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RESULTS FOR ECONOMIC COMPARATIVE STATICS OF STEADY-STATES OF HIGHER-ORDER DISCRETE DYNAMIC SYSTEMS

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ABSTRACT

Non-linear systems of difference equations of various orders arise naturally in many economic models in which the dynamics is explicit. In such contexts, economists often have potential interest in the comparative statics of locally stable steady-states, with respect to the system's parameters.

This paper presents some intuitive and directly usable results for such comparative statics. That is, they establish some usable consequences of qualitative assumptions or information concerning the features of the original dynamic system on the signs and magnitudes of the resulting expressions for comparative statics.

RESULTS FOR ECONOMIC COMPARATIVE STATICS OF STEADY-STATES OF HIGHER-ORDER DISCRETE DYNAMIC SYSTEMS

Consider the following system of K non-linear difference equations where the maximum order of an equation is L .

$$(1) \qquad y(T) = \begin{bmatrix} y^1(T) \\ y^j(T) \\ \vdots \\ y^K(T) \end{bmatrix} = \begin{bmatrix} f^1(y(T-1), \ldots, y(T-L), \theta^1) \\ f^j(y(T-1), \ldots, y(T-L), \theta^j) \\ \vdots \\ f^K(y(T-1), \ldots, y(T-L), \theta^K) \end{bmatrix}$$

In (1), $K \geq 1$, $L \geq 1$, T is time, and θ^j is a parameter affecting the j-th equation.[1] Let the vector $Y = [Y^1 \ldots Y^j \ldots Y^K]$ denote a (locally) stable steady-state value of the set of y .[2] At this steady-state, system (1) can be written in reduced-form as

$$(2) \qquad Y^j = F^j(Y^1, \ldots, Y^k, \ldots, Y^K, \theta^j) , \quad \text{for } j = 1 \text{ to } K .$$

Non-linear systems of difference equations of various orders, such as (1), arise naturally in many economic models in which the dynamics is explicit.[3] In such contexts, economists often are potentially interested in the comparative statics of one or more stable steady-states. That is, let $\partial Y^j/\partial\theta^k$ denote the derivative of a steady-state value of variable y^j with respect to a sustained small change in parameter θ^k .[4] Then, economists often have potential interest in assessing the sign and magnitude of $\partial Y^j/\partial\theta^k$, based on some qualitative information or assumptions concerning the original dynamic system (1).

This paper presents some intuitive and directly usable results for such comparative statics. To my knowledge, these results have not been previously reported, at least not in the accessible and directly usable form in which this paper obtains them. In presenting the results below, I have kept mathematical details to the minimum level necessary, so as to keep the paper brief and to focus on the qualitative economic aspects of the results.

The paper is organized as follows. Section I presents some theorems and a corollary, which are later used to derive comparative statics results. (Brief proofs of these theorems are given in the Appendix.) The results for comparative statics are presented and interpreted in Section II. The paper concludes with brief explanatory remarks.

I. PRELIMINARY RESULTS

In the immediate vicinity of the steady-state under consideration, define the following derivatives, using (1) and (2) respectively.

(3) $$f^j_{k,\ell} = \frac{\partial y^j(T)}{\partial y^k(T-\ell)}, \quad \text{and} \quad F^j_k = \frac{\partial F^j}{\partial Y^k}.$$

Define f_ℓ as a $(K \times K)$ matrix whose $(j \times k)$ element is $f^j_{k,\ell}$. Define F as a $(K \times K)$ matrix whose $(j \times k)$ element is F^j_k. Then (1), (2) and (3) imply

(4) $$F = \sum_\ell f_\ell .$$

Define matrix M as

(5) $$M = I_K - F ,$$

where I_K is an identity matrix of order $(K \times K)$. Let $|M|$ denote the determinant of M . Let C_{kj} denote the co-factor corresponding to the $(k \times j)$ element of M .

Next, define the following two features of system (1), independently of one another.

(C1) $$f^j_{k,\ell} \geq 0 .$$

(C2) $$F^j_k = F_k .$$

I refer to a matrix as "stable" if all of its eigenvalues are smaller than unity in absolute value. The following three theorems are established in the Appendix.

THEOREM 1.

(6) $$|M| > 0 .$$

THEOREM 2. <u>If (C1) holds, then</u> F <u>is stable</u>.

THEOREM 3. <u>If (C2) holds, then</u>

(7) $$C_{kk} - C_{kj} = |M| , \quad \text{for } k \neq j .$$ [5]

Several corollaries of Theorem 2 can be obtained by combining it with properties of stable matrices. The following corollary, established in the Appendix, is the one which I use later.

COROLLARY 1. (a) If (C1) holds, then

(8) $C_{kj} \geq 0$, and

(9) $C_{kk} \geq |M|$.

