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The Effects of Mortality Changes on Fertility Choice and Individual Welfare: Some Theoretical Predictions

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Abstract: The study of the effects of changes in child mortality on individual fertility decisions has been a cornerstone of the economic analysis of population. Empirical studies have overwhelmingly shown that a lower mortality rate leads to lower fertility. Yet, in even the simplest theoretical models of fertility choice, it has not been possible to satisfactorily analyze this relationship. This paper attempts to reduce this long-standing gap between theory and the empirical literature. The paper shows that a set of simple and plausible conditions is sufficient to yield the typically observed effect of mortality changes on fertility choice.

Another concern of this paper is to examine the effects of mortality changes on individual welfare. Though such welfare assessments are important for certain types of policy evaluations, they do not appear to have received attention in the literature. This paper presents some new and robust results on this issue. The analysis captures the dynamic stochastic feature of fertility choice, and also subsumes other endogenous choices (e.g., the quality of the children). The number of children is treated as a discrete variable; this added realism is in fact important for obtaining the results.

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I. INTRODUCTION

An ongoing concern in the economic study of population has been to understand the effects of changes in child mortality rates on individual fertility decisions. This concern stems partly from the set of historically unprecedented changes that have been observed, at different times in different groups of countries, in the mortality rates as well as in the fertility choices of individuals. Empirical studies have overwhelmingly shown that the number of children produced by a couple declines as the mortality rate declines.[1] This relationship also appears quite intuitive.[2] Yet, thus far it has not been possible to satisfactorily analyze this relationship in even the simplest theoretical models of fertility choice. This paper attempts to reduce this long-standing gap between theory and the empirical literature.

For example, consider the following bare-bones model. An individual (couple) makes a one-time decision concerning the number of births. This decision maximizes the expected utility, inclusive of all costs and benefits, from different numbers of surviving children. Even using such a simple single-stage choice model, and using additional strong assumptions, what has typically been shown is that a decline in the mortality rate can lead, with equal plausibility, to an increase or a decrease in fertility.

Among the assumptions that previous theoretical studies dealing with this issue have had to make are the following: (i) Expected utility depends on the expected number of surviving children (see the early and important contribution of Ben-Porath and Welch (1972) and Ben-Porath (1976)). As is explained later, this assumption is either inconsistent with individual choice under uncertainty, or it predicts an outcome that contradicts the typically observed pattern. (ii) Ex-post utility is quadratic in the number of surviving children. The limitation of the quadratic assumption is well-known; there are no special reasons that make this assumption less limiting in the fertility context.[3] (iii) Only the polar outcomes matter; for example, when all children who are born survive; and when none survives. The utilities of other outcomes, when some of the

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[1]See Schultz (1981) for a review of the economic literature. See Freedman (1975) for a summary and a bibliography of the demographic and sociological literature. Freedman and Schultz cite scores of empirical studies showing that a lower mortality rate lowers the number of births. There are some exceptions as well. For example, Dyson and Murphy (1985) point out that, in some cases, a decline in the morality rate might have been accompanied by a brief increase in the economy-wide fertility, primarily due to the effects of such contemporaneous changes as more woman marrying or marrying earlier than before, a decrease in widowhood, and less intensive breastfeeding. However, this increase was quickly overwhelmed by the direct effect of a lower child mortality rate, namely, a rapid decrease in the fertility of couples. The present paper focuses on the direct effect.

[2]See Becker's pioneering paper on the economics of fertility (1960, p. 212).

[3]Newman (1988) has recently used this assumption in a theoretical model of fertility. His analysis of the effect of mortality focuses on the marginal replacement behavior (that is, on the change in the fertility choice due to one extra death of a child) rather than on the expected number of total births.

children die but others survive, do not matter. As described later, O'Hara (1975) uses this assumption in his analysis of fertility.

Since the theoretical predictability established to date is so weak in as simple a model as the single-stage model noted above, it is not surprising that this predictability is no better in more general models. An important aspect of fertility decisions is that they are dynamic and stochastic. Past fertility choices stochastically influence, through mortality, the number and the age-composition of children currently alive. This, in turn, influences current and future fertility decisions. Thus, fertility choice is best modeled as a stochastic dynamic program.[4]

This paper presents what may be viewed as a more fruitful analysis, so far as theoretical predictability is concerned, of the effect of mortality changes on fertility. Using a single-stage choice model, I first present a set of simple conditions under which the number of births declines if the mortality rate declines. I then analyze some more general models. For example, in a two-stage stochastic dynamic model of fertility choice, I examine the effects of a decline in mortality rates on the number of births in each of the two periods, as well as on the expected number of total births. I then show how this analysis extends to a multi-stage choice model.

Another concern of this paper is to examine the effects of mortality changes on individual welfare. I show, for instance, that a lower mortality rate raises individual welfare. Although this result is highly intuitive, it is, to my knowledge, new. In fact, the assessment of the welfare effects of mortality changes does not appear to have received attention in the literature. This lacuna is noteworthy because such assessments are a necessary component of an economic evaluation of government programs aimed at reducing child mortality. The welfare results presented in this paper are quite robust: they do not depend on the properties of the utility function; they are a consequence solely of the optimizing behavior. At the same time, the results are not obvious; for example, they cannot be obtained from the envelope theorem or stochastic dominance arguments alone.

In this paper, the number of children, born or surviving, is represented as non-negative integers. This is obviously the correct representation of the reality. Also, a continuous representation of an intrinsically discrete variable may be a greater source of error in the present context (because, in some cases, none or only one of the children born may survive) than in many other contexts (such as a factory producing

[4]See Heckman and Willis (1975) for an early formulation and empirical demonstration of this approach. See Wolpin (1984) for an estimation which emphasizes child mortality, and also for a useful discussion of some of the limitations that are currently faced in the empirical implementation of this approach.

millions of widgets) in which the variable has a large value. In contrast, most previous theoretical studies have employed a continuous representation, perhaps because of the seeming tractability of such a representation. It turns out, however, that a discrete representation yields a crisper analysis in the present context. More important, as is shown later, this paper's results cannot be obtained, using a set of assumptions comparable to those which I have used, if a continuous representation is employed. This is because a continuous representation leads to a substantial loss of usable information. For the problem at hand, therefore, realism and tractability coincide.

To focus on the fertility choice, other individual choices are kept in the background. The analysis subsumes other endogenous choices (for example, child care and quality, parental human capital formation, time allocation, and labor market participation) by assuming that these choices are made optimally for every fertility choice. Consequently, the results presented in this paper hold if these other choices and the associated budget constraints are analyzed simultaneously along with the fertility choice.

The paper is organized as follows. A single-stage model of fertility choice is analyzed in Section II. Section III analyzes a two-stage model, and discusses its extension to a multi-stage model. The last section presents some remarks. The advantage of beginning with a simple model is that the key aspects and intuition of the analysis can be understood in an uncluttered context.

It is important to emphasize the specific aim of this paper. The paper's objective is to use simple but realistic models to extract some predictability concerning the effects of mortality changes on fertility choice and individual welfare. If no predictability can be established in simpler models, it is unlikely to be established in more general models. On the other hand, if some predictability can be established within simpler models, as it turns out to be the case, then the same approach might be useful in other models.

II. A SINGLE-STAGE MODEL OF FERTILITY CHOICE

The number of children produced is denoted by the integer variable n. The random variable N denotes the number of children who will survive. The possible values of N are $0, 1, \ldots, n$. We assume that the survival of each child is an independent event with probability s, where $1 > s > 0$. A larger value of s thus represents a regime of lower mortality.[5] For simplicity, the present model is formulated in terms

[5] Several modifications of this aspect are possible. One is to treat the survival of the i-th child as an independent event with probability s_i. In this case, a regime of lower mortality may be represented by positing that s_i changes to $s_i + \theta_i$, for $i = 1$ to n, where $\theta_i \geq 0$ for all i, and $\theta_i > 0$ for at least one i. However, such a distinction among children may not be appropriate within the single-stage model under consideration here, where the underlying simplification is that all births take place at the same time, and that all deaths take place at the same time in the future. If one wishes to highlight the distinctions among children that arise from the time-sequence of births, then it is perhaps better to deal with this issue in a dynamic framework, as is done in the next

of the survival probability rather than the mortality rate, $1 - s$, but it can obviously be interpreted in either way. It is assumed that the mortality rate is an exogenous parameter to the individual. However, the analysis remains unaffected if, instead, the mortality rate is endogenously determined by the individual through a production function. In this case, one would examine the fertility effects of a change in parameters (such as preventive health technology, or the prices of relevant inputs) that could potentially alter the individual choice. Note that the exogenous parameters are specific to the individual, who is being considered in isolation from the rest of the economy. Thus, the probability that N out of n children will survive is the binomial density

