3-1-1974

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ABSTRACT

The investigation is based on a dynamic, development-planning model which assumes open unemployment of unskilled labor. The standard deviation of capital holdings across households enters as a state variable and feeds back on the system through its effect on demand composition and the labor-intensity of the output mix. The system is controlled by varying the average level and progressivity of taxes and subsidies which depend on capital income. From this model, the behavior of summary measures of capital and income inequality along a consumption-optimal trajectory is determined. For example, conclusions are drawn concerning possibly divergent changes in capital and income inequality and the relationship between these dispersion measures and output growth. Also investigated are changes in the marginal tax rate on income from human and non-human capital.

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*The authors are grateful to Professor Kenneth Arrow for helpful comments and criticisms.

INTRODUCTION

Of late economists have focussed considerable attention on the relationship of income dispersion[1] to questions relating to the general efficiency of resource allocation. A major area of concern has been the factor opportunity cost effect [5]. It is contended that a reduction in income dispersion will increase the output share of commodities with low factor opportunity costs. So, it is argued, demand for goods which are intense in the relatively abundant factors will rise, while the demand for goods intense in the relatively scarce factors will fall.

There is also the more traditional argument that a more even distribution of income causes a decline in the aggregate savings rate, thereby retarding growth. (See [9], [14], [16].) This is open to question. There is little doubt that the marginal propensity to save out of corporate profits is greater than that out of wage income. But, to some extent, the association of income dispersion with the rate of capital formation is based on the assumption that the household marginal propensity to save varies either with the kind or the level of income earned. These are points of considerable empirical and theoretical disagreement, as indicated in [6], [7] and [8]. Moreover, even if the marginal propensity to save did change as hypothesized, the reduction in private savings could be offset by increases in tax revenues and public savings.

A more important theoretical question, not considered hitherto, is: To what extent can increases in the capital-labor ratio affect the size distribution of income in economies where a shortage of capital has led to considerable open unemployment?

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[1] In this paper we use the term 'dispersion' as a measure of inequities in the distribution of the variable under discussion.

This paper is confined to examining the factor opportunity cost effect and the interaction of capital accumulation with employment and income dispersion. The existing literature utilizes closed economy models which assume full employment; further, it considers income distribution by kind (e.g. relative factor shares) rather than size. In these respects, our paper represents a significant departure from existing work. We consider the case of labor surplus economies, where income redistribution appears to be a particularly crucial policy objective. Open unemployment is assured in our system through an institutionally-determined floor on the minimum real wage. Finally, the closed economy assumption is abandoned, and we utilize the static relationships[2] between income dispersion and other macroeconomic variables derived elsewhere [13].

On the basis of these assumptions, we are able to obtain a description of the changes in capital intensity and capital (and income) dispersion in the context of consumption-optimal growth theory. A noteworthy feature of our formulation is the absence of any assumptions regarding the form of the income and capital distribution functions, beyond the requirement of a finite second moment.

The steady state equilibrium solution of the problem exhibits various interesting characteristics. In particular, we are able to demonstrate the existence of situations wherein income equity and a more even distribution of assets are not consistent objectives. In these cases, for example, decreases in the level and distribution of capital and in the employment rate are accompanied by <u>increases</u> in income dispersion. Alternatively, should the employment rate fall as capital intensity increases, paradoxical situations

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[2] These relationships are obtained in [13] as simple extensions of the type of rigid-wage trade model described by Brecher [3].

may well arise in which increasing capital causes further inequities in income distribution despite a more even distribution of capital holdings. We also show that consumption-optimal program associated with a decrease in human and non-human capital dispersion (as measured by the coefficient of variation) may well imply a lower marginal tax rate.

Dynamic equations describing the change of capital intensity and capital dispersion are derived in Section I. Section II outlines the main arguments in the derivation of a production-demand relationship, which relates output to capital intensity and dispersion. This is an important prerequisite to any considerations of optimal growth, inasmuch as capital and income dispersion have a significant impact on output through demand composition. Equilibrium conditions are obtained in Sections III and IV under alternative assumptions regarding the form of the consumption function. The concluding section examines the policy and growth implications of our model.

I. THE DYNAMIC SYSTEM

It will be convenient to assume that the economy is divided into m groups of varying size but growing at the same rate. The capital (human and non-human) of a household is assumed to be spread evenly among its heirs; the number of heirs is assumed to be the same for each household, so that the capital holdings of each of the households within a group remain equalized.

Assuming a total population of m groups whose household are growing at rate n, let us define the following quantities for a representative household in the j^{th} group:

(1.1) $c_j \triangleq$ consumption

(1.2) $s_j \triangleq$ savings

(1.3) $k_j \triangleq$ capital stock (including human capital)

(1.4) $\hat{w}_j \triangleq$ wage income of unskilled labor assumed allocated completely to consumption

(1.5) $\Pi_j \triangleq$ capital income before taxes and subsidies

(1.6) z_j = disposable capital income.

Then we have savings, s_j equal to the rate of change of capital, i.e.

(1.7) s_j = gross changes in capital stock.

But

(1.8) s_j = total disposable income-consumption = $z_j + \hat{w}_j - c_j$.

We assume that the function determining the consumption per household of the j^{th} group is of the form

(1.9) $$c_j = \alpha k_j + \hat{w}_j$$

where

$$0 \leq \alpha \leq 1$$

i.e. all unskilled wage income--which is untaxed--and a constant fraction of capital income are allocated completely to consumption. The effects on optimal growth results of relaxing the assumption that the marginal propensity to consume out of unskilled wage income is unity and that out of disposable capital income zero are considered later in the paper.

The expression for savings s_j now becomes, from (1.7) and (1.8)

$$s_j = z_j - \alpha k_j \,. \tag{1.10}$$

Denote aggregate domestic savings per household by s, disposable capital income by z, and capital stock per household by k. Then, from (1.9) we obtain the following aggregate savings function

$$s = z - \alpha k \,. \tag{1.11}$$

Then from (1.8) assuming a constant (exponential) depreciation rate δ and a rate of household growth n, we have

$$\frac{dk_j}{dt} = \text{net change in capital stock per household} \tag{1.12}$$

$$= z_j - (\alpha + \delta + n)k_j \,.$$

Denote taxes (net of subsidies) levied on the j^{th} household by NT_j. Taxes are assessed only on capital income, which includes the income of skilled labor. We now have

$$z_j = \Pi_j - NT_j \tag{1.13}$$

$$= rk_j - NT_j$$

where r is the return to capital (assumed constant).

One possible tax function is

$$NT_j = a_0'' + a_1''(\Pi_j - \Pi) \tag{1.14}$$

where Π is the mean capital income per household for the entire society. This can be written as

$$(1.15) \qquad NT_j = a_0'' + a_1''r(k_j - k) \ .$$

The coefficient a_0'' determines the revenue impact of the tax, while a_1'' determines the redistributive effects.

