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INCOME DISTRIBUTION AND OPTIMAL GROWTH:

THE CASE OF OPEN UNEMPLOYMENT

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## Income Distribution and Optimal Growth:

### The Case of Open Unemployment\*

#### INTRODUCTION

Normative questions relating to income equity have long dominated any discussion of income distribution. Of late, however, economists have focussed considerable attention on the relationship of income dispersion to general efficiency of resource allocation, e.g., [6], [11], and [15]. Two main issues have been raised. The first is the factor opportunity cost effect. It is contended that altering income dispersion in a specific direction will increase the output share of commodities with low factor opportunity costs. The demand for goods which are intense in the relatively abundant factors will rise and the demand for goods which are intense in the relatively scarce factors will decrease. The second issue concerns the belief that income distribution affects import demands. To the extent that a change in income dispersion will reduce aggregate import demand, the scarcity value of a limited supply of foreign exchange will be decreased.<sup>1</sup>

In addition to these two effects, there is the argument that a more even distribution of income causes a decline in the aggregate savings rate and thereby retards growth. This link between income distribution and economic growth is more controversial than the others. It depends entirely on a change in the demand for capital goods, whereas the foreign-exchange and factor-opportunity-cost effects include this and other changes in demand composition.

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\*We would like to thank Professors Kenneth J. Arrow and Richard A. Brecher for invaluable comments and criticism.

<sup>1</sup>For empirical evidence on these two effects in Latin American countries, see Cline [6].

The association of income distribution with the rate of capital formation is based on the assumption that the household marginal propensity to save varies with either the level or kind of income earned. These are points of considerable empirical and theoretical disagreement, as indicated in [7], [8], and [9]. Moreover, even if the marginal propensity to save did vary in the hypothesized manner, a reduction in private savings due to income equalization could be offset by increases in tax revenues and public savings. Under these conditions, a reduced dispersion of income will cause the social cost of savings to increase only if (a) government and private consumption are imperfect substitutes in their contribution to social welfare and (b) a change in the public-private consumption breakdown is necessary to keep aggregate savings constant.

For these reasons, the purpose of this paper is confined to analyzing the factor opportunity cost and foreign exchange effects of income distribution. The possibility of income distribution influencing the social cost of savings is examined in a companion paper [16]. The model we present attempts to fill a significant vacuum in the existing literature on the relationship of income distribution to economic growth, which includes [10], [17], and [20]. This literature generally involves closed economy models which assume full employment. Further, it is the distribution of income by kind (e.g., relative factor shares), rather than level, which affects output composition and capital formation in these models. Our model, built upon a foundation of conventional theory, represents a significant departure from growth literature in these areas. The closed economy assumption is abandoned, and the static relationships are derived as a simple extension of the type of neoclassical trade model described by Kemp [12]. The analysis is undertaken in the context of labor surplus economies where redistribution appears to be a particularly crucial

policy objective. Open unemployment is produced in the system by assuming that there is an institutionally-determined floor on the minimum real wage. With this constraint binding, investigation is made of the output and employment effects of changes in the composition of demand. These changes are induced by varying the dispersion of household expenditure.

These are static considerations. Dynamic elements are introduced into the model by equations determining capital accumulation and the change in capital dispersion over time. Our thesis is that the time paths of the capital distribution is more effectively controlled through income taxation than it is through other policies affecting factor payments such as import tariffs. Therefore, the coefficients of the household tax function make up the policy instruments in the model.

The optimal growth problem consists of maximizing an integral of instantaneous welfare subject to two dynamic equations and initial and terminal conditions on the capital-labor ratio and the distribution of capital. We assume that the main basis for differences in income among households is differences in the amount of capital owned. For this reason, commodity demands are implicit functions of capital dispersion, and the problem is designed to provide direct insight into the optimal trajectory of the standard deviation of the distribution of capital. Once this trajectory has been determined, along with that of the income tax schedule and the capital labor ratio, inferences may be drawn about changes in the standard deviation of disposable income. For reasons of simplicity, the term capital refers in most cases only to land and human capital; however, in analyzing our final results, we do consider the effect of modifying the definition to include human capital.

The contributions of this paper are the following:

- (i) We consider income dispersion by household rather than by income classes.
- (ii) We make no restrictions on the precise form of the income distribution other than it possess finite first and second moments.
- (iii) We are able to determine, under fairly general conditions, the consumption optimal growth paths of the distribution of the capital labor ratio, the employment rate, GDP per laborer, and the standard deviation of disposable income per laborer.

In section I, a static model is described, along with the effects of changes in the standard deviation of the distribution of capital and the capital-labor ratio on real income and employment. In section II, this static formulation is incorporated into a dynamic optimization model. Section III provides the derivation of first-order conditions and an appraisal of their policy implications. The last section discusses the extension of the analysis.

## I. THE STATIC MODEL

### 1. Commodity demand functions and savings

Denote consumption per laborer of the  $j$ th household by  $c^j$ , capital per laborer by  $k^j$ , and Wage income per laborer by  $\hat{w}^j$ . (All variables are deflated by the commodity 1 price index.) Then the function determining the consumption-labor ratio of the  $j$ th household may be written as

$$(1.1) \quad c^j = \alpha k^j + \hat{w}^j$$

$$0 \leq \alpha \leq 1$$

This function is consistent with a number of theories of consumption behavior. If it is assumed that the ratio of real cash balances to capital assets remains constant, then the relationship is similar to one proposed by Tobin [17]

which makes consumption proportional to real wealth. If, on the other hand, individual households save in order to maintain a fixed ratio of capital assets to normal income, then, given no adjustment lag,  $\alpha$  may be interpreted as the product of the reciprocal of this ratio and the marginal propensity to consume out of normal income. Wage income is untaxed and allocated completely to consumption expenditure. This assumption may be easily relaxed without qualitatively affecting our results, provided that the tax rate on wage income is constant.

By subtracting consumption per laborer from total household income per laborer, we obtain the function

$$(1.2) \quad s^j = y_D^j + \hat{v}^j - c^j \\ = y_D^j - \alpha k^j$$

where  $s^j$  is savings per laborer of the  $j$ th household and  $y_D^j$  is disposable capital income per laborer of the  $j$ th household, (both deflated by the commodity 1 price index). Denote aggregate domestic savings per laborer by  $s$ , disposable capital income per laborer by  $y_D$ , and capital intensity by  $k$ . Then taking the expected value of (1.2) yields the aggregate savings function

$$(1.3) \quad s = y_D - \alpha k$$

For the sake of simplicity, net foreign capital inflow (which may be easily incorporated by adding an intercept) is set equal to zero and the relationship

$$(1.4) \quad i = s$$

where  $i$  is gross investment per laborer, is assumed to hold as an identity.

