Optimal Accumulation with Learning by Doing", in Karl Shell ed. "Essays on the Theory of Optimal Economic Growth", M.I.T. Press, 1967.

Solow, R. (1956), "A Contribution to the Theory of Economic Growth", *The Quarterly Journal of Economic*, 70, February 1956: 65-94.

Solow, R. M., "Investment and Technical Progress", in Arrow et al., eds, "*Mathematical Methods in the Social Sciences*", Stanford University Press, 1959.

Spence, M., (1976), "Product Selection, Fixed Costs, and Monopolistic Competition", *Review of Economic Studies*, 43: 217-236.

Stokey, N. L., "Human Capital, Product Quality, and Growth", mimeo Northwestern University, March 1990.

Swan, T. W. (1956), "Economic growth and capital accumulation", *Economic Record*, 32, p. 334-61.

Uzawa, H., "Neutral Inventions and the Stability of Growth Equilibrium", *Review of Economic Studies*, February 1961.

Uzawa, I. (1965), "Optimum technical change in an aggregative model of economic growth", *International Economic Review*, 6, p. 18-31.

Young, A., "Learning by Doing and the Dynamic Effects of International Trade", mimeo Columbia University, November 1989.

FIGURE 1: ENDOGENOUS GROWTH MODELS

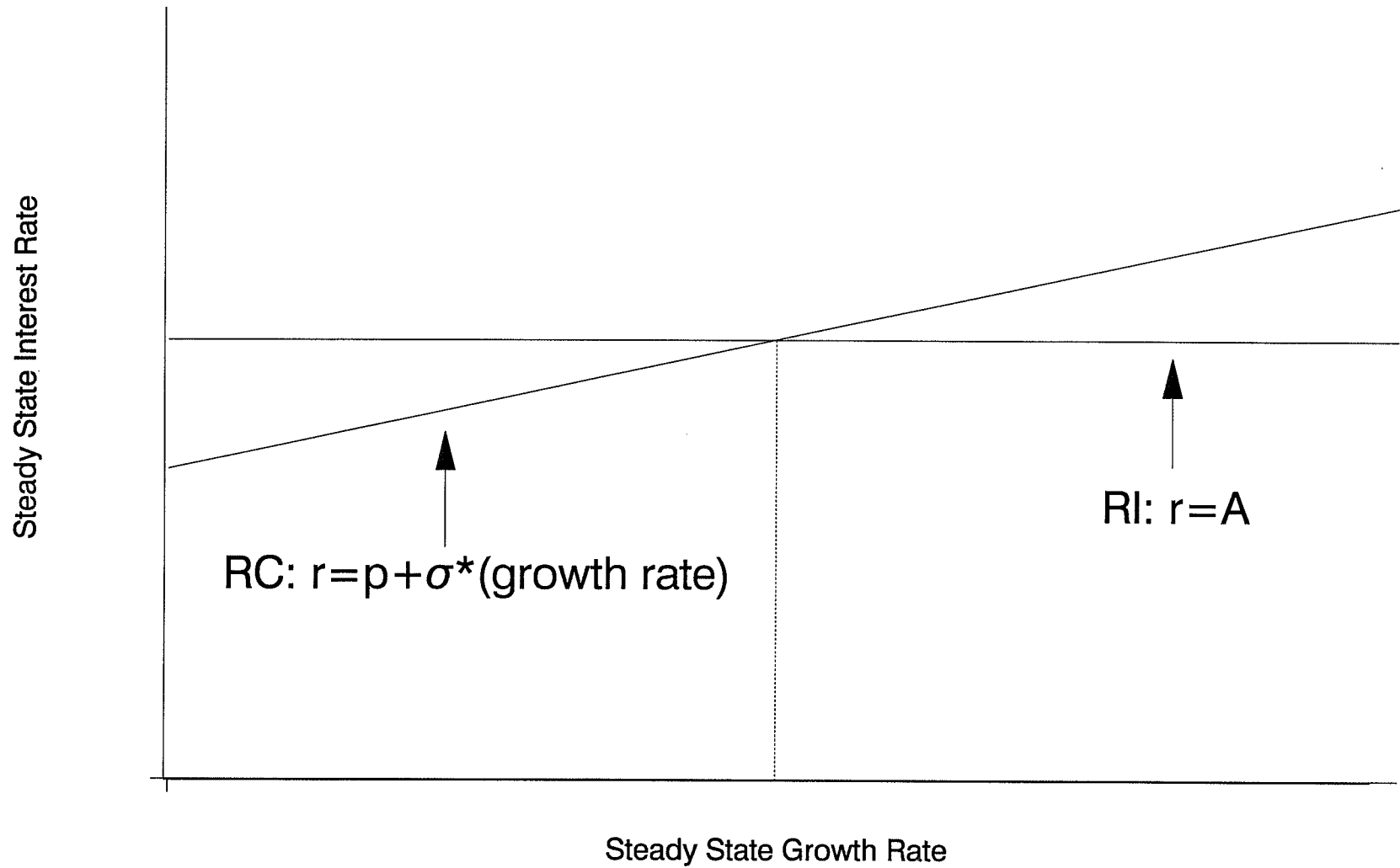


FIGURE 2: EXOGENOUS PRODUCTIVITY GROWTH MODELS

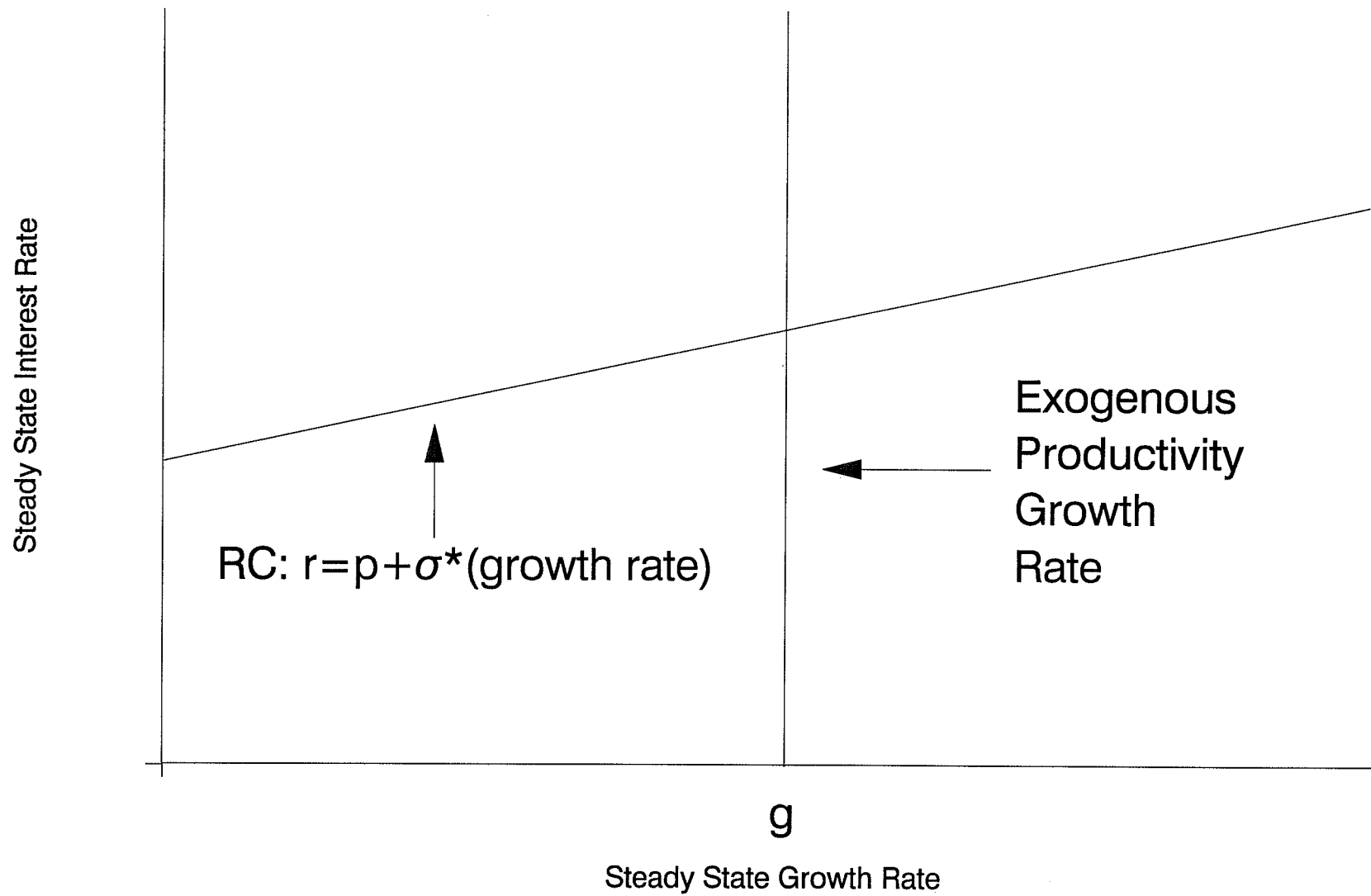


FIGURE 3: BARRO MODEL

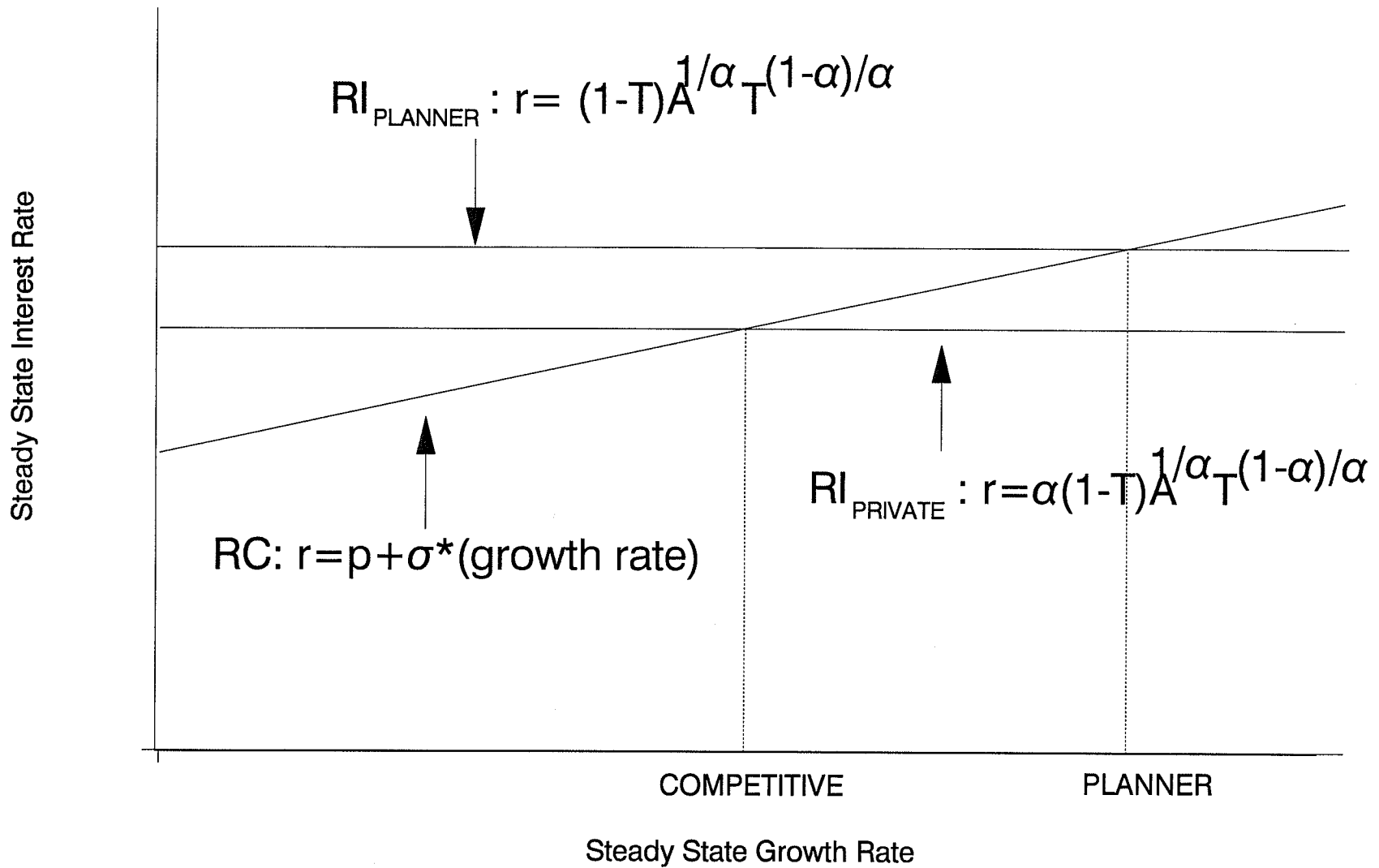


FIGURE 4: ARROW-SHESHINSKI-ROMER MODEL