From (1.13), (1.15), equation (1.12) can be written as

$$(1.16) \qquad \frac{dk_j}{dt} = r\left[k_j - \frac{a_0''}{r} - a_1''(k_j - k) \right] - (\alpha + \delta + n)k_j \ .$$

Define aggregate quantities as below

$$(1.17) \qquad k \triangleq \frac{1}{N} \sum_{j=1}^{m} k_j N_j \ ,$$

N = total number of households, N_j = the number of households in the j^{th} group, and m is the number of groups

$$(1.18) \qquad \sigma^2 = \frac{1}{N} \sum_{j=1}^{m} N_j (k_j - k)^2$$

where k is the weighted average of capital per household in each group and σ^2 is the weighted variance. We have, assuming an exponential growth rate n for the number of households

$$(1.19) \qquad \frac{dk}{dt} = r\left(k - \frac{a_0''}{r} \right) - (n + \delta + \alpha)k$$

and

$$(1.20) \qquad \frac{d\sigma}{dt} = \{r(1 - a_1'') - (n + \alpha + \delta)\}\sigma \ .$$

These are the basic dynamic equations describing the manner in which capital

intensity and capital dispersion vary over time. Note that, in the derivation of these equations, no assumptions have been made regarding the form of the distribution of k_j. It is only necessary that the quantities in (1.17) and (1.18) exist.

II. THE PRODUCTION-DEMAND RELATIONSHIP

The effects of income or asset distribution on the level of output are generally excluded from consideration in the literature on consumption-optimal growth. Such effects, it is felt, are negligible and of little consequence in the determination of the steady-state equilibrium solutions for the economic models usually considered.

Inasmuch as we are interested in examining the impact of changes in capital intensity on income dispersion through its effect upon the employment rate, we must therefore consider possibility that changes in dispersion feed back on employment, however small the magnitude of this effect. This closes the causal chain and permits us to formulate a well-posed problem.

We shall make the following assumptions; these can be eased considerably as shown in [13].

A1: There are two commodities in the system. Commodity 1 is used for both investment and consumption. Commodity (or good) 2 is used for consumption only.

A2: The production functions for the two commodities are differentiable and homogeneous of degree one, with unskilled labor and capital being the factors of production.

A3: Commodity 2 is more labor intensive compared to Commodity 1.

A4: Open unemployment exists in the system even in equilibrium; either the minimum real wage is (exogenously) specified or the rental rate on capital is determined by prices in the international market. Each household has one laborer (either skilled or unskilled).

A5: All government tax revenues are expended on Commodity 1.

and, provisionally[3]

A6: A closed economy.

From A5, we see that the total expenditure on Commodity 1, per household, other than private consumption, is determined by

$$G + i = (r_1 - \alpha)k \tag{2.1}$$

where $G \triangleq$ government expenditure per household

$i \triangleq$ gross investment per household

and $r_1 \triangleq$ rental rate on capital (expressed in terms of good 1).

This expression indicates that the sum of investment and government expenditure depends only on the capital intensity k, since r_1 and α are specified. Consequently, the only relevant changes in output mix are those effected by changes in the composition of private consumer demand. Since good 2 is labor intensive relative to good 1 (A3), these changes in the bill of goods demanded will influence the level of aggregate output and

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[3] The closed economy assumption (A6) has been introduced only to simplify the exposition of the general equilibrium analysis. It is abandoned later The detailed analysis, with trade included, is presented in [13].

employment. The importance of capital dispersion as a variable affecting growth therefore depends on its relationship to the composition of consumer demand.

We will therefore first derive expressions for the aggregate commodity demand functions. We will then solve a general equilibrium model to determine the employment rate and capital intensity, as well as the relative shares of the two goods, when demand equals supply. Solution of the general equilibrium system will yield expressions for the GDP per laborer (y) and the employment rate $(\tilde{e})$ in terms of the capital intensity (k), and the dispersion of capital (σ).

The Commodity Demand Functions

Aggregate private consumption per household for Commodity i may be written in the form

$$c_i = c_i(c, \sigma_c, P) \tag{2.2}$$

where σ_c is the standard deviation of the distribution and P is the commodity-price ratio (good 2 to good 1). Exact derivations of the functional form of (2.2) exist in the case of the quadratic and the exponential demand functions for the individual household. See Klein [3]. Substituting the aggregate version of (1.9) in (2.2) we have

$$c_i = c_i'(\hat{w}, k, \sigma_c, P) . \tag{2.3}$$

Well-behaved household demand functions relating consumption of good i to expenditure are monotone over the range being considered, and pass

through the origin. It can be shown that the expenditure elasticities are greater than (less than) unity according as the functions are convex (concave) to the origin. Denoting the expenditure elasticity of demand of the ℓ^{th} household for the i^{th} commodity by η_i^ℓ, we have

$$\frac{\partial c_i}{\partial \sigma_c} \gtreqless 0 \quad \text{according as} \quad \eta_i^\ell \gtreqless 1 \tag{2.4}$$

i.e., according as the household's commodity demand functions are convex or concave.

The standard deviation of consumption σ_c is related to measures of income dispersion through the variance-covariance matrix of the joint distribution of household capital holdings and employment. (Every household is assumed to face the same wage rate for unskilled labor.) The unskilled labor employment rate of a particular household may well be an increasing function of its capital stock, particularly if family enterprises exist. However, for simplicity, we make the assumption

A7: Employment of unskilled labor is distributed independently of capital holdings and the employment rate of skilled labor remains fixed at unity.

This implies that the household unskilled employment rate is independent of capital holdings and is binomially distributed. (From A4, workers are either employed or unemployed.) Finally, we make the crucial assumption

A8: The ratio of skilled to unskilled laborers (and thus the distribution of human capital) is fixed.