We assume that there are two commodities, labelled 1 and 2. The impact of capital dispersion on economic variables in our two-good model depends critically on the form of the commodity demand functions. The function determining private consumption per laborer of commodity  $i$  may be written as

$$(1.5) \quad c_i = c_i(c, \sigma_c, P)$$

where  $\sigma_c$  is the standard deviation of distribution of household consumption per laborer, and  $P$  is the ratio of the price of good 2 to the price of good 1. An exact derivation of this function exists in the case of a quadratic demand function.<sup>1</sup> Suppose that the household demand function has the form

$$(1.6) \quad c_i^j = a_{i0} + a_{i1} c^j + a_{i2} (c^j)^2 + a_{i3} (c^j \cdot P) + a_{i4} P + a_{i5} P^2$$

where the  $j$  superscripts designate the value of consumption per laborer of the  $j$ th household. As Klein [14] has shown, taking expected values of this expression yields

$$(1.7) \quad c_i = a_{i0} + a_{i1} c + a_{i2} (c^2 + \sigma_c^2) + a_{i3} (c \cdot P) + a_{i4} P + a_{i5} P^2$$

By substituting the expression for private consumption per laborer, into the commodity demand function, (1.5) we obtain

$$(1.8) \quad c_i = c_i(\hat{w}, k, \sigma_c, P)$$

Consumption per laborer is a linear function of  $w$  and  $k$ , and every household is assumed to face the same wage rate. Therefore,  $\sigma_c$  may be obtained from the variance-co-variance matrix of the bivariate distribution for the capital-labor ratio and the employment rate. In the case where capital is defined to include human capital, the employment of particular household may well be an increasing function of its capital-labor ratio; skilled laborers not only tend to get laid off after unskilled laborers but they are also in a position

<sup>1</sup>The aggregate commodity demand function may also be derived from a household demand function of the form

$$c_i^j = A_i (c^j)^{\eta_{ic}} (P)^{\eta_{ip}}$$

where the exponents represent partial elasticities, which are assumed constant. In this case, the form of the distribution of expenditure per laborer must be restricted to be log normal. It can be shown that, under these assumptions, the function determining the aggregate value of  $c_i$  may be written as

$$c_i = e^{\mu/2} A_i (c)^{\eta_{ic}} (P)^{\eta_{ip}}$$

where

$$\mu = \eta_{ic} (\eta_{ic} - 1) \left(\frac{\sigma_c}{c}\right)^2$$



to bump unskilled laborers. In this situation, we would expect the variance of the employment rate to be increasing functions of the variance of capital. Clearly, the higher the aggregate employment rate, the less significance this effect will have.

With this reasoning in mind, the function for the standard deviation of private consumption per laborer can be written as

$$(1.9) \quad \sigma_c = \phi(e, \sigma)$$

where  $\phi_e < 0$ ,  $\phi_\sigma > 0$ , and  $\phi_{\sigma e} < 0$

Throughout much of the subsequent analysis, however, we shall consider only the simple case where capital does not include human capital and wage income per laborer is uniformly distributed. Since the expression for consumption per laborer is linear in  $k$ , we have

$$(1.11) \quad \sigma_c = \alpha \sigma$$

Under these conditions, the commodity demand function may be written in the form

$$(1.11) \quad c_i = c_i''(w, k, P, \sigma)$$

### Aggregate Output and Employment

The static part of the system is summarized by functions relating the aggregate employment rate and gross domestic product per laborer to the aggregate capital-labor ratio and a measure of capital dispersion. These functions represent the reduced form solution of a general equilibrium trade model.

In the version of the trade model presented here, two factors, capital and labor, are considered, as well as two commodities. It is assumed that the real wage expressed in terms of the labor intensive commodity remains constant at an exogenously-specified minimum and that the home country is incompletely specialized. Brecher [3, pp. 32-5] has demonstrated that, under these conditions,

the home offer curve is of the straight-line Ricardian variety (in the region of incomplete specialization).<sup>1</sup> This curve is represented by the line segment  $U_1 A_1 A_2 U_2$  in Figure 1, whereas the foreign offer curve, which has a conventional shape, is represented by OF. The intersection at point S gives the equilibrium level of imports and exports. (Commodity 2 is assumed to be the exportable and commodity 1 the import-competing good.) The equilibrium production point is shown to be point D in Figure 2, which is part of the straight-line transformation surface. With the real wage (expressed in terms of commodity 2) held fixed, we know by the Stolper-Samuelson theorem [18] that the slope of the commodity price line remains constant, provided that there is incomplete specialization. The Engel curve corresponding to the constant commodity price ratio and a specified value for the standard deviation of the distribution of capital,  $\sigma^1$ , is depicted as  $r_1 r_2$  in Figure 2. (The derivation of this curve from commodity demand functions is discussed in the Appendix.)

The offer triangle Dmd giving the equilibrium levels of imports and exports is constructed from a price line p-p which intersects the transformation surface at the equilibrium production point, D, and the Engel curve at point d. The dimensions of this triangle corresponds to those of the triangle OSJ shown in the offer curve diagram (Figure 1).

The rigid wage transformation surface, corresponding to the familiar Rybszynski line in trade theory, is made up of the locus of tangencies between the price line and the production possibility curves.<sup>2</sup> (The latter are based

<sup>1</sup>See also Brecher [9, Part I, Section C].

<sup>2</sup>The production possibilities curves are based on the assumption that the sector i production function is of the form

$$X_i = F^i(K_i, L_i)$$

$$F_{K_i}^i > 0, F_{L_i}^i > 0$$

where  $X_i$  is output,  $K_i$  is capital and  $L_i$  is employed labor in sector i. These functions are presumed to be homogenous of degree 1 in capital and labor and strictly quasi-concave. For a detailed derivation of the transformation surface from the production possibility curves, see Brecher [3, pp. 6-30].