(1) $$b(N, n, s) \equiv \binom{n}{N} s^N (1 - s)^{n-N} .$$

To analyze the individual's optimal choice, we need to describe the expected utility from different values of n. It is convenient to begin by considering the ex-post benefits and costs, and by temporarily abstracting from ex-ante benefits and costs (such as childbearing costs) that depend on the number of births, but not on how many of them survive. Let $u(N)$ denote the ex-post net utility, incorporating all benefits and costs, from N surviving children. One would expect $u(N)$ to first increase and then decrease with N. To see this, one may write $u(N)$ as $u(N) \equiv w(g(N) - h(N))$, where $g(N)$ denotes the benefit from N surviving children expressed in terms of a numeraire (say, dollars), $h(N)$ denotes the corresponding cost, and the function w translates the net benefit into utility. The standard assumption concerning the benefit $g(N)$ is that it is increasing and concave, if not strictly concave, in N. The cost $h(N)$ includes expenditures on children as well as the value of household inputs (such as parents' time) which are available in limited supply and which cannot be adequately substituted by inputs bought from the market. The importance of such aspects of post-birth costs has been pointed by empirical studies (see Schultz (1976, pp. 102-4), Schultz (1988, pp. 424-37) and references therein). It is thus appropriate to assume that $h(N)$ is increasing and strictly convex in N. Next, assuming that the individual is risk-averse or risk-neutral (that is, w is concave in its argument), it follows that $u(N)$ is strictly concave in N. We assume this property of $u(N)$ in the analysis below, although, as we shall see, this assumption can be weakened.

The expected utility from n births, for a given s, is denoted by $U(n, s)$. Thus,

(2) $$U(n, s) \equiv \Sigma_N b(N, n, s) u(N) .$$

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section. Another possible modification is to let s depend on the number of births. In this case, a regime of lower mortality may be represented by positing that $s(n)$ changes to $s(n) + \theta(n)$, where $\theta(n) \geq 0$ for all n, and $\theta(n) > 0$ for at least one n. Given the objective of the paper, described at the end of the last section, we will not deal with such modifications.

Throughout the paper, we suppress the range of the index over which a summation is being taken, whenever the summation is over the entire range of the index. For instance, in the right-hand side of (2), the summation is over $N = 0$ to n. The individual's welfare level is described by the indirect utility $V(s)$. That is,

(3) $$V(s) \equiv \mathrm{Max}_n : U(n, s) \equiv U(n(s), s) .$$

In (3) and below, $n(s)$ denotes the largest optimal value of n.

As was noted earlier, for expositional simplicity, the above model abstracts from the ex-ante costs of producing children. These costs are incorporated in the last part of this section. Also, though this analysis uses the number of births as the choice variable, it can be restated using the conception probability as the choice variable.[6]

For later use, denote the marginal utility of a surviving child by $u_N(N) \equiv u(N + 1) - u(N)$. Denote the change in this marginal utility due to one more surviving child by $u_{NN}(N) \equiv u_N(N + 1) - u_N(N)$. These are respectively the discrete equivalents of the first and the second derivatives of the utility function. The marginal expected utility of an additional birth is denoted by $U_n(n, s) \equiv U(n + 1, s) - U(n, s)$. The change in this marginal expected utility due to one more birth is denoted by $U_{nn}(n, s) \equiv U_n(n + 1, s) - U_n(n, s)$. Clearly, U_n and U_{nn} are defined for $n \geq 0$.

The following relationships, established in the Appendix, play a central role in the analysis.

(4) $$U_n(n, s) = s\, \Sigma_N\, b(N, n, s) u_N(N) .$$

(5) $$U_{nn}(n, s) = s^2\, \Sigma_N\, b(N, n, s) u_{NN}(N) .$$

(6) $$\frac{\partial}{\partial s} U_n(n, s) = \frac{1}{s} \{U_n(n, s) + nU_{nn}(n - 1, s)\} .$$

(7) $$\frac{\partial}{\partial s} U(n, s) = \frac{n}{s} U_n(n - 1, s) .$$

The strict concavity of $u(N)$ in N means that $u_{NN}(N) < 0$. Thus, (5) yields

(8) $$U_{nn}(n, s) < 0 .$$

Also, expression (4) has an intuitive interpretation. Compare the expected utility under two alternatives: $n + 1$ births versus n births. Consider $n + 1$ states-of-the-world in which, respectively, $N = 0, 1, ..., n$

[6] Further, as was remarked earlier, formulation (2) subsumes all choices other than the fertility choice. To see this, let the vector x denote all other choices. The decision problem is: $\mathrm{Max}_{x,n} : A(x) + \Sigma_N\, b(N, n, s) a(N, x)$, where $A(x)$ is the part of the ex-ante utility that depends only on x, and $a(N, x)$ is the ex-post utility that depends on x as well as on N. Then defining $u(N) \equiv \mathrm{Max}_x : A(x) + a(N, x)$ as the maximized value of the ex-post utility for each N, we get formulation (2).

children out of n births survive. Now, if the $(n + 1)$st child does not survive, then the individual has the same utility in each of the states under the two alternatives. If the $(n + 1)$st child survives, then the individual has one more child in each of the states under the first alternative than in the second. The resulting difference in the utility, summed over all states, is $\Sigma_N b(N, n, s)u_N(N)$. Further, since s is the probability that the $(n + 1)$st child will survive, the marginal expected utility of the $(n + 1)$st birth is given by (4).

Properties of the Optimal Choice

Expression (8) shows that: *The expected utility* U *is strictly concave in the number of births,* n .
In turn, this yields

PROPOSITION 1. *The optimal number of births is either unique, or there are two neighboring numbers that are both optimal.*

To prove this result, recall that n(s) denotes the largest optimal value of n . Thus, it must satisfy

(9a) $$U_n(n(s) - 1, s) \geq 0 \text{ , and}$$

(9b) $$U_n(n(s), s) < 0 \text{ .}$$

Using (8) and the definition $U_{nn}(n, s) \equiv U_n(n + 1, s) - U_n(n, s)$, it follows that (9a) and (9b) respectively yield

(10) $$U_n(n, s) > 0 \text{ if } n < n(s) - 1 \text{ , and } U_n(n, s) < 0 \text{ if } n \geq n(s) \text{ .}$$

Expression (10) has the following implication. If the inequality in (9a) is strict, then n(s) is the unique optimal choice. If (9a) holds with equality, then n(s) and n(s) – 1 are the only two optimal choices. Note that, for brevity, these and several other results below are proven for interior values of n(s) ; that is, for $n(s) \geq 1$.

The Effect of a Change in the Mortality Rate on the Number of Births

We first show that: *A lower mortality rate does not increase the number of births.* That is,

(11) $$n(s') \leq n(s) \text{ for } s' > s \text{ .}$$

This result can be established as follows. Since expression (6) holds for any fixed n , we may evaluate it at $n = n(s)$. From (8), U_{nn} is negative for all values of n . From (9b), $U_n(n(s), s) < 0$. Thus, the evaluation of (6) yields

(12) $$\frac{\partial}{\partial s} U_n(n = n(s), s) < 0 \text{ ,}$$

where "n = n(s)" indicates that the value of n is kept unchanged at n(s) in computing this derivative.

Now, let s' denote a survival probability slightly larger than s . Then, (12) implies that $U_n(n(s), s') < U_n(n(s), s)$. In turn, from (9b),

(13) $$U_n(n = n(s), s') < 0 .$$

From (8), U_n is decreasing in n . Thus, it follows from (13) that

(14) $$U_n(n, s') < 0 , \text{ for } n \geq n(s) .$$

Expression (14) implies that $U(n(s), s') > U(n(s) + 1, s') > \cdots$. That is, $U(n, s')$ is larger at $n = n(s)$ than at $n > n(s)$. Consequently, a value of n larger than n(s) is not optimal at s' .

We have thus shown that n(s) is locally non-increasing in s . The global counterpart of this result is (11), and, as shown in the Appendix, it is obtained from the local result, by using standard continuity arguments.

Next, we rule out the uninteresting case in which the number of births remains entirely unchanged throughout the range of mortality rates. Thus, (11) yields

PROPOSITION 2. *The number of births is an increasing integer function of the mortality rate.*

Note that the sole assumption concerning the ex-post utility u(N) employed in the above analysis is that it is strictly concave in N . Even this reasonable assumption can be relaxed, since the only role it plays is to show that U_{nn} is negative, as noted in (8). For example, Proposition 2 can be shown to hold even if u(N) is concave in N , but is strictly concave only at N = 0 .

Finally, suppose that the ex-post utility does not have the above concavity property. Even in this case, Proposition 2 holds for local changes in the mortality rate. To see this, begin with n(s) , which is the largest optimal choice at s . Now, consider all those slightly larger survival probabilities s' for which the optimal choice changes at most by one. That is, the candidates for the optimal choice at s' are n(s) – 1 , n(s) or n(s) + 1 . Then,

(15) n(s) + 1 *is not optimal at* s' .