This assumption is similar to one made by Sheshinsky [16] except that the fixed skill distribution implies a fixed human capital distribution in our

model, but not in his. Under these conditions, σ_w^2, the variance of unskilled wage income per household is seen to be

$$(2.5)\quad \sigma_w^2 = w_1^2(\tilde{e} - \tilde{e}^2) = \underbrace{\lambda w_1^2 (e_u)(1 - e_u)}_{\substack{\text{(sum of the weighted variances of unskilled income within groups)}}} + \underbrace{\left[\frac{(w_1\tilde{e})^2}{\lambda} - (w_1\tilde{e})^2\right]}_{\substack{\text{(weighted variance of unskilled wage income between groups)}}},$$

where w_1 is the wage rate in terms of good 1, $\tilde{e}$ is the ratio of unskilled laborers employed to total households, $\lambda \in [0,1]$ is the fixed ratio of unskilled to total laborers, and e_u $(= \tilde{e}/\lambda)$ is the unskilled employment rate. This expression is derived from the total variance formula for two mutually-exclusive groups--households with skilled and households with unskilled labor. Denote average capital per household for the unskilled-labor group by k_u and the variance of household consumption by σ_c^2. Then, by means of the total-variance formula for grouped data, we obtain, from the household consumption function and (2.5),

$$(2.6)\quad \sigma_c^2 = \sigma_w^2 + \alpha^2\sigma^2 + 2w_1\tilde{e}\alpha(k_u - k) .$$

Given the function for capital accumulation of a representative household the percentage change in $(k_u - k)^2$ can be shown to be the same as the percentage change in σ^2. This implies that $(k_u - k)$ is a fixed proportion of σ. Thus (2.6) may be rewritten as

$$(2.7)\quad \sigma_c^2 = \sigma_w^2 + \alpha^2\sigma^2 + 2w_1\tilde{e}\alpha\beta\sigma$$

where $-1 < \beta < 0$. We assume

(2.8a) $$\sigma \geq -\frac{w_1 \tilde{e} \beta}{\alpha}, \quad \text{implying} \quad \frac{\partial \sigma^2 c}{\partial \tilde{e}} \geq 0$$

and

(2.8b) $$\frac{.5}{(1-\beta^2)\lambda} < e_u = \tilde{e}/\lambda, \quad \text{implying} \quad \frac{\partial \sigma^2 c}{\partial \tilde{e}} < 0 .$$

We can therefore write (2.3) in the form

(2.9) $$c_i = c_i''(k, P, \sigma, \tilde{e})$$

with

(2.10) $$\frac{\partial c_i}{\partial \sigma} \geq 0 \quad \text{if} \quad \eta_i^{\ell} \geq 1$$

(2.11) $$\frac{\partial c_i}{\partial \tilde{e}} > 0 \quad \text{if} \quad c_i > 0$$

(in the case of a quadratic demand function).

Demand and Supply Equilibrium

How, then, do changes in the standard deviation of the distribution of capital, σ, affect the demand curve and the equilibrium point? Assume that the expenditure elasticity of good 1, η_1^{ℓ} is greater than unity and that $\eta_2^{\ell} < 1$ and that $\partial \sigma c^2 / \partial \sigma > 0$. Then a reduction in the household asset dispersion from σ' to σ'' $(\sigma'' < \sigma')$ will cause the demand curve D_2D_1 to shift towards good 2 as shown in Figure 2.[4] This is seen from

[4] Note that each point on D_2D_1 represents a fixed asset dispersion σ, but a different wage income distribution due to changes in unskilled employment per household $\tilde{e}$.

(2.10). The equilibrium point will shift upwards along the transformation surface (from 1 to 2); this transformation surface is linear because of the specified factor price. The new equilibrium point, 2, will represent a higher level of GNP per household and a higher unskilled employment per household, $\tilde{e}$, than those attained at point 1.[5]

By solving such a system we may derive expressions for the GNP per household (y) and average unskilled employment per household

(2.12) $$y = f(k, \sigma; \theta)$$

(2.13) $$\tilde{e} = g(k, \sigma; \theta)$$

where θ is the exogenously specified factor price expressed in terms of one of the goods. Noting that $y = r_1 k + w_1 \tilde{e}$, we have

(2.14) $$\frac{\partial f}{\partial \sigma} \gtreqless 0 \quad \text{as} \quad \frac{\partial g}{\partial \sigma} \gtreqless 0$$

and

(2.15) $$\frac{\partial f}{\partial k} \gtreqless r_1 \quad \text{as} \quad \frac{\partial g}{\partial k} \gtreqless 0$$

where r_1 is the rental rate on capital and w_1 the wage rate in terms of good 1.

From the equilibrium analysis carried out above, we see that, as long as σ lies above the point where $\partial\sigma_c/\partial\sigma = 0$.

$$\frac{\partial \tilde{e}}{\partial \sigma} \equiv \frac{\partial g}{\partial \sigma} \gtreqless 0 \quad \text{according as} \quad \frac{\partial c_1}{\partial \sigma_c} \lesseqgtr 0$$

i.e.,

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[5] Introducing trade into model does not affect the qualitative results, provided there are no intermediate good imports. See [13].

(2.16) $$\frac{\partial \tilde{e}}{\partial \sigma} = \frac{\partial g}{\partial \sigma} \gtreqless 0 \quad \text{according as} \quad \eta_1^{\ell} \lesseqgtr 1 \ .$$

If $\partial \sigma_c / \partial \sigma = 0$, then $\partial \tilde{e} / \partial \sigma = 0$.

The compatibility of output and employment objectives implied by (2.14) and (2.15) may well break down when the good intensive in unskilled labor is inferior. In this situation, under certain conditions, $\partial g / \partial k$ is negative and increases in capital intensity may cause employment to decline even as GDP per household increases. The details are presented in McCabe [13].

III. EQUILIBRIUM GROWTH PATHS

Denoting total consumption (both private and public) by $\hat{c}$, instantaneous welfare is given by

(3.1) $$U(\hat{c}) = Nu(\hat{c})$$

$$\hat{c} = f(k,\sigma) - r_1 \ k - \frac{a_0''}{r_1} + \alpha k$$

where $u(\hat{c})$ determines aggregate utility per household. The possible effects of changes in expenditure distribution on social welfare are not considered in this function.[6] We are concentrating on the analysis of possible growth-equity conflicts along a consumption-optimal growth path.

Then the optimal growth paths are those that maximize discounted future consumption per household, i.e., the quantity

(3.2) $$\int_0^T e^{-\rho t} N_0 e^{nt} u(\hat{c}) dt = \int_0^T e^{-\gamma t} U(\hat{c}) dt$$

[6] Since the commodity price ratio is fixed, private consumption (in terms of good 1) is treated as a single good; public and private consumption expenditures are perfectly substitutable in this function.

where ρ is the rate of social discount, n is the rate of household growth (size constant), γ $(= \rho - n)$ the net rate of social discount, N_0 is the initial number of households and T is the planning horizon--assumed finite for simplicity.