Figure 1

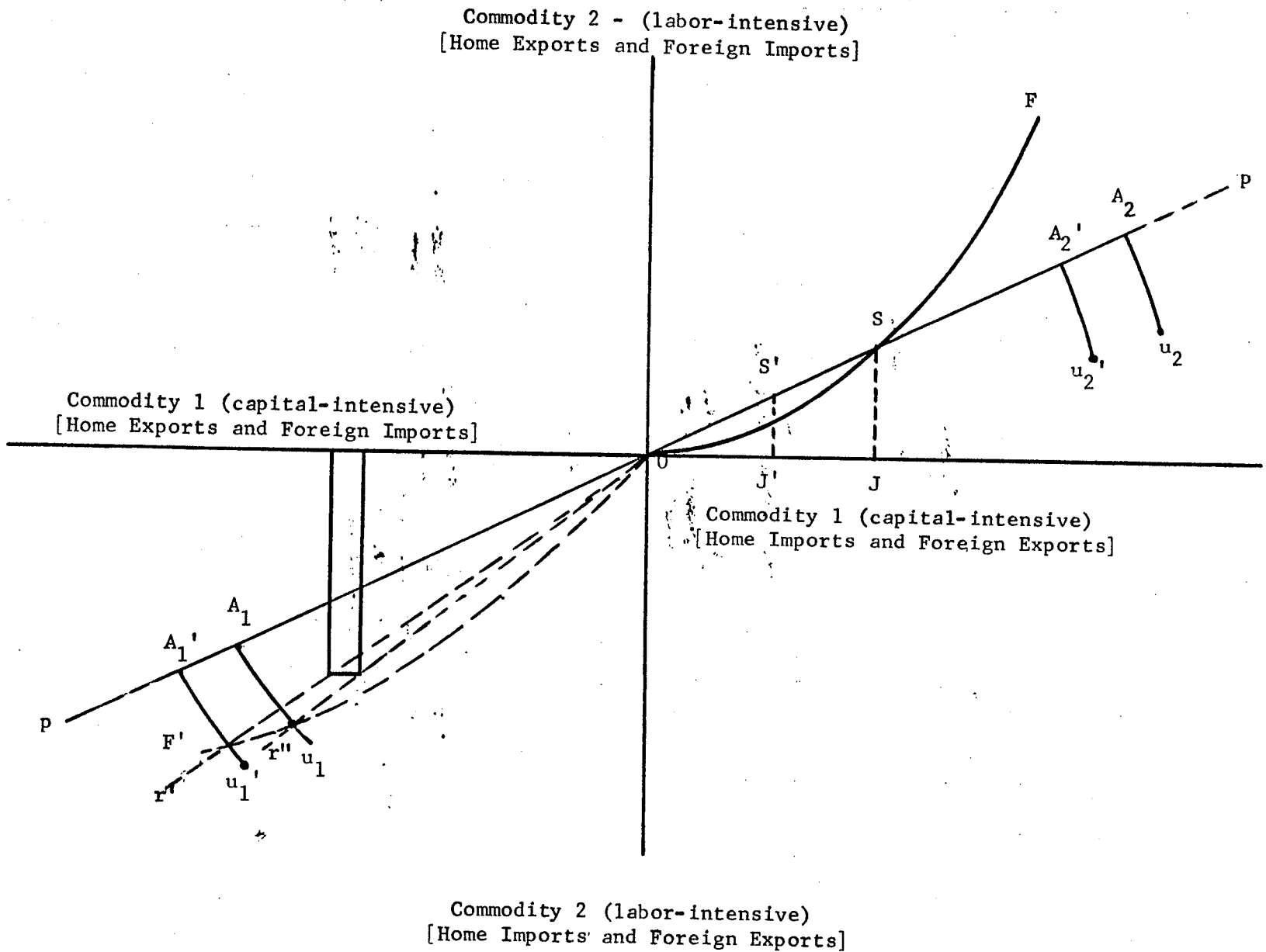
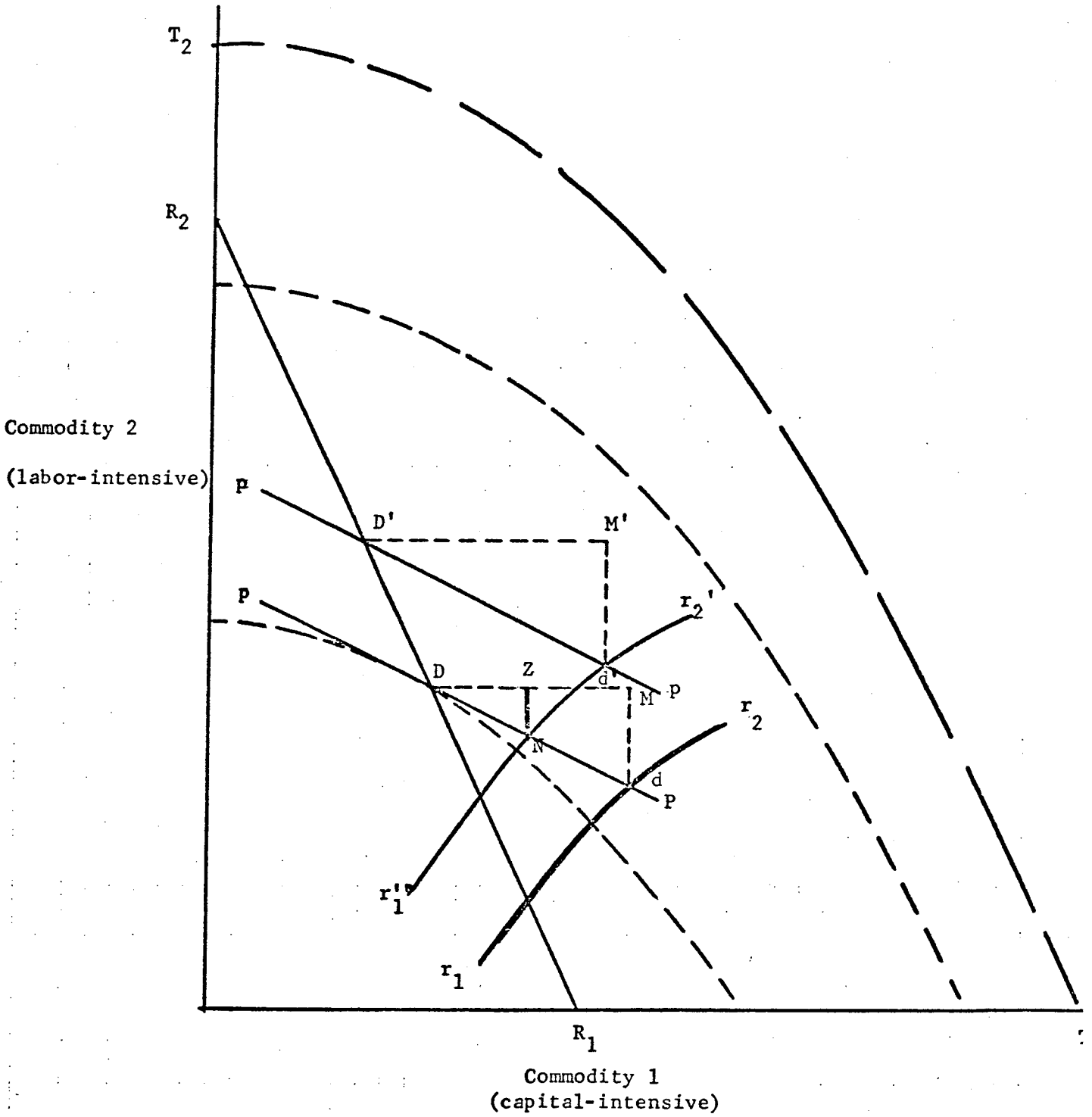


Figure 2



on the conventional assumption of flexible factor prices.) Each production possibility curve corresponds to a different level of employment; the full employment curve is depicted by  $T_1T_2$  in Figure 2. Since commodity 2 is assumed to have a lower capital-labor ratio than commodity 1 at all sets of factor prices, movements along the transformation surface from  $R_1$  to  $R_2$  correspond to increases in both employment and constant-price GDP.

Now consider a decline in the standard deviation of the distribution of capital from  $\sigma'$  to  $\sigma''$ . If we assume that the partial derivative of the commodity 1 (2) demand function with respect to  $\sigma$  is positive (negative), then such an equalization of capital holdings will cause the Engel curve to shift to the left. The level of wage income consistent with a fixed level of commodity 1 demand will increase, whereas the demand for commodity 2 will rise at all levels of wage income. If domestic production is held constant at point D on the transformation surface, the dimensions of the offer triangle will change from Dmd to DZN in Figure 2. This change implies a reduction in the quantity of good 2 exports. Consequently, there will be a disequilibrium in international markets, since the point on the home offer curve,  $S'$ , corresponding to the new offer triangle, DZN, will not intersect the foreign offer curve in Figure 1. Brecher has shown that such an excess demand for an exportable which is relatively labor intense will be cleared by increased domestic production of that commodity [3]. This takes the form of a leftward movement along the transformation surface from D to a new equilibrium point  $D'$ , implying an increase in both constant-price GDP and employment.

By solving such a system we may derive expressions for GDP per laborer ( $y$ ) and the employment rate ( $e$ ) which take the form

$$(1.12) \quad y = f(k, \sigma; \bar{w}_2)$$

$$(1.13) \quad e = g(k, \sigma; \bar{w}_2)$$

where

$k$  = capital-labor ratio

$\sigma$  = standard deviation of the distribution of capital

$\bar{w}_2$  = exogenously-specified real wage

The partial derivatives of (1.12) are linear transforms of those of (1.13).

Denote the real rental rate on capital (expressed in terms of commodity 1) by  $r_1$ . Then the expression for the value of GDP per laborer (expressed in terms of commodity 1) may be written as.<sup>1</sup>

$$(1.14) \quad y = w_1 e + r_1 k$$

where

$$w_1 = P \bar{w}_2$$

$$\text{and } r_1 = r_2 P$$

Since  $P$  and  $r_2$  are uniquely determined by  $\bar{w}_2$  (in the region of incomplete specialization), this is a linear function in  $e$  and  $k$  with constant coefficients.