The proof of (15), which is also used later, is as follows. Recalling that $U_{nn}(n, s) \equiv U_n(n + 1, s) - U_n(n, s)$, expression (6) can be rewritten as

(16) $$\frac{\partial}{\partial s} U_n(n, s) = \frac{1}{s} \{(n + 1)U_n(n, s) - nU_n(n - 1, s)\} .$$

Expressions (9a), (9b) and (16) imply that $\frac{\partial}{\partial s} U_n(n = n(s), s) < 0$. Therefore, $U_n(n(s), s') < U_n(n(s), s)$. The preceding expression and (9b) imply that $U_n(n(s), s') < 0$. This yields (15).

The Effect of a Change in the Mortality Rate on Individual Welfare

This effect is described by

PROPOSITION 3. *Individual welfare does not decrease if the mortality rate decreases. Moreover, individual welfare is strictly higher if the fertility choice is non-trivially different at a lower mortality rate.*

To prove this proposition, note from (7) and (9a) that $\frac{\partial}{\partial s} U(n = n(s), s) \geq 0$. Thus, for a value of s' slightly larger than s ,

(17) $$U(n(s), s') \geq U(n(s), s) .$$

Let n(s') denote the largest optimal choice at s' . Then, definition (3) of the optimum implies

(18) $$U(n(s'), s') \geq U(n(s), s') .$$

Expressions (17) and (18) and the definition of individual welfare, (3), yield $V(s') \geq V(s)$. That is, V is locally non-decreasing in s . Once again, by using continuity arguments, the global counterpart of this result is obtained.

The above proof has an intuitive interpretation. Expression (17) shows that an individual is no worse-off at a lower mortality rate if he were to make a choice that was optimal at a higher mortality rate. Thus, the individual's welfare does not decrease at his actual optimal choice at a lower mortality rate. A stronger result is obtained if the choice at a lower mortality rate differs non-trivially from that at a higher mortality rate. Here, by a non-trivial difference we mean that a choice that is optimal at s is not optimal at s' . In this case, inequality (18) is strict. Thus, individual welfare is strictly higher at the lower mortality rate.

These results on the welfare effect of a change in the mortality rate are quite robust. They are a consequence solely of the optimizing behavior, and they do not depend on the properties of the ex-post utility function u(N) . However, the results are not obvious because, for instance, they require the relationship in (7).

By extending this welfare analysis, we can assess the magnitude of the gain to an individual from a decrease in the mortality rate. For example, let M denote the current income of the individual, and let ΔM denote a hypothetical increase in his income that has the same value to him as an increase in the survival probability from s to s' . We know from Proposition 3 that ΔM is not negative, and that it will typically be positive. Further, for any particular specification of the individual's utility function, the value of ΔM can be calculated from the equality: $V(s, M + \Delta M) = V(s', M)$. The same approach is useful in models in which the mortality rate is determined endogenously by the individual, and in which we wish to assess the individual's welfare gain from a change in a parameter such as preventive health technology.

The usefulness of such welfare assessments is as follows. Governments often undertake programs aimed at reducing child mortality. Such programs are common in most developing countries, but they have also been undertaken in some developed countries for particular social groups that experience relatively high child mortality. In an economic evaluation of such programs, a necessary component is the assessment of individuals' welfare gains of the type described above.

Ex-ante Costs

To incorporate the ex-ante costs and benefits, let $C(n)$ denote the net utility cost that depends on the number of births, n, but not on how many of them survive, N. Then (3) is replaced by $V'(s) \equiv Max_n$: $U'(n, s)$, where

(19) $$U'(n, s) = U(n, s) - C(n) ,$$

and U is given by (2). Define $U'_n(n, s) \equiv U'(n + 1, s) - U'(n, s)$ and $C_n(n) \equiv C(n + 1) - C(n)$. Define $U'_{nn}(n, s)$ and $C_{nn}(n)$ accordingly. Assume that $C_n \geq 0$ and $C_{nn} \geq 0$; that is, the marginal ex-ante utility cost of births is non-negative and non-decreasing in the number of births. From (19),

(20) $$U'_{nn}(n, s) = U_{nn}(n, s) - C_{nn}(n) ,$$

where U_{nn} is given by (5). The optimality conditions, (9a) and (9b), now become $U'_n(n(s) - 1, s) \geq 0 > U'_n(n(s), s)$. However, $\frac{\partial}{\partial s} U'_n$ and $\frac{\partial}{\partial s} U'$ continue to be described by the right-hand sides of (6) and (7) respectively.

It is then straightforward to verify that Propositions 1 and 3 remain unaffected. To examine Proposition 2, we need some additional derivations. First, if

(21) $$\frac{\partial}{\partial s} U'_n(n = n(s), s) \leq 0 ,$$

then, by going through the steps following expression (12), we can verify that (11) holds, and, hence, Proposition 2 holds. To evaluate the left-hand side of (21), we derive the following in the Appendix:

(22) $$\frac{\partial}{\partial s} U'_n(n, s) = \Sigma_{N=0}^{n-1} \phi(N, n - 1, s)u_{NN}(N) + u_N(n) , \text{ where}$$

(23) $$\phi(N, n - 1, s) \equiv (n + 1)sb(N, n - 1, s) - B(N, n - 1, s) ,$$

and $B(N, n, s) \equiv \Sigma_{j=0}^{N} b(j, n, s)$ denotes the cumulative probability of N or fewer survivals out of n births. To evaluate the sign of (22), we need the signs of the ϕ, defined in (23). These signs can be assessed numerically for the limited range of possible values of n and s that are relevant in most fertility contexts, or they can be assessed analytically. As an example of the latter, we show in the Appendix that a set of sufficient conditions for all of the ϕ to be positive is: $n \leq 12$ and $s > 0.81$. A survival probability smaller than 0.81 and a total number of births larger than a dozen are not particularly relevant in many fertility contexts. Thus, for brevity, we treat the ϕ as positive in the results presented below. However, analogous results can be obtained under weaker conditions on n and s than those just noted.

Now, since $u_{NN}(N) < 0$, it follows from (22) that (21) is satisfied if $u_N(n) \leq 0$. That is: *A sufficient condition for Proposition 2 to hold is that the marginal ex-post utility from an extra surviving child is non-positive if all of the children from an optimally chosen number of births survive.*

A related observation is as follows. Since the ϕ in the right-hand side of (22) are positive, (21) is likely to be satisfied if the $u_{NN}(N)$ are sufficiently large negative numbers. Thus: Proposition 2 is likely to hold if the ex-post utility $u(N)$ is sufficiently concave *in* N.

It is useful to briefly consider situations in which the result in (11) is reversed; that is

(24) $$n(s') \geq n(s) \quad \text{for} \quad s' > s .$$

Using arguments similar to that employed earlier, it can be shown that (24) holds if we have

(25) $$\frac{\partial}{\partial s} U_n'(n = n(s) - 1, s) \geq 0 .$$

Now, consider, for a moment, the polar assumption that the ex-post utility $u(N)$ is linear and increasing in n (recalling the discussion in the beginning of this section, however, this assumption is not easily justified). Then, in the right-hand side of (22), $u_{NN}(N) = 0$, and $u_N(n)$ is positive for all n. This yields (25) and, hence, (24).

Next, we look at the role that the ex-ante cost of births, $C(n)$, plays in whether the result in (11) or the result in (24) holds. Recall our assumption that $u(N)$ is first increasing and then decreasing in N. That is, $u_N(N)$ is positive up to some value of N, and negative beyond that. If the values of $C_n(n)$ are larger, then the optimal number of births, $n(s)$, will be smaller. Since $u_N(N)$ is decreasing in N, the values of $u_N(n(s))$ and $u_N(n(s) - 1)$ will be larger at a smaller value of $n(s)$. Consider, then, the case in which $u_N(n(s) - 1)$ is a positive number. If the concavity of $u(N)$ is sufficiently mild (that is, if the $u_{NN}(N)$ are sufficiently small negative numbers), then it is apparent from (22) that (25) is likely to hold, and, thus, the result in (24) is likely to be obtained. Note, however, that the outcome is critically dependent on how concave $u(N)$ is. If the concavity of $u(N)$ is sufficiently strong (that is, if the $u_{NN}(N)$ are sufficiently large negative numbers), then, as discussed earlier, the result in (11) is likely to hold.