Alternatively: The inequalities in (8) and (9) are strict if, in addition to (C1): (b) matrix F is indecomposable, or if (c)

(10) $f^j_{k,\ell} > 0$, for at least one ℓ .

II. RESULTS FOR COMPARATIVE STATICS OF STEADY-STATES

If $F^k_\theta = \partial F^k/\partial\theta^k$, then a perturbation of (2) yields

(11) $$\frac{dY^j}{d\theta^k} = \frac{C_{kj}F^k_\theta}{|M|} .$$

This section shows how the earlier theorems yield some directly usable assessments of (11).

For interpretations of the results to be derived, note that F^k_θ can be viewed as representing the "first-round impact" of a change in parameter θ^k ; that is, it is the derivative of F^k calculated at the pre-change values of variables. By contrast, $dY^j/d\theta^k$ can be viewed as representing the "final steady-state impact" on variable y^j . That is, $dY^j/d\theta^k$ is the derivative of the difference between the post- and pre-change steady-state values of variable y^j , with respect to a change in parameter θ^k . Also, recall that in formulation (1), the direct effect of a change in parameter θ^k is felt only on the k-th variable; all other variables are affected by indirect dynamic effects. Thus, $dY^k/d\theta^k$ can be viewed as the "direct steady-state effect" of a change in parameter θ^k. On the other hand, for $j \neq k$, $dY^j/d\theta^k$ can be viewed as "indirect steady-state effects" on different variables.

Now, Theorem 1, in combination with (11) yields

$$\text{(12)} \qquad \operatorname{sgn}\left(\frac{dY^j}{d\theta^k}\right) = \operatorname{sgn}(C_{kj}F_\theta^k) .$$

Further, if (C1) and (10) hold, then (12) and Corollary 1(c) show

$$\text{(13)} \qquad \operatorname{sgn}\left(\frac{dY^j}{d\theta^k}\right) = \operatorname{sgn}(F_\theta^k) , \quad \text{and}$$

$$\text{(14)} \qquad \left|\frac{dY^k}{d\theta^k}\right| > |F_\theta^k| .$$

That is: (i) the sign of the final impact on any variable is the same as the sign of the first-round impact of a parameter change, and (ii) the direct steady-state effect on a variable has a magnitude larger than that of the first-round impact of a parameter change.

It is apparent from Corollary 1 that results (13) and (14), or result (17) to be derived below, can be restated in different ways. For instance, if (10) does not hold, then Corollary 1(a) yields weaker results: (i) $\operatorname{sgn}(dY^j/d\theta^k)$ = zero or $\operatorname{sgn}(F_\theta^k)$, and (ii) $|dY^k/d\theta^k| \geq |F_\theta^k|$. On the other hand, Corollary 1(b) yields (13) and (14) even if (10) does not hold, provided F is indecomposable.

Next, consider the evaluation of $dY^j/d\theta^k$ in the vicinity of the case where (C2) holds. To see a situation in which such a condition arises in an economic model, suppose one is interested in studying a dynamic system in which the function f^j is the same for all equations in (1). An example is a collection of many interacting sub-economies, in which each sub-economy has the same response function but faces a different set of parameters. Further, suppose that one is interested in evaluating the impact of a change in a parameter facing one of the sub-economies, in the vicinity of the case where all sub-economies face the same set of parameters. That is, one is interested in assessing the case in which one of a set of similar sub-economies is slightly perturbed. For such cases, (11) is evaluated using (C2).

Let D_{jk} denote the corresponding expression for comparative statics. That is

(15) $$D_{jk} = \left.\frac{\partial Y^j}{\partial \theta^k}\right|_{F^j_k = F_k} .$$

Then (7) and (11) yield

(16) $$D_{kk} - D_{jk} = F^k_\theta , \quad \text{for } j \neq k .$$

That is: The difference between the change in the steady-state value of a directly affected variable and that of <u>any</u> indirectly affected variable equals the first-round impact of a parameter change.

If conditions (C1) and (10) hold in addition, then (7), (11) and Corollary 1(c) yield

(17) $$|D_{kk}| > |D_{jk}| , \quad \text{for } j \neq k .$$

That is: The change in the steady-state value of a directly affected variable has a larger magnitude than that of <u>any</u> indirectly affected variable.

Finally, note that in formulation (1) of the dynamic system, a parameter affects only one equation. The results, however, can be used to some extent for the comparative statics of other formulations. For instance, suppose θ is a parameter which affects all equations in (1). Then, by defining $\theta^k = \theta$, and using (11), one obtains $dY^j/d\theta = \sum_k dY^j/d\theta^k = [\sum_k C_{kj}F^k_\theta]/|M|$. Evaluation of such expressions can, in turn, be helped by earlier results.