Thus the paths of optimal capital accumulation and dispersion are given by the solution of the following optimal control problem:

$$\underset{u_1,\, u_2}{\text{Maximize}} \int_0^T e^{-\gamma t} U[f(k,\sigma) - r_1(k - u_1) + \alpha k]dt \tag{3.3}$$

subject to

$$\frac{dk}{dt} = r_1(k - u_1) - (n + \alpha + \delta)k \tag{3.4}$$

$$\frac{d\sigma}{dt} = (r_1 u_2 - \alpha - \delta - n)\sigma \tag{3.5}$$

and the constraints

$$k \geq \underline{k} > 0 \,, \quad \sigma \geq \underline{\sigma} \geq 0 \tag{3.6}$$

$$0 \leq u_1 \leq (r_1 - \alpha/r_1)k \tag{3.7}$$

$$0 < A \leq u_2 \tag{3.8}$$

$$u_1 \leq (1 - u_2)k \,. \tag{3.9}$$

Here u_1 and u_2, the control variables have been defined as (equation (1.20)).

$$u_1 = \frac{a_0''}{r_1} \,; \quad u_2 = 1 - a_1'' \,. \tag{3.10}$$

The upper bound on u_1 given in (3.7) is obtained from (2.1), as gross investment per household has to exceed zero [$G = a_0''$] . The lower bound on u_1 given in (3.8) takes note of the fact that government expenditures are non-negative. The lower bound, A , ensures that the marginal net tax rate a_1'' will be less than unity, thereby retaining the incentive to save. Restrictions such as (3.9) must also be placed on the control variables to ensure that no household is taxed more than its gross capital income.[7] Although such control constraints may be binding durint the transition to an equilibrium point, they will not affect the values of k and σ once an optimal steady-state has been attained. As for the state constraints, the lower bound on k is determined by the product of a proportionality constant (the value of skilled laborer's human capital) and $(1-\lambda)$; that on σ by the product of the same constant and $\sqrt{\lambda(1-\lambda)}$.

The above optimal control problem is linear in the controls u_1 and u_2 ; thus the optimal policies will be of the "bang-singular-bang" type [4, pp. 261-265], i.e., the control variables will move between their boundary values and an interior value(s) corresponding to the singular arc(s).

We will only consider the case where, for all feasible values of k and σ

$$\frac{\partial f(k,\sigma)}{\partial \sigma} < 0 \quad . \tag{3.11}$$

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[7] From expressions (3.17) and (3.18) derived later in the text, it is clear that this condition will always hold as an equality in steady-state equilibrium. In situations where k is rising and σ is falling, this constraint will not be binding. However, if k is falling, the lower bound on disposable income may dominate the upper bound on u_1 , derived from the non-negativity constraint on gross investment. In any case, our main qualitative results will not be altered, provided, as is the case in the problem being considered, the transition to the optimal steady state always implies a decline in u_2 .

The function $f(k,\sigma)$ is only concave if the output share of the labor-intensive good declines as k increases [13]. Denote the undiscounted costate variables by λ_k and λ_σ respectively; then the Hamiltonian has the form

$$H \triangleq e^{-\gamma t}[U(\hat{c}) + \lambda_k \dot{k} + \lambda_\sigma \dot{\sigma}] \quad . \tag{3.12}$$

Then the optimality conditions yield ([4], pp. 261-265)

$$\frac{\partial H}{\partial u_1} = 0 = U_c \cdot r_1 - r_1 \cdot \lambda_k \tag{3.13}$$

$$\frac{\partial H}{\partial u_2} = 0 = r_1 \cdot \sigma \cdot \lambda_\sigma \tag{3.14}$$

and

$$-\dot{\lambda}_k = U_c(f_k - r_1 + \alpha) + \lambda_k(r_1 - \alpha - n - \delta - \gamma) \tag{3.15}$$

$$-\dot{\lambda}_\sigma = U_c(f_\sigma) - \lambda_\sigma(n + \delta + \gamma - r_1 u_2) \quad . \tag{3.16}$$

In addition, we have the dynamics which are given by (3.4) and (3.5). In equilibrium, $\dot{k} = \dot{\sigma} = \dot{\lambda}_k = \dot{\lambda}_\sigma = 0$. This gives

$$u_1 = \left\{1 - \frac{n + \alpha + \delta}{r_1}\right\} k \tag{3.17}$$

$$u_2 = \frac{n + \delta + \alpha}{r_1} \quad . \tag{3.18}$$

Note that $n + \delta + \alpha/r_1 > A$ for a steady-state to exist. Since $r_1 > 0$ and $\sigma > 0$ [from (3.5), (3.8) and $T < \infty$),

(3.19) $\lambda_\sigma = 0$

(3.20) $\lambda_k = U_c$.

The two remaining variables k and σ can be obtained from (3.15) and (3.16)

(3.21) $$\frac{\partial f(k,\sigma)}{\partial k} = n + \delta + \gamma = \rho + \delta$$

(3.22) $$\frac{\partial f(k,\sigma)}{\partial \sigma} = 0 \ .$$

Given that $g_k > 0$, condition (3.21) will hold only if the net social rate of discount (γ) exceeds the private rate of return (r_1) , from (2.17). Otherwise, g_k must be negative. This may well be the case, as indicated in Section II, if commodity 2 is inferior at the equilibrium point. Since $f_\sigma < 0$ by assumption, (3.22) cannot hold. The constrained optimum value of σ is given by the lower bound $\underline{\sigma}$. The optimum value of k is given by (3.21) at a point where $\sigma = \underline{\sigma}$.[8] By comparing these turnpike values of k and σ with the initial conditions, the direction of change of these variables along the optimal trajectory may be established.

As yet we have not determined the standard deviation of the distribution of income. Total income for the j^{th} household, $\hat{y}_j$, is given by

(3.23) $$\begin{aligned}\hat{y}_j &= \text{net capital income and wage income} \\ &= r_1 k_j - NT_j + \hat{w}_j \ .\end{aligned}$$

From the formula for the variance of grouped data, we have from (2.5) and (3.18)

(3.24) $$\sigma_{\hat{y}} = \sqrt{(n+\delta+\alpha)^2\sigma^2 + w_1^2\tilde{e}(1-\tilde{e}) + 2(n+\delta+\alpha)\beta\sigma w_1\tilde{e}} \ .$$

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[8] Writing the dynamic equation for σ as below

$$d\sigma/\sigma = [r_1 u_2 - \alpha - n - \delta]dt$$

we see that since u_2 will be at its lower bound, σ will decline to $\underline{\sigma}$ at a constant rate.

With σ greater than or equal to its optimum value, it can be shown that $\partial\sigma_{\hat{y}}/\partial\sigma > 0$; further, if $\partial\sigma_c/\partial\tilde{e} < 0$, then $\partial\sigma_{\hat{y}}/\partial\tilde{e} < 0$.

We can now plot the movement of key target variables along the optimal trajectory. For the case $g_k > 0$ (neither good inferior), the curves of constant GDP per household, employment rate and $\sigma_{\hat{y}}$, the standard deviation of the distribution of after-tax income are as shown[9] (for a specified equilibrium point (k^*,σ^*)) in Figure 3. The behavior of the various quantities k , σ , $\sigma_{\hat{y}}$, e and GDP per household along the optimal trajectory clearly depends upon the location of the initial values of k and σ (say) k_0 , σ_0 . This is summarized in Table 1 for the cases where $\sigma_0 > \sigma^* = \underline{\sigma}$.