Finally, the analysis presented in this subsection can be extended to a formulation in which the ex-ante cost is not separable as in (19). In a more general formulation, we would replace the $u(N)$ in the right-hand side of (2) by a function $u'(n, N)$. A special case of this formulation is $u'(n, N) \equiv u(N) - C(n)$, which yields (19). In the more general case, we can obtain results analogous to those presented above, by using methods employed in the preceding analysis. For instance, consider Proposition 3. The cost of an extra birth is now defined as $-\{u'(n + 1, N) - u'(n, N)\}$ which depends on the realized value of N. If this cost is assumed to be non-negative, then it is easily verified that Proposition 3 remains unaffected.

III. A DYNAMIC STOCHASTIC MODEL

As noted earlier, the natural formulation of fertility choice is as a stochastic dynamic program. This section examines a two-stage model that captures this aspect. A multi-stage model is then discussed. The present section has the same objective as the previous one, namely, to extract some predictability concerning the fertility effects of a change in the mortality rate. Naturally, one would not expect the results in a dynamic model to be as crisp as those in the preceding section. Yet, as we shall see, a significant part of the earlier analysis carries over.

A Two-stage Model

Since we are considering a two-stage model, it is useful to assess the effect of a change in the mortality rate on three different but related fertility variables: (i) The number of births in the first period. (ii) The number of births in the second period. This is a random number, in general, because it depends on the number of surviving children from the first period. (iii) The expected number of total births; that is, the number of births in the first period plus the expected value of that in the second period.

Let the integer variables n_1 and n_2 denote the number of births in the two periods. The corresponding number of surviving children are denoted by the random variables N_1 and N_2, where $N_1 = 0, 1, ..., n_1$, and $N_2 = 0, 1, ..., n_2$. The individual observes N_1 before choosing n_2. The net ex-post utility, after the number of surviving children from the second period is observed, is denoted by $u(N_1, N_2)$. This and other utilities discussed later subsume intertemporal discounting. Let s_1 and s_2 denote respectively the survival probabilities of a child born in the first and second periods. Correspondingly, $1 - s_1$ and $1 - s_2$ are the mortality rates for a first-period and a second-period child. A special case of this model is when s_1 equals s_2, but this assumption is not employed because it does not simplify the analysis. If the individual's welfare level is described by the indirect utility $V^1(s_1, s_2)$, then

(26) $$V^1(s_1, s_2) \equiv \text{Max}_{n_1} : \Sigma_{N_1} b(N_1, n_1, s_1) \text{ Max}_{n_2} : \Sigma_{N_2} b(N_2, n_2, s_2)u(N_1, N_2) .$$

Once again, this model is highly simplified. It abstracts from those costs that depend on the number of births but not on the number of surviving children. It abstracts from the interim costs or benefits of the children surviving from the first period, before the outcome of the second period choice is known. In addition, it abstracts from the potential fertility effects of possible deaths of the first-period children after the second period choice has been made. These aspects are discussed in the last part of this section.

For later use, let $u_1(N_1, N_2)$ and $u_2(N_1, N_2)$ respectively denote the marginal utilities of a surviving child from the two periods; that is, $u_1(N_1, N_2) \equiv u(N_1 + 1, N_2) - u(N_1, N_2)$, and $u_2(N_1, N_2)$ is defined similarly. Denote the changes in these marginal utilities due to one more surviving child by $u_{11}(N_1, N_2) \equiv u_1(N_1 + 1, N_2) - u_1(N_1, N_2)$, and by $u_{12}(N_1, N_2)$ and $u_{22}(N_1, N_2)$, defined similarly. The surviving children from the two periods are assumed to be substitutes; that is, $u_{12} \equiv u_{21} < 0$. Also, u is assumed to be strictly concave in N_2; that is, $u_{22} < 0$.

Choice in the Second Period

We begin by considering the individual's choice after the outcome of his choice in the first period, N_1, has been observed. Define

(27) $$U^2(N_1, n_2, s_2) \equiv \Sigma_{N_2} b(N_2, n_2, s_2)u(N_1, N_2) .$$

Denote the maximized value of U^2 by

(28) $$V^2(N_1, s_2) \equiv \text{Max}_{n_2} : U^2(N_1, n_2, s_2) \equiv U^2(N_1, n_2(N_1, s_2), s_2) .$$

In (28) and below, $n_2(N_1, s_2)$ denotes the largest optimal value of n_2.

This choice problem is quite similar to the single-stage problem analyzed in the last section; the main difference is that the present problem is parameterized by the observed value of N_1. Define $U_n^2(N_1, n_2, s_2) \equiv U^2(N_1, n_2 + 1, s_2) - U^2(N_1, n_2, s_2)$. Define $U_{nn}^2(N_1, n_2, s_2)$ accordingly. Then, analogous to (5), (6), (7) and (8) we obtain

(29) $$U_{nn}^2(N_1, n_2, s_2) = (s_2)^2 \Sigma_{N_2} b(N_2, n_2, s_2)u_{22}(N_1, N_2) < 0 .$$

(30) $$\frac{\partial}{\partial s_2} U_n^2(N_1, n_2, s_2) = \frac{1}{s_2} \{U_n^2(N_1, n_2, s_2) + n_2 U_{nn}^2(N_1, n_2 - 1, s_2)\} .$$

(31) $$\frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2) = \frac{n_2}{s_2} U_n^2(N_1, n_2 - 1, s_2) .$$

Also, similar to (9a) and (9b), an optimum is characterized by

(32) $$U_n^2(N_1, n_2(N_1, s_2) - 1, s_2) \geq 0 > U_n^2(N_1, n_2(N_1, s_2), s_2) .$$

The analysis of this optimum closely follows that of the single-stage problem of the last section. To avoid repetition, therefore, we only summarize the following three results, which are the analogues of Propositions 1, 2 and 3. These results are also used later.

PROPOSITION 4

(i) The optimal number of births in the second period is either unique, or there are two neighboring numbers that are both optimal.

(ii) The number of births in the second period is an increasing integer function of the mortality rate of a second-period child.

(iii) If $s_2' > s_2$, then $V^2(N_1, s_2') \geq V^2(N_1, s_2)$. *The preceding inequality is strict if a change in the mortality rate of a second-period child alters non-trivially the second-period fertility choice.*

The effect of the number of surviving children from the first period on the number of births in the second period. A new question that arises in the present case is, how does N_1 affect $n_2(N_1, s_2)$? To ascertain this effect, the following expression is established in the Appendix:

(33) $$U_n^2(N_1 + 1, n_2, s_2) < U_n^2(N_1, n_2, s_2) .$$

Next, using expressions (29), (33) and the second part of (32), we obtain

(34) $$U_n^2(N_1 + 1, n_2, s_2) < 0 \quad \text{for} \quad n_2 \geq n_2(N_1, s_2) .$$

From (34), a value of n_2 larger than $n_2(N_1, s_2)$ is not optimal for $N_1 + 1$. In other words, $n_2(N_1, s_2)$ is non-increasing in N_1. Next, we rule out the case in which the optimal n_2 is entirely unaffected by N_1. In this uninteresting case, individual choice is completely separable between the periods; the second period's choice could be made optimally without observing the number of surviving children from the first period. We thus obtain

PROPOSITION 5. *The number of births in the second period is a decreasing integer function of the number of surviving children from the first period.*

Choice in the First Period

The analysis of the choice in the first period differs from that described above. Let the expected utility from a given number of births in the first period be denoted by

(35) $$U^1(n_1, s_1, s_2) \equiv \Sigma_{N_1} b(N_1, n_1, s_1)V^2(N_1, s_2) ,$$

where V^2 was defined in (28). Then (26) can be stated as

(36) $$V^1(s_1, s_2) \equiv \mathrm{Max}_{n_1} : U^1(n_1, s_1, s_2) \equiv U^1(n_1(s_1, s_2), s_1, s_2) ,$$

where $n_1(s_1, s_2)$ denotes the largest optimal value of n_1 .

Let the marginal expected utility from an additional birth in the first period be denoted by $U^1_n(n_1, s_1, s_2) \equiv U^1(n_1 + 1, s_1, s_2) - U^1(n_1, s_1, s_2)$. Define $U^1_{nn}(n_1, s_1, s_2)$ accordingly. Then, analogous to (29), (30) and (31), we obtain

(37) $$U^1_{nn}(n_1, s_1, s_2) = (s_1)^2 \Sigma_{N_1} b(N_1, n_1, s_1)\{V^2_N(N_1 + 1, s_2) - V^2_N(N_1, s_2)\} ,$$ where

$$V^2_N(N_1, s_2) \equiv V^2(N_1 + 1, s_2) - V^2(N_1, s_2) .$$

(38) $$\frac{\partial}{\partial s_1} U^1_n(n_1, s_1, s_2) = \frac{1}{s_2} \{U^1_n(n_1, s_1, s_2) + n_1 U^1_{nn}(n_1 - 1, s_1, s_2)\} .$$

(39) $$\frac{\partial}{\partial s_1} U^1(n_1, s_1, s_2) = \frac{n_1}{s_1} U^1_n(n_1 - 1, s_1, s_2) .$$

An optimum is characterized by

(40) $$U^1_n(n_1(s_1, s_2) - 1, s_1, s_2) \geq 0 > U^1_n(n_1(s_1, s_2), s_1, s_2) .$$

The concavity property of the expected utility U^1 , with respect to n_1 , cannot be established at the present level of generality. Therefore, unlike Proposition 4(i), the results concerning the uniqueness of the optimal n_1 cannot be easily demonstrated. For this reason and to keep the paper brief, we will consider only local perturbations in the optimal choice. That is, we will consider those small changes in the parameters that alter the optimal n_1 by at most one, and then examine whether the optimal choice decreases or increases.