From this result, it follows immediately that

$$(1.15) \quad \begin{aligned} f_{\sigma} & \begin{matrix} > \\ < \end{matrix} \sigma \text{ as } g_{\sigma} \begin{matrix} > \\ < \end{matrix} 0 \\ \text{and } f_{k} & \begin{matrix} > \\ < \end{matrix} r_1 \text{ as } g_k \begin{matrix} > \\ < \end{matrix} 0 \end{aligned}$$

For this reason, the effect of changes in the standard deviation of the distribution of capital on the employment rate is of particular interest to us. This effect depends on the sign of the partial derivative of the demand function for the import-competing good with respect to the standard deviation of the distribution of expenditure. In the Appendix, it is shown that in the region of incomplete specialization

$$(1.16) \quad g_{\sigma} \begin{matrix} > \\ < \end{matrix} \text{ as } \frac{\partial c_1}{\partial \sigma} \begin{matrix} < \\ > \end{matrix} 0$$

---

<sup>1</sup>It makes no difference which commodity price index is used to deflate GDP. Relative commodity prices are fixed, and the units of measurement may be chosen so as to make  $P$  equal unity without loss of generality.

That is to say, reducing (increasing) capital dispersion will cause the employment rate to increase (decrease) if and only if the aggregate demand for commodity 1 per laborer is an increasing (decreasing) function of the standard deviation of expenditure per laborer. Recall the relationship between the aggregate (1.7) and the household demand function (1.6). From this, it is clear that the qualitative effect of changes in capital dispersion on the employment rate is critically related to the properties of the household demand functions for commodity 1. In particular when this function is quadratic, the sign of the partial derivative of the aggregate demand function with respect to  $\sigma_c$  will be the same as the sign of the second partial derivative of the household demand function for commodity 1 with respect to expenditure per laborer.

The foreign exchange effect of a change in  $\sigma$  is neutral in the sense that condition (1.16) will hold even if the labor intensive good (commodity 2) is imported provided that the home country remains incompletely specialized. Under these conditions, the slope of the terms of trade line p-p in Figure 1 will not change; therefore, it is clear that the equilibrium exchange rate, which represents the opportunity cost of foreign exchange will be constant.<sup>1</sup> (The international price of the importable and the domestic price of the exportable are assumed to be fixed.)

The social rate of return on capital,  $f_k$ , does not equal the private rate of return on capital,  $r_1$ . An increase in  $k$  has a direct effect on  $y$  reflected in  $r_1$  and an indirect effect resulting from its influence on the

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<sup>1</sup>This is not true if the home country is completely specialized in commodity 1. In this case the foreign offer curve  $OF'$  intersects the domestic offer curve along the segment  $U_1 A_1$  in Figure 1, and movements in the home offer curve induced by changes in  $\sigma$  will affect the terms of trade. When an increase in  $\sigma$  causes  $P$  to fall, as is the case when the offer curve moves from  $U_1' A_1' A_2' U_2'$  to  $U_1 A_1 A_2$ , the wage rate expressed in terms of commodity 1 will fall.<sup>2</sup> These changes are associated with an increase in employment and a decline in the opportunity cost of foreign exchange.

employment rate. In the appendix, we derive necessary and sufficient conditions for

$$(1.17) \quad g_k > 0, \quad g_{kk} < 0, \quad \text{and} \quad g_{\sigma\sigma} < 0$$

which imply that

$$(1.18) \quad f_k > r_1, \quad f_{kk} < 0, \quad \text{and} \quad f_{\sigma\sigma} < 0$$

A positive value for the Hessian determinant of (1.13), when combined with the above conditions on the second partials of this function, implies that (1.12) and (1.13) are strictly concave. Assuming the household demand function has the quadratic form discussed earlier, the Hessian of (1.12) will always be positive if  $g_{kk}$  and  $g_{\sigma\sigma}$  are both negative. These conditions on the second partials of (1.13) require that the second expenditure partial of the household demand function for good 1, and hence  $\partial c_1 / \partial \sigma$ , be positive.

But, if this requirement is met, then by (1.16)

$$(1.19) \quad g_{\sigma} < 0 \quad \longrightarrow \quad f_{\sigma} < 0.$$

Strict concavity not only confines us to the case where equalization of capital holdings increases output and employment; it also violates a necessary condition for  $f(k, \sigma)$  to have an interior maximum. It is still, however, perfectly legitimate for us to consider a corner maximum where condition (1.19) is met and  $\sigma$  is determined by its lower bound. Consequently, our dynamic analysis will be confined to the case where  $g_{\sigma}$  and hence  $f_{\sigma}$  are negative for all feasible values of  $k$  and  $\sigma$ .

## II. THE DYNAMIC MODEL

Thus far, we have shown the direction of influence of  $k$  and  $\sigma$  on GDP per laborer and the employment rate. But this analysis, pertaining only to a static situation does not provide insight into the mechanism by which changes in  $k$  and  $\sigma$  take place over time. More specifically, intemporal



relationships between these variables and the policy instruments designed to influence them remain to be formulated.

### The Dynamics

The net change in the capital-labor ratio of the individual household is determined by its gross saving per laborer less the change in the capital labor ratio due to depreciation and population growth alone. Consequently, since household saving per laborer is given by (1.2) and fixed rates of depreciation and population growth are assumed, we may write

$$(2.1) \quad \dot{k}^j = y_D^j - (\alpha + n + \delta)k^j$$

where  $\dot{k}^j$  is the time derivative of  $k^j$ ,  $n$  is the rate of population growth, and  $\delta$  is the rate of depreciation.

Denote taxes net of subsidies levied on the  $j$ -th household by  $NT^j$  and capital income per laborer gross of taxes by  $y_K^j$ . Now as

$$(2.2) \quad y_D^j = rk^j - NT^j$$

we have

$$(2.3) \quad \dot{k}^j = (rk^j - NT^j) - (\alpha + n + \delta)k^j$$

One possible tax function is

$$(2.4) \quad NT^j = a_0'' + a_1'' (y_K^j - y_k)$$

which can be written as

$$(2.5) \quad NT^j = a_0'' + a_1'' \cdot r_1 \cdot (k^j - k)$$

The coefficient  $a_0''$  determines the revenue impact of the tax and the coefficient  $a_1''$  determines the re-distributive effect of the tax. Substituting (2.5) into (2.3) yields

$$(2.6) \quad \dot{k}^j = r_1 \left( k^j - \frac{a_0''}{r_1} - a_1'' (k^j - k) \right) - (\alpha + n + \delta) k^j$$

Let  $a_0 = \frac{a_0''}{r_1}$  and  $a_1'' = a_1$  and  $a_1'' = a_1$

then using

$$(2.7) \quad \dot{k} \stackrel{\Delta}{=} E(k^j)$$

$$(2.8) \quad \dot{\sigma} \stackrel{\Delta}{=} E(\dot{\sigma}^j)$$

and

$$(2.9) \quad \dot{\sigma} \stackrel{\Delta}{=} \frac{1}{\sigma} E(k^j - k) \cdot \frac{d}{dt} (k^j - k)$$

we have

$$(2.10) \quad \dot{k} = r_1 (k - a_0) - (\alpha + n + \delta) k$$

$$(2.11) \quad \dot{\sigma} = r_1 (1 - a_1) \sigma - (\alpha + n + \delta) \sigma$$

We define the control variable  $u_2$  by the equation

$$(2.12) \quad 1 - a_1 = u_2$$

The other control variable is given by the relationship

$$(2.13) \quad a_0 = u_1 = \frac{a''_0}{r_1}$$

Note that  $u_1$  is only a pseudo-control variable. In fact, it is the intercept of the net tax function (i.e., the expected net tax  $a''_0$ ) which the government controls, not its capitalized value.

The complete dynamics are then

$$(2.14) \quad \dot{k} = r_1 (k - u_1) - (\alpha + n + \delta) k$$

$$(2.15) \quad \dot{\sigma} = [r_1 u_2 - (\alpha + n + \delta)] \sigma$$

These differential equations are characterized by the lack of any assumption regarding the underlying distribution function beyond its having finite first and second moments.