From Table 1 it is clear that, in the case when $g_k > 0$, equity and growth are always consistent objectives, in the sense that increases in k do not bring about increases in the dispersion of after-tax income, as measured by $\sigma_{\hat{y}}$. However, it is worth noting that in Region III, where k and $\sigma_{\hat{y}}$ both decline, $\sigma_{\hat{y}}$ may well decline more slowly, especially so, since the employment rate $\tilde{e}$, is tending to reverse this decline. In this situation, the income dispersion as measured by the coefficient of variation $\sigma_{\hat{y}}/\hat{y}$ may increase.[10]

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[9]

$$\text{GDP} = \text{constant} = \left(\frac{d\sigma}{dk}\right)_{\text{GDP}} = \left|\frac{g_k}{g_\sigma}\right|\left(1 + g_k \frac{r_1}{w_1}\right) \quad ; \quad g_k > 0$$

$$\tilde{e} = \text{constant} = \left(\frac{d}{dk}\right)_{\tilde{e}} = |g_k/g_\sigma|$$

$$\sigma_y = \text{constant} = \left(\frac{d\sigma}{dk}\right)_y = \left|\frac{g_k}{g_\sigma}\right| \frac{1}{1+\theta} \quad ; \quad \theta > 0$$

TABLE 1

Region	k	σ	GDP	$\tilde{e}$	$\sigma_{\hat{y}}$
I	⊕	↑	⊕	⊕	⊖
II	⊖	⊖	⊖	⊕	⊖
III	⊖		⊖	⊖	⊖
IV	⊖	↓	⊖	⊖	⊕

⊕ increasing

⊖ decreasing

In the case when $g_k < 0$, owing (say) to the good intensive in unskilled labor being inferior, the location of the level curves is for the most part reversed, as shown in Figure 4. The behavior of the key variables is summarized in Table 2. Here, growth and equity are not consistent objectives over a significant portion of the (k,σ) plane. In Region IV, $\sigma_{\hat{y}}$ is unequivocally increasing as k increases. Thus, while capital intensity is increasing, GDP per laborer, the employment rate and income equity are decreasing. In Regions II and III, there may exist situations where $\hat{y}$ is declining faster than $\sigma_{\hat{y}}$; this dispersion, as measured by the coefficient of variation, $\sigma_{\hat{y}}/\hat{y}$, is increasing along with k and possibly GDP per household.

[10] The expression for aggregate disposable income per household in steady-state equilibrium, derived from (3.13), is

$$\hat{y} = y - r_1 u_1 = f(k,\sigma) - r_1 - (n + \alpha + \delta)k$$

$$\partial\hat{y}/\partial k = w_1 g_k + (n + \alpha + \delta) .$$

TABLE 2

$\varepsilon_k < 0$

Region	k	σ	GDP	$\tilde{e}$	$\sigma_{\hat{y}}$
I	⊖	↑	⊕	⊕	⊖
II	⊕	⊖	⊕	⊖	⊖
III	⊕	⊖	⊖	⊖	⊖
IV	⊕	↓	⊖	⊖	⊕

⊕ $\triangleq$ increasing

⊖ $\triangleq$ decreasing

IV. THE CASE OF A KEYNESIAN SAVINGS FUNCTION

The model in the form presented thus far is based on the assumption that the marginal propensity to save out of unskilled labor income is zero. This assumption implies that cet par changes in capital per household between steady states will affect neither σ nor the slope $r(1 - u_2)$ and intercept $[ru_1 - r(1 - u_2)k]$ of the tax function. On the other hand, if the marginal propensity to save out of unskilled wage income were positive, an increase in capital per household would cause the savings of poorer households to increase and the variance of capital per household to decline. To keep σ constant, a higher marginal tax rate would be required.

Let us analyze this effect formally and show the existence of a steady state optimal program in the case of a Keynesian savings function.

Suppose that the savings function for a representative household in the j^{th} group takes the form

$$s_j = -\alpha_0 + \alpha_1(y_j - NT_j) \tag{4.1}$$

where y_j is the total gross income of the j^{th} household and NT_j is the net tax. Then the change in the capital per household of the j^{th} group will be determined by the relationship

$$(4.2) \qquad \frac{dk_j}{dt} = -\alpha_0 + \alpha_1\{r_1[k_j - a_0''/r_1 - a_1''(k_j - k)] + w_1\tilde{e}_j\} - (\delta+n)k_j \ .$$

Let us assume that a_0'' is held constant and government savings per household, s_g, is controlled instead. Then the accumulation equation for aggregate capital per household, $\hat{k}$, may be written

$$(4.3) \qquad \frac{d\hat{k}}{dt} = -\alpha_0 + \alpha_1 r_1 k_p + \alpha_1 w_1\tilde{e} + s_g - (n+\delta)\hat{k}$$

$$= -\alpha_0 + \alpha_1 r_1\hat{k} + \alpha_1 w_1\tilde{e} + s_g - (n+\delta)\hat{k}$$

where k_p is private capital per household assumed equal to $\hat{k}$ [11] and $\alpha^* = \alpha_0 + \alpha_1\bar{a}_0''$. The percentage change in the variance of total capital is given by the following expression (provided each household receives the same amount of capital from the government):

$$(4.4) \qquad (d\sigma^2/dt)/\sigma^2 = 2\cdot\{[(1 - a_1'')\alpha_1 - (n+\delta)] + \alpha_1 w_1 (k_u - k_p)/\sigma^2\}$$

where k_u is average private capital per household for the unskilled-labor group. The expression for the percentage change in $(k_u - k_p)$ is given by

$$(4.5) \quad \frac{d(k_u - k_p)^2}{dt} \Big/ (k_u - k_p)^2 = 2\left\{[(1 - a_1'')\alpha_1 - (n+\delta)] + \frac{w_1(1/\lambda - 1)\tilde{e}\alpha_1}{(k_u - k_p)^2}\right\} \ .$$

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[11] The simplifying assumption that $\hat{k} = k_p$ makes sense if government savings finances public investments (e.g., commuter transport) yielding free services otherwise obtained through purchased durables such as automobiles.