The effect of a change in the mortality rate of a first-period child on the first-period choice. The effect of a change in s_1 on $n_1(s_1, s_2)$ is described by the proposition below. The proof is omitted because it is identical to that of (15). This can be verified using (38) and (40).

PROPOSITION 6. *The number of births in the first period is non-decreasing in the mortality rate of a first-period child.*

The effect of a change in the mortality rate of a second-period child on the first-period choice. To analyze this effect, we need the following two results from an envelope theorem for integer choice variables (see Sah and Zhao (1989) for these and related results).

(i) Consider the optimization in (28). If the optimal value of n_2 is unique, then the derivative $\frac{\partial}{\partial s_2} V^2$ exists and the standard envelope theorem holds:

(41) $$\frac{\partial}{\partial s_2} V^2(N_1, s_2) = \frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2)\Big|_{n_2=n_2(N_1,s_2)} .$$

The same theorem holds even if there are two optimal values of n_2, provided

(42) $$\frac{\partial}{\partial s_2} U_n^2(N_1, n_2 - 1, s_2)\Big|_{n_2=n_2(N_1,s_2)} = 0 .$$

(ii) However, when there are two optimal values of n_2, the standard envelope theorem may not hold because the derivative in (41) may not exist. On the other hand, the right-handed and the left-handed derivatives, denoted respectively by $\frac{\partial}{\partial s_2} V^{2+}$ and $\frac{\partial}{\partial s_2} V^{2-}$, always exist, and the corresponding envelope theorems are:

(43) $$\frac{\partial}{\partial s_2} V^{2+}(N_1, s_2) = \frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2)\Big|_{n_2=n_2(N_1,s_2)-1} , \text{ and}$$
$$\frac{\partial}{\partial s_2} V^{2-}(N_1, s_2) = \frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2)\Big|_{n_2=n_2(N_1,s_2)} .$$

Using these two results, the following proposition is proved in the Appendix.

PROPOSITION 7. *The number of births in the first period is non-decreasing in the mortality rate of a second-period child if:*

(44) *A unit decrease in the number of surviving children from the first period does not induce more than a unit increase in the maximum or the minimum number of optimal births in the second period,* and

(45) $$u_{12}(N_1, N_2) \le s_2 u_{22}(N_1, N_2) .$$

Condition (44) has an intuitive interpretation. We know from Proposition 5 that the number of births in the second period does not decrease if one fewer child from the first period survives. Condition (44) restricts how large the increase in the number of births in the second period can be. This restriction is consistent with most empirical studies, which show that the increase in births per death is considerably smaller than one (see Schultz (1981, Ch. 5) and Schultz (1988, pp. 444-5)).

Condition (45) also has a simple interpretation. Recall that $u_{22} < 0$ and $u_{12} < 0$. Thus, (45) can be restated as

(46) $$|u_{12}(N_1, N_2)| \ge s_2 |u_{22}(N_1, N_2)| .$$

That is, the decrease in the marginal ex-post utility of a surviving child from the second period, due to one more surviving child from the first period, is not significantly smaller than the corresponding decrease due to one more surviving child from the second period.

Note that conditions (44) and (45) are sufficient but not necessary for Proposition 7, and that they can be weakened. For example, it can be shown that condition (45) can be omitted if the maximum number of births in the second period is two.

The Effect of a Change in the Mortality Rate on the Expected Number of Total Births

Recall that, ex-ante, the number of births in the second period, $n_2(N_1, s_2)$, is a random variable contingent upon N_1. At the beginning of fertility decisions, therefore, the number of total births is random. The expected number of total births, denoted by $e(s_1, s_2)$, is

$$(47) \qquad e(s_1, s_2) = n_1(s_1, s_2) + \Sigma_{N_1=0}^{n_1(s_1,s_2)} b(N_1, n_1(s_1, s_2), s_1)n_2(N_1, s_2) .$$

A question that arises, then, is: what are the effects of changes in s_1 and s_2 on $e(s_1, s_2)$? This question is not fully answered by the preceding analysis. For instance, even though the number of births in each of the two periods, $n_1(s_1, s_2)$ and $n_2(N_1, s_2)$, are non-increasing in s_2, it does not follow that e is non-increasing in s_2. This is so because a smaller number of births in the first period can reduce the number of first-period children who survive. In turn, this can raise the number of births in the second period.

The analysis below examines those changes in s_1 and s_2 that induce local perturbations in the optimal choice of n_1 and n_2. First consider the effect of a change in s_1 on e. Note from (47) that s_1 affects e in two ways: (i) it directly affects the probability density b in the second term in the right-hand side of (47), and (ii) it affects $n_1(s_1, s_2)$. The latter effect influences the first term in the right-hand side of (47), and also the density b in the second term.

Consider these two effects separately. The first effect of a larger s_1 is to induce a first-order stochastic improvement in the density b. Further, from Proposition 5, $n_2(N_1, s_2)$ is non-increasing in N_1. Therefore, a standard result concerning stochastic dominance (see Ingersoll (1987, pp. 137-9)) yields

$$(48) \qquad \frac{\partial}{\partial s_1} e(s_1, s_2)\Big|_{n_1=n_1(s_1,s_2)} \leq 0 .$$

The second effect of s_1 on e is through $n_1(s_1, s_2)$. Proposition 6 shows that the optimal n_1 is non-increasing in s_1. Now, if a change in s_1 does not affect $n_1(s_1, s_2)$, then e is not influenced by the effect of s_1 under consideration at present. Therefore, we examine the case where $n_1(s_1', s_2) = n_1(s_1, s_2) - 1$, for a value of s_1' larger than s_1. For notational brevity, denote $n_1(s_1, s_2)$ by q. Then, using (47), it is shown in the Appendix that

$$(49) \qquad e(s_1', s_2) - e(s_1, s_2) = -1 + s_1 \Sigma_{N_1=0}^{q-1} b(N_1, q-1, s_1)\{n_2(N_1, s_2) - n_2(N_1 + 1, s_2)\} .$$

Now, assume that (44) holds; that is, a unit increase in N_1 induces no more than a unit decrease in

$n_2(N_1, s_2)$. Then, the right-hand side of (40) is negative, because $s_1 < 1$. Putting the two effects of s_1 on e together, we conclude that e is non-increasing in s_1, if (44) holds.

Next, consider the effect of a change in s_2 on e. Recalling (47), s_2 has two effects as well: it affects $n_2(N_1, s_2)$ and $n_1(s_1, s_2)$. The analysis of the first effect is straightforward. Proposition 4(ii) shows that $n_2(N_1, s_2)$ is non-increasing in s_2. Thus (47) yields

$$(50) \qquad \frac{\partial}{\partial s_2} e(s_1, s_2)\Big|_{n_1 = n_1(s_1,s_2)} \leq 0 .$$

To analyze the second effect, recall from Proposition 7 that $n_1(s_1, s_2)$ is non-increasing in s_2, provided (44) and (45) hold. The analysis of this effect is thus similar to the earlier analysis of the effect of s_1 on e due to the induced change in $n_1(s_1, s_2)$. Hence, due to this effect, a larger s_2 does not raise e if (44) and (45) hold. Putting the two effects together, e is non-increasing in s_2, if (44) and (45) hold.

Proposition 8 summarizes the above results.[7]

PROPOSITION 8. *The expected number of total births is non-decreasing in the mortality rate of either a first-period child or a second-period child, if (44) and (45) hold.*

The Effect of a Change in the Mortality Rate on Individual Welfare

Recall from (36) that the individual's welfare level is described by the indirect utility $V^1(s_1, s_2)$. It is affected by s_1 and s_2 as follows.

PROPOSITION 9. *Individual welfare is non-increasing in the mortality rate of either a first-period child or a second-period child. Moreover, individual welfare strictly decreases if an increase in either mortality rate induces a non-trivial change in either the first-period or the second-period fertility choice.*

The proof of the effect of s_1 on V^1 is similar to the proof of Proposition 3. This can be established using (36), (39) and (40).