The form of equation (2.15) indicates that the time path of  $\sigma$  depends only on  $u_2$ . Other policies affecting the employment rate (e.g., an export subsidy) will not influence the time path of  $\sigma$  provided that the parameters of the net tax and savings functions are unchanged. At this point, it is important to note that (2.14) and (2.15) are derived under the assumption of zero

marginal propensity to save out wage income, together with the uniform distribution of human capital (which is excluded from total capital). The  $\sigma$  equation is still valid under the assumption of a non-zero marginal propensity to save out of wage income, provided that this income is evenly distributed. Alternatively, we could assume that human capital is included in total capital and that wage income is unevenly distributed due to variation in the employment rate. Under these circumstances, it is legitimate to assume that the marginal propensity to save out of unskilled wage income is zero and equation (2.15) remains intact. (The effects of combining human and physical capital are considered in further detail in section III.)

#### The Criterion Function

Denote consumption (public and private) per laborer by  $c$ , and the total labor force by  $L$ . Instantaneous welfare,  $U$ , is given by the function

$$(2.16) \quad U = L u(c)$$

The function  $u(c)$  determines aggregate utility per laborer. Since the commodity price ratio is fixed, private consumption (expressed in terms of commodity 1) may be treated as a single good. It is assumed that public and private consumption expenditure are perfect substitutes yielding identical marginal benefits to the households.

The possible effects of changes in expenditure distribution on social welfare are not taken into account in this function. The inclusion of  $\sigma_c$  as well as  $c$  in the social welfare function could be justified on the basis that the social welfare function represents an aggregation of individual utility functions. But this approach involves the usual pitfalls associated with cardinal utility and the assumption that utility is divisible [13].

Further, a precise aggregation of household utility functions would require that moments of the expenditure distribution higher than the second be included in the social welfare function.<sup>1</sup> Finally, the only simple alternative to the utility aggregation approach--a preference ordering for  $c$  and  $\sigma_c$  obtained by voting--seems to have equal, if not greater, defects [2].

The justification for the simple welfare function (2.16) goes beyond the fact that a more general function may be difficult to derive or analytically intractable. When concentrating on the efficiency aspects of capital and income re-distribution, it seems reasonable to assume that distributional considerations by themselves do not influence social choice.<sup>2</sup>

The criterion function itself may be written as

$$(2.17) \quad \int_0^T e^{-\gamma t} U(c) dt = \int_0^T e^{-\rho t} e^{nt} L_0 u(e) dt$$

where  $\rho$  is the rate of social discount,  $n$  is the rate of population growth,  $\gamma$  is the discount rate net of population growth,  $T$  is the planning horizon, and  $L_0$  is the initial labor force. Thus the paths of optimal capital accumulation and distribution are given by the solutions of the following optimal control problem:

$$(2.18) \quad \text{Max}_{u_1, u_2} \int_0^T e^{-\gamma t} U(c) dt$$

---

<sup>1</sup>The appearance of the second moment alone is justified only in the case where the household utility function is quadratic. Contrary to the assumptions of our model, the demand functions implied by such a utility function are linear in expenditure.

<sup>2</sup>In the special case where lump-sum transfers of consumer goods can be effected, an increase in  $c$  may be interpreted as making everyone better off.

where

$$(2.19) \quad \dot{k} = r_1 (k - u_1) - (\alpha + n + \delta) k$$

$$(2.20) \quad \dot{\sigma} = [r_1 u_2 - (\alpha + n + \delta)] \sigma$$

and the constraints

$$(2.21) \quad k \geq 0, \sigma \geq 0$$

$$(2.22) \quad 0 \leq u_1 \leq \frac{(r_1 - \alpha)}{r_1} k$$

$$(2.23) \quad A \leq u_2 \leq B$$

The upper bound on  $u_1$  reflects the fact that gross investment per laborer cannot be less than zero. The constraints on  $u_2$  indicate politically-determined upper and lower bounds on the re-distribution coefficient of the tax function.

### III. EQUILIBRIUM GROWTH PATHS

The above optimal control problem is linear in the controls  $u_1$  and  $u_2$ ; thus the optimal policies will be of the "bang-singular-bang" type [5, pp. 261-65], i.e., the controls will move between their boundary values and an interior value (s) corresponding to the singular arc (s).

We shall consider only the case where for all feasible values of  $k$  and  $\sigma$

$$f_{\sigma} (k, \sigma) < 0$$

since in the other cases the function  $f(k, \sigma)$  is not concave. It will now be shown that this first partial derivative of this function with respect to  $\sigma$  will be negative in the case of a steady-state optimum only if  $\sigma$  is equal to its lower bound.

Suppose that an interior maximum exists. Denote the costate variables corresponding to  $k$  and  $\sigma$  by  $\lambda k$  and  $\lambda \sigma$  respectively. Then, on the singular arc, when  $H_u = 0$ , the necessary conditions for optimality are derived from a Hamiltonian of the form

$$(3.1) \quad H = e^{-\gamma t} [U(c) + \lambda_k \dot{k} + \lambda_\sigma \dot{\sigma}]$$

The form of this function indicates that the costate variables may be interpreted as the imputed values (shadow prices) of increases in  $k$  and  $\sigma$ , measured in terms of utility. From (3.1), we derive

$$(3.2) \quad H_{u_1} = 0 = U_c \cdot r_1 - r_1 \lambda_k$$

$$(3.3) \quad H_{u_2} = 0 = r_1 \sigma \lambda_\sigma$$

and

$$(3.4) \quad -\dot{\lambda}_k = U_c (f_k - r_1 + \alpha) + \lambda_k (r_1 - \alpha - n - \delta - \gamma)$$

$$(3.5) \quad -\dot{\lambda}_\sigma = U_c (f_\sigma) - \lambda_\sigma (\alpha + n + \delta + \gamma)$$

In addition, we have the dynamics which are given by equations (2.14) and (2.15). In equilibrium,  $\dot{k} = \dot{\sigma} = \dot{\lambda}_k = \dot{\lambda}_\sigma = 0$ . This gives

$$(3.6) \quad u_1 = \left[ 1 - \frac{(\alpha + n + \delta)}{r_1} \right] k$$

and

$$(3.7) \quad u_2 = \frac{n + \delta + \alpha}{r_1}$$

Since  $r_1$  and  $\sigma$  are positive, the shadow price of  $\sigma$  is given by

$$(3.8) \quad \lambda_\sigma = 0$$

also,

$$(3.9) \quad \lambda_k = U_c$$

The two remaining variables are  $k$  and  $\sigma$ . These can be obtained from (3.4) and (3.5) giving

$$(3.10) \quad f_k(k, \sigma) = n + \delta + \gamma$$

$$(3.11) \quad f_\sigma(k, \sigma) = 0$$

As long as  $n$  and  $\delta$  are non-negative, the upper-bound on  $u_2$  will not be exceeded at the equilibrium point. To insure that the lower-bound restriction on  $u_1$  is met, the inequality

$$r_1 > n + \delta + \alpha > 0$$

must hold.<sup>1</sup> This also implies that the steady-state equilibrium value of  $u_2$  will be positive but less than one. Such an interval may well represent the limits of political acceptability. It means that the tax will have some redistributive effect, but precludes an income-distribution reversal, (i.e., an inverse relationship between pre-tax and after-tax income). We assume that open unemployment exists along the optimal trajectory; otherwise, (1.12) would be a discontinuous function and thereby violate the regularity conditions of the optimal control problem. If the full-employment constraint is never binding, condition (3.10) will be met either if

(a) commodity 2 is sufficiently inferior at the equilibrium point to make  $g_k(k, \sigma)$  negative,<sup>2</sup> or if

(b) the inequality

$$r_1 < n + \delta + \gamma = f_k(k, \sigma)$$

holds. The latter condition implies that

$$\gamma > \alpha$$

and that the net marginal product of capital in the private sector ( $r_1 - \delta$ ) be less than the social rate of discount.<sup>3</sup>

Since condition (3.11) cannot hold, an interior maximum is impossible. This leaves us only with a boundary maximum. It can be shown that as  $t$  approaches infinity,  $\sigma$  will approach its lower bound along the optimal trajectory and  $k$  will be determined by the modified golden rule condition (3.10). That there is an asymptotic turnpike associated with these values of  $k$  and  $\sigma$

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<sup>1</sup>A negative  $u_1$  may not be unrealistic in that it means that a steady state optimum can be attained **only** if there are "forced savings." If the model is modified to include positive net foreign capital inflow, the above condition is sufficient but not necessary for  $u_1$  to be non-negative.