III. CONCLUDING REMARKS

The objective of this paper has been to trace the implications of some possible qualitative features of the original dynamic system on the comparative statics expressions; that is, on the derivatives of steady-state values of variables with respect to the parameters.[6] This requires establishing relationships between the features of the original dynamic system and the properties of the "relevant" Jacobian matrix associated with the reduced-form of the original dynamic system, when the system is evaluated at the steady-state under consideration. In this paper, this Jacobian was denoted as M , and was defined by (2), (5), and the second half of (3).

The theorems presented in this paper are useful illustrations of such relationships, even though they obviously do not exhaust the set of potentially useful relationships. Theorem 1 shows that the relevant Jacobian always has a positive determinant, given that the steady-state under consideration is stable. Theorem 2 shows that the relevant Jacobian is a stable matrix if, in addition, the original system has the feature that the current values of variables are affected non-negatively by the past values. By combining these two theorems with some properties of stable matrices, then, it becomes possible (as shown in Section II) to derive several comparative statics results. Theorem 3 states some additional properties of the relevant Jacobian, when it is evaluated in the vicnity of the special case in which all equations of the original dynamic system are identical. This result is useful, for example, when two or more identical mutually interacting sub-economies are under consideration, and one is interested in assessing the steady-state consequences of a small perturbation in one of these sub-economies.

APPENDIX

To prove Theorems 1 and 2, I first use a standard procedure [see Grandmont (1987b, p. 47), for example] to transform system (1) into a first-order system. Define a $(KL \times 1)$ vector $z(t) = [z_1(t) \ \dots \ z_{KL}(t)]$ such that $z_{(\ell-1)K+k}(t-1) = y^k(t-\ell)$. Thus, $z_{(\ell-1)K+k}(t) = z_{(\ell-2)K+k}(t-1)$, for $\ell = 2$ to L . The system (1) can then be rewritten as the first-order system

$$\text{(A1)} \qquad z(t) = g(z(t-1)) \ .$$

If A is the corresponding $(KL \times KL)$ matrix of derivatives $\partial z(t)/\partial z(t-1)$, evaluated in the vicinity of the steady-state under consideration, then

$$\text{(A2)} \qquad A = \begin{bmatrix} f_1 & f_2 & \cdots & f_{L-1} & f_L \\ I_K & 0_K & \cdots & 0_K & 0_K \\ \vdots & & & & \\ 0_K & 0_K & \cdots & 0_K & 0_K \\ 0_K & 0_K & \cdots & I_K & 0_K \end{bmatrix} ,$$

where 0_K is a null matrix of order $(K \times K)$.

PROOF OF THEOREM 1. Matrix A is stable because the steady-state under consideration is stable. A necessary Shur-Cohn condition for A to be stable [see LaSalle (1986, p. 27)] is that

$$\text{(A3)} \qquad |I_{KL} - A| > 0 \ .$$

Theorem 1 follows from (5) and (A3) if one established that

(A4) $$|I_{KL} - A| = |I_K - F| .$$

To prove (A4), construct $I_{KL} - A$ from (A2), and then consider the following steps in that order: (i) for $k = 1$ to K, add to column k, columns $(\ell - 1)K + k$ where $\ell = 2$ to L, (ii) for $m = 2$ to L, and for $j = 1$ to K, add row $(m - 1)K + j$ to row $mK + j$. The resulting matrix is

$$\begin{bmatrix} I_K - F & -f_2 & \cdots & -f_L \\ 0_K & I_K & \cdots & 0 \\ \vdots & & & \\ 0_K & 0_K & \cdots & I_K \end{bmatrix} .$$

(A4) follows because the determinant of the above matrix is $|I_K - F|$.

PROOF OF THEOREM 2. From (C1) and (A2), A is a non-negative matrix (that is, it has non-negative elements). A necessary and sufficient condition for a non-negative matrix to be stable is that it exhibits row dominance [see Gandolfo (1980, pp. 138-39) for this result]. The row dominance of A implies that there exist positive numbers $(S_1, \ldots, S_{KL})$ such that

(A5) $$S_{(m-1)K+j} > \sum_\ell \sum_k S_{(\ell-1)K+k} A_{(m-1)K+j,(\ell-1)K+k} ,$$

where $A_{b,c}$ denotes the $(b \times c)$ element of A. For $m = 1$, (A2) and (A5) yield

(A6) $$S_j > \sum_\ell \sum_k S_{(\ell-1)K+k} f^j_{k,\ell} .$$

For $m = 2$ to L, (A2) and (A5) yield

(A7) $$S_{(m-1)K+j} > S_{(m-2)K+j} .$$

(A7) implies, in turn, that $S_{(\ell-1)K+k} > S_k$, for $\ell = 2$ to L. The last observation, along with (4), (C1) and (A6) yields

(A8) $$S_j > \sum_k S_k F_k^j .$$

That is, matrix F exhibits row dominance. Theorem 2 follows.