To examine the welfare effect of s_2, consider (35) and (36). Suppose for a moment that the optimal choice of n_1 is left unchanged, although the value of s_2 has increased to s_2'. Then, it is clear from (35), (36) and Proposition 4(iii) that V^1 does not decrease, and that V^1 strictly increases if the higher value of s_2 alters non-trivially the choice of n_2 for even one value of N_1. Compared to this outcome, the actual welfare of the individual is not lower because he will also choose n_1 optimally, given the changed value

[7] These results can be strengthened in several ways. For instance, e is strictly lowered by a larger s_1, provided the second-period choice, $n_2(N_1, s_2)$, is not entirely insensitive to N_1. This is because inequality (48) is strict if $n_2(N_1, s_2)$ is decreasing in N_1 for even one value of N_1. Similarly, e is strictly lowered by a larger s_2 provided it lowers $n_2(N_1, s_2)$ for even one value of N_1. This is because inequality (50) is strict in this case.

of s_2. We thus conclude that $V^1(s_1, s_2') \geq V^1(s_1, s_2)$, and that the preceding inequality is strict if the change in s_2 alters non-trivially the choice of n_1, or any one of the choices of n_2 that the individual might make in the future.

Extensions

Interim utility and mortality. A simplification employed in the above two-stage model was that the mortality of the children born in the current period, insofar as it is a critical determinant of the current fertility decision, is revealed before the next set of decisions is made. This can easily be modified to incorporate age-specific mortality. It is also straightforward to incorporate the net utility (once again, inclusive of all costs and benefits) that the individual derives in a particular period from the surviving children from a previous period, some of whom might die in the near future. As an illustration, let the random variable N_1 denote the number of first-period children surviving at the end of the period, when the second-period choice, n_2, is made. Let the random variable $N_{1,2}$ denote the number, out of N_1, that survive until the end of the second period. Let $s_{1,2}$ denote the corresponding probability of each survival. Denote the utility beyond the second-period by $u(N_{1,2}, N_2)$, and the interim (age-specific) utility during the second period by $v(N_1)$. Then the second-period choice continues to be represented by (27) and (28), provided the u in the right-hand side of (27) is replaced by $\Sigma_{N_{1,2}} b(N_{1,2}, N_1, s_{1,2})u(N_{1,2}, N_2)$. The first-period choice also continues to be described by (35) and (36), provided the V^2 in the right-hand side of (35) is replaced by $v(N_1) + V^2(N_1, s_2)$. Further, the ex-ante costs of births in each of the two periods can be incorporated into the analysis. In this case, as the analysis presented at the end of last section indicates, additional conditions will be needed for some of the results.

Multiple Stages of Choice. Consider, briefly, the following multi-stage extension of the simple two-stage choice model analyzed earlier. Let $t = 1, 2, ..., T$ denote the different periods of choice, where $t = 1$ denotes the first period, and $t = T$ denotes the last. Let $n_t = 1$ or 0 denote whether there is a birth or not in period t. Assume that, to the extent a child's mortality is critical to future fertility decisions, it is experienced within one period after the child's birth. Such an assumption is often employed in the context of developing countries, because a large portion of child mortality is experienced within the very early phase of the life. In any case, this assumption can be modified to incorporate age-specific mortality, as discussed in the previous paragraph. Let the random variable N_t denote the number of children surviving out of n_t. Let s_t denote the survival probability of a child born in period t. Define the vectors $\bar{N}_t \equiv (N_1, ..., N_{t-1})$ and $\bar{s}_t \equiv (s_t, ..., s_T)$. Then the individual choice is described by

(51) $$V^t(\bar{N}_t, \bar{s}_t) \equiv \text{Max}_{n_t} : \Sigma_{N_t} b(N_t, n_t, s_t)V^{t+1}(\bar{N}_{t+1}, \bar{s}_{t+1}) ,$$

for $t = 1$ to T. In (51), $V^{T+1} \equiv u(\bar{N}_{T+1})$ is the ex-post utility; interim utilities can be included as discussed earlier. The individual's welfare level is described by the indirect utility $V^1(\bar{s}_1)$. Let $n_t(\bar{N}_t, \bar{s}_t)$ denote the largest optimal value of n_t.

The foregoing problem can be analyzed using the methods developed in this section. For instance, the following results can be established. (i) $n_t(\bar{N}_t, \bar{s}_t)$ *is non-increasing in* s_t. (ii) *Individual welfare,* V^1, *is non-decreasing in* $(s_1, ..., s_T)$, *and it is strictly increasing in each* s_t *if a larger value of* s_t *has a non-trivial effect on any of the fertility choices.*

IV. CONCLUDING REMARKS[8]

Remarks on some earlier models. Ben-Porath and Welch (1972) and Ben-Porath (1976) have examined a single-stage choice model with the following specification of the expected utility of n births:

(52) $$U(n, s) = G(ns, I - pn) ,$$

where p denotes the ex-ante cost per birth, I denotes the individual's income, and n is treated as a continuous variable. A motivation that they suggest for (52) is that the individual is concerned about the expected number of surviving children, ns. Variants of this model considered by them do not alter the particular aspects that are of interest in the present discussion. Let η_{ns} and η_{np} denote respectively the elasticities of n with respect to s and p. Then (52) yields $\eta_{ns} = -(1 + \eta_{np})$. This relationship implies, in general, an ambiguous fertility effect of a change in the mortality rate (see Heckman and Willis (1975) and Schultz (1976), among others, for discussions of this model). On the other hand, the expected number of surviving children, denoted by $E \equiv ns$, is larger if s is larger. To see this, let η_{Es} and η_{Ep} respectively denote the elasticities of E with respect to s and p. Then, $\eta_{Es} = 1 + \eta_{ns} = -\eta_{Ep} = -\eta_{np} > 0$, under the reasonable assumption that E has the property of a normal good (that is, $\eta_{Ep} < 0$).[9]

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[8] I have benefited from discussions with James Heckman on the material presented below.

[9] In a set of important papers, Barro and Becker have analyzed some of the determinants of population within dynamic models based on dynastic individual utility (see Becker and Barro (1988) and Barro and Becker (1989)). They focus on the expected number of surviving children, E, rather than on the number of births, n, which is the focus of the present paper and of other papers cited in this subsection. A part of their analysis deals with the effect of a change in s on E, using a utility function similar to that in (52) in which the utility depends on E. They use the effect just noted in the text, that E is increasing in s. This effect, however, may arise in other models as well. In our analysis, for instance, n is a decreasing integer function of s. Thus, depending on the value of s, E may be locally increasing or decreasing in s.

The specification in (52) is in general inconsistent with individual choice under uncertainty. The version of (52) which is consistent with choice under uncertainty, but which has not been analyzed by Ben-Porath and Welch, is

(53) $$U(n, s) = G_1 ns + G_2(l - pn) ,$$

where the parameter G_1 does not depend on n, and G_2 is a function. The specification in (53) predicts, contrary to the typically observed pattern, that fertility increases if the mortality rate declines; that is, $\eta_{ns} > 0$. Also, (53) implies that the ex-post net utility from N surviving children is linear in N. As discussed in the beginning of Section II, this assumption is not easily justified.

O'Hara (1975) has examined a single-stage model, in which the ex-post utility is described by $u(Z, N, Q)$, where Z denotes parental consumption, and Q denotes the quality of the children. The quality of the children yields parental benefits only if the children survive beyond some stage; for brevity, call this stage "maturity." His analysis is based on the maximization of the following expected utility: $p_1u(Z, O, O) + p_2u(Z, n, O) + p_3u(Z, n, Q)$, subject to a standard budget constraint. Clearly the only three relevant outcomes, or the states-of-the world, in this specification are: (i) when *all* n children die after birth, (ii) when *all* n children survive before maturity but *none* survives to maturity, and (iii) when *all* n children survive to maturity. The respective probabilities of these outcomes are denoted by p_1, p_2 and p_3. This model assigns no utility to all those outcomes in which some of the children die while others survive. Another problem with this model is that it treats the probabilities p_1, p_2 and p_3 as exogenous parameters. It overlooks the fact that these probabilities must depend on the number of births, n, which is a choice variable. For instance, p_1 is $(1 - s)^n$ in our notation.

Remarks on the use of a discrete representation. It is self-evident that a discrete representation of the number of children, born or surviving, is more realistic than a continuous representation. We now illustrate the reasons why a discrete representation also yields crisper and better results in the present context. For this, we reconsider Propositions 1, 2 and 3, using the simple single-stage model described in the beginning of Section II. Let $f(N, n, s)$ denote the probability density of N survivals out of n births, where N and n are now treated as continuous variables. The expected utility is now $U(n, s) = \int_0^n u(N)f(N, n, s)dN$, instead of (2). Let a subscript denote the variable with respect to which a partial derivative is being taken. Assume that the optimal value of n, denoted by $n(s)$, is interior. Then, instead of (9a) and (9b), an optimality condition now is: $U_n(n, s) = 0$ at $n = n(s)$. To establish the continuous versions of Propositions 1, 2 and 3, we need to show, respectively, the following:

(54) (i) $U_{nn}(n, s) < 0$ for all n ;

(55) (ii) $U_{ns}(n, s)/U_{nn}(n, s) \geq 0$ at $n = n(s)$; and (iii) $U_s(n, s) \geq 0$ at $n = n(s)$.