<sup>2</sup>See appendix for the relationship between inferiority and the sign of  $g_k(k, \sigma)$ .

<sup>3</sup>Leland [22] has shown that, with a certain type of risk aversion, there will be a precautionary demand for savings. His work indicates that it is possible to have positive savings when the household, as well as the social, rate of time preference is less than the net private returns on capital. Further, even after allowing for a reasonable risk premium, this return may be high relative to the rate of time of preference of individual foreign lenders.

may be demonstrated as follows. Note that the equation determining  $\sigma$  may be integrated directly as

$$\frac{d\sigma}{\sigma} = [r_1 u_2 - \alpha - u - \delta] dt$$

and  $r_1$  is assumed to be constant. Integration indicate that if  $\sigma$  approaches zero, then  $u_2$  approaches negative infinity provided that the time horizon is less than infinite. But  $u_2$  is bounded from below by B. Therefore,  $\sigma$  must approach zero asymptotically.

By comparing the asymptotic turnpike values of  $k$  and  $\alpha$  with the initial conditions, the direction of change of these variables along the optimal trajectory may be established. Initial and steady-state optimal values of GDP per laborer and the employment rate may also be calculated from (3.10) and the lower bound on  $\alpha$ . As yet, however, we have not presented a method for determining the standard deviation of the distribution of income.



The standard deviation of the distribution of disposable income per laborer ( $\sigma_y$ ), in a steady-state equilibrium is a linear function of  $\sigma$ . For, from (1.25), we see that the total disposable income (which is the sum of income from capital  $y_D$ , and wage income  $w$ ) is given by

$$\text{Total disposable income} = r \cdot k + w - \text{Expected net tax.}$$

Then, noting the fact that wage income per laborer is uniformly distributed, we see that

$$(3.12) \quad \sigma_y = (1 - a_1) \sigma r_1 = u_2 \cdot \sigma \cdot r_1$$

Then, from (3.7),

$$(3.13) \quad \sigma_y = (n + \delta + \alpha) \sigma$$

The direction of movement of key target variables in the model along the optimal trajectory is illustrated in Figure 3. Level curves are plotted representing the different combinations of  $k$  and  $\sigma$  which will yield the values of the employment rate, GDP per laborer, and the standard deviation of income per laborer existing at the equilibrium point. Since the values of  $\sigma$  are plotted on the vertical axis and the values of  $k$  on the horizontal axis, the curve corresponding to the equilibrium  $\sigma_y$  is a horizontal line  $vv'$  coinciding with the horizontal axis. (Here it is assumed that the lower bound for  $\sigma_y$  is zero.) The curve corresponding the equilibrium value of GDP per laborer,  $yy'$ , clearly has a steeper slope than the one corresponding to the equilibrium employment rate,  $ee'$ .<sup>1</sup> If the initial values of  $k$  and  $\sigma$  are in

<sup>1</sup>The slope of the level curve for equilibrium GDP per laborer is given by the expression

$$d\sigma/dk = \dots \frac{g_k + r_1/w_1}{g_\sigma}$$

The absolute value of the right hand side of this expression is greater than the ratio  $g_k/g_\sigma$ . This ratio equals the absolute value of the slope of the level curve corresponding to a constant employment rate.

Figure 3

standard deviation of the distribution of the capital-labor ratio ( $\sigma$ )

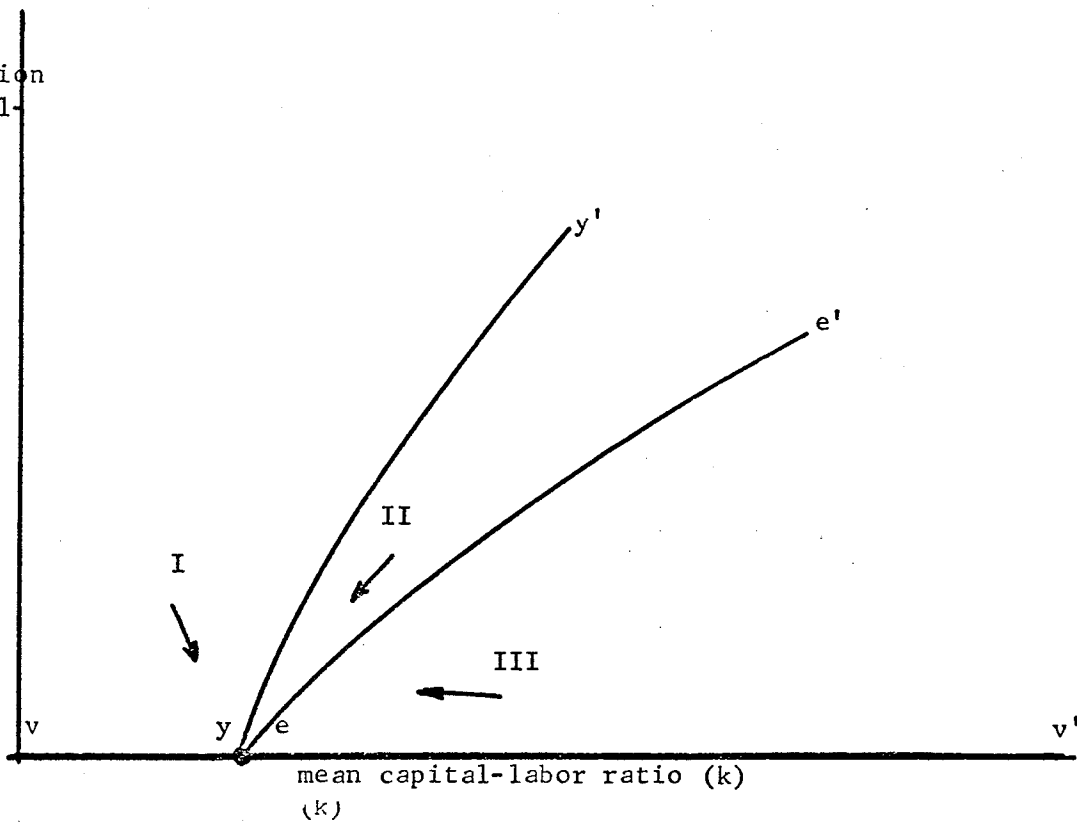
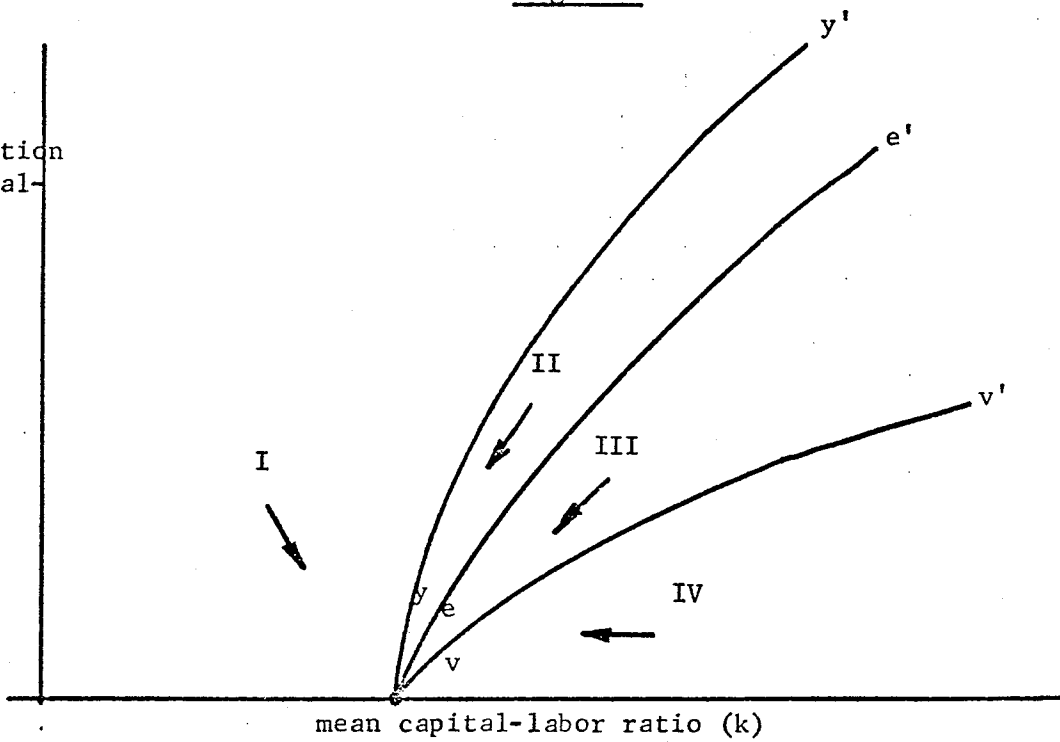