It can be shown that a stable equilibrium, independent of a_1'', will be attained where the percentage change in σ^2 is equal to the percentage change in $(k_u - k_p)^2$. This will occur where

$$(4.6) \qquad (1/\lambda - 1) = \frac{(k_u - k_p)^2}{\sigma^2} .$$

Therefore, with $(k_u - k_p) = -(1/\lambda - 1)^{1/2} \cdot \sigma$, we may write

$$(4.7) \qquad \dot{\sigma} = [(1 - a_1'')\alpha_1 r_1 - (n+\delta)]\sigma - w_1\tilde{e}(1/\lambda - 1)^{1/2} .$$

The sum of government expenditure plus private savings is given by the relationship

$$(4.8) \qquad c_g + s_g + s_p = (1 - \bar{a}_0'')\alpha_0 + \alpha_0 + \alpha_1(r_1 k + w_1\tilde{e})$$

where $\bar{a}_0''$ is the fixed value of a_0'', where s_p is private savings per household and c_g is government consumption per household. Define the control variables as

$$u_1 = s_g$$

$$u_2 = 1 - a_1'' .$$

If all the expenditures on the left-hand side of (4.8) are included in the final demand for the same good, then a function determining average unskilled employment per household from k and $u_2\sigma$ may be derived in a manner similar to the one outlined in Section II. With

$$\tilde{e} = g(k, u_2\sigma)$$

and the dynamics given by (4.3) and (4.7), the steady-state optimum solution involves the modified golden-rule condition and an equation

$$g_2(k,\ u_2\sigma) = 0 \tag{4.9}$$

which only give the optimum value of the product $u_2 \cdot \sigma$.[12] From the steady state equilibrium expression for u_2 derived from (4.7),

$$u_2 = \frac{(n+\delta)}{\alpha_1 r_1} + \frac{w_1 g(k,\ u_2 \cdot \sigma)(1/\lambda - 1)^{1/2}}{\alpha_1 r_1 \sigma} \tag{4.10}$$

it can be shown that[13]

---

[12] The condition (4.9) is obtained from the optimality condition

$$\frac{\partial H}{\partial u_2} = U_c(1 - \alpha_1)g_2\sigma + \lambda_k \alpha_1 g_2 \sigma + \lambda_\sigma\{\alpha_1 r_1 \sigma - w_1 g_2 \sigma (1/\lambda - 1)^{1/2}\} = 0$$

by proving that $\lambda_k = U_c$ and $\lambda_\sigma = 0$. The equality between λ_k and U_c is obvious from the condition

$$\frac{\partial H}{\partial u_1} = -U_c + \lambda_k = 0 \ .$$

It follows from the optimality condition $\partial H/\partial u_2 = 0$ that, if $\lambda_k = U_c$,

$$g_2 U_c = \lambda_\sigma\{\alpha_1 r_1 - w_1 g_2 (1/\lambda - 1)^{1/2}\} \ .$$

The adjoint equation multiplied by $(1/u_2)$ may be written as

$$\partial H/\partial\sigma(1/u_2) = U_c g_2 + \lambda_\sigma\{\alpha_1 r_1 - w_1 g_2 (1/\lambda - 1)^{1/2}\} - \lambda_\sigma \frac{(n+\delta+\gamma)}{u_2} = 0 \ .$$

Substituting the expression for $g_2 U_c$ into this equation yields

$$\lambda_\sigma = 0 \quad \text{if} \quad \frac{(n+\delta+\gamma)}{u_2} > 0 \ .$$

[13] Multiplying both sides of (4.10) by σ yields

$$u_2\sigma = \frac{(n+\delta)}{\alpha_1 r_1}\sigma + \frac{w_1 g(k,\ u_2\sigma)(1/\lambda - 1)^{1/2}}{\alpha_1 r_1}$$

(4.11) $$u_2\sigma = \psi(\sigma,k) \quad , \quad \psi_\sigma > 0 \, , \quad \psi_k > 0 \, .$$

Thus, σ will tend to decline along the optimal trajectory until (4.11) is met. From (4.10) it is clear that the effect of a change in the steady-state value of $\tilde{e}$ is given by

(4.12) $$\frac{\partial \tilde{e}}{\partial k} = g_1 + g_2\sigma \frac{\partial u_2}{\partial k}$$

$$= g_1(1+Q)$$

where $$Q = \frac{g_2 w_1 (1/\lambda - 1)^{1/2}}{\alpha_1 r_1 - w_1 (1/\lambda - 1)^{1/2} g_2} \, .$$

If $g_2 < 0$, then $Q > -1$ and $\partial\tilde{e}/\partial k \gtreqless 0$ as $g_1 \gtreqless 0$. The change in the steady-state equilibrium value of $\tilde{e}$ with respect to σ is simply $u_2 g_2$, which is assumed to be negative. Since k and σ affect the steady-state value of $\tilde{e}$ in the same manner as they did in Section III when $g_k > 0$ or $g_k < 0$, the relative slopes of the level curves in the (k,σ) plane will be qualitatively similar to those shown in Figures 3 and 4, and the value of the absolute slopes will be positive if $g_k > 0$ and negative if $g_k < 0$.

By differentiating this equation totally with respect to σ and $u_2\sigma$ and rearranging, we obtain

$$\frac{\partial(u_2\sigma)}{\partial\sigma} = \frac{(n+\delta)}{\alpha_1 r_1 - w_1 g_2(k,\, u_2\sigma)(1/\lambda - 1)^{1/2}} > 0$$

if $g_2(k,\, u_2\sigma) < 0$. In a similar manner, we derive the expression

$$\frac{\partial(u_2\sigma)}{\partial k} = \frac{w_1 g_1 (1/\lambda - 1)^{1/2}}{\alpha_1 r_1 \sigma - w_1 g_2 (1/\lambda - 1)^{1/2}} > 0$$

if $g_2(k,\, u_2\sigma) < 0$.

It is possible in this variant of the model for the coefficients of variation of both capital and capital income to decline from one steady-state equilibrium to another, even though the marginal tax rate decreases. From (4.11), it is clear that, if $\sigma = \sigma^* = \sigma_0$, an increase in k from one steady-state equilibrium to another will cause u_2 to increase, implying a lower marginal tax rate. Given that $\psi(\sigma,k)$ is concave and $\psi(\sigma,0) = 0$, the elasticity of $u_2\sigma$ with respect to k will be less than unity. Therefore, coefficient of variation of disposable income will decrease as its mean level rises. The intercept of the tax function $\bar{a}_0'' - r_1 k(1 - u_2)$ will also decrease. In our system this implies both first-order and Lorenz domination shifts [1, 15] in the distribution of disposable capital income, as well as unskilled wage income, as k rises. In general, the closer the initial value of σ is to its lower bound and the more k_0 falls short of its optimal value, the more likely it is that the optimal steady-state will involve a lower marginal tax rate and a more even distribution disposable than the initial one.