These expressions are examined in the Appendix, where it is shown that they do not follow from a set of assumptions that are either intuitive or comparable to those employed in our discrete analysis.

A reason for this difference between a discrete and a continuous representation is as follows. To analyze the problem at hand, we need to evaluate the induced changes in the probabilities of various numbers of survivals (and the induced changes in expressions containing these probabilities) when n and s change. In the discrete case, these induced changes need to be evaluated only at integer values of N . In the continuous case, one needs to evaluate these changes on the entire real line representing N . Moreover, the evaluation of these induced changes is greatly simplified in the discrete case when the survival probabilities are described by a functional form such as the binomial density. This is because the binomial density has highly tractable properties that are lost to a significant degree even when a comparable continuous density (for instance, a normal approximation of the binomial density) is used. Also, it is apparent from the analysis in this paper that a discrete representation can be helpful in other, more complex models of fertility choice. Thus, in the present context, tractability and realism go hand in hand.

APPENDIX

Derivation of (4) and (5). The relationship described in (A1) is used repeatedly below. This "partial summation formula" (see Rudin (1976, p. 70)) is the discrete equivalent of integration by parts. Let x_i's and y_i's denote any set of numbers. Define $X_i \equiv \Sigma_{j=0}^{i} x_j$. Then,

(A1) $$\Sigma_{i=0}^{n} x_i y_i = -\Sigma_{i=0}^{n-1} X_i(y_{i+1} - y_i) + X_n y_n .$$

Also, for later use, denote the cumulative binomial density by $B(N, n, s) \equiv \Sigma_{j=0}^{N} b(j, n, s)$.

To derive (4), note that (2), (A1) and the definition $B(n, n, s) = 1$ yield (A2). In turn, (A3) follows.

(A2) $$U(n, s) = -\Sigma_{N=0}^{n-1} B(N, n, s)u_N(N) + u(n) .$$

(A3) $$U_n(n, s) = -\Sigma_{N=0}^{n-1}\{B(N, n + 1, s) - B(N, n, s)\}u_N(N) + \{1 - B(n, n + 1, s)\}u_N(n) .$$

To evaluate (A3), we need two identities. First,

(A4) $$B(N, n + 1, s) - B(N, n, s) = -sb(N, n, s) ;$$

see Sah (1989) for a general version of this identity. Second, by definition, $1 - B(n, n + 1, s) = s^{n+1} = sb(n, n, s)$. Using these, (A3) can be expressed as

(A5) $$U_n(n, s) \equiv U(n + 1, s) - U(n, s) = s\Sigma_{N=0}^{n} b(N, n, s)u_N(N) ,$$

which is (4). Next, we use (A5) again, but substitute U_n in the place of U. This yields (5).

Derivation of (6). A property of the binomial cumulative density is

(A6) $$\frac{\partial}{\partial s} B(N, n, s) = -nb(N, n - 1, s) .$$

Also, (A1) and (A5) yield $U_n(n, s) = s\{-\Sigma_{N=0}^{n-1} B(N, n, s)u_{NN}(N) + u_N(n)\}$. Using (A6) and (5), the derivative of the preceding expression, with respect to s, can be rearranged to yield (6).

Derivation of (7). Expressions (A2) and (A6) yield: $\frac{\partial}{\partial s} U(n) = n\Sigma_{N=0}^{n-1} b(N, n - 1, s)u_N(N)$. A rearrangement of the preceding expression, using (A5), yields (7).

Proof of (11). For a formal statement of the local result we have just established, define a function $g^{n(s)}(s^*) \equiv \frac{\partial}{\partial s^*} U_n(n = n(s), s^*)$ for $s^* \in (0,1)$. Consequently, we may state (12) as: $g^{n(s)}(s) < 0$. Let $g^{n(s)}$ be uniformly continuous on (0,1), and let $\inf_s g^{n(s)}(s)$ be non-zero. If we define $\varepsilon \equiv -\frac{1}{2} \inf_s g^{n(s)}(s)$, then $\varepsilon > 0$. From the uniform continuity of $g^{n(s)}$, there exists an $h(n(s), \varepsilon) > 0$ such that $g^{n(s)}(s') < \varepsilon + g^{n(s)}(s) < 0$, if $s + h(n(s), \varepsilon) > s' > s$. Thus, recalling the steps in (13) and (14), the local result proved in the text is: (11) holds if $s + h(n(s), \varepsilon) > s' > s$.

There is a finite number of possible values of $n(s)$; in fact, the largest feasible value of $n(s)$ is the biological maximum number of births. Hence, there is a finite number of values of $h(n(s), \varepsilon)$, no matter what s might be. Therefore, $h' \equiv \text{Min}_s\, h(n(s), \varepsilon)$ exists and is positive. In turn,

(A7) $\qquad n(s') \leq n(s)$ if $s + h' \geq s' \geq s$.

Now, consider a value of s' arbitrarily larger than s . Define $\tau \equiv (s' - s)/m > 0$, where m is an integer chosen such that $h' \geq \tau$. Then, $n(s') - n(s) = \Sigma_{j=0}^{m-1} [n(s' - j\tau) - n(s' - (j + 1)\tau)]$. Given (A7), the expression in the square bracket inside the preceding summation is non-positive for each j . Thus, (11) follows.

Derivation of (22) and Evaluation of (23). From (A1) and (A5), $U_n(n, s) = s\{-\Sigma_{N=0}^{n-1} B(N, n, s)u_{NN}(N) + u_N(n)\}$. Using this and (5), and recalling that $\frac{\partial}{\partial s} U_n'(n, s)$ is given by the right-hand side of (6), we obtain (22), where $\phi(N, n - 1, s) = nsb(N, n - 1, s) - B(N, n, s)$. Next, from (A4), $-B(N, n, s) = -B(N, n - 1, s) + sb(N, n - 1, s)$. Thus, (23) follows.

To evaluate ϕ , we first show that ϕ is single-peaked in N . Define $\phi_N(N, n - 1, s) \equiv \phi(N + 1, n - 1, s) - \phi(N, n - 1, s)$. From (23), $\phi_N(N, n - 1, s) = \{(n + 1)s - 1\}b(N + 1, n - 1, s) - (n + 1)sb(N, n - 1, s)$. From this and the definition of b , we obtain

(A8) $\qquad \phi_N(N, n - 1, s) \gtrless 0$ if and only if $\alpha \gtrless \beta(N)$,

where $\alpha \equiv \{(n + 1)s - 1\}/(n + 1)s$, and $\beta(N) \equiv (N + 1)(1 - s)/(n - 1 - N)s$. Since $\beta(N + 1) > \beta(N)$, using (A8), we can show that: $\phi_N(N - 1, n - 1, s) > 0$ if $\phi_N(N, n - 1, s) \geq 0$; and $\phi_N(N + 1, n - 1, s) < 0$ if $\phi_N(N, n - 1, s) \leq 0$. Thus, ϕ is single-peaked in N .

This single-peakedness implies that $\phi(N, n - 1, s)$ achieves a minimum at $N = 0$ or $N = n - 1$. Thus, if $\phi(0, n - 1, s)$ and $\phi(n - 1, n - 1, s)$ are positive, then

(A9) $\qquad \phi(N, n - 1, s) > 0$ for $N = 0$ to $n - 1$.

Our objective now is to find a set of sufficient conditions under which $\phi(0, n - 1, s)$ and $\phi(n - 1, n - 1, s)$ are positive. From (23), $\phi(0, n - 1, s) = \{(n + 1)s - 1\}(1 - s)^{n-1}$. Thus, since $n \geq 1$, it follows that $\phi(0, n - 1, s) > 0$ if $s > 0.5$. Next, from (23), $\phi(n - 1, n - 1, s) = (n + 1)s^n - 1$, which is positive if $s > \gamma(n)$, where $\gamma(n) \equiv \exp\{-\{\ell n(n + 1)\}/n\}$. Further $\gamma(n) > \gamma(n - 1)$. Hence, if $s > \gamma(n^*)$, then $s > \gamma(n)$ for $n \leq n^*$. Since $\gamma(12) = 0.81$, it follows that $\phi(n - 1, n - 1, s) > 0$ if $s > 0.81$ and $n \leq 12$.