Figure 4

standard deviation of the distribution of the capital-labor ratio ( $\sigma$ )



region I, which lies above the  $yy'$  curve, then both GDP per laborer and the employment rate will increase as the equilibrium point is approached. Initial values of  $k$  and  $\sigma$  which lie between the two curves in region II represent a case in which there is a decline in GDP per laborer and an increase in the employment rate along the optimal trajectory. This is attributable to a decline in the capital labor ratio which has a greater effect on GDP per laborer than it does on the employment rate. If the initial values of  $k$  and  $\sigma$  lie in region III, then both GDP per laborer and the employment rate will decline along the optimal trajectory.

Defining capital to include human capital does not alter the qualitative features of the optimal growth trajectory a great deal. Under these conditions, it is no longer legitimate to assume that a household's employment rate is independent of its capital-labor ratio. As already noted in Section I, the variance and co-variance parameters of the bivariate distribution of the capital-labor-ratio and the employment rate depend on  $\sigma$  and  $e$ .

and the same return on capital,  $r_1$ , the standard deviation of the distribution of income per laborer is determined by the parameters of this distribution and  $u_2$ . Consequently,  $\sigma_y$  is an implicit function of  $\sigma$  and  $e$ , and we may write

$$(3.14) \sigma_y = \psi(\sigma, e, u_2)$$

In steady-state equilibrium we have

$$(3.15) \sigma_y = \psi^*(\sigma, e)$$

since  $u_2$  may be taken as a constant. Because the employment advantage of a household with a high relative capital-labor ratio diminishes with an increase in  $e$ ,

$$\psi_e^* < 0 \quad \text{and} \quad \psi_\sigma^* > 0.^1$$

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<sup>1</sup>For empirical evidence of an inverse relationship between income dispersion and the aggregate employment rate, see Schultz [21].

Given that the first partials of these functions have the same sign as those derived in Section I, for the case of incomplete specialization, then (3.15) may be written as

$$(3.16) \quad \sigma_y = \Psi^{**}(\sigma, k)$$

where

$$\begin{aligned} \Psi_{\sigma}^{**} &> 0 \\ \Psi_k^{**} &< 0 \end{aligned}$$

The  $vv'$  curve in this case is shown in Figure 4. This curve is no longer perfectly horizontal, as it was in the case where  $\sigma_y$  depended on  $\sigma$  alone. Instead, it now has a positive slope. If the initial values of  $k$  and  $\sigma$  lie below this curve, in region IV, the standard deviation of the distribution of income will increase along the optimal trajectory even though  $\sigma$  declines. In all the other regions, however, the dispersion measures will change in the same direction.

#### IV. CONCLUSIONS

In this paper, we have developed (1) a static model showing the relationship between the distribution of the capital labor ratio, the aggregate employment rate and GDP per laborer; and (2) a system of dynamic equations determining the inter-temporal behavior of the capital distribution. Under certain conditions, we have derived the consumption optimal long run behavior of this distribution, the output and employment variables, and a measure of the dispersion of per-capita disposable income. In this section, we examine some of the implications of our results in greater detail.

1. The optimal growth problem is restricted to the good-things-coming-together case where GDP per laborer and the employment rate

are decreasing functions of the standard deviation of the capital-labor ratio,  $\sigma$ . Under these conditions,  $\sigma$  asymptotically approaches its lower bound along the optimal trajectory. The optimal trajectories of  $k$  and  $\sigma$  are uniquely-determined by the first-order condition only in this boundary-maximum situation. Nonetheless, it is still possible to examine alternative time paths even if output and employment are increasing functions (or independent) of  $\sigma$ . While optimization presents difficulties in this case, we may determine the  $k$  and  $\sigma$  trajectories associated with given instrument paths, from the dynamic equations, (2.10) and (2.11). From these trajectories, the effect of specified changes in the parameters of the net tax function on such target variables as the employment rate, income distribution and consumption may be analyzed.

2. It may be argued that the objectives involving these variables will be achieved by taxes and subsidies on foreign trade as well as income taxation.<sup>1</sup> Assuming that the Metzler paradox conditions do not hold, a tariff reduction (or export subsidy) will shift the offer curve to the left and increase the output share of the exportable commodity. See [3, Chapter 8] and [4, Part 4]. With  $k$  and  $\sigma$  constant, this change will increase the employment rate and GDP (measured at constant domestic prices). However, even though commercial policy has the same effect on these variables as a change in  $\sigma$  the impact on income distribution and GDP measured at world prices may be considerably different. From (3.14) and (3.15), it is clear that  $\sigma$  effects income distribution directly as well as through its influence on the employment rate. By contrast, tariff policy can decrease the standard deviation of the distribution of income

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<sup>1</sup> Another possible alternative to income taxation is the wage subsidy. However, given that this is an effective policy instrument, it is not clear why open unemployment should exist at all.

only in the case where his parameter depends on the employment rate, Otherwise, it has no effect. A tariff reduction (or export subsidy) unlike a change in  $\sigma$  will always cause the terms of trade to shift against the home country; hence, its effect on GDP measured at world prices is ambiguous. For these reasons a combination of capital re-distribution and tariff policies may be warranted when the employment rate is a decreasing function of  $\sigma$ . In this case, it may be desirable to improve the terms of trade short of the standard optimal tariff point or at least use tariff policy to prevent highly adverse terms of trade change.<sup>1</sup> The resulting decline or lack of growth in employment could be offset by increasing the re-distribution coefficient of the income tax function and thereby decreasing  $\sigma$ . Of course, if the employment rate is an increasing function of  $\sigma$ , alterations in the distribution of capital per laborer cannot be used to increase the flexibility of tariff policy, without conflicting with a goal of greater equity.