PROOF OF THEOREM 3. I show below that when matrix M is simplified using (C2), then:

(A9) $$|M| = 1 - \sum_i F_i ,$$

(A10) $$C_{kk} = 1 - \sum_{i \neq k} F_i , \quad \text{and}$$

(A11) $$C_{kj} = F_k , \quad \text{for } k \neq j .$$

An immediate consequence of these identities is (7).

To prove (A9), define: (i) vector h as the $(K \times 1)$ vector with unity elements, (ii) vector e_k as the k-th column of identity matrix I_K, and (iii) vector M_k as the k-th column of matrix M. Now, in matrix M, add to column M_1, each of columns M_2 to M_K. The resulting matrix is $[\sum_k M_k \;\; M_2 \ldots M_K]$. Noting that each element of vector $\sum_k M_k$ is $1 - \sum_i F_i$, therefore

(A12) $$|M| = (1 - \sum_i F_i)|T^1|$$

where matrix $T^1 = [h \;\; M_2 \ldots M_K]$. Next, in matrix T^1, multiply first column by F_k and add it to k-th column. Repetition of this step for $k = 2$ to K yields the matrix $[h \quad e_2 \ldots e_K]$. The determinant of the last matrix is unity. Thus, (A9) follows from (A12). The proof of (A10) is identical.

To prove (A11), let B_{kj} denote the matrix obtained by deleting the k-th row and j-th column of M. That is

$$\text{(A13)} \qquad C_{kj} = (-1)^{k+j}|B_{kj}| .$$

Now, first consider the case where $k > j$. In matrix B_{kj}, subtract the j-th row from each of the other rows. Call the resulting matrix T^2. Expand the determinant of T^2 along its $(k - 1)$ row. This gives $|B_{kj}| = |T^2| = (-1)^{k+j}F_k$. Thus, (A11) follows from (A13). The proof of (A11) for the case where $k < j$ is analogous.

PROOF OF COROLLARY 1. Let the $(K \times K)$ matrix C denote the matrix whose $(k \times j)$ element is C_{kj}. Let C^t be the transpose of C. Since $|M| > 0$ from (6), a standard result of matrix algebra is: $C^t = |M|[I - F]^{-1}$. Also, since F is a stable matrix from Theorem 2, $I + F + F^2 + \ldots$ converges to $[I - F]^{-1}$. Therefore

$$\text{(A14)} \qquad C^t = |M|(1 + F + F^2 + \ldots) .$$

Now, $|M| > 0$ from (6), and F is a non-negative matrix from (C1). Therefore (A14) yields (8) and (9). Further, if (10) holds, then (4) implies that F is a positive matrix. From (A14), therefore: $C_{kj} > 0$, and $C_{kk} > |M|$. The preceding strict inequalities hold even if F is non-negative, provided F is indecomposable, because in this case, some of the powers of F in (A14) are positive matrices.

FOOTNOTES

1. As noted later, the results obtained below are also useful for formulations in which a parameter affects more than one dynamic equation.
2. See Hirsch and Smale (1974, pp. 278-81) or LaSalle (1986, Ch. 1) for the standard definitions of (local) stability. I assume that system (1) has at least one stable steady-state.
3. See, for example, Gandolfo (1980), Grandmont (1987a), Samuelson (1947) and Sargent (1987).
4. It is assumed throughout that derivatives $dY^j/d\theta^k$, as well as other derivatives to be used later, are well-defined in the immediate vicinity of the steady-state under consideration. Also, for brevity, I use the following convention concerning the indices. Unless stated otherwise, $i = 1$ to K, $j = 1$ to K, $k = 1$ to K, $\ell = 1$ to L, and $m = 1$ to L.
5. This theorem is meaningful only if $K \geq 2$. Also, it can be seen from the Appendix that this theorem (and, therefore, the corresponding comparative statics result, (16), to be derived later) does not require the steady-state under consideration to be stable.
6. It might be noted that this objective is different from the one pursued in those previous economically-motivated studies of stable steady-states of difference equation systems, which have attempted to devise statements of stability conditions (that is, a set of necessary and, or, sufficient conditions for the stability of the original dynamic system) which can be interpreted as economically

meaningful restrictions on the original system. For illustrations and an articulation of difficulties inherent in operationalizing the latter objective, see the compendium of stability conditions (for first-order multiple equation systems, and for higher-order single equation systems) in Gandolfo (1985, pp. 108-15, 136-39).

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