Derivation of (33). The expression analogous to (A5) for the second-period choice is $U_n^2(N_1, n_2, s_2) = s_2\, \Sigma_{N_2=0}^{n_2} b(N_2, n_2, s_2)u_2(N_1, N_2)$. Hence, the definition of $u_{12}(N_1, N_2)$ yields

(A10) $$U_n^2(N_1 + 1, n_2, s_2) - U_n^2(N_1, n_2, s_2) = s_2 \Sigma_{N_2=0}^{n_2} b(N_2, n_2, s_2)u_{12}(N_1, N_2) .$$

In turn, (A10) yields (33), since $u_{12} < 0$.

Proof of Proposition 7. The expression analogous to (A5) for the first-period choice is

(A11) $$U_n^1(n_1, s_1, s_2) = s_1 \Sigma_{N_1=0}^{n_1} b(N_1, n_1, s_1)V_N^2(N_1, s_2) ,$$

where $V_N^2(N_1, s_2)$ is defined in (37) and, in turn, V^2 is defined in (28). The former definition yields

(A12) $$\frac{\partial}{\partial s_2} V_N^{2\pm}(N_1, s_2) = \frac{\partial}{\partial s_2} V^{2\pm}(N_1 + 1, s_2) - \frac{\partial}{\partial s_2} V^{2\pm}(N_1, s_2) .$$

The two one-sided derivatives, $\frac{\partial}{\partial s_2} V_N^{2\pm}$, are obviously identical if the derivative, $\frac{\partial}{\partial s_2} V_N^2$, exists. Now, in Step 1 below, we show that (A12) is non-positive if (44) and (45) hold. Step 2 completes the proof.

Step 1. Recall that n_2 has either one or two optimal values at N_1 . The same is true for the optimal value of n_2 at $N_1 + 1$. Thus there are four possible combinations for which we need to show that (A12) is non-positive. Since the proof is similar in these four cases, we analyze here only the case in which the optimal n_2 is unique at N_1 and at $N_1 + 1$. In this case, using (31) and (41), (A12) becomes

(A13) $$\frac{\partial}{\partial s_2} V_N^2(N_1, s_2) = \frac{1}{s_2} \{mU_n^2(N_1 + 1, m - 1) - m'U_n^2(N_1, m' - 1)\} ,$$

where, for brevity, we have used the notations $m \equiv n_2(N_1 + 1, s_2)$ and $m' \equiv n_2(N_1, s_2)$, and have suppressed s_2 . Note that (44) implies $m' = m$ or $m + 1$. Also, by definition, $m \geq 0$.

Now, if $m' = m$, then, using (33), we conclude that (A13) is non-positive. Next, consider the case $m' = m + 1$. Then the right-hand side of (A13) is

(A14) $$\frac{m}{s_2} [\{U_n^2(N_1 + 1, m - 1) - U_n^2(N_1, m - 1)\} + \{U_n^2(N_1, m - 1) - U_n^2(N_1, m)\}] - \frac{1}{s_2} U_n^2(N_1, m) .$$

The term inside the square bracket in the above expression can be rewritten, using (29) and (A10), as $s_2 \Sigma_{N_2=0}^{m-1} b(N_2, m - 1, s_2)\{u_{12}(N_1, N_2) - s_2 u_{22}(N_1, N_2)\}$. The preceding expression is non-positive if (45) holds. In turn, (A14) is non-positive because $m \geq 0$, and because, from (32), $U_n^2(N_1, m) \equiv U_n^2(N_1, n_2(N_1, s_2) - 1) \geq 0$. Thus, (A13) is non-positive if (44) and (45) hold.

Step 2. Since (A12) is non-positive, the corresponding derivatives of (A11) with respect to s_2 are also non-positive. Therefore, from arguments which are familiar by now, the optimal n_1 is non-increasing in s_2 , if (44) and (45) hold. This proves Proposition 7.

Derivation of (49). Define $Y(q) \equiv \Sigma_{N_1=0}^{q} b(N_1, q, s_1)n_2(N_1, s_2)$. Thus, using (47), $e(s_1', s_2) - e(s_1, s_2) = -1 + Y(q - 1) - Y(q)$. Next, using (A5), but substituting $Y(q)$ in the place of $U(n, s)$, we obtain $Y(q) - Y(q - 1) = s_1 \Sigma_{N_1=0}^{q-1} b(N_1, q - 1, s_1)\{n_2(N_1 + 1, s_1) - n_2(N_1, s_1)\}$. Expression (49) follows.

Evaluation of (54) and (55). Let $F(N, n, s) \equiv \int_0^n f(j, n, s)dj$ denote the cumulative density of N survivals. Define $G(N, n, s) \equiv \int_0^N F(K, n, s)dK$. Thus, $G_N(n, n, s) = 1$, and $G_{Nn}(n, n, s) = G_{NN}(n, n, s) = 0$. Using these expressions, and integration by parts, we obtain

(A15) $$U_{nn}(n, s) = \int_0^n G_{nn}(N, n, s)u_{NN}(N)dN - G_{nn}(n, n, s)u_N(n) .$$

Now, consider (54). To establish (54), we need to show that (A15) is negative. Note that though the $u_{NN}(N)$ are negative, they can have any magnitude for different values of N. Also, depending on the value of n, $u_N(n)$ can be a positive or negative number of arbitrary magnitude. Thus, (A15) will be negative, in general, only if

(A16) $$G_{nn}(n, n, s) = 0 \text{ ; and } G_{nn}(N, n, s) \geq 0 \text{ for } n > N \geq 0 .$$

Clearly, (A16) does not represent any intuitive property of the survival probabilities. It does not describe a property of stochastic dominance, of any order, for the survival probabilities. Next, an assumption concerning $f(N, n, s)$ that is comparable to the binomial density, (1), is the one in which the binomial density is approximated by a normal density. For example, $f(N, n, s) = z((N - sn)/(ns(1 - s))^{1/2})$, where z is the unit normal density. It can be easily ascertained that this or other similar approximations do not yield (A16). Similar difficulties arise in establishing the expressions in (55).

REFERENCES

Barro, Robert J. and Gary S. Becker, 1989, "Fertility Choice in a Model of Economic Growth," *Econometrica* 57, 481-501.

Becker, Gary S., 1960, "An Economic Analysis of Fertility," in National Bureau of Economic Research, ed., *Demographic and Economic Change*, Princeton University Press, Princeton, NJ, 209-240.

Becker, Gary S. and Robert J. Barro, 1988, "A Reformulation of the Economic Theory of Fertility," *Quarterly Journal of Economics* 103, 1-25.

Ben-Porath, Yoram, 1976, "Fertility Response to Child Mortality: Micro Data from Israel," *Journal of Political Economy* 84, S163-78.

Ben-Porath, Yoram and Finis Welch, 1972, "Chance, Child Traits, and Choice of Family Size," Report No. R-1117-NIH/RF, Rand Corporation, Santa Monica, CA.

Dyson, Tim and Mike Murphy, 1985, "The Onset of Fertility Transition," *Population and Development Review* 11, 399-440.

Freedman, Ronald, 1975, *The Sociology of Human Fertility*, Irvington Publishers, New York, NY.

Heckman, James J. and Robert J. Willis, 1975, "Estimation of a Stochastic Model of Reproduction: An Econometric Approach," in Terlecky, Nestor E., ed., *Household Production and Consumption*, Columbia University Press, New York, NY, 99-138.

Ingersoll, John E., 1987, *Theory of Financial Decision Making*, Rowman and Littlefield, Totowa, NJ.

Newman, John L., 1988, "A Stochastic Dynamic Model of Fertility," in Schultz, T. Paul, ed., *Research in Population Economics*, Vol. 6, Jai Press, Greenwich, CT, 41-68.

O'Hara, Donald J., 1975, "Microeconomic Aspects of the Demographic Transition," *Journal of Political Economy* 83, 1203-16.

Rudin, Walter, 1976, *Principles of Mathematical Analysis*, McGraw-Hill, New York, NY.

Sah, Raaj K., 1989, "Comparative Properties of Sums of Independent Binomials with Independent Parameters," forthcoming, *Economic Letters*.

Sah, Raaj K. and Jingang Zhao, 1989, "An Envelope Theorem for Integer and Discrete Choice Variables," working paper, Yale University, New Haven, CT.

Schultz, T. Paul, 1988, "Economic Demography and Development," in Ranis, Gustav and T. Paul Schultz, eds., *The State of Development Economics*, Basil Blackwell, Oxford, UK, 416-451.

Schultz, T. Paul, 1981, *Economics of Population*, Addison-Wesley, Reading, MA.

Schultz, T. Paul, 1976, "Determinants of Fertility: A Micro-economic Model of Choice," in Coale, Ansley J., ed., *Economic Factors in Population Growth*, John Wiley, New York, NY, 89-124.

Wolpin, Kenneth I., 1984, "An Estimable Dynamic Stochastic Model of Fertility and Child Mortality," *Journal of Political Economy* 92, 852-874.

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