3. Our model may be easily modified to include the effects of once-and-for-all changes in production function parameters. These changes are particularly important in the case where the lower bound on  $\sigma$  is positive due, say, to some persons being uneducable. Under these conditions, the parameter variations will have a significant impact on the long-run optimal income distribution. This impact may well be considerably different from that obtained by comparative static analysis with  $k$  and  $\sigma$  constant. For example, such analysis indicates that, under certain conditions, a Hicks-neutral productivity

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<sup>1</sup>Standard optimal tariff theory, e.g., Kemp [12, pp. 296-363], is not applicable in this case since it is based on the assumption of full employment.

increase in the capital-intensive sector will cause the employment rate to decline [3, chapter VI]. Given (3.14), this implies a more uneven distribution of income. On the other hand, the asymptotic turnpike values of these variables may change in the opposite direction if the increase in productivity (and hence  $r_1$  and  $w_1$ ) is associated with a rise in the optimal value of  $k$ .

## Appendix

### Notation:

$P$  = the ratio of the price of commodity 2 to the price of commodity 1.

$w_i$  = the real wage (expressed in terms of commodity i)

$r_i$  = rental rate on capital (expressed in terms of commodity i)

$w$  = wage income per laborer expressed in terms of commodity 1.

$e$  = the employment rate

$k$  = the capital labor ratio

$\sigma$  = the standard deviation of the capital labor ratio

$M_{1c}$  = the marginal propensity to consume commodity 1 out of total expenditure.

$B_{1e}$  = the first commodity's share in the general expansion of output due solely to an increase in the employment rate

$B_{1k}$  = the first commodity's share in the general expansion of output due solely to an increase in the capital labor ratio

$M_{1cc}$  = the second partial derivative of the commodity 1 demand function with respect to private consumption per laborer.

$M_{1\sigma\sigma}$  = the second partial derivative of the commodity 1 demand function with respect to  $\sigma$

$M_{1c\sigma}$  = the cross partial derivative of the commodity 1 demand function with respect to  $c$  and  $\sigma$ .

$L$  = the specified labor force in the home country

$L^*$  = the specified labor force in the rest of the world.

Final demand per laborer is equal to the sum of private consumption (c), public consumption (g), and private investment (i) per laborer. Since the latter two components are assumed to affect only commodity 1,<sup>1</sup> and the relationship

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<sup>1</sup>This assumption may be relaxed. Allocating a fixed proportion of (g + i) to a non-traded goods sector (including construction and government services) will not change our qualitative results.



$$(A.1) \quad g + i = (r_1 - \alpha) k$$

holds, the functions determining final demand per laborer in the two sectors may be written in the form

$$(A.2) \quad y_1 = c_1''(k, \sigma, \hat{w}, p) + (r_1 - \alpha) k$$

$$(A.3) \quad y_2 = c_2''(k, \sigma, w, p)$$

Sectoral final demands per laborer are implicit function of the employment rate, since  $\hat{w}$  is given by

$$(A.4) \quad \hat{w} = w_1 e$$

With  $p$ ,  $k$ , and  $\alpha$  held constant,  $y_1$  is uniquely determined by  $y_2$  through the employment rate variable; hence, the modified Engel curve, shown in Figure 2 of the text, may be derived.

Recall that commodity 1 is assumed to be the import-competing good. The output of this commodity per laborer,  $x_1$ , depends on the commodity price ratio, the employment rate, and the capital-labor ratio. Net imports of commodity 1 per laborer,  $z_1$ , are given by the expression

$$(A.5) \quad z_1 = y_1 - x_1 = c_1''(k, \sigma, \hat{w}, 1/p) + (r_1 - \alpha) k - x_1(1/p, e, k)$$

By substituting the expression for  $\hat{w}$  into this relationship, we obtain

$$(A.6) \quad z_1 = z_1(w_2, 1/p, k, \sigma, e)$$

The balance of payments condition may be written as

$$(A.7) \quad Lz_1(\bar{w}_2, 1/p, k, \sigma, e) - L^* pZ_2^*(p) = 0$$

where the function  $z_2^*(p)$  determines the rest of the world's net imports of commodity 2 per laborer.<sup>1</sup>

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<sup>1</sup>This formulation is a simple extension of Kemp's [12, chapter 4] and <sup>(his general)</sup> approach to comparative static analysis applies here. We have retained his assumptions and notation as much as possible.

It can be shown that the endogenous variables in this equation will be unaffected by labor force growth, provided that the ratio of  $L^*$  to  $L$  remains fixed.

By differentiating this expression totally, we obtain expressions for the partial derivatives of the employment rate with respect to  $k$  and  $\sigma$ :

$$(A.8) \quad \frac{\partial e}{\partial k} = \frac{-r_1 (z - B_{1k})}{w_1 (M_{1c} - B_{1e})}$$

$$(A.9) \quad \frac{\partial e}{\partial \sigma} = \frac{-M_{1\sigma}}{w_1 (M_{1c} - B_{1e})}$$

where  $z = \frac{\alpha M_{1c}}{r_1} + \frac{(r_1 - \alpha)}{r_1}$

Consequently, provided that the marginal propensity to consume commodity 1 out of total expenditure  $M_{1c}$ , is non-negative but less than unity, the partial derivative of the employment rate with respect to the capital-labor ratio ( $\partial e / \partial k$ ) will be positive.<sup>1</sup> Moreover, given this type of non-inferiority,

$$\frac{\partial e}{\partial \sigma} > 0 \quad \text{as} \quad \frac{\partial c_1}{\partial \sigma} = M_{1\sigma} < 0$$

since  $(M_{1c} - B_{1e})$  is negative.

The higher order partial derivatives take the form

$$(A.10) \quad \frac{\partial^2 e}{\partial k^2} = \frac{-[\alpha^2 M_{1cc} \Delta - w_1 \alpha M_{1cc} r_1 (z - B_{1k})]}{\Delta^2}$$

$$(A.11) \quad \frac{\partial^2 e}{\partial \sigma \partial k} = \frac{-[\alpha M_{1c\sigma} \Delta - w_1 M_{1c\sigma} r_1 (z - B_{1k})]}{\Delta^2}$$

$$(A.12) \quad \frac{\partial^2 e}{\partial \sigma^2} = \frac{-[M_{1\sigma\sigma} \Delta - w_1 (M_{1c\sigma}) M_{1\sigma}]}{\Delta^2}$$

If the household demand function is quadratic in expenditure, the cross partial derivative  $M_{1c\sigma}$  will be zero. See (1.6). This implies that the cross partial derivative  $\partial^2 e / \partial k \partial \sigma$  will be zero. Consequently, if the coefficient for squared expenditure in the household demand function is positive,

<sup>1</sup>It can be shown that  $z$  in equation (A.8) is less than one if and only if  $M_{1c}$  is less than one, and Kemp [12, p. 110] has proven that

$$B_{1k} > 1 \quad \text{and} \quad B_{1e} < 0.$$

when commodity 1 is relatively capital intensive.

then the Hessian for the function determining the employment rate will be negative definite. The second partials  $M_{lcc}$  and  $M_{l\sigma\sigma}$  will both be positive and therefore

$$(A.13) \quad \partial^2 e / \partial \sigma^2 < 0 \text{ and } \partial^2 e / \partial k^2 < 0, \text{ since } (z - B_{lk}) < 0$$

Further, given that  $\partial^2 e / \partial k \partial \sigma = 0$ , the condition

$$(A.14) \quad (\partial^2 c / \partial k^2) \cdot ( (\partial^2 e / \partial \sigma^2) - (\partial^2 e / \partial k \partial \sigma) (\partial^2 e / \partial \sigma \partial k) ) > 0$$

will hold.

These conditions will not hold at a point where

$$(A.15) \quad \partial e / \partial \sigma = 0 \text{ and } M_{l\sigma} = 0$$

In the case of the quadratic demand function described, such zero partials imply

$$(A.16) \quad M_{lcc} = M_{l\sigma\sigma} = 0$$

At this point, the value of  $\sigma$  which maximizes the employment rate is not unique, since the Hessian vanishes when the first-order condition (A.15) is met.

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