V. SUMMARY AND CONCLUSIONS

In this paper we have developed a normative model for the size distribution of capital and income dispersion, essentially free of assumptions regarding the underlying distributions. We proceeded to examine the impact of taxes and subsidies upon capital intensity, capital and income dispersion, employment rate and GDP per household, along consumption-optimal growth paths. We found cases in which growth and income equity were not consistent objectives; in these cases, decreases in the level and distribution of capital and in the employment rate are accompanied by increases in income dispersion. Alternatively, in situations where increasing capital intensity lowers the employment rate, there were cases wherein increasing output causes further

inequities in income distribution despite a more even distribution of capital holdings. The inverse relationship between the employment rate and capital intensity can exist under the assumptions made only in the case where the relatively labor-intensive good is inferior. Nonetheless, the pattern of growth, in which increases in capital intensity are associated with a decline in the employment rate and a rise in income inequity has been observed in a number of countries during the early stages of development, as indicated in [2] and [12].

We demonstrate that, given a positive marginal propensity to save out of unskilled wage income an increase in the disposable income of the poorest household may well be associated with a lower marginal tax rate if the optimal steady-state k rises. This association arises because change in the marginal tax rate is necessary to satisfy a positive lower bound on capital inequality (due to assumed differences in human capital absorbtion), not because the marginal tax rate has a direct effect on aggregate capital accumulation.[14]

The major assumption, explicitly stated in Section III, is that $\partial g/\partial \sigma < 0$, implying that the income elasticity of good 1, the capital-intensive good, is greater than unity, with $\eta_1 > \eta_2$ and $\eta_2 < 1$ and that the Engel curve is non-linear. Only in these cases [14] is $f(k,\sigma)$ concave and the optimal control problem well-posed. We do not feel that these are overly restrictive conditions, inasmuch as all that is required is that g_σ be negative, however small it may be in magnitude. It is also consistent with the observed historical trend that, in most economies, output share of

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[14] Sheshinsky [16] presents a model in which a decrease in the marginal tax rate has a direct positive effect on aggregate capital accumulation and under certain conditions, tends to arise disposable income of the poorest persons in the population.

industrial (capital-intensive) sector goes up while that of agricultural (labor-intensive) sector goes down.

A similar optimal control problem can be derived from a neo-classical system like that of Stiglitz [17], in which real wage flexibility ensures full employment. In this case, national income will be independent of demand composition and hence of capital dispersion. But if σ is assumed to affect social welfare directly, the alternative model will produce qualitative results quite similar to ours.

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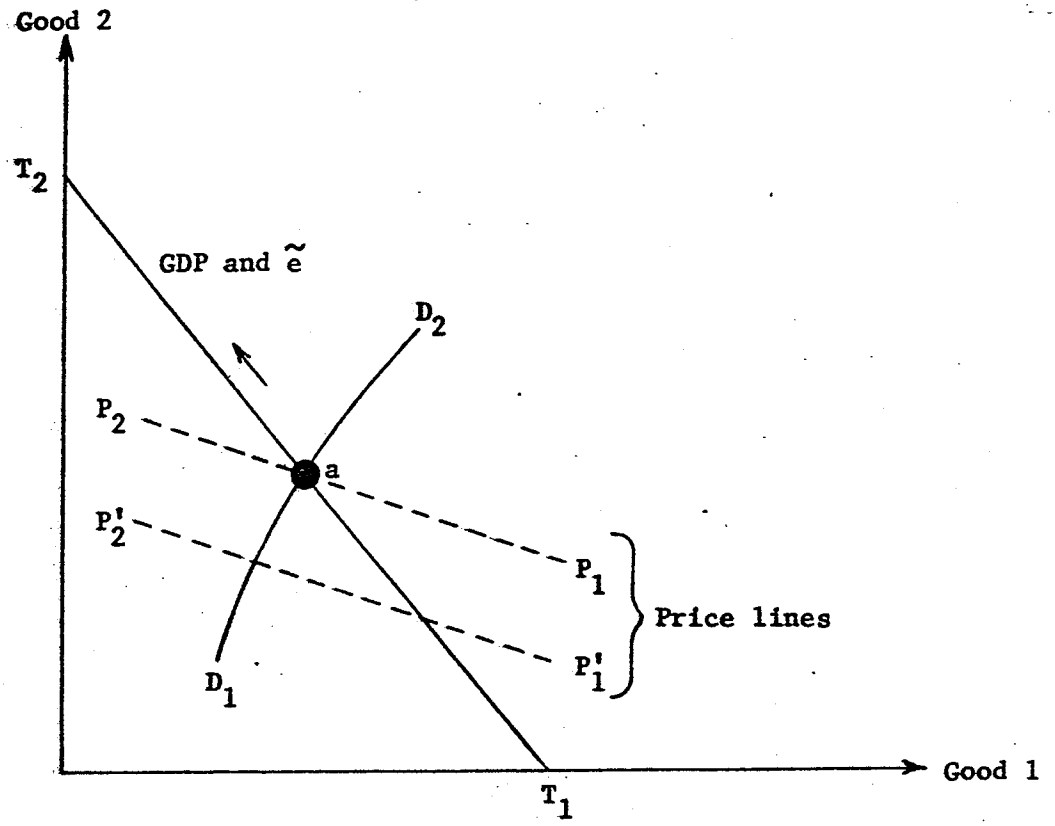
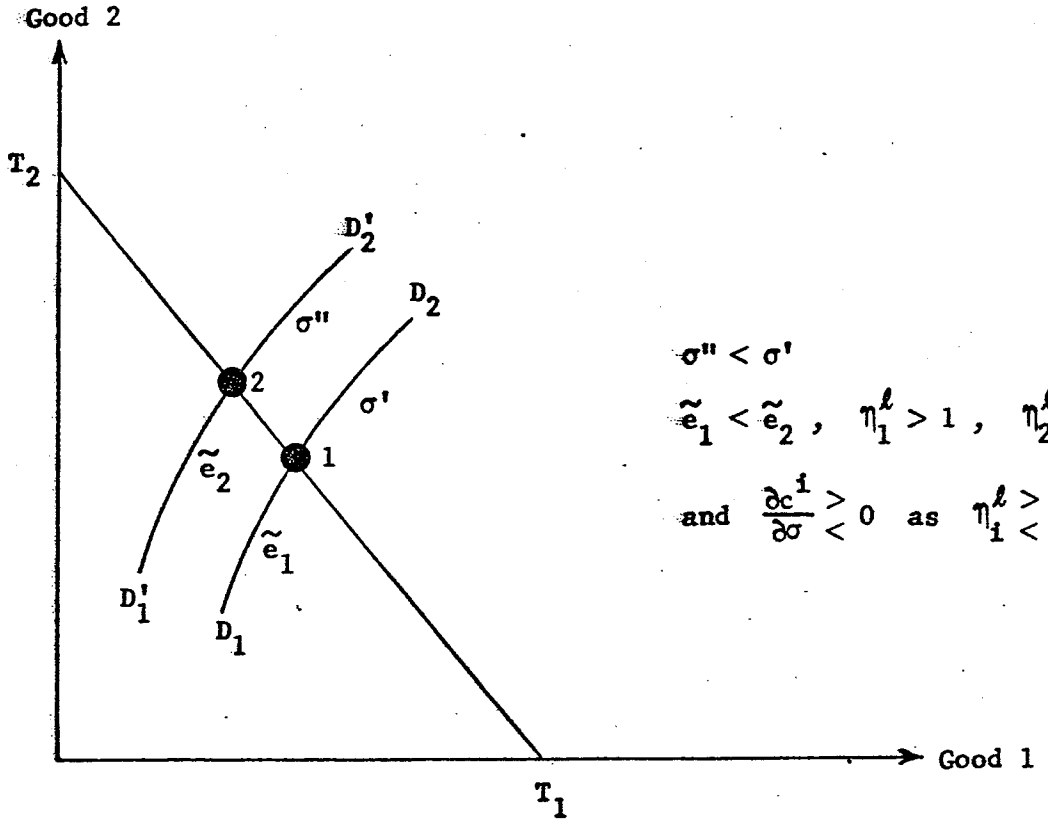


FIGURE 1



$\sigma'' < \sigma'$

$\tilde{e}_1 < \tilde{e}_2$, $\eta_1^{\ell} > 1$, $\eta_2^{\ell} < 1$

and $\frac{\partial c^i}{\partial \sigma} \gtrless 0$ as $\eta_i^{\ell} \gtrless 1$

FIGURE 2

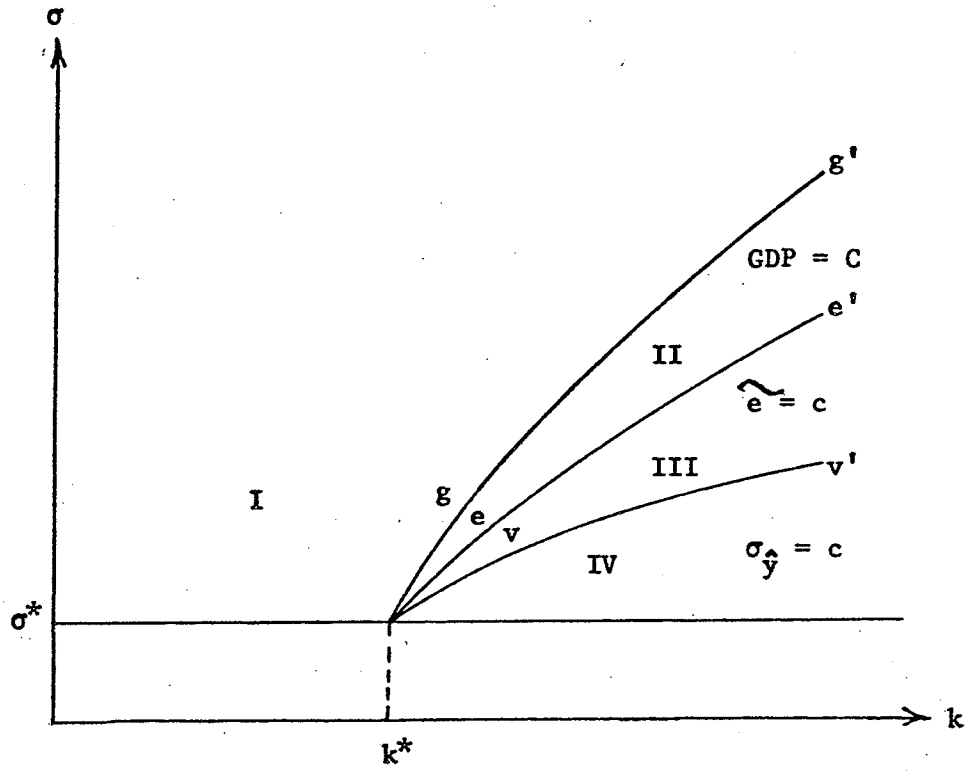


FIGURE 3. $g_k > 0$

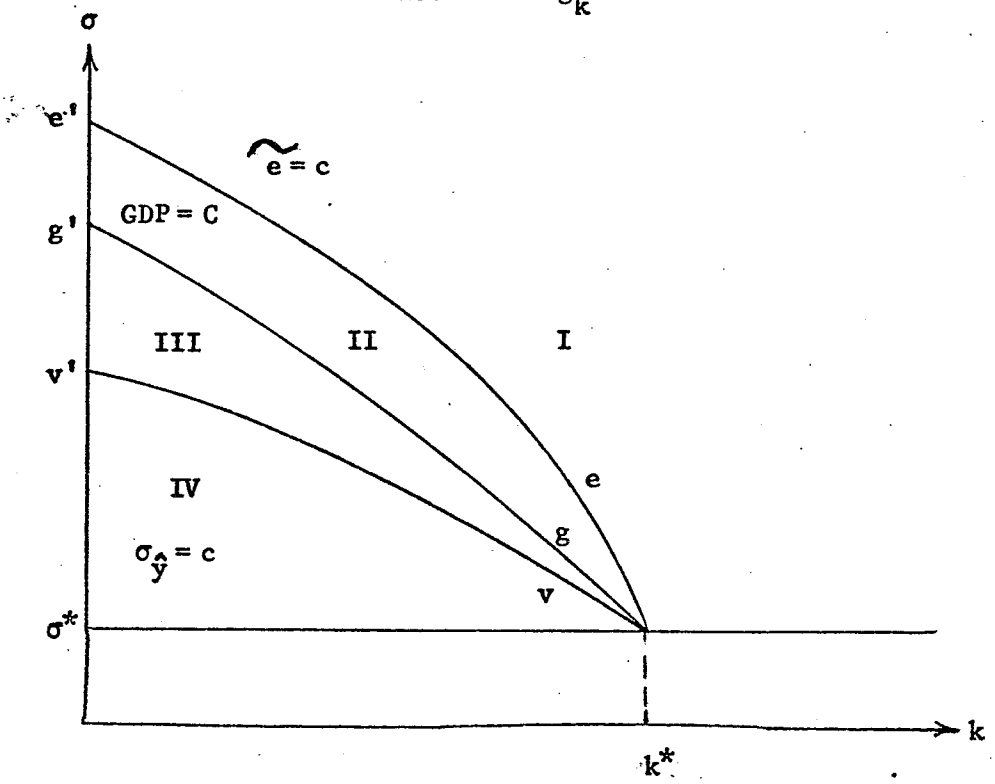


FIGURE 4. $g_k < 0$